A Second Course in Elementary Differential Equations: Problems and Solutions

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Contents

28 Calculus of Matrix-Valued Functions of a Real Variable

Problem 28.1
Consider the following matrices

$$\mathbf{A}(t) = \begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}, \ \mathbf{B}(t) = \begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix}, \ \mathbf{c}(t) = \begin{bmatrix} t+1 \\ -1 \end{bmatrix}$$

(a) Find 2**A**(t) - 3t**B**(t)
(b) Find **A**(t)**B**(t) - **B**(t)**A**(t)
(c) Find **A**(t)**c**(t)
(d) Find det(**B**(t)**A**(t))

Solution.
(a)

$$\begin{aligned} 2\mathbf{A}(t) - 3t\mathbf{B}(t) &= 2\begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix} - 3t\begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix} \\ &= \begin{bmatrix} 2t-2 & 2t^2 \\ 4 & 4t+2 \end{bmatrix} - \begin{bmatrix} 3t^2 & -3t \\ 0 & 3t^2+6t \end{bmatrix} \\ &= \begin{bmatrix} 2t-2-3t^2 & 2t^2+3t \\ 4 & 2-2t-3t^2 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{A}(t)\mathbf{B}(t) - \mathbf{B}(t)\mathbf{A}(t) &= \begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix} - \begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix}\begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix} \\ &= \begin{bmatrix} t^2-t & t^3+2t^2-t+1 \\ 2t & 2t^2+5t \end{bmatrix} - \begin{bmatrix} t^2-t-2 & t^3-2t-1 \\ 2t+4 & 2t^2+5t+2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2t^2+t+2 \\ -4 & -2 \end{bmatrix} \end{aligned}$$

(c)

$$\mathbf{A}(t)\mathbf{c}(t) = \begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} t+1 \\ -1 \end{bmatrix} = \begin{bmatrix} (t-1)(t+1)+t^2(-1) \\ 2(t+1)+(2t+1)(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(d)

$$\begin{aligned} det(\mathbf{B}(t)\mathbf{A}(t)) &= \left|\left[\begin{array}{cc} t & -1 \\ 0 & t+2 \end{array}\right]\left[\begin{array}{cc} t-1 & t^2 \\ 2 & 2t+1 \end{array}\right]\right| = \left|\left[\begin{array}{cc} t^2-t-2 & t^3-2t-1 \\ 2t+4 & 2t^2+5t+2 \end{array}\right]\right| \\ &= -(t^3+3t^2+2t) \blacksquare \end{aligned}$$

Problem 28.2
Determine all values t such that $\mathbf{A}(t)$ is invertible and, for those t-values, find $\mathbf{A}^{-1}(t)$.

$$\mathbf{A}(t) = \left[\begin{array}{cc} t+1 & t \\ t & t+1 \end{array}\right]$$

Solution.
We have $det(\mathbf{A}(t)) = 2t+1$ so that $\mathbf{A}$ is invertible for all $t \neq -\frac{1}{2}$. In this case,

$$\mathbf{A}^{-1}(t) = \frac{1}{2t+1}\left[\begin{array}{cc} t+1 & -t \\ -t & t+1 \end{array}\right] \blacksquare$$

Problem 28.3
Determine all values t such that $\mathbf{A}(t)$ is invertible and, for those t-values, find $\mathbf{A}^{-1}(t)$.

$$\mathbf{A}(t) = \left[\begin{array}{cc} \sin t & -\cos t \\ \sin t & \cos t \end{array}\right]$$

Solution.
We have $det(\mathbf{A}(t)) = 2\sin t \cos 2 = \sin 2t$ so that $\mathbf{A}$ is invertible for all $t \neq \frac{n\pi}{2}$ where n is an integer. In this case,

$$\mathbf{A}^{-1}(t) = \frac{1}{\sin 2t}\left[\begin{array}{cc} \cos t & \cos t \\ -\sin t & \sin t \end{array}\right] \blacksquare$$

Problem 28.4
Find

$$\lim_{t\to 0}\left[\begin{array}{ccc} \frac{\sin t}{t} & t\cos t & \frac{3}{t+1} \\ e^{3t} & \sec t & \frac{2t}{t^2-1} \end{array}\right]$$

Solution.

$$\lim_{t\to 0}\left[\begin{array}{ccc} \frac{\sin t}{t} & t\cos t & \frac{3}{t+1} \\ e^{3t} & \sec t & \frac{2t}{t^2-1} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 3 \\ 1 & 1 & 0 \end{array}\right] \blacksquare$$

Problem 28.5
Find

$$\lim_{t\to 0}\begin{bmatrix} te^{-t} & \tan t \\ t^2-2 & e^{\sin t}\end{bmatrix}$$

Solution.

$$\lim_{t\to 0}\begin{bmatrix} te^{-t} & \tan t \\ t^2-2 & e^{\sin t}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 1\end{bmatrix} \blacksquare$$

Problem 28.6
Find $\mathbf{A}'(t)$ and $\mathbf{A}''(t)$ if

$$\mathbf{A}(t) = \begin{bmatrix} \sin t & 3t \\ t^2+2 & 5\end{bmatrix}$$

Solution.

$$\mathbf{A}'(t) = \begin{bmatrix} \cos t & 3 \\ 2t & 0\end{bmatrix}$$

$$\mathbf{A}''(t) = \begin{bmatrix} -\sin t & 0 \\ 2 & 0\end{bmatrix} \blacksquare$$

Problem 28.7
Express the system

$$\begin{array}{rcl} y_1' & = & t^2y_1 + 3y_2 + \sec t \\ y_2' & = & (\sin t)y_1 + ty_2 - 5 \end{array}$$

in the matrix form

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{g}(t)$$

Solution.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix},\ \mathbf{A}(t) = \begin{bmatrix} t^2 & 3 \\ \sin t & t\end{bmatrix},\ \mathbf{g}(t) = \begin{bmatrix} \sec t \\ -5\end{bmatrix} \blacksquare$$

Problem 28.8
Determine $\mathbf{A}$(t) where

$$\mathbf{A}'(t) = \begin{bmatrix} 2t & 1 \\ \cos t & 3t^2\end{bmatrix},\quad \mathbf{A}(0) = \begin{bmatrix} 2 & 5 \\ 1 & -2\end{bmatrix}$$

Solution.
Integrating componentwise we find

$$\mathbf{A}(t) = \begin{bmatrix} t^2 + c_{11} & t + c_{12} \\ \sin t + c_{21} & t^3 + c_{22} \end{bmatrix}$$

Since

$$\mathbf{A}(0) = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

by equating componentwise we find $c_{11} = 2,\ c_{12} = 5,\ c_{21} = 1$, and $c_{22} = -2$. Hence,

$$\mathbf{A}(t) = \begin{bmatrix} t^2 + 2 & t + 5 \\ \sin t + 1 & t^3 + -2 \end{bmatrix} \blacksquare$$

Problem 28.9
Determine $\mathbf{A}$(t) where

$$\mathbf{A}''(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}(0) = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}(1) = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

Solution.
Integrating componentwise we find

$$\mathbf{A}'(t) = \begin{bmatrix} t + c_{11} & \frac{t^2}{2} + c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Integrating again we find

$$\mathbf{A}(t) = \begin{bmatrix} \frac{t^2}{2} + c_{11}t + d_{11} & \frac{t^3}{6} + c_{12}t + d_{12} \\ c_{21}t + d_{21} & c_{22}t + d_{22} \end{bmatrix}$$

But

$$\mathbf{A}(0) = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

so by equating componentwise we find $d_{11} = 1,\ d_{12} = 1,\ d_{21} = -2$, and $d_{22} = 1$. Thus,

$$\mathbf{A}(t) = \begin{bmatrix} \frac{t^2}{2} + c_{11}t + 1 & \frac{t^3}{6} + c_{12}t + 1 \\ c_{21}t + -2 & c_{22}t + 1 \end{bmatrix}$$

Since

$$\mathbf{A}(1) = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + c_{11} & \frac{7}{6} + c_{12} \\ -2 + c_{21} & 1 + c_{22} \end{bmatrix}$$

we find $c_{11} = -\frac{5}{2},\ c_{12} = \frac{5}{6},\ c_{21} = 0,\ c_{22} = 2$. Hence,

$$\mathbf{A}(t) = \begin{bmatrix} \frac{t^2}{2} - \frac{5}{2}t + 1 & \frac{t^3}{6} + \frac{5}{6}t + 1 \\ -2 & 2t + 1 \end{bmatrix} \blacksquare$$

Problem 28.10
Calculate $\mathbf{A}(t) = \int_0^t \mathbf{B}(s)ds$ where

$$\mathbf{B}(s) = \begin{bmatrix} e^s & 6s \\ \cos 2\pi s & \sin 2\pi s \end{bmatrix}$$

Solution.
Integrating componentwise we find

$$\mathbf{A}(t) = \begin{bmatrix} \int_0^t e^s ds & \int_0^t 6s ds \\ \int_0^t \cos 2\pi s ds & \int_0^t \sin 2\pi s ds \end{bmatrix} = \begin{bmatrix} e^t - 1 & 3t^2 \\ \frac{\sin 2\pi t}{2\pi} & \frac{1 - \cos 2\pi t}{2\pi} \end{bmatrix} \blacksquare$$

Problem 28.11
Construct a 2×2 nonconstant matrix function $\mathbf{A}$(t) such that $\mathbf{A}^2(t)$ is a constant matrix.

Solution.
Let

$$\mathbf{A}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

Then

$$\mathbf{A}^2(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \blacksquare$$

Problem 28.12
(a) Construct a 2×2 differentiable matrix function $\mathbf{A}$(t) such that

$$\frac{d}{dt}\mathbf{A}^2(t) \neq 2\mathbf{A}\frac{d}{dt}\mathbf{A}(t)$$

That is, the power rule is not true for matrix functions.
(b) What is the correct formula relating $\mathbf{A}^2(t)$ to $\mathbf{A}$(t) and $\mathbf{A}$'(t)?

Solution.
(a) Let

$$\mathbf{A}(t) = \begin{bmatrix} 1 & t \\ t^2 & 0 \end{bmatrix}$$

Then

$$\mathbf{A}^2(t) = \begin{bmatrix} 1+t^3 & t \\ t^2 & t^3 \end{bmatrix}$$

so that

$$\frac{d}{dt}\mathbf{A}^2(t) = \begin{bmatrix} 3t^2 & 1 \\ 2t & 3t^2 \end{bmatrix}$$

On the other hand,

$$2\mathbf{A}(t)\frac{d}{dt}\mathbf{A}(t) = 2\begin{bmatrix} 1 & t \\ t^2 & 0 \end{bmatrix}\begin{bmatrix} 2+4t^3 & 2t+2t^4 \\ 2t^2+2t^5 & 2t^3 \end{bmatrix} = \begin{bmatrix} 1 & t \\ t^2 & 0 \end{bmatrix} \blacksquare$$

(b) (b) The correct formula is $\frac{d}{dt}\mathbf{A}^2(t) = \mathbf{A}(t)\mathbf{A}'(t) + \mathbf{A}'(t)\mathbf{A}(t)$ ■

Problem 28.13
Transform the following third-order equation

$$y''' - 3ty' + (\sin 2t)y = 7e^{-t}$$

into a first order system of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

Solution.
Let $x_1 = y,\ x_2 = y', x_3 = y''$. Then by letting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin 2t & 3t & 0 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 7e^{-t} \end{bmatrix}$$

then the differential equation can be presented by the first order system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \blacksquare$$

Problem 28.14
By introducing new variables x_1 and x_2, write $y'' - 2y + 1 = t$ as a system of two first order linear equations of the form $\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{b}$

Solution.
By letting $x_1 = y$ and $x_2 = y'$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ t-1 \end{bmatrix} \blacksquare$$

Problem 28.15
Write the differential equation $y'' + 4y' + 4y = 0$ as a first order system.

Solution.
By letting $x_1 = y$ and $x_2 = y'$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \blacksquare$$

Problem 28.16
Write the differential equation $y'' + ky' + (t-1)y = 0$ as a first order system.

Solution.
By letting $x_1 = y$ and $x_2 = y'$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ t-1 & k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \blacksquare$$

Problem 28.17
Change the following second-order equations to a first-order system.

$$y'' - 5y' + ty = 3t^2, \quad y(0) = 0, \; y'(0) = 1$$

Solution.
If we write the problem in the matrix form

$$\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{y}_0$$

then

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -5 & t \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 0 \\ 3t^2 \end{bmatrix}, \; \mathbf{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \blacksquare$$

Problem 28.18
Consider the following system of first-order linear equations.

$$\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \cdot \mathbf{x}$$

Find the second-order linear differential equation that $\mathbf{x}$ satisfies.

Solution.
The system is

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

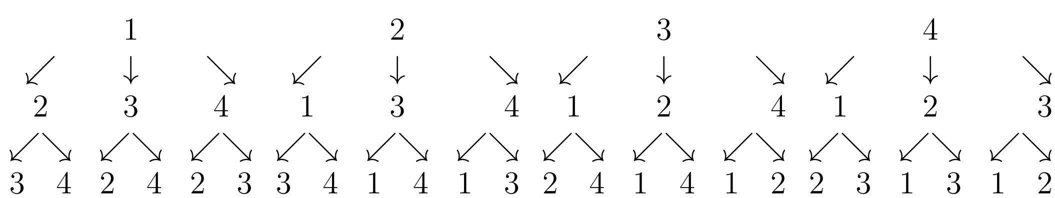
It follows that

$$x_1' + 2x_2' = 5x_1 \text{ or } x_1 = \frac{x_1' + 2x_2'}{5}$$

so we let $w = \frac{x_1+2x_2}{5}$ so that $w' = x_1$. Thus, $x_1' = 3x_1+2x_2 = 3x_1+(5w-x_1) = 2x_1 + 5w$. Hence, $x_1'' = 2x_1' + 5w' = 2x_1' + 5x_1$ or $x_1'' - 2x_1' - 5x_1 = 0$ ■

Problem 28.19
List all the permutations of $S = \{1, 2, 3, 4\}$.

Solution.



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Problem 28.20
List all elementary products from the matrices
(a)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(b)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Solution.
(a) The only elementary products are $a_{11}a_{22}, a_{12}a_{21}$.
(b) An elementary product has the form $a_{1*}a_{2*}a_{3*}$. Since no two factors come from the same column, the column numbers have no repetitions; consequently they must form a permutation of the set $\{1, 2, 3\}$. The $3! = 6$ permutations yield the following elementary products:
$a_{11}a_{22}a_{33}, a_{11}a_{23}a_{32}, a_{12}a_{23}a_{31}, a_{12}a_{21}a_{33}, a_{13}a_{21}a_{32}, a_{13}a_{22}a_{31}$ ■

Problem 28.21
Find $det(A)$ if
(a)

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(b)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Solution.
By using the definition of a determinant and Exercise **??** we obtain
(a) $|A| = a_{11}a_{22} - a_{21}a_{12}$.
(b) $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$
■

29 nth Order Linear Differential Equations: Exsitence and Uniqueness

For Problems 29.1 - 29.3, use Theorem 29.1 to find the largest interval $a < t < b$ in which a unique solution is guaranteed to exist.

Problem 29.1

$$y''' - \frac{1}{t^2 - 9}y'' + \ln(t+1) + (\cos t)y = 0, \;\; y(0) = 1, \;\; y'(0) = 3, \;\; y''(0) = 0$$

Solution.
The coefficient functions are all continuous for $t \neq -3, -1, 3$. Since $t_0 = 0$, the largest interval of existence is $-1 < t < 3$ ■

Problem 29.2

$$y''' + \frac{1}{t+1}y' + (\tan t)y = 0, \;\; y(0) = 0, \;\; y'(0) = 1, \;\; y''(0) = 2$$

Solution.
The coefficient functions are all continuous for $t \neq -1$ and $t \neq (2n+1)\frac{\pi}{2}$ where n is an integer. Since $t_0 = 0$, the largest interval of existence is $-1 < t < \frac{\pi}{2}$ ■

Problem 29.3

$$y'' - \frac{1}{t^2 + 9}y'' + \ln(t^2+1)y' + (\cos t)y = 0, \;\; y(0) = 1, \;\; y'(0) = 3, \;\; y''(0) = 0$$

Solution.
The coefficient functions are all continuous for t so that the interval of existence is $-\infty < t < \infty$ ■

Problem 29.4
Determine the value(s) of r so that $y(t) = e^{rt}$ is a solution to the differential equation

$$y''' - 2y'' - y' + 2y = 0$$

Solution.
Inserting y and its derivatives into the equation we find

$$\begin{aligned} 0 &= y''' - 2y'' - y' + 2y \\ &= (r^3 - 2r^2 - r + 2)e^{rt} \end{aligned}$$

Since $e^{rt} > 0$, we must have $0 = r^3 - 2r^2 - r + 2 = r^2(r-2) - (r-2) = (r-2)(r^2-1) = (r-2)(r-1)(r+1)$. Hence, $r = -1, 1, 2$ ■

Problem 29.5
Transform the following third-order equation

$$y''' - 3ty' + (\sin 2t)y = 7e^{-t}$$

into a first order system of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

Solution.
Let $x_1 = y,\ x_2 = y', x_3 = y''$. Then by letting

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin 2t & 3t & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 7e^{-t} \end{bmatrix}$$

then the differential equation can be presented by the first order system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \ ■$$

30 The General Solution of nth Order Linear Homogeneous Equations

In Problems 30.1 - 30.3, show that the given solutions form a fundamental set for the differential equation by computing the Wronskian.

Problem 30.1

$$y''' - y' = 0,\ y_1(t) = 1,\ y_2(t) = e^t,\ y_3(t) = e^{-t}$$

Solution.
We have

$$\begin{aligned} W(t) &= \left|\begin{array}{ccc} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{array}\right| \\ &= (1)\left|\begin{array}{cc} e^t & -e^{-t} \\ e^t & e^{-t} \end{array}\right| - e^t\left|\begin{array}{cc} 0 & -e^{-t} \\ 0 & e^{-t} \end{array}\right| + e^{-t}\left|\begin{array}{cc} 0 & e^t \\ 0 & e^t \end{array}\right| = 2 \neq 0 \blacksquare \end{aligned}$$

Problem 30.2

$$y^{(4)} + y'' = 0,\ y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = \cos t,\ y_4(t) = \sin t$$

Solution.
We have

$$W(t) = \left|\begin{array}{cccc} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{array}\right| = \cos^2 t + \sin^2 t = 1 \neq 0 \blacksquare$$

Problem 30.3

$$t^2 y''' + ty'' - y' = 0,\ y_1(t) = 1,\ y_2(t) = \ln t,\ y_3(t) = t^2$$

Solution.
We have

$$\begin{aligned} W(t) &= \begin{vmatrix} 1 & \ln t & t^2 \\ 0 & \frac{1}{t} & 2t \\ 0 & -\frac{1}{t^2} & 2 \end{vmatrix} \\ &= (1)\begin{vmatrix} t^{-1} & 2t \\ -t^{-2} & 2 \end{vmatrix} - \ln t \begin{vmatrix} 0 & 2t \\ 0 & 2 \end{vmatrix} + t^2 \begin{vmatrix} 0 & t^{-1} \\ 0 & -t^{-2} \end{vmatrix} = 3t^{-1} \neq 0,\ t > 0 \ \blacksquare \end{aligned}$$

Use the fact that the solutions given in Problems 30.1 - 30.3 for a fundamental set of solutions to solve the following initial value problems.

Problem 30.4

$$y''' - y' = 0,\ y(0) = 3,\ y'(0) = -3,\ y''(0) = 1$$

Solution.
The general solution is $y(t) = c_1 + c_2 e^t + c_3 e^{-t}$. With the initial conditions we have

$$\begin{array}{rcrcr} y(0) = 3 & \Longrightarrow & c_1 + c_2 + c_3 & = & 3 \\ y'(0) = -3 & \Longrightarrow & c_2 - c_3 & = & -3 \\ y''(0) = 1 & \Longrightarrow & c_2 + c_3 & = & 1 \end{array}$$

Solving these simultaneous equations gives $c_1 = 2,\ c_2 = -1$ and $c_3 = 2$ and so the unique solution is

$$y(t) = 2 - e^t + 2e^{-t} \ \blacksquare$$

Problem 30.5

$$y^{(4)} + y'' = 0,\ y(\tfrac{\pi}{2}) = 2 + \pi,\ y'(\tfrac{\pi}{2}) = 3,\ y''(\tfrac{\pi}{2}) = -3,\ y'''(\tfrac{\pi}{2}) = 1.$$

Solution.
The general solution is $y(t) = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$. With the initial conditions we have

$$\begin{array}{rcrcr} y(\frac{\pi}{2}) = 2 + \pi & \Longrightarrow & c_1 + \frac{\pi}{2}c_2 + c_4 & = & 2 + \pi \\ y'(\frac{\pi}{2}) = 3 & \Longrightarrow & c_2 - c_3 & = & 3 \\ y''(\frac{\pi}{2}) = -3 & \Longrightarrow & -c_4 & = & -3 \\ y'''(\frac{\pi}{2}) = 1 & \Longrightarrow & c_3 & = & 1 \end{array}$$

Solving these simultaneous equations gives $c_1 = -(\pi + 1),\ c_2 = 4,\ c_3 = 1$ and $c_4 = 3$. Hence, the unique solution is

$$y(t) = -(\pi + 1) + 4t + \cos t + 3\sin t \blacksquare$$

Problem 30.6

$$t^2y''' + ty'' - y' = 0, ,\ y(1) = 1,\ y'(1) = 2,\ y''(1) = -6$$

Solution.
The general solution is $y(t) = c_1 + c_2 \ln t + c_3t^2$. With the initial conditions we have

$$\begin{array}{rcrcr} y(1) = 1 & \Longrightarrow & c_1 + c_3 & = & 1 \\ y'(1) = 2 & \Longrightarrow & c_2 + 2c_3 & = & 2 \\ y''(1) = -6 & \Longrightarrow & -c_2 + 2c_3 & = & -6 \end{array}$$

Solving these simultaneous equations gives $c_1 = 2,\ c_2 = 4$ and $c_3 = -1$ and so the unique solution is

$$y(t) = 2 + 4\ln t - t^2 \blacksquare$$

Problem 30.7
In each question below, show that the Wronskian determinant $W(t)$ behaves as predicted by Abel's Theorem. That is, for the given value of t_0, show that

$$W(t) = W(t_0)e^{-\int_{t_0}^t p_{n-1}(s)ds}$$

(a) $W(t)$ found in Problem 30.1 and $t_0 = -1$.
(b) $W(t)$ found in Problem 30.2 and $t_0 = 1$.
(c) $W(t)$ found in Problem 30.3 and $t_0 = 2$.

Solution.
(a) For the given differential equation $p_{n-1}(t) = p_2(t) = 0$ so that Abel's theorem predict $W(t) = W(t_0)$. Now, for $t_0 = -1$ we have $W(t) = W(-1) =$constant. From Problem 28.1, we found that $W(t) = 2$.
(b) For the given differential equation $p_{n-1}(t) = p_3(t) = 0$ so that Abel's theorem predict $W(t) = W(t_0)$. Now, for $t_0 = 1$ we have $W(t) = W(1) =$constant.

From Problem 28.2, we found that $W(t) = 1$.
(c) For the given differential equation $p_{n-1}(t) = p_2(t) = \frac{1}{t}$ so that Abel's theorem predict $W(t) = W(2)e^{-\int_2^t \frac{ds}{s}} = W(2)e^{\ln(\frac{2}{t})} = \frac{2}{t}W(2)$. From Problem 28.3, we found that $W(t) = \frac{3}{t}$ so that $W(2) = \frac{3}{2}$ ■

Problem 30.8
Determine $W(t)$ for the differential equation $y''' + (\sin t)y'' + (\cos t)y' + 2y = 0$ such that $W(1) = 0$.

Solution.
Here $p_{n-1}(t) = p_2(t) = \sin t$. By Abel's Theorem we have

$$W(t) = W(1)e^{-\int_1^t \sin s ds} \equiv 0 \ ■$$

Problem 30.9
Determine $W(t)$ for the differential equation $t^3y''' - 2y = 0$ such that $W(1) = 3$.

Solution.
Here $p_{n-1}(t) = p_2(t) = 0$. By Abel's Theorem we have

$$W(t) = W(1)e^{-\int_1^t 0 ds} = W(1) = 3 \ ■$$

Problem 30.10
Consider the initial value problem

$$y''' - y' = 0, \ y(0) = \alpha, \ y'(0) = \beta, \ y''(0) = 4.$$

The general solution of the differential equation is $y(t) = c_1 + c_2e^t + c_3e^{-t}$.
(a) For what values of α and β will $\lim_{t\to\infty} y(t) = 0$?
(b) For what values α and β will the solution $y(t)$ be bounded for $t \geq 0$, i.e., $|y(t)| \leq M$ for all $t \geq 0$ and for some $M > 0$? Will any values of α and β produce a solution $y(t)$ that is bounded for all real number t?

