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A note on elimination of imaginaries for pairs of fields

Stefano Leonesi
Dipartimento di Matematica e Informatica, Università di Camerino, Via Madonna delle Carceri, 62032 Camerino (MC) Italy.
stefano.leonesi@unicam.it

Sonia L'Innocente [i]
Dipartimento di Matematica e Informatica, Università di Camerino, Via Madonna delle Carceri, 62032 Camerino (MC) Italy.
sonia.linnocente@unicam.it

Carlo Toffalori
Dipartimento di Matematica e Informatica, Università di Camerino, Via Madonna delle Carceri, 62032 Camerino (MC) Italy.
carlo.toffalori@unicam.it

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Abstract. We show that, if B is a field and $A \neq B$ is any algebraically closed field extension of B, then the theory of (A, B) eliminates imaginaries if and only if A is a finite extension of B (and so if and only if (A, B) is elementary equivalent to $(\mathbb{C}, \mathbb{R})$).

Keywords: Elimination of imaginaries, pair of fields.

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Introduction

Resorting to quotient structures is a customary tool in Algebra (and general Mathematics). Indeed, from a model theoretic point of view, the classes of an equivalence relation E $\emptyset$-definable in a structure $\mathcal{A}$ can be viewed as "imaginary" elements of A (or of a cartesian power A^n of A), accompanying the "real" elements (or tuples) in A, and the Shelah construction of A^{eq} shows how to make these imaginary points true elements of the structure.

However, this resort to ^{eq} and imaginary elements is sometimes unnecessary. This is essentially the case when, for any $\emptyset$-definable equivalence relation E on A^n, there is a $\emptyset$-definable function f_E from A^n into some A^m such that two tuples $\vec{a}$, $\vec{a'} \in A^n$ are equivalent in E if and only they have the same image by

[i]and Dipartimento di Matematica, Università di Siena, Via del Capitano, 53100 Siena, Italy.

f_E, $f_E(\vec{a}) = f_E(\vec{a'})$: so the imaginary class $E(A^n, \vec{a})$ can be definably replaced by the "real" tuple $f_E(\vec{a}) \in A^m$.

When T is a complete (countable) first order theory and imaginary elements can be forgotten in any model $\mathcal{A}$ of T in the sense we have just explained, one says that T eliminates the imaginaries. The correct definition actually requires that any definable set X in the monster model Ω of T has a canonical base, that is a definably closed C such that, for every automorphism σ of Ω, σ fixes X setwise if and only if σ acts identically on C (see [5], for instance). This definition is equivalent to the one sketched above (directly referring to definable equivalence relations) when the language of T has at least two constant symbols (in particular when we deal with fields). There is an alternative way of introducing our notion, still involving equivalence relations E. In fact, a complete theory T is said to have elimination of imaginaries if, for every $\emptyset$-definable $E = E(\vec{v}, \vec{w})$ in Ω^n for some n and every $\vec{a} \in \Omega^n$, there are a formula $\psi(\vec{v}, \vec{z})$ and a unique $\vec{b}$ in Ω for which $E(\Omega^n, \vec{a}) = \psi(\Omega^n, \vec{b})$. When there are finitely many $\vec{b}$'s with this property, T is said to have weak elimination of imaginaries; when $\vec{b}$ is unique and the formula $\psi(\vec{v}, \vec{z})$ does not depend on $\vec{a}$ (but only on E), T is said to have uniform elimination of imaginaries. Anyway, when dealing with fields,

weak elimination of imaginaries = elimination of imaginaries =
= uniform elimination of imaginaries

(actually, the second equality holds whenever structures with at least two constants for different elements are involved).

Complete theories admitting elimination of imaginaries include algebraically closed fields (of any characteristic) and real closed fields, but exclude separably closed fields of any imperfection degree ≥ 1 (see Messmer's contribution to [5]).

