A Note on Discounting and Funding Value Adjustments for Derivatives

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Abstract: In this paper, valuation of a derivative partially collateralized in a specific foreign currency defined in its credit support annex traded between default-free counterparties is studied. Two pricing approaches – by hedging and by expectation – are presented to obtain the same valuation formulae. Our findings show that the current marking-to-market value of such a derivative consists of three components: the price of the perfectly collateralized derivative (a.k.a. price by collateral rate discounting), the value adjustment due to different funding spreads between the payoff currency and the collateral currency, and the value adjustment due to funding requirements of the uncollateralized exposure. These results generalize previous works on discounting for fully collateralized derivatives and on funding value adjustment for partially collateralized or uncollateralized derivatives.

Keywords: CSA, collateral, foreign collateral, derivative pricing, hedging, martingale pricing, FVA, funding cost, funding and discounting

1 Introduction

The impact of collateralization to valuation of over-the-counter (OTC) derivatives is well recognized and observed in the market, in particular when the borrowing rate of the derivative desk is significantly higher than the return rate of the collateral (a.k.a. *collateral rate*) designated in its credit support annex (CSA) since the recent credit and liquidity crunch. The conventional LIBOR-OIS[1] spread is usually regarded as an indicator of such a gap. This impact has been extensively investigated in practice and in theory (e.g., [1, 2, 3]). As a consequence, the approach to discounting projected cashflows with the collateral rate, a.k.a. *collateral rate discounting* or *CSA discounting*, is addressed. Collateral rate discounting for a derivative with its payoff in a single currency, however, implies several model assumptions [2, 3], including:

1. Full collateralization, i.e., the posted collateral amount equals to the marking-to-market (MtM) of the derivative;
2. Bilateral collateralization with the same collateral rate for both counterparties, i.e., each counterparty posts collateral when the derivative has a negative MtM from its view (out of the money) and receives the same return rate on the collateral;
3. Continuous settlement, i.e., the collateral adjustment is settled immediately when MtM changes;
4. Domestic collateralization, i.e., collateral in the same currency as that of the derivative payoff;
5. Cash-equivalent collateral, i.e., the posted collateral must have the highest quality and be "risk-free".

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[1]Overnight index swap.

Intuitively, under above assumptions, a derivative may be regarded as "secured" and the counterparty credit risk (CCR) becomes negligible. In this paper, a collateralized derivative with all above assumptions being fulfilled is referred as *perfectly collateralized*, whereas the term *full collateralization* refers to the relaxation of perfect collateralization with collateral currency being allowed different from payoff currency. As shown in [3, 4, 5, 6, 7], the value of such a derivative depends on the collateral currency even in the fully collateralized case. We here restrict ourselves within the case of a specific foreign collateral currency for any derivative, so the embedded cheapest-to-deliver (CTD) option of collateral posting for some CSAs that allow more than one collateral currency is out of the scope of this paper. A derivative with its payoff in a single currency is called *domestic collateralized* if it is (possibly partially) collateralized in its payoff currency, and *foreign collateralized* if its specific collateral currency is different from the payoff currency.

It is also worth noting that an implicit assumption widely made for pricing by replication is that the unsecured borrowing rate and unsecured lending rate of the derivative desk are the same. This assumption might be regarded as counter-intuitive. However, the derivative desk has to borrow cash from its funding source (e.g., treasury desk) to start trading, so it is usually in debt regarding cash positions and needs to pay its borrowing rate as well. With any extra cash, the derivative desk tends to reduce its borrowing positions of cash, if it cannot lend it with a higher rate. Therefore, it is safe to make such an assumption, and our results could be extend to the case without this assumption. In our framework, this borrowing/lending rate is referred as the (unsecured) *funding rate* of this counterparty, and the spread between its funding rate and the collateral rate determined in CSA is named *funding spread*.

The collateral settled in a daily basis is the most common practice, in particular, in consistent with the requirements by clearing houses (e.g., LCH). Therefore, in many cases the collateral rate is defined to be the overnight index rate of the collateral currency in accordance with the settlement frequency. In such a case, the collateral rate discounting is equivalent to the overnight rate discounting, a.k.a. *OIS discounting*. In addition, eligible collateral assets may not be limited to cash, and government bonds in payoff currency with minimum sovereign risk are frequently agreed for collateral. It also occurs that risky assets are posted as collateral with certain hair-cut. Again, in our theoretical framework it is assumed that collateral is posted only in cash.

Despite of collateralization in foreign currency, partial collateralization is also considered in this paper[2]. Therefore, the presented results generalize many previous works, and the value adjustments due to collateral currency and funding cost/benefit incurred by partial collateralization[3] are both included. Similar to [2], counterparties of the derivative are both assumed default free, and the extension of our results to defaultable counterparties will be a topic of our future research.

1.1 Related Works

The theoretical foundation of valuation for derivatives partially collateralized in domestic currency is developed in the seminal work [2] by a replication and PDE approach, where both counterparties are assumed default free. As special cases, the approaches of collateral rate discounting and funding rate discounting are presented for the perfectly collateralized case and uncollateralized case, respectively. Furthermore, the funding value adjustment (FVA) due to partial collateralization is also implied in [2]. A small gap in the theory in [2] is pointed out and filled in [8], and is acknowledged in the Correction Note at the end of [7], while the results in [2] are valid. Alternatively, two different valuation approaches by expectation for perfectly collateralized derivatives are proposed in [3] to obtain the collateral rate discounting results as well as the application in interest rate curve building. These works are further extended to the case of fully collateralized case with foreign collateral currency [3, 4, 5] to build the multiple discounting framework. It is also worth noting that the valuation methodologies in [2] and in [3, 4, 5] may be under different measures. Such a difference is addressed in this paper, as well as the link between them.

Prior to this paper, attempt is made by a research team of the 16th IMA Workshop on Mathematical Modeling in Industry for Graduate Students [9] to develop valuation methodologies for derivatives partially collateralized in foreign currency, where some similar results to this work are reported.

For uncollateralized derivative traded between defaultable counterparties, the comprehensive valuation methodologies are studied in [10, 11] by replication and in [12] by expectation, to include both bilateral credit value adjustment (CVA)[4] and funding cost. The impact of collateral currency is not covered in these works. The replication

[2] That is, the assumptions 1 and 4 of perfect collateralization are both relaxed.

[3] Intuitively, this cost or benefit happens when a perfectly collateralized derivative is employed to hedge a partially collateralized derivative to match the cashflows. The extra posted or received collateral for the hedging position may be borrowed or lent with a rate higher than the collateral rate, resulting in such a cost or a benefit.

[4] A.k.a. CVA and DVA (debt/debit/default value adjustment).

approach is further applied in [6, 13] to capture the impact of collateral and its currency for partially collateralized derivative and the results reported there are similar to part of results in this paper, where this adjustment is termed as liquidity value adjustment (LVA) in those works. On the other hand, a collateral rate adjustment (CRA) is proposed on top of OIS discounting results in [14, 15] for perfectly collateralized derivatives with collateral rate different from OIS rate, while only very limited technical details are provided.

A comprehensive list of literature on new discounting theory due to collateralization as well as CVA, DVA and FVA can be found in [16], while we here restrict ourselves within the framework without CCR as in [2], and give only a few previous works directly related to our work in above.

1.2 Our Contribution

We study the derivative with payoff in a single currency and partially collateralized in a specific foreign currency traded between two default free counterparties. To calculate its present value with respect to the impact of collateral, two types of approaches are employed.

In the first type of approach, following the ideas in [2], a portfolio including underlying asset of the derivative and cash positions with various funding sources and return rates is constructed to replicate the value of the derivative, which might be regarded as a generalization of the Black-Scholes-Merton's framework as well. With the similar analysis on self-financing condition to [8], a PDE is formulated. Applying Feynman-Kac formula yields our main results on the value of such a derivative under a measure that each underlying asset follows a Wiener process with drift equivalent to its actual funding cost (which could be either a rate secured by the asset or the unsecured funding rate of the derivative desk).

In the second type of approach, valuation methodologies by expectation under risk neutral measure with risk free rate equivalent to the unsecured funding rate of the derivative desk are developed, similar to [3, 4, 5]. Within this type of approach, the expectation is calculated either of a self-financing portfolio of the derivative and cash positions, or of all the future cashflows including both the derivative payoff and re-investment return of collateral. The resulting valuation formulae are consistent with those of our first type of approach in the case that positions of underlying assets in the replication portfolio are maintained by unsecured funding rate, i.e., the underlying assets are not eligible for collateral. Thus, a uniform valuation framework is developed.

The current MtM value of such a derivative can be further decomposed into three components: the pricing by discounting derivative payoff with the return rate as if it was collateralized in payoff currency[5], a value adjustment due to the mismatch of funding spreads[6] of the payoff (domestic) currency and the collateral (foreign) currency, and the value adjustment resulting from the uncollateralized portion of the derivative value which is further partitioned into two parts due to the mismatch of the MtM value of the derivative and due to the mismatch (shortfall) of the collateral. Several special cases for either domestic collateral or fully collateralization are discussed, and consistent results to those in [2, 3, 4, 5] are reported.

The remainder of this paper is organized as follows. The model setup and notations are given in Section 2, the valuation methodology by replication and PDE approach is presented in Section 3 and the methodology by expectation in Section 4. These results are further discussed in Section 5. Finally, Section 6 concludes this paper.

2 Model Setup and Notations

In a domestic currency d market, let us consider a derivative which matures at $T > 0$ with a given payoff of V_T^{d} in d-currency[7]. This derivative is collateralized in a specific foreign currency f with currency exchange rate X_t[8] at time $t \geqslant 0$, which is expressed as the number of units in d per one unit in f. The (cash-equivalent) collateral amount C_t^{f} in f-currency at time $t \in [0, T)$ against the derivative is assumed depending on the CSA definition and the value of the derivative at t, and may differ from the derivative value denominated in f-currency in general (partially collateralized) cases.

Assume that the derivative is on a set of underlying assets whose prices $\boldsymbol{S}_t = (S_t^{(1)}, \cdots, S_t^{(n)})^\top \in \mathbb{R}_+^n$[9] are

[5] This may be different from the actual collateral currency defined in the CSA as the derivative could be foreign collateralized. This component is in fact the collateral rate discounting result, and as a special case, the OIS discounting result if the assumed domestic collateral rate is its overnight index rate.

[6] As the spread between the unsecured funding rate of the derivative desk and the return rate of the collateral defined in CSA.

[7] Assume no intermediate cashflow of the derivative within the time interval (t, T).

[8] To simplify analysis, we only use the concept of the instantaneous currency exchange rate in this paper.