Solution.
(a) Since $y(0) = \alpha$, $c_1 + c_2 + c_3 = \alpha$. Since $y'(t) = c_2e^t - c_3e^{-t}$ and $y'(0) = \beta$, $c_2 - c_3 = \beta$. Also, since $y''(t) = c_2e^t + c_3e^{-t}$ and $y''(0) = 4$ we have $c_2 + c_3 = 4$. Solving these equations for c_1, c_2, and c_3 we find $c_1 = \alpha - 4, \ c_2 = \beta/2 + 2$ and $c_3 = -\beta/2 + 2$. Thus,

$$y(t) = \alpha - 4 + (\beta/2 + 2)e^t + (-\beta/2 + 2)e^{-t}.$$

If $\alpha = 4$ and $\beta = -4$ then

$$y(t) = 4e^{-t}$$

and

$$\lim_{t\to\infty} 4e^{-t} = 0.$$

(b) In the expression of $y(t)$ w know that e^{-t} is bounded for $t \ge 0$ whereas e^t is unbounded for $t \ge 0$. Thus, for $y(t)$ to be bounded we must choose $\beta/2 + 2 = 0$ or $\beta = -4$. The number α can be any number. Now, for the solution $y(t)$ to be bounded on $-\infty < t < \infty$ we must have simultaneously $\beta/2 + 2 = 0$ and $-\beta/2 + 2 = 0$. But there is no β that satisfies these two equations at the same time. Hence, $y(t)$ is always unbounded for any choice of α and β ■

Problem 30.11

Consider the differential equation $y''' + p_2(t)y'' + p_1(t)y' = 0$ on the interval $-1 < t < 1$. Suppose it is known that the coefficient functions $p_2(t)$ and $p_1(t)$ are both continuous on $-1 < t < 1$. Is it possible that $y(t) = c_1 + c_2t^2 + c_3t^4$ is the general solution for some functions $p_1(t)$ and $p_2(t)$ continuous on $-1 < t < 1$?

(a) Answer this question by considering only the Wronskian of the functions $1, t^2, t^4$ on the given interval.

(b) Explicitly determine functions $p_1(t)$ and $p_2(t)$ such that $y(t) = c_1 + c_2t^2 + c_3t^4$ is the general solution of the differential equation. Use this information, in turn, to provide an alternative answer to the question.

Solution.

(a) The Wronskian of $1, t^2, t^4$ is

$$W(t) = \begin{vmatrix} 1 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{vmatrix} = 16t^3$$

Since 0 is in the interval $-1 < t < 1$ and $W(0) = 0$, $\{1, t^2, t^4\}$ cannot be a fundamental set and therefore the general solution cannot be a linear combination of $1, t^2, t^4$.

(b) First notice that $y = 1$ is a solution for any p_1 and p_2. If $y = t^2$ is a solution then substitution into the differential equation leads to $tp_1 + p_2 = 0$. Since $y = t^4$ is also a solution, substituting into the equation we find $t^2p_1 + 3tp_2 + 6 = 0$. Solving for p_1 and p_2 we find $p_1(t) = \frac{3}{t^2}$ and $p_2(t) = -\frac{3}{t}$. Note that both functions are not continuous at $t = 0$ ■

Problem 30.12
(a) Find the general solution to $y''' = 0$.
(b) Using the general solution in part (a), construct a fundamental set $\{y_1(t), y_2(t), y_3(t)\}$ satisfying the following conditions

$$\begin{array}{lll} y_1(1) = 1, & y_1'(1) = 0, & y_1''(1) = 0. \\ y_2(1) = 0, & y_1'(1) = 1, & y_1''(1) = 0. \\ y_1(1) = 0, & y_1'(1) = 0, & y_1''(1) = 1. \end{array}$$

Solution.
(a) Using antidifferentiation we find that $y(t) = c_1 + c_2 t + c_3 t^2$.
(b) With the initial conditions of y_1 we obtain the following system

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 1 \\ & & c_2 & + & 2c_3 & = & 0 \\ & & & & 2c_3 & = & 0 \end{array}$$

Solving this system we find $c_1 = 1,\ c_2 = 0,\ c_3 = 0$. Thus, $y_1(t) = 1$. Repeating this argument for y_2 we find the system

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 0 \\ & & c_2 & + & 2c_3 & = & 1 \\ & & & & 2c_3 & = & 0 \end{array}$$

Solving this system we find $c_1 = -1,\ c_2 = 1,\ c_3 = 0$. Thus, $y_2(t) = t - 1$. Repeating this argument for y_3 we find the system

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 0 \\ & & c_2 & + & 2c_3 & = & 0 \\ & & & & 2c_3 & = & 1 \end{array}$$

Solving this system we find $c_1 = -\frac{1}{2},\ c_2 = -1,\ c_3 = \frac{1}{2}$. Thus, $y_3(t) = \frac{1}{2}(t-1)^2$ ■

31 Fundamental Sets and Linear Independence

Problem 31.1
Determine if the following functions are linearly independent

$$y_1(t) = e^{2t}, \quad y_2(t) = \sin(3t), \quad y_3(t) = \cos t$$

Solution.
First take derivatives

$$\begin{array}{lll} y_1'(t) = 2e^{2t} & y_2'(t) = 3\cos(3t) & y_3'(t) = -\sin t \\ y_1''(t) = 4e^{2t} & y_2''(t) = 9\sin(3t) & y_3''(t) = -\cos t \end{array}$$

The Wronskian is

$$\begin{aligned} W(t) &= \begin{vmatrix} e^{2t} & \sin 3t & \cos t \\ 2e^{2t} & 3\cos(3t) & -\sin t \\ 4e^{2t} & 9\sin(3t) & -\cos t \end{vmatrix} \\ &= e^{2t}(-3\cos 3t\cos t - 9\sin 3t\sin t) - \sin 3t(-2e^t\cos t + 4e^{2t}\sin t) \\ &+ \cos t(-18e^{2t}\sin 3t - 12e^{2t}\cos 3t) \end{aligned}$$

Thus,$W(0) = -15$. Since this is a nonzero number, we can conclude that the three functions are linearly independent ■

Problem 31.2
Determine whether the three functions : $f(t) = 2, g(t) = \sin^2 t, h(t) = \cos^2 t$, are linearly dependent or independent on $-\infty < t < \infty$

Solution.
Computing the Wronskian

$$\begin{aligned} W(t) &= \begin{vmatrix} 2 & \sin^2 t & \cos^2 t \\ 0 & \sin 2t & -\sin 2t \\ 0 & 2\cos 2t & -2\cos 2t \end{vmatrix} \\ &= 2[\sin(2t)(-2\cos 2t) - (-\sin 2t(2\cos 2t)) = 0 \end{aligned}$$

So the functions are linearly depedent ■

Problem 31.3
Determine whether the functions, $y_1(t) = 1; y_2(t) = 1 + t; y_3(t) = 1 + t + t^2$; are linearly dependent or independent. Show your work.

Solution.

$$\begin{aligned} 0 &= ay_1 + by_2 + cy_3 \\ &= a(1) + b(1+t) + c(1+t+t^2) \\ &= (a+b+c) + (b+c)t + ct^2 \end{aligned}$$

Equating coefficients we find

$$\begin{aligned} a+b+c &= 0 \\ b+c &= 0 \\ c &= 0 \end{aligned}$$

Solving this system we find that $a = b = c = 0$ so that $y_1, y_2,$ and y_3 are linearly independent ■

Problem 31.4
Consider the set of functions $\{y_1(t), y_2(t), y_3(t)\} = \{t^2+2t, \alpha t+1, t+\alpha\}$. For what value(s) α is the given set linearly depedent on the interval $-\infty < t < \infty$?

Solution.

$$\begin{aligned} 0 &= ay_1 + by_2 + cy_3 \\ &= a(t^2+2t) + b(\alpha t+1) + c(t+\alpha) \\ &= b + \alpha c + (2a + \alpha b + c)t + at^2 \end{aligned}$$

Equating coefficients we find

$$\begin{aligned} b + \alpha c &= 0 \\ 2a + \alpha b + c &= 0 \\ a &= 0 \end{aligned}$$

Solving this system we find that $a = b = c = 0$ provided that $\alpha \neq \pm 1$. In this case, $y_1, y_2,$ and y_3 are linearly independent ■

Problem 31.5
Determine whether the set $\{y_1(t), y_2(t), y_3(t)\} = \{t|t|+1, t^2-1, t\}$ is linearly independent or linearly dependent on the given interval
(a) $0 \le t < \infty$.
(b) $-\infty < t \le 0$.
(c) $-\infty < t < \infty$.

Solution.
(a) If $t \geq 0$ then $y_1(t) = t|t| + 1 = t^2 + 1$. In this case, we have

$$\begin{array}{rcl} 0 & = & ay_1 + by_2 + cy_3 \\ & = & a(t^2+1) + b(t^2-1) + c(t) \\ & = & (a+b)t^2 + ct + a - b \end{array}$$

Equating coefficients we find

$$\begin{array}{rcl} a+b & = & 0 \\ c & = & 0 \\ a-b & = & 0 \end{array}$$

Solving this system we find that $a = b = c = 0$. This shows that y_1, y_2, and y_3 are linearly independent.
(b) If $t \leq 0$ then $y_1(t) = -t^2 + 1 = -(t^2 - 1) = -y_2(t) + 0y_3(t)$. Thus, y_1, y_2, and y_3 are linearly dependent.
(c) Since y_1, y_2, and y_3 are linearly independent on the interval $0 \leq t < \infty$, they are linearly independent on the entire interval $-\infty < t < \infty$ ■

In Problems 31.6 - 31.7, for each differential equation, the corresponding set of functions $\{y_1(t), y_2(t), y_3(t)\}$ is a fundamental set of solutions.
(a) Determine whether the given set $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set to the differential equation.
(b) If $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set then find the coefficient matrix $\mathbf{A}$ such that

$$\left[\begin{array}{c} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \end{array}\right] = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]$$

(c) If $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set, determine whether it is a fundamental set by calculating the determinant of $\mathbf{A}$.

Problem 31.6

$$y''' + y'' = 0$$
$$\{y_1(t), y_2(t), y_3(t)\} = \{1, t, e^{-t}\}$$
$$\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{1 - 2t, t + 2, e^{-(t+2)}\}$$

Solution.
(a) Since $\overline{y}_1(t) = 1-2t$, $\overline{y}_1'(t) = -2$ and $\overline{y}_1''(t) = \overline{y}_1'''(t) = 0$. Thus, $\overline{y}_1''' + \overline{y}_1'' = 0$. Similarly, since $\overline{y}_2(t) = t+2$, $\overline{y}_2'(t) = 1$ and $\overline{y}_2''(t) = \overline{y}_2'''(t) = 0$. Thus, $\overline{y}_2''' + \overline{y}_2'' = 0$. Finally, since $\overline{y}_3(t) = e^{-(t+2)}$, $\overline{y}_3'(t) = -e^{-(t+2)}, \overline{y}_3''(t) = e^{-(t+2)}$ and $\overline{y}_3'''(t) = -e^{-(t+2)}$. Thus, $\overline{y}_3''' + \overline{y}_3'' = 0$. It follows, that $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{1-2t, t+2, e^{-(t+2)}\}$ is a solution set.
(b) Since $\overline{y}_1 = 1y_1 - 2y_2 + 0y_3$, $\overline{y}_2 = 2y_1 + 1y_2 + 0y_3$, and $\overline{y}_3 = 0y_1 + 0y_2 + e^{-2}y_3$ we have

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & e^{-2} \end{bmatrix}$$

(c) Since $det(\mathbf{A}) = 5e^{-2} \neq 0$, $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{1-2t, t+2, e^{-(t+2)}\}$ is a fundamental set of solutions ■

Problem 31.7

$$t^2y''' + ty'' - y' = 0, t > 0$$
$$\{y_1(t), y_2(t), y_3(t)\} = \{t, \ln t, t^2\}$$
$$\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{2t^2 - 1, 3, \ln(t^3)\}$$

Solution.
(a) Since $\overline{y}_1(t) = 2t^2 - 1$, $\overline{y}_1'(t) = 4t$, $\overline{y}_1''(t) = 4$, and $\overline{y}_1'''(t) = 0$. Thus, $t^2\overline{y}_1''' + t\overline{y}_1'' - \overline{y}_1' = 0$. Similarly, since $\overline{y}_2(t) = 3$, $\overline{y}_2'(t) = \overline{y}_2''(t) = \overline{y}_2'''(t) = 0$. Thus, $t^2\overline{y}_2''' + t\overline{y}_2'' - \overline{y}_2' = 0$. Finally, since $\overline{y}_3(t) = \ln t^3$, $\overline{y}_3'(t) = \frac{3}{t}, \overline{y}_3''(t) = -\frac{3}{t^2}$ and $\overline{y}_3'''(t) = \frac{6}{t^3}$. Thus, $t^2\overline{y}_3''' + t\overline{y}_3'' - \overline{y}_3' = 0$. It follows, that $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{2t^2 - 1, 3, \ln t^3\}$ is a solution set.
(b) Since $\overline{y}_1 = -1y_1 + 0y_2 + 2y_3$, $\overline{y}_2 = 3y_1 + 0y_2 + 0y_3$, and $\overline{y}_3 = 0y_1 + 3y_2 + 0y_3$ we have

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

(c) Since $det(\mathbf{A}) = 18 \neq 0$, $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{2t^2 - 1, 3, \ln t^3\}$ is a fundamental set of solutions ■

32 Higher Order Homogeneous Linear Equations with Constant Coefficients

Problem 32.1
Solve $y''' + y'' - y' - y = 0$

Solution.
The characteristic equation is

$$r^3 + r^2 - r - 1 = 0.$$

Factoring this equation using the method of grouping we find

$$r^2(r+1) - (r+1) = (r+1)^2(r-1) = 0$$

Hence, $r = -1$ is a root of multiplicity 2 and $r = 1$ is a of multiplicity 1. Thus, the general solution is given by

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^t \blacksquare$$

Problem 32.2
Find the general solution of $16y^{(4)} - 8y'' + y = 0$.

Solution.
The characteristic equation is

$$16r^4 - 8r^2 + 1 = 0.$$

This is a complete square

$$(4r^2 - 1)^2 = 0$$

Hence, $r = -\frac{1}{2}$ and $r = \frac{1}{2}$ are roots of multiplicity 2. Thus, the general solution is given by

$$y(t) = c_1e^{-\frac{t}{2}} + c_2te^{-\frac{t}{2}} + c_3e^{\frac{t}{2}} + c_4te^{\frac{t}{2}} \blacksquare$$

Problem 32.3
Solve the following constant coefficient differential equation :

$$y''' - y = 0.$$

Solution.
In this case the characteristic equation is $r^3 - 1 = 0$ or $r^3 = 1 = e^{2k\pi i}$. Thus, $r = e^{\frac{2k\pi i}{3}}$ where k is an integer. Replacing k by 0,1, and 2 we find

$$\begin{array}{rcl} r_0 &=& 1 \\ r_1 &=& -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ r_2 &=& -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{array}$$

Thus, the general solution is

$$y(t) = c_1e^t + c_2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}t}{2} + c_3e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}t}{2} \blacksquare$$

Problem 32.4
Solve $y^{(4)} - 16y = 0$

Solution.
In this case the characteristic equation is $r^4 - 16 = 0$ or $r^4 = 16 = 16e^{2k\pi i}$. Thus, $r = 2e^{\frac{k\pi i}{2}}$ where k is an integer. Replacing k by 0,1,2 and 3 we find

$$\begin{array}{rcl} r_0 &=& 2 \\ r_1 &=& 2i \\ r_2 &=& -2 \\ r_3 &=& -2i \end{array}$$

Thus, the general solution is

$$y(t) = c_1e^{2t} + c_2e^{-2t} + c_3\cos(2t) + c_4\sin(2t) \blacksquare$$

Problem 32.5
Solve the initial-value problem

$$y''' + 3y'' + 3y' + y = 0, \ y(0) = 0, \ y'(0) = 1, \ y''(0) = 0.$$

Solution.
We have the characteristic equation

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0$$

Which has a root of multiplicity 3 at $r = -1$. We use what we have learned about repeated roots roots to get the general solution. Since the multiplicity of the repeated root is 3, we have

$$y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}, \quad y_3(t) = t^2e^{-t}.$$

The general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t}.$$

Now Find the first three derivatives

$$\begin{array}{rcl} y'(t) & = & -c_1e^{-t} + c_2(1-t)e^{-t} + c_3(2t-t^2)e^{-t} \\ y''(t) & = & c_1e^{-t} + c_2(-2+t)e^{-t} + c_3(t^2-4t+2)e^{-t} \end{array}$$

Next plug in the initial conditions to get

$$\begin{array}{rcl} 0 & = & c_1 \\ 1 & = & c_2 \\ 0 & = & -2+2c_3 \end{array}$$

Solving these equations we find $c_1 = 0,\ c_2 = 1$, and $c_3 = 1$. The unique solution is then

$$y(t) = te^{-t} + t^2e^{-t} \blacksquare$$

Problem 32.6
Given that $r = 1$ is a solution of $r^3 + 3r^2 - 4 = 0$, find the general solution to

$$y''' + 3y'' - 4y = 0$$

Solution.
Since $r = 1$ is a solution then using synthetic division of polynomials we can write $(r-1)(r+2)^2 = 0$. Thus, the general solution is given by

$$y(t) = c_1e^t + c_2e^{-2t} + c_3te^{-2t} \blacksquare$$

Problem 32.7
Given that $y_1(t) = e^{2t}$ is a solution to the homogeneous equation, find the general solution to the differential equation

$$y''' - 2y'' + y' - 2y = 0$$

Solution.
The characteristic equation is given by

$$r^3 - 2r^2 + r - 2 = 0$$

Using the method of grouping we can factor into

$$(r-2)(r^2+1)=0$$

The roots are $r_1=2,\ r_2=-i$, and $r_3=i$. Thus, the general solution is

$$y(t)=c_1e^{2t}+c_3\cos t+c_4\sin t \blacksquare$$

Problem 32.8
Suppose that $y(t)=c_1\cos t+c_2\sin t+c_3\cos(2t)+c_4\sin(2t)$ is the general solution to the equation

$$y^{(4)}+a_3y'''+a_2y''+a_1y'+a_0y=0$$

Find the constants a_0, a_1, a_2, and a_3.

Solution.
The characteristic equation is of order 4. The roots are given by $r_1=i,\ r_2=-i,\ r_3=-2i$, and $r_4=2i$. Hence the characteristic equation is $(r^2+1)(r^2+4)=0$ or $r^4+5r^2+4=0$. Comparing coefficients we find $a_0=4,\ a_1=0,\ a_2=5$, and $a_3=0$. Thus, the differential equation $y^{(4)}+5y''+4y=0$ $\blacksquare$

Problem 32.9
Suppose that $y(t)=c_1+c_2t+c_3\cos 3t+c_4\sin 3t$ is the general solution to the homogeneous equation

$$y^{(4)}+a_3y'''+a_2y''+a_1y'+a_0y=0$$

Determine the values of $a_0,\ a_1,\ a_2$, and a_3.

Solution.
The characteristic equation is of order 4. The roots are given by $r=0$ (of multiplicity 2), $r=-3i$ and $r=3i$. Hence the characteristic equation is $r^2(r^2+9)=0$ or $r^4+9r^2=0$. Comparing coefficients we find $a_0=a_1=0,\ a_2=9$, and $a_3=0$. Thus, the differential equation $y^{(4)}+9y''=0$ $\blacksquare$

Problem 32.10
Suppose that $y(t)=c_1e^{-t}\sin t+c_2e^{-t}\cos t+c_3e^t\sin t+c_4e^t\cos t$ is the general solution to the homogeneous equation

$$y^{(4)}+a_3y'''+a_2y''+a_1y'+a_0y=0$$

Determine the values of $a_0,\ a_1,\ a_2$, and a_3.

Solution.
The characteristic equation is of order 4. The roots are given by $r_1 = -1 - i,\ r_2 = -1+i,\ r_3 = 1-i$, and $r_4 = 1+i$. Hence $(r-(-1-i))(r-(-1+i)) = r^2+2r+2$ and $(r-(1-i))(r-(1+i)) = r^2-2r+2$ so that the characteristic equation is $(r^2+2r+2)(r^2-2r+2) = r^4+4 = 0$. Comparing coefficients we find $a_0 = 4,\ a_1 = a_2 = a_3 = 0$. Thus, the differential equation $y^{(4)} + 4y = 0$ ■

Problem 32.11
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t,\ y_2(t) = e^t, y_3(t) = \cos t$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for n for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

Solution.
The fundamental set must contain the functions $y_4(t) = 1$ and $y_5(t) = \sin t$. Thus, the smallest value of n is 5 and the fundamental set in this case is $\{1, t, \cos t, \sin t, e^t\}$ ■

Problem 32.12
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t^2 \sin t,\ y_2(t) = e^t \sin t$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for n for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

Solution.
The fundamental set must contain the functions $y_3(t) = \sin t,\ y_4(t) = \cos t,\ y_5(t) = t\sin t,\ y_6(t) = t\cos t,\ y_7(t) = t^2\cos t$, and $y_8(t) = e^t\cos t$. Thus, the smallest value of n is 8 and the fundamental set in this case is $\{\sin t, \cos t, t\sin t, t\cos t, t^2\sin t, t^2\cos t, e^t\sin t, e^t\cos t\}$ ■

Problem 32.13
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t^2,\ y_2(t) = e^{2t}$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for n for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

Solution.
The fundamental set must contain the functions $y_3(t) = 1$ and $y_4(t) = t$ Thus, the smallest value of n is 4 and the fundamental set in this case is $\{1, t, t^2, e^{2t}\}$ ■

33 Non Homogeneous nth Order Linear Differential Equations

Problem 33.1
Consider the nonhomogeneous differential equation

$$t^3y''' + at^2y'' + bty' + cy = g(t), \quad t > 0$$

Determine a, b, c, and $g(t)$ if the general solution is given by $y(t) = c_1t + c_2t^2 + c_3t^4 + 2\ln t$

Solution.
Since t, t^2, t^4 are solutions to the homogeneous equation then

$$\begin{array}{rcc} 0+0+bt+ct=0 & \Longrightarrow & b+c=0 \\ 0+at^2(2)+bt(2t)+ct^2=0 & \Longrightarrow & 2a+2b+c=0 \\ t^3(24t)+at^2(12t^2)+bt(4t^3)+ct^4=0 & \Longrightarrow & 12a+4b+c=-24 \end{array}$$

Solving the system of equations we find $a = -4,\ b = 8,\ c = -8$. Thus,

$$t^3y''' - 4t^2y'' + 8ty' - 8y = g(t)$$

But $2\ln t$ is a particular solution so that

$$g(t) = t^3(4t^{-3} - 4t^2(-2t^{-2}) + 8t(2t^{-1}) - 16\ln t = 28 - 16\ln t \blacksquare$$

Problem 33.2
Consider the nonhomogeneous differential equation

$$y''' + ay'' + by' + cy = g(t), \quad t > 0$$

Determine a, b, c, and $g(t)$ if the general solution is given by $y(t) = c_1 + c_2t + c_3e^{2t} + 4\sin 2t$

Solution.
The characteristic equation is $r^2(r-2) = 0$ so that the associated homogeneous equation is $y''' - 2y'' = 0$. Thus, $a = -1,\ b = c = 0$. The particular solution is $y_p(t) = 4\sin 2t$. Inserting into the equation we find

$$g(t) = -32\cos 2t - 2(-16\sin 2t) = -32\cos 2t + 32\sin 2t \blacksquare$$

Problem 33.3
Solve

$$y^{(4)} + 4y'' = 16 + 15e^t$$

Solution.
We first find the solution to the homogeneous differential equation. The characteristic equations is

$$r^4 + 4r^2 = 0 \text{ or } r^2(r^2 + 4) = 0$$

The roots are

$$r = 0 \text{ (repeated twice)}, \, r = 2i, \, r = -2i$$

The homogeneous solution is

$$y_h(t) = c_1 + c_2 t + c_3 \sin(2t) + c_4 \cos(2t)$$

Since $g(t)$ is a sum of two terms, we can work each term separately. The trial function for the function $g(t) = 16$ is $y_p = 1$. Since this is a solution to the homogeneous equation, we multiply by t to get

$$y_p(t) = t$$

This is also a solution to the homogeneous equation, so multiply by t again to get

$$y_p(t) = t^2$$

which is not a solution of the homogeneous equation. We write

$$y_{p_1} = At^2, \; y'_{p_1} = 2At, \; y''_{p_1} = 2A, \; y'''_{p_1} = 2A, \; y^{(4)}_{p_1} = 0$$

Substituting back in, we get $0 + 4(2A) = 16$ or $A = 2$. Hence

$$y_{p_1}(t) = 2t^2$$

Now we work on the second piece. The trial function for $g(t) = 15e^t$ is e^t. Since this in not a solution to the homogeneous equation, we get

$$y_{p_2} = Ae^t, \; y'_{p_2} = Ae^t, \; y''_{p_2} = Ae^t, \; y'''_{p_2} = Ae^t, \; y^{(}_{p_2}4) = Ae^t$$

Plugging back into the original equation gives $Ae^t + 4Ae^t = 15e^t$ and this implies $A = 3$. Hence

$$y_{p_2}(t) = 3e^t$$

The general solution to the nonhomogeneous differential equation is

$$y(t) = c_1 + c_2 t + c_3 \sin(2t) + c_4 \cos(2t) + 2t^2 + 3e^t \; \blacksquare$$

Problem 33.4
Solve: $y^{(4)} - 8y'' + 16y = -64e^{2t}$

Solution.
The characteristic equation $r^4 - 8r^2 + 16 = (r^2-4)^2 = 0$ has two double roots at $r = -2$ and $r = 2$. The homogeneous solution is then

$$y_h(t) = c_1e^{-2t} + c_2te^{-2t} + c_3e^{2t} + c_4te^{2t}$$

Since the nonhomogeneous term is an exponential function, we use the method of undetermined coefficients to find a particular solution of the form

$$y_p(t) = At^2e^{2t}$$

Plug this into the equation and get

$$(48A + 64At + 16At^2)e^{2t} - 8(2A + 8At + 4At^2)e^{2t} + 16At^2e^{2t} = -64e^{2t}$$

from which we see $32A = -64$ or $A = -2$. The general solution is

$$y(t) = c_1e^{-2t} + c_2te^{-2t} + c_3e^{2t} + c_4te^{2t} - 2t^2e^{2t} \blacksquare$$

