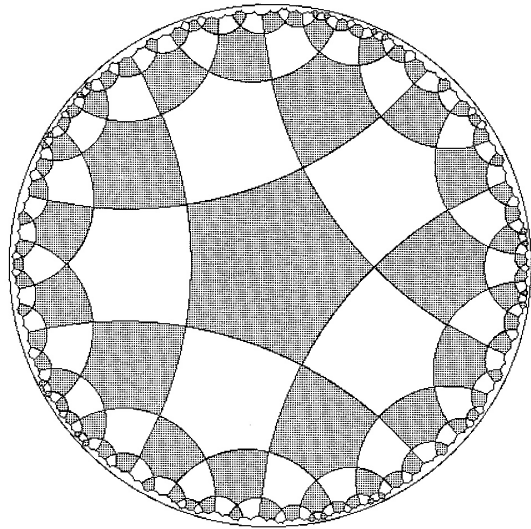


A Quick Introduction to Non-Euclidean Geometry



A Tiling of the Poincare Plane
From *Geometry: Plane and Fancy*, David
Singer, page 61.

Dr. Robert Gardner
Presented at Science Hill High School
March 22, 2006

Euclidean Geometry



Euclid (325 BCE – 265 BCE)

Note. (From *An Introduction to the History of Mathematics*, 5th Edition, Howard Eves, 1983.) Alexander the Great founded the city of Alexandria in the Nile River delta in 332 BCE. When Alexander died in 323 BCE, one of his military leaders, Ptolemy, took over the region of Egypt. Ptolemy made Alexandria the capitol of his territory and started the University of Alexandria in about 300 BCE. The university had lecture rooms, laboratories, museums, and a library with over 600,000 papyrus scrolls. Euclid, who may have come from Athens, was made head of the department of mathematics. Little else is known about Euclid.



The eastern Mediterranean from
“The World of the *Decameron*” website.

Note. Euclid’s *Elements* consists of 13 books which include 465 propositions. American high-school geometry texts contain much of the material from Books I, III, IV, VI, XI, and XII. No copies of the *Elements* survive from Euclid’s time. Modern editions are based on a version prepared by Theon of Alexandria, who lived about 700 years after Euclid. No work, except for the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influence on scientific thinking.

Note. The definitions given in Euclid's *Elements* are not at all modern. Some examples are:

- A *point* is that which has no part.
- A *line* is breadthless length.
- A *straight line* is a line which lies evenly with the points on itself.
- *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Note. The postulates of Euclidean geometry are (as stated in *The Elements* and a restatement in more familiar language):

1. *To draw a straight line from any point to any point.* There is one and only one straight line through any two distinct points.
2. *To produce a finite straight line continuously in a straight line.* A line segment can be extended beyond each endpoint.
3. *To describe a circle with any center and distance.* For any point and any positive number, there exists a circle with the point as center and the positive number as radius.
4. *That all right angles are equal to one another.*
5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Note. Euclid started with ideas of points and lines as we draw them on flat pieces of paper, and then tried to set up definitions that are consistent with the behavior of the paper models. This is not at all the modern way that mathematicians view things (at least philosophically). Many mathematicians start with definitions and axioms which have no meaning at all beyond the meaning given by the axioms and definitions (this is called the *formalist* approach to math).

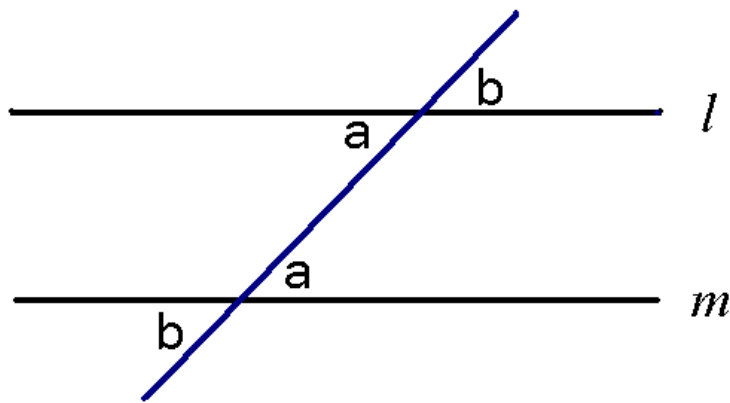


David Hilbert (1862-1943)