[9] $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

denominated in d-currency, where $n \geqslant 1$ is an integer. The underlying assets may generate continuous cashflows[10] with short rates $\boldsymbol{r}_t^{\mathrm{D}} = (r_t^{\mathrm{D},1}, \cdots, r_t^{\mathrm{D},n})^\top \in \mathbb{R}_+^n$. If the i-th underlying asset is eligible for repo collateral, the funding rate secured with this asset (repo rate) is denoted as $r_t^{\mathrm{R},i}$, for $i \in \{1, \ldots, n\}$, and let $\boldsymbol{r}_t^{\mathrm{R}} = (r_t^{\mathrm{R},1}, \cdots, r_t^{\mathrm{R},n})^\top \in \mathbb{R}_+^n$. Further denote by $r_t^{\mathrm{F,d}}$ and by $r_t^{\mathrm{F,f}}$ the short rates of unsecured domestic funding and unsecured foreign funding, respectively. The short rate of the foreign currency collateral designated in the CSA is referred as $r_t^{\mathrm{C,f}}$, while $r_t^{\mathrm{C,d}}$ is the short rate of the domestic currency collateral if the derivative was domestic collateralized. Further, let us define

$$\lambda^{\mathsf{d}} := r^{\mathrm{F,d}} - r^{\mathrm{C,d}} , \quad \lambda^{\mathsf{f}} := r^{\mathrm{F,f}} - r^{\mathrm{C,f}} , \tag{2.1}$$

which are called domestic and foreign funding spreads, respectively.

At any time $t \in [0, T)$, denote by $V_t^{\mathsf{d,f}}$ the d-value of the derivative *partially* collateralized in f-currency, and by $\bar{V}_t^{\mathsf{d,f}}$ the d-value of the derivative *fully* collateralized in f-currency, respectively. Note that $C_t^{\mathsf{f}} \neq V_t^{\mathsf{d,f}} / X_t$ in general, while $C_t^{\mathsf{f}} \equiv \bar{V}_t^{\mathsf{d,f}} / X_t$. If this derivative was *partially* collateralized in its domestic currency d, its d-value is denoted as $V_t^{\mathsf{d,d}}$, where $V_t^{\mathsf{d,d}} \neq C_t^{\mathsf{d}}$ in general for domestic collateral amount C_t^{d}. Finally, the d-value of the derivative is $\bar{V}_t^{\mathsf{d,d}}$ if it was *perfectly* collateralized, where $\bar{V}_t^{\mathsf{d,d}} \equiv C_t^{\mathsf{d}}$. Here $\bar{V}_t^{\mathsf{d,d}}$ is actually the price of the derivative by OIS discounting in case the domestic collateral rate is defined as the overnight index rate of d currency. In all these cases, we always have the following boundary conditions

$$V_T^{\mathsf{d},\xi} := V_{T-}^{\mathsf{d},\xi} = V_T^{\mathsf{d}} = \bar{V}_{T-}^{\mathsf{d},\xi} =: \bar{V}_T^{\mathsf{d},\xi} , \quad \forall \xi \in \{\mathsf{d}, \mathsf{f}\} . \tag{2.2}$$

Conventionally, the term FVA refers to the difference of the value of a derivative against its price by OIS discounting, as the perfectly collateralized version of a derivative is the most liquid hedging instrument[11] without introducing additional counterparty credit risk and liquidity risk in the current market. Thus, FVA of a partially collateralized derivative is $V_t^{\mathsf{d,f}} - \bar{V}_t^{\mathsf{d,d}}$.

In this paper, we focus on the derivative with a single payoff in d currency which is partially collateralized in f currency. The counterparties trading this derivatives are assumed default free.

3 Pricing by Replication

To replicate the derivative, we may consider a trading strategy which contains following components: the underlying assets and their funding positions, the collateral account and an unsecured funding account. Let us elaborate. Denote

$$\boldsymbol{\theta}_t^{\mathrm{A}} = (\theta_t^{(1)}, \cdots, \theta_t^{(n)})^\top \in \mathbb{R}^n$$

the holding position of the assets at the time $t \in [0, T]$. Then,

- Amount $\theta_t^{(i)} S_t^{(i)}$ is needed to finance long or short of the i-th underlying asset with a short rate $r_t^{\mathrm{R},i}$ secured by the underlying asset, for $i \in \{1, \ldots, n\}$, if the underlying is eligible as collateral; otherwise, an unsecured funding short rate $r_t^{\mathrm{F,d}}$is needed to finance the position[12];
- Dividend cashflow is generated by the i-th underlying at the short rate $r_t^{\mathrm{D},i}$ which is paid to the buyer of the repo contract[13], for $i \in \{1, \ldots, n\}$;
- Collateral amount C_t^{f} is posted at the time of t with a corresponding collateral short rate $r_t^{\mathrm{C,f}}$;
- Amount $V_t^{\mathsf{d,f}} - X_t C_t^{\mathsf{f}}$ is to be financed with the unsecured domestic funding short rate $r_t^{\mathrm{F,d}}$.

Similar to the discussion in [8], the first component of the trading strategy, denoted as A, is the portfolio of n repo contracts of the underlying assets. Let $\boldsymbol{v}^A$ and $\boldsymbol{g}^A$ be the price and the gain (or the yield [17]) processes of A in the d-currency. As the repo contract can be terminated at zero additional cost, we have

$$\boldsymbol{v}_t^A = \mathbf{0} \in \mathbb{R}^n , \quad \forall t \in [0, T]; , \tag{3.1}$$

[10] For instance, dividend of underlying stocks.

[11] And it is easily obtained from a clearing house

[12] More discussion of this case can be found in Section 5.

[13] If the repo contract defines that the dividend is paid to the seller, then the short rate of this repo should be $r_t^{\mathrm{R},i} - r_t^{\mathrm{D},i}$ according to the non-arbitrage arguments, and our results are still valid.

while the gain process satisfies the following equation

$$\mathrm{d}g_t^A = \mathrm{d}\boldsymbol{S}_t + \mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{R}}\big)\boldsymbol{S}_t\,\mathrm{d}t\,, \quad \forall t \in (0,T]\,, \tag{3.2}$$

where

$$\mathrm{diag}(\boldsymbol{a}) := \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}, \quad \forall \boldsymbol{a} = (a_1,\cdots,a_n)^\top \in \mathbb{R}^n\,.$$

The second component, denoted as C, is the collateral account. Let v^C and g^C be the price and the gain processes of C in the d-currency. As the collateral amount in the f-currency is C^{f}, we have

$$v_t^C = X_t\,C_t^{\mathrm{f}}\,, \quad \forall t \in [0,T]\,, \tag{3.3}$$

and the gain process satisfies

$$\mathrm{d}g_t^C = r_t^{\mathrm{C,f}} X_t\,C_t^{\mathrm{f}}\,\mathrm{d}t + C_t^{\mathrm{f}}\,\mathrm{d}X_t\,, \quad \forall t \in (0,T]\,. \tag{3.4}$$

The last component of the trading strategy, denoted as F, is the unsecured domestic funding account. Let v^F and g^F be the price and the gain processes of F in the d-currency. Similar to the second component, we have

$$v_t^F = V_t^{\mathrm{d,f}} - X_t\,C_t^{\mathrm{f}}\,, \quad \forall t \in [0,T]\,, \tag{3.5}$$

and

$$\mathrm{d}g_t^F = r_t^{\mathrm{F,d}}\left(V_t^{\mathrm{d,f}} - X_t\,C_t^{\mathrm{f}}\right)\mathrm{d}t\,, \quad \forall t \in (0,T]\,. \tag{3.6}$$

Let us assume that there exists a function

$$\pi^{\mathrm{d,f}} : (\boldsymbol{s},c,x,t) \in \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \mapsto \pi^{\mathrm{d,f}}(\boldsymbol{s},c,x,t) \in \mathbb{R}$$

such that the value of the aforementioned derivative can be written as

$$V_t^{\mathrm{d,f}} = \pi^{\mathrm{d,f}}(\boldsymbol{S}_t,\, C_t^{\mathrm{f}},\, X_t,\, t)\,, \quad t \in [0,T]\,. \tag{3.7}$$

On the other hand, consider a strategy $\left(\boldsymbol{\theta}_t^{\mathrm{A}\,\top},\, 1,\, 1\right)^\top$ on (A, C, F). Clearly from (3.1), (3.3) and (3.5), the strategy gives a replication portfolio, denoted as Π_t, i.e., for $t \in [0,T]$,

$$\Pi_t = \boldsymbol{\theta}_t^{\mathrm{A}\,\top} \cdot \boldsymbol{v}^A + 1\cdot v_t^C + 1\cdot v_t^F = V_t^{\mathrm{d,f}}\,, \tag{3.8}$$

and we further assume that the portfolio is of self-finance [17], then, from (3.2), (3.4) and (3.6), we have

$$\begin{aligned} \mathrm{d}\Pi_t \;=\;& \boldsymbol{\theta}_t^{\mathrm{A}\,\top}\cdot\left(\mathrm{d}\boldsymbol{S}_t + \mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{R}}\big)\boldsymbol{S}_t\,\mathrm{d}t\right) + \\ & 1\cdot\left(r_t^{\mathrm{C,f}} X_t\,C_t^{\mathrm{f}}\,\mathrm{d}t + C_t^{\mathrm{f}}\,\mathrm{d}X_t\right) + 1\cdot\left(r_t^{\mathrm{F,d}}(V_t^{\mathrm{d,f}} - X_t\,C_t^{\mathrm{f}})\,\mathrm{d}t\right). \end{aligned} \tag{3.9}$$

From (3.7) and by using Ito's lemma, we have

$$\begin{aligned} \mathrm{d}V_t^{\mathrm{d,f}} \;=\;& \mathrm{d}\pi^{\mathrm{d,f}}(\boldsymbol{S}_t,\, C_t^{\mathrm{f}},\, X_t;\, t) \\ =\;& \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial t}\right\}\mathrm{d}t + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial \boldsymbol{s}}\right\}\mathrm{d}\boldsymbol{S}_t + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial c}\right\}\mathrm{d}C_t^{\mathrm{f}} + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial x}\right\}\mathrm{d}X_t + \\ & \frac{1}{2}\sum_{\alpha,\beta\in\{s_1,\cdots,s_n,c,x\}}\left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial\alpha\,\partial\beta}\right\}\mathrm{d}[\zeta(\alpha),\,\zeta(\beta)]_t\,, \end{aligned} \tag{3.10}$$

where[14] $\boldsymbol{s} = (s_1,\cdots,s_n)^\top$, the mapping ζ is defined by $\zeta(s_i) = S^{(i)}$, $i \in \{1,\ldots,n\}$, $\zeta(c) = C^{\mathrm{f}}$ and $\zeta(x) = X$, and $[\cdot\,,\,\cdot]_t$ is a quadratic co-variation/variation process. From (3.8), we have

$$\mathrm{d}\Pi_t = \mathrm{d}V_t^{\mathrm{d,f}}\,, \quad t \in [0,T]\,, \tag{3.11}$$

[14] $\{\pi^{\mathrm{d,f}}\}$ means $\pi^{\mathrm{d,f}}(\boldsymbol{S}_t,\, C_t^{\mathrm{f}},\, X_t;\, t)$. Similar meaning applies to $\left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial t}\right\}$, and etc.

and then substituting (3.9) and (3.10) into (3.11), we must have

$$\begin{aligned} & \boldsymbol{\theta}_t^{\mathrm{A}\top} \,\mathrm{d}\boldsymbol{S}_t + C_t^{\mathrm{f}}\,\mathrm{d}X_t + \Big[\boldsymbol{\theta}_t^{\mathrm{A}\top}\mathrm{diag}(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{R}})\boldsymbol{S}_t + r_t^{\mathrm{C,f}} X_t\, C_t^{\mathrm{f}} + r_t^{\mathrm{F,d}}(V_t^{\mathrm{d,f}} - X_t\, C_t^{\mathrm{f}})\Big]\,\mathrm{d}t \\ = \;& \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial \boldsymbol{s}}\right\}\mathrm{d}\boldsymbol{S}_t + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial c}\right\}\mathrm{d}C_t^{\mathrm{f}} + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial x}\right\}\mathrm{d}X_t + \\ & \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial t}\right\}\mathrm{d}t + \frac{1}{2}\sum_{\alpha,\beta\in\{s_1,\cdots,s_n,c,x\}}\left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial\alpha\,\partial\beta}\right\}\mathrm{d}[\zeta(\alpha),\,\zeta(\beta)]_t\,,\quad t\in[0,T]\,. \end{aligned} \tag{3.12}$$