Problem 33.5
Given that $y_1(t) = e^{2t}$ is a solution to the homogeneous equation, find the general solution to the differential equation,

$$y''' - 2y'' + y' - 2y = 12\sin 2t$$

Solution.
The characteristic equation is $r^3 - 2r^2 + r - 2 = 0$. We know that $r = 2$ is a root of this equation. Using synthetic division we can write

$$r^3 - 2r^2 + r - 2 = (r-2)(r^2+1) = 0$$

So the roots are $r = 2,\ r = -i,\ r = i$ and the general solution is

$$y_h(t) = c_1e^{2t} + c_2\cos t + c_3\sin t$$

To find a particular solution we use the method of undetermined coefficients by considering the trial function $y_p(t) = A\cos 2t + B\sin 2t$. In this case, we have

$$\begin{aligned} y_p'(t) &= -2A\sin 2t + 2B\cos 2t \\ y_p''(t) &= -4A\cos 2t - 4b\sin 2t \end{aligned}$$

Inseting into the differential equation we find

$$\begin{array}{rcl} 12\sin 2t & = & y_p''' - 2y_p'' + y_p' - 2y_p \\ & = & 8A\sin 2t - 8B\cos 2t - 2(-4A\cos 2t - 4B\sin 2t) \\ & + & (2B\cos 2t - 2A\sin 2t) - 2(A\cos 2t + B\sin 2t) \\ & = & (6A - 6B)\cos 2t + (6A + 6B)\sin 2t \end{array}$$

Equating coefficients we find $6A - 6B = 0$ and $6A + 6B = 12$. Solving we find $A = B = 1$ and so the general solution is

$$y(t) = c_1e^{2t} + c_2\cos t + c_3\sin t + \cos 2t + \sin 2t \blacksquare$$

Problem 33.6
Find the general solution of the equation

$$y''' - 6y'' + 12y' - 8y = \sqrt{2t}e^{2t}$$

Solution.
The characteristic equation $r^3 - 6r^2 + 12r - 8 = (r-2)^3 = 0$ has a triple root at $r = 2$. Hence, the homogeneous solution is

$$y_h(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t}$$

We use the method of variation of paramemters to find the particular solution

$$y_p(t) = u_1e^{2t} + u_2te^{2t} + r_3t^2e^{2t}$$

The Wronskian is

$$W(t) = \left|\begin{array}{ccc} e^{2t} & te^{2t} & t^2e^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} & 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{array}\right| = 2e^{6t}$$

Also,

$$W_1(t) = \left|\begin{array}{ccc} 0 & te^{2t} & t^2e^{2t} \\ 0 & e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 1 & 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{array}\right| = t^2e^{4t}$$

$$W_2(t) = \left|\begin{array}{ccc} e^{2t} & 0 & t^2e^{2t} \\ 2e^{2t} & 0 & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} & 1 & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{array}\right| = -2te^{4t}$$

$$W_3(t) = \begin{vmatrix} e^{2t} & te^{2t} & 0 \\ 2e^{2t} & e^{2t}+2te^{2t} & 0 \\ 4e^{2t} & 4e^{2t}+4te^{2t} & 1 \end{vmatrix} = e^{4t}$$

Hence,

$$\begin{array}{rcl} u_1(t) & = & \int \frac{W_1(t)}{W(t)} g(t)dt = \int \frac{\sqrt{2}}{2} t^{\frac{5}{2}} dt = \frac{\sqrt{2}}{7} t^{\frac{7}{2}} \\ u_2(t) & = & \int \frac{W_2(t)}{W(t)} g(t)dt = \int -\sqrt{2} t^{\frac{3}{2}} dt = -\frac{2\sqrt{2}}{5} t^{\frac{5}{2}} \\ u_3(t) & = & \int \frac{W_3(t)}{W(t)} g(t)dt = \int \frac{\sqrt{2}}{2} t^{\frac{1}{2}} dt = \frac{\sqrt{2}}{3} t^{\frac{3}{2}} \end{array}$$

Hence, the general solution is

$$\begin{array}{rcl} y(t) & = & c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + \frac{\sqrt{2}}{7} t^{\frac{7}{2}} e^{2t} - \frac{2\sqrt{2}}{5} t^{\frac{7}{2}} e^{2t} + \frac{\sqrt{2}}{3} t^{\frac{7}{2}} e^{2t} \\ & = & c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + \frac{8\sqrt{2}}{105} t^{\frac{7}{2}} e^{2t} \blacksquare \end{array}$$

Problem 33.7
(a) Verify that $\{t, t^2, t^4\}$ is a fundamental set of solutions of the differential equation

$$t^3y''' - 4t^2y'' + 8ty' - 8y = 0$$

(b) Find the general solution of

$$t^3y''' - 4t^2y'' + 8ty' - 8y = 2\sqrt{t}, \quad t > 0$$

Solution.
(a) Let $y_1(t) = t,\ y_2(t) = t^2,\ y_3(t) = t^4$. Then

$$\begin{array}{rcl} t^3y_1''' - 4t^2y_1'' + 8ty_1' - 8y_1 & = & t^3(0) - 4t^2(0) + 8t(1) - 8t = 0 \\ t^3y_2''' - 4t^2y_2'' + 8ty_2' - 8y_2 & = & t^3(0) - 4t^2(2) + 8t(2t) - 8t^2 = 0 \\ t^3y_3''' - 4t^2y_3'' + 8ty_3' - 8y_3 & = & t^3(24t) - 4t^2(12t^2) + 8t(4t^3) - 8t^4 = 0 \end{array}$$

$$W(t) = \begin{vmatrix} t & t^2 & t^4 \\ 1 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{vmatrix} = 6t^4 \neq 0, t > 0$$

Since $W(1) \neq 0$ then $\{y_1, y_2, y_3\}$ is a fundamental set of solutions.
(b) Using the method of variation of parameters we look for a solution of the form

$$y_p(t) = u_1t + u_2t^2 + u_3t^4$$

where

$$\begin{array}{rcl}
u_1'(t) & = & \dfrac{\begin{vmatrix} 0 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ 1 & 2 & 12t^2 \end{vmatrix} 2t^{-\frac{5}{2}}}{6t^4} = \frac{2}{3}t^{-\frac{3}{2}} \\
u_2'(t) & = & \dfrac{\begin{vmatrix} t & 0 & t^4 \\ 1 & 0 & 4t^3 \\ 0 & 1 & 12t^2 \end{vmatrix} 2t^{-\frac{5}{2}}}{6t^4} = -t^{-\frac{5}{2}} \\
u_3'(t) & = & \dfrac{\begin{vmatrix} t & t^2 & 0 \\ 1 & 2t & 0 \\ 0 & 2 & 1 \end{vmatrix} 2t^{-\frac{5}{2}}}{6t^4} = \frac{1}{3}t^{-\frac{9}{2}}
\end{array}$$

Thus,

$$\begin{array}{rcl}
u_1(t) & = & \int \frac{2}{3}t^{-\frac{3}{2}}dt = -\frac{4}{3}t^{-\frac{1}{2}} \\
u_2(t) & = & \int -t^{-\frac{5}{2}}dt = \frac{2}{3}t^{-\frac{3}{2}} \\
u_3(t) & = & \int \frac{1}{3}t^{-\frac{9}{2}}dt = -\frac{2}{21}t^{-\frac{7}{2}}
\end{array}$$

Thus, the general solution is

$$y(t) = c_1t + c_2t^2 + c_3t^4 - \frac{4}{3}t^{\frac{1}{2}} + \frac{2}{3}t^{\frac{1}{2}} - \frac{2}{21}t^{\frac{1}{2}} = c_1t + c_2t^2 + c_3t^4 - \frac{16}{21}t^{\frac{1}{2}} \blacksquare$$

Problem 33.8
(a) Verify that $\{t, t^2, t^3\}$ is a fundamental set of solutions of the differential equation

$$t^3y''' - 3t^2y'' + 6ty' - 6y = 0$$

(b) Find the general solution of by using the method of variation of parameters

$$t^3y''' - 3t^2y'' + 6ty' - 6y = t, \quad t > 0$$

Solution.
(a) Let $y_1(t) = t,\ y_2(t) = t^2,\ y_3(t) = t^3$. Then

$$\begin{array}{rcl}
t^3y_1''' - 3t^2y_1'' + 6ty_1' - 6y_1 & = & t^3(0) - 3t^2(0) + 6t(1) - 6t = 0 \\
t^3y_2''' - 3t^2y_2'' + 6ty_2' - 6y_2 & = & t^3(0) - 3t^2(2) + 6t(2t) - 6t^2 = 0 \\
t^3y_3''' - 3t^2y_3'' + 6ty_3' - 6y_3 & = & t^3(6) - 3t^2(6t) + 6t(3t^2) - 6t^3 = 0
\end{array}$$

$$W(t) = \begin{vmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 2t^3 \neq 0, t > 0$$

Since $W(1) \neq 0$ then $\{y_1, y_2, y_3\}$ is a fundamental set of solutions for $t > 0$.
(b) Using the method of variation of parameters we look for a solution of the form

$$y_p(t) = u_1 t + u_2 t^2 + u_3 t^3$$

where

$$\begin{array}{rcl} u_1'(t) & = & \dfrac{\left|\begin{array}{ccc} 0 & t^2 & t^3 \\ 0 & 2t & 3t^2 \\ 1 & 2 & 6t \end{array}\right| t^{-2}}{2t^3} = \frac{1}{2t} \\ u_2'(t) & = & \dfrac{\left|\begin{array}{ccc} t & 0 & t^3 \\ 1 & 0 & 3t^2 \\ 0 & 1 & 6t \end{array}\right| t^{-2}}{2t^3} = -\frac{1}{t^2} \\ u_3'(t) & = & \dfrac{\left|\begin{array}{ccc} t & t^2 & 0 \\ 1 & 2t & 0 \\ 0 & 2 & 1 \end{array}\right| t^{-2}}{2t^3} = \frac{1}{2t^3} \end{array}$$

Thus,

$$\begin{array}{rcl} u_1(t) & = & \int \frac{1}{2t} dt = \frac{1}{2} \ln t \\ u_2(t) & = & \int -\frac{1}{t^2} dt = \frac{1}{t} \\ u_3(t) & = & \int \frac{1}{2t^3} dt = -\frac{1}{4t^2} \end{array}$$

Thus, the general solution is

$$y(t) = c_1 t + c_2 t^2 + c_3 t^3 + \frac{t}{2} \ln t + \frac{3}{4} t = c_1 t + c_2 t^2 + c_3 t^3 + \frac{t}{2} \ln t$$

since $\frac{3}{4}t$ is a solution to the homogeneous equation ■

Problem 33.9
Solve using the method of undetermined coefficients: $y''' - y' = 4 + 2\cos t$

Solution.
We first solve the homogeneous differential equation

$$y''' - y' = 0$$

The characteristic equation is

$$r^3 - r = 0$$

Factoring gives

$$r(r-1)(r+1) = 0$$

Solving we find $r = 0,\ r = -1$ and $r = 1$. The homogeneous solution is

$$y_h(t) = c_1 + c_2e^t + c_3e^{-t}$$

The trial function generated by $g(t) = 4 + 2\cos(2t)$ is

$$y_p(t) = At + B\cos(2t) + C\sin(2t)$$

Then

$$\begin{array}{rcl} y_p' &=& A - 2B\sin(2t) + 2C\cos(2t) \\ y_p'' &=& -4B\cos(2t) - 4C\sin(2t) \\ y_p''' &=& 8B\sin(2t) - 8C\cos(2t) \end{array}$$

Plugging back into the original differential equation gives

$$[8B\sin(2t) - 8C\cos(2t)] - [A - 2B\sin(2t) + 2C\cos(2t)] = 4 + 2\cos(2t)$$

Combining like terms gives

$$-10C\cos(2t) + 10B\sin(2t) - A = 4 + 2\cos(2t)$$

Equating coefficients gives

$$\begin{array}{rcl} -10C &=& 2 \\ 10B &=& 0 \\ -A &=& 4 \end{array}$$

Solving we find $A = -4,\ B = 0$, and $C = -\frac{1}{5}$. The general solution is thus

$$y(t) = c_1 + c_2e^t + c_3e^{-t} - 4t - \frac{1}{5}\sin(2t) \blacksquare$$

Problem 33.10
Solve using the method of undetermined coefficients: $y''' - y' = -4e^t$

Solution.
We first solve the homogeneous differential equation

$$y''' - y' = 0$$

The characteristic equation is

$$r^3 - r = 0$$

Factoring gives

$$r(r-1)(r+1) = 0$$

Solving we find $r = 0,\ r = -1$ and $r = 1$. The homogeneous solution is

$$y_h(t) = c_1 + c_2e^t + c_3e^{-t}$$

The trial function generated by $g(t) = -4e^t$ is

$$y_p(t) = Ate^t$$

Then

$$\begin{array}{rcl} y_p' & = & Ae^t + Ate^t \\ y_p'' & = & 2Ae^t + Ate^t \\ y_p''' & = & 3Ae^t + Ate^t \end{array}$$

Plugging back into the original differential equation gives

$$(3Ae^t + Ate^t) - (Ae^t + Ate^t) = -4e^t$$

Combining like terms gives

$$2Ae^t = -4e^t$$

Solving we find $A = -2$. The general solution is thus

$$y(t) = c_1 + c_2e^t + c_3e^{-t} - 2te^t \blacksquare$$

Problem 33.11
Solve using the method of undetermined coefficients: $y''' - y'' = 4e^{-2t}$

Solution.
The characteristic equation is

$$r^3 - r^2 = 0$$

Factoring gives

$$r^2(r-1) = 0$$

Solving we find $r = 0$ (double root) and $r = 1$. The homogeneous solution is

$$y_h(t) = c_1 + c_2 t + c_3 e^t$$

The trial function generated by $g(t) = 4e^{-2t}$ is

$$y_p(t) = Ae^{-2t}$$

Then

$$\begin{array}{rcl} y_p' &=& -2Ae^{-2t} \\ y_p'' &=& 4Ae^{-2t} \\ y_p''' &=& -8Ae^{-2t} \end{array}$$

Plugging back into the original differential equation gives

$$(-8e^{-2t}) - (4Ae^{-2t}) = 4e^{-2t}$$

Combining like terms gives

$$-12Ae^{-2t} = 4e^{-2t}$$

Solving we find $A = -\frac{1}{3}$. The general solution is thus

$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} - \frac{1}{3}e^{-2t} \blacksquare$$

Problem 33.12
Solve using the method of undetermined coefficients: $y''' - 3y'' + 3y' - y = 12e^t$.

Solution.
We first solve the homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0$$

Factoring gives

$$(r-1)^3 = 0$$

Solving we find $r = 1$ of multiplicity 3. The homogeneous solution is

$$y_h(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

The trial function generated by $g(t) = 12e^t$ is

$$y_p(t) = At^3e^t$$

Then

$$\begin{array}{lcl} y_p' &=& 3At^2e^t + At^3e^t \\ y_p'' &=& 6Ate^t + 6At^2e^t + At^3e^t \\ y_p''' &=& 6Ae^t + 18Ate^t + 9At^2e^t + At^3e^t \end{array}$$

Plugging back into the original differential equation gives

$$(6Ae^t+18Ate^t+9At^2e^t+At^3e^t)-3(6Ate^t+6At^2e^t+At^3e^t)+3(3At^2e^t+At^3e^t)-At^3e^t = 12e^t$$

Combining like terms gives

$$6Ae^t = 12e^t$$

Solving we find $A = 2$. The general solution is thus

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + 2t^3e^t \blacksquare$$

Problem 33.13
Solve using the method of undetermined coefficients: $y''' + y = e^t + \cos t$.

Solution.
The characteristic equation is

$$r^3 + 1 = 0$$

Factoring gives

$$(r+1)(r^2 - r + 1) = 0$$

Solving we find $r = -1,\ r = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $r = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The homogeneous solution is

$$y_h(t) = c_1e^{-t} + c_2e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + c_3e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

The trial function generated by $g(t) = e^t + \cos t$ is

$$y_p(t) = Ae^t + B\cos t + C\sin t$$

Then

$$\begin{array}{lcl} y_p' &=& Ae^t - B\sin t + C\cos t \\ y_p'' &=& Ae^t - B\cos t - C\sin t \\ y_p''' &=& Ae^t + B\sin t - C\cos t \end{array}$$

Plugging back into the original differential equation gives

$$(Ae^t + B\sin t - C\cos t) + (Ae^t + B\cos t + C\sin t) = e^t + \cos t$$

or

$$2Ae^t + (B + C)\sin t + (B - C)\cos t = e^t + \cos t$$

Equating coefficients we find

$$\begin{array}{rcl} 2A & = & 1 \\ B + C & = & 0 \\ B - C & = & 1 \end{array}$$

Solving we find $A = B = \frac{1}{2}$ and $C = -\frac{1}{2}$. Thus, the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t + c_3 e^{\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t + \frac{1}{2}(e^t + \cos t - \sin t \blacksquare$$

In Problems 33.14 and 33.15, answer the following two questions.
(a) Find the homogeneous general solution.
(b) Formulate an appropriate for for the particular solution suggested by the method of undetermined coefficients. You need not evaluate the undetermined coefficients.

Problem 33.14
$y''' - 3y'' + 3y' - y = e^t + 4e^t \cos 3t + 4$

Solution.
(a) The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r - 1)^3 = 0$$

So $r = 1$ is a root of multiplicity 3 so that the homogeneous general solution is

$$y_h(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t.$$

(b) The trial function for the right-hand function $g(t) = e^t + 4e^t \cos 3t + 4$ is $y_p(t) = At^3 e^t + Be^t \cos 3t + Ce^t \sin 3t + D.$ $\blacksquare$

Problem 33.15
$y^{(4)} + 8y'' + 16y = t\cos 2t$

Solution.
(a) The characteristic equation is

$$r^4 + 8r^2 + 16 = (r^2 + 4)^2 = 0$$

So $r = -2i$ and $r = 2i$ are roots of multiplicity 2 so that the homogeneous general solution is

$$y_h(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t.$$

(b) The trial function for the right-hand function $g(t) = t \cos 2t$ is $y_p(t) = t^2(At + B) \cos 2t + t^2(Ct + D) \sin 2t.$ ■

Consider the nonhomogeneous differential equation

$$y''' + ay'' + by' + cy = g(t)$$

In Problems 33.16 - 33.17, the general solution of the differential equation is given, where c_1, c_2, and c_3 represent arbitrary constants. Use this information to determine the constants a, b, c and the function $g(t)$.

Problem 33.16
$y(t) = c_1 + c_2 t + c_3 e^{2t} + 4 \sin 2t.$

Solution.
The roots of the characteristic equation are $r = 0$ of multiplicity 2 and $r = 2$. Thus, the equation is

$$r^2(r - 2) = r^3 - 2r^2 = 0.$$

The corresponding differential equation is $y''' - 2y'' = 0$. Comparing coefficients we find $a = -2,\ b = c = 0$. The function $y_p(t) = 4 \sin 2t$ is a particular solution to the nonhomogeneous equation. Taking derivatives we find $y_p'(t) = 8 \cos 2t,\ y_p''(t) = -16 \sin 2t,\ y_p'''(t) = -32 \cos 2t$. By plugging into the equation we find $-32 \cos 2t + 32 \sin 2t = g(t)$ ■

Problem 33.17
$y(t) = c_1 + c_2 t + c_3 t^2 - 2t^3$

Solution.
The roots of the characteristic equation are $r = 0$ of multiplicity 3. Thus, the equation is

$$r^3 = 0.$$

The corresponding differential equation is $y''' = 0$. Comparing coefficients we find $a = b = c = 0$. The function $y_p(t) = -2t^3$ is a particular solution to the nonhomogeneous equation. Taking derivatives we find $y_p'(t) = -6t^2,\ y_p''(t) = -12t,\ y_p'''(t) = -12$. By plugging into the equation we find $-12 = g(t)$ ■

Problem 33.18
Consider the nonhomogeneous differential equation

$$t^3y''' + at^2y'' + bty' + cy = g(t), \quad t > 0$$

Suppose that $y(t) = c_1t + c_2t^2 + c_3t^4 + 2\ln t$ is the general solution to the above equation. Determine the constants a, b, c and the function $g(t)$

Solution.
The functions $y_1(t) = t,\ y_2(t) = t^2$, and $y_3(t) = t^4$ are solutions to the homogeneous equation. Substituting into the equation we find

$$\begin{array}{rcc} 0 + 0 + bt + ct = 0 & \longrightarrow & b + c = 0 \\ 0 + 2at^2 + 2bt^2 + ct^2 = 0 & \longrightarrow & 2a + 2b + c = 0 \\ 24t^4 + 12at^4 + 4bt^4 + ct^4 = 0 & \longrightarrow & 12a + 4b + c = -24 \end{array}$$

Solving this system of equations we find $a = -4,\ b = 8$, and $c = -8$. Thus,

$$t^3y''' - 4t^2y'' + 8ty' - 8y = g(t).$$

Since $y_p(t) = 2\ln t$ is a particular solution to the nonhomogeneous equation then by substitution we find $g(t) = t^3(\frac{4}{t^3}) - 4t^2(-\frac{2}{t^2}) + 8t(\frac{2}{t}) - 16\ln t = 28 - 16\ln t$ ■

34 Existence and Uniqueness of Solution to Initial Value First Order Linear Systems

Problem 34.1
Consider the initial value problem

$$\begin{array}{rcl} (t+2)y_1' & = & 3ty_1+5y_2, \quad y_1(1)=0 \\ (t-2)y_2' & = & 2y_1+4ty_2, \quad y_2(1)=2 \end{array}$$

Determine the largest t-interval such that a unique solution is guaranteed to exist.

Solution.
All the coefficient functions and the right-hand side functions are continuous for all $t \neq \pm 2$. Since $t_0 = 1$, the $t-$interval of existence is $-2 < t < 2$ ■

Problem 34.2
Verify that the functions $y_1(t) = c_1e^t \cos t + c_2e^t \sin t$ and $y_2(t) = -c_1e^t \sin t + c_2e^t \cos t$ are solutions to the linear system

$$\begin{array}{rcl} y_1' & = & y_1+y_2 \\ y_2' & = & -y_1+y_2 \end{array}$$

Solution.
Taking derivatives we find $y_1'(t) = (c_1+c_2)e^t \cos t + (c_2-c_1)e^t \sin t$ and $y_2'(t) = -(c_1+c_2)e^t \sin t + (c_2-c_1)e^t \cos t$. But $y_1+y_2 = (c_1+c_2)e^t \cos t + (c_2-c_1)e^t \sin t = y_1'(t)$ and $-y_1+y_2 = -(c_1+c_2)e^t \sin t + (c_2-c_1)e^t \cos t = y_2'(t)$ ■

Problem 34.3
Consider the initial value problem

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{y}(0) = \mathbf{y}_0$$

where

$$\mathbf{A} = \left[\begin{array}{cc} 3 & 2 \\ 4 & 1 \end{array}\right], \quad \mathbf{y}_0 = \left[\begin{array}{c} -1 \\ 8 \end{array}\right]$$

(a) Verify that $\mathbf{y}(t) = c_1e^{5t}\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + c_2e^{-t}\left[\begin{array}{c} -1 \\ 2 \end{array}\right]$ is a solution to the first order linear system.
(b) Determine c_1 and c_2 such that $\mathbf{y}$(t) solves the given initial value problem.

Solution.
(a) We have

$$\mathbf{y}'(t) = c_1 e^{5t}\begin{bmatrix} 5 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and

$$\begin{aligned}\mathbf{A}\mathbf{y}(t) &= c_1 e^{5t}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= c_1 e^{5t}\begin{bmatrix} 5 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \mathbf{y}'(t)\end{aligned}$$

(b) We need to solve the system

$$\begin{aligned} c_1 - c_2 &= -1 \\ c_1 + 2c_2 &= 8 \end{aligned}$$

Solving this system we find $c_1 = 2$ and $c_2 = 3$. Therefore, the unique solution to the system is

$$\mathbf{y}(t) = \begin{bmatrix} 2e^{5t} - 3e^{-t} \\ 2e^{5t} + 6e^{-t} \end{bmatrix} \blacksquare$$

Problem 34.4
Rewrite the differential equation $(\cos t)y'' - 3ty' + \sqrt{t}y = t^2 + 1$ in the matrix form $\mathbf{y}(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t)$.

Solution.
Rewriting the equation in the form $y'' - 3t\sec ty' + \sqrt{t}\sec t = (t^2 + 1)\sec t$ we find

$$\begin{aligned}\mathbf{y}' &= \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\sqrt{t}\sec t & 3t\sec t \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ (t^2+1)\sec t \end{bmatrix} \\ &= \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t) \blacksquare\end{aligned}$$

Problem 34.5
Rewrite the differential equation $2y'' + ty + e^{3t} = y''' + (\cos t)y'$ in the matrix form $\mathbf{y}(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t)$.

Solution.
Rewriting the given equation in the form $y''' - 2y'' + (\cos t)y' - ty = e^{3t}$ we find

$$\begin{aligned} \mathbf{y}' &= \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -\cos t & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} + \begin{bmatrix} 0 & 0 & e^{3t} \end{bmatrix} \\ &= \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t) \blacksquare \end{aligned}$$

Problem 34.6
The initial value problem

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 2\cos(2t) \end{bmatrix}, \quad \mathbf{y}(-1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

was obtained from an initial value problem for a higher order differential equation. What is the corresponding scalar initial value problem?

Solution.
Carrying the matrix arithmetic we find $y'' = -3y + 2y' + 2\cos 2t$. Thus, the initial value problem is

$$y'' + 3y - 2y' = 2\cos 2t, \ y(-1) = 1, \ y'(-1) = 4 \blacksquare$$

Problem 34.7
The initial value problem

$$\mathbf{y}'(t) = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_2 + y_3 \sin y_1 + y_3^2 \end{bmatrix}, \quad \mathbf{y}(1) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

was obtained from an initial value problem for a higher order differential equation. What is the corresponding scalar initial value problem?

