

**A Practical Introduction to  
Differential Forms**

**Alexia E. Schulz**

and

**William C. Schulz**

August 12, 2013

Transgalactic Publishing Company  
Flagstaff, Vienna, Cosmopolis



# Contents

<b>1</b>	<b>Introduction and Basic Applications</b>	<b>1</b>
1.1	INTRODUCTION . . . . .	2
1.2	Some Conventions . . . . .	3
1.3	Some Formulas to Recall . . . . .	4
1.4	Coordinate systems . . . . .	6
1.5	The Algebra of Differential Forms . . . . .	6
1.6	The Operator $d$ . . . . .	9
1.7	Orientation . . . . .	10
1.8	Differential Forms and Vectors . . . . .	12
1.9	grad, curl and div . . . . .	12
1.10	The Poincaré Lemma and it's Converse . . . . .	14
1.11	Boundaries . . . . .	16
1.12	Integrals of Forms . . . . .	18
1.13	Variable Changes . . . . .	19
1.14	Surface integrals . . . . .	20
1.15	The Generalized Stokes Theorem . . . . .	23
1.16	Curvilinear Coordinates I: preliminary formulas . . . . .	26
1.17	Curvilinear Coordinates II: the calculations . . . . .	30
1.18	Surfaces and Manifolds . . . . .	36
1.19	The Dualizing Operator $*$ . . . . .	40
1.20	The Codifferential $\delta$ . . . . .	47
1.21	The Laplacian . . . . .	50
1.22	Maxwell's Equations in 3-space . . . . .	56
1.23	Indefinite Inner Product Spaces . . . . .	61
1.24	Maxwell's equations in Space-Time . . . . .	65
1.25	Energy Flow and Energy Density . . . . .	71
<b>2</b>	<b>Mathematical Theory</b>	<b>73</b>
2.1	INTRODUCTION . . . . .	74
2.2	Manifolds Mappings and Pullbacks . . . . .	74
2.3	Proofs of Stokes Theorem . . . . .	74
2.4	Proofs of the Converse of the Poincare Lemma . . . . .	74
2.5	Permutations . . . . .	75
2.6	The operator $\Phi$ . . . . .	76



## Chapter 1

# Introduction and Basic Applications

## 1.1 INTRODUCTION

These notes began life as an introduction to differential forms for a mathematical physics class and they still retain some of that flavor. Thus the material is introduced in a rather formal manner and the mathematical complexities are put off to later sections. We have tried to write so that those whose primary interest is in the applications of differential forms can avoid the theoretical material provided they are willing to accept the formulas that are derived in the mathematical sections, which are clearly marked as such. Those who wish may read the mathematical sections as they occur, or later, or indeed may put them off to a more convenient time, perhaps in a future life, without loss to the continuity of the applied thread. Anyway, such is my hope. But we want to also emphasize that those who wish will find all the mathematical details available, at a level of rigor usual to the better mathematical physics books. The treatment is mostly local, and what little manifold theory is needed is quietly developed as we go. We have tried to introduce abstract material in circumstances where it is useful to do so and we have also tried to avoid introducing a lot of abstract mathematical material all at one time.

The two areas most completely addressed in these notes, besides the foundational material, are coordinate changes and Maxwell's equations since we feel that these illustrate the power of differential forms quite well. We treat Maxwell's equations in both three and four dimensions in separate sections. We will also look at a few other things.

Notation has been carefully chosen to be consistent with standard tensor notation to facilitate comparison with such treatments, and to facilitate learning basic differential geometry.

The treatment of Maxwell's equations requires the derivation of the potential equations. Although not strictly necessary, we have introduced the codifferential  $\delta$  and the Laplace operator  $\Delta d\delta + \delta d$  since this is the natural route using modern mathematics. For example we point out that the condition of Lorenz can be expressed instantly and easily in terms of the codifferential in four dimensions. And as long as we have it available we can look at a couple of other applications of the Laplace operator on forms.

A justified criticism of these notes might be that many things are done twice, which is not efficient. We have sacrificed efficiency for convenience to the reader who may wish to deal with only one particular thing, and so would like a relatively complete treatment in the section without having to read five others. Similarly, many formulas are repeated at the beginning of sections where they are used, rather than referred to in previous sections. The increase in paper is rather small, and for those getting it electronically there is no waste at all. It is difficult for a mathematician to resist the call of generality but since one of us is a physicist the brakes have been applied, and we hope that the product is a reasonable compromise between the siren song of mathematics and the needs of practical physics.

## 1.2 Some Conventions

Here we will introduce some conventions that will be used throughout these notes. The letter  $A$  will be used for a region of 2-dimensional space, for example the unit disk consisting of points whose distance from the origin is less than or equal to 1. Its boundary would be the unit circle consisting of points whose distance from the origin is exactly 1. We will use the symbol  $\partial$  to indicate the boundary. Thus if  $A$  is the unit disk  $A = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$  then the boundary of  $A$  is  $\partial A = \{x \in \mathbb{R}^2 \mid |x| = 1\}$  which is the unit circle. Notice carefully the difference between the terms DISK and CIRCLE. (DISK and CIRCLE are often confused in common speech.)

The letter  $M$  will be used for a (solid) region of 3 dimensional space, for example the unit ball,  $M = \{x \in \mathbb{R}^3 \mid |x| \leq 1\}$  whose boundary is the unit sphere  $\partial M = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ . (The terms BALL and SPHERE are often confused in common speech, particularly in cases like a beach ball or a basketball since they are filled with air.)

The letter  $S$  will be used for a (2 dimensional) surface in three dimensional space, for example the upper half of the unit sphere. The boundary of this  $S$  would be a circle in the  $x, y$  plane.

If we do not wish to specify dimension, we will use the letter  $K$ . The use of  $K$  indicates that the formula will work in any dimension, and this usually means *any* dimension, not just 1, 2 or 3 dimensional space. Naturally  $\partial K$  means the boundary of  $K$ .

The ball and sphere have analogs in every dimension. It is customary to refer to the ball in  $\mathbb{R}^n$  is the  $n$ -ball and its boundary as the  $(n - 1)$ -sphere. For example, the unit disk is the 2-ball and its boundary, the unit circle, is the 1-sphere. Note that the  $m$ -sphere lives in  $\mathbb{R}^{m+1}$ . It is called the  $m$ -sphere because it requires  $m$  variables to describe it, like latitude and longitude on the 2-sphere.

Also useful to know are the terms open and closed. This is a tricky topological concept, so we will treat it only intuitively.  $K$  is *closed* if it includes its boundary. Thus the unit disk and unit ball are closed. If we remove the boundary  $\partial K$  from  $K$  the resulting set  $K^\circ$  is called *open*. Thus for the unit ball in  $\mathbb{R}^3$  we have

$$\begin{aligned} M &= \{x \in \mathbb{R}^3 \mid |x| \leq 1\} && \text{closed 3-ball} \\ M^\circ &= \{x \in \mathbb{R}^3 \mid |x| < 1\} && \text{open 3-ball} \\ \partial M &= \{x \in \mathbb{R}^3 \mid |x| = 1\} && \text{2-sphere} \end{aligned}$$

We want to give a real world example here but remember it must be inexact since real world objects are granular (atomic) in constitution, so can only approximate the perfect mathematical objects. Some people prefer to eat the closed peach (with fuzzy skin), some people prefer the open peach (fuzzy skin removed, peach $^\circ$ ) and the boundary of the peach,  $\partial\text{peach}$ , is the fuzzy skin. Perhaps this will help you remember. Deeper knowledge of these matters can be found in the wonderful book [2] and also [4].

For functions we will use a slightly augmented variant of the physics convention. When we write  $f : S \rightarrow \mathbb{R}$  we mean a function whose input is a point  $p \in S$  and whose output is a real number. This is theoretically useful but not suitable for calculation. When we wish to calculate, we need to introduce coordinates. If we are dealing with the upper half of the unit sphere (set of points in  $\mathbb{R}^3$  whose distance from the origin is exactly one and for which  $z \geq 0$ ) then we might write  $f(x, y)$  if we choose to represent points in the  $x, y$  coordinate system. Notice, and this is an important point, that the coordinate  $x$  takes as input  $p \in S$  and outputs a real number, it's  $x$  coordinate. Hence the coordinates  $x$  and  $y$  are functions just like  $f$ . If  $S$  is the upper half of the unit sphere in  $\mathbb{R}^3$  then  $x$  and  $y$  are not really good coordinates. It would be better to use longitude and colatitude for my coordinates and then we would write  $f(\phi, \theta)$ .<sup>1</sup> Note use of the same letter  $f$  no matter what the coordinate system, because the  $f$  represents a quantity in physics, whereas in math it represents a functional relationship and we would not use the same letter for different coordinates. Note also that  $f(.5, .5)$  is ambiguous in physics unless you have already specified the coordinate system. Not so with the math convention.

Finally, we will almost always use the letters  $f, g, h$  for functions on  $A, M, S, K$ . Mostly these will occur in coordinate form, for example  $f(x, y, z)$  for a function on  $M$ .

### 1.3 Some Formulas to Recall

You are all familiar with the  $dx, dy, dz$  which occur in the derivative notation  $\frac{dy}{dx}$  and the integral notation

$$\int_M f(x, y) dx dy$$

$$\int_M f(x, y, z) dx dy dz$$

and you recall the Green, divergence and Stokes theorems, which I list here for convenience:

#### Green's theorem

$$\int_{\partial A} f(x, y) dx + g(x, y) dy = \int_A \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dx dy$$

#### The divergence theorem or Gauss's theorem

$$\int_{\partial M} f(x, y, z) dy dz + g(x, y, z) dz dx + h(x, y, z) dx dy = \int_M \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} dx dy dz$$

---

<sup>1</sup>BEWARE.  $\phi$  is longitude in physics but colatitude in mathematics.  $\theta$  is colatitude in physics but longitude in math.



**The Classical Stokes' theorem**

$$\begin{aligned} & \int_{\partial S} f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz \\ &= \int_M \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy \end{aligned}$$

You might be more familiar with the last two in the vector forms

$$\int_{\partial M} \mathbf{v} \cdot d\mathbf{S} = \int_M \operatorname{div} \mathbf{v} dV$$

and

$$\int_{\partial S} \mathbf{v} \cdot d\boldsymbol{\ell} = \int_S \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}$$

There are some conventions on integrals that we will mention now. In former times when integrating over a three dimensional object we would write  $\int \int \int_M \operatorname{div} \mathbf{v} dV$ . This is now completely antiquated, and we will not do it.

On the other hand, there is a convention that when integrating around curves or surfaces that have no boundary we put a small circle on the integral, so that we write

$$\oint_{\partial M} \mathbf{v} \cdot d\mathbf{S} \quad \text{for} \quad \int_{\partial M} \mathbf{v} \cdot d\mathbf{S}$$

Since this is favored by the physics community we will mostly use it. Notice that if a geometric object is the boundary of something, then it itself has no boundary, and so we will use the circled integral almost exclusively with boundaries.

For our purposes we will define a differential form to be an object like

$$f(x, y) dx \quad f(x, y, z) dydz \quad f(x, y, z) dxdydz$$

which we find as integrands in the written out forms of the Green, divergence and Stokes theorem above. If  $\omega$  is a sum of such objects it turns out that the three theorems collapse to one mighty theorem, called *the generalized Stokes theorem*, which is valid for all dimensions:

$$\oint_{\partial S} \omega = \int_S d\omega$$

To use this theorem and for other purposes it is only necessary to

1. Learn the algebra that the  $dx, dy, dz$  satisfy which is almost the same as ordinary algebra with one important exception.
2. Learn the rule for the operator  $d$  which is almost trivial.

Once these are learned differential forms can be manipulated easily and with confidence. It is also useful to learn how various things that happen in vector analysis can be mimicked by differential forms, and we will do this, naively at first and then in much more detail.

If you are concerned about what differential forms ARE, the answer is a little tricky and we are going to put it off for the moment. Later we will discuss the surprisingly dull answer to this question. Incidentally, the difficulty in explaining what they really are is one reason they have not become more common in elementary textbooks despite their extreme usefulness.

Just to give a tiny hint of the geometrical interpretation of differential forms. A two form measures the density of lines of force of a field, as introduced by James Faraday a century and a half ago. For more on this subject see [1] or [8]. We will discuss it a bit more when we have more equipment.

## 1.4 Coordinate systems

Our first job is to talk a little about coordinates. You already know most of this so we can do it quickly. As already specified, we will use the notation  $A$  for a finite region in  $\mathbb{R}^2$  (which is the standard notation for ordinary two dimensional real space). We will use  $M$  for a three dimensional finite region of  $\mathbb{R}^3$  and  $S$  for a curved surface in  $\mathbb{R}^3$ . Standard coordinates for  $A$  would be  $x, y$ , but we might want to use polar coordinates  $r, \theta$  or even more general coordinates  $u, v$ . The important thing for  $A$  is that there be *two* coordinates and that they be "independent", so that  $v$  is not a function of  $u$ .

In the case of  $M$ , a three dimensional region of  $\mathbb{R}^3$ , we will need three coordinates  $x, y, z$  or  $r, \theta, \phi$  or more generally  $u, v, w$  to describe it.

Since  $S$  is a surface in  $\mathbb{R}^3$ , it will be described by two coordinates. In elementary cases this will often be  $x, y$  but in more complicated situations it is often wise to Taylor<sup>2</sup> the coordinates to the geometric object, for example if one is working with the Unit Sphere in  $\mathbb{R}^3$  then the appropriate coordinates would be  $\theta, \phi$  (longitude and colatitude). Proper choice of coordinates can make a nasty problem much more pleasant.

It is important to be able to move from one coordinate system for a geometric situation to a different coordinate system and we will discuss this later.

## 1.5 The Algebra of Differential Forms

We now begin our discussion of the algebra of differential forms. The type of multiplication we will use is called *exterior multiplication*. The term goes back to Grassmann and he invented it to contrast with interior multiplication, which is what we now refer to as the *inner product*. The multiplication symbol used is  $\wedge$  as in  $dx \wedge dy$  but this is very often omitted. We will use it for a while, and then omit it when we get bored with it, but return to using it whenever we think it clarifies matters.

Let us start with a function in  $f(x, y, z)$  on  $\mathbb{R}^3$ . You already know how to

---

<sup>2</sup>bad pun

form  $df$ :

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Recalling that, like  $f$ , the coordinate  $x$  is also a function on  $\mathbb{R}^3$  the previous formula writes the differential of  $f$  in terms of the differentials of the three *special* functions  $x, y, z$ . So we note that there is not much difference between  $df$  and  $dx$ ; they are the same kind of object. All objects of this type (differentials of functions) are collected together in the set

$$\Lambda^1(\mathbb{R}^3)$$

and are called 1-forms.

If we multiply a couple of these together we get objects like

$$f dx \wedge g dy = fg dx \wedge dy \in \Lambda^2(\mathbb{R}^3)$$

Note that functions  $f$  commute with  $dx$ :  $f dx = dx f$ ; see below. Linear combinations of such objects are called 2-forms. And of course there are 3-forms

$$f dx \wedge dy \wedge dz \in \Lambda^3(\mathbb{R}^3)$$

To complete the system we will place the functions in the basement of the building:  $f \in \Lambda^0(\mathbb{R}^3)$ . It is customary to omit the wedge when multiplying by a function; we write

$$f dx \wedge dy \quad \text{for} \quad f \wedge dx \wedge dy$$

There is no significance to this; it is *just* convention.

The algebra of these objects is just like ordinary algebra except for the changes caused by the rule

$$dg \wedge df = -df \wedge dg \quad (\text{anti-commutativity})$$

for the 1-forms  $df$  and  $dg$ . An algebra satisfying this rule is called an exterior or Grassmann algebra. This algebra was invented by Hermann Guenther Grassmann about 1840 in an attempt to find an algebra mirroring elementary geometry. It is sufficient to postulate this rule for the coordinate differentials only,

$$dy \wedge dx = -dx \wedge dy \quad \text{etc.}$$

since the general rule will follow by linearity.

Thus the exterior algebra is *not* commutative. Our rule is often called anti-commutative and is the simplest generalization of commutative, but it has many consequences which seem strange to the beginner. For example, if we substitute  $f$  for  $g$  in the rule  $dg \wedge df = -df \wedge dg$  we get

$$df \wedge df = -df \wedge df$$

so moving the right side to the left side by the usual algebraic processes which are all valid here we get

$$\begin{aligned}df \wedge df + df \wedge df &= 0 \\2 df \wedge df &= 0 \\df \wedge df &= 0\end{aligned}$$

Thus *the product of a one form with itself is 0*, which is very important. Let's look at another example

$$\begin{aligned}(f dx + g dy) \wedge (f dx + g dy) &= f f dx \wedge dx + f g dx \wedge dy + g f dy \wedge dx + g g dy \wedge dy \\&= 0 + f g(dx \wedge dy + dy \wedge dx) + 0 \\&= f g 0 = 0\end{aligned}$$

as promised.

We also see from this that there are no four forms in 3-space, since if we multiply four of the objects  $dx, dy, dz$  together there will be a repetition which will kill the form:

$$dx \wedge dy \wedge dz \wedge dx = -dx \wedge dy \wedge dx \wedge dz = dx \wedge dx \wedge dy \wedge dz = 0 \wedge dy \wedge dz = 0$$

In general, for a space  $K$  of dimension  $n$  there will be forms  $\omega \in \Lambda^j(K)$  for  $j = 0, \dots, n$ . It is not true in general that for  $r$ -forms  $\omega$  with  $r \geq 2$  that  $\omega \wedge \omega = 0$  although this is fortuitously true for dimensions  $n \leq 3$ . There is an example in dimension 4 where  $\omega \wedge \omega \neq 0$  in the problems.

Now let us start to deal a little more abstractly, so we will use a region  $K$  of dimension  $n$ , and consider  $\omega \in \Lambda^j(K)$  and  $\eta \in \Lambda^k(K)$ . Then a little thought shows that

$$\eta \wedge \omega = (-1)^{jk} \omega \wedge \eta$$

For example, with  $\omega = dx \in \Lambda^1(\mathbb{R}^3)$  and  $\eta = dy \wedge dz \in \Lambda^2(\mathbb{R}^3)$  we have

$$\begin{aligned}(dy \wedge dz) \wedge dx &= dy \wedge dz \wedge dx \\&= -dy \wedge dx \wedge dz \\&= dx \wedge dy \wedge dz\end{aligned}$$

and if you look at how this special case works you will see why the general case works.

Note that nothing prevents us from mixing things up as in

$$2 dx + 3 dx \wedge dy$$

but such things do not occur in practise. Forms where each term has the same number of differentials (forms of the same *degree*) are called homogeneous, and we almost always use homogeneous expressions.

## 1.6 The Operator $d$

Our next project is the differential operator  $d$ , which we introduce according to the following four rules, where  $K$  is a space with coordinates  $u^1, \dots, u^n$ :

$d$  is the unique operator that satisfies the following laws

1.  $d$  is a linear operator
2. On the functions  $\Lambda^0(K)$ ,  $df$  is given by the advanced calculus formula

$$df = \frac{\partial f}{\partial u^1} du^1 + \dots + \frac{\partial f}{\partial u^n} du^n$$

3. if  $\omega \in \Lambda^j(K)$  and  $\eta \in \Lambda^k(K)$  then (Leibniz's Rule)

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^j \omega \wedge d\eta$$

4.  $dd = 0$  (and thus  $ddf = 0$  and  $ddu^i = 0$ )

We have used coordinates  $u^1, \dots, u^n$  rather than  $x^1, \dots, x^n$  to emphasize that these coordinates need not be rectangular or even orthogonal.

Let's look at some examples in 3-space of how these rules make everything work. First we examine  $d(fdx)$ . Since  $f \in \Lambda^0(\mathbb{R}^3)$  we have, using rule 3,

$$d(fdx) = df \wedge dx + (-1)^0 f \wedge ddx = df \wedge dx + 0$$

we also used rule 4,  $ddx = 0$ , in the last equation. This derives the practical rule in a special case, and the general case (see problems) will be the same.

$$d(f dx^{i_1} \wedge \dots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

This is the practical rule for  $d$  and the one you will use for almost everything you do, so learn it well.

Now let  $\omega = f dx$  and  $\eta = g dy$ . Then we have

$$\begin{aligned} d(f dx \wedge g dy) &= d(f dx) \wedge g dy + (-1)^1 f dx \wedge d(g dy) \\ &= (df \wedge dx) \wedge g dy - f dx \wedge (dg \wedge dy) \\ &= df g \wedge dx \wedge dy + f dg \wedge dx \wedge dy \\ &= (df g + f dg) \wedge dx \wedge dy \\ &= d(fg) dx \wedge dy \end{aligned}$$

just as we expected from the practical rule since  $f dx \wedge g dy = fg dx \wedge dy$ . This also illustrates how rule 2 is a generalization of Leibniz's formula for the derivative of a product.

## 1.7 Orientation

At this point we must deal with one of the less pleasant aspects of elementary geometry which is orientation. An orientation, to speak loosely, is a sense of twist in a space. For example, in  $\mathbb{R}^2$  our standard sense of twist is counterclockwise; we measure angles from the  $x$  axis in the direction of the  $y$  axis. If we reverse either axis, we get the opposite twist. If the  $y$  axis went *down* instead of *up* then we would measure angles in a clockwise direction and  $\mathbb{R}^2$  would have the *opposite* orientation. If we reverse both axes then the sense of twist returns to counterclockwise.

In  $\mathbb{R}^3$  the standard orientation is given by the following rule: if you place the fingers of your **RIGHT** hand so that the fingers curl from the  $x$  to the  $y$  coordinate axes then your thumb points in the direction of the  $z$  axis. This is called the *right hand rule*. It has become standard to use this in Calculus books only since the 1940's, and the opposite convention is still found in Italian books, so it is wise to check. To appreciate the subtlety of this concept, think of trying to communicate it to inhabitants of a planet in another galaxy. Since nature is almost symmetric in terms of left and right, the only way we know to clarify this is certain non-symmetric aspects of beta decay. This is referred to in physics as parity. Hopefully parity properties remain the same from galaxy to galaxy.

This idea of orientation, though subtle, is strongly coupled with differential forms and is the reason for the anticommutativity. It also is a prime source of mistakes, and great care must be taken to keep things in correct order. In  $\mathbb{R}^2$  (and its subspaces) the correct order, which expresses the orientation properly, is  $dx \wedge dy$ . Thus  $dy \wedge dx$  is in incorrect order as indicated by the minus sign in

$$dy \wedge dx = -dx \wedge dy$$

Recall that when using Green's theorem

$$\oint_{\partial A} f(x, y) dx + g(x, y) dy = \int_A \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dx dy$$

it is *critical* that the direction around the boundary of the left integral be *counterclockwise*. If it is taken clockwise then the two sides of the equation will have opposite signs. This is again due to the orientation which is built into  $\mathbb{R}^2$  but which we seldom notice explicitly. There are similar worries in the use of the divergence theorem and Stokes theorem.

In applications, the principal place where orientation occurs in  $\mathbb{R}^n$  is in  $n$ -forms and  $(n - 1)$ -forms. We will first tell you the general formula and then give you practical methods to make orientation (relatively) easy to deal with. Let the variables used in  $\mathbb{R}^n$  be  $u^1, u^2, \dots, u^n$ . (We have switched from  $x^1, x^2, \dots, x^n$  to  $u^1, u^2, \dots, u^n$  to emphasize that the variables can be general; not necessarily rectangular or orthogonal.) The use of superscripts to number the variables is to conform to tensor analysis standards and we don't need to go into the reasons for it here; just do it! And remember  $u^3$  is the third variable,

not the cube of  $u$ . If we choose an *order* for the variables, which we did by numbering them, this chooses one of the two orientations. Then

$$\begin{aligned} du^1 \wedge du^2 \wedge \dots \wedge du^n &\text{ is in correct order} \\ du^2 \wedge du^1 \wedge \dots \wedge du^n &\text{ is in incorrect order} \end{aligned}$$

because

$$du^2 \wedge du^1 \wedge \dots \wedge du^n = -du^1 \wedge du^2 \wedge \dots \wedge du^n$$

As you can see with a little practice, interchanging any two of the  $du^i$  reverses the sign and changes correct to incorrect order or incorrect to correct order. If you are familiar with permutations, odd permutations of  $du^1 \wedge du^2 \wedge \dots \wedge du^n$  give incorrect order and even permutations give correct order.

That part is easy. The tricky part is the  $(n-1)$ -forms. Here the correct order is (with  $du^i$  missing from the list)

$$(-1)^{i-1} du^1 \wedge \dots \wedge du^{i-1} \wedge du^{i+1} \wedge \dots \wedge du^n$$

The reason for this choice is so that

$$du^i \wedge ((-1)^{i-1} du^1 \wedge \dots \wedge du^{i-1} \wedge du^{i+1}) = du^1 \wedge \dots \wedge du^n$$

which is correct because the  $du^i$  must hop over the  $n-1$  elements  $du^1, \dots, du^{i-1}$  in order to get back into correct order and each hop contributes a minus sign.