Our interest in this paper is on the elimination of imaginaries for pairs (A, B) of fields, with B a subfield of A. The corresponding first order language L' enlarges the usual signature for fields $L = \{+, \cdot, -, 0, 1\}$ by a unary relation symbol P (to be interpreted in the subfield B). From a model theoretic point of view, pairs of fields include some very classical examples, such as (A, B) with $A \supsetneqq B$ and A, B both algebraically closed or real closed; these structures were the matter of A. Robinson's analysis in [8]. By the way, in the former (algebraically closed) case, the theory of the involved (A, B)'s promptly recalls the beautiful pairs introduced by Poizat in [7]. In fact, for every complete theory T, a pair (A, B) of models of T with A an elementary extension of B is said to be beautiful if B is ω_1-saturated and, for any tuple $\vec{a}$ in A, every type over $B \cup \vec{a}$ is realized in A. Poizat showed that, for a stable T, two beautiful pairs of models of T are always elementary equivalent; furthermore, if T does not have the finite cover property, then the (complete) theory T' of beautiful pairs is also stable. In the particular case of the theory T of algebraically closed fields (of a given characteristic p), T' is just the theory of all pairs (A, B) with $A \supsetneqq B$ algebraically closed of characteristic p. In this sense, our analysis partly overlaps the recent work of Pillay and Vassiliev [6] on elimination of imaginaries for beautiful pairs, showing that, for a stable theory T without the finite cover property, the theory T' of beautiful pairs has elimination of imaginaries if and only if no infinite group is definable in a model of T. This clearly implies , as an immediate corollary, that no theory of pairs of algebraically closed fields $A \supsetneqq B$ can eliminate imaginaries.

However, our perspective in this paper is a little oblique and slanting and concerns complete theories of pairs of fields (A, B) where A, B may not be algebraically closed, or stable, or even models of the same theory. There are several noteworthy examples in this setting; for instance, think of $(\mathbb{R}, \mathbb{R}_0)$ - the other basic case treated by A. Robinson -, or $(\mathbb{C}, \mathbb{Q})$, and so on. In particular, we are interested in the case when A is algebraically closed. Then, there is a theorem of Keisler [3] -complementing A. Robinson's result and extending some other partial contributions of A. Robinson himself - saying that, for every complete (possibly unstable) theory T of fields, the theory T' of the pairs (A, B), with $B \models T$ and A an algebraically closed extension of B, is also complete. Moving inside this setting, we will show that, for A algebraically closed, the complete theory of a pair (A, B) of fields eliminates imaginaries if and only if A is a finite extension of B (which restricts the positive cases to the pairs where $A = B$, or B is real closed and $A = B(\sqrt{-1})$).

We refer to [2] for algebra and to [1, 5, 4] for model theory and in particular elimination of imaginaries.

1 Some trivial and less trivial cases

First let us briefly discuss some trivial situations, where the elimination of imaginaries is quite obvious. In fact, let A be any field whose theory has the elimination property, and let B be $\emptyset$-definable in A. Then the theory of (A, B) does eliminate imaginaries (since it essentially equals the theory of A). This is what happens, in particular, when

- $\star$ $A = B$
- $\star$ A has a prime characteristic p, and $B = A^p$

or also when

- $\star$ B is finite

(in fact B is definable without parameters in A by the formula $v^{p^k} = v$ where p is the characteristic of A and p^k is the cardinality of B). With respect to the last case, it may be worth emphasizing that, at least for $A \neq B$ and $|B| > 2$, elimination of imaginaries fails once we forget multiplication and hence we restrict our attention to the additive structure.

In fact, A becomes in this way a vectorspace of dimension ≥ 2 over the finite field B of size ≥ 3, and it is known that, in this framework, the elimination of imaginaries gets lost.

Now let us propose some less trivial results, still concerning arbitrary fields A and B. The first one is likely to apply to larger settings and to exceed the specific case of fields. However let us state and show it in our particular framework.

1 Theorem. *If the theory of (A, B) eliminates imaginaries in L', then the L-theory of B does.*

PROOF. We can assume (A, B) -hence B- sufficiently saturated. Let $\varphi(\vec{v})$ be an L-formula with parameters, we are looking for a canonical base C of B. Relativize $\varphi(\vec{v})$ to P and get a new L'-formula $\varphi'(\vec{v})$. As the theory of (A, B) eliminates imaginaries, $\varphi'(\vec{v})$ has a canonical base C' in (A, B): C' is definably closed in L' -in particular it is a subfield of A- and, for all automorphisms σ' of (A, B), σ' fixes $\varphi'(A, B)$ setwise if and only if it acts identically on C'. Look at $B(C')$ as an extension of B. In this perspective, $B(C')$ does not include any element b transcendental over B. Otherwise