Solution.
Let

$$\mathbf{y}(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix}$$

Then

$$\mathbf{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \\ y^{(4)} \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ y''' \\ y' + y'' \sin y + (y'')^2 \end{bmatrix}$$

Equating components we find

$$y^{(4)} = y' + y'' \sin y + (y'')^2.$$

Thus, the initial value problem is

$$y^{(4)} = y' + y'' \sin y + (y'')^2, \; y(1) = y'(1) = 0, \; y''(1) = -1, \; y'''(1) = 2 \blacksquare$$

Problem 34.8
Consider the system of differential equations

$$\begin{array}{rcl} y'' & = & tz' + y' + z \\ z'' & = & y' + z' + 2ty \end{array}$$

Write the above system in the form

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ z(t) \\ z'(t) \end{bmatrix}$$

Identify $\mathbf{P}$(t) and $\mathbf{g}$(t).

Solution.

$$\begin{array}{rcl} \mathbf{y}' & = & \begin{bmatrix} y' \\ y'' \\ z' \\ z'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 \\ 2t & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ z \\ z' \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ & = & \mathbf{P}(t)\mathbf{y} + \mathbf{G}(t) \blacksquare \end{array}$$

Problem 34.9
Consider the system of differential equations

$$\begin{aligned} y'' &= 7y' + 4y - 8z + 6z' + t^2 \\ z'' &= 5z' + 2z - 6y' + 3y - \sin t \end{aligned}$$

Write the above system in the form

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ z(t) \\ z'(t) \end{bmatrix}$$

Identify $\mathbf{P}$(t) and $\mathbf{g}$(t).

Solution.

$$\begin{aligned} \mathbf{y}' &= \begin{bmatrix} y' \\ y'' \\ z' \\ z'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 7 & -8 & 6 \\ 0 & 0 & 0 & 1 \\ 3 & -6 & 2 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \\ z \\ z' \end{bmatrix} \begin{bmatrix} 0 \\ t^2 \\ 0 \\ -\sin t \end{bmatrix} \\ &= \mathbf{P}(t)\mathbf{y} + \mathbf{G}(t) \blacksquare \end{aligned}$$

35 Homogeneous First Order Linear Systems

In Problems 35.1 - 35.3 answer the following two questions.
(a) Rewrite the given system of linear homogeneous differential equations as a homogeneous linear system of the form $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.
(b) Verify that the given function $\mathbf{y}$(t) is a solution of $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.

Problem 35.1

$$\begin{array}{rcl} y_1' &=& -3y_1 - 2y_2 \\ y_2' &=& 4y_1 + 3y_2 \end{array}$$

and

$$\mathbf{y}(t) = \left[\begin{array}{c} e^t + e^{-t} \\ -2e^t - e^{-t} \end{array}\right]$$

Solution.
(a)

$$\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]' = \left[\begin{array}{cc} -3 & -2 \\ 4 & 3 \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$$

(b) We have

$$\mathbf{y}' = \left[\begin{array}{c} e^t - e^{-t} \\ -2e^t + e^{-t} \end{array}\right]$$

and

$$\mathbf{P}(t)\mathbf{y} = \left[\begin{array}{cc} -3 & -2 \\ 4 & 3 \end{array}\right]\left[\begin{array}{c} e^t + e^{-t} \\ -2e^t - e^{-t} \end{array}\right] = \left[\begin{array}{c} e^t - e^{-t} \\ -2e^t + e^{-t} \end{array}\right] = \mathbf{y}' \blacksquare$$

Problem 35.2

$$\begin{array}{rcl} y_1' &=& y_2 \\ y_2' &=& -\frac{2}{t^2}y_1 + \frac{2}{t}y_2 \end{array}$$

and

$$\mathbf{y}(t) = \left[\begin{array}{c} t^2 + 3t \\ 2t + 3 \end{array}\right]$$

Solution.
(a)

$$\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]' = \left[\begin{array}{cc} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$$

(b) We have

$$\mathbf{y}' = \left[\begin{array}{c} 2t+3 \\ 2 \end{array}\right]$$

and

$$\mathbf{P}(t)\mathbf{y} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{array}\right]\left[\begin{array}{c} t^2+3t \\ 2t+3 \end{array}\right] = \left[\begin{array}{c} 2t+3 \\ 2 \end{array}\right] = \mathbf{y}' \blacksquare$$

Problem 35.3

$$\begin{array}{rcl} y_1' & = & 2y_1 + y_2 + y_3 \\ y_2' & = & y_1 + y_2 + 2y_3 \\ y_3' & = & y_1 + 2y_2 + y_3 \end{array}$$

and

$$\mathbf{y}(t) = \left[\begin{array}{c} 2e^t + e^{4t} \\ -e^t + e^{4t} \\ -e^t + e^{4t} \end{array}\right]$$

Solution.
(a)

$$\left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]' = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]$$

(b) We have

$$\mathbf{y}' = \left[\begin{array}{c} 2e^t + 4e^{4t} \\ -e^t + 4e^{4t} \\ -e^t + 4e^{4t} \end{array}\right]$$

and

$$\mathbf{P}(t)\mathbf{y} = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{array}\right]\left[\begin{array}{c} 2e^t + e^{4t} \\ -e^t + e^{4t} \\ -e^t + e^{4t} \end{array}\right] = \left[\begin{array}{c} 2e^t + 4e^{4t} \\ -e^t + 4e^{4t} \\ -e^t + 4e^{4t} \end{array}\right] = \mathbf{y}' \blacksquare$$

In Problems 35.4 - 35.7
(a) Verify the given functions are solutions of the homogeneous linear system.

(b) Compute the Wronskian of the solution set. On the basis of this calculation can you assert that the set of solutions forms a fundamental set?
(c) If the given solutions are shown in part(b) to form a fundamental set, state the general solution of the linear homogeneous system. Express the general solution as the product $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{\Psi}(t)$ is a square matrix whose columns are the solutions forming the fundamental set and $\mathbf{c}$ is a column vector of arbitrary constants.
(d) If the solutions are shown in part (b) to form a fundamental set, impose the given initial condition and find the unique solution of the initial value problem.

Problem 35.4

$$\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} 2e^{3t} - 4e^{-t} \\ 3e^{3t} - 10e^{-t} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} 4e^{3t} + 2e^{-t} \\ 6e^{3t} + 5e^{-t} \end{bmatrix}$$

Solution.
(a) We have

$$\mathbf{y}_1' = \begin{bmatrix} 6e^{3t} + 4e^{-t} \\ 9e^{3t} + 10e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\begin{bmatrix} 2e^{3t} - 4e^{-t} \\ 3e^{3t} - 10e^{-t} \end{bmatrix} = \begin{bmatrix} 6e^{3t} + 4e^{-t} \\ 9e^{3t} + 10e^{-t} \end{bmatrix} = \mathbf{y}_1'$$

Similarly,

$$\mathbf{y}_2' = \begin{bmatrix} 12e^{3t} - 2e^{-t} \\ 18e^{3t} - 5e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\begin{bmatrix} 4e^{3t} + 2e^{-t} \\ 6e^{3t} + 5e^{-t} \end{bmatrix} = \begin{bmatrix} 12e^{3t} - 2e^{-t} \\ 18e^{3t} - 5e^{-t} \end{bmatrix} = \mathbf{y}_2'$$

(b) The Wronskian is given by

$$W(t) = \begin{vmatrix} 2e^{3t} - 4e^{-t} & 4e^{3t} + 2e^{-t} \\ 3e^{3t} - 10e^{-t} & 6e^{3t} + 5e^{-t} \end{vmatrix} = 20e^{2t}$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(c) The general solution is

$$\mathbf{y}(t) = c_2\mathbf{y}_1 + c_2\mathbf{y}_2 = \begin{bmatrix} 2e^{3t} - 4e^{-t} & 4e^{3t} + 2e^{-t} \\ 3e^{3t} - 10e^{-t} & 6e^{3t} + 5e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} -2 & 6 \\ -7 & 11 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving this system we find $c_1 = -0.3,\ c_2 = -0.1$. Therefore the solution to the initial value problem is

$$\mathbf{y}(t) = -0.3\begin{bmatrix} 2e^{3t} - 4e^{-t} \\ 3e^{3t} - 10e^{-t} \end{bmatrix} - 0.1\begin{bmatrix} 4e^{3t} + 2e^{-t} \\ 6e^{3t} + 5e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + e^{-t} \\ -1.5e^{3t} + 2.5e^{-t} \end{bmatrix} \blacksquare$$

Problem 35.5

$$\mathbf{y}' = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{y}_1(t) = \begin{bmatrix} -5e^{-2t}\cos 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) \end{bmatrix},$$

$$\mathbf{y}_2(t) = \begin{bmatrix} -5e^{-2t}\sin 3t \\ e^{-2t}(3\cos 3t + \sin 3t) \end{bmatrix}$$

Solution.
(a) We have

$$\mathbf{y}_1' = \begin{bmatrix} 5e^{-2t}(2\cos 3t + 3\sin 3t) \\ -11e^{-2t}(\cos 3t + \sin 3t) \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -5e^{-2t}\cos 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) \end{bmatrix} = \begin{bmatrix} 5e^{-2t}(2\cos 3t + 3\sin 3t) \\ -11e^{-2t}(\cos 3t + \sin 3t) \end{bmatrix} = \mathbf{y}_1'$$

Similarly,

$$\mathbf{y}_2' = \begin{bmatrix} 5e^{-2t}(2\sin 3t - 3\cos 3t) \\ e^{-2t}(-3\cos 3t - 11\sin 3t) \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -5e^{-2t}\sin 3t \\ e^{-2t}(3\cos 3t + \sin 3t) \end{bmatrix} = \begin{bmatrix} 5e^{-2t}(2\sin 3t - 3\cos 3t) \\ e^{-2t}(-3\cos 3t - 11\sin 3t) \end{bmatrix} = \mathbf{y}_2'$$

(b) The Wronskian is given by

$$W(t) = \begin{vmatrix} -5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) & e^{-2t}(3\cos 3t - \sin 3t) \end{vmatrix} = -15e^{-4t}$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(c) The general solution is

$$\mathbf{y}(t) = c_2\mathbf{y}_1 + c_2\mathbf{y}_2 = \begin{bmatrix} -5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) & e^{-2t}(3\cos 3t - \sin 3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} -5 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Solving this system we find $c_1 = -1,\ c_2 = 1$. Therefore the solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} -5e^{-2t}\cos 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) \end{bmatrix} - \begin{bmatrix} -5e^{-2t}\sin 3t \\ e^{-2t}(3\cos 3t + \sin 3t) \end{bmatrix} = \begin{bmatrix} 5e^{-2t}(\cos 3t - \sin 3t) \\ e^{-2t}(2\cos 3t + 4\sin 3t) \end{bmatrix} \blacksquare$$

Problem 35.6

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \mathbf{y}(-1) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^{3t} \\ -2e^{3t} \end{bmatrix}$$

Solution.
(a) We have

$$\mathbf{y}_1' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{y}_1'$$

Similarly,

$$\mathbf{y}_2' = \begin{bmatrix} 3e^{3t} \\ -6e^{-3t} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} \\ -2e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ -6e^{-3t} \end{bmatrix} = \mathbf{y}_2'$$

(b) The Wronskian is given by

$$W(t) = \begin{vmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{vmatrix} = -3e^{-3t}$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(c) The general solution is

$$\mathbf{y}(t) = c_2\mathbf{y}_1 + c_2\mathbf{y}_2 = \begin{bmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} 1 & e^{-3} \\ 1 & -2e^{-3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Solving this system we find $c_1 = 0,\ c_2 = -2e^3$. Therefore the solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} -2e^{3(t+1)} \\ 4e^{3(t+1)} \end{bmatrix} \blacksquare$$

Problem 35.7

$$\mathbf{y}' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 0 \\ 2e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix}$$

$$\mathbf{y}_3(t) = \begin{bmatrix} 0 \\ 2e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$

Solution.
(a) We have

$$\mathbf{y}_1' = \begin{bmatrix} -2e^{-2t} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ 0 \\ 0 \end{bmatrix} = \mathbf{y}_1'$$

Similarly,

$$\mathbf{y}_2' = \begin{bmatrix} 0 \\ 2e^t(\cos 2t - 2\sin 2t) \\ -e^t(\sin 2t + 2\cos 2t) \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t(\cos 2t - 2\sin 2t) \\ -e^t(\sin 2t + 2\cos 2t) \end{bmatrix} = \mathbf{y}_2'$$

$$\mathbf{y}_3' = \begin{bmatrix} 0 \\ 2e^t(\sin 2t + 2\cos 2t) \\ e^t(\cos 2t - 2\sin 2t) \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t(\sin 2t + 2\cos 2t) \\ e^t(\cos 2t - 2\sin 2t) \end{bmatrix} = \mathbf{y}_3'$$

(b) The Wronskian is given by

$$W(t) = \begin{vmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t \cos 2t & 2e^t \sin 2t \\ 0 & -e^t \sin 2t & e^t \cos 2t \end{vmatrix} = 2.$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ forms a fundamental set of solutions.
(c) The general solution is

$$\mathbf{y}(t) = c_2\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t \cos 2t & 2e^t \sin 2t \\ 0 & -e^t \sin 2t & e^t \cos 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Solving this system using Cramer's rule we find $c_1 = 3,\ c_2 = 2,\ c_3 = -2$. Therefore the solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} 3e^{-2t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^t \cos 2t \\ -2e^t \sin 2t \end{bmatrix} - \begin{bmatrix} 0 \\ 4e^t \sin 2t \\ 2e^t \cos 2t \end{bmatrix} = \begin{bmatrix} 3e^{-2t} \\ 4e^t(\cos 2t - \sin 2t) \\ -2e^t(\sin 2t + \cos 2t) \end{bmatrix} \blacksquare$$

In Problems 35.8 - 35.9, the given functions are solutions of the homogeneous linear system.

(a) Compute the Wronskian of the solution set and verify the set is a fundamental set of solutions.
(b) Compute the trace of the coefficient matrix.
(c) Verify Abel's theorem by showing that, for the given point t_0, $W(t) = W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds}$.

Problem 35.8

$$\mathbf{y}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix} \mathbf{y}, \mathbf{y}_1(t) = \begin{bmatrix} 5e^{-t} \\ -7e^{-t} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}, \ t_0 = -1, \ -\infty < t < \infty$$

Solution.
(a) The Wronskian is

$$W(t) = \begin{vmatrix} 5e^{-t} & e^t \\ -7e^{-t} & -e^t \end{vmatrix} = 2.$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(b) $tr(\mathbf{P}(t)) = 6 - 6 = 0$.
(c) $W(t) = 2$ and $W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds} = 2e^{\int_{-1}^t 0ds} = 2$ ■

Problem 35.9

$$\mathbf{y}' = \begin{bmatrix} 1 & t \\ 0 & -t^{-1} \end{bmatrix} \mathbf{y}, \mathbf{y}_1(t) = \begin{bmatrix} -1 \\ t^{-1} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \ t_0 = -1, \ t \neq 0, \ 0 < t < \infty$$

Solution.
(a) The Wronskian is

$$W(t) = \begin{vmatrix} -1 & e^t \\ t^{-1} & 0 \end{vmatrix} = -t^{-1}e^t.$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(b) $tr(\mathbf{P}(t)) = 1 - t^{-1}$
(c) $W(t) = -t^{-1}e^t$ and $W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds} = -e \cdot e^{\int_1^t (1-s^{-1})ds} = -t^{-1}e^t$ ■

Problem 35.10
The functions

$$\mathbf{y}_1(t) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}$$

are known to be solutions of the homogeneous linear system $\mathbf{y}' = \mathbf{P}\mathbf{y}$, where $\mathbf{P}$ is a real 2×2 constant matrix.
(a) Verify the two solutions form a fundamental set of solutions.
(b) What is tr($\mathbf{P}$)?
(c) Show that $\mathbf{\Psi}(t)$ satisfies the homogeneous differential equation $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$, where

$$\mathbf{\Psi}(t) = [\mathbf{y}_1(t) \;\; \mathbf{y}_2(t)] = \left[\begin{array}{cc} 5 & 2e^{3t} \\ 1 & e^{3t} \end{array}\right]$$

(d) Use the observation of part (c) to determine the matrix $\mathbf{P}$.[Hint: Compute the matrix product $\mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t)$. It follows from part (a) that $\mathbf{\Psi}^{-1}(t)$ exists.] Are the results of parts (b) and (d) consistent?

Solution.
(a) The Wronskian is given by

$$W(t) = \left|\begin{array}{cc} 5 & 2e^{3t} \\ 1 & e^{3t} \end{array}\right| = 3e^{3t}$$

Since

$W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(b) Since $W'(t) - tr(\mathbf{P}(t))W(t) = 0$, $9e^{3t} - 3tr(\mathbf{P}(t))e^{3t} = 0$. Thus, $tr(\mathbf{P}(t)) = 3$.
(c) We have

$$\mathbf{\Psi}'(t) = [\mathbf{y}_1' \; \mathbf{y}_2'] = [\mathbf{P}(t)\mathbf{y}_1 \; \mathbf{P}(t)\mathbf{y}_2] = \mathbf{P}(t)[\mathbf{y}_1 \; \mathbf{y}_2] = \mathbf{P}(t)\mathbf{\Psi}(t)$$

(d) From part(c) we have

$$\begin{array}{rcl} \mathbf{P}(t) & = & \mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t) = \left[\begin{array}{cc} 0 & 6e^{3t} \\ 0 & 3e^{3t} \end{array}\right] \cdot \frac{1}{3e^{3t}} \left[\begin{array}{cc} e^{3t} & -2e^{3t} \\ -1 & 5 \end{array}\right] \\ & = & \left[\begin{array}{cc} -2 & 10 \\ -1 & 5 \end{array}\right] \end{array}$$

The results in parts (b) and (d) are consistent since $tr(\mathbf{P}(t)) = -2 + 5 = 3$ ■

Problem 35.11
The homogeneous linear system

$$\mathbf{y}' = \left[\begin{array}{cc} 3 & 1 \\ -2 & \alpha \end{array}\right]\mathbf{y}$$

has a fundamental set of solutions whose Wronskian is constant, $W(t) = 4, \ -\infty < t < \infty$. What is the value of α?

Solution.
We know that $W(t)$ satisfies the equation $W'(t) - tr(\mathbf{P}(t))W(t) = 0$. But $tr(\mathbf{P}(t)) = 3 + \alpha$. Thus, $W'(t) - (3 + \alpha)W(t) = 0$. Solving this equation we find $W(t) = W(0)e^{\int_0^t (3+\alpha)ds}$. Since $W(t) = W(0) = 4$, $e^{\int_0^t (3+\alpha)ds} = 1$. Evaluating the integral we find $e^{(3+\alpha)t} = 1$. This implies that $3 + \alpha = 0$ or $\alpha = -3$ ■

36 First Order Linear Systems: Fundamental Sets and Linear Independence

In Problems 36.1 - 36.4, determine whehter the given functions are linearly dependent or linearly independent on the interval $-\infty < t < \infty$.

Problem 36.1

$$\mathbf{f}_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \ \mathbf{f}_2(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$$

Solution.
Suppose

$$k_1 \begin{bmatrix} t \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $k_1 t + k_2 t^2 = 0$ and $k_1 + k_2 = 0$ for all t. In particular, for $t = -1$ we have $-k_1 + k_2 = 0$. But $k_1 + k_2 = 0$. These two equations imply that $k_1 = k_2 = 0$. Hence, $\{\mathbf{f}_1(t), \mathbf{f}_2(t)\}$ is a linearly independent set ■

Problem 36.2

$$\mathbf{f}_1(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}, \ \mathbf{f}_2(t) = \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}, \ \mathbf{f}_3(t) = \begin{bmatrix} \frac{e^t - e^{-t}}{2} \\ 0 \end{bmatrix}$$

Solution.
Note that

$$\frac{1}{2}\begin{bmatrix} e^t \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{e^t - e^{-t}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This shows that $\mathbf{f}_1(t), \mathbf{f}_2(t)$, and $\mathbf{f}_3(t)$ are linearly dependent ■

Problem 36.3

$$\mathbf{f}_1(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}, \ \mathbf{f}_2(t) = \begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}, \ \mathbf{f}_3(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution.
Note that

$$0\begin{bmatrix}1\\t\\0\end{bmatrix}+0\begin{bmatrix}0\\1\\t^2\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This shows that $\mathbf{f}_1(t), \mathbf{f}_2(t)$, and $\mathbf{f}_3(t)$ are linearly dependent ■

Problem 36.4

$$\mathbf{f}_1(t)=\begin{bmatrix}1\\\sin^2 t\\0\end{bmatrix},\ \mathbf{f}_2(t)=\begin{bmatrix}0\\2(1-\cos^2 t)\\-2\end{bmatrix},\ \mathbf{f}_3(t)=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Solution.
Note that

$$\begin{bmatrix}1\\\sin^2 t\\0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\2(1-\cos^2 t)\\-2\end{bmatrix}-\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This shows that $\mathbf{f}_1(t), \mathbf{f}_2(t)$, and $\mathbf{f}_3(t)$ are linearly dependent ■

Problem 36.5
Consider the functions

$$\mathbf{f}_1(t)=\begin{bmatrix}t^2\\0\end{bmatrix},\quad \mathbf{f}_2(t)=\begin{bmatrix}2t\\1\end{bmatrix}$$

(a) Let $\mathbf{\Psi}(t)=[\mathbf{f}_1(t)\ \mathbf{f}_2(t)]$. Determine $det(\mathbf{\Psi}(t))$.
(b) Is it possible that the given functions form a fundamental set of solutions for a linear system $\mathbf{y}'=\mathbf{P}(t)\mathbf{y}$ where $\mathbf{P}$(t) is continuous on a t-interval containing the point $t=0$? Explain.
(c) Determine a matrix $\mathbf{P}$(t) such that the given vector functions form a fundamental set of solutions for $\mathbf{y}'=\mathbf{P}(t)\mathbf{y}$. On what t-interval(s) is the coefficient matrix $\mathbf{P}$(t) continuous?(Hint: The matrix $\mathbf{\Psi}(t)$ must satisfy $\mathbf{\Psi}'(t)=\mathbf{P}(t)\mathbf{\Psi}(t)$ and $det(\mathbf{\Psi}(t))\neq 0$.)

Solution.
(a) We have

$$F(t)=\begin{vmatrix}t^2 & 2t\\0 & 1\end{vmatrix}=t^2.$$

(b) Since $F(0)=0$, the given functions do not form a fundamental set for a linear system $\mathbf{y}'=\mathbf{P}(t)\mathbf{y}$ on any $t-$interval containing 0.
(c) For $\mathbf{\Psi}(t)$ to be a fundamental matrix it must satisfy the differential equation $\mathbf{\Psi}'(t)=\mathbf{P}(t)\mathbf{\Psi}(t)$ and the condition $det(\mathbf{\Psi}(t))\neq 0$. But $det(\mathbf{\Psi}(t))=t^2$ and this is not zero on any interval not containing zero. Thus, our coefficient matrix $\mathbf{P}$(t) must be continuous on either $-\infty<t<0$ or $0<t<\infty$. Now, from the equation $\mathbf{\Psi}'(t)=\mathbf{P}(t)\mathbf{\Psi}(t)$ we can find $\mathbf{P}(t)=\mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t)$. That is,

$$\begin{aligned}\mathbf{P}(t) &= \mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t)=\tfrac{1}{t^2}\left[\begin{array}{cc}2t & 2\\ 0 & 0\end{array}\right]\left[\begin{array}{cc}1 & -2t\\ 0 & t^2\end{array}\right]\\ &= \left[\begin{array}{cc}2t^{-1} & -2\\ 0 & 0\end{array}\right]\ \blacksquare\end{aligned}$$

Problem 36.6
Let

$$\mathbf{y}'=\left[\begin{array}{ccc}1 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 2\end{array}\right]\mathbf{y},\ \mathbf{\Psi}(t)=\left[\begin{array}{ccc}e^t & e^{-t} & 4e^{2t}\\ 0 & -2e^{-t} & e^{2t}\\ 0 & 0 & 3e^{2t}\end{array}\right],\ \overline{\mathbf{\Psi}}(t)=\left[\begin{array}{ccc}e^t+e^{-t} & 4e^{2t} & e^t+4e^{2t}\\ -2e^{-t} & e^{2t} & e^{2t}\\ 0 & 3e^{2t} & 3e^{2t}\end{array}\right]$$

(a) Verify that the matrix $\mathbf{\Psi}(t)$ is a fundamental matrix of the given linear system.
(b) Determine a constant matrix $\mathbf{A}$ such that the given matrix $\overline{\mathbf{\Psi}}(t)$ can be represented as $\overline{\mathbf{\Psi}}(t)=\mathbf{\Psi}(t)\mathbf{A}$.
(c) Use your knowledge of the matrix $\mathbf{A}$ and assertion (b) of Theorem 36.4 to determine whether $\overline{\mathbf{\Psi}}(t)$ is also a fundamental matrix, or simply a solution matrix.