Based on (3.12), we impose

$$\left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial \boldsymbol{s}}\right\} = \boldsymbol{\theta}_t^{\mathrm{A}}\,,\quad \frac{\partial \pi^{\mathrm{d,f}}}{\partial c} = 0\,,\quad \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial x}\right\} = C_t^{\mathrm{f}}\,, \tag{3.13}$$

by which, it suggests that, instead of (3.7), we should have

$$V_t^{\mathrm{d,f}} \,=\, \pi^{\mathrm{d,f}}\Big|_{(s,x,t)=\big(\boldsymbol{S}_t,\,X_t,\,t\big)}\,,\quad t\in[0,T]\,. \tag{3.14}$$

Hence (3.12) can be re-written as[15]

$$\begin{aligned} & \left[\left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial \boldsymbol{s}}\right\}\mathrm{diag}(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{R}})\boldsymbol{S}_t + r_t^{\mathrm{F,d}} V_t^{\mathrm{d,f}} + \left(r_t^{\mathrm{C,f}} - r_t^{\mathrm{F,d}}\right) X_t\, C_t^{\mathrm{f}}\right]\,\mathrm{d}t \\ = \;& \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial t}\right\}\mathrm{d}t + \frac{1}{2}\sum_{\alpha,\beta\in\{s_1,\cdots,s_n,x\}}\left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial\alpha\,\partial\beta}\right\}\mathrm{d}[\zeta(\alpha),\,\zeta(\beta)]_t\,,\quad t\in[0,T]\,, \end{aligned}$$

or

$$\begin{aligned} & \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial t}\right\}\mathrm{d}t + \left\{\frac{\partial \pi^{\mathrm{d,f}}}{\partial \boldsymbol{s}}\right\}\mathrm{diag}(\boldsymbol{r}_t^{\mathrm{R}} - \boldsymbol{r}_t^{\mathrm{D}})\boldsymbol{S}_t\,\mathrm{d}t + \\ & \frac{1}{2}\left[\sum_{i,j=1}^{n}\left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial s_i\,\partial s_j}\right\}\mathrm{d}[S^{(i)},S^{(j)}]_t + 2\sum_{i=1}^{n}\left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial s_i\,\partial x}\right\}\mathrm{d}[S^{(i)},X]_t + \left\{\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial^2 x}\right\}\mathrm{d}[X,X]_t\right] \\ = \;& \left[r_t^{\mathrm{F,d}} V_t^{\mathrm{d,f}} + \left(r_t^{\mathrm{C,f}} - r_t^{\mathrm{F,d}}\right) X_t\, C_t^{\mathrm{f}}\right]\,\mathrm{d}t\,,\quad t\in[0,T]\,. \end{aligned} \tag{3.15}$$

Let us introduce dynamics for the asset price $\boldsymbol{S}_t$ and the FX rate X_t. Let $\boldsymbol{\mu}^{\mathrm{A}}$ and $\boldsymbol{\sigma}^{\mathrm{A}}$ be $\mathbb{R}^n$-valued and $\mathbb{R}_+^n$-valued processes, respectively, μ^{X} and σ^{X} be $\mathbb{R}$-valued and $\mathbb{R}_+$-valued processes, respectively. Assume that, under a given measure, $\boldsymbol{S}_t$ and X_t satisfy the following dynamics

$$\mathrm{d}\begin{pmatrix}\boldsymbol{S}_t\\ X_t\end{pmatrix} = \begin{pmatrix}\boldsymbol{\mu}^{\mathrm{A}}\\ \mu^{\mathrm{X}}\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & 0\\ 0 & \sigma^{\mathrm{X}}\end{pmatrix}\mathrm{d}\begin{pmatrix}\boldsymbol{W}_t^{\mathrm{A}}\\ W_t^{\mathrm{X}}\end{pmatrix}\,, \tag{3.16}$$

where $(\boldsymbol{W}^{\mathrm{A}\top}, W^{\mathrm{X}})^\top$ is some $\mathbb{R}^{n+1}$-valued correlated Wiener process with

$$\mathrm{d}\left[\begin{pmatrix}\boldsymbol{W}^{\mathrm{A}}\\ W^{\mathrm{X}}\end{pmatrix}\right]_t = \rho\,\mathrm{d}t\,,\quad \rho = \begin{pmatrix}\rho^{\mathrm{A}} & \rho^{\mathrm{X}}\\ \rho^{\mathrm{X}\top} & 1\end{pmatrix}\,, \tag{3.17}$$

and $[\rho]_{(n+1)\times(n+1)}$ is a given correlation matrix. From (3.17), we also have

$$\mathrm{d}\left[\begin{pmatrix}\boldsymbol{S}\\ X\end{pmatrix}\right]_t = \begin{pmatrix}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}})\rho^{\mathrm{A}}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & \mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}})\rho^{\mathrm{X}}\sigma^{\mathrm{X}}\\ \sigma^{\mathrm{X}}\rho^{\mathrm{X}\top}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & \sigma^{\mathrm{X}}\sigma^{\mathrm{X}}\end{pmatrix}\mathrm{d}t\,. \tag{3.18}$$

Then one may find a measure, denoted as $\mathcal{Q}$, such that, under $\mathcal{Q}$, the dynamics (3.16) can be written as

$$\mathrm{d}\begin{pmatrix}\boldsymbol{S}_t\\ X_t\end{pmatrix} = \begin{pmatrix}\tilde{\boldsymbol{\mu}}^{\mathrm{A}}\\ \tilde{\mu}^{\mathrm{X}}\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & 0\\ 0 & \sigma^{\mathrm{X}}\end{pmatrix}\mathrm{d}\begin{pmatrix}\tilde{\boldsymbol{W}}_t^{\mathrm{A}}\\ \tilde{W}_t^{\mathrm{X}}\end{pmatrix}\,,\quad \begin{pmatrix}\tilde{\boldsymbol{\mu}}^{\mathrm{A}}\\ \tilde{\mu}^{\mathrm{X}}\end{pmatrix} := \begin{pmatrix}\mathrm{diag}(\boldsymbol{r}_t^{\mathrm{R}} - \boldsymbol{r}_t^{\mathrm{D}})\boldsymbol{S}_t\\ (r_t^{\mathrm{F,d}} - r_t^{\mathrm{F,f}})X_t\end{pmatrix} \tag{3.19}$$

[15] From the second equation in (3.13), we have $\frac{\partial^2 \pi^{\mathrm{d,f}}}{\partial\alpha\partial c} = 0$, $\forall \alpha \in \{s_1,\cdots,s_n,c,x\}$.

where $(\tilde{\boldsymbol{W}}^{\mathrm{A}\top}, \tilde{W}^{\mathrm{X}})^\top$ is some $\mathbb{R}^{n+1}$-valued ρ-correlated Wiener process under $\mathcal{Q}$[16]. With the consideration of (3.18) and (3.19) together with the third equation in (3.13) and also by using (2.1), we re-visit (3.15), which can be now further re-written as

$$
\begin{aligned}
& \left\{\frac{\partial \pi^{\mathsf{d,f}}}{\partial t}\right\} \mathrm{d}t + \left\{\frac{\partial \pi^{\mathsf{d,f}}}{\partial \boldsymbol{s}}\right\} \mathrm{diag}\left(\boldsymbol{r}_t^{\mathrm{R}} - \boldsymbol{r}_t^{\mathrm{D}}\right) \boldsymbol{S}_t \, \mathrm{d}t + \left\{\frac{\partial \pi^{\mathsf{d,f}}}{\partial x}\right\} \left(r_t^{\mathrm{F,d}} - r_t^{\mathrm{F,f}}\right) X_t \, \mathrm{d}t + \\
& \frac{1}{2}\left[\sum_{i,j=1}^{n} \left\{\frac{\partial^2 \pi^{\mathsf{d,f}}}{\partial s_i \, \partial s_j}\right\} \sigma_i^{\mathrm{A}} \rho_{i,j}^{\mathrm{X}} \sigma_j^{\mathrm{A}} + 2 \sum_{i=1}^{n} \left\{\frac{\partial^2 \pi^{\mathsf{d,f}}}{\partial s_i \, \partial x}\right\} \sigma_i^{\mathrm{A}} \rho_i^{\mathrm{X}} \sigma^{\mathrm{X}} + \left\{\frac{\partial^2 \pi^{\mathsf{d,f}}}{\partial^2 x}\right\} \sigma^{\mathrm{X}} \sigma^{\mathrm{X}}\right] \mathrm{d}t \\
= \; & \left[r_t^{\mathrm{F,d}} V_t^{\mathsf{d,f}} + \left(r_t^{\mathrm{C,f}} - r_t^{\mathrm{F,d}}\right) X_t \, C_t^{\mathsf{f}}\right] \mathrm{d}t + \left\{\frac{\partial \pi^{\mathsf{d,f}}}{\partial x}\right\} \left(r_t^{\mathrm{F,d}} - r_t^{\mathrm{F,f}}\right) X_t \, \mathrm{d}t \\
= \; & \left[r_t^{\mathrm{F,d}} V_t^{\mathsf{d,f}} + \left(r_t^{\mathrm{C,f}} - r_t^{\mathrm{F,d}}\right) X_t \, C_t^{\mathsf{f}}\right] \mathrm{d}t + \left(r_t^{\mathrm{F,d}} - r_t^{\mathrm{F,f}}\right) X_t \, C_t^{\mathsf{f}} \, \mathrm{d}t \\
= \; & \left[r_t^{\mathrm{F,d}} \left\{\pi^{\mathsf{d,f}}\right\} - \lambda_t^{\mathsf{f}} \, X_t \, C_t^{\mathsf{f}}\right] \mathrm{d}t \,, \quad t \in [0, T] \,.
\end{aligned}
\tag{3.20}
$$

