A Note on the Reliability of Redundant Systems

F. E. Udwadia

Department of Civil Engineering
University of Southern California
Los Angeles, California 90007

Transmitted by Robert Kalaba

ABSTRACT

This paper attempts to study the effect of the order of redundancy of a system on its instantaneous failure rate, and its life expectancy. Closed form solutions are presented for types of systems which are commonly met with in engineering practice. Both subsystem redundancy and component redundancy are investigated and the latter is shown to be, in general, superior.

INTRODUCTION

It is well known that the reliability of engineering systems can be enhanced by increasing the level of redundancy used. The term redundancy is generally used to connote the replacement of a subsystem (or component) that has failed, by another subsystem (or component) which is in working order. Systems that perform just as "well," despite the failure of one or more of their subsystems (or components), may in this sense, then, be termed redundant. Often, several subsystems in an engineering system may be constructed and put into operation in a "parallel" configuration, thereby ensuring the continued functioning of the composite assembly in spite of the possible failure of one or more (but not all) of these similar redundant units. Examples of the use of this philosophy can be found in numerous and diverse areas of engineering such as computer technology, nuclear engineering, guidance and navigation of aerosystems, structural design, and lifeline engineering.

Though several investigators (e.g. [1, 2]) have studied the effect of redundancies on survival probabilities of parallel configurations, few if any

52 Vanderbilt Ave.,
New York, NY 10017 0096-3003/81/010035+15$02.50

have obtained closed form solutions for the system life expectancies and the associated variances. By using an instantaneous failure rate of the form $\mu_0 t^k$, $k>0$, in this note we explicitly compare the survival functions for various k values as well as the closed form results for the life expectancies. Furthermore, the concept of component redundancy is considered, and it is shown that the composite system reliability obtained with component redundancy is, in general, higher than that achievable with subsystem redundancy.

SUBSYSTEM REDUNDANCY

Consider a "parallel" system configuration having a total of N similar subsystems. We shall refer to $N-1$ as the order of the redundancy. Let the lifetime of each subsystem be represented by a random variable T_0. We define, for each of the subsystems, the survival function $s_0(t)$ as [1]

$$s_0(t)=\Pr\{T_0>t\}. \tag{1}$$

Thus the probability $q_0(t)\,dt$ that a subsystem will fail in $]t,\ t+dt]$ is given by the relation

$$q_0(t)=-\frac{ds_0}{dt}. \tag{2}$$

Defining by $\lambda_0(t)$ the instantaneous failure rate, we have [2]

$$\lambda_0(t)\,dt \triangleq \Pr\{\text{failure occurs in}\,]t,\ t+dt]\,|\ \text{system performs well at time } t\}$$

$$\triangleq \frac{\Pr\{t<T\leqslant t+dt\}}{\Pr\{T>t\}}$$

$$=\frac{q_0(t)\,dt}{s_0(t)}=-\frac{s_0'(t)}{s_0(t)}dt. \tag{3}$$

The function $\lambda_0(t)$ is nonnegative.

We note that the probabilities $\lambda_0(t)\,dt$ and $q_0(t)\,dt$ are quite different from one another. The latter is the *a priori* probability that failure occurs in $]t,\ t+dt]$, whereas the former is the conditional probability that failure occurs in $]t,\ t+dt]$, conditioned upon the subsystem functioning adequately up to time t.

Noting (3), we then have

$$\frac{ds_0}{dt}+\lambda_0(t)s_0(t)=0, \qquad s_0(0)=1, \tag{4}$$

whose solution is

$$s_0(t)=\exp\left(-\int_0^t \lambda_0(\tau)\,d\tau\right). \tag{5}$$

Thus the survival function of the subsystem can be ascertained if its instantaneous failure rate is known. Statistically, $\lambda_0(t)\Delta t$ may be estimated by the ratio of the number of subsystems that fail in $]t, t+\Delta t]$ to those that are in adequate condition at time t.

In this sequel we shall assume $\lambda_0(t)$ to have the form

$$\lambda_0(t)=\mu_0^{(k)}t^k, \qquad k\geqslant 0, \tag{6}$$

where $\mu_0^{(k)}$ is a constant.

The form of the relation (6), in particular, encompasses three types of systems which are commonly encountered in engineering practice. They are represented by:

(a) $k=0$, representing subsystems with a constant failure rate, such as many electronic components and some electromechanical systems;

(b) $0<k<1$, representing subsystems whose failure rate increases with time, though the rate of increase decreases with increasing time; and

(c) $k\geqslant 1$, representing subsystems with a progressively increasing failure rate, the rate of increase of $\lambda_0(t)$ being nondecreasing with increasing time.