Solution.
(a) Since

$$\mathbf{\Psi}'(t)=\left[\begin{array}{ccc}e^t & -e^{-t} & 8e^{2t}\\ 0 & 2e^{-t} & 2e^{2t}\\ 0 & 0 & 6e^{2t}\end{array}\right]$$

and

$$\mathbf{P}(t)\mathbf{\Psi}(t)=\left[\begin{array}{ccc}1 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 2\end{array}\right]\left[\begin{array}{ccc}e^t & e^{-t} & 4e^{2t}\\ 0 & -2e^{-t} & e^{2t}\\ 0 & 0 & 3e^{2t}\end{array}\right]=\left[\begin{array}{ccc}e^t & -e^{-t} & 8e^{2t}\\ 0 & 2e^{-t} & 2e^{2t}\\ 0 & 0 & 6e^{2t}\end{array}\right]$$

Thus, $\mathbf{\Psi}$ is a solution matrix. To show that $\mathbf{\Psi}(t)$ is a fundamental matrix we need to verify that $det(\mathbf{\Psi}(t)) \neq 0$. Since $det(\mathbf{\Psi}(t)) = -6e^{2t} \neq 0$, $\mathbf{\Psi}(t)$ is a fundamental matrix.
(b) Note that

$$\overline{\mathbf{\Psi}}(t) = \left[\begin{array}{ccc} e^t + e^{-t} & 4e^{2t} & e^t + 4e^{2t} \\ -2e^{-t} & e^{2t} & e^{2t} \\ 0 & 3e^{2t} & 3e^{2t} \end{array}\right] = \left[\begin{array}{ccc} e^t & e^{-t} & 4e^{2t} \\ 0 & -2e^{-t} & e^{2t} \\ 0 & 0 & 3e^{2t} \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Thus,

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

(c) Since $det(\mathbf{A}) = 1$, $\overline{\mathbf{\Psi}}(t)$ is a fundamental matrix ■

Problem 36.7
Let

$$\mathbf{y}' = \left[\begin{array}{cc} 1 & 1 \\ 0 & -2 \end{array}\right]\mathbf{y}, \quad \mathbf{\Psi}(t) = \left[\begin{array}{cc} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{array}\right]$$

where the matrix $\mathbf{\Psi}(t)$ is a fundamental matrix of the given homogeneous linear system. Find a constant matrix $\mathbf{A}$ such that $\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$ with $\overline{\mathbf{\Psi}}(0) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Solution.
We need to find a matrix $\mathbf{A}$ such that $\overline{\mathbf{\Psi}}(0) = \mathbf{\Psi}(0)\mathbf{A}$ or $\mathbf{A} = \mathbf{\Psi}^{-1}(0)\overline{\mathbf{\Psi}}(0) = \mathbf{\Psi}^{-1}(0) = -\frac{1}{3}\left[\begin{array}{cc} -3 & -1 \\ 0 & 1 \end{array}\right]$ ■

37 Homogeneous Systems with Constant Coefficients

In Problems 37.1 - 37.3, a 2×2 matrix $\mathbf{P}$ and vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are given.
(a) Decide which, if any, of the given vectors is an eigenvector of $\mathbf{P}$, and determine the corresponding eigenvalue.
(b) For the eigenpair found in part (a), form a solution $\mathbf{y}_k(t)$, where $k = 1$ or $k = 2$, of the first order system $\mathbf{y}' = \mathbf{P}\mathbf{y}$.
(c) If two solution are found in part (b), do they form a fundamental set of solutions for $\mathbf{y}' = \mathbf{P}\mathbf{y}$.

Problem 37.1

$$\mathbf{P} = \left[\begin{array}{cc} 7 & -3 \\ 16 & -7 \end{array}\right], \ \mathbf{x}_1 = \left[\begin{array}{c} 3 \\ 8 \end{array}\right], \ \mathbf{x}_2 = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

Solution.
(a) We have

$$\mathbf{P}\mathbf{x}_1 = \left[\begin{array}{cc} 7 & -3 \\ 16 & -7 \end{array}\right]\left[\begin{array}{c} 3 \\ 8 \end{array}\right] = -1\mathbf{x}_1.$$

Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $r_1 = -1$. Similarly,

$$\mathbf{P}\mathbf{x}_2 = \left[\begin{array}{cc} 7 & -3 \\ 16 & -7 \end{array}\right]\left[\begin{array}{c} 1 \\ 2 \end{array}\right] = 1\mathbf{x}_2.$$

Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $r_1 = 1$.
(b) Solutions to the system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ are $\mathbf{y}_1(t) = e^{-t}\mathbf{x}_1$ and $\mathbf{y}_2(t) = e^t\mathbf{x}_2$.
(c) The Wronskian is

$$W(t) = \left|\begin{array}{cc} 3e^{-t} & e^t \\ 8e^{-t} & 2e^t \end{array}\right| = -2 \neq 0$$

so that the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions ■

Problem 37.2

$$\mathbf{P} = \left[\begin{array}{cc} -5 & 2 \\ -18 & 7 \end{array}\right], \ \mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 3 \end{array}\right], \ \mathbf{x}_2 = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

Solution.
(a) We have

$$\mathbf{P}\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\mathbf{x}_1.$$

Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $r_1 = 1$. Similarly,

$$\mathbf{P}\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$

Since the right-hand side cannot be a scalar multiple of $\mathbf{x}_2$, $\mathbf{x}_2$ is not an eigenvector of $\mathbf{P}$.
(b) Solution to the system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ is $\mathbf{y}_1(t) = e^t\mathbf{x}_1$.
(c) The Wronskian is not defined for this problem ■

Problem 37.3

$$\mathbf{P} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution.
(a) We have

$$\mathbf{P}\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4\mathbf{x}_1.$$

Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $r_1 = 4$. Similarly,

$$\mathbf{P}\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0\mathbf{x}_2.$$

Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $r_1 = 0$.
(b) Solutions to the system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ are $\mathbf{y}_1(t) = e^{4t}\mathbf{x}_1$ and $\mathbf{y}_2(t) = \mathbf{x}_2$.
(c) The Wronskian is

$$W(t) = \begin{vmatrix} e^{4t} & 1 \\ -2e^{4t} & 2 \end{vmatrix} = 4e^{4t} \neq 0$$

so that the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions ■

In Problems 37.4 - 37.6, an eigenvalue is given of the matrix $\mathbf{P}$. Determine a corresponding eigenvector.

Problem 37.4

$$\mathbf{P} = \begin{bmatrix} 5 & 3 \\ -4 & -3 \end{bmatrix}, \quad r = -1$$

Solution.
We have

$$(\mathbf{P}+\mathbf{I})\mathbf{x} = \begin{bmatrix} 6 & 3 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 6x_1 + 3x_2 \\ -4x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -2x_1$. Letting $x_1 = 1$ then $x_2 = -2$ and an eigenvector is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \blacksquare$$

Problem 37.5

$$\mathbf{P} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}, \quad r = -4$$

Solution.
We have

$$(\mathbf{P}+4\mathbf{I})\mathbf{x} = \begin{bmatrix} 5 & -7 & 3 \\ -1 & 3 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 5x_1 - 7x_2 + 3x_3 \\ -x_1 + 3x_2 + x_3 \\ 4x_1 - 4x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = 2x_2$ and $x_3 = -x_2$. Letting $x_2 = 1$ then $x_1 = 2$ and $x_3 = -1$. Thus, an eigenvector is

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \blacksquare$$

Problem 37.6

$$\mathbf{P} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix}, \quad r = 5$$

Solution.
We have

$$(\mathbf{P}-5\mathbf{I})\mathbf{x} = \begin{bmatrix} -4 & 3 & 1 \\ 2 & -4 & 2 \\ 4 & 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} -4x_1+3x_2+x_3 \\ 2x_1-4x_2+2x_3 \\ 4x_1+3x_2-7x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = x_2 = x_3$. Letting $x_3 = 1$ then $x_1 = 1$ and $x_2 = 1$. Thus, an eigenvector is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \blacksquare$$

In Problems 37.7 - 37.10, Find the eigenvalues of the matrix **P**.

Problem 37.7

$$\mathbf{P} = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} -5-r & 1 \\ 0 & 4-r \end{vmatrix} = (r+5)(r-4) = 0$$

Thus, the eigenvalues are $r = -5$ and $r = 4$ $\blacksquare$

Problem 37.8

$$\mathbf{P} = \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 3-r & -3 \\ -6 & 6-r \end{vmatrix} = r(r-9) = 0$$

Thus, the eigenvalues are $r = 0$ and $r = 9$ $\blacksquare$

Problem 37.9

$$\mathbf{P} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 5-r & 0 & 0 \\ 0 & 1-r & 3 \\ 0 & 2 & 2-r \end{vmatrix} = (5-r)(r+1)(r-4) = 0$$

Thus, the eigenvalues are $r = 5,\ r = 4$, and $r = -1$ ■

Problem 37.10

$$\mathbf{P} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 1-r & -7 & 3 \\ -1 & -1-r & 1 \\ 4 & -4 & -r \end{vmatrix} = -r(r-4)(r+4) = 0$$

Thus, the eigenvalues are $r = 0,\ r = 4$, and $r = -4$ ■

In Problems 37.11 - 37.13, the matrix $\mathbf{P}$ has distinct eigenvalues. Using Theorem 35.4 determine a fundamental set of solutions of the system $\mathbf{y}' = \mathbf{P}\mathbf{y}$.

Problem 37.11

$$\mathbf{P} = \begin{bmatrix} -0.09 & 0.02 \\ 0.04 & -0.07 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} -0.09-r & 0.02 \\ 0.04 & -0.07-r \end{vmatrix} = r^2 + 0.16r + 0.0055 = 0$$

Solving this quadratic equation we find $r = -0.11$ and $r = -0.05$. Now,

$$(\mathbf{P} + 0.11\mathbf{I})\mathbf{x} = \begin{bmatrix} 0.02 & 0.02 \\ 0.04 & 0.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 0.02x_1 + 0.02x_2 \\ 0.04x_1 + 0.04x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -x_2$. Letting $x_2 = 1$ then $x_1 = -1$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly,

$$(\mathbf{P} + 0.05\mathbf{I})\mathbf{x} = \begin{bmatrix} -0.04 & 0.02 \\ 0.04 & -0.02 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} -0.04x_1 + 0.02x_2 \\ 0.04x_1 - 0.02x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 = x_2$. Letting $x_1 = 1$ then $x_2 = 2$. Thus, an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^{-0.11t}\mathbf{x}_1,\ e^{-0.05t}\mathbf{x}_2\}$ ■

Problem 37.12

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 1-r & 2 & 0 \\ -4 & 7-r & 0 \\ 0 & 0 & 1-r \end{vmatrix} = (r-1)(r-3)(r-5) = 0$$

Solving this equation we find $r_1 = 1,\ r_2 = 3$, and $r_3 = 5$. Now,

$$(\mathbf{P}-\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 2 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 2x_2 \\ -4x_1+6x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = x_2 = 0$, and x_3 is arbitrary. Letting $x_3 = 1$, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly,

$$(\mathbf{P}-3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 2 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} -2x_1+2x_2 \\ -4x_1+4x_2 \\ -2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = x_2$ and $x_3 = 0$. Letting $x_1 = x_2 = 1$, an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(\mathbf{P}-5\mathbf{I})\mathbf{x} = \begin{bmatrix} -4 & 2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} -4x_1+2x_2 \\ -4x_1+2x_2 \\ -4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 = x_2$, and $x_3 = 0$. Letting $x_1 = 1$, an eigenvector is

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^t\mathbf{x}_1,\ e^{3t}\mathbf{x}_2,\ e^{5t}\mathbf{x}_3\}$ ■

Problem 37.13

$$\mathbf{P} = \begin{bmatrix} 3 & 1 & 0 \\ -8 & -6 & 2 \\ -9 & -9 & 4 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 3-r & 1 & 0 \\ -8 & -6-r & 2 \\ -9 & -9 & 4-r \end{vmatrix} = (r-1)(r-2)(r+2) = 0$$

Solving this equation we find $r_1 = -2,\ r_2 = 1$, and $r_3 = 2$. Now,

$$(\mathbf{P}+2\mathbf{I})\mathbf{x} = \begin{bmatrix} 5 & 1 & 0 \\ -8 & -4 & 2 \\ -9 & -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 5x_1 + x_2 \\ -8x_1 - 4x_2 + 2x_3 \\ -9x_1 - 9x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -5x_1$. Letting $x_1 = 1$, we find $x_2 = -5$ and $x_3 = -6$. An eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$$

Similarly,

$$(\mathbf{P}-\mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 1 & 0 \\ -8 & -7 & 2 \\ -9 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 2x_1 + x_2 \\ -8x_1 - 7x_2 + 2x_3 \\ -9x_1 - 9x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -2x_1$. Letting $x_1 = 1$, we find $x_2 = -2$ and $x_3 = -3$. An eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$(\mathbf{P}-2\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 \\ -8 & -8 & 2 \\ -9 & -9 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ -8x_1 - 8x_2 + 2x_3 \\ -9x_1 - 9x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $-x_1 = x_2$. Letting $x_1 = 1$, we find $x_2 = -1,\ x_3 = 0$. An eigenvector is

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^{-2t}\mathbf{x}_1,\ e^{t}\mathbf{x}_2,\ e^{2t}\mathbf{x}_3\}$ ■

Problem 37.14
Solve the following initial value problem.

$$\mathbf{y}' = \begin{bmatrix} 5 & 3 \\ -4 & -3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 5-r & 3 \\ -4 & -3-r \end{vmatrix} = (r+1)(r-3) = 0$$

Solving this quadratic equation we find $r_1 = -1$ and $r_2 = 3$. Now,

$$(\mathbf{P}+\mathbf{I})\mathbf{x} = \begin{bmatrix} 6 & 3 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} 6x_1+3x_2 \\ -4x_1-2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -2x_1$. Letting $x_1 = 1$ then $x_2 = -2$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly,

$$(\mathbf{P}-3\mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} 2x_1+3x_2 \\ -4x_1-6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 = -3x_2$. Letting $x_1 = 3$ then $x_2 = -2$. Thus, an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^{-t}\mathbf{x}_1,\ e^{3t}\mathbf{x}_2\}$. The general solution is then

$$\mathbf{y}(t) = c_1e^{-t}\mathbf{x}_1 + c_2e^{3t}\mathbf{x}_2.$$

Using the initial condtion we find $c_1e^{-1}+3c_2e^3 = 2$ and $-2c_1e^{-1}-2c_2e^3 = 0$. Solving this system we find $c_1 = -e$ and $c_2 = e^{-3}$. Hence, the unique solution is given by

$$\mathbf{y}(t) = -e^{1-t}\mathbf{x}_1 + e^{3(t-1)}\mathbf{x}_2 \blacksquare$$

Problem 37.15
Solve the following initial value problem.

$$\mathbf{y}' = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 4-r & 2 & 0 \\ 0 & 1-r & 3 \\ 0 & 0 & -2-r \end{vmatrix} = (r+2)(r-1)(r-4) = 0$$

Solving this equation we find $r_1 = -2,\ r_2 = 1$, and $r_3 = 4$. Now,

$$(\mathbf{P}+2\mathbf{I})\mathbf{x} = \begin{bmatrix} 6 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 6x_1 + 2x_2 \\ 3x_2 + 3x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -3x_1$. Letting $x_1 = 1$, we find $x_2 = -3$ and $x_3 = 3$. An eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Similarly,

$$(\mathbf{P}-\mathbf{I})\mathbf{x} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 3x_1 + 2x_2 \\ 3x_3 \\ -3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $3x_1 + 2x_2 = 0$ and $x_3 = 0$. Letting $x_1 = 2$, we find $x_2 = -3$. An eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

$$(\mathbf{P}-4\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 2x_2 \\ -3x_2 + 3x_3 \\ -6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_3 = x_2 = 0$ and x_1 arbitrary. Letting $x_1 = 1$, an eigenvector is

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^{-2t}\mathbf{x}_1,\ e^t\mathbf{x}_2,\ e^{4t}\mathbf{x}_3\}$. The general solution is

$$\mathbf{y}(t) = c_1e^{-2t}\mathbf{x}_1 + c_2e^t\mathbf{x}_2 + c_3e^{4t}\mathbf{x}_3.$$

Using the initial condition we find $c_1+2c_2+c_3 = -1,\ -3c_1-3c_2 = 0,\ 3c_1 = 3$. Solving this system we find $c_1 = 1,\ c_2 = -1,\ c_3 = 0$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = e^{-2t}\mathbf{x}_1 - e^t\mathbf{x}_2 + 0e^{4t}\mathbf{x}_3 \blacksquare$$

Problem 37.16
Find α so that the vector $\mathbf{x}$ is an eigenvector of $\mathbf{P}$. What is the corresponding eigenvalue?

$$\mathbf{P} = \begin{bmatrix} 2 & \alpha \\ 1 & -5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution.
We must have $\mathbf{Pu} = r\mathbf{u}$ for some value r. That is

$$\begin{bmatrix} 2 & \alpha \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = r\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Equating components we find $2 - \alpha = r$ and $1 + 5 = -r$. Solving we find $r = -6$ and $\alpha = 8$ $\blacksquare$

Problem 37.17
Find α and β so that the vector $\mathbf{x}$ is an eigenvector of $\mathbf{P}$ corresponding the eigenvalue $r = 1$.

$$\mathbf{P} = \begin{bmatrix} \alpha & \beta \\ 2\alpha & \beta \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution.
We must have $\mathbf{Pu} = \mathbf{u}$. That is

$$\begin{bmatrix} \alpha & \beta \\ 2\alpha & -\beta \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Equating components we find $-\alpha + \beta = -1$ and $2\alpha - \beta = 1$. Solving we find $\alpha = 0$ and $\beta = -1$ $\blacksquare$

38 Homogeneous Systems with Constant Coefficients: Complex Eigenvalues

Problem 38.1
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} -r & -9 \\ 1 & -r \end{vmatrix} = r^2 + 9 = 0$$

Solving this quadratic equation we find $r_1 = -3i$ and $r_2 = 3i$. Now,

$$(\mathbf{P} + 3i\mathbf{I})\mathbf{x} = \begin{bmatrix} 3i & -9 \\ 1 & 3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 3ix_1 - 9x_2 \\ x_1 + 3ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -3ix_2$. Letting $x_2 = i$ then $x_1 = 3$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ i \end{bmatrix}$$

An eigenvector corresponding to the eigenvalue $3i$ is then

$$\mathbf{x}_2 = \begin{bmatrix} 3 \\ -i \end{bmatrix} \blacksquare$$

Problem 38.2
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 3-r & 1 \\ -2 & 1-r \end{vmatrix} = r^2 - 4r + 5 = 0$$

Solving this quadratic equation we find $r_1 = 2 - i$ and $r_2 = 2 + i$. Now,

$$(\mathbf{P} - (2-i)\mathbf{I})\mathbf{x} = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} (1+i)x_1 + x_2 \\ -2x_1 - (1-i)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $(1+i)x_1 = -x_2$. Letting $x_1 = 1-i$ then $x_2 = -2$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1-i \\ -2 \end{bmatrix}$$

An eigenvector corresponding to the eigenvalue $2 - i$ is then

$$\mathbf{x}_2 = \begin{bmatrix} 1+i \\ -2 \end{bmatrix} \blacksquare$$

Problem 38.3
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 1-r & -4 & -1 \\ 3 & 2-r & 3 \\ 1 & 1 & 3-r \end{vmatrix} = -r^3 + 6r^2 - 21r = 26 = 0$$

Using the rational root test one finds that $r = 2$ is a solution so that the characteristic equation is $(r-2)(r^2 - 4r + 13) = 0$. Solving this equation we find $r_1 = 2,\ r_2 = 2 - 3i$, and $r_3 = 2 + 3i$. Now,

$$(\mathbf{P} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} -x_1 - 4x_2 - x_3 \\ 3x_1 + 3x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -x_3$ and $x_2 = 0$. Letting $x_3 = -1$ then $x_1 = 1$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Next,

$$(\mathbf{P}-(2-3i)\mathbf{I})\mathbf{x} = \left[\begin{array}{ccc} -1+3i & -4 & -1 \\ 3 & 3i & 3 \\ 1 & 1 & 1+3i \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]\left[\begin{array}{c} (-1+3i)x_1 - 4x_2 - x_3 \\ 3x_1 + 3ix_2 + 3x_3 \\ x_1 + x_2 + (1+3i)x_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

Solving this system we find $3ix_1 = (4-i)x_2$ and $x_3 = (-1+3i)x_1 - 4x_2$. Letting $x_2 = 3i$ then $x_1 = 4-i$ and $x_3 = -1+i$. Thus, an eigenvector is

$$\mathbf{x}_2 = \left[\begin{array}{c} 4-i \\ 3i \\ -1+i \end{array}\right]$$

An eigenvector corresponding to the eigenvalue $2+3i$ is then

$$\mathbf{x}_3 = \left[\begin{array}{c} 4+i \\ -3i \\ -1-i \end{array}\right] \blacksquare$$

In Problems 38.4 - 38.6, one or more eigenvalues and corresponding eigenvectors are given for a real matrix $\mathbf{P}$. Determine a fundamental set of solutions for $\mathbf{y}' = \mathbf{P}\mathbf{y}$, where the fundamental set consists entirely of real solutions.

Problem 38.4
$\mathbf{P}$ is a 2×2 matrix with an eigenvalue $r = i$ and corresponding eigenvector

$$\mathbf{x} = \left[\begin{array}{c} -2+i \\ 5 \end{array}\right]$$

Solution.
From the given information, a solution to the system is given by

$$\begin{aligned} \mathbf{y}(t) &= e^{it}\left[\begin{array}{c} -2+i \\ 5 \end{array}\right] = \left[\begin{array}{c} (\cos t + i\sin t)(-2+i) \\ (\cos t + i\sin t)5 \end{array}\right] \\ &= \left[\begin{array}{c} (-2\cos t - \sin t) + (\cos t - 2\sin t)i \\ 5\cos t + 5i\sin t \end{array}\right] \\ &= \left[\begin{array}{c} -2\cos t - \sin t \\ 5\cos t \end{array}\right] + i\left[\begin{array}{c} \cos t - 2\sin t \\ 5\sin t \end{array}\right] \end{aligned}$$

Thus, a fundamental set of solution consists of the vectors

$$\mathbf{y}_1(t) = \left[\begin{array}{c} -2\cos t - \sin t \\ 5\cos t \end{array}\right], \quad \mathbf{y}_2(t) = \left[\begin{array}{c} \cos t - 2\sin t \\ 5\sin t \end{array}\right] \blacksquare$$

Problem 38.5
$\mathbf{P}$ is a 2×2 matrix with an eigenvalue $r = 1+i$ and corresponding eigenvector

$$\mathbf{x} = \left[\begin{array}{c} -1+i \\ i \end{array}\right]$$

Solution.
From the given information, a solution to the system is given by

$$\begin{aligned}
\mathbf{y}(t) &= e^{(1+i)t}\left[\begin{array}{c} -1+i \\ i \end{array}\right] = \left[\begin{array}{c} (e^t\cos t + ie^t\sin t)(-1+i) \\ (e^t\cos t + ie^t\sin t)i \end{array}\right] \\
&= \left[\begin{array}{c} (-e^t\cos t - e^t\sin t) + (e^t\cos t - e^t\sin t)i \\ -e^t\sin t + ie^t\cos t \end{array}\right] \\
&= \left[\begin{array}{c} -e^t\cos t - e^t\sin t \\ -e^t\sin t \end{array}\right] + i\left[\begin{array}{c} e^t\cos t - e^t\sin t \\ e^t\cos t \end{array}\right]
\end{aligned}$$

Thus, a fundamental set of solution consists of the vectors

$$\mathbf{y}_1(t) = \left[\begin{array}{c} -e^t\cos t - e^t\sin t \\ -e^t\sin t \end{array}\right], \quad \mathbf{y}_2(t) = \left[\begin{array}{c} e^t\cos t - e^t\sin t \\ e^t\cos t \end{array}\right] \blacksquare$$

Problem 38.6
$\mathbf{P}$ is a 4×4 matrix with eigenvalues $r = 1+5i$ with corresponding eigenvector

$$\mathbf{x} = \left[\begin{array}{c} i \\ 1 \\ 0 \\ 0 \end{array}\right]$$

and eigenvalue $r = 1+2i$ with corresponding eigenvector

$$\mathbf{x} = \left[\begin{array}{c} 0 \\ 0 \\ i \\ 1 \end{array}\right]$$

Solution.
From the given information, a solution to the system is given by

$$\begin{aligned}
\mathbf{y}(t) &= e^{(1+5i)t}\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (e^t\cos 5t + ie^t\sin 5t)i \\ (e^t\cos 5t + ie^t\sin 5t) \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (-e^t\sin 5t + ie^t\cos 5t) \\ e^t\cos 5t + e^t\sin 5t \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -e^t\sin 5t \\ e^t\cos 5t \\ 0 \\ 0 \end{bmatrix} + i\begin{bmatrix} e^t\cos 5t \\ e^t\sin 5t \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

This yields the two solutions

$$\mathbf{y}_1(t) = \begin{bmatrix} -e^t\sin 5t \\ e^t\cos 5t \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} e^t\cos 5t \\ e^t\sin 5t \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$\begin{aligned}
\mathbf{y}(t) &= e^{(1+2i)t}\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (e^t\cos 2t + ie^t\sin 2t)i \\ e^t\cos 2t + ie^t\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ -e^t\sin 2t + ie^t\cos 2t \\ e^t\cos 2t + ie^t\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ -e^t\sin 2t \\ e^t\cos 2t \end{bmatrix} + i\begin{bmatrix} 0 \\ 0 \\ e^t\cos 2t \\ e^t\sin 2t \end{bmatrix}
\end{aligned}$$

This yields the two solutions

$$\mathbf{y}_3(t) = \begin{bmatrix} 0 \\ 0 \\ -e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}, \quad \mathbf{y}_4(t) = \begin{bmatrix} 0 \\ 0 \\ e^t \cos 2t \\ e^t \sin 2t \end{bmatrix}$$

Thus, a fundamental set of solutions consists of the vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ ■

Problem 38.7
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Solution.
By Problem 36.1, an eigenvector corresponding to the eigenvalue $r = -3i$ is

$$\mathbf{x} = \begin{bmatrix} 3 \\ i \end{bmatrix}$$

Thus, a solution corresponding to this eigenvector is

$$\begin{aligned} \mathbf{y}(t) &= e^{-3it} \begin{bmatrix} 3 \\ i \end{bmatrix} = \begin{bmatrix} (\cos 3t - i \sin 3t)(3) \\ (\cos 3t - i \sin 3t)i \end{bmatrix} \\ &= \begin{bmatrix} 3\cos 3t \\ \sin 3t \end{bmatrix} + i \begin{bmatrix} -3\sin 3t \\ \cos 3t \end{bmatrix} \end{aligned}$$

This yields the two solutions

$$\mathbf{y}_1(t) = \begin{bmatrix} 3\cos 3t \\ \sin 3t \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} -3\sin 3t \\ \cos 3t \end{bmatrix}$$