Now from (3.20), we may conclude that if the derivative price $V_t^{\mathsf{d,f}}$ has the form of (3.14), then the function $\pi^{\mathsf{d,f}}$ is a solution of the following PDE's solution

$$
\mathcal{D}.\pi^{\mathsf{d,f}} = r^{\mathrm{F,d}} \, \pi^{\mathsf{d,f}} - \lambda^{\mathsf{f}} \, x \, C^{\mathsf{f}} \,, \tag{3.21}
$$

with a terminal condition for $\pi^{\mathsf{d,f}}(\boldsymbol{s}, x, T)$ which is given by the derivative matured payoff, i.e.,

$$
\pi^{\mathsf{d,f}}(\boldsymbol{s}, x, T)\Big|_{(\boldsymbol{s},x)=(\boldsymbol{S}_T, X_T)} = V_T^{\mathsf{d}} \,, \tag{3.22}
$$

where

$$
\begin{aligned}
\mathcal{D} \;\; := \;\; & \frac{\partial \cdot}{\partial t} + \frac{\partial \cdot}{\partial \boldsymbol{s}} \mathrm{diag}\left(\boldsymbol{r}^{\mathrm{R}} - \boldsymbol{r}^{\mathrm{D}}\right) \boldsymbol{s} + \frac{\partial \cdot}{\partial x} \left(r^{\mathrm{F,d}} - r^{\mathrm{F,f}}\right) x + \\
& \frac{1}{2}\left[\sum_{i,j=1}^{n} \sigma_i^{\mathrm{A}} \rho_{i,j}^{\mathrm{A}} \, \sigma_j^{\mathrm{A}} \, \frac{\partial^2 \cdot}{\partial s_i \, \partial s_j} + 2 \sum_{i=1}^{n} \sigma_i^{\mathrm{A}} \rho_i^{\mathrm{X}} \sigma^{\mathrm{X}} \, \frac{\partial^2 \cdot}{\partial s_i \, \partial x} + \sigma^{\mathrm{X}} \, \sigma^{\mathrm{X}} \, \frac{\partial^2 \cdot}{\partial^2 x}\right] \,,
\end{aligned}
\tag{3.23}
$$

which is called the Dynkin or Kolmogorov backward operator. We also assume that $\boldsymbol{\sigma}^{\mathrm{A}}$, σ^{X}, ρ, $\boldsymbol{r}^{\mathrm{R}}$, $\boldsymbol{r}^{\mathrm{D}}$, $r^{\mathrm{F,d}}$, λ_t^{f}, and C^{f} are all functions of $(\boldsymbol{S}_t, X_t, t)$. According to Feynman-Kac formula (e.g., Theorem 5.7.6 of [18] or Appendix E of [17]), the following theorem holds about the solution to (3.21)-(3.22):

Theorem 3.1. *With regular conditions for* (3.21)-(3.22), *its unique solution with sub-exponential growth admits the following stochastic representation:*

$$
V_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}} \, \mathrm{d}u} \, V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F,d}} \, \mathrm{d}v} \lambda_u^{\mathsf{f}} \, X_u \, C_u^{\mathsf{f}} \, \mathrm{d}u\right] \,, \tag{3.24}
$$

where $\mathcal{Q}$ is the measure introduced in (3.19). Particularly, we have the following special results:
(I.1) if it is partially collateralized in the d*-currency, i.e., in the domestic collateral, setting $X \equiv 1$ and replacing* f *in (3.24) by* d*, then as in [2],*

$$
V_t^{\mathsf{d,d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}} \, \mathrm{d}u} \, V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F,d}} \, \mathrm{d}v} \lambda_u^{\mathsf{d}} \, C_u^{\mathsf{d}} \, \mathrm{d}u\right] \,; \tag{3.25}
$$

(I.2) if it is fully collateralized in f*-currency, i.e., $X \, C^{\mathsf{f}} = \bar{V}^{\mathsf{d,f}}$ in (3.24), then as in [3, 4, 5],*

$$
\bar{V}_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}} \, \mathrm{d}u} \, V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F,d}} \, \mathrm{d}v} \lambda_u^{\mathsf{f}} \, \bar{V}_u^{\mathsf{d,f}} \, \mathrm{d}u\right] \,; \tag{3.26}
$$

---

[16] Actually, we may first get

$$
\boldsymbol{\lambda} := \begin{pmatrix} \mathrm{diag}\left(\boldsymbol{\sigma}^{\mathrm{A}}\right) & 0 \\ 0 & \sigma^{\mathrm{X}} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\mu}^{\mathrm{A}} - \mathrm{diag}\left(\boldsymbol{r}^{\mathrm{R}} - \boldsymbol{r}^{\mathrm{D}}\right) \boldsymbol{S} \\ \mu^{\mathrm{X}} - \left(r^{\mathrm{F,d}} - r^{\mathrm{F,f}}\right) X \end{pmatrix} .
$$

We further assume that $\boldsymbol{\lambda}$ satisfies some regular conditions such that the measure $\mathcal{Q}$ can be obtained by the Girsanov transformation with the kernel of $\boldsymbol{\lambda}$ and

$$
\mathrm{d}\begin{pmatrix} \tilde{\boldsymbol{W}}_t^{\mathrm{A}} \\ \tilde{W}_t^{\mathrm{X}} \end{pmatrix} = \mathrm{d}\begin{pmatrix} \boldsymbol{W}_t^{\mathrm{A}} \\ W_t^{\mathrm{X}} \end{pmatrix} + \boldsymbol{\lambda} \, \mathrm{d}t \,.
$$

(I.3) if it is perfectly collateralized, i.e., $\mathsf{C}^{\mathsf{d}} = \bar{V}^{\mathsf{d,d}}$ *in (3.25) or replacing* f *in (3.26) by* d*, then as in [2],*

$$\bar{V}_t^{\mathsf{d,d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F,d}}\,\mathrm{d}v} \lambda_u^{\mathsf{d}}\, \bar{V}_u^{\mathsf{d,d}}\,\mathrm{d}u\right] \ ; \quad \textit{and} \tag{3.27}$$

(I.4) finally, if it is uncollateralized, i.e., $\mathsf{C}^{\mathsf{f}} \equiv 0$ *in (3.24) or* $\mathsf{C}^{\mathsf{d}} \equiv 0$ *in (3.25), then as in [2],*

$$V_t^{\mathsf{d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}}\right] \ , \tag{3.28}$$

where V_t^{d} *is the price without collateral.*

As similarly pointed in [2], we may express (3.24) in the following way[17]:

Theorem 3.2. *The solution (3.24) has another equivalent form:*

$$V_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}} - \int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left(\lambda_u^{\mathsf{d}}\, V_u^{\mathsf{d,f}} - \lambda_u^{\mathsf{f}}\, X_u\, \mathsf{C}_u^{\mathsf{f}}\right)\mathrm{d}u\right]. \tag{3.29}$$

Similarly, we also have the following special results:
(II.1) if it is partially collateralized in d*-currency, then as in [2],*

$$V_t^{\mathsf{d,d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}} - \int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\, \lambda_u^{\mathsf{d}}(V_u^{\mathsf{d,d}} - \mathsf{C}_u^{\mathsf{d}})\,\mathrm{d}u\right] \ ; \tag{3.30}$$

(II.2) if it is fully collateralized in f*-currency, then*

$$\bar{V}_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}} - \int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}(\lambda_u^{\mathsf{d}} - \lambda_u^{\mathsf{f}})\, \bar{V}_u^{\mathsf{d,f}}\,\mathrm{d}u\right] \ ; \tag{3.31}$$

(II.3) if it is perfectly collateralized, then as in [2, 3],

$$\bar{V}_t^{\mathsf{d,d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}}\right] \ ; \quad \textit{and} \tag{3.32}$$

(II.4) finally, if it is uncollateralized, then as in [2],

$$V_t^{\mathsf{d}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u}\, V_T^{\mathsf{d}} - \int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\, \lambda_u^{\mathsf{d}}\, V_u^{\mathsf{d}}\,\mathrm{d}u\right] \ . \tag{3.33}$$

From equations (3.11) and by substituting the dynamics (3.19) into (3.9), we have, under the measure $\mathcal{Q}$,

$$\begin{aligned} \mathrm{d}V_t^{\mathsf{d,f}} \quad &= \quad \left[r_t^{\mathrm{F,d}}\, V_t^{\mathsf{d,f}} - \left(r_t^{\mathrm{F,f}} - r_t^{\mathrm{C,f}}\right) X_t\, \mathsf{C}_t^{\mathsf{f}}\right]\mathrm{d}t + (\cdots)\,\mathrm{d}\begin{pmatrix}\tilde{\boldsymbol{W}}_t^{\mathrm{A}} \\ \tilde{W}_t^{\mathrm{X}}\end{pmatrix} \\ \text{(or)} \quad &=: \quad \left(r_t^{\mathrm{F,d}} V_t^{\mathsf{d,f}} - \lambda_t^{\mathsf{f}}\, X_t\, \mathsf{C}_t^{\mathsf{f}}\right)\mathrm{d}t + \Sigma_t^{\mathrm{drv}}\,\mathrm{d}W_t^{\mathrm{drv}} \ , \end{aligned} \tag{3.34}$$

where Σ^{drv} is the diffusion term of the derivative price $V^{\mathsf{d,f}}$ and W^{drv} is some $\mathcal{Q}$-Wiener process. By the discussion in Chapter 5 of [17], the conditional expected rate of change of the derivative value at time t becomes[18]

$$\lim_{\tau\to t}\frac{\mathrm{d}}{\mathrm{d}\tau_+}\mathrm{E}_t^{\mathcal{Q}}\left[V_\tau^{\mathsf{d,f}}\right] = \left(r_t^{\mathrm{F,d}} V_t^{\mathsf{d,f}} - \lambda_t^{\mathsf{f}}\, X_t\, \mathsf{C}_t^{\mathsf{f}}\right) ,$$

[17] If we claim that the solution $\pi^{\mathsf{d,f}}$ to (3.21)-(3.22) exists, then it also satisfies (3.29), which may be obtained by re-arranging the right hand side of (3.21) to be

$$r^{\mathrm{C,d}}\,\pi^{\mathsf{d,f}} - \left[-(r^{\mathrm{F,d}} - r^{\mathrm{C,d}})\,\pi^{\mathsf{d,f}} + \lambda^{\mathsf{f}}\,x\,\mathsf{C}^{\mathsf{f}}\right] = r^{\mathrm{C,d}}\,\pi^{\mathsf{d,f}} - \left[-\lambda^{\mathsf{d}}\,\pi^{\mathsf{d,f}} + \lambda^{\mathsf{f}}\,x\,\mathsf{C}^{\mathsf{f}}\right] \ ,$$

and applying Theorem 5.7.6 of [18] again. We provide another rigorous proof in Appendix A.

[18] "$\mathrm{d}/\,\mathrm{d}\tau_+$" is the right derivative at τ.

or by using the associated abuses of notation, we may write

$$\mathrm{E}_t^{\mathcal{Q}}\left[\mathrm{d}V_t^{\mathsf{d},\mathsf{f}}\right]=\left(r_t^{\mathrm{F},\mathsf{d}}V_t^{\mathsf{d},\mathsf{f}}-\lambda_t^{\mathsf{f}}\,X_t\,C_t^{\mathsf{f}}\right)\mathrm{d}t\,.$$

Thus, the growth rate of the derivative (under the measure $\mathcal{Q}$) is the domestic funding rate $r_t^{\mathrm{F},\mathsf{d}}$ applied to its value less the foreign funding spread λ_t^{f} applied to the d-currency equivalent collateral.