Many mechanical and structural systems fall into categories (b) or (c). The superscript k on the $\mu_0^{(k)}$ is used to indicate that for each k the dimension of $\mu_0^{(k)}$ (and therefore its physical interpretation) changes.

Using (5), then

$$s_0^{(k)}=\exp\left(-\mu_0^{(k)}\frac{t^{k+1}}{k+1}\right)\triangleq\exp\left(-f^{(k)}(t)\right). \tag{7}$$

The statistical moments of the subsystem lifetime, T_0, become

$$E_k[T_0^n]=-\int_0^\infty t^n\,ds_0^{(k)}(t) \tag{8}$$

The subscripts and superscripts k will be used to correspond to the exponent of Eq. (6).

Now consider $N+1$ such subsystems connected in a "parallel" configuration. The survival function, $s_N^{(k)}(t)$, of the composite system corresponding to an order of redundancy of N would then become

$$s_N^{(k)}(t)=1-\left[1-s_0^{(k)}(t)\right]^{N+1}, \tag{9}$$

and the normalized instantaneous composite system failure rate, $\overline{\lambda_N^{(k)}(t)}$, would become by (3)

$$\overline{\lambda_N^{(k)}(t)} \triangleq \frac{\lambda_N^{(k)}(t)}{\lambda_0^{(k)}(t)}=(N+1)\frac{s_0^{(k)}(t)}{1-s_0^{(k)}(t)}\cdot\frac{1}{\left[1-s_0^{(k)}(t)\right]^{-N-1}-1}. \tag{10}$$

We note that even if the instantaneous failure rate for each subsystem were a constant ($k=0$), the instantaneous failure rate for the composite system would indeed be a function of time.

Figure 1(a)–(d) shows the survival functions $s_N^{(k)}(\bar{t}^{(k)})$ for $k=0$, $1/2$, 1, and 2. The time scale ($\bar{t}^{(k)} \triangleq t/E_k[T_0]$) is normalized with respect to the expected life $E_k[T_0]$ of the system with zero order redundancy. This normalization, besides being physically meaningful, yields results which are independent of the choice of the parameter value $\mu_0^{(k)}$, for using the expression for $E_k[T_0]$ (which is derived a little later), the argument, $f^{(k)}(t)$, of the exponential as defined by relation (7), becomes

$$f^{(k)}(t)=\left[\bar{t}\,\Gamma\left(\frac{k+2}{k+1}\right)\right]^{k+1}.$$

We observe that the survival functions $s_N^{(k)}(\bar{t}^{(k)})$ differ quite substantially from $s_0^{(k)}(\bar{t}^{(k)})$ when $\bar{t}^{(k)}<1$. This indicates that for periods of time less than the expected life of each of the individual subsystems, the increase in the redundancy diminshes the *a priori* failure probability of the composite system considerably. Furthermore, comparing the curves for different k values, we find that the effect of redundancy for $\bar{t}^{(k)}<1$ is more beneficial for larger k values. Clearly for large enough $\bar{t}^{(k)}(\bar{t}^{(k)}\gg 1)$, the survival function will approach that of a single subsystem, each of the subsystems in the composite having far exceeded their indivdual life expectancies at such large times. The more rapidly the instantaneous failure rate of each subsystem increases with time, the sooner (in time) the survival function of the composite will reach that of the single subsystem.