$$\sigma'_{\restriction \overline{B}} = id_{\restriction \overline{B}}, \quad \sigma'(b) = b + 1$$

enlarges to an automorphism of (A, B) preserving $\varphi'(\vec{v})$ (actually $\varphi(\vec{v})$), but moving some point in C'; hence b cannot occur in C'. Similarly, no element b separable algebraic over B (and out of B) can lie in $B(C')$; in fact the minimum polynomial of b over B has degree > 1, and again

$$\sigma'_{\restriction B} = id_{\restriction B}, \quad \sigma'(b) \neq b \text{ coniugate of } b \text{ over } B$$

defines an automorphism of B and even of (A, B); σ fixes B, hence $\varphi'(\vec{v})$, but it is not identical on C'.

So $B(C')$ is a purely inseparable extension of B. Let $b \in C' - B$. Then the characteristic of B -and A- is a prime p, and $b \in B^{p^h} - B^{p^{h-1}}$ for some positive integer h (in particular b is the only p^h-root of some $b' \in B$). As C' is definably closed, $b \in C'$. So b can be taken away from C' (as it is anyway represented by b', and any automorphism σ' of (A, B) fixes b if and only if it fixes b').

In conclusion, we can assume $C' \subseteq B$ and, unless replacing C' by a suitable extension in B, C' is definably closed in B. Now observe that any automorphism σ of B can be easily extended to an automorphism σ' of (A, B). Moreover σ

fixes C' pointwise if and only if σ' does, hence if and only if σ' preserves $\varphi'(\vec{v})$ and, in conclusion, if and only if σ preserves $\varphi(\vec{v})$. QED

Hence, whenever the theory of B cannot eliminate imaginaries in L, the theory of (A, B) inherits this negative feature in L'. For instance, if B is any separably closed field of prime characteristic and imperfection degree ≥ 1 and A is any extension of B (possibly the algebraic closure $\overline{B}$ itself), then the theory of the resulting pairs does not eliminate imaginaries. We are going to show a much more general negative result later.

2 The main analysis

We are moving now to the main part of this paper. In fact, we are going to show that, for A an algebraically closed field and (A, B) a proper field extension,

the complete theory of (A, B) eliminates imaginaries
if and only if
A is a finite extension of B

(which means B a real closed field and $A = B(\sqrt{-1})$). In detail, first we will show that our condition is sufficient (Theorem 2), and then we will see that it is also necessary (Theorem 3).

2 Theorem. *The complete theory of the pairs (A, B) where B is a real closed field and $A = B(i)$ (so A is an algebraically closed field of characteristic 0) eliminates imaginaries (here i denotes $\sqrt{-1}$, as usual).*

Proof. As a preliminary step, let us show that the theory of (A, B, i) with B real closed and $A = B(i)$ eliminates imaginaries in a language with an additional constant symbol for i. The crucial point here is that (A, B, i) is $\emptyset$-definable in B as follows:

- ⋆ A is given by B^2 (so identifying any $a \in A$ with the ordered pair $(a_0, a_1) \in B^2$ corresponding to the canonical composition $a = a_0 + ia_1$),
- ⋆ B is recovered as $\{(a_0, 0) : a_0 \in B\}$, i as $(0, 1)$,
- ⋆ finally, the addition and multiplication of A are defined in B in the usual way ruled by i.

Now take a sufficiently saturated (A, B, i). Let $X \subseteq A^m$ be definable in (A, B, i); when referring to the interpretation of (A, B, i) inside B^2, X can be represented by a suitable $X' \subseteq B^{2m}$ definable in B (indeed, for every formula $\varphi(\vec{v})$ in the language of (A, B, i) -or even of (A, B)- there is a corresponding formula $\varphi'(\vec{v}(0), \vec{v}(1))$ in the language of B such that, for $\vec{a} = \vec{a}(0) + i\vec{a}(1)$ a tuple in A,

$(A, B, i) \models \varphi(\vec{a})$ if and only if $B \models \varphi'(\vec{a}(0), \vec{a}(1)))$. As real closed fields eliminate imaginaries, X' has a canonical base C in B: for every automorphism σ' of B, σ' fixes X setwise if and only if it acts identically on C (and C is definably closed). Pick an automorphism σ of (A, B, i). Note that σ determines an automorphism σ' of B. Indeed this is true for every automorphism σ of (A, B). However, when enlarged to B^2, $\sigma'(0, 1) = (0, 1)$, and $(0, 1)$ is the element interpreting i in B; so, in order to make our machinery work, we have to restrict our analysis to the σ's with $\sigma(i) = i$ (and this is the reason why we have included i in the language at the beginning of the proof). In fact, in our setting, σ fixes X setwise if and only if $\sigma'(X') = X'$, and hence if and only if σ' (and σ) acts identically on C.