Let us consider three special cases, in which two of them are "boundary" cases. The first "boundary" case is that the derivative is uncollateralized, i.e., $C_t^{\mathsf{f}}\equiv 0$, then, from (3.34), we have

$$\mathrm{E}_t^{\mathcal{Q}}\left[\mathrm{d}V_t^{\mathsf{d}}\right]=r_t^{\mathrm{F},\mathsf{d}}V_t^{\mathsf{d}}\,\mathrm{d}t\,,$$

i.e., $e^{\int_0^t -r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\,V_t^{\mathsf{d}}$ is a $\mathcal{Q}$-martingale. Then time t MtM value of the uncollateralized derivative can be simply written as

$$V_t^{\mathsf{d}}=\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}}\right]\,,\tag{3.35}$$

which is consistent with the traditional funding rate discounting [2][19]. In the other "boundary" case, the derivative is fully collateralized in f-currency. Then (3.34) gives

$$\mathrm{E}_t^{\mathcal{Q}}\left[\mathrm{d}\bar{V}_t^{\mathsf{d},\mathsf{f}}\right]=\left(r_t^{\mathrm{F},\mathsf{d}}-\lambda_t^{\mathsf{f}}\right)\bar{V}_t^{\mathsf{d},\mathsf{f}}\,\mathrm{d}t\,,$$

which implies $e^{\int_0^t -(r_u^{\mathrm{F},\mathsf{d}}-\lambda_u^{\mathsf{f}})\,\mathrm{d}u}\,\bar{V}_t^{\mathsf{d},\mathsf{f}}$ is a $\mathcal{Q}$-martingale. Therefore, the time t MtM value of the fully foreign collateralized derivative becomes

$$\bar{V}_t^{\mathsf{d},\mathsf{f}}=\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T (r_u^{\mathrm{F},\mathsf{d}}-\lambda_u^{\mathsf{f}})\,\mathrm{d}u}V_T^{\mathsf{d}}\right]\,,\tag{3.36}$$

which is equivalent to those in [3, 5]. In the two "boundary" cases, we clearly see that the current value of the derivative is the expectation of its matured payoff with an appropriate "discounting". In other words, the time t value of the derivative is indifferent to a path towards V_T^{d}.

In the last case, we introduce a *collateral-ratio* process γ such that

$$\gamma_t\,V_t^{\mathsf{d},\mathsf{f}}:=X_t\,C_t^{\mathsf{f}}\,,\quad\forall t\in[0,T]\,,\tag{3.37}$$

which is a generalization of that in [6, 13]. Then, we similarly have

$$\mathrm{E}_t^{\mathcal{Q}}\left[\mathrm{d}V_t^{\mathsf{d},\mathsf{f}}\right]=\left(r_t^{\mathrm{F},\mathsf{d}}-\gamma_t\,\lambda_t^{\mathsf{f}}\right)V_t^{\mathsf{d},\mathsf{f}}\,\mathrm{d}t\,,$$

which also implies $e^{\int_0^t -(r_u^{\mathrm{F},\mathsf{d}}-\gamma_u\,\lambda_u^{\mathsf{f}})\,\mathrm{d}u}\,V_t^{\mathsf{d},\mathsf{f}}$ is a $\mathcal{Q}$-martingale, and hence the time t MtM value of the partially foreign collateralized derivative becomes

$$V_t^{\mathsf{d},\mathsf{f}}=\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T (r_u^{\mathrm{F},\mathsf{d}}-\gamma_u\,\lambda_u^{\mathsf{f}})\,\mathrm{d}u}V_T^{\mathsf{d}}\right]\,.\tag{3.38}$$

One, however, should not be misled by the expression of (3.38). Since the collateral ratio process (3.37) may depend on the value process $V^{\mathsf{d},\mathsf{f}}$, hence in general the expectation (3.38) may be subject to a distribution of value paths in $\{V_t^{\mathsf{d},\mathsf{f}}\,:\,t\in[t,T]\}$.

4 Pricing by Expectation

In [3, 5], the impact of collateralization on the derivative pricing has been studied with all conditions for perfect collateralization except (4). In this section, by relaxing condition (1) for perfect collateralization as well to allow different types of collateralization, we elaborate two different approaches to generalize the result by [3, 5].

[19] Conventional "LIBOR discounting" when the funding rate is assumed LIBOR rate. This result may also be directly obtained by (3.24).

4.1 First Approach - Cashflow Analysis

Cashflows of a collateralized European style derivative consist of a final payoff at its maturity plus intermediate collateral account flows throughout the life of the trade. For the final payoff, at maturity T, the derivative pays V_T^{d}, hence the time t PV of the final payoff is

$$\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u} V_T^{\mathsf{d}}\right] \tag{4.1}$$

in the $\mathsf{d}-$currency, where the conditional expectation is under the domestic risk neutral measure $\mathcal{Q}^{\mathsf{d}}$, which corresponds to the unsecured domestic funding rate $r_u^{\mathrm{F},\mathsf{d}}$.

For intermediate cashflows in the collateral account, let $\Pi_m = \{t_{0,m}, t_{1,m}, \ldots, t_{m,m}\}$ be a partition of $[t, T]$, i.e.,

$$t = t_{0,m} \le t_{1,m} \le \cdots \le t_{m,m} = T, \quad m \geqslant 1.$$

Define $\|\Pi_m\| := \max_{i=1,\ldots,m} \Delta t_i^{(m)}$ where $\Delta t_i^{(m)} = t_{i,m} - t_{i-1,m}$, $1 \leqslant i \leqslant m$. We consider a sequence of infinitesimal fine partitions $\{\Pi_m : m = 1, 2, \cdots\}$ with $\|\Pi_m\| \to 0$ as $m \to \infty$. For any sufficient large $m \gg 1$, consider the time interval $[t_{i-1,m}, t_{i,m}]$ for any $i = 1, \ldots, m$. At time $t_{i-1,m}$, if the MtM of the derivative is positive, i.e. $V_{t_{i-1,m}}^{\mathsf{d},\mathsf{f}} > 0$, then the buyer of the derivative contract will receive collateral $C_{t_{i-1,m}}^{\mathsf{f}}$ in the f-currency from his counterparty. Once the collateral is received, the buyer can lend it out in the foreign money market to earn the unsecured foreign funding rate $r_{t_{i-1,m}}^{\mathrm{F},\mathsf{f}}$. Meantime, according to CSA, the buyer has to pay his counterparty interest on the collateral at the foreign collateral rate $r_{t_{i-1,m}}^{\mathrm{C},\mathsf{f}}$. If $V_{t_{i-1,m}}^{\mathsf{d},\mathsf{f}} < 0$, due to the bilateral collateralization assumption, the buyer has to borrow $-C_{t_{i-1,m}}^{\mathsf{f}}$ in the f-currency at the unsecured foreign funding rate $r_{t_{i-1,m}}^{\mathrm{F},\mathsf{f}}$ to post collateral to his counterparty,[20] and earn the same foreign collateral rate $r_{t_{i-1}}^{\mathrm{C},\mathsf{f}}$ on the posted collateral. In either case, at the end of the infinitesimal fine time interval, $t_{i,m}$, the net cashflow in the collateral account for the buyer is $\lambda_{t_{i-1,m}}^{\mathsf{f}} C_{t_{i-1,m}}^{\mathsf{f}} \Delta t_i^{(m)}$ in the f-currency, where $\lambda_{t_{i-1,m}}^{\mathsf{f}} = r_{t_{i-1,m}}^{\mathrm{F},\mathsf{f}} - r_{t_{i-1,m}}^{\mathrm{C},\mathsf{f}}$. Therefore, the PV of total intermediate cashflows in the collateral account at time t, as $m \to \infty$, becomes

$$\mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\lim_{m\to\infty}\sum_{i=1}^{m} e^{-\int_t^{t_{i,m}} r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v} \lambda_{t_{i-1,m}}^{\mathsf{f}} C_{t_{i-1,m}}^{\mathsf{f}} \Delta t_i^{(m)}\right] \quad = \quad \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v} \lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\,\mathrm{d}u\right]$$

in the f-currency, or equivalently,

$$X_t \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v} \lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\,\mathrm{d}u\right] \tag{4.2}$$

in the d-currency, and the conditional expectation is under the foreign risk neutral measure $\mathcal{Q}^{\mathsf{f}}$, which corresponds to the unsecured foreign funding rate $r_u^{\mathrm{F},\mathsf{f}}$. Combining (4.1) and (4.2), the time t PV of the collateralized derivative is

$$V_t^{\mathsf{d},\mathsf{f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u} V_T^{\mathsf{d}}\right] + X_t \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v} \lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\,\mathrm{d}u\right]. \tag{4.3}$$

Notice that, expectations in (4.3) are under different measures, which is unpleasant. Theorem 4.1 shows that the pricing formula (4.3) can be transformed to be under the single domestic risk neutral measure $\mathcal{Q}^{\mathsf{d}}$. A proof of Theorem 4.1 is attached in Appendix B.

Theorem 4.1. *The current MtM value of a partially foreign collateralized derivative in (4.3) is equivalent to:*

$$V_t^{\mathsf{d},\mathsf{f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u} V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v} \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\,\mathrm{d}u\right]. \tag{4.4}$$

The pricing formula (4.4) is conceptually intuitive, but, similar to (3.24), both are not convenient to use in practice, since they are recursive formulas. To simplify (4.4), we begin with the following theorem:

[20] Here we assume that $C_{t_{i-1,m}}^{\mathsf{f}}$ and $V_{t_{i-1}}^{\mathsf{d},\mathsf{f}}$ have the same sign.

Theorem 4.2. *Let*

$$M_t := e^{-\int_0^t r_u^{\mathrm{F,d}}\,\mathrm{d}u} V_t^{\mathsf{d,f}} + \int_0^t e^{-\int_0^s r_v^{\mathrm{F,d}}\,\mathrm{d}v} \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\,\mathrm{d}u,$$

then M_t is a martingale under $\mathcal{Q}^{\mathsf{d}}$, and $V_t^{\mathsf{d,f}}$ follows the following stochastic process

$$\mathrm{d}V_t^{\mathsf{d,f}} = \left(r_t^{\mathrm{F,d}} V_t^{\mathsf{d,f}} - \lambda_t^{\mathsf{f}} X_t C_t^{\mathsf{f}}\right)\,\mathrm{d}t + e^{\int_0^t r_u^{\mathrm{F,d}}\,\mathrm{d}u}\,\mathrm{d}M_t. \tag{4.5}$$

A proof of Theorem (4.2) is presented in Appendix C. From Theorem 4.2, we have the following two observations. First of all, if we let

$$Y_s := e^{-\int_t^s r_u^{\mathrm{C,d}}\,\mathrm{d}u} V_s^{\mathsf{d,f}} - \int_t^s e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left(\lambda_u^{\mathsf{d}} V_u^{\mathsf{d,f}} - \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\right)\,\mathrm{d}u$$

for $s \in [t, T]$, then

$$\mathrm{d}Y_s = -r_s^{\mathrm{C,d}} e^{-\int_t^s r_u^{\mathrm{C,d}}\,\mathrm{d}u} V_s^{\mathsf{d,f}}\,\mathrm{d}s + e^{-\int_t^s r_u^{\mathrm{C,d}}\,\mathrm{d}u}\,\mathrm{d}V_s^{\mathsf{d,f}} - e^{-\int_t^s r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left(\lambda_s^{\mathsf{d}} V_s^{\mathsf{d,f}} - \lambda_s^{\mathsf{f}} X_s C_s^{\mathsf{f}}\right)\,\mathrm{d}s. \tag{4.6}$$