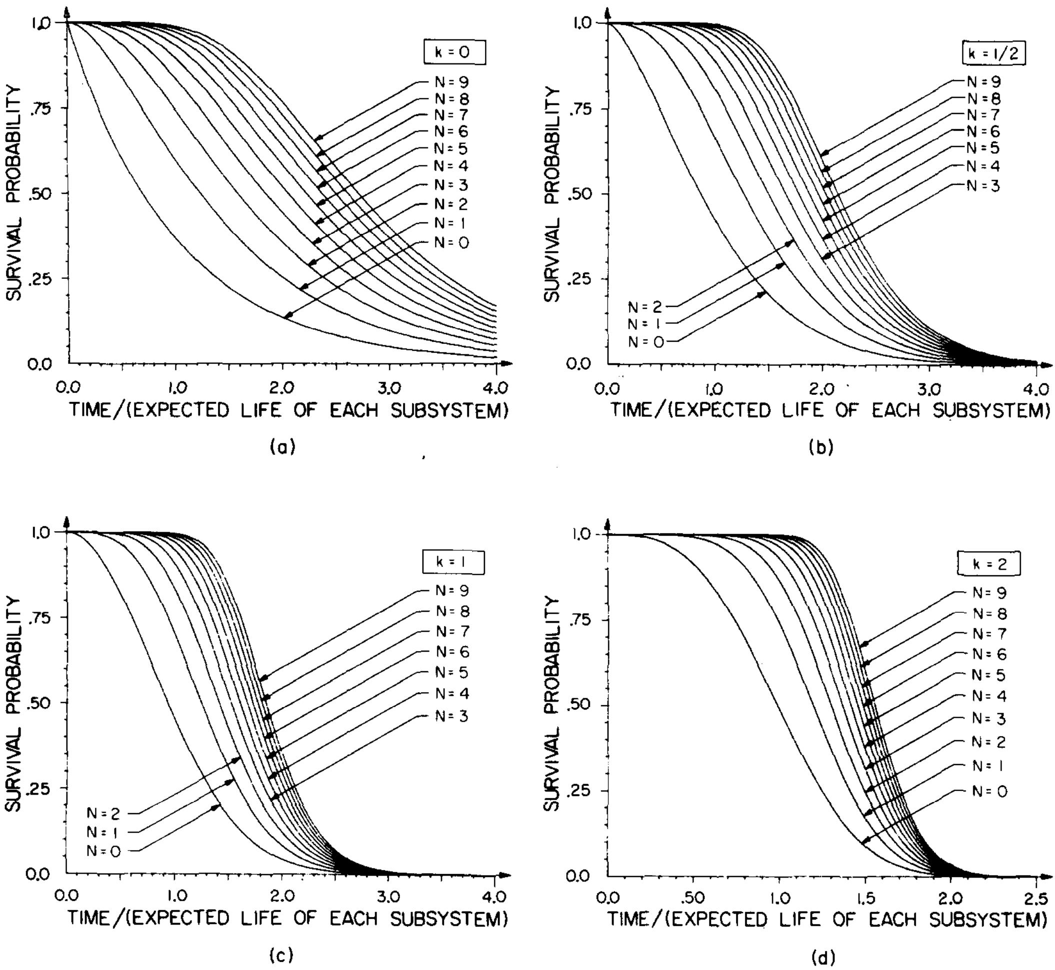

FIG. 1.

Figure 2(a)–(d) shows the normalized instantaneous failure rates, $\bar{\lambda}_N^{(k)}(\bar{t}^{(k)})$, for $k=0$, $\frac{1}{2}$, 1, and 2. For values of $\bar{t}^{(k)}<1$, there is a considerable drop in the value of $\lambda_N^{(k)}(\bar{t}^{(k)})$. For $N\gg1$, Eq. (10) yields

$$\ln\left[\bar{\lambda}_N^{(k)}(t)\right] \sim \ln[N+1] + (N+1)\ln\left[1-s_0^{(k)}(t)\right]. \tag{11}$$