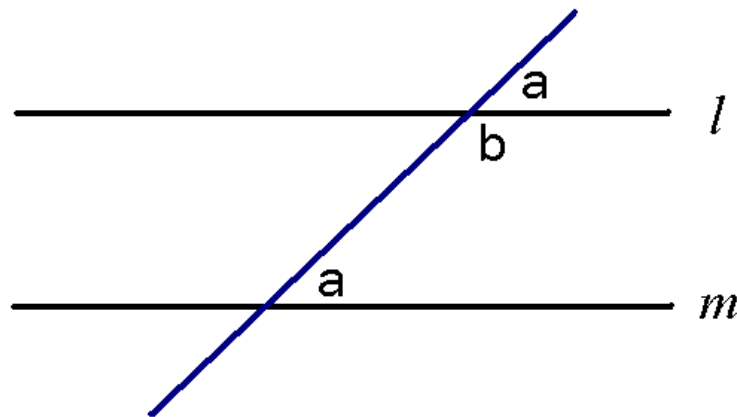
A famous quote by 20th century geometer David Hilbert is: “One must be able to say at all times – instead of points, straight lines, and planes – tables, chairs, and beer mugs.” Hilbert’s point here is that we should not put any meaning into the words used in mathematics beyond the meaning given by the definitions of mathematics. Put in contemporary terms, the drawings of points and lines on paper are not points and lines, but form a *model* for Euclidean geometry.

Note. Euclid seems to avoid the use of the parallel postulate and proves 28 propositions without using the parallel postulate. Two such results include:

Proposition 27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

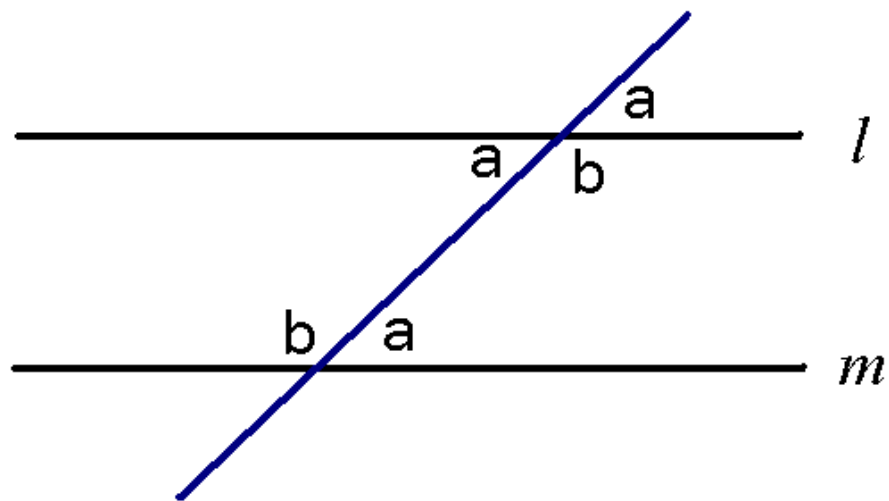


Proposition 28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.



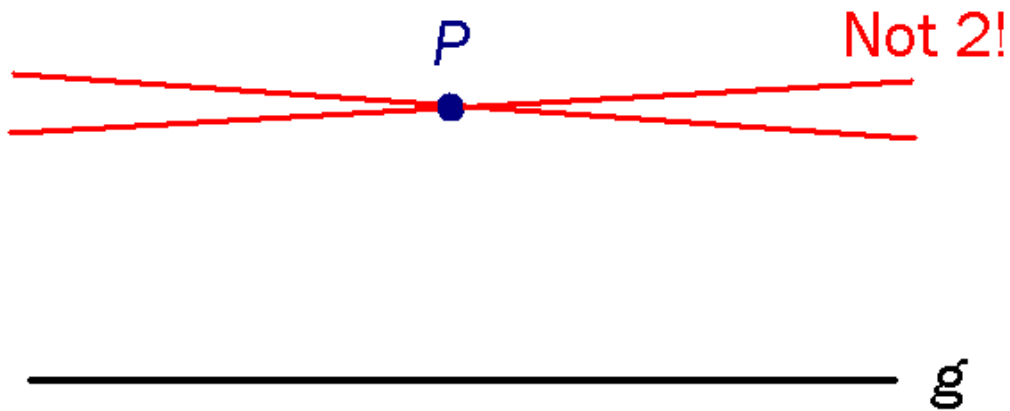
Note. The first result in the *Elements* which uses the parallel postulate in its proof is:

Proposition 29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

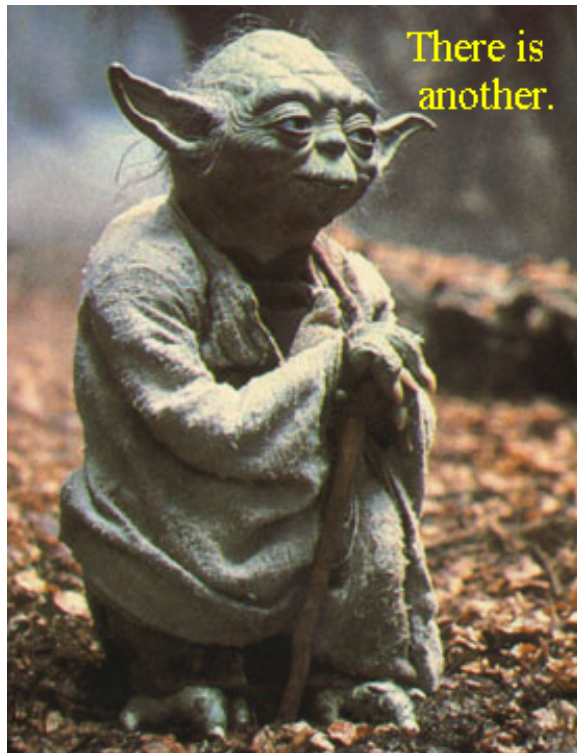


Note. Some results from Euclidean geometry which are equivalent to the parallel postulate are:

1. Two lines that are parallel to the same line are parallel to each other.
2. A line that meets one of two parallels also meets the other.
3. There exists a triangle whose angle-sum is two right angles.
4. Parallel lines are equidistant from one another.
5. Similar triangles exist which are not congruent.
6. Any two parallel lines have a common perpendicular.
7. **Playfair's Theorem.** For a given line g and a point P not on g , there exists a unique line through P parallel to g .



Note. None of the above results are surprising to those of us familiar with Euclidean geometry. The only thing that is maybe surprising is that we would go to such lengths to *prove* these obvious results! However, these results are only “obvious” in Euclidean geometry and they are familiar to us because of our *model* (points and lines on paper) of Euclidean geometry. However, there are models of geometry where each of the above results is not only not obvious, but not even true! We have only considered one model for geometry. But...



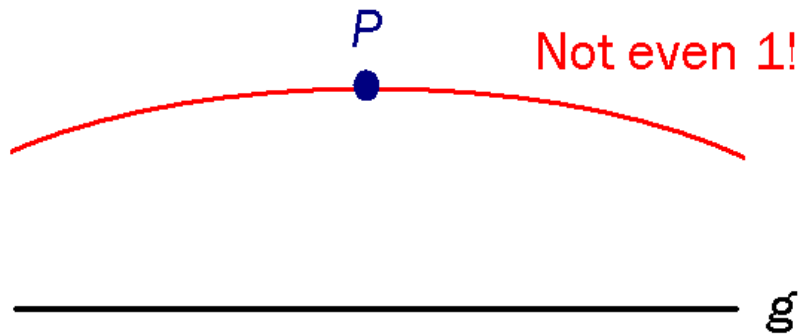
Non-Euclidean Geometry

Note. Playfair's Theorem is equivalent to the Parallel Postulate:

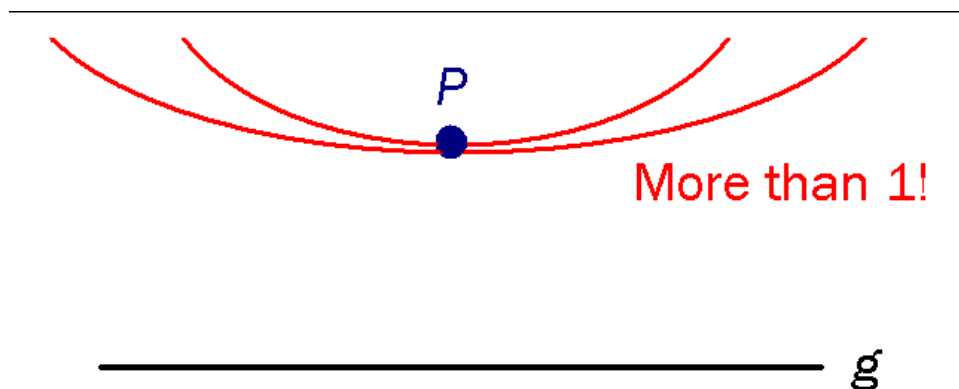
Playfair's Theorem. *For a given line g and a point P not on g , there exists a unique line through P parallel to g .*

If we negate it, we get a version of non-Euclidean geometry. There are two options:

Parallel Postulate for Spherical Geometry. *For a given line g and a point P not on a , there are no lines through P parallel to a .*