Substituting (4.5) into (4.6) gives

$$\begin{aligned}
\mathrm{d}Y_s &= -r_s^{\mathrm{C,d}} e^{-\int_t^s r_u^{\mathrm{C,d}}\,\mathrm{d}u} V_s^{\mathsf{d,f}}\,\mathrm{d}s + e^{-\int_t^s r_u^{\mathrm{C,d}}\,\mathrm{d}u}\left(\left(r_s^{\mathrm{F,d}} V_s^{\mathsf{d,f}} - \lambda_s^{\mathsf{f}} X_s C_s^{\mathsf{f}}\right)\,\mathrm{d}s + e^{\int_0^s r_u^{\mathrm{F,d}}\,\mathrm{d}u}\,\mathrm{d}M_s\right) \\
&\quad - e^{-\int_t^s r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left(\lambda_s^{\mathsf{d}} V_s^{\mathsf{d,f}} - \lambda_s^{\mathsf{f}} X_s C_s^{\mathsf{f}}\right)\,\mathrm{d}s \\
&= e^{\int_0^t r_u^{\mathrm{F,d}}\,\mathrm{d}u} e^{\int_t^s \lambda_u^{\mathsf{d}}\,\mathrm{d}u}\,\mathrm{d}M_s.
\end{aligned} \tag{4.7}$$

Then (4.7) implies Y_s is also a $\mathcal{Q}^{\mathsf{d}}$-martingale. By $Y_t = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}[Y_T]$, we have

$$V_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u} V_T^{\mathsf{d}} - \int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left(\lambda_u^{\mathsf{d}} V_u^{\mathsf{d,f}} - \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\right)\,\mathrm{d}u\right]. \tag{4.8}$$

Secondly, if the derivative is collateralized with the collateral-ratio γ_t as defined in (3.37), then from (4.9), we have

$$\mathrm{d}V_t^{\mathsf{d,f}} = \left(r_t^{\mathrm{F,d}} - \gamma_t \lambda_t^{\mathsf{f}}\right) V_t^{\mathsf{d,f}}\,\mathrm{d}t + e^{\int_0^t r_u^{\mathrm{F,d}}\,\mathrm{d}u}\,\mathrm{d}M_t, \tag{4.9}$$

which implies that $e^{-\int_0^t \left(r_u^{\mathrm{F,d}} - \lambda_u^{\mathsf{f}} \gamma_u\right)\mathrm{d}u} V_t^{\mathsf{d,f}}$ is a $\mathcal{Q}^{\mathsf{d}}$-martingale. Therefore, the time t PV of the derivative becomes

$$V_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T \left(r_u^{\mathrm{F,d}} - \gamma_u \lambda_u^{\mathsf{f}}\right)\mathrm{d}u} V_T^{\mathsf{d}}\right]. \tag{4.10}$$

Based on the value of γ_t, we have the following special cases:

- Without collateralization: $\gamma_t = 0$. In this case, the pricing formula (4.10) is simplified to

$$V_t^{\mathsf{d}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{F,d}}\,\mathrm{d}u} V_T^{\mathsf{d}}\right]. \tag{4.11}$$

- Full collateralization: $\gamma_t = 1$. In this case, the pricing formula (4.10) is simplified to

$$\bar{V}_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T \left(r_u^{\mathrm{F,d}} - \lambda_u^{\mathsf{f}}\right)\mathrm{d}u} V_T^{\mathsf{d}}\right], \tag{4.12}$$

  which is the equation (36) in [3].

One may find that formulae (4.4) and (4.8) are very much similar to (3.24) and (3.29), respectively. Clearly (4.10), (4.11), (4.12) are also similar to (3.38), (3.35) and (3.36), respectively. However, in general, the measure $\mathcal{Q}$ used in Theorem 3.1 and 3.2 may be different from $\mathcal{Q}^{\mathsf{d}}$ in Theorem 4.1 and 4.2. More discussion on measures $\mathcal{Q}$ and $\mathcal{Q}^{\mathsf{d}}$ is in Section 5.

4.2 Second Approach - Self-Financing Strategy

Using an infinitesimal fine partition $\Pi_m = \{t_{0,m}, t_{1,m}, \ldots, t_{m,m}\}$ defined in the last subsection, consider the following trading strategy:

> *Long $\theta^{\mathrm{TS}}_{t_0}$ units of derivative at time t_0, and keep it until $t_{1,m}$; At time $t_{1,m}$, close the existing derivative position, and immediately reinvest all available funds, including the fund earned by closing the derivative position and cash earned in the collateral account, in $\theta^{\mathrm{TS}}_{t_{1,m}}$ units of the same derivative. Repeat this strategy in each infinitesimal time interval $[t_{i-1,m}, t_{i,m}]$ until the maturity $t_{m,m} = T$.*

By choosing an appropriate θ^{TS}, this trading strategy is self-financing with the initial payment of $\theta^{\mathrm{TS}}_t V^{\mathrm{d,f}}_t$ in d-currency and the terminal payoff of $\theta^{\mathrm{TS}}_T V^{\mathrm{d}}_T$ in d-currency, and there are no intermediate cash flows during (t, T). Therefore, by the martingale pricing theory, under the domestic risk neutral measure $\mathcal{Q}^{\mathrm{d}}$,

$$\theta^{\mathrm{TS}}_t V^{\mathrm{d,f}}_t = \mathrm{E}^{\mathcal{Q}^{\mathrm{d}}}_t \left[e^{-\int_t^T r^{\mathrm{F,d}}_u \,\mathrm{d}u} \theta^{\mathrm{TS}}_T V^{\mathrm{d}}_T \right]. \tag{4.13}$$

The holding position of the derivative, θ^{TS}, is determined to make the trading strategy self-financing. As we discussed in the previous subsection, the collateral account grows at rate of λ^{f}_t, therefore, in the continuous model setting, the gain process of one unit of derivative, denoted as g^V, is

$$\mathrm{d}g^V_s = \mathrm{d}V^{\mathrm{d,f}}_s + \lambda^{\mathrm{f}}_s X_s C^{\mathrm{f}}_s \,\mathrm{d}s\,, \quad s \geqslant t\,. \tag{4.14}$$

The self-financing condition requires

$$\theta^{\mathrm{TS}}_s \,\mathrm{d}g^V_s = \mathrm{d}\left(\theta^{\mathrm{TS}}_s V^{\mathrm{d,f}}_s\right). \tag{4.15}$$

Substituting (4.14) into (4.15) gives

$$\theta^{\mathrm{TS}}_s \lambda^{\mathrm{f}}_s X_s C^{\mathrm{f}}_s \,\mathrm{d}s = V^{\mathrm{d,f}}_s \,\mathrm{d}\theta^{\mathrm{TS}}_s + \mathrm{d}[\theta^{\mathrm{TS}}, V^{\mathrm{d,f}}]_s.$$

If we further assume that the derivative is collateralized with the collateral-ratio as defined in (3.37), then

$$\theta^{\mathrm{TS}}_s \lambda^{\mathrm{f}}_s \gamma_s V^{\mathrm{d,f}}_s \,\mathrm{d}s = V^{\mathrm{d,f}}_s \,\mathrm{d}\theta^{\mathrm{TS}}_s + \mathrm{d}[\theta^{\mathrm{TS}}, V^{\mathrm{d,f}}]_s. \tag{4.16}$$

It is not difficult to show that one solution to (4.16) is

$$\theta^{\mathrm{TS}}_s = e^{\int_t^s \lambda^{\mathrm{f}}_u \gamma_u \,\mathrm{d}u}, \quad s \in [t, T], \tag{4.17}$$

since $\mathrm{d}\theta^{\mathrm{TS}}_s = \lambda^{\mathrm{f}}_s \gamma_s \theta^{\mathrm{TS}}_s \mathrm{d}s$ and $\mathrm{d}[\theta^{\mathrm{TS}}, V^{\mathrm{d,f}}]_s = 0$. Clearly, $\theta^{\mathrm{TS}}_t = 1$. Substitute (4.17) into (4.13), we have

$$V^{\mathrm{d,f}}_t = \mathrm{E}^{\mathcal{Q}^{\mathrm{d}}}_t \left[e^{-\int_t^T r^{\mathrm{F,d}}_u \,\mathrm{d}u} e^{\int_t^s \lambda^{\mathrm{f}}_u \gamma_u \,\mathrm{d}u} V^{\mathrm{d}}_T \right],$$

which is the pricing formula (4.10) we derived in the last subsection.

5 Further Discussion

5.1 Relationship Between the Values of Two Approaches

As mentioned at the end of Section 4.1, pricing result (4.8) is almost the same as (3.29), except that they are under two possibly different measures, $\mathcal{Q}^{\mathrm{d}}$ in Section 4 and $\mathcal{Q}$ in Section 3. (The same situation for (4.4) and (3.24).) Recall from footnote 16, the measure $\mathcal{Q}$ is obtained by the Girsanov transformation with an appropriate kernel $\boldsymbol{\lambda}$ such that the resulting dynamics of $(\boldsymbol{S}, X)$ has the drift term given in (3.19), while the measure $\mathcal{Q}^{\mathrm{d}}$ is corresponding to the numeraire B^{d}, which is defined in Appendix B, such that $(B^{\mathrm{d}}, \mathcal{Q}^{\mathrm{d}})$ is a numeraire pair (see also Chapter 6 in [17] and Chapter 7 in [19]). Clearly $\mathcal{Q}$ is not come from an equivalent martingale measure (EMM) for a given numeraire, which is actually not needed in the approach in Section 3. Hence, in general, $\mathcal{Q}$ is different from $\mathcal{Q}^{\mathrm{d}}$, and the MtM value by (3.29) may not be equal to that by (4.8), particularly in an incomplete market.

However, we may consider another situation in the original portfolio A of n asset repo contracts in Section 3. If none of the underlying assets is acceptable for collateral as in repo contracts, then to construct the replication portfolio Π_t, one has to finance the long or short $\theta_t^{(i)}$ positions of the i-th asset with its unsecured funding short rate $r_t^{\mathrm{F,d}}$, for $i \in \{1, \ldots, n\}$. Instead of considering the repo contracts as in Section 3, we study the sub-portfolio denoted as A' of the n underlying assets and their funding positions. This portfolio always has value zero because $\theta_t^{(i)} S_t^{(i)}$ amount is funded for the i-th position, i.e.,

$$v_t^{A'} = \mathbf{0} \in \mathbb{R}^n \ , \ \forall t \in [0, T] \ , \tag{5.1}$$

and the gain process

$$\mathrm{d}g_t^{A'} = \mathrm{d}S_t + \mathrm{diag}\left(r_t^{\mathrm{D}} - r_t^{\mathrm{F,d}} \mathbf{1}_n\right) S_t \, \mathrm{d}t \ , \ \ \forall t \in (0, T] \ , \tag{5.2}$$

where $\mathbf{1}_n = (1, \cdots, 1)^\top \in \mathbb{R}^n$. Comparing the last equation in (3.19), in this case, we have the drift term of (S, X) dynamics under $\mathcal{Q}$ as follows

$$\begin{pmatrix} \mathrm{diag}\left(r_t^{\mathrm{F,d}} \mathbf{1}_n - r_t^{\mathrm{D}}\right) S_t \\ \\ \left(r_t^{\mathrm{F,d}} - r_t^{\mathrm{F,f}}\right) X_t \end{pmatrix} , \tag{5.3}$$

which tells us that, for the basic asset $(S, X 1_{\mathrm{f}}, B^{\mathrm{d}})$[21], their B^{d}-deflated gain processes are $\mathcal{Q}$-martingales. Thus both $\mathcal{Q}$ and $\mathcal{Q}^{\mathrm{d}}$ are EMMs. Under the complete market assumption, we have that the two measures agree at least on the σ-information at T[22], and therefore, in this case, the MtM value given by (3.29) also agrees on that given by (4.8).