In fact, if the order of the redundancy is increased from N to M, the

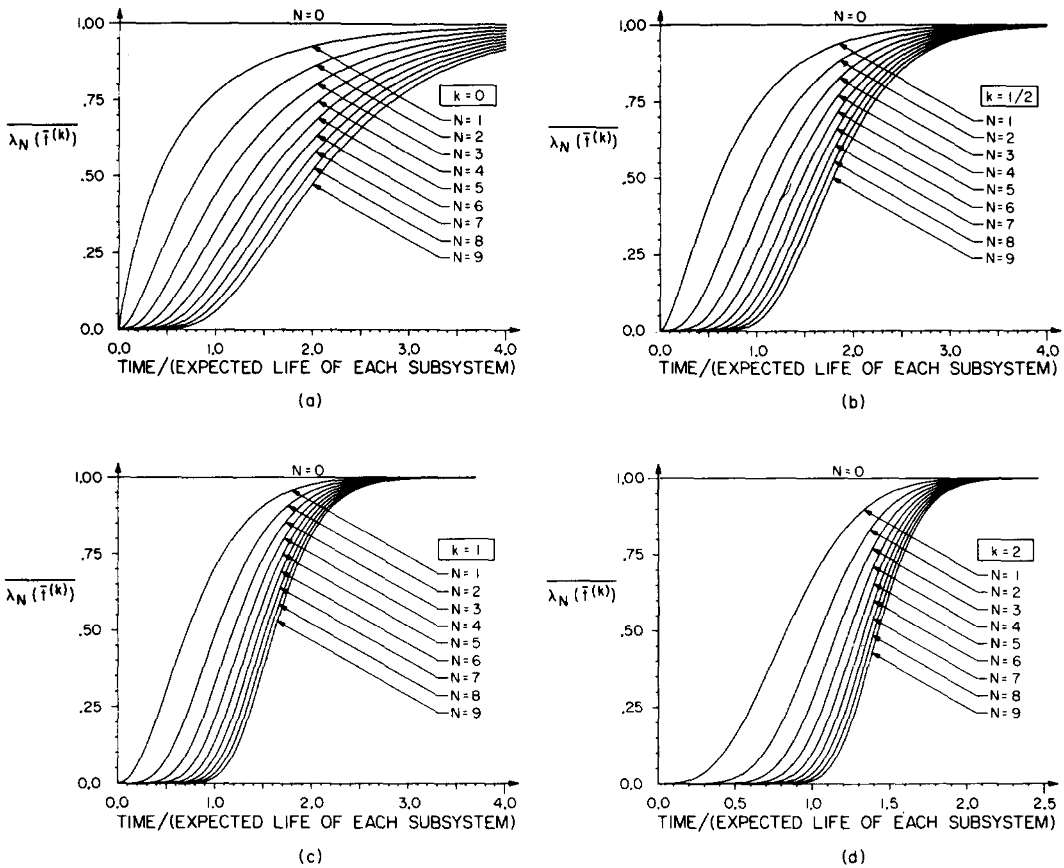

FIG. 2.

logarithm of the ratio (for large N and M), becomes

$$\ln\left[\frac{\lambda_N^{(k)}(t)}{\lambda_M^{(k)}(t)}\right] \sim \ln\left[\frac{N+1}{M+1}\right] + \left(\frac{N+1}{M+1}\right)\ln\left[1 - s_0^{(k)}(t)\right]. \tag{12}$$

Clearly the benefit of increasing the redundancy decreases as N (and therefore M) increases. In the range of principal interest, namely $\bar{t}^{(k)} < 1$, the expressions (11) and (12) are reasonably accurate for $N \gtrsim 5$. For $\bar{t}^{(k)} \gg 1$, we have $s^{(k)}(\bar{t}^{(k)}) \ll 1$, so that the second term in (11) yields a small contribution to reducing the instantaneous system failure rate. In the range $\bar{t}^{(k)} < 1$, the higher k values yield, in general, lower instantaneous failure rates. However, the rapid increase of the subsystem instantaneous failure rate with time, associated with the higher k values, causes the $\lambda_N^{(k)}(\bar{t}^{(k)})$ to rapidly approach $\lambda_0^{(k)}(\bar{t}^{(k)})$ for $\bar{t}^{(k)} \gg 1$.

The expected life of the composite system, $E_k[T_N]$, having an order of redundancy of N, can be expressed as

$$E_k[T_N] = -\int_0^\infty t\, ds_N^{(k)}(t)$$

$$= (N+1)\int_0^\infty t\left[1 - s_0^{(k)}(t)\right]^N s_0^{(k)}(t)\lambda_0^{(k)}(t)\, dt. \tag{13}$$

Using (6) and (7), this yields

$$E_k[T_N] = (N+1)E_k[T_0]\left\{1 + \sum_{m=1}^{N} \binom{N}{m}(-1)^m (m+1)^{-(k+2)/(k+1)}\right\}, \tag{14}$$

where $E_k[T_0]$ is the expected life of each subsystem and is given by

$$E_k[T_0] = \left[\frac{k+1}{\mu_0^{(k)}}\right]^{1/(k+1)} \Gamma\left(\frac{k+2}{k+1}\right). \tag{15}$$

Figure 3 shows the variation of the normalized life expectancy of the composite system, defined as

$$\bar{E}_k[T_N] \triangleq \frac{E_k[T_N]}{E_k[T_0]}, \tag{16}$$

with changing orders of redundancy, N. The normalization once again makes the results invariant with respect to $\mu_0^{(k)}$.

We observe that whereas the instantaneous failure rates $\lambda_N^{(k)}(\bar{t}^{(k)})$ are dramatically reduced, for $\bar{t}^{(k)} < 1$, by increasing the order of the redundancy, the expected life of the redundant system is less substantially altered. By way of example, for $k=1$, going from zero redundancy to that of order 5, while reducing the instantaneous failure rate ($\bar{t}^{(1)} < 1$) by at least a factor of 7, changes the life expectancy by only a factor of 1.2. As the order of redundancy increases, this difference becomes more and more pronounced.