The general solution is then given by

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \begin{bmatrix} 3c_1 \cos 3t - 3c_2 \sin 3t \\ c_1 \sin 3t + c_2 \cos 3t \end{bmatrix}$$

Using the initial condition we find $c_1 = 2$ and $c_2 = 2$. Hence, the unique solution is

$$\mathbf{y}(t) = \begin{bmatrix} 6\cos 3t - 6\sin 3t \\ 2\sin 3t + 2\cos 3t \end{bmatrix}$$ ■

Problem 38.8
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Solution.
By Problem 36.2, an eigenvector corresponding to the eigenvalue $r = 2 - i$ is

$$\mathbf{x} = \begin{bmatrix} 1-i \\ -2 \end{bmatrix}$$

Thus, a solution corresponding to this eigenvector is

$$\begin{aligned} \mathbf{y}(t) &= e^{(2-i)t}\begin{bmatrix} 1-i \\ -2 \end{bmatrix} = \begin{bmatrix} e^{2t}(\cos t - i\sin t)(1-i) \\ e^{2t}(\cos t - i\sin t)(-2) \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}(\cos t - \sin t \\ -2e^{2t}\cos t \end{bmatrix} + i\begin{bmatrix} -e^{2t}(\cos t + \sin t) \\ 2e^{2t}\sin t \end{bmatrix} \end{aligned}$$

This yields the two solutions

$$\mathbf{y}_1(t) = \begin{bmatrix} e^{2t}(\cos t - \sin t \\ -2e^{2t}\cos t \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} -e^{2t}(\cos t + \sin t) \\ 2e^{2t}\sin t \end{bmatrix}$$

The general solution is then given by

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \begin{bmatrix} c_1e^{2t}(\cos t - \sin t) - c_2e^{2t}(\cos t + \sin t) \\ -2c_1e^{2t}\cos t + 2c_2e^{2t}\sin t \end{bmatrix}$$

Using the initial condition we find $c_1 = -3$ and $c_2 = -11$. Hence, the unique solution is

$$\mathbf{y}(t) = e^{2t}\begin{bmatrix} 8\cos t + 14\sin t \\ 6\cos t - 22\sin t \end{bmatrix} \blacksquare$$

Problem 38.9
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 9 \\ 4 \end{bmatrix}$$

Solution.
By Problem 36.3, an eigenvector corresponding to the eigenvalue $r = 2$ is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Thus, a solution corresponding to this eigenvector is

$$\mathbf{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \\ -e^{2t} \end{bmatrix}$$

An eigenvector corresponding to the eigenvalue $r = 2 - 3i$ is

$$\mathbf{x}_2 = \begin{bmatrix} 4-i \\ 3i \\ -1+i \end{bmatrix}$$

Thus, a solution corresponding to this eigenvector is

$$\begin{aligned}\mathbf{y}(t) &= e^{(2-3i)t}\begin{bmatrix} 4-i \\ 3i \\ -1+i \end{bmatrix} = \begin{bmatrix} e^{2t}(\cos 3t - i\sin 3t)(4-i) \\ e^{2t}(\cos 3t - i\sin 3t)(3i) \\ e^{2t}(\cos 3t - i\sin 3t)(-1+i) \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}(4\cos 3t - \sin 3t \\ 3e^{2t}\sin 3t \\ e^{2t}(-\cos 3t + \sin 3t \end{bmatrix} + i\begin{bmatrix} -e^{2t}(\cos 3t + 4\sin 3t) \\ 3e^{2t}\cos 3t \\ e^{2t}(\cos 3t + \sin 3t) \end{bmatrix}\end{aligned}$$

This yields the two solutions

$$\mathbf{y}_2(t) = \begin{bmatrix} e^{2t}(4\cos 3t - \sin 3t \\ 3e^{2t}\sin 3t \\ e^{2t}(-\cos 3t + \sin 3t) \end{bmatrix}, \quad \mathbf{y}_3(t) = \begin{bmatrix} -e^{2t}(\cos 3t + 4\sin 3t) \\ 3e^{2t}\cos 3t \\ e^{2t}(\cos 3t + \sin 3t) \end{bmatrix}$$

The general solution is then given by

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t) = \begin{bmatrix} c_1e^{2t} + c_2e^{2t}(4\cos 3t - \sin 3t) - c_3e^{2t}(\cos 3t + 4\sin 3t) \\ 3c_2e^{2t}\sin 3t + 3c_3e^{2t}\cos 3t \\ -c_1e^{2t} + c_2e^{2t}(-\cos 3t + \sin 3t) + c_3e^{2t}(\cos 3t + \sin 3t) \end{bmatrix}$$

Using the initial condition we find $c_1 = -2$, $c_2 = 1$, and $c_3 = 3$. Thus, the unique solution is

$$\mathbf{y}(t) = e^{2t}\begin{bmatrix} -2 + \cos 3t - 13\sin 3t \\ 3\sin 3t + 9\cos 3t \\ 2 + 2\cos 3t + 4\sin 3t \end{bmatrix} \blacksquare$$

39 Homogeneous Systems with Constant Coefficients: Repeated Eigenvalues

In Problems 39.1 - 39.4, we consider the initial value problem $\mathbf{y}' = \mathbf{P}\mathbf{y},\ \mathbf{y}(0) = \mathbf{y}_0$.
(a) Compute the eigenvalues and the eigenvectors of $\mathbf{P}$.
(b) Construct a fundamental set of solutions for the given differential equation. Use this fundamental set to construct a fundamental matrix $\mathbf{\Psi}(t)$.
(c) Impose the initial condition to obtain the unique solution to the initial value problem.

Problem 39.1

$$\mathbf{P} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} 3-r & 2 \\ 0 & 3-r \end{vmatrix} = (r-3)^2 = 0$$

and has a repeated root $r = 3$. We find an eigenvector as follows.

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_2 = 0$ and x_1 is arbitrary. Letting $x_1 = 1$ then an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) The above eigenvector yields the solution

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}$$

But we need two linearly independent solutions to form the general solution of the given system and we only have one. We look for a solution of the form

$$\mathbf{y}_2(t) = e^{3t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + te^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where

$$(\mathbf{P}-3\mathbf{I})\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = \frac{1}{2}$ and x_1 arbitrary. Let $x_1 = 0$ then a second solution is

$$\mathbf{y}_2(t) = e^{3t}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + te^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & \frac{e^{3t}}{2} \end{bmatrix}$$

(c) Since $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, $\mathbf{\Psi}(0)\mathbf{c} = \mathbf{y}_0$ or

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solving this system we find $c_1 = 4$ and $c_2 = 2$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = e^{3t}\begin{bmatrix} 2t+4 \\ 1 \end{bmatrix} \blacksquare$$

Problem 39.2

$$\mathbf{P} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} 3-r & 0 \\ 1 & 3-r \end{vmatrix} = (r-3)^2 = 0$$

and has a repeated root $r = 3$. We find an eigenvector as follows.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = 0$ and x_2 is arbitrary. Letting $x_2 = 1$ then an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(b) The above eigenvector yields the solution

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$$

But we need two linearly independent solutions to form the general solution of the given system and we only have one. We look for a solution of the form

$$\mathbf{y}_2(t) = e^{3t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + te^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where

$$\begin{aligned} (\mathbf{P} - 3\mathbf{I}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Solving this system we find $x_1 = 1$ and x_2 arbitrary. Let $x_2 = 0$ then a second solution is

$$\mathbf{y}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{bmatrix} 0 & e^{3t} \\ e^{3t} & te^{3t} \end{bmatrix}$$

(c) Since $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, $\mathbf{\Psi}(0)\mathbf{c} = \mathbf{y}_0$ or

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solving this system we find $c_1 = -3$ and $c_2 = 2$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = e^{3t} \begin{bmatrix} 2 \\ 2t - 3 \end{bmatrix} \blacksquare$$

Problem 39.3

$$\mathbf{P} = \begin{bmatrix} -3 & -36 \\ 1 & 9 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} -3-r & -36 \\ 1 & 9-r \end{vmatrix} = (r-3)^2 = 0$$

and has a repeated root $r = 3$. We find an eigenvector as follows.

$$\begin{bmatrix} -6 & -36 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6x_1 - 36x_2 \\ x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = -6x_2$. Letting $x_1 = 6$ then an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

(b) The above eigenvector yields the solution

$$\mathbf{y}_1 = \begin{bmatrix} 6e^{3t} \\ -e^{3t} \end{bmatrix}$$

But we need two linearly independent solutions to form the general solution of the given system and we only have one. We look for a solution of the form

$$\mathbf{y}_2(t) = e^{3t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + te^{t} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

where

$$\begin{aligned} (\mathbf{P} - 3\mathbf{I}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -6 & -36 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -6x_1 - 36x_2 & x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \end{aligned}$$

Solving this system we find $x_1 + 6x_2 = -1$. Let $x_2 = 0$ so that $x_1 = -1$. Then a second solution is

$$\mathbf{y}_2(t) = e^{3t} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + te^{t} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

A fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{bmatrix} 6e^{3t} & (6t-1)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix}$$

(c) Since $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, $\mathbf{\Psi}(0)\mathbf{c} = \mathbf{y}_0$ or

$$\begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Solving this system we find $c_1 = -2$ and $c_2 = -12$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = e^{3t} \begin{bmatrix} -72t \\ 12t+2 \end{bmatrix} \blacksquare$$

Problem 39.4

$$\mathbf{P} = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} 6-r & 1 \\ -1 & 4-r \end{vmatrix} = (r-5)^2 = 0$$

and has a repeated root $r = 5$. We find an eigenvector as follows.

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_2 = -x_1$. Letting $x_1 = 1$ then $x_2 = -1$. An eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) The above eigenvector yields the solution

$$\mathbf{y}_1 = \begin{bmatrix} e^{5t} \\ -e^{5t} \end{bmatrix}$$

But we need two linearly independent solutions to form the general solution of the given system and we only have one. We look for a solution of the form

$$\mathbf{y}_2(t) = e^{5t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + te^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where

$$\begin{aligned}(\mathbf{P}-5\mathbf{I})\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

Solving this system we find $x_1 + x_2 = 1$. Let $x_2 = 0$ then a second solution is

$$\mathbf{y}_2(t) = e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{bmatrix} e^{5t} & (t+1)e^{5t} \\ -e^{5t} & -te^{5t} \end{bmatrix}$$

(c) Since $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, $\mathbf{\Psi}(0)\mathbf{c} = \mathbf{y}_0$ or

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Solving this system we find $c_1 = 4$ and $c_2 = 0$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = e^{5t}\begin{bmatrix} 4 \\ -4 \end{bmatrix} \blacksquare$$

Problem 39.5
Consider the homogeneous linear system

$$\mathbf{y}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{y}$$

(a) Write the three component differential equations of $\mathbf{y}' = \mathbf{P}\mathbf{y}$ and solve these equations sequentially, first finding $y_3(t)$, then $y_2(t)$, and then $y_1(t)$.
(b) Rewrite the component solutions obtained in part (a) as a single matrix equation of the form $\mathbf{y} = \mathbf{\Psi}(t)\mathbf{c}$. Show that $\mathbf{\Psi}(t)$ is a fundamental matrix.

Solution.
(a) We have

$$\begin{array}{rcl} y_1' & = & 2y_1 + y_2 \\ y_2' & = & 2y_2 + y_3 \\ y_3' & = & 2y_3 \end{array}$$

Solving the last equation we find $y_3(t) = c_3 e^{2t}$. Substituting this into the second equation we find

$$y_2' - 2y_2 = c_3 e^{2t}.$$

Solving this equation using the method of integrating factor we find

$$y_2(t) = c_3 t e^{2t} + c_2 e^{2t}.$$

Substituting this into the first equation we find

$$y_1' - 2y_1 = c_3 t e^{2t} + c_2 e^{2t}.$$

Solving this equation we find

$$y_1(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t}.$$

(b)

$$\mathbf{y}(t) = \left[\begin{array}{ccc} e^{2t} & te^{2t} & t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right]$$

Since

$$W(0) = det(\mathbf{\Psi}(0)) = \left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| = 1$$

we have $\mathbf{\Psi}(t)$ is a fundamental matrix ■

In Problems 39.6 - 39.8, Find the eigenvalues and eigenvectors of **P**. Give the geometric and algebraic multiplicity of each eigenvalue. Does **P** have a full set of eigenvectors?

Problem 39.6

$$\mathbf{P} = \left[\begin{array}{ccc} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 5 \end{array}\right]$$

Solution.
The characteristic equation is

$$\left|\begin{array}{ccc} 5-r & 0 & 0 \\ 1 & 5-r & 0 \\ 1 & 0 & 5-r \end{array}\right| = (5-r)^3 = 0$$

and has a repeated root $r = 5$. We find an eigenvector as follows.

$$\left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ x_1 \\ x_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

It follows that $x_1 = 0$, x_2 and x_3 are arbitrary. Thus,

$$\mathbf{x} = x_2\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] + x_3\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

so that two linearly independent eigenvectors are

$$\mathbf{x}_1 = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], \quad \mathbf{x}_2\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

Hence, $r = 5$ has algebraic multiplicity 3 and geometric multiplicity 2. Hence, $\mathbf{P}$ is defective ■

Problem 39.7

$$\mathbf{P} = \left[\begin{array}{ccc} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{array}\right]$$

Solution.
The characteristic equation is

$$\left|\begin{array}{ccc} 5-r & 0 & 0 \\ 0 & 5-r & 0 \\ 0 & 0 & 5-r \end{array}\right| = (5-r)^3 = 0$$

and has a repeated root $r = 5$. We find an eigenvector as follows.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It follows that x_1, x_2 and x_3 are arbitrary. Thus,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so that the three linearly independent eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $r = 5$ has algebraic multiplicity 3 and geometric multiplicity 3. Hence, $\mathbf{P}$ has a full set of eigenvectors ■

Problem 39.8

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 2-r & 0 & 0 & 0 \\ 0 & 2-r & 0 & 0 \\ 0 & 0 & 2-r & 0 \\ 0 & 0 & 1 & 2-r \end{vmatrix} = (2-r)^4 = 0$$

and has a repeated root $r = 2$. We find an eigenvector as follows.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It follows that $x_3 = 0$, x_1, x_2 and x_4 are arbitrary. Thus,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so that the three linearly independent eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $r = 2$ has algebraic multiplicity 4 and geometric multiplicity 3. Hence, $\mathbf{P}$ defective ■

Problem 39.9
Let $\mathbf{P}$ be a 2×2 real matrix with an eigenvalue $r_1 = a + ib$ where $b \neq 0$. Can $\mathbf{P}$ have a repeated eigenvalue? Can $\mathbf{P}$ be defective?

Solution.
$\mathbf{P}$ have the two distinct eigenvalues $r_1 = a + ib$ and $r_2 = a - ib$. Thus, $\mathbf{P}$ has a full set of eigenvectors ■

Problem 39.10
Dtermine the numbers x and y so that the following matrix is real and symmetric.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & x \\ y & 2 & 2 \\ 6 & 2 & 7 \end{bmatrix}$$

Solution.
Since $\mathbf{P}$ is a real symmetric matrix, $\mathbf{P}^T = \mathbf{P}$. That is,

$$\mathbf{P}^T = \begin{bmatrix} 0 & y & 6 \\ 1 & 2 & 2 \\ x & 2 & 7 \end{bmatrix} \mathbf{P} = \begin{bmatrix} 0 & 1 & x \\ y & 2 & 2 \\ 6 & 2 & 7 \end{bmatrix}$$

Equating entries we find $x = 6$ and $y = 1$ ■

Problem 39.11
Dtermine the numbers x and y so that the following matrix is Hermitian.

$$\mathbf{P} = \begin{bmatrix} 2 & x+3i & 7 \\ 9-3i & 5 & 2+yi \\ 7 & 2+5i & 3 \end{bmatrix}$$

Solution.
Since $\mathbf{P}$ is a Hermitian matrix, $\overline{\mathbf{P}}^T = \mathbf{P}$. That is,

$$\overline{\mathbf{P}}^T = \begin{bmatrix} 2 & 9+3i & 7 \\ x-3i & 5 & 2-5i \\ 7 & 2-yi & 3 \end{bmatrix} \mathbf{P} = \begin{bmatrix} 2 & x+3i & 7 \\ 9-3i & 5 & 2+yi \\ 7 & 2+5i & 3 \end{bmatrix}$$

Equating entries we find $x = 9$ and $y = -5$ ■

Problem 39.12
(a) Give an example of a 2×2 matrix $\mathbf{P}$ that is not invertible but have a full set of eigenvectors.
(b) Give an example of a 2×2 matrix $\mathbf{P}$ that is invertible but does not have a full set of eigenvectors.

Solution.
(a) Consider the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $det(\mathbf{P}) = 0$ so that $\mathbf{P}$ is not invertible. The characterisitc equation of this matrix is

$$\begin{vmatrix} 1-r & 0 \\ 0 & -r \end{vmatrix} = r(r-1) = 0$$

so that the eigenvalues are $r_1 = 0$ and $r_2 = 1$. Hence, the set $\mathbf{P}$ has a full set of eigenvectors.
(b) Consider the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then $det(\mathbf{P}) = 1$ so that $\mathbf{P}$ is invertible. The characterisitc equation of this matrix is

$$\begin{vmatrix} 1-r & 1 \\ 0 & 1-r \end{vmatrix} = (r-1)^2 = 0$$

so that the $r = 1$ is a repeated eigenvalue. An eigenvector is found as follows.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_2 = 0$ and x_1 is arbitrary. Thus, an eigenvector is given

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It follows that $\mathbf{P}$ is defective ■

40 NonHomogeneous First Order Linear Systems

In Problems 40.1 - 40.3, we consider the initial value problem $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(t_0) = \mathbf{y}_0$.
(a) Find the eigenpairs of the matrix $\mathbf{P}$ and form the general homogeneous solution of the differential equation.
(b) Construct a particular solution by assuming a solution of the form suggested and solving for the undetermined constant vectors $\mathbf{a}$,$\mathbf{b}$, and $\mathbf{c}$.
(c) Form the general solution of the nonhomogeneous differential equation.
(d) Find the unique solution to the initial value problem.

Problem 40.1

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Try $\mathbf{y}_p(t) = \mathbf{a}$.

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = (r+3)(r+1) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = -3$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = x_1$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, for $r_2 = -3$ we have

$$(\mathbf{P}+3\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence,

$$\mathbf{y}_h(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) Inserting the suggested function $\mathbf{y}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ into the differential equation leads to $\mathbf{y}_p' = \mathbf{P}\mathbf{y} + \mathbf{g}(t)$ or

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $a_1 = a_2 = 1$. Thus, $\mathbf{y}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) The general solution is given by

$$\mathbf{y}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-3t} + 1 \\ c_1 e^{-t} - c_2 e^{-3t} + 1 \end{bmatrix}$$

(d) Imposing the initial condition we find $c_1 + c_2 + 1 = 3$ and $c_1 - c_2 + 1 = 1$. Solving this system we find $c_1 = c_2 = 1$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} e^{-t} + e^{-3t} + 1 \\ e^{-t} - e^{-3t} + 1 \end{bmatrix} \blacksquare$$

Problem 40.2

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} t \\ -1 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Try $\mathbf{y}_p(t) = t\mathbf{a} + \mathbf{b}$.

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} -r & 1 \\ 1 & -r \end{vmatrix} = (r-1)(r+1) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 1$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 1$ we have

$$(\mathbf{P}-\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = x_1$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence,

$$\mathbf{y}_h(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Inserting the suggested function $\mathbf{y}_p = \begin{bmatrix} ta_1 + b_1 \\ ta_2 + b_2 \end{bmatrix}$ into the differential equation leads to $\mathbf{y}_p' = \mathbf{P}\mathbf{y} + \mathbf{g}(t)$ or

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} ta_1 + b_1 \\ ta_2 + b_2 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Letting $b_1 = b_2 = 0$ we find $a_1 = 0$ and $a_2 = -1$. Thus, $\mathbf{y}_p = \begin{bmatrix} 0 \\ -t \end{bmatrix}$.

(c) The general solution is given by

$$\mathbf{y}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -t \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{t} \\ c_1 e^{-t} - c_2 e^{t} - t \end{bmatrix}$$

(d) Imposing the initial condition we find $c_1 + c_2 = 2$ and $c_1 - c_2 = -1$. Solving this system we find $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{3}{2}e^{t} \\ \frac{1}{2}e^{-t} - \frac{3}{2}e^{t} - t \end{bmatrix} \blacksquare$$

Problem 40.3

$$\mathbf{y}' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} \sin t \\ 0 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try $\mathbf{y}_p(t) = (\sin t)\mathbf{a} + (\cos t)\mathbf{b}$.

Solution.
(a) The characteristic equation is

$$\begin{vmatrix} -3-r & -2 \\ 4 & 3-r \end{vmatrix} = (r-1)(r+1) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 1$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 - 2x_2 \\ 4x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 1$ we have

$$(\mathbf{P}-\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -2x_1$. Letting $x_1 = 1$ we find $x_2 = -2$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Hence,

$$\mathbf{y}_h(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(b) Inserting the suggested function $\mathbf{y}_p = \begin{bmatrix} \sin ta_1 + \cos tb_1 \\ \sin ta_2 + \cos tb_2 \end{bmatrix}$ into the differential equation leads to $\mathbf{y}_p' = \mathbf{Py} + \mathbf{g}(t)$ or

$$\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} \sin ta_1 + \cos tb_1 \\ \sin ta_2 + \cos tb_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ 0 \end{bmatrix} = \begin{bmatrix} \cos ta_1 - \sin tb_1 \\ \cos ta_2 - \sin tb_2 \end{bmatrix}$$

Multiplying and equating coefficients we find

$$\begin{array}{rcl} a_1 & = & -2b_2 - 3b_1 \\ -b_1 & = & -3a_1 - 2a_2 + 1 \\ a_2 & = & 4b_1 + 3b_2 \\ -b_2 & = & 4a_1 + 3a_2 \end{array}$$

Solving this system we find $a_1 = \frac{3}{2}$, $a_2 = -2$, $b_1 = -\frac{1}{2}$, $b_2 = 0$. Thus, $\mathbf{y}_p = \begin{bmatrix} 3.5\sin t - 0.5\cos t \\ -2\sin t \end{bmatrix}$.

(c) The general solution is given by

$$\begin{array}{rcl} \mathbf{y}(t) & = & c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1.5\sin t - 0.5\cos t \\ -2\sin t \end{bmatrix} \\ & = & \begin{bmatrix} c_1 e^{-t} + c_2 e^{t} + 1.5\sin t - 0.5\cos t \\ -c_1 e^{-t} - 2c_2 e^{t} - 2\sin t \end{bmatrix} \end{array}$$

(d) Imposing the initial condition we find $c_1 + c_2 = 0.5$ and $-c_1 - 2c_2 = 0$. Solving this system we find $c_1 = 1$ and $c_2 = -0.5$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} e^{-t} - 0.5e^{t} + 1.5\sin t - 0.5\cos t \\ -e^{-t} + e^{t} - 2\sin t \end{bmatrix} \blacksquare$$

Problem 40.4
Consider the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}\left(\frac{\pi}{2}\right) = \mathbf{y}_0.$$

Suppose we know that

$$\mathbf{y}(t) = \begin{bmatrix} 1 + \sin 2t \\ e^{t} + \cos 2t \end{bmatrix}$$

is the unique solution. Determine $\mathbf{g}(\mathrm{t})$ and $\mathbf{y}_0$.

Solution.
We have

$$\mathbf{y}_0 = \mathbf{y}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 + \sin\pi \\ e^{\frac{\pi}{2}} + \cos\pi \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\frac{\pi}{2}} - 1 \end{bmatrix}$$

Since $\mathbf{y}$(t) is a solution, it satisfies the differential equation so that

$$\begin{aligned}\mathbf{g}(t) &= \begin{bmatrix} 2\cos 2t \\ e^t - 2\sin 2t \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1+\sin 2t \\ e^t + \cos 2t \end{bmatrix} \\ &= \begin{bmatrix} -2e^t \\ e^t + 2 \end{bmatrix} \blacksquare\end{aligned}$$

Problem 40.5
Consider the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & t \\ t^2 & 1 \end{bmatrix}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Suppose we know that

$$\mathbf{y}(t) = \begin{bmatrix} t+\alpha \\ t^2+\beta \end{bmatrix}$$

is the unique solution. Determine $\mathbf{g}$(t) and the constants α and β.

Solution.
We have

$$\begin{bmatrix} 1+\alpha \\ 1+\beta \end{bmatrix} = \mathbf{y}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus, $1+\alpha = 2$ and $1+\beta = -1$. Solving these two equations we find $\alpha = 1$ and $\beta = -2$. Now, inserting $\mathbf{y}$ into the differential equation we find

$$\begin{aligned}\mathbf{g}(t) &= \begin{bmatrix} 1 \\ 2t \end{bmatrix} - \begin{bmatrix} 1 & t \\ t^2 & 1 \end{bmatrix}\begin{bmatrix} t+1 \\ t^2-2 \end{bmatrix} \\ &= \begin{bmatrix} -t^3+t \\ -t^3-2t^2+2t+2 \end{bmatrix} \blacksquare\end{aligned}$$

Problem 40.6
Let $\mathbf{P}$(t) be a 2×2 matrix with continuous entries. Consider the differential equation $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$. Suppose that $\mathbf{y}_1(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$ is the solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} e^t \\ -1 \end{bmatrix}$ is the solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \begin{bmatrix} e^t \\ -1 \end{bmatrix}$. Determine $\mathbf{P}$(t). Hint: Form the matrix equation $[\mathbf{y}_1'\ \mathbf{y}_2'] = \mathbf{P}[\mathbf{y}_1\ \mathbf{y}_2] + [\mathbf{g}_1\ \mathbf{g}_2]$.