So much for theory. In  $\mathbb{R}^3$  correct order is

$$dx \wedge dy \wedge dz \quad \text{correct order}$$

and for 2-forms we have

$$dy \wedge dz, \quad -dx \wedge dz, \quad dx \wedge dy \quad \text{correct order}$$

For practical use it is much better to write these in cyclic order.<sup>3</sup>

$$dy \wedge dz, \quad dz \wedge dx, \quad dx \wedge dy \quad \text{correct order}$$

and the correct order can be easily remembered by writing

$$dxdydzdxdydz$$

and noting that the order of a wedge of two differentials is *correct* if it occurs in this list, for example  $dz \wedge dx$  is correct but  $dx \wedge dz$  is incorrect since  $dxdz$  does *not* occur in the list. Other incorrects are  $dy \wedge dx$  and  $dz \wedge dy$ . The use of differential forms in  $\mathbb{R}^3$  relies critically on writing things with correct order.

---

<sup>3</sup>Cyclic order is a 3-space concept and does not generalize to n-space at all well.

## 1.8 Differential Forms and Vectors

Standard 3-dimensional vector analysis was cobbled together by Josiah Willard Gibbs in the 1890s using pieces from a variety of sources. While it works well for many practical purposes it has large deficiencies from a theoretical standpoint. Those parts of it which concern the dot (scalar) product are fine, but those parts which concern the cross (vector product  $\mathbf{v} \times \mathbf{w}$ ) are mathematically clumsy. To see this, consult any textbook for a proof of the vector triple product law

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

It is often said that the cross product cannot be generalized to higher dimensions but this is not true; what is true is that the analog of the cross product in  $n$  dimensions involves not two but  $n - 1$  vectors. Thus the elementary geometric applications of the cross product can often be reproduced, but the physical applications not so much, which is the reason that for relativity (4 dimensions and space-time rather than just space) we must fall back on Tensor Analysis.

In 3 dimensions there are many formal analogies between differential forms and vector analysis. That is, differential forms will act like vectors in many ways. It is a little difficult to find good mathematical reasons for these analogies and we will not concern ourselves here with what these reasons might be, although we will return to the matter later. The practical consideration is that things will work well if we keep in mind the following rules. For 1-forms we have the correspondences

$$dx \leftrightarrow \hat{\mathbf{i}} \quad dy \leftrightarrow \hat{\mathbf{j}} \quad dz \leftrightarrow \hat{\mathbf{k}}$$

For 2-forms we have the correspondences

$$dy \wedge dz \leftrightarrow \hat{\mathbf{i}} \quad dz \wedge dx \leftrightarrow \hat{\mathbf{j}} \quad dx \wedge dy \leftrightarrow \hat{\mathbf{k}}$$

Note that in the second case we have been careful to place the 2-forms in proper order. This is critical. If you screw this up you will get the wrong sign. Watch  $dz \wedge dx$  term particularly carefully.

With these correspondences kept in mind, we can easily derive many formulas of vector analysis in simple ways. Many things which appear different in vector analysis can be treated in a unified way with differential forms.

## 1.9 grad, curl and div

In this section we show how the three vector operators grad, curl and div occur naturally in the context of differential forms. Since we are in  $\mathbb{R}^3$  we have 0-forms (functions), 1-forms, 2-forms and 3-forms. The  $d$  operator vanishes on 3-forms as previously discussed. Hence we look at  $d$  on 0-forms, 1-forms and 2-forms.

First 0-forms. Let  $f$  be a function (0-form) of the coordinates  $x, y, z$ . Then  $df$  is just the old Calculus  $df$ :

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$



and we see here the components of the gradient

$$\text{grad}f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Next we do 1-forms where we omit terms that are 0:

$$\begin{aligned} d(f dx + g dy + h dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \wedge dx \\ &\quad + \left( \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right) \wedge dy \\ &\quad + \left( \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial z} dz \right) \wedge dz \\ &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx \\ &\quad + \frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial g}{\partial z} dz \wedge dy \\ &\quad + \frac{\partial h}{\partial x} dx \wedge dz + \frac{\partial h}{\partial y} dy \wedge dz \\ &= \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy \wedge dz \\ &\quad + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dz \wedge dx \\ &\quad + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \end{aligned}$$

Now recall that if  $\mathbf{v} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$  then

$$\begin{aligned} \text{curl } \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \end{aligned}$$

Thus we see we have counterfeited the curl on vectors provided we keep the 2-forms in correct order (so the signs come out right) and we use the correspondence

$$dy \wedge dz \leftrightarrow \mathbf{i} \quad dz \wedge dx \leftrightarrow \mathbf{j} \quad dx \wedge dy \leftrightarrow \mathbf{k}$$

Finally, we want to see the differential forms turn up the divergence. Let once again  $\mathbf{v} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$  and using the above correspondence let us apply  $d$  to the form  $f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$ . Then we get

$$d(f dy \wedge dz + g dz \wedge dx + h dx \wedge dy) = df \wedge dy \wedge dz + dg \wedge dz \wedge dx + dh \wedge dx \wedge dy$$

$$\begin{aligned}
&= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial g}{\partial y} dy \wedge dz \wedge dx \\
&+ \frac{\partial h}{\partial z} dz \wedge dx \wedge dy \\
&= \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \wedge dy \wedge dz
\end{aligned}$$

and we see  $\operatorname{div} \mathbf{v}$  as the coefficient of  $dx \wedge dy \wedge dz$ . Of course  $dx \wedge dy \wedge dz$  corresponds to  $d\mathbf{V}$  in vector analysis. We will have much more to say about the relations between these operators and differential forms in the section on change of coordinates.

## 1.10 The Poincaré Lemma and its Converse

Of the many theorems about differential forms, the three most important are the converse of the Poincaré lemma, Stokes' theorem and the Frobenius theorem. Here we treat the Poincaré lemma; Stokes's theorem will be treated in the next section.

The Poincaré lemma was used by Poincaré in his work on the Calculus of Variations. It is very simple; it says that if  $\omega$  is a differential form then  $d\omega = 0$ . Let us see why this works for functions. We will do it in  $\mathbb{R}^3$  space but it works the same in any dimension.

$$\begin{aligned}
df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\
ddf &= \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) dy \wedge dz + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) dz \wedge dx \\
&+ \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) dx \wedge dy \\
&= 0
\end{aligned}$$

because of the equality of mixed partials:  $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ . It is worth noticing that this is one of the places where we need the functions to have two continuous derivatives to guarantee this equality.

So much for the easy stuff. The converse of the Poincaré Lemma says that if  $\omega$  is a differential form and  $\omega \in \Lambda^r(K)$  then there exists a differential form  $\alpha \in \Lambda^{r-1}(K)$  for which  $d\alpha = \omega$ . Sadly the converse is not true in general. To be sure it is true, we need to know that  $K$  is not too complicated a region. Specifically we need to know that  $K$  is *simply connected*. This will be a new concept for most of you. We will try to clarify it with some examples which are of importance in themselves.

First let us look at the unit disk in  $\mathbb{R}^2$  which is all the points in  $\mathbb{R}^2$  whose distance from the origin is less than or equal to 1. We will call this  $D_0(1)$ . Let us draw a curve (say a circle of radius 1/2) then we may "shrink" the curve in a continuous manner until it is down to just a point. (The technical terminology is

that the curve is homotopic to a point.) A *simple closed curve* is a continuous curve with no end points and no intersections with itself. It should be intuitively clear that any simple closed curve in  $D_0(1)$  can be shrunk to a point.

For contrast, consider the *annulus* consisting of points in  $\mathbb{R}^2$  that have distance from the origin less than or equal to 1 and greater than or equal to  $1/2$ . For this region  $K$  some simple closed curves can be shrunk to points and others (those that go round the central "hole") cannot be shrunk to points. As a third example consider the unit disk with just the center removed. Then, exactly like the annulus where a whole disk was removed, a curve round the origin cannot be shrunk to a point, because the point we need, the origin, is not in  $K$ .

**Def** A region  $K$  is *simply connected* if and only if any simple closed curve in  $K$  can be shrunk to a point in  $K$ .

A few more examples, this time in  $\mathbb{R}^3$ . A ball (solid sphere) is simply connected, as is also a ball with the center removed. However, a ball around the origin with the  $z$  axis removed is *not* simply connected and neither is the whole of  $\mathbb{R}^3$  with the  $z$  axis removed. The unit sphere (points in  $\mathbb{R}^3$  at unit distance from origin) *is* simply connected. (Note the contrast to the unit circle in  $\mathbb{R}^2$  which is *not* simply connected.)

These kinds of considerations are called *topological*, and there is a wonderful branch of mathematics which studies such things called topology. Due to the enormous amount of material needed in an engineering curriculum, it is not customary for topology to be included, but if you wish to learn more we recommend the books [2] and [4], which are specifically written with the engineer and physicist in mind.

Now that we know some topology, we can state the theorem:

**Theorem** (Converse of Poincaré Lemma) Let  $K$  be a simply connected region and  $\omega \in \Lambda^r(K)$  and  $d\omega = 0$ . Then there is an  $\alpha \in \Lambda^{r-1}(K)$  such that  $d\alpha = \omega$ .

It should be mentioned that the  $\alpha$  is highly non-unique; there are many  $\alpha$ 's that will work.

In the problems we will see examples where the region is not simply connected and, though  $d\omega = 0$  there is no  $\alpha$  for which  $d\alpha = \omega$ . Practically speaking, this usually takes the form of  $\alpha$  being multi-valued, like  $\sqrt{x}$ , and thus not being a proper function. In this case, we can often manage to get some use out of the multi-valued function provided we restrict our attention to a simply connected subregion. The vector potential of the magnetic field of an infinite vertical current carrying wire is the paradigm example and we will look at it in the problems.

Next we will derive two popular theorems of vector analysis (in  $\mathbb{R}^3$ ) from the converse of the Poincaré lemma.

For the first we again let  $\mathbf{v} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$  and we suppose that  $\text{curl } \mathbf{v} = 0$ . We then form the corresponding differential 1-form  $\omega = f dx + g dy + h dz$ . From the results of the last section, we see that the condition  $\text{curl } \mathbf{v} = 0$  translates into the condition  $d\omega = 0$ . Assuming the region  $K \subseteq \mathbb{R}^3$  is simply connected

we may conclude that there is a 0-form (function)  $k$  for which  $dk = \omega$ , that is

$$dk = \frac{\partial k}{\partial x} dx + \frac{\partial k}{\partial y} dy + \frac{\partial k}{\partial z} dz = f dx + g dy + h dz$$

Translating back into vector notation we have

$$\begin{aligned} \frac{\partial k}{\partial x} \mathbf{i} + \frac{\partial k}{\partial y} \mathbf{j} + \frac{\partial k}{\partial z} \mathbf{k} &= f \mathbf{i} + g \mathbf{j} + h \mathbf{k} \\ \text{grad } k &= \mathbf{v} \end{aligned}$$

We have thus proved the vector analysis theorem

**Corollary** Over a simply connected region  $K \subseteq \mathbb{R}^3$  a curl free vector is a gradient.

There is a very similar theorem which you will prove in the problems. It reads

**Corollary** Over a simply connected region  $K \subseteq \mathbb{R}^3$  a divergence free vector is a curl.

More explicitly, if  $\text{div } \mathbf{v} = 0$  then there is a vector  $\mathbf{w}$  so that  $\text{curl } \mathbf{w} = \mathbf{v}$ .

Although formulas for finding the objects whose existence is asserted exist, they are of no great practical utility and we will not discuss them.

Other formulas which are easily derived at this point and which you will derive in the problems are

$$\begin{aligned} \text{curl grad } k &= 0 \\ \text{div curl } \mathbf{v} &= 0 \end{aligned}$$

These are both consequences of  $dd = 0$  which you probably already figured out. Also available are results about path independent line integrals which we will look at in the problems.

## 1.11 Boundaries

Before we can deal with Stokes theorem we must talk a bit about boundaries and their orientation, and also how to compute integrals of differential forms.

The boundary of a area or surface is simply its edge. We will be satisfied with an intuitive understanding of this and not go into the topological details. The symbol of the boundary of a region or surface  $K$  is  $\partial K$ . For example the boundary of the unit disk in  $\mathbb{R}^2$  is the unit circle, and the boundary of the upper half of the unit sphere in  $\mathbb{R}^3$  is the unit circle in the  $x, y$ -plane. The boundary of the unit ball in  $\mathbb{R}^3$  is the unit sphere. It is common for there to be no boundary; the boundary of the unit sphere in  $\mathbb{R}^3$  is empty, which we write  $\partial S(0, 1) = \emptyset$  where  $\emptyset$  is the empty set.

For Stokes theorem it is important that the boundary  $\partial K$  be oriented correctly relative to  $K$  itself. We will do this with examples, but the methods are

selected so that they work in any dimension. If  $K$  is  $n$ -dimensional in  $n$ -space, like  $A$  or  $M$  then it inherits an orientation from the  $n$ -space, that determined by the order  $x, y$  for  $A$  or  $x, y, z$  for  $M$  and analogously for higher dimensions. To orient the boundary of such regions, which necessarily is described by  $n - 1$  parameters, one uses the exterior normal vector  $\hat{\mathbf{n}}$  to  $K$  so that  $\hat{\mathbf{n}}$  followed by the parameters has the *same* orientation as  $K$ . (We will clarify this with examples.) This gives an order to the parameters which is then the order used below in the integrals. There is no way to get around the fact that this is a bit tricky.

For our first example let  $A$  be the unit disk in  $\mathbb{R}^2$ , with boundary the unit circle and parameter  $t$  for the unit circle. Let  $\mathbf{r}(t)$  trace out the unit circle. Then

$$\hat{\mathbf{T}} = \frac{\frac{d\mathbf{r}}{dt}}{\left| \frac{d\mathbf{r}}{dt} \right|}$$

is the *unit tangent vector*. There are two directions one may go along a curve, and the parameterization must be chosen so that  $\hat{\mathbf{n}}, \hat{\mathbf{T}}$  has the same orientation as  $x, y$ . This amounts to, first, the angle from  $\hat{\mathbf{n}}$  to  $\hat{\mathbf{T}}$  is a *positive* right angle, and second that as  $t$  increases we trace the boundary *counterclockwise* around  $A$ . So this comes down to just going around  $A$  counterclockwise.

For  $M$  the situation is more difficult to see since the boundary will be two dimensional. Suppose the parameters are  $u$  and  $v$ . Let the surface be given by  $\mathbf{r}(u, v)$ . Then when we increase  $u$  and hold  $v$  fixed, a curve will be traced out in certain direction and it will have tangent vector  $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$  pointing in that direction. Similarly for  $\mathbf{r}_v$ . Then the requirement is that  $\hat{\mathbf{n}}, \mathbf{r}_u, \mathbf{r}_v$  have the same orientation as  $x, y, z$ . If the orientation is opposite, reverse the order of the parameters  $u$  and  $v$ .

On a practical note, one usually finds the  $\hat{\mathbf{n}}$  for  $\partial M$  by forming

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

so that one really has only to check whether this  $\hat{\mathbf{n}}$  is pointing *in* or *out* of  $M$  and if it is pointing *in* change the order of  $u$  and  $v$ . Also, one does not actually have to *calculate*  $\mathbf{r}_u \times \mathbf{r}_v$ , only figure out its direction.

As an example, take the upper half of the unit sphere and let us use for coordinates longitude  $\phi$  and colatitude  $\theta$ . A tangent vector to a *phi* curve goes to the right (looking at the sphere from outside) and a tangent vector to the  $\theta$  curve goes down. Crossing these gives a vector pointing *in* so we have the wrong order: the order should be  $\theta, \phi$  not  $\phi, \theta$ . Any 2-forms that occur should thus be presented as  $f(\theta, \phi) d\theta \wedge d\phi$ . This will be critically important when we form integrals.

Also as a practical matter, for  $\partial M \subset \mathbb{R}^3$  one often uses two of  $x, y, z$  as parameters. If this is the case, the proper orders are  $dy \wedge dz, dz \wedge dx, dx \wedge dy$  as one can determine by using the above rules.

The last case we are going to discuss is  $S$ , the surface in  $\mathbb{R}^3$ . If the surface has no boundary then the situation is just that of  $\partial M$  above (practically speaking) and we have already covered the situation; an example would be the unit sphere.

However, if the surface has a boundary, then there is no obvious way to decide which way the normal should point. Mathematically one just has to choose oneself either a direction for the normal  $\hat{\mathbf{n}}$  or, equivalently, an order for the coordinates  $u, v$ . Physically there may be an obvious way to make the choice. Make sure that your  $\hat{\mathbf{n}}$  is consistent with your choice of  $u, v$  so that  $\hat{\mathbf{n}}, \mathbf{r}_u, \mathbf{r}_v$  is oriented like  $x, y, z$ . Once this is settled it only remains to orient the boundary  $\partial S$ . Since  $\partial S$  is one dimensional, it is only a question of choosing which way to go around it. Here is the rule.

Walk around the boundary of the surface  $S$  with your body pointing the same way as  $\hat{\mathbf{n}}$  and your **LEFT** hand pointing toward the surface. The direction you are walking is the correct orientation of the boundary.

This is the orientation you must use for the correct application of Stokes theorem. If you mess it up the sign will be wrong.

Of course in physics the sign often comes out wrong, and one just reverses it at the end. Still, it is reassuring when it comes out right without human intervention.

## 1.12 Integrals of Forms

This is extremely simple. First, the integrals of forms must be taken over regions of the same dimension as the degree of the form. Thus

- a 1-form must be integrated over a curve  $C$
- a 2-form must be integrated over a surface  $S$  or  $A$
- a 3-form must be integrated over a region  $M$

To calculate the integral we recall that in a multiple integral the order of the differentials doesn't matter but for forms the order matters, as the sign will change if two differentials are interchanged. The calculation of an integral is reduced to three steps.

1. Rearrange the integral so that the differentials are in correct order.
2. Remove the wedges to get an ordinary multiple integral
3. Calculate the multiple integral by the appropriate iterated integral

This should become clear from an example. We wish to calculate the integral

$$\int_A x \, dy \wedge dx$$

where  $A$  is the unit disk in  $\mathbb{R}^2$ . Step 1 is to realize that the differentials are not in the right order. We fix.

$$\int_A x \, dy \wedge dx = - \int_A x \, dx \wedge dy$$

Now the differentials under the integral are in the correct order. Step 2 is to throw out the wedges

$$-\int_A x dx \wedge dy = -\int_A x dx dy$$

where the last integral is an ordinary double integral from your advanced calculus course. Step 3 is then to go over to the an iterated integral

$$-\int_A x dx dy = -\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dx dy$$

Step 4, which we did not mention above, is then to plug it into your fancy calculator and get the answer. Another method is to calculate the double integral by switching to polar coordinates

$$-\int_A x dx dy = -\int_0^{2\pi} \int_0^1 (r \cos \theta) r dr d\theta$$

from which we see immediately that the result is 0.

This pretty much is the whole story. With a modest amount of care this will come out right.

## 1.13 Variable Changes

We begin this section with the following observations. Suppose we are calculating

$$\int_A f(x, y) dx \wedge dy$$

Now suppose we find it convenient to change to polar coordinates or some other coordinates  $u, v$ . Then the old  $x, y$  coordinates are functions of the new  $u, v$  coordinates.

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned}$$

Differential Forms allow us to convert the integral immediately. We have

$$\begin{aligned} f(x, y) &= f(x(u, v), y(u, v)) = \tilde{f}(u, v) \\ dx \wedge dy &= \left( \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left( \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \\ &= \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv \end{aligned}$$

Thus

$$\int_A f(x, y) dx \wedge dy = \int_A \tilde{f}(u, v) \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv$$

Naturally there are generalizations to higher dimensions. Before we deal with that, though, we should systematize what we have here. The cleverer students, or those with really good memories, will recognize that expression in brackets as the determinant of the matrix

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

which is called the *Jacobian matrix*<sup>4</sup> of the variable change. Thus we can write the above equation, with  $\det(\cdot)$  signifying the determinant, as

$$\begin{aligned} \int_A f(x, y) dx \wedge dy &= \int_A \tilde{f}(u, v) \det\left(\frac{\partial(x, y)}{\partial(u, v)}\right) du \wedge dv \\ &= \int_A \tilde{f}(u, v) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du \wedge dv \end{aligned}$$

Naturally the description of  $A$  will be quite different in the  $x, y$  and  $u, v$  variables. If we now go over to *unoriented integrals*, that is ordinary double integrals, we will have, assuming that  $x, y$  and  $u, v$  give the same orientation to  $A$ ,

$$\int_A f(x, y) dx dy = \int_A \tilde{f}(u, v) \det\left(\frac{\partial(x, y)}{\partial(u, v)}\right) du dv$$

Since  $x, y$  and  $u, v$  give the same orientation to  $A$ , the determinant in the integral will have a positive value. If the orientations are reversed, it will all be taken care of in the oriented integrals but in the ordinary integrals we need to make the determinant positive by putting in an absolute value, so we have

$$\int_A f(x, y) dx dy = \int_A \tilde{f}(u, v) \left| \det\left(\frac{\partial(x, y)}{\partial(u, v)}\right) \right| du dv$$

This is the famous change of variable formula. In most Calculus books they do not prove it because the proof without differential forms is somewhat difficult. Using differential forms it is quite easy, as you have seen. The reason for this is that differential forms keep control of the orientation, whereas old techniques have to deal with it in an ad hoc and confusing way.

Everything we have done here in 2 dimensions will work perfectly well in  $n$ -dimensions. Enthusiasts may wish to work it out for 3 dimensions. You will then see the connection between differential forms and determinants. In fact, we can assert, with certainty of contradiction from some member of the audience, that determinants are important *because* they are the coefficients in Grassmann Algebra calculations. We will see more of this.

## 1.14 Surface integrals

It is convenient to work out the connection between surface integrals in vector form and differential forms at this point, so we have it available for Stokes

---

<sup>4</sup>properly pronounced Yacobian, although we know it's hopeless



theorem. We will work it out for surfaces in  $\mathbb{R}^3$  but the techniques applied will work just as well for  $(n-1)$ -surfaces in  $\mathbb{R}^n$ . This is one of the places in the theory which, while not difficult, is a little tricky.

While it may not be possible to cover the surface with a single coordinate patch, it is always possible to break up the integral into integrals over each patch and add them. Hence we only need to work out the formula for a single coordinate patch.

The standard thing to have to work out for surface integrals is

$$\int_S P dy \wedge dz + Q dz \wedge dx + R dy \wedge dx$$

If we let  $\mathbf{v} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$  then the above is a disguised form of our old friend from advanced calculus

$$\int_S \mathbf{v} \cdot \hat{\mathbf{n}} dS = \int_S \mathbf{v} \cdot d\mathbf{S}$$

First we need to get some idea of what  $d\mathbf{S}$  should be. We will be using the variables  $u, v$  as coordinates on the surface. If a normal is already available be sure to pick the order of the variables so that  $\hat{\mathbf{n}}, du, dv$  has the same orientation as  $x, y, z$ . The surface will then be described by  $\mathbf{r}(u, v)$ . If we think of moving from a point  $p$  in the  $u$  and  $v$  directions we will have vectors

$$\begin{aligned} \mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v) &\approx \frac{\partial \mathbf{r}}{\partial u} \Delta u \\ \mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v) &\approx \frac{\partial \mathbf{r}}{\partial v} \Delta v \end{aligned}$$

and so the bit of oriented area can be obtained from the cross product

$$d\mathbf{S} = \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \Delta u \Delta v$$

Now remembering that

$$\mathbf{r}(u, v) = x(u, v)\hat{\mathbf{i}} + y(u, v)\hat{\mathbf{j}} + z(u, v)\hat{\mathbf{k}}$$

we have

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \frac{\partial x}{\partial u} \hat{\mathbf{i}} + \frac{\partial y}{\partial u} \hat{\mathbf{j}} + \frac{\partial z}{\partial u} \hat{\mathbf{k}} \\ \frac{\partial \mathbf{r}}{\partial v} &= \frac{\partial x}{\partial v} \hat{\mathbf{i}} + \frac{\partial y}{\partial v} \hat{\mathbf{j}} + \frac{\partial z}{\partial v} \hat{\mathbf{k}} \\ \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) \hat{\mathbf{i}} + \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right) \hat{\mathbf{j}} \\ &\quad + \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right) \hat{\mathbf{k}} \\ &= \frac{\partial(y, z)}{\partial(u, v)} \hat{\mathbf{i}} + \frac{\partial(z, x)}{\partial(u, v)} \hat{\mathbf{j}} + \frac{\partial(x, y)}{\partial(u, v)} \hat{\mathbf{k}} \end{aligned}$$

Going over from  $\Delta$  to  $d$  we have at last

$$\begin{aligned} \mathbf{dS} &= \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv \\ &= \left( \frac{\partial(y, z)}{\partial(u, v)} \hat{\mathbf{i}} + \frac{\partial(z, x)}{\partial(u, v)} \hat{\mathbf{j}} + \frac{\partial(x, y)}{\partial(u, v)} \hat{\mathbf{k}} \right) dudv \end{aligned}$$

Now we also have

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}$$

Finally we have

$$\begin{aligned} &\int_S P dy \wedge dz + P dz \wedge dx + P dx \wedge dy \\ &= \int_S P \frac{\partial(y, z)}{\partial(u, v)} du \wedge dv + Q \frac{\partial(z, x)}{\partial(u, v)} du \wedge dv + R \frac{\partial(x, y)}{\partial(u, v)} du \wedge dv \\ &= \int_S (P \hat{\mathbf{i}} + Q \hat{\mathbf{j}} + R \hat{\mathbf{k}}) \cdot \left( \frac{\partial(y, z)}{\partial(u, v)} \hat{\mathbf{i}} + \frac{\partial(z, x)}{\partial(u, v)} \hat{\mathbf{j}} + \frac{\partial(x, y)}{\partial(u, v)} \hat{\mathbf{k}} \right) du \wedge dv \\ &= \int_S (P \hat{\mathbf{i}} + Q \hat{\mathbf{j}} + R \hat{\mathbf{k}}) \cdot \left( \frac{\partial(y, z)}{\partial(u, v)} \hat{\mathbf{i}} + \frac{\partial(z, x)}{\partial(u, v)} \hat{\mathbf{j}} + \frac{\partial(x, y)}{\partial(u, v)} \hat{\mathbf{k}} \right) dudv \\ &= \int_S \mathbf{v} \cdot \mathbf{dS} \end{aligned}$$

In the next to the last step we switched from an oriented to an unoriented integral since everything was carefully set up to be correctly oriented.