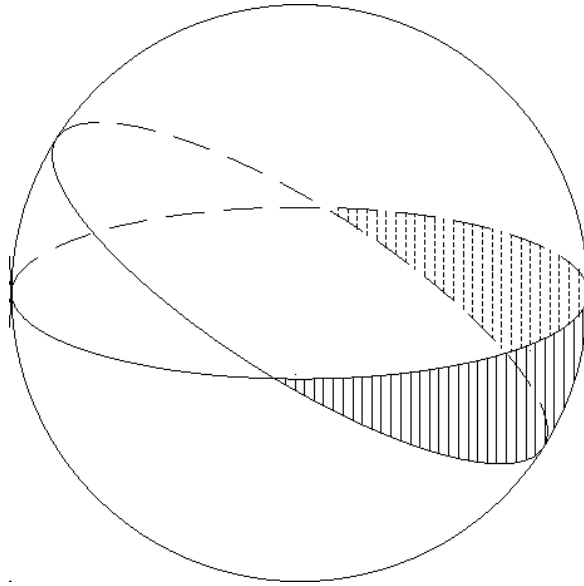


Parallel Postulate for Hyperbolic Geometry. *For a given line g and a point P not on g , there is more than one line through P parallel to g .*



We now consider models for each of these geometries. We can then use these models to illustrate some properties of non-Euclidean geometry.

Note. We model spherical geometry by using the surface of a sphere to represent a plane and using great circles of the sphere to represent lines (and points on the sphere to represent points).

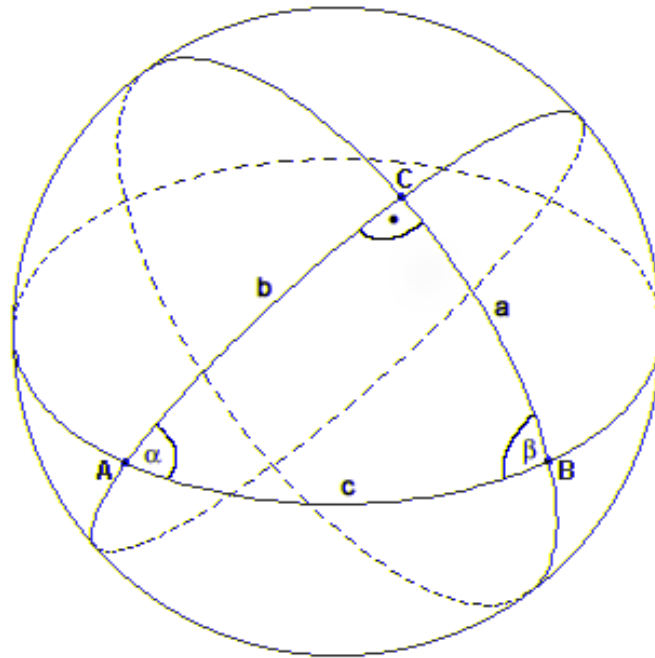


From David Royster's webpage on spherical geometry.

Note. Some properties of spherical geometry include:

- There are no parallel lines.

- The sums of the angles of a triangle are always greater than 180° . Small triangles have angle sums slightly greater than 180° and large triangles have angle sums much more than 180° (but always less than 900°).



A spherical triangle, from [Answers.com](https://www.answers.com).

- Triangles are only similar (i.e., the same shape) when they are congruent (i.e., the same size).

Note. This model of non-Euclidean geometry is easy to visualize and one wonders why it took so long to recognize this as a valid model geometry (in fact, this was not recognized until the 1850s with the work of Georg Bernhard Riemann). Historically, there are two problems. The first is that Euclid assumes (both explicitly and implicitly) that lines are infinite in extent and so since “lines” on the sphere are always finite in length (namely, $2\pi r$), then this spherical geometry does not count as a viable option. The second problem is philosophical — it was simply not believed that there were geometries other than the one described by Euclid. In fact, many of the properties of spherical geometry were studied in the second and first centuries BCE by Theodosius in *Sphaerica*. However, Theodosius’ study was entirely based on the sphere as an object embedded in Euclidean space, and never considered it in the non-Euclidean sense.

Note. Now here is a much less tangible model of a non-Euclidean geometry. Although hyperbolic geometry is about 200 years old (the work of Karl Frederich Gauss, Johann Bolyai, and Nicolai Lobachevsky), this model is only about 100 years old!