Solution.
Following the hint we can write

$$\begin{bmatrix} 0 & e^t \\ -e^{-t} & 0 \end{bmatrix} = \mathbf{P}(t) \begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix} + \begin{bmatrix} -2 & e^t \\ 0 & -1 \end{bmatrix}$$

or

$$\mathbf{P}(t) \begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix}$$

Solving for $\mathbf{P}(\mathrm{t})$ we find

$$\mathbf{P}(t) = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix} \begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix} (-0.5) \begin{bmatrix} -1 & -e^t \\ -e^{-t} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^t \\ 0 & -1 \end{bmatrix} \blacksquare$$

Problem 40.7
Consider the linear system $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ where $\mathbf{P}$ is a constant matrix and $\mathbf{b}$ is a constant vector. An **equilibrium solution, $\mathbf{y}(\mathrm{t})$**, is a constant solution of the differential equation.
(a) Show that $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ has a unique equilibrium solution when $\mathbf{P}$ is invertible.
(b) If the matrix $\mathbf{P}$ is not invertible, must the differential equation $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ possess an equilibrium solution? If an equilibrium solution does exist in this case, is it unique?

Solution.
(a) If $\mathbf{P}$ is invertible and $\mathbf{y}$ is a constant solution then we must have $\mathbf{P}\mathbf{y} + \mathbf{b} = \mathbf{0}$ or $\mathbf{y} = -\mathbf{A}^{-1}\mathbf{b}$. This is the only equilibrium solution.
(b) Since $\mathbf{P}\mathbf{y} + \mathbf{b} = \mathbf{0}$, $\mathbf{P}\mathbf{y} = -\mathbf{b}$. This system has a unique solution only when $\mathbf{P}$ is invertible. If $\mathbf{P}$ is not invertible then either this system has no solutions or infinitely many solutions. That is, either no equilibrium solution or infinitely many equilibrium solutions $\blacksquare$

Problem 40.8
Determine all the equilibrium solutions (if any).

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution.
Since $det(\mathbf{P}) = 1$, the coefficient matrix is invertible and so there is a unique equilibrium solution given by

$$\mathbf{y} = -\mathbf{P}^{-1}\mathbf{b} = -\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \blacksquare$$

Problem 40.9
Determine all the equilibrium solutions (if any).

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Solution.
Since $det(\mathbf{P}) = -1$, the coefficient matrix is invertible and so there is a unique equilibrium solution given by

$$\mathbf{y} = -\mathbf{P}^{-1}\mathbf{b} = -\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} \blacksquare$$

Consider the homogeneous linear system $\mathbf{y}' = \mathbf{P}\mathbf{y}$. Recall that any associated fundamental matrix satisfies the matrix differential equation $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$. In Problems 40.10 - 40.12, construct a fundamental matrix that solves the matrix initial value problem $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}, \quad \mathbf{\Psi}(t_0) = \mathbf{\Psi}_0$.

Problem 40.10

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{\Psi}, \quad \mathbf{\Psi}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.
We first find a fundamental matrix of the linear system $\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{y}$.
The characteristic equation is

$$\begin{vmatrix} 1-r & -1 \\ -1 & 1-r \end{vmatrix} = r(r-2) = 0$$

and has eigenvalues $r_1 = 0$ and $r_2 = 2$. We find an eigenvector corresponding to $r_1 = 0$ as follows.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = x_2$. Letting $x_1 = 1$ then $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

An eigenvector corresponding to $r_2 = 2$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving we find $x_1 = -x_2$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, a fundamental matrix is

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}.$$

But $\mathbf{\Psi} = \overline{\mathbf{\Psi}}\mathbf{C}$. Using the initial condition we find $\mathbf{I} = \mathbf{\Psi}(1) = \overline{\mathbf{\Psi}}(1)\mathbf{C}$ and therefore

$$\mathbf{C} = \overline{\mathbf{\Psi}}^{-1} = \begin{bmatrix} 1 & e^2 \\ 1 & -e^2 \end{bmatrix}^{-1} = 0.5 \begin{bmatrix} 1 & 1 \\ e^{-2} & -e^{-2} \end{bmatrix}.$$

Finally,

$$\begin{aligned} \mathbf{\Psi} &= \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix} (0.5) \begin{bmatrix} 1 & 1 \\ e^{-2} & -e^{-2} \end{bmatrix} \\ &= 0.5 \begin{bmatrix} 1 + e^{2(t-1)} & 1 - e^{2(t-1)} \\ 1 - e^{2(t-1)} & 1 + e^{2(t-1)} \end{bmatrix} \end{aligned}$$ ■

Problem 40.11

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{\Psi}, \quad \mathbf{\Psi}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Solution.
From the previous problem, a fundamental matrix is

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}.$$

But $\mathbf{\Psi} = \overline{\mathbf{\Psi}}\mathbf{C}$. Using the initial condition we find $\mathbf{\Psi}(0) = \overline{\mathbf{\Psi}}(0)\mathbf{C}$ and therefore

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 0.5 \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}.$$

Finally,

$$\begin{aligned} \mathbf{\Psi} &= \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix} (0.5) \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \\ &= 0.5 \begin{bmatrix} 3 - e^{2t} & 1 - e^{2t} \\ 3 + e^{2t} & 1 + e^{2t} \end{bmatrix} \end{aligned}$$

■

Problem 40.12

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{\Psi}, \quad \mathbf{\Psi}\left(\frac{\pi}{4}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.
We first find a fundamental matrix of the linear system $\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{y}$. The characteristic equation is

$$\begin{vmatrix} 1-r & 4 \\ -1 & 1-r \end{vmatrix} = r^2 - 2r + 5 = 0$$

and has eigenvalues $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. We find an eigenvector corresponding to $r_1 = 1 + 2i$ as follows.

$$\begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2ix_1 + 4x_2 \\ -x_1 - 2ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = -2ix_2$. Letting $x_2 = 1$ then $x_1 = -2i$ and an eigenvector is

$$\mathbf{x} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

Thus a solution to the system is

$$\begin{aligned}\mathbf{y}(t) &= e^t(\cos t + i\sin t)\begin{bmatrix} -2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^t\sin 2t \\ e^t\cos 2t \end{bmatrix} + i\begin{bmatrix} -2e^t\cos 2t \\ e^t\sin 2t \end{bmatrix}\end{aligned}$$

Thus, a fundamental matrix is

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} 2e^t\sin 2t & -2e^t\cos 2t \\ e^t\cos 2t & e^t\sin 2t \end{bmatrix}.$$

But $\mathbf{\Psi} = \overline{\mathbf{\Psi}}\mathbf{C}$. Using the initial condition we find $\mathbf{I} = \mathbf{\Psi}\left(\frac{\pi}{4}\right) = \overline{\mathbf{\Psi}}\left(\frac{\pi}{4}\right)\mathbf{C}$ and therefore

$$\mathbf{C} = \overline{\mathbf{\Psi}}^{-1} = \begin{bmatrix} 2e^{\frac{\pi}{4}} & 0 \\ 0 & e^{\frac{\pi}{4}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2}e^{\frac{-\pi}{4}} & 0 \\ 0 & e^{-\frac{\pi}{4}} \end{bmatrix}.$$

Finally,

$$\begin{aligned}\mathbf{\Psi} &= \begin{bmatrix} 2e^t\sin 2t & -2e^t\cos 2t \\ e^t\cos 2t & e^t\sin 2t \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{\frac{-\pi}{4}} & 0 \\ 0 & e^{-\frac{\pi}{4}} \end{bmatrix} \\ &= \begin{bmatrix} e^{t-\frac{\pi}{4}}\sin 2t & -2e^{t-\frac{\pi}{4}}\cos 2t \\ \frac{1}{2}e^{t-\frac{\pi}{4}}\cos 2t & e^{t-\frac{\pi}{4}}\sin 2t \end{bmatrix} \blacksquare\end{aligned}$$

In Problems 40.13 - 40.14, use the method of variation of parameters to solve the given initial value problem.

Problem 40.13

$$\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Solution.

We first find a fundamental matrix of the linear system $\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y}$. The characteristic equation is

$$\begin{vmatrix} 9-r & -4 \\ 15 & -7-r \end{vmatrix} = (r-3)(r+1) = 0$$

and has eigenvalues $r_1 = -1$ and $r_2 = 3$. We find an eigenvector corresponding to $r_1 = 0 - 1$ as follows.

$$\begin{bmatrix} 10 & -4 \\ 15 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 - 4x_2 \\ 15x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = \frac{2}{5}x_2$. Letting $x_2 = 5$ then $x_1 = 2$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

An eigenvector corresponding to $r_2 = 3$

$$\begin{bmatrix} 6 & -4 \\ 15 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - 4x_2 \\ 15x_1 - 10x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving we find $x_1 = \frac{2}{3}x_2$. Letting $x_2 = 3$ we find $x_1 = 2$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus, a fundamental matrix is

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} 2e^{-t} & 2e^{3t} \\ 5e^{-t} & 3e^{3t} \end{bmatrix}.$$

Therefore,

$$\overline{\mathbf{\Psi}}^{-1} = -0.25 \begin{bmatrix} -3e^{t} & 2e^{t} \\ 5e^{-3t} & -2e^{-3t} \end{bmatrix}.$$

But the variation of parameters formula is

$$\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{y}(0) + \mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds.$$

Thus,

$$\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds = 0.25\int_0^t \begin{bmatrix} -3e^{2s} \\ 5e^{-2s} \end{bmatrix} ds = -0.125 \begin{bmatrix} 3(e^{2t} - 1) \\ -5(e^{-2t} - 1) \end{bmatrix}$$

and

$$\mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds = \begin{bmatrix} 0.75e^{-t} - 2e^{t} + 1.25e^{3t} \\ 1.875e^{-t} - 3.75e^{t} + 1.875e^{3t} \end{bmatrix}.$$

Hence,

$$\begin{aligned}
\mathbf{y}(t) &= \left[\begin{array}{cc} 2e^{-t} & 2e^{3t} \\ 5e^{-t} & 3e^{3t} \end{array}\right](-0.25)\left[\begin{array}{cc} -3 & 2 \\ 5 & -2 \end{array}\right]\left[\begin{array}{c} 2 \\ 5 \end{array}\right] + \left[\begin{array}{c} 0.75e^{-t} - 2e^{t} + 1.25e^{3t} \\ 1.875e^{-t} - 3.75e^{t} + 1.875e^{3t} \end{array}\right] \\
&= \left[\begin{array}{c} 2.75e^{-t} - 2e^{t} + 1.25e^{3t} \\ 6.875e^{-t} - 3.75e^{t} + 1.875e^{3t} \end{array}\right] \blacksquare
\end{aligned}$$

Problem 40.14

$$\mathbf{y}' = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solution.
We first find a fundamental matrix of the linear system $\mathbf{y}' = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]\mathbf{y}$. The characteristic equation is

$$\left|\begin{array}{cc} 1-r & 1 \\ 0 & 1-r \end{array}\right| = (r-1)^2 = 0$$

and has repeated eigenvalue $r = 1$. We find an eigenvector corresponding to $r = 1$ as follows.

$$\left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} x_2 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

It follows that $x_2 = 0$ and x_1 arbitrary. Letting $x_1 = 1$ an eigenvector is

$$\mathbf{x} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

A corresponding solution is

$$\mathbf{y}_1(t) = \left[\begin{array}{c} e^{t} \\ 0 \end{array}\right].$$

A second solution has the form

$$\mathbf{y}_2(t) = te^{t}\left[\begin{array}{c} e^{t} \\ 0 \end{array}\right] + e^{t}\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$$

Inserting this into the differential equation and solving for a_1 and a_2 we find $a_1 = 0$ and $a_2 = 1$. Thus,

$$\mathbf{y}_2(t) = \left[\begin{array}{c} te^{t} \\ e^{t} \end{array}\right]$$

A fundamental matrix is

$$\overline{\mathbf{\Psi}} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

Therefore,

$$\overline{\mathbf{\Psi}}^{-1} = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

But the variation of parameters formula is

$$\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{y}(0) + \mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds.$$

Thus,

$$\begin{aligned}\mathbf{y}(t) &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int_0^t \begin{bmatrix} e^{-s}(1-s) \\ e^{-s} \end{bmatrix} ds \\ &= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} te^{-t} \\ 1-e^{-t} \end{bmatrix} = \begin{bmatrix} te^t \\ e^t - 1 \end{bmatrix} \blacksquare\end{aligned}$$

41 Solving First Order Linear Systems with Diagonalizable Constant Coefficients Matrix

In Problems 41.1 - 41.4, the given matrix is diagonalizable. Find matrices $\mathbf{T}$ and $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{P}\mathbf{T} = \mathbf{D}$.

Problem 41.1

$$\mathbf{P} = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 3-r & 4 \\ -2 & -3-r \end{vmatrix} = (r-1)(r+1) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 1$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + 4x_2 \\ -2x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 1$ we have

$$(\mathbf{P}-\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ -2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -2x_2$. Letting $x_1 = 2$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Therefore

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \blacksquare$$

Problem 41.2

$$\mathbf{P} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 2-r & 3 \\ 2 & 3-r \end{vmatrix} = r(r-5) = 0$$

Thus, the eigenvalues are $r_1 = 0$ and $r_2 = 5$. An eigenvector corresponding to $r_1 = 0$ is found as follows

$$(\mathbf{P} + 0\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 + 3x_2 = 0$. Letting $x_1 = 3$ we find $x_2 = -2$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Similarly, for $r_2 = 5$ we have

$$(\mathbf{P} - 5\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 3x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = x_2$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \blacksquare$$

Problem 41.3

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\left|\begin{array}{cc} 1-r & 2 \\ 2 & 1-r \end{array}\right| = (r+1)(r-3) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 3$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 2x_1+2x_2 \\ 2x_1+2x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$$

Similarly, for $r_2 = 3$ we have

$$(\mathbf{P}-3\mathbf{I})\mathbf{x}_2 = \left[\begin{array}{cc} -2 & 2 \\ 2 & -2 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} -2x_1+2x_2 \\ 2x_1-2x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $x_1 = x_2$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Therefore

$$\mathbf{D} = \left[\begin{array}{cc} -1 & 0 \\ 0 & 3 \end{array}\right], \quad \mathbf{T} = \left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right] \blacksquare$$

Problem 41.4

$$\mathbf{P} = \left[\begin{array}{cc} -2 & 2 \\ 0 & 3 \end{array}\right]$$

Solution.
The characteristic equation is

$$\left|\begin{array}{cc} -2-r & 2 \\ 0 & 3-r \end{array}\right| = (r+2)(r-3) = 0$$

Thus, the eigenvalues are $r_1 = -2$ and $r_2 = 3$. An eigenvector corresponding to $r_1 = -2$ is found as follows

$$(\mathbf{P} + 2\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = 0$. Letting $x_1 = 1$ an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, for $r_2 = 3$ we have

$$(\mathbf{P} - 3\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5x_1 + 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $5x_1 - 2x_2 = 0$. Letting $x_1 = 2$ we find $x_2 = 5$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Therefore

$$\mathbf{D} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \blacksquare$$

In Problems 41.5 - 41.6, you are given the characteristic polynomial for the matrix $\mathbf{P}$. Determine the geometric and algebraic multiplicities of each eigenvalue. If the matrix $\mathbf{P}$ is diagonalizable, find matrices $\mathbf{T}$ and $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{P}\mathbf{T} = \mathbf{D}$.

Problem 41.5

$$\mathbf{P} = \begin{bmatrix} 7 & -2 & 2 \\ 8 & -1 & 4 \\ -8 & 4 & -1 \end{bmatrix}, \quad p(r) = (r-3)^2(r+1).$$

Solution.

The algebraic multiplicity of $r_1 = -1$ is 1. An eigenvector corresponding to this eigenvalue is found as follows.

$$(\mathbf{P} + \mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 8 & -2 & 2 \\ 8 & 0 & 4 \\ -8 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8x_1 - 2x_2 + 2x_3 \\ 8x_1 + 4x_3 \\ -8x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = 2x_1$ and $x_3 = -2x_1$. Letting $x_1 = 1$ an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Hence, the geometric multiplicity of $r_1 = -1$ is 1.
The algebraic multiplicity of $r_2 = 3$ is 2. An eigenvector corresponding to this eigenvalue is found as follows.

$$(\mathbf{P} - 3\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 4 & -2 & 2 \\ 8 & -4 & 4 \\ -8 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 + 2x_3 \\ 8x_1 - 4x_2 + 4x_3 \\ -8x_1 + 4x_2 - 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 - x_2 + x_3 = 0$. Letting $x_1 = 1$ we find

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, two linearly independent eigenvectors are

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $r_2 = 3$ has geometric multiplicity 2. It follows that $\mathbf{P}$ is diagonalizable with

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ -2 & 1 & 0 \end{bmatrix} \blacksquare$$

Problem 41.6

$$\mathbf{P} = \begin{bmatrix} 5 & -1 & 1 \\ 14 & -3 & 6 \\ 5 & -2 & 5 \end{bmatrix}, \quad p(r) = (r-2)^2(r-3).$$

Solution.
The algebraic multiplicity of $r_1 = 3$ is 1. An eigenvector corresponding to this eigenvalue is found as follows.

$$(\mathbf{P} - 3\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 2 & -1 & 1 \\ 14 & -6 & 6 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ 14x_1 - 6x_2 + 6x_3 \\ 5x_1 - 2x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = 0$ and $x_3 = x_2$. Letting $x_2 = 1$ an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the geometric multiplicity of $r_1 = 3$ is 1.
The algebraic multiplicity of $r_2 = 2$ is 2. An eigenvector corresponding to this eigenvalue is found as follows.

$$(\mathbf{P} - 2\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 3 & -1 & 1 \\ 14 & -5 & 4 \\ 5 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 + x_3 \\ 14x_1 - 5x_2 + 6x_3 \\ 5x_1 - 2x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = 4x_1$ and $x_3 = x_1$. Letting $x_1 = 1$ we find

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Hence, $r_2 = 2$ has geometric multiplicity 1 so the matrix $\mathbf{P}$ is nondiagonalizable ■

Problem 41.7
At leat two (and possibly more) of the following four matrices are diagonalizable. You should be able to recognize two by inspection. Choose them and give a reason for your choice.

$$(a) \begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}, \ (b) \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}, \ (c) \begin{bmatrix} 3 & 0 \\ 3 & -4 \end{bmatrix}, \ (d) \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

Solution.
The matrix (b) is diagonalizable since it is a real symmetric matrix. The matrix (c) is diagonalizable since it is a lower triangular matrix with distinct eigenvalues 3 and 4 ■

Problem 41.8
Solve the following system by making the change of variables $\mathbf{y} = \mathbf{Tz}$.

$$\mathbf{y}' = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} - 2e^t \\ e^{-2t} + e^t \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} -4-r & -6 \\ 3 & 5-r \end{vmatrix} = (r+1)(r-2) = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 2$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 - 6x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -2x_2$. Letting $x_2 = -1$ we find $x_1 = 2$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 2$ we have

$$(\mathbf{P}-2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 6 & -6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - 6x_2 \\ 3x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -x_2$. Letting $x_2 = -1$ we find $x_1 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Thus,

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

Letting $\mathbf{y} = \mathbf{T}\mathbf{z}$ we obtain

$$\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -e^t \\ e^{2t} \end{bmatrix}$$

That is,

$$\begin{aligned} z_1' &= -z_1 - e^t \\ z_2' &= 2z_2 + e^{2t} \end{aligned}$$

Solving this system we find

$$\mathbf{z}(t) = \begin{bmatrix} -\frac{1}{2}e^t + c_1 e^{-t} \\ te^{2t} + c_2 e^{2t} \end{bmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = \mathbf{T}\mathbf{z}(t) = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}e^t + c_1 e^{-t} \\ te^{2t} + c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} & e^{2t} \\ -e^{-t} & -e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -e^t + te^{2t} \\ \frac{1}{2}e^t - te^{2t} \end{bmatrix} \blacksquare$$

Problem 41.9
Solve the following system by making the change of variables $\mathbf{y} = \mathbf{T}\mathbf{z}$.

$$\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 4t + 4 \\ -2t + 1 \end{bmatrix}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 3 - r & 2 \\ 1 & 4 - r \end{vmatrix} = (r - 5)(r - 2) = 0$$

Thus, the eigenvalues are $r_1 = 2$ and $r_2 = 5$. An eigenvector corresponding to $r_1 = 2$ is found as follows

$$(\mathbf{P} - 2\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -2x_2$. Letting $x_2 = -1$ we find $x_1 = 2$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 5$ we have

$$(\mathbf{P} - 5\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = x_2$. Letting $x_2 = 1$ we find $x_1 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Thus,

$$\mathbf{T}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

Letting $\mathbf{y} = \mathbf{Tz}$ we obtain

$$\mathbf{z}' = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 2t+1 \\ 2 \end{bmatrix}$$

That is,

$$\begin{aligned} z_1' &= 2z_1 + 2t + 1 \\ z_2' &= 5z_2 + 2 \end{aligned}$$

Solving this system we find

$$\mathbf{z}(t) = \begin{bmatrix} -t - 1 + c_1e^{2t} \\ -\frac{2}{5} + c_2e^{5t} \end{bmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = \mathbf{Tz}(t) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -t - 1 + c_1e^{2t} \\ -\frac{2}{5} + c_2e^{5t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{5t} \\ -e^{2t} & e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -2t - \frac{12}{5} \\ t + \frac{3}{5} \end{bmatrix}$$ ■

Problem 41.10
Solve the following system by making the change of variables $\mathbf{x} = \mathbf{Tz}$.

$$\mathbf{x}'' = \begin{bmatrix} 6 & 7 \\ -15 & -16 \end{bmatrix}\mathbf{x}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 6-r & 7 \\ -15 & -16-r \end{vmatrix} = (r+1)(r+9) = 0$$

Thus, the eigenvalues are $r_1 = -9$ and $r_2 = -1$. An eigenvector corresponding to $r_1 = -9$ is found as follows

$$(\mathbf{P} + 9\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 15 & 7 \\ -15 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15x_1 + 7x_2 \\ -15x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $15x_1 = -7x_2$. Letting $x_1 = 7$ we find $x_2 = -15$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 7 \\ -15 \end{bmatrix}$$

Similarly, for $r_2 = -1$ we have

$$(\mathbf{P} + 1\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 7 & 7 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 + 7x_2 \\ -15x_1 - 15x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -x_2$. Letting $x_2 = -1$ we find $x_1 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix}$$

Letting $\mathbf{x} = \mathbf{Tz}$ we obtain

$$\mathbf{z}'' + \begin{bmatrix} -9 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{z} = \mathbf{0}$$

That is,

$$\begin{aligned} z_1'' &= 9z_1 \\ z_2'' &= z_2 \end{aligned}$$

Solving we find

$$\mathbf{z}(t) = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{3t} \\ k_1 e^{-t} + k_2 e^t \end{bmatrix}.$$

The general solution is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{Tz} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t} + c_2 e^{3t} \\ k_1 e^{-t} + k_2 e^t \end{bmatrix} \\ &= \begin{bmatrix} 7(c_1 e^{-3t} + c_2 e^{3t}) + k_1 e^{-t} + k_2 e^t \\ -15(c_1 e^{-3t} + c_2 e^{3t}) - (k_1 e^{-t} + k_2 e^t) \end{bmatrix} \end{aligned}$$ ■

Problem 41.11
Solve the following system by making the change of variables $\mathbf{x} = \mathbf{Tz}$.

$$\mathbf{x}'' = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

Solution.
The characteristic equation is

$$\begin{vmatrix} 4-r & 2 \\ 2 & 1-r \end{vmatrix} = r(r-5) = 0$$

Thus, the eigenvalues are $r_1 = 0$ and $r_2 = 5$. An eigenvector corresponding to $r_1 = 0$ is found as follows

$$(\mathbf{P} - 0\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = -2x_1$. Letting $x_1 = 1$ we find $x_2 = -2$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly, for $r_2 = 5$ we have

$$(\mathbf{P} - 5\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = 2x_2$. Letting $x_2 = 1$ we find $x_1 = 2$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Letting $\mathbf{x} = \mathbf{T}\mathbf{z}$ we obtain

$$\mathbf{z}'' + \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{z} = \mathbf{0}$$

That is,

$$\begin{aligned} z_1'' &= 0 \\ z_2'' &= 5z_2 \end{aligned}$$

Solving we find

$$\mathbf{z}(t) = \begin{bmatrix} c_1 t + c_2 \\ k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t) \end{bmatrix}.$$

The general solution is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{T}\mathbf{z} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 t + c_2 \\ k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t) \end{bmatrix} \\ &= \begin{bmatrix} (c_1 t + c_2) + 2[k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t)] \\ -2(c_1 t + c_2) + [k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t)] \end{bmatrix}\end{aligned} \quad \blacksquare$$

42 Solving First Order Linear Systems Using Exponential Matrix

Problem 42.1
Find $e^{\mathbf{P}(t)}$ if $\mathbf{P} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Solution.
We find
$$\mathbf{P}^2 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -4\mathbf{I}.$$
Hence,
$$\mathbf{P}^4 = (-1)^2 4^2 \mathbf{I},\ \mathbf{P}^6 = (-1)^3 4^3 \mathbf{I},\ \mathbf{P}^{2n} = (-1)^n 4^n \mathbf{I}.$$
It follows that
$$\mathbf{P}^{2n+1} = \mathbf{P}^{2n}\mathbf{P} = (-1)^n 4^n \mathbf{P}.$$
Now we split the terms of the power series expansion of $e^{\mathbf{P}t}$ into even powers and odd powers of **P.** We get

$$\begin{array}{rcl} e^{\mathbf{P}t} & = & \sum_{n=0}^{\infty} \frac{(\mathbf{P}t)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\mathbf{P}t)^{2n+1}}{(2n+1)!} \\ & = & \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}(-1)^n 4^n \mathbf{I} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(-1)^n 4^n \mathbf{P} \\ & = & \mathbf{I}\sum_{n=0}^{\infty}(-1)^n \frac{(2t)^{2n}}{(2n)!} + \frac{\mathbf{P}}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2t)^{2n+1}}{(2n+1)!} \\ & = & \mathbf{I}\cos 2t + \frac{\mathbf{P}}{2}\sin 2t \\ & = & \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix} \blacksquare \end{array}$$

Problem 42.2
Consider the linear differential system
$$\mathbf{y}' = \mathbf{P}\mathbf{y}, \quad \mathbf{P} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$
(a) Calculate $e^{\mathbf{P}t}$. Hint: Every square matrix satisfies its characteristic equation.
(b) Use the result from part (a) to find two independent solutions of the differential system. Form the general solution.