We can sweat a little more out of this. The scalar element of area is of course the absolute value of  $\mathbf{dS}$  so that

$$dS = |\mathbf{dS}| = \sqrt{\left( \frac{\partial(y, z)}{\partial(u, v)} \right)^2 + \left( \frac{\partial(z, x)}{\partial(u, v)} \right)^2 + \left( \frac{\partial(x, y)}{\partial(u, v)} \right)^2} dudv$$

This is itself a very important formula. Also since

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}$$

we have

$$\mathbf{dS} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} dudv = \hat{\mathbf{n}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv = \hat{\mathbf{n}} dS$$

we can write

$$\int_S \mathbf{v} \cdot \mathbf{dS} = \int_S \mathbf{v} \cdot \hat{\mathbf{n}} dS$$

It would be wise to note that the integrals we have been working with in this section are often called *flux integrals*. In contrast, there is another kind of

surface integral which has little to do with what we are doing. This kind of integral is not vectorial in nature. They look like

$$\int_S f(x, y, z) dS$$

A problem that utilized this kind of integral would be finding the center of mass of a hemispherical shell. We have developed the necessary tools to evaluate these integrals, although for us this is a side issue and we only mention it since students should be aware of the two distinct kinds of surface integral.

To evaluate this kind of integral, choose parameter  $u, v$  for the surface, (or a portion of the surface,) and then use the above above formula for  $dS$  to get

$$\begin{aligned} & \int_S f(x, y, z) dS \\ &= \int f(x(u, v), y(u, v), z(u, v)) \sqrt{\left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2} dudv \end{aligned}$$

These kinds of integrals are very sensitive to the choice of parameters, and an integral that might be extremely difficult with parameters  $x, y$  (so that  $z = z(x, y)$ ) might be quite easy with, say,  $\theta, \phi$  as parameters.

## 1.15 The Generalized Stokes Theorem

The generalized Stokes theorem is one of the triumphs of elementary mathematics, subsuming as it does the fundamental theorem of Calculus, Green's theorem, the divergence theorem (= Gauss's theorem), and the ordinary Stokes's theorem. Moreover, it is dimension independent; it works as well in 26 dimensions as in 2 or 3.

**Theorem** (Generalized Stokes Theorem)

Let  $K$  be a oriented subset of  $\mathbb{R}^d$  of dimension  $n$  and let  $\partial K$  be the properly oriented boundary of  $K$ . Let  $\omega$  be an  $(n-1)$ -form. Then

$$\int_K d\omega = \oint_{\partial K} \omega$$

One of the important qualities of this theorem is that it is very easy to remember. Hence it can be used to write down the Green, divergence and Stokes' theorems even if you don't precisely remember where the signs go; it figures that out for you. We will now look at the Green, divergence and Stokes' theorems one at a time and see how they fall out of the generalized Stokes theorem. At the end, for fun, we will come back and look at the fundamental theorem of Calculus, which is tricky because it is too simple.

**Green's theorem** Let  $A \subset \mathbb{R}^2$  be a region with boundary  $\partial A$  oriented counterclockwise. Then

$$\oint_{\partial A} P(x, y) dx + Q(x, y) dy = \int_K \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy$$

**Proof** Let  $\omega = P(x, y) dx + Q(x, y) dy$ . Applying the general Stokes' theorem we have

$$\begin{aligned}
 \oint_{\partial A} P(x, y) dx + Q(x, y) dy &= \int_{\partial A} \omega \\
 &= \oint_K d\omega \\
 &= \int_K d(P(x, y) dx + Q(x, y) dy) \\
 &= \int_K \left( \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \right) \\
 &= \int_K \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \\
 &= \int_K \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy
 \end{aligned}$$

where we used  $dy \wedge dx = -dx \wedge dy$  and in the last line we used the rule from converting from oriented integrals to ordinary double integrals.

Notice that it was not necessary for us to remember where the sign goes in Green's theorem; our methodology automatically puts it in the correct place.

We will do Stokes' theorem next since it looks so similar in many ways to Green's theorem.

**Stokes' Theorem** Let  $S$  be a surface in  $\mathbb{R}^3$  with definite choice of normal  $\hat{\mathbf{n}}$  and correctly oriented boundary  $\partial S$ . Let  $v = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ . Then

$$\begin{aligned}
 \oint_{\partial S} P dx + Q dy + R dz &= \\
 &= \int_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \int_S \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy
 \end{aligned}$$

or in vector notation

$$\oint_{\partial S} \mathbf{v} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{v} \cdot \hat{\mathbf{n}} dS$$

Recall that the boundary of  $S$  is oriented so that when you walk around the boundary with your body in the direction of  $\hat{\mathbf{n}}$  and your left hand reaching in toward the surface then you are walking in the direction of the orientation. (This can be reformulated in terms of fingers and thumb but we find this less confusing.)

**Proof:** Let  $\omega$  be the 1-form

$$\omega = P dx + Q dy + R dz$$

Then applying Stokes' theorem

$$\oint_{\partial S} P dx + Q dy + R dz =$$

$$\begin{aligned}
&= \oint_{\partial S} \omega \\
&= \int_S d\omega \\
&= \int_S d(P dx + Q dy + R dz) \\
&= \int_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \int_S \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \\
&= \int_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \int_S \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
&= \int_S \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}
\end{aligned}$$

where we have used the rules governing the transition from oriented to unoriented (ordinary) integrals. Recall that in unoriented integrals the order of the differentials does not matter, but we have left them in cyclic form as a matter of good practise. The transition to vector form was discussed in the previous section.

Now to the divergence theorem. This is quite easy.

**The Divergence theorem (Gauss's theorem)** Let  $M$  be a region of  $\mathbb{R}^3$  and  $\partial M$  the surface which is its correctly oriented boundary. Then

$$\begin{aligned}
\oint_{\partial M} P dy dz + Q dz dx + P dx dy &= \\
&= \int_M \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz
\end{aligned}$$

or in vector notation

$$\oint_{\partial M} \mathbf{v} \cdot \hat{\mathbf{n}} dS = \int_M \operatorname{div} \mathbf{v} dx dy dz$$

**Proof** Let  $\omega$  be the 2-form

$$\omega = P dy \wedge dz$$

Then applying the generalized Stokes' theorem

$$\begin{aligned}
\oint_{\partial M} P dx dy &= \oint_{\partial M} P dx \wedge dy \\
&= \oint_{\partial M} \omega \\
&= \int_M d\omega \\
&= \int_M d(P dy \wedge dz)
\end{aligned}$$

$$\begin{aligned}
&= \int_M \frac{\partial P}{\partial x} dx \wedge dy \wedge dz \\
&= \int_M \frac{\partial P}{\partial x} dx dy dz
\end{aligned}$$

The other two terms are handled similarly. Notice that we have proved slightly more than the theorem states; we have proved the analog of the theorem *for each term*. This can occasionally be useful.

## 1.16 Curvilinear Coordinates I: preliminary formulas

In this section we will derive formulas for the gradient, curl, divergence and Laplacian in curvilinear coordinates. This is not nearly as straightforward as one might expect, but you will not see these troubles because of good organization. The central trick in this development is to express each of the differential operators in terms of operators that are manifestly coordinate independent. Each of these operators has a coordinate formulation and when they are put together the desired expression arises.

Unfortunately the derivation of some of the basic formulas is a little more conceptual and difficult than most of the material in these notes. To spare those who are interested only in the results, we have put these derivations into Chapter 2 (Mathematical Theory). There is also a lower level introduction to the  $*$  operator in the section after the Curvilinear coordinate sections.

To express grad, divergence and curl invariantly we need two functions,  $\Phi : V \rightarrow V^*$  and  $*$  :  $\Lambda_r(V) \rightarrow \lambda^{n-r}(V)$ . Both of these functions are independent of the choice of the coordinate system, as can be seen in Chapter 2. Although we are just using these formulas to derive the formulas for curvilinear coordinates, they have much wider application in modern mathematics, and effort put into learning them may well pay off in other circumstances. And you will be glad to know that in our circumstances there is not much difficulty.

We will write the formulas we need in  $n$ -space because it is no more difficult than writing them in 3-space, and indeed some things are actually clearer. Also it is important to know which things work generally and which things, like curl, work only in three space.

Let  $u^1, u^2, \dots, u^n$  be curvilinear coordinates. Then we have a position vector  $\mathbf{r}$  and its derivatives which are more or less the base vectors. They require normalization later, but it would be a big mistake to normalize now.

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}(u^1, \dots, u^n) \\
\mathbf{e}_i &= \frac{\partial \mathbf{r}}{\partial u^i} \\
g_{ij} &= (\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j \\
(g^{ij}) &= (g_{ij})^{-1}
\end{aligned}$$

We are going to mostly use the parentheses notation  $(\mathbf{e}_i, \mathbf{e}_j)$  instead of  $\mathbf{e}_i \cdot \mathbf{e}_j$  because it keeps things grouped together better. It is just notation. The  $g_{ij}$  are called the *metric coefficients* and allow one to work with distance and angles in the coordinates  $u^1, u^2, \dots, u^n$ . We think of vectors as being based at particular points which are determined by  $\mathbf{r}$ . This is an elementary form of the concept *tangent space* which we are not going to use.

Any vector  $\mathbf{v}$  based at  $\mathbf{r}(u^1, \dots, u^n)$  can be expressed in terms of the basis vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  which are also thought of as being based at  $\mathbf{r}$ . Thus

$$\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_i = v^i \mathbf{e}_i \quad (\text{sum sign omitted})$$

The sum sign is often omitted when the same letter appears both up and down; since invented by Einstein it is probably a good idea, but you do need to keep it in mind. This simplifies the notation considerably.

Next we need a function where you input the vector  $\mathbf{v}$  and it outputs the  $i^{\text{th}}$  coordinate  $v^i$ . You have known this function for years, but never knew what it was. It is  $du^i$ . Thus

$$du^i(\mathbf{v}) = v^i$$

This is indeed what  $du^i$  really is, but with the current organization of the math curriculum it is not convenient to explain this at the entry level. However, now you know! Incidentally, it *is* possible to make this consistent with the idea that  $du^i$  is a little bit of  $u^i$ . It works like this. Nearby points are connected by vectors which are very short. If  $\mathbf{v}$  is such a short vector, then  $v^i = du^i(\mathbf{v})$  is also small. So if we abbreviate  $du^i(\mathbf{v})$  by  $du^i$  it will behave as a small number, sort of. It's surprising that this works as well as it does.

Since  $du^i : V \rightarrow \mathbb{R}$  (and is linear) it is a linear functional and thus  $du^i \in V^*$ , where  $V^*$  is the *vector space of linear functionals on  $V$* . A wedge product of three  $du^i$ , for example  $du^1 \wedge du^3 \wedge du^5$  would then be in  $\Lambda^3(V^*)$ .

The  $du^1, \dots, du^n$  are the *dual basis* to  $e_1, \dots, e_n$  and form a basis for  $V^*$ .

Now we discuss  $\Phi : V \rightarrow V^*$  and  $\Phi^{-1} : V^* \rightarrow V$ . We have

$$\begin{aligned} \Phi(\mathbf{v}) &= \Phi(v^i e_i) = g_{ij} v^i du^j \\ \Phi^{-1}(\omega) &= \Phi^{-1}(\lambda_i du^i) = g^{ij} \omega_i e_j \end{aligned}$$

Here  $\omega$  is an arbitrary element of  $V^*$  and the  $\lambda_i$  are its coordinates in the basis  $du^1, \dots, du^n$ .

Recall that the  $g_{ij}$  give an inner (dot) product for  $V$  and in Chapter two we show that this may be "lifted" to  $V^*$ . If

$$\begin{aligned} \lambda &= \lambda_i du^i & \lambda &\in V^* & \lambda_i &\in \mathbb{R} \\ \mu &= \mu_i du^i & \mu &\in V^* & \mu_i &\in \mathbb{R} \end{aligned}$$

then

$$(\lambda, \mu) = g^{ij} \lambda_i \mu_j$$

just like

$$(\mathbf{u}, \mathbf{v}) = g_{ij}u^i v^j$$

but using the inverse matrix.

Now we digest the information on the  $*$  operator, which is derived in subsequent section and in Chapter 2.

$$\begin{aligned} *du^i &= \sum_k (-1)^{k-1} g^{ik} \sqrt{g} du^1 \wedge \dots \wedge du^{k-1} \wedge du^{k+1} \wedge \dots \wedge du^n \\ **\omega &= (-1)^{r(n-r)} \omega \quad \text{for } \omega \in \Lambda^r(V^*) \\ \Omega_0 &= \sqrt{g} du^1 \wedge \dots \wedge du^n \\ *\Omega_0 &= 1 \\ *1 &= \Omega_0 \end{aligned}$$

If the coordinates are orthogonal then things simplify. Setting

$$\begin{aligned} (\mathbf{e}_i, \mathbf{e}_j) &= g_{ij} = h_i^2 \delta_{ij} = \begin{cases} h_i^2 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \\ (du^i, du^j) &= g^{ij} = \frac{1}{h_i^2} \delta_{ij} = \begin{cases} \frac{1}{h_i^2} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \\ \sqrt{g} &= h_1 h_2 \dots h_n \end{aligned}$$

We know that a basis for  $\Lambda^r(V^*)$  is given by

$$du^{i_1} \wedge du^{i_2} \wedge \dots \wedge du^{i_r} \quad \text{where } i_1 < i_2 < \dots < i_r$$

Let  $\{k_1, \dots, k_{n-r}\} = \{1, 2, \dots, n\} - \{i_1, \dots, i_r\}$  so that we can form the permutation

$$\left( \begin{array}{ccc|ccc} 1 & \dots & r & r+1 & \dots & n \\ i_1 & \dots & i_r & k_1 & \dots & k_{n-r} \end{array} \right)$$

of  $\{1, 2, \dots, n\}$ . The *sign* of this permutation is  $(-1)^s$  where  $s$  is the number of interchanges necessary to rearrange the bottom line into the top line (or vice versa). The vertical line is just for the convenience of the reader; it divides the first part of the permutation from the second. Then

$$*(du^{i_1} \wedge du^{i_2} \wedge \dots \wedge du^{i_r}) = (-1)^s \frac{h_{k_1} \dots h_{k_{n-r}}}{h_{i_1} \dots h_{i_r}} du^{k_1} \wedge du^{k_2} \wedge \dots \wedge du^{k_{n-r}}$$

as we show in the Chapter 2.

Now it is time for two examples. The results will just be listed but it is easy to verify they are correct; mostly they can be done in ones head. *We will now begin to omit the wedges between differentials; the reader may assume that if differentials are being multiplied that wedges have been omitted for ease of reading.*



Example 1.  $n = 3$  and coordinates are  $x, y, z$ .

$$\begin{aligned}\mathbf{r} &= (x, y, z) \\ \mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial x} = (1, 0, 0) \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial y} = (0, 1, 0) \\ \mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) \\ g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = (e_i, e_j) &= \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}\end{aligned}$$

Thus we have orthogonal coordinates and also  $h_i^2 = g_{ii} = 1$ . Thus our formulas give

$$\begin{aligned} *dx &= dy \wedge dz & *dy &= (-1)dx \wedge dz & *dz &= dx \wedge dy \\ *dy \wedge dz &= dx & *dx \wedge dz &= (-1)dy & *dx \wedge dy &= dz \end{aligned}$$

Naturally in practise we prefer  $dz \wedge dx$  (cyclic order) to  $(-1)dx \wedge dz$ .

Example 2.  $n = 3$  and coordinates are  $\rho, \theta, \phi$ .

Here  $\theta$  (physics standard) is the angle off the  $z$ -axis.

$$\begin{aligned}\mathbf{r} &= (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta) \\ \mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial \rho} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) & h_1^2 = g_{11} &= 1 \\ \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \theta} &= (\rho \cos \theta \cos \phi, \rho \cos \theta \sin \phi, -\rho \sin \theta) & h_2^2 = g_{22} &= \rho^2 \\ \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial \phi} &= (-\rho \sin \theta \sin \phi, \rho \sin \theta \cos \phi, 0) & h_3^2 = g_{33} &= \rho^2 \sin^2 \theta\end{aligned}$$

$$\begin{aligned}g &= \det(g_{ij}) = h_1^2 h_2^2 h_3^2 \\ \sqrt{g} &= h_1 h_2 h_3 = \rho^2 \sin \theta \\ \Omega_0 &= \sqrt{g} d\rho d\theta d\phi = \rho^2 \sin \theta d\rho d\theta d\phi \\ *d\rho &= \frac{h_2 h_3}{h_1} d\theta d\phi = \rho^2 \sin \theta d\theta d\phi \\ *d\theta &= -\frac{h_1 h_3}{h_2} d\rho d\phi = \frac{h_1 h_3}{h_2} d\phi d\rho = \sin \theta d\phi d\rho \\ *d\phi &= \frac{h_1 h_2}{h_3} d\rho d\theta = \frac{1}{\sin \theta} d\rho d\theta \\ *(d\theta d\phi) &= \frac{1}{\rho^2 \sin \theta} d\rho \\ *(d\phi d\rho) &= \frac{1}{\sin \theta} d\theta \\ *(d\rho d\theta) &= \sin \theta d\phi\end{aligned}$$

For the last three equations use  $**\omega = (-1)^{r(n-r)}\omega$

## 1.17 Curvilinear Coordinates II: the calculations

Now we must consider the strategy of how we will find the formulas for grad, div and curl in curvilinear coordinates. This is quite simple once you know how. We look at the formulas in  $x, y, z$  coordinates and figure how to express the  $x, y, z$  expressions in terms of the operators  $\Phi$  and  $*$ . Since these are coordinate independent, we can use the formulas we find in *any* coordinate system. The rest is just relatively easy calculations.

We will start with curl. Recall that the input and output of curl are vectors and that if we set  $\omega = v^1 dx + v^2 dy + v^3 dz$  then

$$d\omega = \left( \frac{\partial v^3}{\partial y} - \frac{\partial v^2}{\partial z} \right) dydz + \left( \frac{\partial v^1}{\partial z} - \frac{\partial v^3}{\partial x} \right) dzdx + \left( \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y} \right) dxdy$$

which looks a lot like curl. Now if we are starting with a vector  $\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + v^3 \mathbf{e}_3$  we can get to  $\omega$  by using  $\Phi$ . Indeed, using the formulas from the last section,

$$\begin{aligned} \Phi(\mathbf{v}) &= g_{ij} v^i du^j \\ &= \delta_{ij} v^i du^j \\ &= v^1 dx + v^2 dy + v^3 dz \\ &= \omega \end{aligned}$$

Then  $d\omega$  gives us the above 2-form. However, we cannot use  $\Phi^{-1}$  to descend again to vectors because  $\Phi^{-1}$  eats only 1-forms. However  $*$  gets us from 2-forms to 1-forms, and we have

$$\begin{aligned} *d\omega &= \left( \frac{\partial v^3}{\partial y} - \frac{\partial v^2}{\partial z} \right) dx + \left( \frac{\partial v^1}{\partial z} - \frac{\partial v^3}{\partial x} \right) dy + \left( \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y} \right) dz \\ \Phi^{-1}(*d\omega) &= \left( \frac{\partial v^3}{\partial y} - \frac{\partial v^2}{\partial z} \right) \vec{e}_1 + \left( \frac{\partial v^1}{\partial z} - \frac{\partial v^3}{\partial x} \right) \vec{e}_2 + \left( \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y} \right) \vec{e}_3 \\ \Phi^{-1}(*d\Phi(\mathbf{v})) &= \text{curl } \mathbf{v} \end{aligned}$$

Thus we see, using  $\circ$  for function composition,

$$\text{curl} = \Phi^{-1} \circ * \circ d \circ \Phi$$

Since  $\Phi$ ,  $*$ , and  $d$  are all independent of the choice of coordinate system, so is  $\text{curl} = \Phi^{-1} \circ * \circ d \circ \Phi$ . Thus we can use this formula for curl in any coordinate system; we just have to be able to calculate the functions  $\Phi$ ,  $*$ , and  $d$  in the new coordinates and this is trivial. We will now use our method to get formulas for curl in any *orthogonal* coordinate system. We could do it in an arbitrary coordinate system also, but this would be beyond the needs of most engineers and scientists, and would also come out messy.

We will find the formula for curl in the orthogonal coordinate system  $u^1, u^2, u^3$ . Although we will eventually have to change over to physical coordinates, we will

begin with  $\mathbf{v}$  expressed in the natural coordinates system

$$\begin{aligned}\mathbf{e}_i &= \frac{\partial \mathbf{r}}{\partial u_i} \\ \mathbf{v} &= v^1 \vec{e}_1 + v^2 \vec{e}_2 + v^3 \vec{e}_3\end{aligned}$$

Since the coordinate system is orthogonal, we have  $g_{ij} = 0$  for  $i \neq j$  and  $g_{ii} = h_i^2$ . Thus we have

$$\begin{aligned}\mathbf{v} &= v^1 \vec{e}_1 + v^2 \vec{e}_2 + v^3 \vec{e}_3 \\ \Phi(\mathbf{v}) &= g_{ij} v^i du^j = h_1^2 v^1 du^1 + h_2^2 v^2 du^2 + h_3^2 v^3 du^3 \\ d\Phi(\mathbf{v}) &= \left( \frac{\partial(h_3^2 v^3)}{\partial u^2} - \frac{\partial(h_2^2 v^2)}{\partial u^3} \right) du^2 du^3 + \text{etc.} \\ *d\Phi(\mathbf{v}) &= \frac{h_1}{h_2 h_3} \left( \frac{\partial(h_3^2 v^3)}{\partial u^2} - \frac{\partial(h_2^2 v^2)}{\partial u^3} \right) du^1 + \text{etc.} \\ \text{curl } \mathbf{v} = \Phi^{-1} * d\Phi(\mathbf{v}) &= \frac{1}{h_1^2} \frac{h_1}{h_2 h_3} \left( \frac{\partial(h_3^2 v^3)}{\partial u^2} - \frac{\partial(h_2^2 v^2)}{\partial u^3} \right) \mathbf{e}_1 + \text{etc.} \\ &= \frac{1}{h_1 h_2 h_3} \left( \frac{\partial(h_3^2 v^3)}{\partial u^2} - \frac{\partial(h_2^2 v^2)}{\partial u^3} \right) \mathbf{e}_1 + \text{etc.}\end{aligned}$$

This is correct but it is in terms of the natural basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  whereas in physics it is usual to express the vector in terms of the normalized vectors

$$\hat{\mathbf{e}}_1 = \frac{1}{h_1} \mathbf{e}_1, \quad \hat{\mathbf{e}}_2 = \frac{1}{h_2} \mathbf{e}_2, \quad \hat{\mathbf{e}}_3 = \frac{1}{h_3} \mathbf{e}_3$$

Then we have

$$\begin{aligned}\mathbf{v} &= v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + v^3 \mathbf{e}_3 \\ &= v^1 h_1 \hat{\mathbf{e}}_1 + v^2 h_2 \hat{\mathbf{e}}_2 + v^3 h_3 \hat{\mathbf{e}}_3 \\ &= \tilde{v}^1 \hat{\mathbf{e}}_1 + \tilde{v}^2 \hat{\mathbf{e}}_2 + \tilde{v}^3 \hat{\mathbf{e}}_3\end{aligned}$$

where

$$\tilde{v}_1 = h_1 v^1 \quad \tilde{v}_2 = h_2 v^2 \quad \tilde{v}_3 = h_3 v^3$$

In terms of the  $\tilde{v}^i$  and the  $\hat{\mathbf{e}}_i$  the formula for curl becomes

$$\begin{aligned}\text{curl } \mathbf{v} &= \frac{1}{h_1 h_2 h_3} \left[ \left( \frac{\partial(h_3^2 v^3)}{\partial u^2} - \frac{\partial(h_2^2 v^2)}{\partial u^3} \right) h_1 \hat{\mathbf{e}}_1 + \text{etc.} \right] \\ &= \frac{1}{h_1 h_2 h_3} \left[ \left( \frac{\partial(h_3 \tilde{v}^3)}{\partial u^2} - \frac{\partial(h_2 \tilde{v}^2)}{\partial u^3} \right) h_1 \hat{\mathbf{e}}_1 + \text{etc.} \right] \\ &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ h_1 \tilde{v}^1 & h_1 \tilde{v}^2 & h_1 \tilde{v}^3 \end{vmatrix}\end{aligned}$$

*This* is the formula you find in reference books.