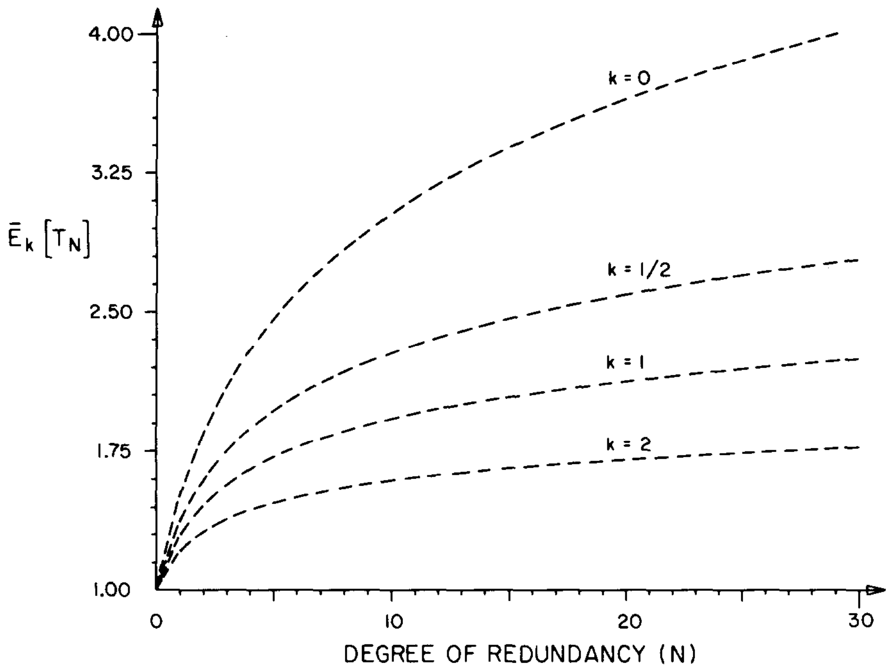

FIG. 3.

The second moment of the system life, T_N, can be expressed using (8), (9), and (3) as

$$E_k[T_N^2] = (N+1)E_k[T_0^2]\left\{1 + \sum_{m=1}^{N}\binom{N}{M}(-1)^m(m+1)^{-(k+3)/(k+1)}\right\}, \tag{17}$$

where $E_k[T_0^2]$ is the second moment of the life T_0 of each subsystem and is defined by the relation

$$E_k[T_0^2] = \left[\frac{k+1}{\mu_0^{(k)}}\right]^{2/(k+1)} \Gamma\left(\frac{k+3}{k+1}\right). \tag{18}$$

The variance of the system's life expectancy can be obtained as

$$\mathrm{Var}_k[T_N] = E_k[T_N^2] - \{E_k[T_N]\}^2. \tag{19}$$

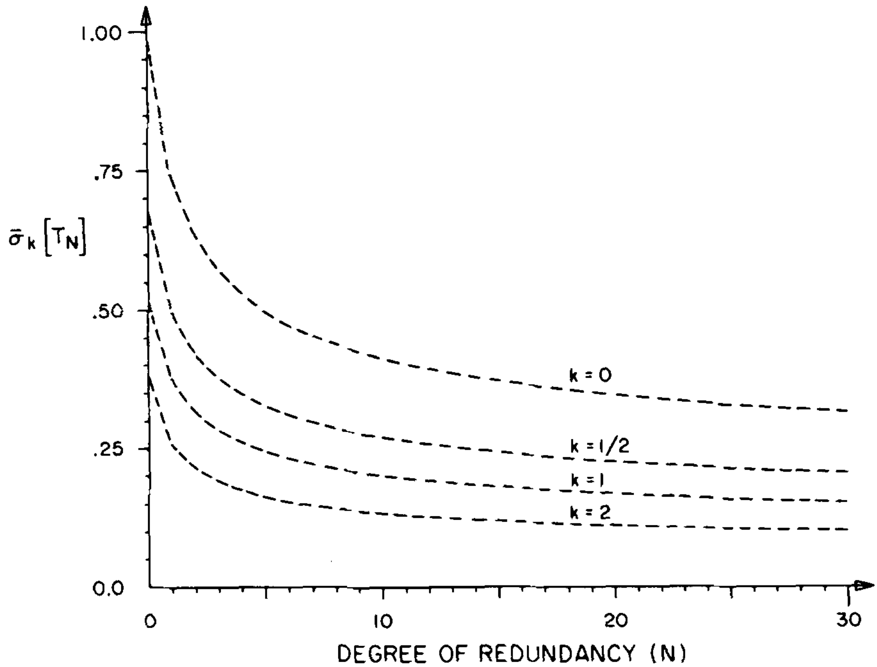

FIG. 4.

Figure 4 shows the normalized standard deviation

$$\overline{\sigma_k[T_N]} = \frac{[\mathrm{Var}_k(T_N)]^{1/2}}{E_k[T_N]} \tag{20}$$

for different orders of redundancy. As N increases the standard deviation $\overline{\sigma_k[T_N]}$ decreases. The larger values of $\bar{E}_k[T_N]$ corresponding to the lower k values are thus accompanied by larger standard deviations.

COMPONENT REDUNDANCY

The concept of subsystem redundancy requires the replacement of one subsystem by another so as to ensure the continued functioning of the composite assembly. Let us now assume that each of the N similar subsystems is composed of k components c_i, $i = 1, 2, \ldots, l$. The subsystems being similar, the component c_i, $i \in (1, l)$, of one subsystem is (statistically) similar to the corresponding component in any of the other systems. For simplicity we assume that the components c_i are in a "series" configuration so that each subsystem, by itself, has zero redundancy.