5.2 Funding Value Adjustments

In this subsection, it is further assumed that both domestic and foreign collateral rates are overnight index rates of the corresponding currencies, unless specified otherwise[23]. It is worth noting that in the MtM value (3.29) for partially foreign collateralized derivative, the first term, which is also (3.32), is the pricing result for the corresponding perfectly collateralized derivative as in [2, 3][24]. The second term, however, is a value adjustment of the price of perfectly collateralized derivative to incorporate the impact of funding mismatching due to imperfection of collateralization, or in the conventional term, FVA, which is defined as follows:

$$\mathrm{FVA}(t) := [V_t^{\mathsf{d,f}}(3.29) - \bar{V}_t^{\mathsf{d,d}}(3.32)] = -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}} \mathrm{d}v} \left(\lambda_u^{\mathsf{d}} V_u^{\mathsf{d,f}} - \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\right) \mathrm{d}u\right] . \tag{5.4}$$

Clearly, the term FVA is naturally defined as the difference of the value of partially foreign collateralized derivative less that of the perfectly collateralized derivative. This is consistent with the concepts of FVA in literature, e.g., [10, 11, 12][25]. From the viewpoint of the derivative desk, it has to hedge a partially foreign collateralized derivative with the most liquid instrument in the market, i.e., the corresponding perfectly collateralized derivative, to fulfill the cashflow liability. Hence, any difference between the value $V_t^{\mathsf{d,f}}$ and the price $\bar{V}_t^{\mathsf{d,d}}$ incurs extra funding cost or benefit for the derivative desk.

The FVA term (5.4) can be further decomposed into the following two major components:

$$\mathrm{FVA}(t) = \mathrm{FVA}_1(t) + \mathrm{FVA}_2(t) \ , \tag{5.5}$$

where

$$\begin{aligned} \mathrm{FVA}_1(t) &:= [\bar{V}_t^{\mathsf{d,f}}(3.31) - \bar{V}_t^{\mathsf{d,d}}(3.32)], \\ \mathrm{FVA}_2(t) &:= [V_t^{\mathsf{d,f}}(3.29) - \bar{V}_t^{\mathsf{d,f}}(3.31)]. \end{aligned} \tag{5.6}$$

We see that FVA_1 means the difference of the value of the fully foreign collateralized derivative less that of the perfectly collateralized derivative, while FVA_2 represents the difference of the value of the partially foreign collateralized derivative less that of the fully foreign collateralized derivative. In below these two FVA terms are thoroughly studied.

[21] Term 1_{f} means one unit of f-currency. Hence $X 1_{\mathrm{f}}$ is a tradable asset in the d-currency market.

[22] See, for example, Chapter 7 in [19] and Chapter 6 in [17].

[23] This assumption may be dropped easily by adding another FVA term as the difference between the current MtM value of the fully domestic collateralized derivative with a collateral rate different from the overnight index rate and that of the OIS discounting.

[24] That is the OIS discounting result. In the current market, it is well accepted that most liquid derivatives can be regarded as perfectly collateralized with overnight index rate as the collateral rate. In particular this is the case for derivatives traded in clearing houses or with standard CSA. As a consequence, it is safe to assume that most market quotes are such prices by OIS discounting.

[25] Or the combination of FVA and LVA discussed in [6, 13].

First, substituting (3.31) and (3.32) into the equation of FVA_1 in (5.6) yields

$$\text{FVA}_1(t) = -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v}(\lambda_u^{\mathsf{d}} - \lambda_u^{\mathsf{f}})\bar{V}_u^{\mathsf{d,f}}\,\mathrm{d}u\right], \tag{5.7}$$

which is due to the mismatch of funding cost to fulfill collateral requirements in fully collateralized case[26]. If the funding spreads of payoff and collateral currencies happens to be zero, i.e., $\lambda^{\mathsf{d}} = \lambda^{\mathsf{f}}$, then FVA_1 vanishes even if the payoff and collateral are of different currencies. From hedging point of view, when the derivative desk hedge a fully foreign collateralized derivative with its corresponding perfectly collateralized trade, it receives collateral (or posts collateral, respective) in f currency if it is in the money (or out of the money, respectively) for the original derivative, and posts collateral (or receives collateral, respectively) in d currency for the hedging position, with the corresponding dividend yield λ_u^{d} or λ_u^{f}. The desk's collateral commitment and its different funding cost for d and f currencies leads to the value adjustment in (5.7), which can be a cost or a benefit depending the difference of the funding spreads of the two currencies[27], even though the collateral of the original derivative is always equivalent to its MtM at any time.

Based on (3.36), an equivalent form of the MtM value of fully foreign collateralized derivative reads:

$$\bar{V}_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathsf{C,d}}\,\mathrm{d}u}e^{-\int_t^T(\lambda_u^{\mathsf{d}} - \lambda_u^{\mathsf{f}})\,\mathrm{d}u}V_T^{\mathsf{d}}\right], \tag{5.8}$$

which implies that the fully foreign collateralized derivative would be valued by discounting its payoff with a synthetic discount curve with short rate $r_u^{\mathsf{C,d}} + \lambda_u^{\mathsf{d}} - \lambda_u^{\mathsf{f}}$. This approach leads to the *multiple discounting* framework depending on collateral currency which is widely employed in industry, e.g., [4, 5]. In this way, impact of FVA_1 can be replaced by choosing an appropriate (synthetic) discounting curve, though extra attention should be paid on the correlation between derivative payoff and the short rates/spreads in the general cases.

Furthermore, for a very special single currency case[28], instead of regarding d and f as currencies, let us assume that the superscript d stands for a "standard" collateralization such that the collateral rate $r_u^{\mathsf{C,d}}$ is the overnight index rate of the payoff/collateral currency, and that the superscript f for a "non-standard" collateralization such that its collateral rate $r_u^{\mathsf{C,f}}$ is different from the overnight index rate. For the same derivative desk the funding rate is the same, i.e., $r_u^{\mathsf{F,d}} = r_u^{\mathsf{F,f}}$. Therefore, to value this fully domestic collateralized derivative with collateral rate different from overnight rate, it holds that

$$\bar{V}_t^{\mathsf{d,f}} = \mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_u^{\mathsf{C,f}}\,\mathrm{d}u}V_T^{\mathsf{d}}\right], \tag{5.9}$$

i.e., discounting with the "non-standard" collateral rate, which seems likw the collateral rate adjustment (CRA) proposed in [14, 15] when collateral rate differs from overnight index rate[29]. From our analysis above, it would be more appropriate to make this adjustment on the funding spread rather than simply on the collateral rate. Therefore, this is a *funding spread adjustment*, or, conventionally, FVA.

Second, Substituting (3.29) and (3.31) into the equation of FVA_2 in (5.6) gives

$$\begin{aligned} \text{FVA}_2(t) &= -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v}\lambda_u^{\mathsf{d}}\left(V_u^{\mathsf{d,f}} - \bar{V}_u^{\mathsf{d,f}}\right)\mathrm{d}u\right] \\ &\quad -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}\left(\bar{V}_u^{\mathsf{d,f}} - X_u C_u^{\mathsf{f}}\right)\mathrm{d}u\right] \\ &=: \text{FVA}_2^{\text{MtM}}(t) + \text{FVA}_2^{\text{Collateral}}(t). \end{aligned} \tag{5.10}$$

FVA_2, as the funding value adjustment due to partial collateralization, can be further decomposed into two parts: $\text{FVA}_2^{\text{MtM}}(t)$ due to mismatch between MtM value of the partially collateralized derivative and that of the corresponding fully collateralized trade, as well as $\text{FVA}_2^{\text{collateral}}(t)$ due to mismatch (shortfall) of collateral amount between them. From hedging point of view, the derivative desk hedges a partially foreign collateralized derivative with its corresponding fully foreign collateralized trade. Since there the two derivatives have different MtMs

[26]Notice this is not perfect collateralization because the collateral currency is not the same as the derivative payoff currency.

[27]If the collateral currency can be chosen from a set of different currencies, the derivative desk would choose the one most in its favour to reduce its funding cost when it is out of the money. This leads to the embedded cheapest-to-deliver (CTD) option in collateral management. On the other hand, when it is in the money, it seems to sell such a CTD option to its counterparty. This optionality is not within the scope of this paper.

[28]Where both payoff and collateral are in the same currency.

[29]However, the concrete form of CRA is not provided in [14, 15] so we are unable to make accurate comparison.

$V_u^{\mathsf{d,f}}$ and $\bar{V}_u^{\mathsf{d,f}}$, respectively, the derivative desk's accounting profit or loss $V_u^{\mathsf{d,f}} - \bar{V}_u^{\mathsf{d,f}}$ of the portfolio consisting of the original derivative and its hedging position denominated in d currency on its book, but this MtM profit or loss amount is not realized. Otherwise, the derivative desk would have used this realized profit in cash to earn $r_u^{\mathrm{F},\mathsf{d}}$ return (or have borrowed in $r_u^{\mathrm{F},\mathsf{d}}$ to cover the realized loss, respectively) if it is in the money (or out of the money, respectively) for this portfolio, and would have paid back (or have received, respectively) the accrued interest at next MtM calculation date[30], with the return rate $r_u^{\mathrm{C},\mathsf{d}}$ due to the cash position and the calculation frequency equivalent to the collateral position. However, this proceeds of this unrealized profit or loss cannot be booked, thus $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ occurs. On the other hand, the derivative desk only receives (or posts, respectively) C_u^{f} amount of collateral in f currency, but has to post (or receive, respectively) $\bar{C}_u^{\mathsf{f}} = \bar{V}_u^{\mathsf{d,f}}/X_u$ amount of collateral in f currency, with dividend yield λ_u^{f}. Then $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ for this part follows. Again, $\mathrm{FVA}_2(t)$ could be cost or benefit depending on the MtM dynamics.

In the special case that the collateral is posted in d currency as well, FVA_1 vanishes. In addition, though both $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ and $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ exist, the same funding spread in (5.10) leads to the cancellation of the term with $\bar{V}_u^{\mathsf{d,d}}$, leading to the following effectively FVA amount:

$$\mathrm{FVA} = [V_t^{\mathsf{d,d}}(3.30) - \bar{V}_t^{\mathsf{d,d}}(3.32)] \; = \; -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{d}}(V_u^{\mathsf{d,d}} - C_u^{\mathsf{d}})\,\mathrm{d}u\right], \tag{5.11}$$

which is consistent with the results in [10, 11, 12, 13]. This identical funding spreads in $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ and $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ frequently misleads people simply thinking of the credit exposure being directly used for FVA calculation, like in (5.11).