As an example, let us write the formula for Spherical Coordinates using the material for Spherical coordinates at the end of the previous section. Recall that  $h_1 = 1$ ,  $h_2 = \rho$ ,  $h_3 = \rho \sin \theta$ . Then

$$\operatorname{curl} \mathbf{v} = \frac{1}{\rho^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{e}}_1 & \rho \hat{\mathbf{e}}_2 & \rho \sin \theta \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \tilde{v}^1 & \rho \tilde{v}^2 & \rho \sin \theta \tilde{v}^3 \end{vmatrix}$$

While the curl will only work in 3 dimensions, we can get formulas for grad and div in  $n$  dimensions with no extra effort at all. This is important since we might need either of them in 2 as well as 3 dimensions, and maybe even 4 dimensions for relativity. So we shall do them all at once.

There is no difficulty at all with grad. It inputs a function and outputs a vector so the immediate candidate for grad is  $\Phi^{-1} \circ d$ . This obviously works in rectangular coordinates since then  $\Phi(\mathbf{e}_i) = du^i$ . and thus

$$\begin{aligned} \operatorname{grad} f &= \Phi^{-1} \left( \frac{\partial f}{\partial u^i} du^i \right) \quad (\text{sum on } i \text{ understood}) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial u^i} \mathbf{e}_i \end{aligned}$$

The last is our old friend the advanced calculus gradient. Notice that we could not use the summation convention here since both the  $i$ 's in the last term count as low indices. (Recall that *high* index in a denominator counts as *low*.) Watch carefully in the next developments to see why this has happened.

Next recall that for general (not necessarily orthogonal) coordinates we have

$$\Phi(\mathbf{e}_i) = g_{ij} du^j \quad \Phi^{-1}(du^i) = g^{ij} \mathbf{e}_j$$

and so

$$\begin{aligned} \operatorname{grad} f &= \Phi^{-1}(df) \\ &= \Phi^{-1} \left( \frac{\partial f}{\partial u^i} du^i \right) \\ &= \frac{\partial f}{\partial u^i} g^{ij} \mathbf{e}_j \end{aligned}$$

which is nice and simple. For *orthogonal* coordinates we have

$$g^{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{g_{ii}} = \frac{1}{h_i^2} & \text{if } i = j \end{cases}$$

and thus, and thus with natural basis vectors  $\mathbf{e}_i$  and physical (normalized) basis vectors  $\hat{\mathbf{e}}_i = \frac{1}{h_i} \mathbf{e}_i$  we have

$$\begin{aligned} \operatorname{grad} f &= \sum_{i=1}^n \frac{1}{h_i^2} \frac{\partial f}{\partial u^i} \mathbf{e}_i \\ &= \sum_{i=1}^n \frac{1}{h_i} \frac{\partial f}{\partial u^i} \hat{\mathbf{e}}_i \end{aligned}$$

The second formula, with physical basis vectors  $\hat{\mathbf{e}}_i$ , is what one generally sees in reference books.

Our next task is the divergence. This is harder, since we must use the  $*$  operator, but we can break it down into easy steps. The first observation we make is important for many purposes. Let

$$\omega = \sum_{i=1}^n (-1)^{i-1} v^i du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n$$

It is customary to write this as

$$\omega = \sum_{i=1}^n (-1)^{i-1} v^i du^1 \wedge \cdots \wedge d\hat{u}^i \wedge \cdots \wedge du^n$$

where the hat on the  $d\hat{u}^i$  indicates that it is NOT THERE. This is just notation; it tends to make everything look cleaner, but you must keep your wits about you and watch for it. Now when we find  $d\omega$  it comes out very nice:

$$d\omega = \sum_{i,j=1}^n (-1)^{i-1} \frac{\partial v^i}{\partial u^j} du^j \wedge du^1 \wedge \cdots \wedge d\hat{u}^i \wedge \cdots \wedge du^n$$

Now note that if  $j \neq i$  then the differential contain a repetition and thus give 0. Hence only the terms with  $j = i$  need be retained and we have

$$\begin{aligned} d\omega &= \sum_{i,j=1}^n (-1)^{i-1} \frac{\partial v^i}{\partial u^j} du^j \wedge du^1 \wedge \cdots \wedge d\hat{u}^i \wedge \cdots \wedge du^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial v^i}{\partial u^i} du^i \wedge du^1 \wedge \cdots \wedge d\hat{u}^i \wedge \cdots \wedge du^n \\ &= \sum_{i=1}^n \frac{\partial v^i}{\partial u^i} du^1 \wedge \cdots \wedge du^i \wedge \cdots \wedge du^n \\ &= \left( \sum_{i=1}^n \frac{\partial v^i}{\partial u^i} \right) du^1 \wedge \cdots \wedge du^n \end{aligned}$$

where in one step the  $du^i$  had to hop over  $i - 1$  other differentials to find its proper slot. You will notice has this has the look of divergence about it.

To utilize this formula we must get from our vector to an  $(n - 1)$ -form. We can get from the vector to a 1-form using  $\Phi$  and then to an  $(n - 1)$ -form using  $*$ . From there,  $d$  will take us to an  $n$ -form and then another  $*$  will get us back to a 0-form, or scalar, so  $* \circ d \circ * \circ \Phi$  will get us from a vector to a scalar as we wish for divergence. Thus  $* \circ d \circ * \circ \Phi$  is a good candidate for divergence. Let's try it in rectangular coordinates. In rectangular coordinates the natural basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  is just  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  when  $n = 3$

$$\mathbf{v} = v^i \mathbf{e}_i$$

$$\begin{aligned}
\Phi(\mathbf{v}) &= v^i du^i \\
*\Phi(\mathbf{v}) &= \sum_{i=1}^n v^i (-1)^{i-1} du^1 \wedge \cdots \wedge \hat{du}^i \wedge \cdots \wedge du^n \\
d*\Phi(\mathbf{v}) &= \left( \sum_{i=1}^n \frac{\partial v^i}{\partial u^i} \right) du^1 \wedge \cdots \wedge du^n
\end{aligned}$$

as we saw above. Finally

$$*d*\Phi(\mathbf{v}) = \sum_{i=1}^n \frac{\partial v^i}{\partial u^i} = \operatorname{div} \mathbf{v}$$

which shows us that indeed the invariant form of  $\operatorname{div}$  is  $* \circ d \circ * \circ \Phi$ . If you are worried about the expressions for  $*$ , they follow immediately from the general expressions below.

Before we launch into the final calculation for  $\operatorname{div}$ , let's recall the formulas for the  $*$  operator. For  $\omega = \omega_i du^i$  we have

$$*\omega = \sum_{i,k=1}^n (-1)^{k-1} g^{ik} \sqrt{g} \omega_i du^1 \wedge \cdots \wedge \hat{du}^k \wedge \cdots \wedge du^n$$

where, as before, the hat on  $du^j$  indicates the term is NOT THERE. This is derived in the Chapter 2. Also derived there are

$$\begin{aligned}
\Omega_0 &= \sqrt{g} du^1 \wedge \cdots \wedge du^n \\
*\Omega_0 &= 1 \\
*1 &= \Omega_0
\end{aligned}$$

We also require the formula in general coordinates for  $\Phi$ .

$$\Phi(\mathbf{v}) = \Phi(v^i \mathbf{e}_i) = g_{ij} v^i du^j$$

Now we are ready for the final calculation for  $\operatorname{div}$ .

$$\begin{aligned}
\operatorname{div} \mathbf{v} &= *d*\Phi(\mathbf{v}) \\
&= *d*(g_{ij} v^i du^j) \\
&= *d\left( \sum_{k=1}^n (-1)^{k-1} g^{jk} \sqrt{g} (g_{ij} v^i) du^1 \wedge \cdots \wedge \hat{du}^k \wedge \cdots \wedge du^n \right)
\end{aligned}$$

Since  $g^{jk} g_{ij} = \delta_i^k$ , the only non-zero terms are when  $k = i$ , to the above simplifies to

$$\operatorname{div} \mathbf{v} = *d\left( \sum_{i=1}^n (-1)^{i-1} \sqrt{g} v^i du^1 \wedge \cdots \wedge \hat{du}^i \wedge \cdots \wedge du^n \right)$$

$$\begin{aligned}
&= * \left( \sum_{i,j=1}^n (-1)^{i-1} \frac{\partial(\sqrt{g} v^i)}{\partial u^j} du^j \wedge du^1 \wedge \dots \wedge d\hat{u}^i \wedge \dots \wedge du^n \right) \\
&= * \left( \sum_{i=1}^n \frac{\partial(\sqrt{g} v^i)}{\partial u^i} \wedge du^1 \wedge \dots \wedge du^i \wedge \dots \wedge du^n \right) \\
&= * \left( \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial(\sqrt{g} v^i)}{\partial u^i} \right) \sqrt{g} du^1 \wedge \dots \wedge du^n \\
&= \left( \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial(\sqrt{g} v^i)}{\partial u^i} \right)
\end{aligned}$$

and this is the formula for  $\text{div}$  when  $\mathbf{v}$  is expressed in natural coordinates  $\mathbf{v} = v^i \mathbf{e}_i$ . For orthogonal coordinates we have as usual  $\mathbf{e}_i = h_i \hat{\mathbf{e}}_i$  and

$$\begin{aligned}
\mathbf{v} &= \sum_{i=1}^n v^i \mathbf{e}_i \\
&= \sum_{i=1}^n v^i h_i \hat{\mathbf{e}}_i \\
&= \sum_{i=1}^n \tilde{v}^i \hat{\mathbf{e}}_i
\end{aligned}$$

where  $\tilde{v}^i = h_i v^i$  are the physical coordinates with respect to the orthonormal basis  $\hat{\mathbf{e}}_i$ . Now using

$$\begin{aligned}
\sqrt{g} &= h_1 h_2 \cdots h_n \\
\tilde{v}^i &= h_i v^i \quad \text{no sum on } i
\end{aligned}$$

we have

$$\begin{aligned}
\text{div } \mathbf{v} &= \frac{1}{h_1 h_2 \cdots h_n} \sum_{i=1}^n \frac{\partial}{\partial u^i} (h_1 h_2 \cdots h_n v^i) \\
&= \frac{1}{h_1 h_2 \cdots h_n} \sum_{i=1}^n \frac{\partial}{\partial u^i} (h_1 \cdots h_{i-1} h_{i+1} \cdots h_n \tilde{v}^i)
\end{aligned}$$

Lastly we want to give a formula for the Laplacian. In physics the formula for the Laplacian *in rectangular coordinates* is

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial u^i^2}$$

The normal notation for the Laplacian is  $\Delta f$ . However, in recent years in mathematics there has been a big tendency to use the symbol  $\Delta f$  for the negative

of the sum above. There are very good reasons for this<sup>5</sup>. So to make sure there will be no confusion, we will use the symbol  $\Delta f$  for the Laplacian as customary in physics. It will be awhile before we need to deal with the mathematical Laplacian.

To find the Laplacian of functions in any coordinate system is now absurdly easy, because we have

$$\Delta f = \operatorname{div} \operatorname{grad} f$$

Thus we need only recall the two previously derived formulas for grad and div

$$\begin{aligned} \operatorname{grad} f &= \frac{\partial f}{\partial u^i} g^{ij} \mathbf{e}_j \\ \operatorname{div} \mathbf{v} &= \frac{1}{\sqrt{g}} \sum_{j=1}^n \frac{\partial(\sqrt{g} v^j)}{\partial u^j} \end{aligned}$$

Thus it comes down to substituting

$$v^j = \frac{\partial f}{\partial u^i} g^{ij}$$

into the formula for div  $\mathbf{v}$  to get

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial u^j} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial u^i} \right)$$

One sees immediately that in orthogonal coordinates we have

$$\Delta f = \frac{1}{h_1 \cdots h_n} \sum_{i=1}^n \frac{\partial}{\partial u^i} \left( h_1 \cdots h_{i-1} h_{i+1} \cdots h_n \frac{\partial f}{\partial u^i} \right)$$

## 1.18 Surfaces and Manifolds

Up to this point we have confined ourselves mostly to  $\mathbb{R}^n$  but now it is time to take a more general viewpoint, especially since it will require no additional effort. We have occasionally talked about two dimensional surfaces in  $\mathbb{R}^3$  but there is nothing special about that configuration. A convenient place to start our discussion is  $m$ -dimensional subspaces in  $\mathbb{R}^n$ . The word *surface* is usually used for a 2-dimensional object; the corresponding word for an  $m$ -dimensional subspace of  $\mathbb{R}^n$  is *embedded manifold*. (If  $m = n - 1$  then the embedded manifold is often called a *hypersurface*. This is a special case where things are generally nicer than they are in general.) The *embedded* in *embedded manifold* refers to the surrounding  $n$ -dimensional space. There are manifolds which are not embedded, (for example, the 4-dimensional space of the Universe of General Relativity

---

<sup>5</sup>For example, the mathematical Laplacian has *positive* or 0 eigenvalues; the physics one has negative or 0 eigenvalues



is usually thought of as not embedded in a higher dimensional space,) and we will briefly discuss this later and also in the second chapter. The advantage of embedding is that a) it seems familiar and b) it is easy to define tangent vectors.

Our discussion here will be more or less intuitive, and we will not discuss various pathologies that can occur to make life difficult, as these are not usual in ordinary physics. Also, we will imitate some of the constructions that are used in the more general case, of nonembedded manifolds in order to make the transition easier.

First, it is important that the the surface neither cross itself, nor approach itself. The first is easy to avoid; the second is more tricky. Recall that a ball in  $\mathbb{R}^n$  is the set of points lying within a certain distance of the central point. More explicitly  $B_r(p) = \{x \in \mathbb{R}^n \mid |x - p| < r\}$ . Also recall that a homeomorphism is a one to one onto map from one set to another which is continuous and has a continuous inverse. Now let  $p$  be a point on the manifold. We require that there is an  $r$  small enough so that the the intersection of  $B_r(p)$  with the manifold be homeomorphic to a disk in  $\mathbb{R}^m$ . Put another way, every point  $p$  on the manifold has a neighborhood that looks like a disk in  $\mathbb{R}^m$  (and the neighborhood is found by intersecting the manifold with a  $B_r(p)$ ). That is why the manifold has  $m$  dimensions.

Now we need to superimpose coordinates on this situation. That's easy; the disk in  $\mathbb{R}^m$  has coordinates  $u^1, \dots, u^m$  and if  $q$  on the manifold corresponds to  $y$  in the disk which has coordinates  $u^1, \dots, u^m$ , which we can then use for  $p$ . Home free? By no means. The most important thing comes next. Suppose  $\tilde{p}$  is in the neighborhood of  $p$  (as above) and on the manifold. Then  $\tilde{p}$  also has a neighborhood defined by  $B_s(\tilde{p})$  and a homomorphism of its own to a disk in  $\mathbb{R}^m$ , which will give the neighborhood of  $\tilde{p}$  it's own set of coordinates, say  $\tilde{u}^1, \dots, \tilde{u}^m$ . Then, each point in the overlap of  $B_r(p)$  and  $B_s(\tilde{p})$  will have *two* sets of coordinates  $u^1, \dots, u^m$  and  $\tilde{u}^1, \dots, \tilde{u}^m$ . Each set will be a function of the other:

$$\begin{aligned}\tilde{u}^i &= \tilde{u}^i(u^1, \dots, u^m) \\ u^i &= u^i(\tilde{u}^1, \dots, \tilde{u}^m)\end{aligned}$$

The critical thing here is that these functions have many derivatives. Many means, for physics purposes, at least three continuous derivatives. The manifold is then called a  $C^3$ -manifold. In mathematics it is customary to avoid the boring details and require the functions to have infinitely many continuous derivatives so it is  $C^\infty$ -manifold. This seemingly boring requirement is the key to the whole subject; it is what makes the manifold smooth like a ball rather than unsmooth like a cube. It takes some contemplation to see this but it's true.

One more comment; in classical tensor analysis they are continually talking about coordinate changes as if they made some big difference. They do; the talk of coordinate changes in tensor analysis is just how that subject deals with the material in the previous paragraph<sup>6</sup>.

---

<sup>6</sup>This is not a well known fact even among the tensor users. Reveal it only to trusted friends

There is a little more to the story, because, after all, we have to put the manifold inside  $\mathbb{R}^n$  and this requires a little more to be put into the definition. The requirement is easy to state. Since the manifold is inside  $\mathbb{R}^n$  and  $p$  will thus have coordinates  $x^1, \dots, x^n$ . These will be functions of  $p$ 's disk coordinates  $u^1, \dots, u^m$ ; that is

$$x^i = x^i(u^1, \dots, u^m) \quad i = 1, \dots, n$$

The requirement is that the  $n \times m$  matrix

$$\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^m} \\ \frac{\partial x^2}{\partial u^1} & \cdots & \frac{\partial x^2}{\partial u^m} \\ \vdots & \vdots & \vdots \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^m} \end{pmatrix}$$

have rank  $m$ . Since  $m \leq n$ ,  $m$  is the largest rank it *could* have, so we often say *the matrix has maximal rank*. The reason this is important is that we could set up a coordinate system around any  $p$  for  $\mathbb{R}^n$  that used  $u^1, \dots, u^m$  for the first  $m$  coordinates and then  $n - m$  other coordinates to finish the job, but we do not need to pursue this.

We should also mention that while in theory we have set the system up to use disks in  $\mathbb{R}^m$  to make coordinates, you can actually do it with any regions you like rather than disks. We only did it with disks to make it precise and visual. Things are rather looser in practise.

The test to see that this condition is satisfied is to check that all the  $m \times m$  minors of the matrix are non-zero. It is now time for an example. We will use (as usual) the sphere in 3-space, which illustrates the concepts nicely. The coordinates will be  $\theta, \phi$  where  $\phi$  is the longitude and  $\theta$  is the angle off the  $z$ -axis<sup>7</sup>.

The  $\mathbb{R}^3$  coordinates of the point with coordinates  $\theta, \phi$  are

$$\begin{aligned} x &= \sin \theta \cos \phi \\ y &= \sin \theta \sin \phi \\ z &= \cos \theta \\ \mathbf{R} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \end{aligned}$$

The natural basis of the tangent space is

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial \mathbf{R}}{\partial \theta} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\ \mathbf{e}_2 &= \frac{\partial \mathbf{R}}{\partial \phi} = (-\sin \theta \sin \phi, \sin \theta \cos \phi, 0) \\ \hat{\mathbf{n}} &= \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = (\sin^2 \theta \cos \phi, \sin^2 \theta \sin \phi, \sin \theta \cos \theta) / \sin \theta \\ &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \end{aligned}$$

---

<sup>7</sup>Mathematicians beware; math convention is opposite!

The matrix discussed above is then

$$\begin{pmatrix} \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \\ -\sin \theta & 0 \end{pmatrix}$$

The minors are

$$\begin{aligned} \text{Row 23} &= \begin{pmatrix} \cos \theta \sin \phi & \sin \theta \cos \phi \\ -\sin \theta & 0 \end{pmatrix} = \sin^2 \theta \cos \phi \\ \text{Row 13} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \theta \sin \phi \\ -\sin \theta & 0 \end{pmatrix} = -\sin^2 \theta \sin \phi \\ \text{Row 12} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \end{pmatrix} = \sin \theta \cos \theta \end{aligned}$$

There is a lot that can be learned from this example. Notice first that the minors are all 0 at the poles where  $\theta = 0$  or  $\theta = \pi$ . This is reflected also in  $\mathbf{e}_2$  which is 0 at the poles and thus not much of a basis vector for the tangent space. Note that at the poles  $\phi$  is not defined either. Note that in  $\hat{\mathbf{n}} = (\mathbf{e}_1 \times \mathbf{e}_2)/|\mathbf{e}_1 \times \mathbf{e}_2|$  both numerator and denominator are 0, so  $\hat{\mathbf{n}}$  is not well defined though it seems to be after the 0's disappear because of the cancellation of  $\sin \theta$ .

The proper response to this is to select a second coordinate system, perhaps one with east and west poles, to cover the two offending points with proper coordinates. Needless to say, this is hardly ever done in this and many other cases, because with a little sympathetic treatment it is possible to get around the troubles at the poles. However, it is really important to notice such bad points and to make sure whatever you are doing makes sense at such bad points. Often it does, sometimes it doesn't. Beware.

Now on to other things. Notice in the example how easy it was to get the basis of tangent vectors  $\mathbf{e}_1, \mathbf{e}_2$ , one for each coordinate, at each point of the manifold. The vector space spanned by  $\mathbf{e}_1, \mathbf{e}_2$  is called the Tangent plane, and is denoted by  $T_p(S)$  (where  $S$  is the sphere). In the more general case of an embedded  $m$ -dimensional manifold  $S$  in  $n$ -dimensional space, we will have  $m$  tangent vectors at each point,  $\mathbf{e}_1, \dots, \mathbf{e}_m$ , and they will be a basis for an  $m$ -dimensional space  $T_p(S)$  called the *Tangent Space* to the manifold  $S$  at  $p$ . The condition on the minors guarantees that the  $\mathbf{e}_i$  will be linearly independent.

Consider an  $m$ -dimensional manifold  $S$  with coordinates  $u^1, \dots, u^m$  and basis vectors  $\mathbf{e}_i = \partial \mathbf{R} / \partial u^i$ . Now let  $\mathbf{v} \in T_p(S)$ . Then  $\mathbf{v} = \sum v^i \mathbf{e}_i$  and we need a function whose input is  $\mathbf{v}$  and whose output is  $v^i$ . This function is  $du^i$ . Although the idea is new to you, this is what  $du^i$  actually is. Thus the defining equation is

$$du^i(\mathbf{v}) = v^i$$

(We have seen this before; I'm just reminding you in this new context.) If you are familiar with the the space of linear functionals of a vector space, then  $du^1, \dots, du^m$  form a basis for this space of linear functionals (called the *dual space* and denoted by  $T_p^*(S)$ ).

When you follow this path for awhile, you become aware of what a small role the embedding space  $\mathbb{R}^n$  plays in the theory. Riemann also wondered about this, and eventually showed how to get around it. Let us ask what the important contributions of the embedding space  $\mathbb{R}^n$  are. There are two most important contributions. First, the embedding space provides an inner product for the  $T_p(S)$  which live inside it. Second, it provides us with a convenient way (visible in the sphere example) of finding tangent vectors and thus the tangent space  $T_p(S)$ . The first is a critical idea; the second just requires technical trickery. For the first, Riemann proposed attaching an inner product in the form of an  $m \times m$  positive definite matrix to each point of  $S$ . In doing this Riemann invented the Riemannian Manifold. For the second, the tangent vectors, Tensor Analysis defined a vector as an object that changes in certain ways under coordinate changes. This works but is a bit unsatisfying. The more modern way is to define tangent vectors as directional derivatives on the manifold. We will look at this further in chapter 2 where we will define differentiable manifolds, although most of the work has already been done in this section. For the remainder of this chapter we will stay with embedded manifolds.

We must also mention that the theory we develop subsequently must be modified if the manifold is not *orientable*. Examples of non-orientable manifolds are the Möbius strip and the Klein bottle. The problem is that when a right handed orthonormal basis of the tangent space is slid around the median line of the Möbius strip it returns as a left handed orthonormal basis. The easiest definition of orientable is that an  $n$ -dimensional manifold  $K$  (embedded or not) is orientable if and only if it has a never vanishing form  $\omega \in \Lambda^n(K)$ . Such a form is called a topform. From it one can always manufacture a volume form if  $K$  has an inner product (e.g. is a Riemannian Manifold). We will not consider non-orientable manifolds in what follows and some of the material will not work on them. See Chapter 2 for more discussion of this matter.

## 1.19 The Dualizing Operator $*$

One cannot get very far in differential forms without the dualizing operator  $*$  :  $\Lambda^r(\mathbb{R}) \rightarrow \Lambda^{n-r}(\mathbb{R})$ . This operator reflects analytically certain geometric properties studied in higher geometry courses called duality properties. Sadly we cannot take the time to talk about this geometry, which would take many pages to deal with and anyway is not a matter of great practical importance for most of physics. In this section we will give a brief introduction to  $*$  which will suffice for most of the more elementary applications, as for example Maxwell's equations which we present in a later section.