Johann Bolyai
1802–1860

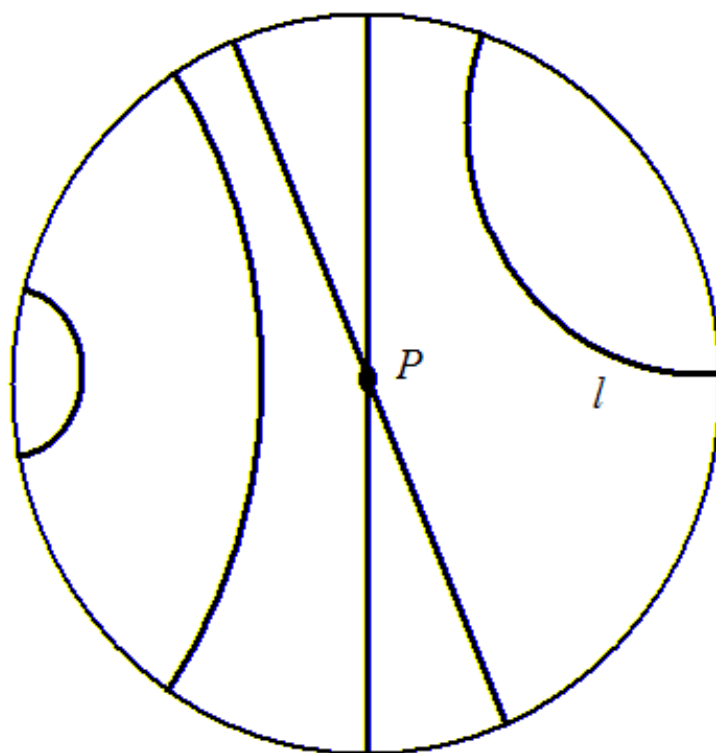


Karl Gauss
1777–1855



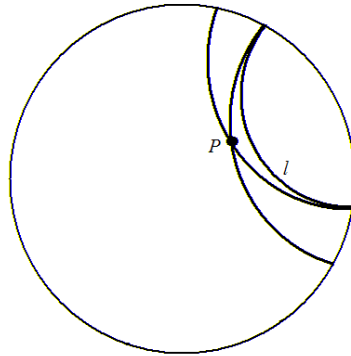
Nicolai Lobachevsky
1793–1856

Definition. The *Poincare disk* model of hyperbolic geometry represents the “plane” as an open unit disk, “points” of the plane are points of the disk, and “lines” are circular arcs which are perpendicular to the boundary of the disk.

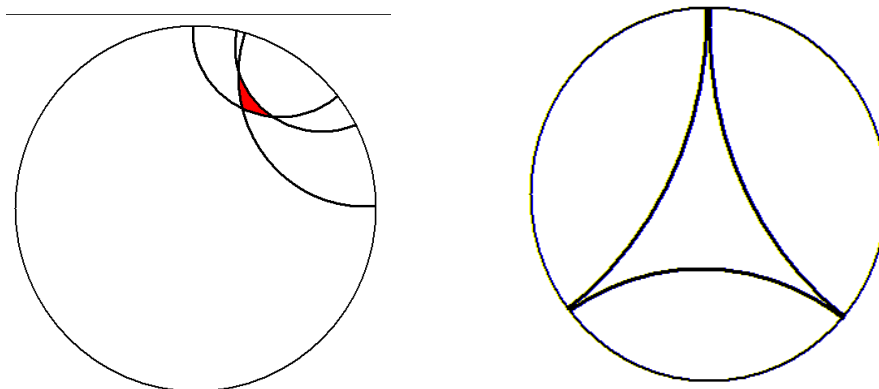


Note. Some properties of hyperbolic geometry are:

Through a point not on a given line, there is more than one line (in fact, an infinite number) through the point parallel to the given line.



The sums of the angles of a triangle is always less than 180° . Small triangles have angle sums slightly less than 180° and large triangles have angle sums much less than 180° (in fact, as close to 0° as we like).



Triangles are only similar (i.e., the same shape) when they are congruent (i.e., the same size).

Note. One apparent shortcoming of the Poincare model is that “lines” are not infinite in length. This is not the case since, in this model, distances are not measured in the usual (i.e., Euclidean) way. If you are familiar with calculus, then we can say the differential of arclength, ds , satisfies $ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$. Now if we consider the diameter D of the disk from $(-1, 0)$ to $(1, 0)$, we have the length

$$\int_D ds = \int_{-1}^1 \frac{dx}{1 - x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \Big|_{-1}^1 = \infty.$$

In general, “lines” in the Poincare disk are infinite (under this measure of arclength).

Conclusion. So we have three possible models for geometry. Some of their properties are:

	Angle Sums of Triangles	Similar Triangles?	Number of Parallel Lines Through P	Curvature
Euclidean	180°	YES	1	0
Spherical	$> 180^\circ$	only when congruent	0	+
Hyperbolic	$< 180^\circ$	only when congruent	∞	-