Instead of subsystem replacement, we now permit component replacement. Thus when the failure of any component c_i belonging to one subsystem occurs, that component is replaced by the corresponding component from another subsystem. Thus we have an ongoing "cannibalization" process where different subsystems are exploited to keep one of them in working condition.

Let $r_i(t)$ be the survival function of the component c_i. If $N+1$ similar subsystems are used, then the survival function for the composite system assembly with component redundancy becomes

$$s_N^*(t)=\prod_{i=1}^{l}\left\{1-\left[1-r_i(t)\right]^{N+1}\right\}. \tag{21}$$

On the other hand, with subsystem redundancy, the survival function $s_N(t)$ becomes

$$s_N(t)=1-\left[1-r_1(t)r_2(t)\cdots r_l(t)\right]^{N+1} \tag{22}$$

Appendix 1 shows that

$$s_N^*(t)\geqslant s_N(t). \tag{23}$$

Consider the case where each subsystem is composed of l components in a "series" configuration, each of which has a constant instantaneous failure rate μ_0/l. Thus,

$$r_i=\exp\left[-\frac{\mu_0}{l}t\right], \tag{24}$$

and the instantaneous failure rate of each subsystem is μ_0.

Figure 5 shows the variation of the survival probability, for $l=2$, with various orders of subsystem redundancy. We observe that component replacement decreases the *a priori* failure probability substantially. This effect is more prominent for large values of N and $\bar{t}^{(0)}<1$.

Figure 6 illustrates the case of a composite system having an order of redundancy of unity. The effect on $s_1^*(t)$ of the number of components, l, in each subsystem is illustrated. The degenerate case $l=1$ corresponds to subsystem redundancy and is shown for comparison. A similar result is

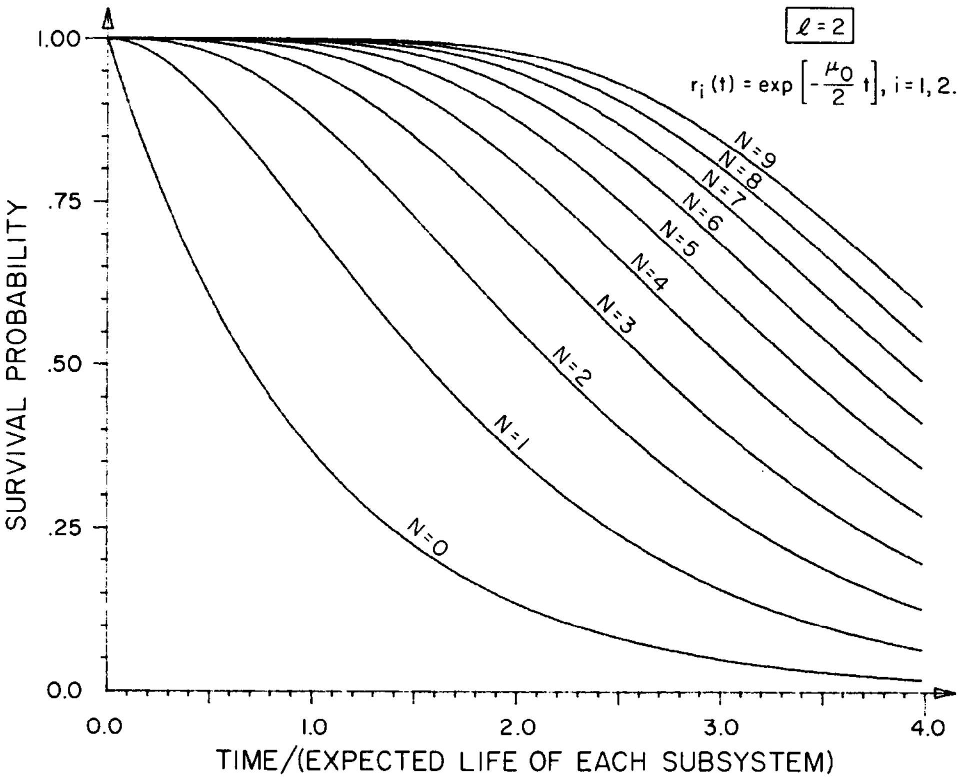

FIG. 5.

shown in Figure 7 for a system whose subsystem redundancy is two ($N=2$). As observed, component redundancy greatly increases the survival probability of the composite system even for $t^{(0)} \sim 2$.

CONCLUSIONS AND DISCUSSION

This paper gives closed form solutions for the instantaneous failure rate, the life expectancy, and the standard deviation of the life of a composite system formed by having several similar redundant subsystems arranged in a "parallel" configuration. The instantaneous failure rate of each subsystem is chosen to be of a form commonly met with in actual engineering practice.