In most elementary applications of  $*$  we deal with the cases

$$\begin{array}{ll} r = 0 & * : \Lambda^0(\mathbb{R}) \rightarrow \Lambda^n(\mathbb{R}) \\ r = n & * : \Lambda^n(\mathbb{R}) \rightarrow \Lambda^0(\mathbb{R}) \end{array}$$

and

$$\begin{aligned} r = 1 & & * : \Lambda^1(\mathbb{R}) & \rightarrow \Lambda^{n-1}(\mathbb{R}) \\ r = n - 1 & & * : \Lambda^{n-1}(\mathbb{R}) & \rightarrow \Lambda^1(\mathbb{R}) \end{aligned}$$

In each case it is only necessary to derive one of the two formulas; the other is then obtained from

$$**\omega = (-1)^{r(n-r)} \quad \text{for } \omega \in \Lambda^r(\mathbb{R})$$

To define  $*$  it is necessary first to put an inner product on each  $\Lambda^r$ . This is easily done; We define the inner product with

$$\omega = \omega_i du^i \quad \text{and} \quad \eta = \eta_i du^i$$

by

$$(\omega, \eta) = g^{ij} \omega_i \eta_j$$

We can now put an inner product on each  $\Lambda^r$  as follows

$$(\omega_1 \wedge \dots \wedge \omega_r, \eta_1 \wedge \dots \wedge \eta_r) = \begin{vmatrix} (\omega_1, \eta_1) & \dots & (\omega_1, \eta_r) \\ \vdots & \ddots & \vdots \\ (\omega_r, \eta_1) & \dots & (\omega_r, \eta_r) \end{vmatrix}$$

and extending by linearity. This defines an inner product on  $\Lambda^r$  for  $1 \leq r \leq n$ , but we also need it on  $\Lambda^0$  which is just the scalars. For these we define

$$(r, s) = rs \quad \text{for } r, s \text{ scalars}$$

For completeness, we remark that if  $\omega \in \Lambda^r$  and  $\eta \in \Lambda^s$  and  $r \neq s$ ,  $0 \leq r, s \leq n$ , then we put

$$(\omega, \eta) = 0$$

Our most important use of this formula is the following case where  $r = n$ :

$$(\omega_1 \wedge \dots \wedge \omega_n, \eta_1 \wedge \dots \wedge \eta_n) = \begin{vmatrix} (\omega_1, \eta_1) & \dots & (\omega_1, \eta_n) \\ \vdots & \ddots & \vdots \\ (\omega_n, \eta_1) & \dots & (\omega_n, \eta_n) \end{vmatrix}$$

Noting that the above formula gives us

$$(du^r, du^s) = (\delta_r^i du^i, \delta_s^j du^j) = g^{ij} \delta_r^i \delta_s^j = g^{rs}$$

we have with the inner product in  $\Lambda^n$

$$\begin{aligned} (du^1 \wedge \dots \wedge du^n, du^1 \wedge \dots \wedge du^n) &= \begin{vmatrix} (du^1, du^1) & \dots & (du^1, du^n) \\ \vdots & \ddots & \vdots \\ (du^n, du^1) & \dots & (du^n, du^n) \end{vmatrix} \\ &= \begin{vmatrix} g^{11} & \dots & g^{1n} \\ \vdots & \ddots & \vdots \\ g^{n1} & \dots & g^{nn} \end{vmatrix} = \det(g^{i,j}) = \frac{1}{g} \end{aligned}$$

Thus

$$(\sqrt{g} du^1 \wedge \dots \wedge du^n, \sqrt{g} du^1 \wedge \dots \wedge du^n) = 1$$

We will define

$$\Omega_0 \stackrel{\text{def}}{=} \sqrt{g} du^1 \wedge \dots \wedge du^n$$

and refer to it as the *normalized topform*. The normalized topform is unique up to sign; it will change sign if two variables interchange their numbers. Choice of a sign is the same as choice of an orientation. The reason it is essentially unique is that  $\Lambda^n$  is one-dimensional, so there are only two elements of size 1 and they are negatives of one another.

If  $M$  is a  $n$ -dimensional region then the volume of  $M$  is

$$\text{vol}(M) = \int_M \Omega_0$$

There are of course subtleties here; for example the sign could come out wrong. And of course you have to prove this works using whatever your definition of volume is. We will ignore these problems. Also we mention that if  $f$  is the density of something, electrical charge for example, or gas, then the amount of the stuff will be

$$\text{Amount of stuff in } M = \int_M f \Omega_0$$

Next we want the official definition of  $*$  which is at last possible since we have the inner product on  $\Lambda^r$ . For any  $\alpha \in \Lambda^{n-r}$  we will always have

$$\omega \wedge \alpha = k \Omega_0$$

for some constant  $k$ . Then for  $\eta \in \Lambda^r$  we define  $*\eta$  as that unique element of  $\Lambda^{n-r}$  for which the constant  $k$  is  $(\omega, \eta)$ , which comes down to

$$\boxed{\omega \wedge *\eta = (\omega, \eta) \Omega_0}$$

This is the most important equation involving  $*$ ; if you remember this you can derive everything else. The existence and uniqueness of  $*\eta$  are derived in Chapter 2, but it is not a difficult matter; it comes down to the representation of a linear functional in an inner product space.

Some equations readily follow from the basic equation. For example, we have

$$\begin{aligned} \omega \wedge *\eta &= (\omega, \eta) \Omega_0 &= (\eta \omega) \Omega_0 &= \eta \wedge *\omega \\ \omega \wedge *\eta &= \eta \wedge *\omega \end{aligned}$$

which has important uses, for example in the next section on the codifferential. Other important equations like

$$**\omega = (-1)^{r(n-r)}\omega \quad \text{for } \omega \in \Lambda^r$$

do not follow so readily from the basic equation. We will have more to say about this equation later.

Next, using the basic equation  $\omega \wedge *\eta = (\omega, \eta)\Omega_0$  we want to derive the formulas for  $*$  for general coordinates and 1-forms. Recall that a hat over a term means that it is *missing*. We have  $\{du^1, \dots, du^n\}$  is a basis for  $\Lambda^1$ . A basis for  $\Lambda^{n-1}$  is

$$du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n \quad 1 \leq i \leq n$$

Thus, since  $du^i \in \Lambda^1$ , we have  $*du^i \in \Lambda^{n-1}$  and then we can express  $*du^i$  as

$$*du^i = \sum_{j=1}^n (-1)^{j-1} a^j du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n$$

where the  $a^j$  are functions of  $u^1, \dots, u^n$  which we must determine and the  $(-1)^{j-1}$  is inserted for convenience. We now compute both sides of the basic equation and setting them equal will give us the  $a^j$ . The basic equation is  $du^k \wedge *du^i = (du^k, du^i)\Omega_0$ . Computing the left side of the basic equation we have

$$\begin{aligned} du^k \wedge *du^i &= du^k \wedge \left( \sum_{j=1}^n (-1)^{j-1} a^j du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \right) \\ &= \sum_{j=1}^n (-1)^{j-1} a^j du^k \wedge du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \end{aligned}$$

All terms on the right side in the sum will be 0 except the one where  $j = k$  because if  $j \neq k$  there will be repetition in the differentials killing the term. Thus

$$du^k \wedge *du^i = a^k du^1 \wedge \dots \wedge du^k \wedge \dots \wedge du^n$$

Notice how the  $(-1)^{j-1}$  was used to return  $du^k$  to its proper place in the product of differentials. This is a good trick to remember, though it is never important, just convenient. Next we compute the right side of the of the basic equation. We have

$$(du^k, du^i)\Omega_0 = g^{ki} \sqrt{g} du^1 \wedge \dots \wedge du^n$$

Comparing the two expressions we see that

$$a^k = g^{ki} \sqrt{g}$$

and thus

$$\begin{aligned} *du^i &= \sum_{k=1}^n (-1)^{k-1} a^k du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n \\ &= \sum_{k=1}^n (-1)^{k-1} g^{ki} \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n \end{aligned}$$

This equation

$$\boxed{*du^i = \sum_{k=1}^n (-1)^{k-1} g^{ki} \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n}$$

is of fundamental importance. You have already seen it used in the derivation of the formulas for curvilinear coordinates.

It is possible to determine  $*du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n$  in the same way but that method leads into a forest of determinants. Instead, we will use the formula  $**\omega = (-1)^{r(n-r)}\omega$  with  $r = 1$  and some trickery involving the inverse matrices  $(g^{ji})$  and  $(g_{ik})$ .

$$\begin{aligned} *du^i &= \sum_{j=1}^n (-1)^{j-1} g^{ji} \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \\ \sum_{i=1}^n g_{ik} *du^i &= \sum_{i,j=1}^n (-1)^{j-1} g^{ji} g_{ik} \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \\ &= \sum_{j=1}^n (-1)^{j-1} \left( \sum_{i=1}^n g^{ji} g_{ik} \right) \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \\ &= \sum_{j=1}^n (-1)^{j-1} \delta_k^j \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n \\ &= (-1)^{k-1} \sqrt{g} du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n \end{aligned}$$

Hence, starring both sides, we have

$$\begin{aligned} (-1)^{k-1} \sqrt{g} * du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n &= \sum_{i=1}^n g_{ik} ** du^i \\ &= \sum_{i=1}^n g_{ik} (-1)^{1(n-1)} du^i \\ *du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n &= (-1)^{-k+1} (-1)^{n-1} \sum_{i=1}^n g_{ik} du^i \end{aligned}$$

giving us our final answer

$$\boxed{*du^1 \wedge \dots \wedge \widehat{du^k} \wedge \dots \wedge du^n = (-1)^{n-k} \sum_{i=1}^n g_{ik} du^i}$$

This is not quite sufficient for our needs. We also need formulas for  $*$  when  $r = 0$  and when  $r = n$ . These are easy. Recall that  $\Lambda^0$  is just the scalars  $\mathbb{R}$  and that the inner product is just the ordinary product in  $\mathbb{R}$  and that a basis for  $\mathbb{R}$  is just the single scalar 1. Then

$$*1 = \Omega_0$$



since

$$1 \wedge *1 = 1 \wedge \Omega_0 = \Omega_0 = (1, 1)\Omega_0$$

as required. Then

$$*\Omega_0 = **1 = (-1)^{0(n-0)}1 = 1$$

We have determined the formulas in generalized coordinates for 0-forms, 1-forms,  $(n-1)$ -forms and  $n$ -forms. It is possible to work out similar formulas for  $r$ -forms, but these are not as important in applied mathematics and to derive them we would have to introduce some equipment involving minors of determinants. Hence we will put this off to Chapter 2 and concentrate here on the more practically oriented formulas for *orthogonal* coordinates which are relatively easy to derive. Recall that for orthogonal coordinates we have for the position vector  $\mathbf{R} = \mathbf{R}(u^1, \dots, u^n)$

$$\begin{aligned} \mathbf{e}_j &= \frac{\partial \mathbf{R}}{\partial u^j} \\ h_i &= (\mathbf{e}_i, \mathbf{e}_i) \\ \hat{\mathbf{e}}_i &= \frac{1}{h_i} \mathbf{e}_i \\ (du^i, du^j) &= g^{ij} = \frac{1}{h_i^2} \\ (h_i du^i, h_j du^j) &= \delta_{ij} \\ E_i &= h_i du^i \end{aligned}$$

so that  $\{h_1 du^1, \dots, h_n du^n\}$  form an orthonormal set. For convenience let us set  $E_i = h_i du^i$ . Then  $(E_i, E_j) = \delta_{ij}$ . Now from the basic equation  $\omega \wedge * \eta = (\omega, \eta)\Omega_0$  we see immediately that

$$*(E_{i_1} \wedge \dots \wedge E_{i_r}) = \text{sgn}(\sigma)(E_{j_1} \wedge \dots \wedge E_{j_{n-r}})$$

where  $\{j_1, \dots, j_{n-r}\} = \{1, \dots, n\} - \{i_1, \dots, i_r\}$  and  $\sigma$  is the permutation

$$\sigma = \left( \begin{array}{ccc|ccc} 1 & \dots & r & r+1 & \dots & n \\ i_1 & \dots & i_r & j_1 & \dots & j_{n-r} \end{array} \right)$$

Here the convention is  $1 \leq i_1 < \dots < i_r \leq n$  and  $1 \leq j_1 < \dots < j_{n-r} \leq n$  but in fact the formula will work just as well with the  $i$ 's and  $j$ 's in any order. Indeed it suffices to check this for  $\omega$  running through a basis  $\{E_{k_1} \wedge \dots \wedge E_{k_r} \mid 1 \leq i_1, \dots, \leq k_r\}$  of  $\Lambda^r$ .

$$(E_{k_1} \wedge \dots \wedge E_{k_r}) \wedge *(E_{i_1} \wedge \dots \wedge E_{i_r}) = E_{k_1} \wedge \dots \wedge E_{k_r} \wedge \text{sgn}(\sigma)(E_{j_1} \wedge \dots \wedge E_{j_{n-r}})$$

Now if  $\{k_1, \dots, k_r\} \neq \{i_1, \dots, i_r\}$  there will be repetitions on the right side and the result will be 0. Hence we will take  $\{k_1, \dots, k_r\} = \{i_1, \dots, i_r\}$  and since

both are in increasing order we have  $i_\ell = k_\ell$  and

$$\begin{aligned}
(E_{i_1} \wedge \cdots \wedge E_{i_r}) \wedge *(E_{i_1} \wedge \cdots \wedge E_{i_r}) &= E_{i_1} \wedge \cdots \wedge E_{i_r} \wedge \operatorname{sgn}(\sigma)(E_{j_1} \wedge \cdots \wedge E_{j_{n-r}}) \\
&= E_1 \wedge \cdots \wedge E_n \\
&= (h_1 du^1) \wedge \cdots \wedge (h_n du^n) \\
&= h_1 \cdots h_n du^1 \wedge \cdots \wedge du^n \\
&= \sqrt{g} du^1 \wedge \cdots \wedge du^n \\
&= \Omega_0
\end{aligned}$$

On the other hand

$$(E_{k_1} \wedge \cdots \wedge E_{k_r}, E_{i_1} \wedge \cdots \wedge E_{i_r})\Omega_0 = \det\left((E_{k_\ell}, E_{i_m})\right)\Omega_0$$

Now if  $\{k_1, \dots, k_r\} \neq \{i_1, \dots, i_r\}$  then there will be a row in the determinant which is entirely 0. Hence we take  $\{k_1, \dots, k_r\} = \{i_1, \dots, i_r\}$  and since both are in increasing order we have  $i_\ell = k_\ell$ , the determinant has 1's on the main diagonal and 0's elsewhere, so the determinant is 1 and the result is

$$(E_{i_1} \wedge \cdots \wedge E_{i_r}, E_{i_1} \wedge \cdots \wedge E_{i_r})\Omega_0 = \Omega_0$$

Hence if  $\{k_1, \dots, k_r\} \neq \{i_1, \dots, i_r\}$  both sides of the fundamental equation are 0 and if  $\{k_1, \dots, k_r\} = \{i_1, \dots, i_r\}$  then both sides are  $\Omega_0$ , proving that

$$\boxed{*(E_{i_1} \wedge \cdots \wedge E_{i_r}) = \operatorname{sgn}(\sigma)(E_{j_1} \wedge \cdots \wedge E_{j_{n-r}})}$$

Replacing the  $E_i$  by  $h_i du^i$  we have

$$\boxed{*(h_{i_1} du^{i_1} \wedge \cdots \wedge h_{i_r} du^{i_r}) = \operatorname{sgn}(\sigma)(h_{j_1} du^{j_1} \wedge \cdots \wedge h_{j_{n-r}} du^{j_{n-r}})}$$

and from this

$$*(du^{i_1} \wedge \cdots \wedge du^{i_r}) = \operatorname{sgn}(\sigma) \frac{h_{j_1} \cdots h_{j_{n-r}}}{h_{i_1} \cdots h_{i_r}} (du^{j_1} \wedge \cdots \wedge du^{j_{n-r}})$$

where  $\sigma$  is the permutation

$$\sigma = \left( \begin{array}{ccc|ccc} 1 & \cdots & r & r+1 & \cdots & n \\ i_1 & \cdots & i_r & j_1 & \cdots & j_{n-r} \end{array} \right)$$

This gives us a formula valid for any  $r$

This is also a convenient moment to prove the formula

$$**\omega = (-1)^{r(n-r)}\omega$$

We can do this using the basis  $\{E_1, \dots, E_n\}$ . We define the *reverse*  $\tilde{\sigma}$  of  $\sigma$  to be, with the above  $\sigma$ ,

$$\tilde{\sigma} = \left( \begin{array}{ccc|ccc} 1 & \dots & n-r & n-r+1 & \dots & n \\ j_1 & \dots & j_{n-r} & i_1 & \dots & i_r \end{array} \right)$$

Now if we move the  $r$   $i$ 's each past the  $n-r$   $j$ 's there will be a total of  $r(n-r)$  hops to get from  $\tilde{\sigma}$  back to  $\sigma$ . Hence if it takes  $s$  hops to return  $\sigma$  to the identity,  $\text{sgn}(\sigma) = (-1)^s$  and

$$\begin{aligned} \text{sgn}(\tilde{\sigma}) &= (-1)^{s+r(n-r)} = (-1)^{r(n-r)} \text{sgn}(\sigma) \\ \text{sgn}(\sigma) \text{sgn}(\tilde{\sigma}) &= (-1)^{r(n-r)} \end{aligned}$$

Then, with  $\omega = E_{i_1} \wedge \dots \wedge E_{i_r}$  we have

$$\begin{aligned} **\omega &= **E_{i_1} \wedge \dots \wedge E_{i_r} \\ &= \text{sgn}(\sigma) * E_{j_1} \wedge \dots \wedge E_{j_{n-r}} \\ &= \text{sgn}(\sigma) \text{sgn}(\tilde{\sigma}) E_{i_1} \wedge \dots \wedge E_{i_r} \\ &= (-1)^{r(n-r)} E_{i_1} \wedge \dots \wedge E_{i_r} \\ &= (-1)^{r(n-r)} \omega \end{aligned}$$

Since this is true for the elements of a basis, it will be true for all  $\omega$  by linearity.

## 1.20 The Codifferential $\delta$

To deal with the Laplacian for forms, we need the codifferential  $\delta : \Lambda^r \rightarrow \Lambda^{r-1}$ . We will use this in our treatment of Maxwell's equations. For example, the condition of Lorenz is naturally expressed in terms of the codifferential.

In contrast to previous constructions,  $\delta$  depends for its very definition on the presence of an inner product for the forms on a manifold  $M$ . Using the  $*$  we can easily define such an inner product. Let  $\omega, \eta \in \Lambda^r$ . Then the inner product is defined by either of the following

$$((\omega, \eta)) = \int_M \omega \wedge *\eta = \int_M (\omega, \eta) \Omega_0$$

Some points to remember. The inner product  $(\omega, \eta)$  is an inner product that lives in  $T_p(M)$  for each  $p \in M$ . It and  $*$  are strictly local, being algebraic constructions on each tangent space  $T_p(M)$  separately. On the other hand,  $((\omega, \eta))$  depends on integrating the information given at each  $p$  over the manifold. It is thus a global object. Second, although we express it in our formulas for a single coordinate system, in many cases it will be necessary to cut  $M$  into separate pieces each of which lives inside the domain of a coordinate system, and to use that coordinate system when integrating over that piece. Sometimes, as for the sphere, we can use a single coordinate system even though that system is bad at specific points, like the North and South pole. This is best considered dumb

luck, and one should always be careful when doing it. But it does work often in applications as long as nothing is discontinuous at the bad points.

Now that we have the inner product on forms, we can consider  $((\omega, d\eta))$  where  $\omega \in \Lambda^r$  and  $\eta \in \Lambda^{r-1}$ . We will consider two possible scenarios

- a** The manifold  $M$  has no boundary;  $\partial M = \emptyset$  Such a manifold is often referred to as *closed*. The idea "no boundary" includes "no boundary at infinity" so that in this case the manifold  $M$  is compact.
- b** Either  $\omega$  or  $\eta$  vanishes off a compact subset  $M_0$  of  $M$ , so the integral will always be finite.

With one of these situations in place, we can do an integration by parts in the way you have seen done in differential equations, and come up with formal adjoint  $\delta$  for  $d$ . After all this preamble, the actual calculation is quite short. We need to recall that, with  $\omega \in \Lambda^{r-1}$  and  $\eta \in \Lambda^r$

$$\begin{aligned}\omega \wedge * \eta &= \eta \wedge * \omega \\ d(\omega \wedge * \eta) &= d\omega \wedge * \eta + (-1)^{r-1} \omega \wedge d * \eta\end{aligned}$$

Also note because of the above assumptions on  $\omega$  and  $\eta$  we have by Stokes theorem

$$\int_K d(\omega \wedge * \eta) = \int_{\partial K} \omega \wedge * \eta = 0$$

We can now derive the formula for the (formal) adjoint of  $d$ .

$$\begin{aligned}((d\omega, \eta)) &= \int_K d\omega \wedge * \eta \\ &= \int_K d(\omega \wedge * \eta) - (-1)^{r-1} \int_K \omega \wedge d * \eta \\ &= 0 + (-1)^r \int_K \omega \wedge d * \eta \\ &= (-1)^r (-1)^{(n-r+1)(r-1)} \int_K \omega \wedge * * d * \eta\end{aligned}$$

Now we must simplify the exponent. Recall that

$$\begin{aligned}k^2 &\equiv k \pmod{2} \\ -k &\equiv k \\ 2k &\equiv 0 \\ k(k-1) &\equiv 0\end{aligned}$$

Thus

$$r + (n-r+1)(r-1) \equiv r + (n-r)(r-1) + r-1 \pmod{2}$$

$$\begin{aligned}
&\equiv (n-r)(r-1) - 1 \\
&\equiv n(r-1) - r(r-1) - 1 \\
&\equiv n(r-1) - 1 + 2 \\
&\equiv n(r-1) + 1
\end{aligned}$$

Hence

$$\begin{aligned}
((d\omega, \eta)) &= (-1)^{n(r-1)+1} \int_K \omega \wedge * * d * \eta \\
&= \int_K \omega \wedge * \left( (-1)^{n(r-1)+1} * d * \eta \right)
\end{aligned}$$

Thus if we set

$$\delta : \Lambda^r \rightarrow \Lambda^{r-1}$$

defined by

$$\boxed{\delta\eta = (-1)^{n(r-1)+1} * d * \eta \quad \eta \in \Lambda^r}$$

we have

$$\boxed{((d\omega, \eta)) = ((\omega, \delta\eta)) \quad \omega \in \Lambda^{r-1}, \eta \in \Lambda^r}$$

as we wanted. Naturally  $\delta$  is only a *formal* adjoint to  $d$  because we have not considered any boundary conditions, but this is not unusual in mathematical physics.

It is worth noticing that the formula for  $\delta$  simplifies if we consider the cases of even and odd dimensional spaces separately. We have

$$n \text{ odd:} \quad \delta\omega = (-1)^r * d * \omega$$

$$n \text{ even:} \quad \delta\omega = - * d * \omega$$

There are some useful identities connecting  $*$ ,  $d$ , and  $\delta$  which are consequences of  $*$  being almost an involution, which we now derive. First we have

$$\begin{aligned}
* \delta\omega &= (-1)^{n(r-1)+1} * * d * \omega \quad \omega \in \Lambda^r \\
&= (-1)^{n(r-1)+1} (-1)^{(n-r+1)(r-1)} d * \omega \quad \omega \in \Lambda^r \\
&= (-1)^r d * \omega \quad \omega \in \Lambda^r
\end{aligned}$$

since

$$\begin{aligned}
n(r-1) + 1 + (n-r+1)(r-1) &\equiv (n+n-r+1)(r-1) + 1 \pmod{2} \\
&\equiv -(r-1)(r-1) + 1 \\
&\equiv (r-1)^2 + 1 \\
&\equiv (r-1) + 1 \\
&\equiv r
\end{aligned}$$

Applying our identity to  $*\omega$  we have

$$\begin{aligned}
 *\delta*\omega &= (-1)^{n-r}d**\omega \\
 &= (-1)^{n-r+r(n-r)}d\omega \\
 &= (-1)^{(n-r)(r+1)}d\omega \\
 **\delta*\omega &= (-1)^{(n-r)(r+1)}*d\omega \\
 (-1)^{(n-r-1)(r+1)}\delta*\omega &= (-1)^{(n-r)(r+1)}*d\omega \\
 \delta*\omega &= (-1)^{(n-r)(r+1)+(n-r-1)(r+1)}*d\omega \\
 \delta*\omega &= (-1)^{(r+1)(n-r+n-r-1)}*d\omega \\
 \delta*\omega &= (-1)^{r+1}*d\omega
 \end{aligned}$$

The  $r$  in these formulas always refers to the degree of  $\omega$ , and this must be carefully remembered when applying the formulas. It is easy to make mistakes applying these formulas.