So the (complete) theory of (A, B, i) with A, B as before admits elimination of imaginaries, and even a uniform elimination of imaginaries, because the underlying language has at least two constants (interpreting different elements). Note that i is in the algebraic closure of $\emptyset$ in (A, B) (but not in the definable closure of $\emptyset$, because it is everywhere accompanied by $-i$). But this means that the theory of (A, B) has weak elimination of imaginaries and (as A is a field) this is enough to ensure elimination of imaginaries even for (A, B).

Let us provide the details. Take an equivalence relation $E(\vec{v}, \vec{w})$ $\emptyset$-definable in (A, B). Working in the language with i, and using the uniform elimination of imaginaries, we can find a formula $\psi(\vec{v}, \vec{z}, i)$ (with a possibly dumb i) and, for every tuple $\vec{a}$ in A, a unique $\vec{b}_+$ also in A such that $(A, B, i) \models \forall \vec{v}(E(\vec{v}, \vec{a}) \leftrightarrow \psi(\vec{v}, \vec{b}_+, i))$. There is no loss of generality for our purposes in replacing $\psi(\vec{v}, \vec{z}, i)$ by its conjunction with $i^2 = -1$. Moreover

$$(A, B, i) \models \forall \vec{w}\, \exists ! \vec{z} \forall \vec{v}\, (E(\vec{v}, \vec{w}) \leftrightarrow \psi(\vec{v}, \vec{z}, i)),$$

as said. Recall that (A, B, i) and $(A, B, -i)$ are isomorphic (by conjugation), hence also $(A, B, -i)$ satisfies the last statement, in particular, given $\vec{a}$, there is a unique $\vec{b}_-$ such that

$$(A, B, i) \models \forall \vec{v}\, (E(\vec{v}, \vec{a}) \leftrightarrow \psi(\vec{v}, \vec{b}_-, -i)),$$

(and $(-i)^2 = -1$, of course). In conclusion, there are at most two different tuples $(\vec{b}_+, i)$, $(\vec{b}_-, -i)$ such that

$$(A, B) \models \forall \vec{v}\, (E(\vec{v}, \vec{a}) \leftrightarrow \psi(\vec{v}, \vec{b}_\pm, \pm i)).$$

So $\psi(\vec{v}, \vec{z}, u)$ $(\wedge u^2 = -1)$ is a formula that weakly eliminates the E-imaginaries. *QED*

Note that the previous argument applies to a more general framework, more precisely to the complete theories of pairs of fields (A, B) where A is a proper finite extension of B and the theory of B eliminates imaginaries. In all these cases, the theory of the pairs inherits this elimination property. In fact, write A as $B(t_0, t_1, \ldots, t_n)$ where n is as small as possible (so $n = 0$ when the characteristic of A is 0 and more generally when the primitive element theorem applies). For $i \leq n$, let $f_i(t_0, \ldots, t_{i-1})$ be the minimum polynomial of t_i over $B(t_0, \ldots, t_{i-1})$, d_i denotes its degree; so the products $\prod_{i \leq n} b_i^{h_i}$ (with $0 \leq h_i < d_i$) form a base of A as a vector space over B. Let $\vec{r}$ be the tuple of the coefficients in B of the decompositions of the elements tt' (t, t' in this base) with respect to the base itself. At this point, show that the theory of $(A, B, t_0, \ldots, t_n)$ eliminate imaginaries. The key fact here is that $(A, B, t_0, \ldots, t_n)$ can be interpreted in $(B, \vec{r})$ as follows:

- A is given by a suitable direct power $B^{\prod_{i \leq n} d_i}$,
- B is recovered as $\{(b, 0, \ldots, 0) : b \in B\}$
- addition, multiplication and the t_i's are defined in the obvious way (requiring $\vec{r}$).