The results show that while such redundancy can greatly reduce the system's instantaneous failure rate, only modest increases in the expected life

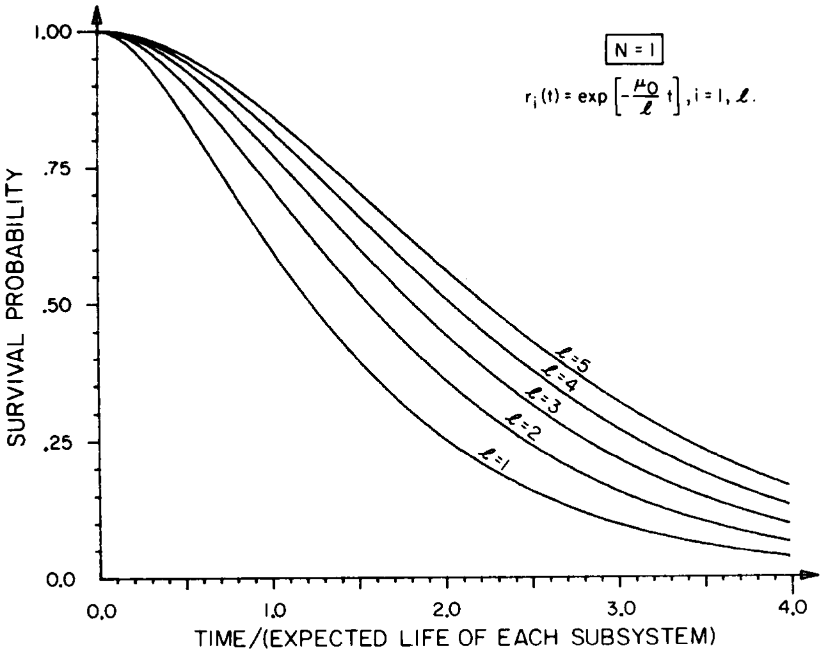
N = 1
$r_i(t) = \exp\left[-\frac{\mu_0}{\ell} t\right], i = 1, \ell.$
SURVIVAL PROBABILITY
1.00
.75
.50
.25
0.0
0.0
1.0
2.0
3.0
4.0
TIME/(EXPECTED LIFE OF EACH SUBSYSTEM)
$\ell=5$
$\ell=4$
$\ell=3$
$\ell=2$
$\ell=1$

FIG. 6.

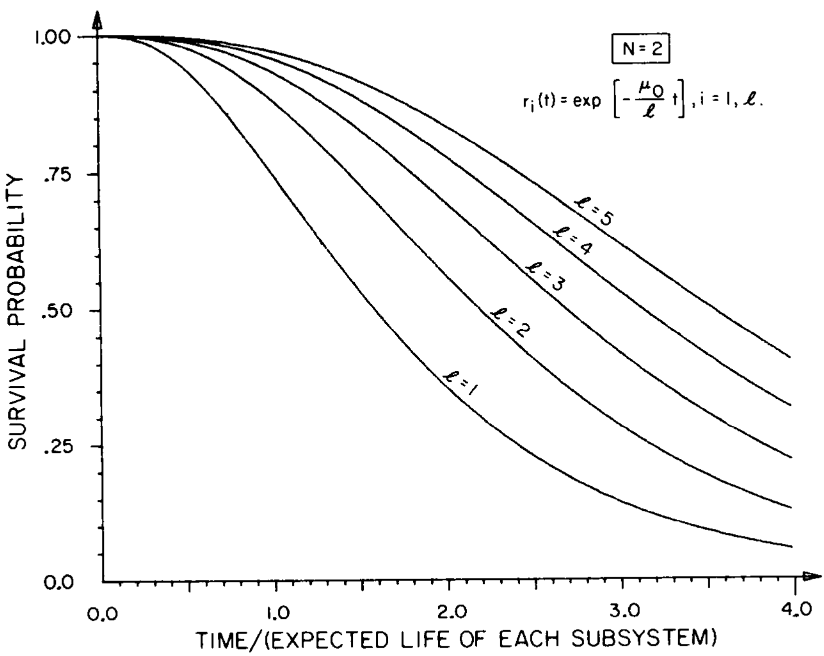
N = 2
$r_i(t) = \exp\left[-\frac{\mu_0}{\ell} t\right], i = 1, \ell.$
SURVIVAL PROBABILITY
1.00
.75
.50
.25
0.0
0.0
1.0
2.0
3.0
4.0
TIME/(EXPECTED LIFE OF EACH SUBSYSTEM)
$\ell=5$
$\ell=4$
$\ell=3$
$\ell=2$
$\ell=1$

FIG. 7.

of the system can be anticipated over periods of time less than the life expectancy of each of the individual subsystems.