## 1.21 The Laplacian

One of the most important operators in mathematical physics is the Laplacian. We have derived formulas for the Laplacian on functions but using the codifferential it is possible to give formulas for the Laplacian on differential forms also. This is important, for example, in Electromagnetic theory where we will need the Laplacian of the one form which expresses the vector potential of the magnetic field.

The Laplacian

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x^{i2}}$$

has a long history in mathematical physics and almost as long a history in pure mathematics. New vistas for the Laplacian were opened by Beltrami in the 1860's when he defined it on a surface. Eventually it was found to have an unexpected geometric meaning when defined on differential forms. We will have the briefest of looks at this much later in the book. In earlier ages the Laplacian was defined by use of Tensors, but in the 20th Century it was realized it could be defined in terms of the  $*$  operator and  $d$ , which made for a more elegant exposition, though problems remain in the calculation of formulas. The mathematical definition on forms is now

**Def**  $\Delta = d\delta + \delta d$

on any Riemannian or pseudo-Riemannian manifold. Note that we must have an inner product on the surface or manifold in order to define  $*$ ; no inner product, no Laplacian! The pseudo in pseudo-Riemannian means that the inner product must be non-degenerate but need not be positive definite as is the case in Relativity. We will go into this in some detail in the section of Electromagnetics in four-dimensional space-time.

Note also the extremely important point that by its definition the Laplacian is independent of the coordinate system. It is even independent of the choice of orientation, since reversing the orientation reverses the sign of each of the two  $*$ 's found in each term of its definition and since there are two minus signs each term will come out the same.

Most annoyingly, the mathematical operator  $\Delta$  does not quite coincide with the operator  $\mathbb{A}$  used in classical mathematical physics and we must be a bit careful about this; the relationship is  $\mathbb{A} = -\Delta$ . In order to keep it absolutely clear *which* Laplacian we are using I have invented the symbol  $\mathbb{A}$  for the familiar **Physics Laplacian**.

There are a couple of things that can be said in favor of the mathematics notation. First, the eigenvalues  $\lambda$  of  $\Delta$  are positive (or 0) while those of  $\mathbb{A}$  are negative (or 0). Connected with this is the fact that  $((\Delta\omega, \omega)) \geq 0$  while  $((\mathbb{A}\omega, \omega)) \leq 0$ . The wave equations are

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \Delta\omega = 0 \qquad \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \mathbb{A}\omega = 0$$

and the Schrödinger equations

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\hbar^2}{2m} \Delta \Psi + V\Psi \qquad i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \mathbb{A} \Psi + V\Psi$$

and Poisson's equation for the potential of charge distribution

$$\Delta \phi = \frac{\rho}{\epsilon} \qquad \mathbb{A} \phi = -\frac{\rho}{\epsilon}$$

with the math forms perhaps marginally more elegant. There may be other reasons too for suspecting that Laplace chose the wrong sign. All we can do at the present time is to be careful about which one we are using.

Notice that  $d : \Lambda^r \rightarrow \Lambda^{r+1}$  and  $\delta : \Lambda^r \rightarrow \Lambda^{r-1}$  so that the combined  $\Delta : \Lambda^r \rightarrow \Lambda^r$ . Notice also that the definition of the Laplacian defines  $\Delta$  on forms of all degrees, not just on functions. This is very important.

Another point of interest is that  $\Delta$  is a sort of *square*. Recall that  $d^2 = dd = 0$  and  $\delta^2 = \delta\delta = 0$ . Thus we have

$$\begin{aligned} (d + \delta)^2 &= (d + \delta)(d + \delta) = dd + d\delta + \delta d + \delta\delta \\ &= d\delta + \delta d = \Delta \end{aligned}$$

However, this is not as interesting as it looks because

$$d + \delta : \Lambda^r \rightarrow \Lambda^{r+1} \oplus \Lambda^{r-1}$$

and it is hard to see how to get practical use out of that.

Before going on it will be instructive to calculate a couple of Laplacians from scratch. Although we could make use of formulas derived in previous sections, I think it is interesting to see the whole calculation in one place. For functions

$f \in \Lambda^0$  we have  $\delta f = 0$  (since  $*f \in \Lambda^n$  and thus  $d*f = 0$ ) which simplifies the calculation greatly. Thus we have

$$\begin{aligned}
\Delta f &= (d\delta + \delta d)f = \delta df \\
&= (-1)^{n(1+1)+1} * d * (df) \\
&= - * d * \left( \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \right) \\
&= - * d \left( \sum_{i=1}^n (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \right) \\
&= - * \left( \sum_{i,j=1}^n (-1)^{i-1} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \right)
\end{aligned}$$

The only non zero term (due to repetitions) is when  $j = i$ , hence

$$\begin{aligned}
\Delta f &= - * \left( \sum_{i=1}^n (-1)^{i-1} \frac{\partial^2 f}{\partial x^{i2}} dx^i \wedge dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \right) \\
&= - * \left( \sum_{i=1}^n \frac{\partial^2 f}{\partial x^{i2}} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \dots \wedge dx^n \right) \\
&= - \sum_{i=1}^n \frac{\partial^2 f}{\partial x^{i2}}
\end{aligned}$$

which is just what we were expecting. A very similar calculation will get  $\Delta f$  for general coordinates, a result which we already have found by other methods, and which you can do in the problems.

For  $r = 1$  the computation is a lot more complicated and we will only do it for  $n = 3$ , which avoids unpleasantness and suffices for our applications. (The big advantage of doing it for  $n = 3$  is that we can exploit cyclic order.) We will first compute  $\delta d\omega$ , then compute  $d\delta\omega$ , and then put them together to get  $\Delta\omega$ . Although there are some interesting features, you might want to just skim this calculation.

$$\begin{aligned}
\omega &= \omega_1 dx + \omega_2 dy + \omega_3 dz \\
d\omega &= \left( \frac{\partial\omega_3}{\partial y} - \frac{\partial\omega_2}{\partial z} \right) dy \wedge dz + \left( \frac{\partial\omega_1}{\partial z} - \frac{\partial\omega_3}{\partial x} \right) dz \wedge dx + \left( \frac{\partial\omega_2}{\partial x} - \frac{\partial\omega_1}{\partial y} \right) dx \wedge dy \\
*d\omega &= \left( \frac{\partial\omega_3}{\partial y} - \frac{\partial\omega_2}{\partial z} \right) dx + \left( \frac{\partial\omega_1}{\partial z} - \frac{\partial\omega_3}{\partial x} \right) dy + \left( \frac{\partial\omega_2}{\partial x} - \frac{\partial\omega_1}{\partial y} \right) dz \\
d*d\omega &= \left[ \frac{\partial}{\partial y} \left( \frac{\partial\omega_2}{\partial x} - \frac{\partial\omega_1}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial\omega_1}{\partial z} - \frac{\partial\omega_3}{\partial x} \right) \right] dy \wedge dz \\
&\quad + \left[ \frac{\partial}{\partial z} \left( \frac{\partial\omega_3}{\partial y} - \frac{\partial\omega_2}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial\omega_2}{\partial x} - \frac{\partial\omega_1}{\partial y} \right) \right] dz \wedge dx
\end{aligned}$$



$$\begin{aligned}
& + \left[ \frac{\partial}{\partial x} \left( \frac{\partial \omega_1}{\partial z} - \frac{\partial \omega_3}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z} \right) \right] dx \wedge dy \\
*d * d\omega & = \left[ \frac{\partial^2 \omega_2}{\partial y \partial x} - \frac{\partial^2 \omega_1}{\partial y^2} - \frac{\partial^2 \omega_1}{\partial z^2} + \frac{\partial^2 \omega_3}{\partial z \partial x} \right] dx \\
& + \left[ \frac{\partial^2 \omega_3}{\partial z \partial y} - \frac{\partial^2 \omega_2}{\partial z^2} - \frac{\partial^2 \omega_2}{\partial x^2} + \frac{\partial^2 \omega_1}{\partial x \partial y} \right] dx \\
& + \left[ \frac{\partial^2 \omega_1}{\partial x \partial y} - \frac{\partial^2 \omega_3}{\partial x^2} - \frac{\partial^2 \omega_3}{\partial y^2} + \frac{\partial^2 \omega_2}{\partial y \partial z} \right] dx
\end{aligned}$$

OK,  $*d * d$  is done. Now we need  $d * d*$ , and then we can put them together. This one is easier.

$$\begin{aligned}
\omega & = \omega_1 dx + \omega_2 dy + \omega_3 dz \\
*\omega & = \omega_1 dy \wedge dz + \omega_2 dz \wedge dx + \omega_3 dx \wedge dy \\
d * \omega & = \left( \frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z} \right) dx \wedge dy \wedge dz \\
*d * \omega & = \left( \frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z} \right) \\
d * d * \omega & = \left( \frac{\partial^2 \omega_1}{\partial x^2} + \frac{\partial^2 \omega_2}{\partial x \partial y} + \frac{\partial^2 \omega_3}{\partial x \partial z} \right) dx + \left( \frac{\partial^2 \omega_1}{\partial y \partial x} + \frac{\partial^2 \omega_2}{\partial y^2} + \frac{\partial^2 \omega_3}{\partial y \partial z} \right) dy \\
& + \left( \frac{\partial^2 \omega_1}{\partial z \partial x} + \frac{\partial^2 \omega_2}{\partial z \partial y} + \frac{\partial^2 \omega_3}{\partial z^2} \right) dz
\end{aligned}$$

$$\begin{aligned}
\delta d\omega & = ((-1)^{3(2+1)+1} * d*)d\omega = *d * d\omega \\
d\delta\omega & = d((-1)^{3(1+1)+1} * d*)\omega = -d * d * \omega \\
\Delta\omega & = (\delta d + d\delta)\omega \\
& = *d * d\omega - d * d * \omega \\
& = - \left( \frac{\partial^2 \omega_1}{\partial x^2} + \frac{\partial^2 \omega_1}{\partial y^2} + \frac{\partial^2 \omega_1}{\partial z^2} \right) dx - \left( \frac{\partial^2 \omega_2}{\partial x^2} + \frac{\partial^2 \omega_2}{\partial y^2} + \frac{\partial^2 \omega_2}{\partial z^2} \right) dy \\
& - \left( \frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2} + \frac{\partial^2 \omega_3}{\partial z^2} \right) dz
\end{aligned}$$

This is certainly an interesting result. We denote the Laplacian on functions momentarily by  $\Delta_0 : \Lambda^0 \rightarrow \Lambda^0$  then the result can be written

$$\Delta\omega = (\Delta_0\omega_1)dx + (\Delta_0\omega_2)dy + (\Delta_0\omega_3)dz$$

How general is this result? It certainly cannot work for general coordinates, but it *does* work for rectangular coordinates on  $\Lambda^r(\mathbb{R}^n)$ . Unfortunately we do not know any better way to prove this than by brute force. For our applications what we have done suffices.

There is a vector form of the above calculation based on a well known formula in three dimensions

$$\operatorname{curl} \operatorname{curl}(\mathbf{v}) = \operatorname{grad} \operatorname{div}(\mathbf{v}) - \nabla^2(\mathbf{v})$$

which we rewrite as

$$\Delta(\mathbf{v}) = \operatorname{grad} \operatorname{div}(\mathbf{v}) - \operatorname{curl} \operatorname{curl}(\mathbf{v})$$

where in rectangular coordinates

$$\nabla^2(\mathbf{v}) = (\nabla^2 v^1) \hat{\mathbf{i}} + (\nabla^2 v^2) \hat{\mathbf{j}} + (\nabla^2 v^3) \hat{\mathbf{k}}$$

In general coordinates and three dimensions we can use the above formula as a definition of  $\Delta(\mathbf{v})$  since we have formulas for  $\operatorname{grad} \operatorname{div}(\mathbf{v})$  and  $\operatorname{curl} \operatorname{curl}(\mathbf{v})$ . If you examine the differential form calculation above carefully, you will be able to see the  $d\delta\omega$  corresponds to  $\operatorname{grad} \operatorname{div}(\mathbf{v})$  and  $\delta d\omega$  corresponds to  $\operatorname{curl} \operatorname{curl}(\mathbf{v})$ .

Soon we will show that  $*\Delta\omega = \Delta*\omega$  in all circumstances. From this we can easily derive that in  $\Lambda^2(\mathbb{R}^3)$

$$\Delta(\omega^1 dydz + \omega^2 dzdx + \omega^3 dxdy) = \Delta_0(\omega^1) dydz + \Delta_0(\omega^2) dzdx + \Delta_0(\omega^3) dxdy$$

and in  $\Lambda^3(\mathbb{R}^3)$

$$\Delta(f dxdydz) = \Delta_0(f) dxdydz$$

which tells us all we need to know about  $\Lambda^r(\mathbb{R}^3)$ .

Now let's develop a bit more theory. First, recall that the global inner product (on all of  $K$ ) is

$$((\omega, \eta)) = \int_K (\omega, \eta) \Omega_0 = \int_K \omega \wedge *\eta \quad \omega, \eta \in \Lambda^r(K)$$

provided the integral is finite, which it will be if the support

$$\operatorname{supp}(\omega) = (\text{closure of } \{x \in K \mid \omega(x) \neq 0\})$$

of  $\omega$  or the support of  $\eta$  is compact. (If  $K$  itself is compact this condition is automatically valid. Compact for our purpose means closed and bounded.) We are assuming here that all functions and coefficients of forms have enough continuous derivatives to make the calculations make sense. In general three is enough<sup>8</sup>

Recall from the chapter on the codifferential that

$$((d\omega, \eta)) = ((\omega, \delta\eta)) \quad \omega \in \Lambda^r, \eta \in \Lambda^{r+1}$$

---

<sup>8</sup>In mathematics these objects are usually taken to have infinitely many derivatives, so as to avoid thinking about such things.

From this we have

$$\begin{aligned} ((\Delta\omega, \eta)) &= ((d\delta\omega, \eta) + ((\delta d\omega, \eta)) \\ &= ((\delta\omega, \delta\eta) + ((d\omega, d\eta)) \end{aligned}$$

This shows that if  $\omega$  has coefficients with two continuous derivatives that

$$((\Delta\omega, \omega)) = 0 \text{ if and only if } \delta\omega = 0 \text{ and } d\omega = 0$$

and this is also clearly equivalent to  $\Delta\omega = 0$ . This leads to the definition

**Def**  $\omega$  (with at least one continuous derivative) is *Harmonic* if and only if  $\delta\omega = 0$  and  $d\omega = 0$ .

and then to the

**Theorem** If  $\omega$  has at least two continuous derivatives then  $\omega$  is Harmonic if and only if  $\Delta\omega = 0$ .

We prefer to use  $\delta\omega = 0$  and  $d\omega = 0$  as the definition of Harmonic and  $\Delta\omega = 0$  as a consequence because of the number of derivatives necessary for these two conditions. The definition as it stands is competing with the old definition  $\Delta\omega = 0$ ; bet on the former.

Next, from the previous equation we have

$$\begin{aligned} ((\Delta\omega, \eta)) &= ((\delta\omega, \delta\eta) + ((d\omega, d\eta)) \\ &= ((\omega, d\delta\eta) + ((\omega, \delta d\eta)) \\ &= ((\omega, \Delta\eta)) \end{aligned}$$

where we are still assuming that  $\text{supp}(\omega)$  or  $\text{supp}(\eta)$  is compact. The equation

$$((\Delta\omega, \eta)) = ((\omega, \Delta\eta))$$

with the condition on the supports means that  $\Delta$  is a formally self adjoint operator. It is standard in elementary mathematical physics to leave things here. To get *real* self adjointness requires consideration of the boundary conditions for the operator, and to deal properly with this requires functional analysis and Sobolev spaces, which is a can of worms we wish to *keep* in the can here.

Our final Laplacian duty is the equation  $*\Delta\omega = \Delta*\omega$  which we can do with a lovely little calculation. Recall from the chapter on the codifferential that for  $\omega \in \Lambda^r(K)$

$$\begin{aligned} d*\omega &= (-1)^r*\delta\omega \\ \delta*\omega &= (-1)^{r+1}*d\omega \end{aligned}$$

Then we have

$$\Delta*\omega = (d\delta + \delta d)*\omega$$

$$\begin{aligned}
&= d(\delta * \omega) + \delta(d * \omega) \\
&= d((-1)^{r+1} * d\omega) + \delta((-1)^r * \delta\omega) \\
&= (-1)^{r+1}(d * d\omega) + (-1)^r(\delta * \delta\omega) \\
&= (-1)^{r+1}(-1)^{r+1}(*\delta d\omega) + (-1)^r(-1)^r(*d\delta\omega) \\
&= *( \delta d\omega + d\delta\omega ) \\
&= * \Delta \omega
\end{aligned}$$

Notice that in going from step 4 to step 5 that  $d\omega \in \Lambda^{r+1}$  and  $\delta\omega \in \Lambda^{r-1}$  which is what makes the signs come out right.

## 1.22 Maxwell's Equations in 3-space

In this section we will show how the standard vector analysis treatment of Maxwell's equations can be recast in the notation of differential forms. Notation has been chosen to make it easy to switch over to a tensor treatment, which we provide in an addendum to the section.

The prerequisites for this section include the  $*$  operator, the codifferential which in *three dimensions* is  $\delta = *d*$  and the Laplacian  $\Delta = \delta d + d\delta$  (math form) and  $\underline{\Delta} = -\Delta$  (physics form).

A standard form of Maxwell's equations using vector formulation is

$$\begin{array}{lll}
\operatorname{div} \mathbf{D} = \rho & \operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} & \mathbf{D} = \epsilon \mathbf{E} \\
\operatorname{div} \mathbf{B} = 0 & \operatorname{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{1}{c} \mathbf{j} & \mathbf{B} = \mu \mathbf{H}
\end{array}$$

The form of these equations indicates how they should be put into differential forms. The divergence is mirrored by  $d$  on 2-forms and the curl by  $d$  on 1-forms. This suggests that  $\mathbf{D}$  and  $\mathbf{B}$  should be 2 forms and  $\mathbf{E}$  and  $\mathbf{H}$  should be 1-forms. We also note that this suggests the e.g.  $\mathbf{D}$  and  $\mathbf{E}$  are *not* trivial variants of one another, which was the attitude expressed by Maxwell and Faraday. We set

$$\begin{array}{ll}
D = D^1 dydz + D^2 dzdx + D^3 dzdx & E = E_1 dx + E_2 dy + E_3 dz \\
B = B^1 dydz + B^2 dzdx + B^3 dzdx & H = H_1 dx + H_2 dy + H_3 dz
\end{array}$$

Since  $\mathbf{j}$  is a current density (to be integrated over a surface) it should be a two form

$$j = j^1 dydz + j^2 dzdx + j^3 dzdx$$

and since  $\rho$  is a charge density it should be integrated over a region and thus should be a three form, for which we will use the letter P, an upper case Greek  $\rho$ .

$$P = \rho \Omega_0$$

In rectangular coordinates  $P = \rho dx dy dz$ . The matter equations can also be written as forms, for which we require the  $*$  operator:

$$\begin{array}{ll}
D &= \epsilon * E \\
B &= \mu * H
\end{array}$$

and then the differential equations become

$$\begin{aligned} dD = P & & dE = -\frac{1}{c} \frac{\partial B}{\partial t} \\ dB = 0 & & dH = \frac{1}{c} \frac{\partial D}{\partial t} + \frac{1}{c} j \end{aligned}$$

The equations of most significance derived from these are the equation of continuity and the potential equations. The equation of continuity is easy:

$$\begin{aligned} dH &= \frac{1}{c} \frac{\partial D}{\partial t} + \frac{1}{c} j \\ 0 = ddH &= \frac{1}{c} \frac{\partial}{\partial t} dD + \frac{1}{c} dj \\ 0 &= \frac{1}{c} \frac{\partial}{\partial t} P + \frac{1}{c} dj \\ 0 &= \frac{\partial P}{\partial t} + dj \end{aligned}$$

This is the form that the equation of continuity takes in our treatment. We note however that if we just consider the coefficient of  $dx dy dz$  from the equation it becomes

$$0 = \frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j}$$

Now we start on the potential equations.

$$dB = 0 \implies B = dA$$

(by the converse of the Poincaré lemma), where  $A$  is a 1-form.  $A$  is not uniquely determined; we can add  $dG$  to  $A$  for any  $G \in \Lambda^0$  (that is, a function), since

$$d(A + dg) = dA + ddG = dA + 0 = B$$

We can use this  $G$  to modify  $A$  and this is called *changing the gauge*. We will return to this matter later on. Next we have

$$\begin{aligned} dE &= -\frac{1}{c} \frac{\partial}{\partial t} B \\ &= -\frac{1}{c} \frac{\partial}{\partial t} dA = -\frac{1}{c} d \left( \frac{\partial A}{\partial t} \right) \\ d \left( E + \frac{1}{c} \frac{\partial A}{\partial t} \right) &= 0 \end{aligned}$$

Since  $E + \frac{1}{c} \frac{\partial A}{\partial t}$  is a 1-form, there must be a 0-form  $-\phi$  for which

$$E + \frac{1}{c} \frac{\partial A}{\partial t} = -d\phi$$

(The minus sign is historical and of no theoretical importance.)  $\phi$  is called the *scalar potential*. Often the letter  $U$  is used for it instead of  $\phi$ .  $A$  is called the *vector potential*.

If  $A$  is changed to  $A' = A + dG$  there will be a corresponding change in  $\phi$  which we now determine.

$$\begin{aligned} E + \frac{1}{c} \frac{\partial A}{\partial t} &= -d\phi \\ E + \frac{1}{c} \frac{\partial A'}{\partial t} &= -d\phi' \\ E + \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} \frac{\partial}{\partial t} dG &= -d\phi' \\ -d\phi + \frac{1}{c} \frac{\partial}{\partial t} dG &= -d\phi' \\ d(\phi' - \phi + \frac{1}{c} \frac{\partial}{\partial t} G) &= 0 \\ \phi' - \phi + \frac{1}{c} \frac{\partial}{\partial t} G &= C \end{aligned}$$

where  $C$  is some constant. Thus

$$\phi' = \phi - \frac{1}{c} \frac{\partial}{\partial t} G + C$$

Digesting, we have

$$\begin{aligned} A' &= A + dG \\ \phi' &= \phi - \frac{1}{c} \frac{\partial}{\partial t} G \end{aligned}$$

where we have set  $C = 0$  as customary.

Our next job is to derive the potential equations. However, there is so much slop in  $A$  that we cannot reasonably expect nice equations without putting some extra conditions on  $A$ . We could simply pull the condition out of the air, but it will be more fun to see it appear in context.

Recall that the physics Laplacian  $\Delta$  is the negative of the mathematical Laplacian  $\triangle = d\delta + \delta d$  where  $\delta$  in three dimensions is

$$\delta\omega = (-1)^r * d * \omega \quad \omega \in \Lambda^r$$

Now we have

$$\begin{aligned} \triangle A &= (\delta d + d\delta)A = \delta dA + d\delta A \\ &= \delta B + d\delta A \\ &= (-1)^2 * d * B + d\delta A \\ &= *d(\mu H) + d\delta A = \mu * dH + d\delta A \\ &= \mu * \left( \frac{1}{c} \frac{\partial D}{\partial t} + \frac{1}{c} j \right) + d\delta A \end{aligned}$$

$$\begin{aligned}
&= \mu \left( \frac{1}{c} \frac{\partial(*D)}{\partial t} + \frac{1}{c} *j \right) + d\delta A \\
&= \mu \left( \frac{1}{c} \frac{\partial(\epsilon E)}{\partial t} + \frac{1}{c} *j \right) + d\delta A \\
&= \frac{\mu}{c} *j + \frac{\epsilon\mu}{c} \frac{\partial E}{\partial t} + d\delta A \\
&= \frac{\mu}{c} *j + \frac{\epsilon\mu}{c} \frac{\partial}{\partial t} \left( -d\phi - \frac{1}{c} \frac{\partial A}{\partial t} \right) + d\delta A \\
&= \frac{\mu}{c} *j - \frac{\epsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2} + d \left( \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right)
\end{aligned}$$

Rearranging and replacing  $\Delta A$  by  $-\Delta A$  we have

$$\frac{\epsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2} - \Delta A = \frac{\mu}{c} *j + d \left( \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right)$$

This would be the familiar wave equation for  $A$

$$\square A = \frac{\epsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2} - \Delta A = \frac{\mu}{c} *j$$

with velocity  $\frac{c}{\sqrt{\epsilon\mu}}$  if not for the term  $d \left( \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right)$ . Hence, using the slop in  $A$ , we will set this to 0.

$$\delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} = 0 \quad \text{Condition of Lorenz}$$

Of course, we need to know that adding a suitable  $dG$  will force the Condition of Lorenz to be true. We will look at this later. (Usually this step is neglected!)