In a summury, at any future time $u \in (t, T)$, let us consider the whole portfolio of a derivative and its collateral. We notice that any component of this portfolio causes funding adjustment if it cannot be hedged/funded/replicated by the portfolio of the corresponding perfectly collateralized hedging position with its collateral. Therefore, a term *funding exposure* is coined here for such a component with a single funding spread. As a result, the generic FVA can be formed as follows:

$$\mathrm{FVA}(t) = -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}\left(\boldsymbol{\lambda}_u^{\top}\cdot\boldsymbol{V}_u^{\mathrm{F}}\right)\,\mathrm{d}u\right], \tag{5.12}$$

where $\boldsymbol{V}_u^{\mathrm{F}} = (V_u^{\mathrm{F},1}, \ldots, V_u^{\mathrm{F},m})^{\top}$ and each $V_u^{\mathrm{F},j}$ is a funding exposure, for $j \in \{1, \ldots, m\}$, and $\boldsymbol{\lambda}_u = (\lambda_u^{(1)}, \ldots, \lambda_u^{(m)})^{\top}$ is the vector of corresponding funding spread. Then in the partially foreign collateralized case, in (5.12)

$$\boldsymbol{V}_u^{\mathrm{F}} = \left(\bar{V}_u^{\mathsf{d,f}},\, V_u^{\mathsf{d,f}} - \bar{V}_u^{\mathsf{d,f}},\, \bar{V}_u^{\mathsf{d,f}} - X_u C_u^{\mathsf{f}}\right)^{\top}, \quad \boldsymbol{\lambda}_u = \left(\lambda_u^{\mathsf{d}} - \lambda_u^{\mathsf{f}},\, \lambda_u^{\mathsf{d}},\, \lambda_u^{\mathsf{f}}\right)^{\top}. \tag{5.13}$$

Even after simplification by cancelling similar terms in (5.12) with (5.13), it still holds that the funding exposures and corresponding funding spreads are vectors in below

$$\boldsymbol{V}_u^{\mathrm{F}} = \left(V_u^{\mathsf{d,f}},\, -X_u C_u^{\mathsf{f}}\right)^{\top}, \quad \boldsymbol{\lambda}_u = \left(\lambda_u^{\mathsf{d}},\, \lambda_u^{\mathsf{f}}\right)^{\top}. \tag{5.14}$$

Give the above generic form (5.12) of FVA as well as the fact that modelling the funding spread of the derivative desk is at least as hard as counterparty's default process, it implies that the complexity of FVA calculation is not less than that of CVA calculation[31]. Also notice that in the fully foreign collateralized case, though the credit exposure is zero, FVA_1 still exists while CVA vanishes. So in the special case of domestic collateralization, there is only one type of funding exposure, which happens to be equivalent to the credit exposure, and the FVA calculation may be similar to CVA calculation.

6 Conclusion

Derivatives partially collateralized in foreign currencies are studied in this paper and the valuation methodologies by replication and by expectations are presented. These two approaches are further unified and the corresponding FVA terms are discussed.

[30] This is also the next collateral calculation/settlement date.

[31] Notice that here the credit exposure is simply $V_u^{\mathsf{d,f}} - X_u C_u^{\mathsf{f}}$ if the counterparties were defaultable. This argument also works for the case of wrong way risk exists, as in that case the correlation between the desk's funding spread and the exposure has to be taken into account.

Extension of our work in this paper to the case of defaultable counterparties is of our particular interest. It is anticipated that CVA and bilateral FVA will be included, and the double counting between funding benefit adjustment and DVA will be naturally avoided.

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A A Direct Proof of Theorem 3.2

Proof. To show (3.29), first from (3.24), we have

$$V_u^{\mathsf{d},\mathsf{f}} = \mathrm{E}_u^{\mathcal{Q}}\left[e^{-\int_u^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,V_T^{\mathsf{d}} + \int_u^T e^{-\int_u^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{f}}\,\mathrm{d}\xi\right],\quad \forall u \in [t,T].$$

With the Law of Total Expectation and Fubini's Theorem, it holds that

$$\begin{aligned}
&\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}\,\lambda_u^{\mathsf{d}} V_u^{\mathsf{d},\mathsf{f}}\,\mathrm{d}u\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}\,\lambda_u^{\mathsf{d}}\,\mathrm{E}_u^{\mathcal{Q}}\left[e^{-\int_u^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,V_T^{\mathsf{d}}\right]\mathrm{d}u\right] + \\
&\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}\,\lambda_u^{\mathsf{d}}\,\mathrm{E}_u^{\mathcal{Q}}\left[\int_u^T e^{-\int_u^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{d}}\,\mathrm{d}\xi\right] du\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v} e^{-\int_u^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,\lambda_u^{\mathsf{d}} V_T^{\mathsf{d}}\,\mathrm{d}u\right] + \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T\int_u^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v} e^{-\int_u^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,\lambda_u^{\mathsf{d}}\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{f}}\,\mathrm{d}\xi\,\mathrm{d}u\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,V_T^{\mathsf{d}}\int_t^T e^{\int_t^u \lambda_v^{\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{d}}\,\mathrm{d}u\right] + \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{f}}\left(\int_t^{\xi} e^{\int_t^u \lambda_v^{\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{d}}\,\mathrm{d}u\right)\mathrm{d}\xi\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,V_T^{\mathsf{d}}\left(e^{\int_t^u \lambda_v^{\mathsf{d}}\,\mathrm{d}v}\right)\Big|_{u=t}^{u=T}\right] + \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{f}}\left(e^{\int_t^u \lambda_v^{\mathsf{d}}\,\mathrm{d}v}\right)\Big|_{u=t}^{u=\xi}\mathrm{d}\xi\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[\left(e^{-\int_t^T r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v} - e^{-\int_t^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\right)V_T^{\mathsf{d}}\right] + \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T\left(e^{-\int_t^{\xi} r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v} - e^{-\int_t^{\xi} r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\right)\lambda_\xi^{\mathsf{f}} X_\xi C_\xi^{\mathsf{f}}\,\mathrm{d}\xi\right].
\end{aligned}$$

Replacing notation ξ by u in the second term and substituting the above equality to the right-hand side of (3.29) yield

$$\begin{aligned}
&\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_v^{\mathrm{C},\mathsf{d}} dv}\,V_T^{\mathsf{d}}\right] - \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v}(\lambda_u^{\mathsf{d}} V_u^{\mathsf{d},\mathsf{f}} - \lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}})\,\mathrm{d}u\right]\\
=\ &\mathrm{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\,V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}} X_u C_u^{\mathsf{f}}\,\mathrm{d}u\right] = V_t^{\mathsf{d},\mathsf{f}},
\end{aligned}$$

which completes the proof of (3.29). ☐

B A Proof of Theorem 4.1

Proof. Let $B_t^{\mathsf{f}} := e^{\int_0^t r_u^{\mathrm{F},\mathsf{f}}\,\mathrm{d}u}$ for $t \in [0,T]$, then B_t^{f} is the numeraire under the foreign risk neutral measure $\mathcal{Q}^{\mathsf{f}}$. Similarly, let $B_t^{\mathsf{d}} := e^{\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}$ for $t \in [0,T]$, then B_t^{d} is the numeraire under the domestic risk neutral measure $\mathcal{Q}^{\mathsf{d}}$. Applying Fubini's Theorem to (4.2) gives

$$\begin{aligned}
X_t\mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\,\mathrm{d}u\right] &= X_t\int_t^T \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\right]\mathrm{d}u\\
&= X_t\int_t^T \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\frac{B_t^{\mathsf{f}}}{B_u^{\mathsf{f}}}\lambda_u^{\mathsf{f}} C_u^{\mathsf{f}}\right]\mathrm{d}u. \qquad \text{(B.1)}
\end{aligned}$$

Changing measure from $\mathcal{Q}^{\mathsf{f}}$ to $\mathcal{Q}^{\mathsf{d}}$, we have

$$\begin{aligned}
\mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\frac{B_t^{\mathsf{f}}}{B_u^{\mathsf{f}}}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\right] &= \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[\frac{B_t^{\mathsf{f}}}{B_u^{\mathsf{f}}}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\cdot\frac{B_u^{\mathsf{f}}/\left(B_u^{\mathsf{d}}/X_u\right)}{B_t^{\mathsf{f}}/\left(B_t^{\mathsf{d}}/X_t\right)}\right] \\
&= \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[\frac{B_t^{\mathsf{d}}}{B_u^{\mathsf{d}}}\frac{X_u}{X_t}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\right] \\
&= \frac{1}{X_t}\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[X_u e^{-\int_t^s r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\right].
\end{aligned} \tag{B.2}$$

Substituting (B.2) into (B.1) gives

$$X_t\mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\,\mathrm{d}u\right] = \int_t^T \mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[X_u e^{-\int_t^s r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\right]\mathrm{d}u.$$

Apply Fubini's Theorem again, we have

$$X_t\mathrm{E}_t^{\mathcal{Q}^{\mathsf{f}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{F},\mathsf{f}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\,\mathrm{d}u\right] = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[\int_t^T X_u e^{-\int_t^s r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\lambda_u^{\mathsf{f}}C_u^{\mathsf{f}}\,\mathrm{d}u\right]. \tag{B.3}$$

Combining (4.1) and (B.3), the time t PV of the collateralized derivative is

$$V_t^{\mathsf{d},\mathsf{f}} = \mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}} + \int_t^T e^{-\int_t^s r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right].$$

in the d-currency. □

C A Proof of Theorem 4.2

Proof. The formula (4.4) is equivalent with

$$\begin{aligned}
V_t^{\mathsf{d},\mathsf{f}} &= e^{\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_0^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}} + \int_t^T e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right] \\
&= e^{\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_0^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}} + \int_0^T e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u - \int_0^t e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right] \\
&= e^{\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\left(\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_0^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}} + \int_0^T e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right] - \int_0^t e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right),
\end{aligned} \tag{C.1}$$

or equivalently,

$$\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[e^{-\int_0^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_T^{\mathsf{d}} + \int_0^T e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u\right] = e^{-\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_t^{\mathsf{d},\mathsf{f}} + \int_0^t e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u. \tag{C.2}$$

If we introduce

$$M_t := e^{-\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_t^{\mathsf{d},\mathsf{f}} + \int_0^t e^{-\int_0^u r_v^{\mathrm{F},\mathsf{d}}\,\mathrm{d}v}\lambda_u^{\mathsf{f}}X_uC_u^{\mathsf{f}}\,\mathrm{d}u, \tag{C.3}$$

and notice $V_T^{\mathsf{d},\mathsf{f}} = V_T^{\mathsf{d}}$ as in (2.2), then substituting (C.3) into (C.2) gives

$$\mathrm{E}_t^{\mathcal{Q}^{\mathsf{d}}}\left[M_T\right] = M_t, \tag{C.4}$$

implying that M_t is a martingale under $\mathcal{Q}^{\mathsf{d}}$. Meanwhile, (C.3) gives

$$\mathrm{d}M_t = -r_t^{\mathrm{F},\mathsf{d}}e^{-\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}V_t^{\mathsf{d},\mathsf{f}}\,\mathrm{d}t + e^{-\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\,\mathrm{d}V_t^{\mathsf{d},\mathsf{f}} + e^{-\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\lambda_t^{\mathsf{f}}X_tC_t^{\mathsf{f}}\,\mathrm{d}t,$$

or equivalently,

$$\mathrm{d}V_t^{\mathsf{d},\mathsf{f}} = \left(r_t^{\mathrm{F},\mathsf{d}}V_t^{\mathsf{d},\mathsf{f}} - \lambda_t^{\mathsf{f}}X_tC_t^{\mathsf{f}}\right)\mathrm{d}t + e^{\int_0^t r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u}\,\mathrm{d}M_t.$$

□