Also recall from [1], p. 122, that the elimination of imaginaries is preserved adding parameters; in particular the theory of $(B, \vec{r})$ still eliminates imaginaries. Furthermore $\vec{r}$ is in the definable closure of $\emptyset$ in $(A, B, t_0, \ldots, t_n)$.

Now deduce that the theory (A, B) has weak (and consequently full) elimination of imaginaries. Basically, proceed as in the case of i. Of course, replace the formula $u^2 = -1$ by $\bigwedge_{i \leq n} f_i(u_0, \ldots, u_{i-1}, u_i) = 0$.

Also notice that, for $(t'_0, \ldots, t'_n) \in A$ satisfying the last condition, $A = B(t'_0, \ldots, t'_n)$ and $(A, B, t_0, \ldots, t_n)$, $(A, B, t'_0, \ldots, t'_n)$ are isomorphic.

Anyway, the only setting to which this generalization applies, when we assume A algebraically closed, is just the original framework (A, B) with B real closed and $A = B(i)$ (and does not need any tuple $\vec{r}$ of new parameters).

As said at the beginnig of this section, Theorem 2 exhausts the only case eliminating imaginaries in our sketched framework (so A algebraically closed and $B \neq A$). In fact, the following proposition holds.

3 Theorem. *If A is an algebraically closed field, B is a subfield of A and A is not a finite extension of B, then the theory of (A, B) does not eliminate imaginaries.*

Proof. As usual, assume (A, B) sufficiently saturated. First, observe that there exists some element $a \in A$ transcendental over$B$. This is obvious when B is algebraically closed or real closed; in the other cases, we can use the fact (mentioned in [3], lemma 3.1) that, for every positive integer n, A contains some elements a_n which are not algebraic of degree $\leq n$ over B, and then apply a straightforward compactness argument to obtain a. At this point, look at the formula

$$\varphi(v) : \exists y\,(P(y) \wedge v = a + y)$$

We claim that $\varphi(v)$ (so $B + a$) cannot admit any canonical base C. Suppose towards a contradiction that a canonical base exists: C is definably closed (in particular it is a subfield of A) and, for every automorphism σ of (A, B), σ preserves $\varphi(v)$ if and only if σ fixes C pointwise.

Case 1: $a \in C$.

Let σ act identically on B (and even on the algebraic closure $\overline{B}$ of B) and send a to $a+b$ where $b \in B$ and $b \neq 0$ (whence the element $a+b$ is still transcendental over B); σ can be extended to an automorphism of (A, B). Furthermore, for every y satisfying P, also $y - b$ satisfies P, and hence σ preserves $\varphi(v)$. In spite of this, $\sigma(a) \neq a$, and so σ does not fix C pointwise.

More generally, assume that a is algebraic over $B(C)$; then there are $a_1, \ldots, a_n \in C$ algebraically independent over B such that a is algebraic over $B(a_1, \ldots, a_n)$. Notice that, as a is transcendental over B, the tuple $(a_1, \ldots, a_n)$ cannot be empty (so $n \geq 1$). Furthermore, we can assume that a is not algebraic any more over $B(a_1, \ldots, a_{n-1})$. The exchange property ensures that a_n is algebraic over $B(a_1, \ldots, a_{n-1}, a)$. At this point, define an automorphism σ of (A, B) such that σ acts identically on $B(a_1, \ldots, a_{n-1})$ (and even on its algebraic closure) but, as before,

$$\sigma(a) = a + b$$

for some non zero element $b \in B$ such that $a + b$ is not a conjugate of a with respect to $B(a_1, \ldots, a_n)$. Notice that $\sigma(a_n)$ has to be different from a_n (otherwise a and $\sigma(a)$ admit the same minimum polynomial over $B(a_1, \ldots, a_n)$). Consequently σ does not fix C pointwise; however σ preserves $\varphi(v)$.

Case 2: a is transcendental over $B(C)$.

Let σ act identically on $\overline{B(C)}$, in particular on C, $\sigma(a)$ be transcendental over $B(C)$, $\sigma(a) - a \notin B$ (for instance, let $\sigma(a) = 2a$ when the characteristic of B is not 2). As said, σ fixes C pointwise; however, when y satisfies P, no y' still satisfying P can make the equality $y' + \sigma(a) = y + a$ true. So σ cannot preserve $\varphi(v)$. In conclusion, in both cases, we have reached a contradiction. Hence, no canonical base C is possible. QED

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