It is also useful to decode the condition of Lorenz. We have

$$\begin{aligned}
(-1)^1 * d * A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} &= 0 \\
- * d(A_1 dydz + A_1 dzdx + A_1 dxdy) - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} &= 0 \\
- * \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dxdydz - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} &= 0 \\
- \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right) &= 0
\end{aligned}$$

We must also have the potential equation for  $\phi$ . This is derived in a similar manner, but easier. It is interesting that it throws up the same extra terms which we eliminate with the Condition of Lorenz. This suggests that the time and space variables might have more of a connection than we expect á priori. This was likely one of the things that induced Lorenz develop the Lorenz-Einstein transformation equations. We imitate the previous calculation, noting that

$\delta\phi = 0$  since  $\delta$  is 0 on  $\Lambda^0$ ,

$$\begin{aligned}
\Delta\phi &= (\delta d + d\delta)\phi = \delta d\phi + d\delta\phi \\
&= \delta\left(-E - \frac{1}{c}\frac{\partial A}{\partial t}\right) + 0 \\
&= -\delta E - \frac{1}{c}\frac{\partial(\delta A)}{\partial t} \\
&= -(-1)^1 * d * E - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\delta A - \frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t}\right) \\
&= *d\frac{1}{\epsilon}D - \frac{\epsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2}
\end{aligned}$$

where we have invoked the Condition of Lorenz  $\delta A - \frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t} = 0$  once again. Continuing

$$\begin{aligned}
\Delta\phi &= \frac{1}{\epsilon} * P - \frac{\epsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} \\
\frac{\epsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi &= \frac{1}{\epsilon} * P = \frac{1}{\epsilon} * (\rho\Omega_0) \\
\frac{\epsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi &= \frac{1}{\epsilon}\rho
\end{aligned}$$

which is the familiar wave equation for  $\phi$  in coordinate independent form (except for orientation considerations and provided that  $\rho$  really *is* the physical charge density and has not been modified to fit into some special coordinate system). Note that we used  $\Delta\phi = -\Delta\phi$  again. Note also that the equation is identical in form to the Dalemberertian equation for  $A$ . And finally note that the Condition of Lorenz did not pop up so naturally in this derivation as it did in the calculation for  $A$ .

Our next job is a look at the Condition of Lorenz. A close examination of the derivation of the wave equation for  $A$  will show that if  $A$  is a solution of the equation then the Condition of Lorenz must hold. The question is, can we always *force* the Condition of Lorenz to hold by choosing an appropriate  $G$  in

$$A' = A + dG \quad \phi' = \phi - \frac{1}{c}\frac{\partial G}{\partial t}$$

The answer, as we will show, is yes. We need

$$\delta A' - \frac{\epsilon\mu}{c}\frac{\partial\phi'}{\partial t} = 0$$

How do we find the  $G$ ? Substituting into this equation we have

$$\delta(A + dG) - \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}\left(\phi - \frac{1}{c}\frac{\partial G}{\partial t}\right) = 0$$



Since  $\delta G = 0$ , this can be rewritten as

$$\begin{aligned} (\delta d + d\delta)G + \frac{\epsilon\mu}{c^2} \frac{\partial^2 G}{\partial t^2} + \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} &= 0 \\ \frac{\epsilon\mu}{c^2} \frac{\partial^2 G}{\partial t^2} + \Delta G &= - \left( \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right) \\ \frac{\epsilon\mu}{c^2} \frac{\partial^2 G}{\partial t^2} - \underline{\Delta} G &= - \left( \delta A - \frac{\epsilon\mu}{c} \frac{\partial \phi}{\partial t} \right) \end{aligned}$$

(recalling that  $\underline{\Delta} = -\Delta$ ) and once again we have a wave equation. We solve for  $G$ , correct the  $A$  and  $\phi$  to  $A'$  and  $\phi'$ , and then solve the wave equations for  $A'$  and  $\phi'$ . Actually we don't have to make the correction, since solving the wave equations for  $A$  and  $\phi$  will work just fine; the  $A$  and  $\phi$  we find will automatically satisfy the Condition of Lorenz. The importance of the above is to show that  $A$  and  $\phi$  we seek actually exist.

## 1.23 Indefinite Inner Product Spaces

This section and the next contain material of far less general usefulness than the preceding sections. They are directed specifically to Relativity, and if you are not interested in this subject then you have little need of these sections.

This first of the two contains the modifications necessary when the inner product is not positive definite, so that there are vectors for which  $(\mathbf{v}, \mathbf{v}) < 0$ . This causes some modifications in the  $*$  operator and some changes in sign in the Laplacian. In this section we will handle the situation in general and in the next the specific material for Relativity. We will concentrate again on Maxwell's equations as an example. Naturally developing the theory again with the additional complications makes for a high degree of repetition, and we will be somewhat more terse this time since you have already seen the material once.

Let  $V$  be an  $n$ -dimensional space on which there is an inner product which is non-degenerate. Thus the inner product satisfies the following

- a.  $(\alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w}) = \alpha(\mathbf{u}, \mathbf{w}) + \beta(\mathbf{v}, \mathbf{w})$       Linearity
- b.  $(\mathbf{v}, \mathbf{w}) = (\mathbf{w}, \mathbf{v})$       Symmetry
- c. If  $(\mathbf{v}, \mathbf{w}) = 0$  for all  $\mathbf{w}$  then  $v = 0$       Non-degeneracy

Notice the c. replaces the usual condition  $(\mathbf{v}, \mathbf{v}) > 0$  for  $\mathbf{v} \neq 0$ . In indefinite inner product spaces there will certainly be vectors for which  $(\mathbf{v}, \mathbf{v}) = 0$ . In Relativity these are called *null vectors* and in math isotropic vectors.

Using the usual methods of linear algebra and being careful to avoid vectors  $\mathbf{v}$  for which  $(\mathbf{v}, \mathbf{v}) = 0$  one can construct without difficulty an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  for  $V$ . Renumbering if necessary they can be arranged in a sequence

$\mathbf{e}_1, \dots, \mathbf{e}_{n-s}, \mathbf{e}_{n-s+1}, \dots, \mathbf{e}_n$  for which

$$\begin{aligned}(\mathbf{e}_i, \mathbf{e}_i) &= 1 \quad \text{for } i = 1, \dots, n-s \\(\mathbf{e}_i, \mathbf{e}_i) &= -1 \quad \text{for } i = n-s+1, \dots, n\end{aligned}$$

Here, *orthonormal* means  $(\mathbf{e}_i, \mathbf{e}_i) = \pm 1$ . Those elements of the basis with  $(\mathbf{e}_i, \mathbf{e}_i) = +1$  will be called *positive basis elements* and those elements with  $(\mathbf{e}_i, \mathbf{e}_i) = -1$  will be called *negative basis elements*. J. J. Sylvester proved early in the history of linear algebra (1852) that the number  $s$  of negative basis elements does not depend on the choice of basis; all orthonormal bases will have the same  $s$ . This is called Sylvester's law of inertia. This  $s$  will metastasize through all the formulas.

It is worth mentioning that since we have assumed that the inner product is non-degenerate there will indeed be  $n$  basis vectors in the orthonormal basis.

For the orthonormal basis defined above we get the usual matrix of metric coefficients  $g_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$  and it and its inverse will be

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

with  $n-s$  ones and  $s$  negative ones.

As far as the exterior or Grassmann product (wedge product) of the vectors goes, there will be no changes, since this apparatus works independently of the inner product. We first begin to get effects from the indefiniteness when we introduce the  $*$  operator.

We will begin work in  $\mathbb{R}^m$  or some  $n$ -dimensional submanifold  $K$  of it with coordinates  $u^1, \dots, u^n$  and an indefinite inner product on the cotangent space  $\Lambda^1(K)$ . At each point the cotangent space is spanned by  $du^1, \dots, du^n$ . The topform will be  $du^1 \wedge \dots \wedge du^n$ , but the inner product of this with itself may well be negative and thus unsuitable for use in finding volumes. If this should happen, (which it often does in applications) we put in a negative sign to fix it. To see the sign, we reason as follows. There is a process in linear algebra by which we can manipulate  $du^1, \dots, du^n$  to give us an orthonormal basis  $E^1, \dots, E^n$  which will have the matrix  $E$  above for its metric. By reverseing the sign of  $E^n$  if necessary we can assure that we can assure that  $du^1, \dots, du^n$  and  $E^1, \dots, E^n$  have the same orientation. Suppose that

$$E^i = \sum \alpha_j^i du^j$$

We know that  $\det(\alpha_j^i) > 0$  because the orientations match. Then we have

$$E = (\alpha_j^i)^\top((du^i, du^j))(\alpha_j^i)$$

and thus

$$\begin{aligned} \det E &= \det [(\alpha_j^i)^\top((du^i, du^j))(\alpha_j^i)] = \det(\alpha_j^i)^\top \det((du^i, du^j)) \det(\alpha_j^i) \\ &(-1)^s = [\det(\alpha_j^i)]^2 \det(g^{ij}) \end{aligned}$$

Since  $[\det(\alpha)]^2$  is positive  $(-1)^s$  and  $\det(g^{ij})$  have the same sign and we have  $(-1)^s \det(g^{ij}) > 0$ . Thus to normalize the topform  $du^1 \wedge \dots \wedge du^n$  it is natural to take

$$\Omega_0 = \frac{1}{\sqrt{(-1)^s \det(g^{ij})}} du^1 \wedge \dots \wedge du^n$$

If we set, as is commonly done,  $g = \det(g_{ij})$  then we have  $\det(g^{ij}) = g^{-1}$  and the formula becomes

$$\Omega_0 = \sqrt{(-1)^s g} \quad du^1 \wedge \dots \wedge du^n$$

There is another useful point. Both  $\Omega_0$  and  $E^1 \wedge \dots \wedge E^n$  are normalized topforms in the one-dimensional space  $\Lambda^n$ . At worst they could be negatives of one another, but this cannot happen since they have the same orientation. Hence we have

$$E^1 \wedge \dots \wedge E^n = \Omega_0$$

Our next job is to revise the formula for  $**\omega$ . We take  $E^1, \dots, E^{n-s}$  be the normalized positive basis elements,  $((E^i, E^i) = +1)$ , and  $E^{n-s+1}, \dots, E^n$  to be the normalized negative basis elements,  $((E^i, E^i) = -1)$ . We now take a typical basis element  $E^{i_1} \wedge \dots \wedge E^{i_r}$  with  $i_1 < i_2 < \dots < i_r$ . To this basis element corresponds a permutation

$$\sigma = \left( \begin{array}{cccc|cccc} 1 & 2 & \dots & r & r+1 & r+2 & \dots & n \\ i_1 & i_2 & \dots & i_r & k_1 & k_2 & \dots & k_{n-r} \end{array} \right)$$

with  $k_1 < k_2 < \dots < k_{n-r}$ . We suspect from our previous work that

$$*E^{i_1} \wedge \dots \wedge E^{i_r} = a E^{k_1} \wedge \dots \wedge E^{k_{n-r}}$$

and we must determine  $a$ .

$$\begin{aligned} \text{Let } s_1 \text{ of } E^{i_1}, \dots, E^{i_r} &\text{ have } (E^i, E^i) = -1 \\ \text{Let } s_2 \text{ of } E^{k_1}, \dots, E^{k_{n-r}} &\text{ have } (E^i, E^i) = -1 \end{aligned}$$

where of course  $s_1 + s_2 = s$ . Using the definition of  $*$  and our guess above about the form of  $*E^{i_1} \wedge \dots \wedge E^{i_r}$  we have

$$E^{i_1} \wedge \dots \wedge E^{i_r} \wedge *(E^{i_1} \wedge \dots \wedge E^{i_r}) = \det((E^{i_j}, E^{i_\ell})) \Omega_0$$

$$\begin{aligned}
E^{i_1} \wedge \dots \wedge E^{i_r} \wedge a E^{k_1} \wedge \dots \wedge E^{k_{n-r}} &= (-1)^{s_1} \Omega_0 \\
a \operatorname{sgn}(\sigma) E^1 \wedge \dots \wedge E^n &= (-1)^{s_1} \Omega_0 \\
a \operatorname{sgn}(\sigma) \Omega_0 &= (-1)^{s_1} \Omega_0 \\
a \operatorname{sgn}(\sigma) &= (-1)^{s_1} \\
a &= (-1)^{s_1} \operatorname{sgn}(\sigma)
\end{aligned}$$

and thus

$$\boxed{* E^{i_1} \wedge \dots \wedge E^{i_r} = (-1)^{s_1} \operatorname{sgn}(\sigma) E^{k_1} \wedge \dots \wedge E^{k_{n-r}}}$$

Now we are in a position to find  $**\omega$ . Let

$$\tilde{\sigma} = \left( \begin{array}{cccc|cccc} 1 & 2 & \dots & n-r & n-r+1 & n-r+2 & \dots & n \\ k_1 & k_2 & \dots & k_{n-r} & i_1 & i_2 & \dots & i_r \end{array} \right)$$

Then, in a similar fashion to above,

$$* E^{k_1} \wedge \dots \wedge E^{k_{n-r}} = (-1)^{s_2} \operatorname{sgn}(\tilde{\sigma}) E^{i_1} \wedge \dots \wedge E^{i_r}$$

Hence

$$\begin{aligned}
** E^{i_1} \wedge \dots \wedge E^{i_r} &= (-1)^{s_1} \operatorname{sgn} \operatorname{sgn}(\sigma) * (E^{k_1} \wedge \dots \wedge E^{k_{n-r}}) \\
&= (-1)^{s_1} \operatorname{sgn}(\sigma) (-1)^{s_2} \operatorname{sgn}(\tilde{\sigma}) E^{i_1} \wedge \dots \wedge E^{i_r} \\
&= (-1)^s (-1)^{r(n-r)} E^{i_1} \wedge \dots \wedge E^{i_r}
\end{aligned}$$

since  $s_1 + s_2 = s$  and  $\operatorname{sgn}(\sigma) \operatorname{sgn}(\tilde{\sigma}) = (-1)^{r(n-r)}$ . Since any element  $\omega \in \Lambda^r$  can be expressed in terms of these basis elements  $E^{i_1} \wedge \dots \wedge E^{i_r}$  we have

$$** \omega = (-1)^{r(n-r)+s} \omega \quad \text{for } \omega \in \Lambda^r$$

If we now write this using differentials as the basis we have

$$* du^{i_1} \wedge \dots \wedge du^{i_r} = (-1)^{s_1} \operatorname{sgn}(\sigma) du^{k_1} \wedge \dots \wedge du^{k_{n-r}}$$

where the notation is that used above.

Our next job is the codifferential  $\delta$ . Because the inner product is no longer positive definite there are some odd effects and operator which corresponds to the Laplacian is now the D'Albertian. We will return to this later. For the moment we will be interested in the formula for the codifferential, which we derive in a similar manner to the positive definite case. Let  $\omega \in \Lambda^{r-1}$  and  $\eta \in \Lambda^r$ . Recall that

$$d(\omega \wedge * \eta) = d\omega \wedge * \eta + (-1)^{r-1} \omega \wedge d * \eta$$

Then

$$((d\omega, \eta)) = \int_K d\omega \wedge * \eta$$

$$\begin{aligned}
&= \int_K d(\omega \wedge * \eta) - (-1)^{r-1} \int_K \omega \wedge d * \eta \\
&= \int_{\partial K} \omega \wedge * \eta - (-1)^{r-1} \int_K \omega \wedge d * \eta \\
&= 0 + (-1)^r \int_K \omega \wedge d * \eta
\end{aligned}$$

by Stoke's theorem, because we are assuming boundary conditions that kill off the boundary intergral, as is usual in this kind of calculation. Continuing we have

$$\begin{aligned}
((d\omega, \eta)) &= (-1)^r (-1)^{(n-r+1)(r-1)+s} \int_K \omega \wedge * * d * \eta \\
&= (-1)^{n(r-1)+s+1} \int_K \omega \wedge * * d * \eta \\
&= \int_K \omega \wedge * \left[ (-1)^{n(r-1)+s+1} * d * \eta \right] \\
&= ((\omega, (-1)^{n(r-1)+s+1} * d * \eta))
\end{aligned}$$

So if we define

$$\boxed{\delta \eta = (-1)^{n(r-1)+s+1} * d * \eta}$$

we have

$$((d\omega, \eta)) = ((\omega, \delta \eta))$$

Finally we want to introduce the analog of the Laplacian. Since the situation is so different from that of positive definite inner products, it seems reasonable to use a different notation, especially since it (more or less) coincides with that usual in physics. We will use the notation  $\square \omega$  for the new operator, which is defined exactly as was the Laplacian in the positive defintate case, namely

$$\text{Def} \quad \square \omega = (d\delta + \delta d)\omega$$

The notation is meant to suggest a relationship with the classical Dalember-tian

$$\square f = \frac{1}{c} \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}$$

## 1.24 Maxwell's equations in Space-Time

In this section we will use the material in the preceding section to work out Maxwell's equations in Space-Time. We will do this twice. First we will work out the the theory in free space where we assume  $\epsilon = \mu = 1$  and then we will work it out in general without the constrains on  $\epsilon$  and  $\mu$ . As is the preceding section this is highly specialized material and needed by a small minority of readers.

For Special Relativity, the coordinate system  $cdt, dx, dy, dz$  forms an orthonormal coordinate system, with that order. Thinking of it as  $dx^0, dx^1, dx^0, dx^1$  for convenience, we have

$$(cdt, cdt) = +1, \quad (dx, dx) = -1, \quad (dy, dy) = -1, \quad (dz, dz) = -1,$$

and

$$(dx^i, dx^j) = 0 \quad \text{for } i \neq j$$

The matrix for this coordinate system is

$$(g^{ij}) = ((dx^i, dx^j)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We can perform the  $*$  operation using the following permutations, all of which are even permutations (have  $\text{sgn}(\sigma) = +1$ ).

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

Using these and the formula

$$* dx^{i_1} \wedge \dots \wedge dx^{i_r} = (-1)^{s_1} \text{sgn}(\sigma) dx^{j_1} \wedge \dots \wedge dx^{j_r}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & r & r+1 & r+2 & \dots & n \\ i_1 & i_2 & \dots & i_r & j_1 & j_2 & \dots & j_{n-r} \end{pmatrix}$$

and  $s_1$  is the number of negative basis elements (that is  $dx, dy, dz$ ) among  $dx^{i_1}, \dots, dx^{i_r}$ , we have the following formulas

$$\begin{aligned} *1 &= cdt \wedge dx \wedge dy \wedge dz \\ *cdt &= dx \wedge dy \wedge dz & *dx \wedge dx \wedge dx &= cdt \\ *dx &= cdt \wedge dy \wedge dz & *cdt \wedge dy \wedge dz &= dx \\ *dy &= cdt \wedge dz \wedge dx & *cdt \wedge dz \wedge dx &= dy \\ *dz &= cdt \wedge dx \wedge dy & *cdt \wedge dx \wedge dy &= dz \\ *cdt \wedge dx &= -dy \wedge dz & *dy \wedge dz &= cdt \wedge dx \\ *cdt \wedge dy &= -dz \wedge dx & *dz \wedge dx &= cdt \wedge dy \\ *cdt \wedge dz &= -dx \wedge dy & *dy \wedge dz &= cdt \wedge dx \\ *cdt \wedge dx \wedge dy \wedge dz &= -1 \end{aligned}$$

For example, for the 2nd entry we have the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

with  $s_1 = 0$  and  $\text{sgn}(\sigma) = +1$ , whereas for the fourth entry we have

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

with  $s_1 = 1$  and  $\text{sgn}(\sigma) = -1$ , this being the the fourth permatation in the above list with the second and third entries swapped. For the seventh entry we have

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

so  $s_1 = 1$  and  $\text{sgn}(\sigma) = 1$ , this being the second permutation in the list. The entries in the second column can be derived from those in the first, but beware since the permutation is reversed. The fourth row second column has

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

and here  $s_1 = 2$  and  $\sigma$  is the third entry in the list of permatations so  $\text{sgn}(\sigma) = 1$ . It is far easier to derive the second column using  $**\omega = (-1)^{r(4-r)+3}\omega$ .

Next we are interested in the codifferential  $\delta$ . We have, as usual, with wedges omitted for ease of reading,

$$A = -\phi cdt + A_1 dx + A_2 dy + A_3 dz$$

Then

$$\begin{aligned} \delta A &= *d*A = *d(-\phi dx dy dz + A_1 cdt dy dz + A_2 cdt dx dy + A_3 cdt dz dx) \\ &= * \left( -\frac{1}{c} \frac{\partial \phi}{\partial t} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) cdt dx dy dz \\ &= \frac{1}{c} \frac{\partial \phi}{\partial t} - \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y} - \frac{\partial A_3}{\partial z} \end{aligned}$$

Cognoscenti will immediately recognize that the familiar condition of Lorenz,  $\text{div } A - \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$ , is here expressed by  $\delta A = 0$ .

Next we will compute the d'Alembertian  $\square f$  of a function. We have  $\delta f = 0$  so

$$\begin{aligned} \square f &= (\delta d + d\delta)f = \delta df + 0 \\ &= *d* \left( \frac{1}{c} \frac{\partial f}{\partial t} cdt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \\ &= *d \left( \frac{1}{c} \frac{\partial f}{\partial t} dx dy dz + \frac{\partial f}{\partial x} cdt dy dz + \frac{\partial f}{\partial y} cdt dz dx + \frac{\partial f}{\partial z} cdt dx dy \right) \\ &= * \left( \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} \right) cdt dx dy dz \\ &= -\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

The minus signs come from swapping differentials around and sometimes from the action of  $*$ . This is the negative of the classical d'Albertian

$$\square f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}$$

An obnoxious calculation shows that if

$$A = A_0 c dt + A_1 dx + A_2 dy + A_3 dz$$

then

$$\square A = (\square A_0) c dt + (\square A_1) dx + (\square A_2) dy + (\square A_3) dz$$

We know of no intelligent way to prove this but in view of its importance we will do the calculation in an appendix to this chapter, which should be skimmed from a distance.

We now want to apply this equipment to the problem of potential equations in four dimensional Space-Time. We first consider the case when  $\epsilon = \mu = 1$  where everything is simple. When these are not 1, there are certain complications which we will discuss later.

For ease of reference we repeat here Maxwell's equations.

$$\begin{aligned} \operatorname{div} \mathbf{D} &= \rho & \operatorname{curl} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} & \mathbf{D} &= \epsilon \mathbf{E} \\ \operatorname{div} \mathbf{B} &= 0 & \operatorname{curl} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} & \mathbf{B} &= \mu \mathbf{H} \end{aligned}$$

along with

$$\operatorname{curl} \mathbf{A} = \mathbf{B} \quad d\phi = -\mathbf{E} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

Let us set as is usual  $F = dA$ . We then have

$$\begin{aligned} A &= -\phi c dt + A_1 dx + A_2 dy + A_3 dz \\ F = dA &= \left( \frac{1}{c} \frac{\partial A_1}{\partial t} + \frac{\partial \phi}{\partial x} \right) c dt dx + \left( \frac{1}{c} \frac{\partial A_2}{\partial t} + \frac{\partial \phi}{\partial y} \right) c dt dy + \left( \frac{1}{c} \frac{\partial A_3}{\partial t} + \frac{\partial \phi}{\partial z} \right) c dt dz \\ &\quad + \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) dy dz + \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) dz dx + \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dx dy \\ &= -E_1 c dt dx - E_2 c dt dy - E_3 c dt dz \\ &\quad + B_1 dy dz + B_2 dz dx + B_3 dx dy \end{aligned}$$

Next we take the  $*$  of both sides. We remember that here we have  $\epsilon = 1$  and  $\mu = 1$  so that  $\mathbf{E} = \mathbf{D}$  and  $\mathbf{B} = \mathbf{H}$ . Then

$$\begin{aligned} \tilde{F} = *F &= E_1 dy dz + E_2 dz dx + E_3 dx dy + B_1 c dt dx + B_2 c dt dy + B_3 c dt dz \\ &= D_1 dy dz + D_2 dz dx + D_3 dx dy + H_1 c dt dx + H_2 c dt dy + H_3 c dt dz \\ d * F &= \left( \frac{1}{c} \frac{\partial D_1}{\partial t} - \left( \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} \right) \right) c dt dy dz + \left( \frac{1}{c} \frac{\partial D_2}{\partial t} - \left( \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} \right) \right) c dt dz dx \\ &\quad + \left( \frac{1}{c} \frac{\partial D_3}{\partial t} - \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \right) c dt dx dy + \left( \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} \right) dx dy dz \end{aligned}$$



$$\begin{aligned}
&= \rho dx dy dz - \frac{1}{c} j_1 c dt dy dz - \frac{1}{c} j_2 c dt dz dx - \frac{1}{c} j_3 c dt dx dy \\
\delta F &= *d*F = \rho c dt - \frac{1}{c} j_1 dx - \frac{1}{c} j_2 dy - \frac{1}{c} j_3 dz
\end{aligned}$$

Now the coup; recall that we may take  $\delta A = 0$  (the condition of Lorenz) so that we have

$$\begin{aligned}
\Box A &= (\delta d + d\delta)A = \delta dA + 0 \\
&= \delta F = \rho c dt - \frac{1}{c} j_1 dx - \frac{1}{c} j_2 dy - \frac{1}{c} j_3 dz
\end{aligned}$$

Using the fact that in rectangular coordinates we have

$$\Box A = -(\Box \phi) c dt + (\Box A_1) dx + (\Box A_2) dy + (\Box A_3) dz$$

we have, with the classical d'Alembertian

$$\Box A_i = \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = -\Box A_i$$

we have the potential equations in four space

$$\Box \phi = \rho$$

$$\Box A_i = \frac{1}{c} j_i$$

which is the same result we got in the section on Maxwell's equations in three dimensional space.