Lastly, the concept of component redundancy has been broached, and it has been shown to be in general more effective than subsystem redundancy in enhancing the reliability of composite redundant systems.

APPENDIX I

To show that $s_N^*(t) \geqslant s_N(t)$, we first prove a useful lemma.

LEMMA. *Given s and t such that* $0 \leqslant s, t \leqslant 1$, *for any integer* $n > 0$,

$$(s+t-st)^n \geqslant s^n + t^n - (st)^n \tag{L-1}$$

PROOF. If s and/or t equals zero, the result is obvious. We shall then prove the result for $0 < s, t \leqslant 1$. The result is correct for $n=1$. We shall assume that it is valid for $n=N$ and thence prove it correct for $n=N+1$. Thus,

$$\begin{aligned}(s+t-st)^{N+1} &= (s+t-st)^N(s+t-st)\\ &\geqslant \left[s^N + t^N - (st)^N \right](s+t-st)\\ &\geqslant s^{N+1} + t^{N+1} + (st)^{N+1}\\ &\quad + st\left[t^{N-1} - t^N \right] + st\left[s^{N-1} - s^N \right]\\ &\quad - (st)^N\left[t+s \right].\end{aligned} \tag{L-2}$$

Then to show (L-1) to be valid for $n=N+1$ it will suffice if we can show that

$$st\left[t^{N-1} - t^N \right] + st\left[s^{N-1} - s^N \right] \geqslant (st)^N\left[t+s-2st \right]. \tag{L-3}$$

Dividing both sides of the inequality by $(st)^N$, (L-3) becomes

$$\frac{1}{s^{N-1}}\left[1-t\right] + \frac{1}{t^{N-1}}\left[1-s\right] \geqslant t+s-2st. \tag{L-4}$$

But

$$\frac{1}{s^{N-1}}[1-t]+\frac{1}{t^{N-1}}[1-s]\geqslant 2-s-t, \tag{L-5}$$

and

$$2-t-s\geqslant t+s-2st, \tag{L-6}$$

because (L-6) implies

$$1+st\geqslant s+t, \tag{L-7}$$

which is obvious, since $1-s\geqslant t(1-s)$. Thus (L-3) is valid and the proof is complete.

We note in passing from (L-5) that the inequality (L-1) becomes stronger with increasing values of N.

THEOREM. *Given $r_i(t)$, $i=1,2,\ldots,l$ such that $0<r_i(t)<1$, for all t, we have*

$$\prod_{i=1}^{l}\left[1-(1-r_i(t))^n\right]\geqslant 1-\left[1-r_1(t)r_2(t)\cdots r_l(t)\right]^n. \tag{A-1}$$

for any integer $n>0$.

PROOF. The proof is done by induction. The result is obvious for all n, with $l=1$.

Assume the result is true for $n=N$ and $l=L-1$, we shall prove that it is true for $n=N$ and $l=L$:

$$\prod_{i=1}^{L}\left\{1-[1-r_i(t)]^N\right\}=\left\{\prod_{i=1}^{L-1}\left\{1-[1-r_i(t)]^N\right\}\right\}\left\{1-[1-r_L(t)]^N\right\}$$

$$\geqslant\left\{1-\left[1-r_1(t)r_2(t)\cdots r_{L-1}(t)\right]^N\right\}$$

$$\times\left\{1-\left[1-r_L(t)\right]^N\right\}.$$

Denoting $1-r_1(t)r_2(t)\cdots r_{L-1}(t)$ by s and $1-r_L(t)$ by t, we then have

$$\prod_{i=1}^{L}\left\{1-\left[1-r_i(t)\right]^N\right\} \geq \left[1-s^N\right]\left[1-t^N\right],$$

where $0 \leq s, t \leq 1$.

Noting that

$$1-\left[1-r_1(t)r_2(t)\cdots r_L(t)\right]^N = 1-(s+t-st)^N,$$

we need now only to show that

$$\left[1-s^N\right]\left[1-t^N\right] \geq 1-(s+t-st)^N. \tag{A-2}$$

Simplifying (A-2) yields

$$(s+t-st)^N \geq s^N + t^N - (st)^N, \tag{A-3}$$

which is true by the lemma.

The author is grateful to Professor M. Gorman, University of Southern California for having read the manuscript. This research was supported by a grant from the National Science Foundation.

REFERENCES

1 W. Feller, *An Introduction to Probability Theory and Its Applications*, Vols. 1, 2, Wiley, New York, 1957.

2 W. H. Von Alven (Ed.), *Reliability Engineering*, Prentice-Hall, 1964.