We now want to extend this result to the case where  $\epsilon$  or  $\mu$  are not one. This is surprisingly difficult. However, we suspect it can be done because the corresponding equations in three space have the same form for the potential  $\phi$  and the vector potential components  $A_i$ . Recall that the form of the equation is

$$\frac{\epsilon\mu}{c^2} \frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = \frac{\mu}{c} j_i$$

This suggests an electromagnetic wave moving at a speed of  $k = \frac{c}{\sqrt{\epsilon\mu}}$ . Now we must steer the boat, like Odysseus, between Charybdis and Skilla. Charybdis refers to the fact that for the d'Alembertian to come out with  $\frac{1}{k^2} = \frac{\epsilon\mu}{c^2}$  in it, we are going to have to modify the  $*$  operator by changing  $c$  to  $k$ . Skilla refers to the fact that Maxwell's equations have  $c$  not  $k$  in them. Thus replacing all the  $c$ 's by  $k$ 's won't work; we must steer more subtly. Nevertheless the fact that the equations for  $\phi$  and  $A_i$  have the same form in 3-space suggests that it is possible to navigate successfully, which we now do.

In the following calculations the  $*$  operator uses the same equations as before *but* the constant  $c$  in those equations is replaced by the constant  $k$ . Except for

some trivial algebra the calculation goes as before; the potential form  $A$  is, as before, recalling that  $k = c/\sqrt{\epsilon\mu}$ ,

$$\begin{aligned} A &= -\phi cdt + A_1 dx + A_2 dx + A_3 dx \\ A &= -\frac{c}{k}\phi kdt + A_1 dx + A_2 dx + A_3 dx \\ A &= -\sqrt{\epsilon\mu}\phi kdt + A_1 dx + A_2 dx + A_3 dx \end{aligned}$$

First we go for the codifferential  $\delta A$ ,

$$\begin{aligned} *A &= -\sqrt{\epsilon\mu}\phi dx dy dz + A_1 kdt dy dz + A_2 kdt dz dx + A_3 kdt dx dy \\ d*A &= \left(-\frac{\sqrt{\epsilon\mu}}{k}\frac{\partial\phi}{\partial t} - \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y} - \frac{\partial A_3}{\partial z}\right) kdt dx dy dz \\ \delta A = *d*A &= \frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = 0 \end{aligned}$$

by the condition of Lorenz, which we are assuming, as before. One of the the positive aspects of the four dimensional treatment is that the condition of Lorenz is so simply expressed:  $\delta A = 0$

Next we have, and note we here use  $c$  not  $k$  so that we can use Maxwell's equations,

$$\begin{aligned} F &= dA \\ &= \left(\frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\partial\phi}{\partial x}\right) cdt dx + \left(\frac{1}{c}\frac{\partial A_2}{\partial t} + \frac{\partial\phi}{\partial y}\right) cdt dy + \left(\frac{1}{c}\frac{\partial A_3}{\partial t} + \frac{\partial\phi}{\partial z}\right) cdt dz \\ &\quad + \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) dy dz + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) dz dx + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx dy \\ &= -E_1 cdt dx - E_2 cdt dy - E_3 cdt dz \\ &\quad + B_1 dy dz + B_2 dz dx + B_3 dx dy \\ &= \sqrt{\epsilon\mu}(-E_1 kdt dx - E_2 kdt dy - E_3 kdt dz) \\ &\quad + B_1 dy dz + B_2 dz dx + B_3 dx dy \\ *F &= \sqrt{\epsilon\mu}(E_1 dy dz + E_2 dz dx + E_3 dx dy) \\ &\quad + B_1 kdt dx + B_2 kdt dy + B_3 kdt dz \\ &= \frac{\sqrt{\epsilon\mu}}{\epsilon}(D_1 dy dz + D_2 dz dx + D_3 dx dy) \\ &\quad + \frac{\mu}{\sqrt{\epsilon\mu}}(H_1 cdt dx + H_2 cdt dy + H_3 cdt dz) \\ &= \sqrt{\frac{\mu}{\epsilon}}\tilde{F} \end{aligned}$$

where

$$\begin{aligned}
\tilde{F} &= \sqrt{\frac{\epsilon}{\mu}} * F \\
&= H_1 c dt dx + H_2 c dt dy + H_3 c dt dz \\
&\quad + D_1 dy dz + D_2 dz dx + D_3 dx dy
\end{aligned}$$

Now we find  $d\tilde{F}$ ,

$$\begin{aligned}
d\tilde{F} &= \left( \frac{1}{c} \frac{\partial D_1}{\partial t} - \left( \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} \right) \right) c dt dy dz + \left( \frac{1}{c} \frac{\partial D_2}{\partial t} - \left( \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} \right) \right) c dt dz dx \\
&\quad + \left( \frac{1}{c} \frac{\partial D_3}{\partial t} - \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \right) c dt dx dy + \left( \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} \right) dx dy dz \\
&= \rho dx dy dz - \frac{1}{c} (j_1 c dt dy dz + j_2 c dt dz dx + j_3 c dt dx dy) \\
&= \rho dx dy dz - \frac{\sqrt{\epsilon\mu}}{c} (j_1 k dt dy dz + j_2 k dt dz dx + j_3 k dt dx dy)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\Box A &= (\delta d + d\delta)A \\
&= \delta dA + 0 \quad (\text{condition of Lorenz } \delta A = 0) \\
&= \delta F = * d * F \\
&= * d \sqrt{\frac{\mu}{\epsilon}} \tilde{F} = \sqrt{\frac{\mu}{\epsilon}} * d\tilde{F} \\
&= \sqrt{\frac{\mu}{\epsilon}} * \left( \rho dx dy dz - \frac{\sqrt{\epsilon\mu}}{c} (j_1 k dt dy dz + j_2 k dt dz dx + j_3 k dt dx dy) \right) \\
&= \sqrt{\frac{\mu}{\epsilon}} \left( \rho k dt - \frac{\sqrt{\epsilon\mu}}{c} (j_1 dx + j_2 dy + j_3 dz) \right) \\
&= \sqrt{\frac{\mu}{\epsilon}} \left( \frac{1}{\sqrt{\epsilon\mu}} \rho c dt - \frac{\sqrt{\epsilon\mu}}{c} (j_1 dx + j_2 dy + j_3 dz) \right) \\
&= \frac{\rho}{\epsilon} c dt - \frac{\mu}{c} (j_1 dx + j_2 dy + j_3 dz)
\end{aligned}$$

This decodes in the usual way, with  $\boxplus = -\Box$ , to

$$\begin{aligned}
\boxplus \phi &= \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = \frac{\rho}{\epsilon} \\
\boxplus A_i &= \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = \frac{\mu}{c} j_i
\end{aligned}$$

## 1.25 Energy Flow and Energy Density

The force on a particle of charge  $e$  and velocity  $\vec{v}$  is

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

Thus the work done in going from  $\vec{R}$  to  $\vec{R} + d\vec{\ell}$  is

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{\ell} = e(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell} \\ &= e(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} dt \\ &= e \vec{E} \cdot \vec{v} dt \end{aligned}$$

Thus

$$\frac{dW}{dt} = e \vec{E} \cdot \vec{v}$$

Now let  $\rho$  be a continuous charge distribution. Then, as usual, we replace  $e$  by  $\rho dt$  and integrate to find  $dW/dt$ . We also need  $\vec{j} = \rho \vec{v}$ . Then we have

$$\begin{aligned} \frac{dW}{dt} &= \int_M (\rho d\tau) \vec{E} \cdot \vec{v} \\ &= \int_M \vec{E} \cdot (\rho \vec{v}) d\tau \\ &= \int_M \vec{E} \cdot \vec{j} d\tau \end{aligned}$$

Rewriting in terms of differential forms we have

$$\frac{dW}{dt} = \int_M E \wedge j$$

Now we can eliminate  $j$  by using the Maxwell equation

$$dH = \frac{1}{c} \frac{\partial D}{\partial t} + \frac{1}{c} j$$

which is Ampère's law. Thus

$$\begin{aligned} \frac{1}{c} j &= -\frac{1}{c} \frac{\partial D}{\partial t} + dH \\ \frac{dW}{dt} &= c \int_M E \wedge \frac{1}{c} j \\ &= c \int_M E \wedge \left( -\frac{1}{c} \frac{\partial D}{\partial t} + dH \right) \end{aligned}$$

For the next bit of the calculation we need the formula

$$E \wedge \frac{\partial D}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (E \wedge D)$$

We use  $D = \epsilon * E$  and the general formula for differential forms  $\lambda \wedge * \mu = \mu \wedge * \lambda$ . We then have

$$\begin{aligned} E \wedge \frac{\partial D}{\partial t} &= E \wedge \frac{\partial}{\partial t} (\epsilon * E) \\ &= \epsilon E \wedge \frac{\partial}{\partial t} (*E) \end{aligned}$$

$$\begin{aligned}
&= \epsilon E \wedge * \frac{\partial E}{\partial t} \\
&= \epsilon \frac{\partial E}{\partial t} \wedge * E \\
&= \frac{\partial E}{\partial t} \wedge D
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2} \frac{\partial}{\partial t} (E \wedge D) &= \frac{1}{2} \left( \frac{\partial E}{\partial t} \wedge D + E \wedge \frac{\partial D}{\partial t} \right) \\
&= \frac{1}{2} \left( E \wedge \frac{\partial D}{\partial t} + E \wedge \frac{\partial D}{\partial t} \right) \\
&= E \wedge \frac{\partial D}{\partial t}
\end{aligned}$$

In an exactly similar calculation we have

$$\frac{\partial B}{\partial t} \wedge H = \frac{1}{2} \frac{\partial}{\partial t} (B \wedge H)$$

Resuming the previous calculation we have

$$\begin{aligned}
\frac{dW}{dt} &= c \int_M E \wedge \left( -\frac{1}{c} \frac{\partial D}{\partial t} + dH \right) \\
&= c \int_M -E \wedge \frac{1}{c} \frac{\partial D}{\partial t} + E \wedge dH \\
&= c \int_M -E \wedge \frac{1}{c} \frac{\partial D}{\partial t} + (dE \wedge H - d(E \wedge H)) \\
&= c \int_M -E \wedge \frac{1}{c} \frac{\partial D}{\partial t} + \left( -\frac{1}{c} \frac{\partial B}{\partial t} \wedge H - d(E \wedge H) \right) \\
&= c \int_M -\frac{1}{2c} \frac{\partial}{\partial t} (E \wedge D) - \frac{1}{2c} \frac{\partial}{\partial t} (B \wedge H) - d(E \wedge H) \\
&= \frac{\partial}{\partial t} \int_M -\frac{1}{2} (E \wedge D) - \frac{1}{2} (B \wedge H) - c \int_{\partial M} E \wedge H
\end{aligned}$$

by Stokes' theorem. If we define the Energy Density  $u$  of the field as

$$u = -\frac{1}{2} (E \wedge d + B \wedge H)$$

and the Flux of energy through the surface  $\partial M$  as

$$S = c E \wedge H \quad \text{Poyting's Flux Form}$$

we can interpret the previous equation as the change in the Energy as due to the decrease in the field energy less the flux of energy through the boundary.



## Chapter 2

# Mathematical Theory

## 2.1 INTRODUCTION

This chapter has two purposes. First we must prove the two major theorems of elementary differential forms, and second we want to extend the theory with some deeper methods. For example, we would like to have formulas for the  $*$  operator on manifolds, where orthonormal coordinates systems are usually not possible, so we need some more flexible methods for that.

We also want discuss the mathematical theories that underlie the development in the first chapter. Also, we will look at the theory in a more general setting; that of a differentiable manifold rather than simply  $n$ -space. A differentiable manifold is like a surface in  $n$ -space but with more dimensions and without the  $n$ -space. If the manifold is actually inside an  $n$ -space the manifold is called an embedded manifold. In point of fact, all manifolds can be embedded in an  $n$ -space of sufficiently high dimension <sup>1</sup>, but it is no longer customary to do this. Also it may not be natural to do this; few people think that the four dimensional Space-Time of our universe is embedded in a five (or more) dimensional Euclidean space  $\mathbb{R}^n$ .

## 2.2 Manifolds Mappings and Pullbacks

In order to have a firm basis on which to proceed it would be wise to spend a bit of time on some technical material of great importance for modern differential geometry and everything that uses it. This material is not very deep but requires a bit of thought. First we want to give a good definition of *manifold*. Then we want to discuss mappings  $f : X \rightarrow Y$  between manifolds and how a differential form on  $Y$  may be "pulled back" to  $X$  using  $f$ . This material is useful in the sections that follow, as well as form a great many other things, and should be studied carefully.

## 2.3 Proofs of Stokes Theorem

We will present in this chapter two proofs of Stokes theorem. The first is a low level proof using the familiar methods of Advanced Calculus, as seen in a third semester Calculus course. The second using more modern mathematical machinery, in particular partitions of unity, which are now common in advanced mathematics.

## 2.4 Proofs of the Converse of the Poincare Lemma

We will present two proofs of the Converse of the Poincare Lemma. The first uses familiar material from Advanced calculus. The second uses slightly more advanced material but is very interesting because a borrows the idea of chain

---

<sup>1</sup>This is accomplished by the Whitney or Nash embedding theorems, depending on the circumstances



homotopy from Algebraic Topology, although no previous knowledge of this is necessary.

## 2.5 Permutations

One cannot get very far using differential forms without the dualizing operator  $*$  which we saw used in the previous section for deriving formulas for curvilinear coordinates. In this section we will give an introduction to the  $*$  operator which will be more of a tour than a treatment; the full treatment will be found in Chapter 2. Thus this section will present the basic ideas and the formulas the physicist will find most useful. Formulas for more sophisticated uses will be found in Chapter 2.

A critical and not particularly well liked part of this is some of the lore of permutations. A permutation is a rearrangement of the set  $\{1, 2, \dots, n\}$ ; for example  $\{3, 5, 6, 1, 2, 4\}$  is a rearrangement of  $\{1, 2, 3, 4, 5, 6\}$ . By interchanging two elements, adjacent or not, a permutation may be brought back to  $\{1, 2, \dots, n\}$ . There are many ways to do this of course, but it turns out that the number of interchanges to do this is always the same mod 2. Thus no matter how the interchanges are done,  $(-1)^s$  will always have the same value, and that value is the *sign* of the permutation. It is convenient for us (though somewhat non-standard) to write a permutation  $\sigma$  as a function with inputs at the top and outputs at the bottom. Thus the permutation above would be written

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$$

Here are a possible sequence of interchanges to get back to the identity.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix} \longrightarrow \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

We used four interchanges;  $s = 4$ . Clearly this can be done in many ways but the number of interchanges  $s$  will always be an even number, and thus

$$\text{sgn}(\sigma) = (-1)^s = +1$$

is well defined.

There is one other useful observation. Suppose that a permutation has the property that it compounded of two *increasing* subsequences of the numbers  $\{1, 2, \dots, n\}$ ; for example

$$\left( \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 7 & 8 & 1 & 2 & 4 & 6 & 9 \end{array} \right)$$

or more generally with  $i_1 < \dots < i_r$  and  $i_{r+1} < \dots < i_n$

$$\sigma = \left( \begin{array}{ccc|ccc} 1 & \dots & r & r+1 & \dots & n \\ i_1 & \dots & i_r & i_{r+1} & \dots & i_n \end{array} \right)$$

Note that the elements after the vertical line are determined by those before it. Now the interesting thing here is that the sign of such a permutation can be found as follows. First set

$$T_r = 1 + 2 + \cdots + r = \frac{r(r+1)}{2}$$

Then for such a  $\sigma$

$$\text{sgn}(\sigma) = (-1)^{\sum_{j=1}^r i_j - T_r}$$

Thus for the example above

$$\text{sgn}(\sigma) = (-1)^{3+5+7+8 - T_4} = (-1)^{23-10} = -1$$

Since this kind of permutation is the most common type in differential forms, this is quite handy.

## 2.6 The operator $\Phi$

In this mathematical appendix we provide mathematical justification for many of the things that were skipped over in the foregoing sections. For example, in some of the sections we showed how differential forms "mimicked" the action of curl on vectors. In this section we show how this can be made mathematically precise.

There are two coordinate independent operators whose theory we must develop. We do not do this in complete generality, but limit ourselves to our particular needs. Nevertheless, we do enough so that anyone who reads and understands this appendix will find large areas of differential geometry familiar.

The first operator, which we call  $\Phi$ , sets up an isomorphism between a vector space  $V$  with an inner product and its dual space, the space of linear functionals on  $V$ , which we call  $V^*$ . This is quite easy and is merely an invariant form of the raising and lowering of indices used in Tensor Analysis.

The second operator is the  $*$  operator, invented by Heinrich Grassmann and popularized by William V. D. Hodge. This is a little more difficult and requires introduction of an inner product on the dual space and also the introduction of a normalized topform, which we explain below.

Let  $V$  be a  $n$ -dimensional vector space with an inner product, which we write  $(\mathbf{v}, \mathbf{w})$  rather than  $\mathbf{v} \cdot \mathbf{w}$ , and let  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  be a basis. We set

$$g_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$$

Then  $\mathbf{v} = \sum_i v^i \mathbf{e}_i$  for any  $\mathbf{v} \in V$  but, following Einstein, we will omit the sum sign when an index is repeated, one up one down, and we will write this  $\mathbf{v} = v^i \mathbf{e}_i$ . Similarly  $\mathbf{w} = w^j \mathbf{e}_j$  and we have

$$\begin{aligned} (\mathbf{v}, \mathbf{w}) &= (v^i \mathbf{e}_i, w^j \mathbf{e}_j) \\ &= v^i w^j (\mathbf{e}_i, \mathbf{e}_j) \\ &= g_{ij} v^i w^j \end{aligned}$$

This connects  $(\mathbf{v}, \mathbf{w})$  whose value does not depend on a choice of basis with quantities  $g_{ij}$ ,  $v^i$  and  $w^j$  whose values *do* depend upon the basis. It is important to be able to go back and forth between *coordinate free* notation and indexed notation where the objects depend upon the choice of basis. Tensor notation is an attempt to live in both worlds, and it does it quite well, at the expense of rather complex looking equations.

Because an inner product is non-degenerate, (which means that if  $(\mathbf{v}, \mathbf{w}) = 0$  for all  $\mathbf{w} \in V$  then  $\mathbf{v} = 0$ ) we must have

$$\det(g_{ij}) \neq 0$$

This is most important.

A linear functional  $\ell$  is defined by the requirement that

$$\ell(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\ell(\mathbf{v}) + \beta\ell(\mathbf{w})$$

The set of all linear functionals is a vector space and is called  $V^*$ .

An important theorem is

**Theorem** Representation of linear functional.

Let  $\ell$  be a linear functional on a vector space with an inner product. There exists a unique  $\mathbf{u} \in V$  for which

$$\ell(\mathbf{v}) = (\mathbf{u}, \mathbf{v})$$

We will prove this shortly. First we give an important example. If  $\mathbf{v} = v^i \mathbf{e}_i$  then we can define a linear functional whose value on  $\mathbf{v}$  is  $v^i$ . This functional is called  $\mathbf{E}^i$  (note upper index). It should be clear that

$$\mathbf{E}^i(\mathbf{e}_j) = \delta_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now let  $\ell \in V^*$  be a linear functional and let  $\lambda_i = \ell(\mathbf{e}_i)$ . Then I claim that  $\ell = \lambda_i \mathbf{E}^i$ . Indeed

$$\begin{aligned} \lambda_i \mathbf{E}^i(\mathbf{v}) &= \lambda_i \mathbf{E}^i(v^j \mathbf{e}_j) \\ &= \lambda_i v^j \mathbf{E}^i(\mathbf{e}_j) \\ &= \lambda_i v^j \delta_j^i \\ &= \lambda_j v^j \\ &= v^j \ell(\mathbf{e}_j) \\ &= \ell(v^j \mathbf{e}_j) \\ &= \ell(\mathbf{v}) \end{aligned}$$

Then it is easy to prove that the  $\mathbf{E}^i$  are linearly independent and thus form a basis for  $V^*$ .

Now back to  $\Phi : V \rightarrow V^*$ . Let  $\ell \in V^*$  be a linear functional and let it be represented by  $\mathbf{u} \in V$ . Then the theorem says that for all  $\mathbf{v} \in V$  we have

$$\ell(\mathbf{v}) = (\mathbf{u}, \mathbf{v})$$

We define  $\Phi$  by

$$\Phi(\mathbf{u}) = \ell$$

Since this has been defined in terms of the inner product, which does not depend upon the basis,  $\Phi$  is independent of the basis.

For our next task we need to define

$$(g^{ij}) = (g_{ij})^{-1}$$

When these matrices are written out the inverse relationship becomes

$$\begin{aligned} g_{ij}g^{jk} &= \delta_i^k \\ g^{ij}g_{jk} &= \delta_k^i \end{aligned}$$

Also note that  $g_{ij} = (\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{e}_j, \mathbf{e}_i) = g_{ji}$  the order of the indices in the above equations is not important.

It is now time to prove the theorem. Since we would like to know the connection between  $\ell$  and  $\mathbf{u}$  in coordinates, it makes sense to prove the theorem using coordinates. Let  $\ell = \lambda_j \mathbf{E}^j$  and define  $\mathbf{u}$  by

$$u^i = g^{ij} \lambda_j \quad \text{and} \quad \mathbf{u} = u^i \mathbf{e}_i$$

We now verify this works:

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) &= (u^i \mathbf{e}_i, v^k \mathbf{e}_k) \\ &= g_{ik} u^i v^k \\ &= g_{ik} g^{ij} \lambda_j v^k \\ &= g_{ki} g^{ij} \lambda_j v^k \\ &= \delta_k^j \lambda_j v^k \\ &= \lambda_k v^k \\ &= \lambda_k \mathbf{E}^k(\mathbf{v}) \\ &= \ell(\mathbf{v}) \end{aligned}$$

Then the formula for  $\Phi$  in coordinates is

$$\Phi(\mathbf{u}) = \ell \quad \text{if and only if} \quad g_{ij} u^j = \lambda_i$$

and of course

$$\Phi^{-1}(\ell) = \mathbf{u} \quad \text{if and only if} \quad g^{ij} \lambda_j = u^i$$

Our next job is to put an inner product on  $V^*$  and we will do this by *exporting* the inner product on  $V$  to  $V^*$  in such a way as to make  $\Phi$  an isometry. That is we define, for  $\lambda, \mu \in V^*$ ,

$$(\lambda, \mu) \stackrel{\text{Def}}{=} (\Phi^{-1}(\lambda), \Phi^{-1}(\mu))$$

Now to do this in coordinates we have, with  $\lambda = \lambda_k \mathbf{E}^k$  and  $\mu = \mu_n \mathbf{E}^n$

$$\begin{aligned}(\lambda, \mu) &= g_{ij} g^{ik} \lambda_k g^{jm} \mu_m \\ &= \delta_j^k \lambda_k g^{jm} \mu_m \\ &= \lambda_j g^{jm} \mu_m \\ &= g^{jm} \lambda_j \mu_m\end{aligned}$$

which is the desired formula for the inner product in coordinates.



# Bibliography

- [1] Misner, Thorne and Wheeler, GRAVITATION, W.H. Freeman, San Francisco, 1973
- [2] Frankel, Theodore THE GEOMETRY OF PHYSICS, 3<sup>rd</sup> edition, Cambridge University Press, Cambridge,etc, 2012
- [3] Gilkey, Peter B. INVARIANCE THEORY, THE HEAT EQUATION AND THE ATIZYA-SINGER INDEX THEOREM, 2nd Edition Boca Raton 1995
- [4] Nakahara, Mikio GEOMETRY, TOPOLOGY, AND PHYSICS, Institute of Physics Publishing, London, 2003
- [5] Lovelock, David and Rund, Hanno TENSORS, DIFFERENTIAL FORMS AND VARIATIONAL PRINCIPLES, Wiley Interscience
- [6] Roe, John, ELLIPTIC OPERATORS, TOPOLOGY AND ASYMPTOTIC METHODS, 2nd Edition, Boca Raton, 1998
- [7] Rosenberg, S. THE LAPLACIAN ON A RIEMANNIAN MANIFOLD, Cambridge, 1997
- [8] Schulz, William, THEORY AND APPLICATION OF GRASSMANN ALGEBRA, <http://www.cefns.nau.edu/~schulz/grassmann.pdf>