Solution.
(a) Since the characteristic equation of $\mathbf{P}$ is $p(r) = (r+1)^2$, $(\mathbf{I}+\mathbf{P})^2 = 0$. But

$$\begin{aligned} e^{\mathbf{P}t} &= e^{(-\mathbf{I}+(\mathbf{I}+\mathbf{P}))t} \\ &= e^{-\mathbf{I}t}e^{(\mathbf{I}+\mathbf{P})t} \\ &= e^{-t}\mathbf{I}(\mathbf{I}+t(\mathbf{I}+\mathbf{P})+t^2\tfrac{(\mathbf{I}+\mathbf{P})^2}{2!}+\cdots) \\ &= e^{-t}(\mathbf{I}+t(\mathbf{I}+\mathbf{P})) \\ &= e^{-t}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\right\} \\ &= e^{-t}\begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \end{aligned}$$

(b) Since $\mathbf{y} = e^{\mathbf{P}t}\mathbf{y}(0)$ is the solution to $\mathbf{y}' = \mathbf{P}\mathbf{y},\ \mathbf{y}(0) = \mathbf{y}_0$, we generate two solutions with

$$\begin{aligned} \mathbf{y}_1 &= e^{\mathbf{P}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-t}\begin{bmatrix} 1+2t \\ -t \end{bmatrix} \\ \mathbf{y}_2 &= e^{\mathbf{P}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-t}\begin{bmatrix} 4t \\ 1-2t \end{bmatrix} \end{aligned}$$

Let $\mathbf{\Psi}(t) = [\mathbf{y}_1\ \mathbf{y}_2]$. Then $det(\mathbf{\Psi}(0)) = det(\mathbf{I}) = 1$ then $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions. Thus, the general solution is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ ■

Problem 42.3
Show that if

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

then

$$e^{\mathbf{D}} = \begin{bmatrix} e^{d_1} & 0 & 0 \\ 0 & e^{d_2} & 0 \\ 0 & 0 & e^{d_3} \end{bmatrix}$$

Solution.
One can easily show by induction on n that

$$\mathbf{D}^n = \begin{bmatrix} d_1^n & 0 & 0 \\ 0 & d_2^n & 0 \\ 0 & 0 & d_3^n \end{bmatrix}.$$

Thus,

$$\begin{aligned}
e^{\mathbf{D}} &= \sum_{n=0}^{\infty}\frac{\mathbf{D}^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} d_1^n & 0 & 0 \\ 0 & d_2^n & 0 \\ 0 & 0 & d_3^n \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=0}^{\infty}\frac{d_1^n}{n!} & 0 & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{d_2^n}{n!} & 0 \\ 0 & 0 & \sum_{n=0}^{\infty}\frac{d_3^n}{n!} \end{bmatrix} \\
&= \begin{bmatrix} e^{d_1} & 0 & 0 \\ 0 & e^{d_2} & 0 \\ 0 & 0 & e^{d_3} \end{bmatrix} \blacksquare
\end{aligned}$$

Problem 42.4
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \mathbf{y}_0$$

Solution.
The solution is given by

$$\mathbf{y}(t) = e^{\mathbf{P}t}\mathbf{y}_0 = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\mathbf{y}_0 \blacksquare$$

Problem 42.5
Show that if r is an eigenvalue of $\mathbf{P}$ then e^r is an eigenvalue of $e^{\mathbf{P}}$.

Solution.
Since r is an eigenvalue of $\mathbf{P}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Px} = r\mathbf{x}$. In this case,

$$\begin{aligned}
e^{\mathbf{P}}\mathbf{x} &= \left(\mathbf{I} + \mathbf{P} + \frac{\mathbf{P}^2}{2!} + \cdots\right)\mathbf{x} \\
&= \mathbf{x} + \mathbf{Px} + \frac{\mathbf{P}^2\mathbf{x}}{2!} + \cdots \\
&= \mathbf{x} + r\mathbf{x} + \frac{r^2}{2!}\mathbf{x} + \cdots \\
&= (1 + r + \frac{r^2}{2!} + \cdots)\mathbf{x} = e^r\mathbf{x} \blacksquare
\end{aligned}$$

Problem 42.6
Show that $det(e^{\mathbf{A}}) = e^{tr(\mathbf{A})}$. Hint: Recall that the determinant of a matrix is equal to the product of its eigenvalues and the trace is the sume of the eigenvalues. This follows from the expansion of the characteristic equation into a polynomial.

Solution.
Suppose r and v are two eigenvalues of $\mathbf{A}$. Then $tr(\mathbf{A}) = r + v$. Hence, $e^{r+v} = e^r \cdot e^v$. But e^r and e^v are eigenvalues of $e^{\mathbf{A}}$. It follows that $e^{tr(\mathbf{A})}$ is the product of the eigenvalues of $e^{\mathbf{A}}$. But $det(e^{\mathbf{A}})$ is the product of eigenvalues of $e^{\mathbf{A}}$ ■

Problem 42.7
Prove: For any invertible $n \times n$ matrix $\mathbf{P}$ and any $n \times n$ matrix $\mathbf{A}$

$$e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} = \mathbf{P}^{-1}e^{\mathbf{A}}\mathbf{P}$$

(Thus, if $\mathbf{A}$ is similar to $\mathbf{B}$; then $e^{\mathbf{A}}$ is similar to $e^{\mathbf{B}}$).

Solution.
One can easily show by induction on n that $(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}$. Thus,

$$\begin{array}{rcl} e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} & = & \sum_{n=0}^{\infty} \frac{(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n}{n!} \\ & = & \sum_{n=0}^{\infty} \frac{(\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P})}{n!} \\ & = & \mathbf{P}^{-1}\left(\sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}\right)\mathbf{P} \\ & = & \mathbf{P}^{-1}e^{\mathbf{A}}\mathbf{P} \ \blacksquare \end{array}$$

Problem 42.8
Prove: If $\mathbf{AB} = \mathbf{BA}$ then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.

Solution.
Since $\mathbf{A}$ and $\mathbf{B}$ commute, the binomial formula is valid. That is

$$(A+B)^n = \sum_{p+q=n} \frac{n!}{p!q!}\mathbf{A}^p\mathbf{B}^q.$$

Here the sum runs over non-negative integers p and q that sum to n. We really need commutativity here, in order to put the $\mathbf{A}$s on one side and $\mathbf{B}$ s on the other.
Now we can compute

$$\begin{array}{rcl} e^{\mathbf{A}+\mathbf{B}} & = & \sum_{n=0}^{\infty} \frac{(\mathbf{A}+\mathbf{B})^n}{n!} \\ & = & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p+q=n} \frac{n!}{p!q!}\mathbf{A}^p\mathbf{B}^q \\ & = & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\mathbf{A}^p\mathbf{B}^q}{p!q!} \\ & = & \left(\sum_{p=0}^{\infty} \frac{\mathbf{A}^p}{p!}\right)\left(\sum_{q=0}^{\infty} \frac{\mathbf{B}^q}{q!}\right) \\ & = & e^{\mathbf{A}}e^{\mathbf{B}} \ \blacksquare \end{array}$$

Problem 42.9
Prove: For any square matrix $\mathbf{A}$, $e^{\mathbf{A}}$ is invertible with $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.

Solution.
For any t and s we have $(t\mathbf{A})(s\mathbf{A}) = (s\mathbf{A})(t\mathbf{A})$. From the previous problem we can write $e^{t\mathbf{A}+s\mathbf{A}} = e^{t\mathbf{A}}e^{s\mathbf{A}}$. Now, let $t = 1$ and $s = -1$ to obtain $\mathbf{I} = e^{\mathbf{0}} = e^{\mathbf{A}}e^{-\mathbf{A}}$. This shows that $e^{\mathbf{A}}$ is invertible and $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$ ■

Problem 42.10
Consider the two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that $\mathbf{AB} \neq \mathbf{BA}$ and $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{A}}$.

Solution.
A simple calculation shows that

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{BA} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Since $\mathbf{A}$ is diagonal, we have

$$e^{\mathbf{A}} = \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}$$

A simple algebra one finds

$$e^{\mathbf{B}} = \begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix}$$

Since

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$e^{\mathbf{A}+\mathbf{B}} = \mathbf{I}$$

On the other hand,

$$e^{\mathbf{A}}e^{\mathbf{B}} = \begin{bmatrix} e\cos 1 & e\sin 1 \\ -e^{-1}\sin 1 & e^{-1}\cos 1 \end{bmatrix} ■$$

43 The Laplace Transform: Basic Definitions and Results

Problem 43.1
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2}dt$ converges. If the integral converges, give its value.

Solution.
We have

$$\begin{array}{rcl}\int_0^\infty \frac{1}{1+t^2}dt &=& \lim_{A\to\infty}\int_0^A \frac{1}{1+t^2}dt = \lim_{A\to\infty}[\arctan t]_0^A\\ &=& \lim_{A\to\infty}\arctan A = \frac{\pi}{2}\end{array}$$

So the integral is convergent ■

Problem 43.2
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2}dt$ converges. If the integral converges, give its value.

Solution.
We have

$$\begin{array}{rcl}\int_0^\infty \frac{t}{1+t^2}dt &=& \frac{1}{2}\lim_{A\to\infty}\int_0^A \frac{2t}{1+t^2}dt = \frac{1}{2}\lim_{A\to\infty}[\ln{(1+t^2)}]_0^A\\ &=& \frac{1}{2}\lim_{A\to\infty}\ln{(1+A^2)} = \infty\end{array}$$

Hence, the integral is divergent ■

Problem 43.3
Determine whether the integral $\int_0^\infty e^{-t}\cos{(e^{-t})}dt$ converges. If the integral converges, give its value.

Solution.
Using substitution we find

$$\begin{array}{rcl}\int_0^\infty e^{-t}\cos{(e^{-t})}dt &=& \lim_{A\to\infty}\int_1^{e^{-A}} -\cos u du\\ &=& \lim_{A\to\infty}[-\sin u]_1^{e^{-A}} = \lim_{A\to\infty}[\sin 1 - \sin{(e^{-A})}]\\ &=& \sin 1\end{array}$$

Hence, the integral is convergent ■

Problem 43.4
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have

$$\begin{array}{rcl}\mathcal{L}[e^{3t} & = & \lim_{A\to\infty}\int_0^A e^{3t}e^{-st}dt = \lim_{A\to\infty} e^{t(3-s)}dt\\ & = & \lim_{A\to\infty}\left[\frac{e^{t(3-s)}}{3-s}\right]_0^A\\ & = & \lim_{A\to\infty}\left[\frac{e^{A(3-s)}}{3-s}-\frac{1}{3-s}\right]\\ & = & \frac{1}{s-3},\ s>3\ \blacksquare\end{array}$$

Problem 43.5
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
Using integration by parts we find

$$\begin{array}{rcl}\mathcal{L}[t-5] & = & \lim_{A\to\infty}\int_0^A(t-5)e^{-st}dt = \lim_{A\to\infty}\left\{\left[\frac{-(t-5)e^{-st}}{s}\right]_0^A+\frac{1}{s}\int_0^A e^{-st}dt\right\}\\ & = & \lim_{A\to\infty}\left\{\frac{-(A-5)e^{-sA}+5}{s}-\left[\frac{e^{-st}}{s^2}\right]_0^A\right\}\\ & = & \frac{1}{s^2}-\frac{5}{s},\ s>0\ \blacksquare\end{array}$$

Problem 43.6
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have

$$\int_0^\infty e^{(t-1)^2}e^{-st}dt = \int_0^\infty e^{(t-1)^2-st}dt.$$

Since $\lim_{t\to\infty}(t-1)^2-st = \lim_{t\to\infty} t^2\left(1-\frac{(2+s)}{t}+\frac{1}{t^2}\right) =$, for any fixed s we can choose a positive C such that $(t-1)^2-st\geq 0$. In this case, $e^{(t-1)^2-st}\geq 1$ and this implies that $\int_0^\infty e^{(t-1)^2-st}dt \geq \int_C^\infty dt$. The integral on the right is divergent so that the integral on the left is also divergent by the comparison theorem of improper integrals. Hence, $f(t)=e^{(t-1)^2}$ does not have a Laplace transform $\blacksquare$

Problem 43.7
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have
$$\mathcal{L}[(t-2)^2] = \lim_{T\to\infty}(t-2)^2e^{-st}dt.$$
Using integration by parts with $u' = e^{-st}$ and $v = (t-2)^2$ we find
$$\int_0^T (t-2)^2e^{-st}dt = -\left[\frac{(t-2)^2e^{-st}}{s}\right]_0^T + \frac{2}{s}\int_0^T (t-2)e^{-st}dt = \frac{4}{s} - \frac{(T-2)^2e^{-sT}}{s} + \frac{2}{s}\int_0^T (t-2)e^{-st}dt.$$
Thus,
$$\lim_{T\to\infty}\int_0^T (t-2)^2e^{-st}dt = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt$$
Using by parts with $u' = e^{-st}$ and $v = t-2$ we find
$$\int_0^T (t-2)e^{-st}dt = \left[-\frac{(t-2)e^{-st}}{s} + \frac{1}{s^2}e^{-st}\right]_0^T.$$
Letting $T\to\infty$ in the above expression we find
$$\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt = -\frac{2}{s} + \frac{1}{s^2}, \quad s > 0.$$
Hence,
$$F(s) = \frac{4}{s} + \frac{2}{s}\left(-\frac{2}{s} + \frac{1}{s^2}\right) = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0 \blacksquare$$

Problem 43.8
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.
$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & t \ge 1 \end{cases}$$

Solution.
We have
$$\mathcal{L}[f(t)] = \lim_{T\to\infty}\int_1^T (t-1)e^{-st}dt.$$

Using integration by parts with $u' = e^{-st}$ and $v = t - 1$ we find

$$\lim_{T\to\infty}\int_1^T (t-1)e^{-st}dt = \lim_{T\to\infty}\left[-\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_1^T = \frac{e^{-s}}{s^2}, \quad s > 0 \blacksquare$$

Problem 43.9
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$

Solution.
We have

$$\mathcal{L}[f(t)] = \int_1^2 (t-1)e^{-st}dt = \left[-\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_1^2 = -\frac{e^{-2s}}{s} + \frac{1}{s^2}(e^{-s} - e^{-2s}), \quad s \ne 0 \blacksquare$$

Problem 43.10
Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st}dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0.$$

Solution.
Let $u' = e^{-st}$ and $v = t^n$. Then $u = -\frac{e^{-st}}{s}$ and $v' = nt^{n-1}$. Hence,

$$\int t^n e^{-st}dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0 \blacksquare$$

Problem 43.11
For $s > 0$ and n a positive integer evaluate the limits

$$\lim_{t\to 0} t^n e^{-st} \qquad \text{(b) } \lim_{t\to\infty} t^n e^{-st}$$

Solution.
(a) $\lim_{t\to 0} t^n e^{-st} = \lim_{t\to 0} \frac{t^n}{e^{st}} = \frac{0}{1} = 0.$
(b) Using L'Hôpital's rule repeatedly we find

$$\lim_{t\to\infty} t^n e^{-st} = \cdots = \lim_{t\to\infty} \frac{n!}{s^n e^{st}} = 0 \blacksquare$$

Problem 43.12
(a) Use the previous two problems to derive the reduction formula for the Laplace transform of $f(t) = t^n$,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0.$$

(b) Calculate $\mathcal{L}[t^k]$, for $k = 1, 2, 3, 4, 5$.
(c) Formulate a conjecture as to the Laplace transform of $f(t), t^n$ with n a positive integer.

Solution.
(a) Using the two previous problems we find

$$\begin{aligned}\mathcal{L}[t^n] &= \lim_{T\to\infty}\int 0^T t^n e^{-st}dt = \lim_{T\to\infty}\left\{-\left[\frac{t^n e^{-st}}{s}\right]_0^T + \frac{n}{s}\int_0 T t^{n-1}e^{-st}dt\right\}\\ &= \frac{n}{s}\lim_{T\to\infty}\int_0^T t^{n-1}e^{-st}dt = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0\end{aligned}$$

(b) We have

$$\begin{aligned}\mathcal{L}[t] &= \frac{1}{s^2}\\ \mathcal{L}[t^2] &= \frac{2}{s}\mathcal{L}[t] = \frac{2}{s^3}\\ \mathcal{L}[t^3] &= \frac{3}{s}\mathcal{L}[t^2] = \frac{6}{s^4}\\ \mathcal{L}[t^4] &= \frac{4}{s}\mathcal{L}[t^3] = \frac{24}{s^5}\\ \mathcal{L}[t^5] &= \frac{5}{s}\mathcal{L}[t^4] = \frac{120}{s^5}\end{aligned}$$

(c) By induction, one can easily shows that for $n = 0, 1, 2, \cdots$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0 \blacksquare$$

From a table of integrals,

$$\begin{aligned}\int e^{\alpha u}\sin\beta u du &= e^{\alpha u}\frac{\alpha\sin\beta u - \beta\sin\beta u}{\alpha^2+\beta^2}\\ \int e^{\alpha u}\cos\beta u du &= e^{\alpha u}\frac{\alpha\cos\beta u + \beta\sin\beta u}{\alpha^2+\beta^2}\end{aligned}$$

Problem 43.13
Use the above integrals to find the Laplace transform of $f(t) = \cos\omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\mathcal{L}[\cos \omega t] = \lim_{T\to\infty} -\left\{e^{-st}\left[\frac{-s\cos \omega t + \omega \sin \omega t}{s^2+\omega^2}\right]_0^T\right\} = \frac{s}{s^2+\omega^2}, \quad s > 0 \blacksquare$$

Problem 43.14
Use the above integrals to find the Laplace transform of $f(t) = \sin \omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\mathcal{L}[\sin \omega t] = \lim_{T\to\infty} -\left\{e^{-st}\left[\frac{-s\sin \omega t + \omega \cos \omega t}{s^2+\omega^2}\right]_0^T\right\} = \frac{\omega}{s^2+\omega^2}, \quad s > 0 \blacksquare$$

Problem 43.15
Use the above integrals to find the Laplace transform of $f(t) = \cos \omega(t-2)$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
Using a trigonometric identity we can write $f(t) = \cos \omega(t-2) = \cos \omega t \cos 2\omega + \sin \omega t \sin 2\omega$. Thus, using the previous two problems we find

$$\mathcal{L}[\cos \omega(t-2)] = \frac{s\cos 2\omega + \omega \sin 2\omega}{s^2+\omega^2}, \quad s > 0 \blacksquare$$

Problem 43.16
Use the above integrals to find the Laplace transform of $f(t) = e^{3t}\sin t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\begin{array}{lll} \mathcal{L}[e^{3t}\sin t] & = & \lim_{T\to\infty}\int_0^T e^{-(s-3)t}\sin t dt \\ & = & \lim_{T\to\infty} -\left\{e^{-(s-3)t}\left[\frac{(s-3)\sin t + \cos t}{(s-3)^2+1}\right]_0^T\right\} \\ & = & \frac{1}{(s-3)^2+1}, \quad s > 3 \blacksquare \end{array}$$

Problem 43.17
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Solution.
We have $\mathcal{L}[e^{-7t}] = \frac{1}{s+7},\ s > -7$, $\mathcal{L}[t] = \frac{1}{s^2},\ s > 0$, and $\mathcal{L}[e^{2t}] = \frac{1}{s-2},\ s > 2$. Hence,

$$\mathcal{L}[5e^{-7t} + t + 2e^{2t}] = 5\mathcal{L}[e^{-7t}] + \mathcal{L}[t] + 2\mathcal{L}[e^{2t}] = \frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2},\ s > 2 \ \blacksquare$$

Problem 43.18
Consider the function $f(t) = \tan t$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Solution.
(a) Since $f(t) = \tan t = \frac{\sin t}{\cos t}$ and this function is discontinuous at $t = (2n+1)\frac{\pi}{2}$. Since this function has vertical asymptotes there it is not piecewise continuous.
(b) The graph of the function does not show that it can be bounded by exponential functions. Hence, no such numbers a and M $\blacksquare$

Problem 43.19
Consider the function $f(t) = t^2e^{-t}$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Solution.
(a) Since t^2 and e^{-t} are continuous everywhere, $f(t) = t^2e^{-t}$ is continuous on $0 \le t < \infty$.
(b) By L'Hôpital's rule one has

$$\lim_{t\to\infty} \frac{t^2}{e^t} = 0$$

Since $f(0) = 0$, $f(t)$ is bounded. Since $f'(t) = (2t - t^2)e^{-t}$, $f(t)$ has a maximum when $t = 2$. The value of this maximum is $f(2) = 4e^{-2}$. Hence, $M = 4e^{-2}$ and $a = 0$ $\blacksquare$

Problem 43.20
Consider the function $f(t) = \frac{e^{t^2}}{e^{2t}+1}$.

(a) Is $f(t)$ continuous on $0 \leq t < \infty$, discontinuous but piecewise continuous on $0 \leq t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \leq Me^{at}$ for $0 \leq t < \infty$?

Solution.
(a) Since et^2 and $e^{2t}+1$ are continuous everywhere, $f(t) = \frac{e^{t^2}}{e^{2t}+1}$ is continuous on $0 \leq t < \infty$.
(b) Since $e^{2t} + 1 \leq e^{2t} + e^{2t} = 2e^{2t}$, $f(t) \geq \frac{1}{2}e^{t^2}e^{-2t} = \frac{1}{2}e^{t^2-2t}$. But for $t \geq 4$ we have $t^2 - 2t > \frac{t^2}{2}$. Hence, $f(t) > \frac{1}{2}e^{\frac{t^2}{2}}$. So $f(t)$ is not of exponential order at infinity ■

Problem 43.21
Consider the floor function $f(t) = \lfloor t \rfloor$, where for any integer n we have $\lfloor t \rfloor = n$ for all $n \leq t < n + 1$.

(a) Is $f(t)$ continuous on $0 \leq t < \infty$, discontinuous but piecewise continuous on $0 \leq t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \leq Me^{at}$ for $0 \leq t < \infty$?

Solution.
(a) The floor function is a piecewise continuous function on $0 \leq t < \infty$.
(b) Since $\lfloor t \rfloor \leq t < e^t$ for $0 \geq t < \infty$ we have $M = 1$ and $a = 1$ ■

Problem 43.22
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Solution.
Since $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a}$, $s > a$ we have

$$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 3e^{2t}, \ t \geq 0 \ ■$$

Problem 43.23
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$.

Solution.
Since $\mathcal{L}[t] = \frac{1}{s^2},\ s > 0$ and $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a},\ s > a$ we have

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right) &= -2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= -2t + e^{-t},\ t \geq 0 \blacksquare\end{aligned}$$

Problem 43.24
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$.

Solution.
We have

$$\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 2(e^{-2t} + e^{2t}),\ t \geq 0 \blacksquare$$

44 Further Studies of Laplace Transform

Problem 44.1
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t+5]$.

Solution.

$$\mathcal{L}[2e^t+5] = 2\mathcal{L}[e^t]+5\mathcal{L}[1] = \frac{2}{s-1}+\frac{5}{s}, \quad s>1 \blacksquare$$

Problem 44.2
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}h(t-1)]$.

Solution.

$$\mathcal{L}[e^{3t-3}h(t-1)] = \mathcal{L}[e^{3(t-1)}h(t-1)] = e^{-s}\mathcal{L}[e^{3t}] = \frac{e^{-s}}{s-3}, \quad s>3 \blacksquare$$

Problem 44.3
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2\omega t]$.

Solution.

$$\mathcal{L}[\sin^2\omega t] = \mathcal{L}[\frac{1-\cos 2\omega t}{2}] = \frac{1}{2}(\mathcal{L}[1]-\mathcal{L}[\cos 2\omega t]) = \frac{1}{2}\left(\frac{1}{s}-\frac{s^2}{s^2+4\omega^2}\right), \quad s>0 \blacksquare$$

Problem 44.4
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t\cos 3t]$.

Solution.

$$\mathcal{L}[\sin 3t\cos 3t] = \mathcal{L}[\frac{\sin 6t}{2} = \frac{1}{2}\mathcal{L}[\sin 6t] = \frac{3}{s^2+26}, \quad s>0 \blacksquare$$

Problem 44.5
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t}\cos 3t]$.

Solution.

$$\mathcal{L}[e^{2t}\cos 3t] = \frac{s-3}{(s-3)^2+9}, \quad s>3 \blacksquare$$

Problem 44.6
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2+3t+5)]$.

Solution.

$$\mathcal{L}[e^{4t}(t^2+3t+5)] = \mathcal{L}[e^{4t}t^2]+3\mathcal{L}[e^{4t}t]+5\mathcal{L}[1] = \frac{2}{(s-4)^3}+\frac{3}{(s-4)^2}+\frac{5}{s-4}, \ \ s>4 \blacksquare$$

Problem 44.7
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25}+\frac{4}{s-3}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{10}{s^2+25}+\frac{4}{s-3}] = 2\mathcal{L}^{-1}[\frac{5}{s^2+25}]+4\mathcal{L}^{-1}[\frac{1}{s-3}] = 2\sin 5t+4e^{3t}, \ \ t\geq 0 \blacksquare$$

Problem 44.8
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{5}{(s-3)^4}] = \frac{5}{6}\mathcal{L}^{-1}[\frac{3!}{(s-3)^4}] = \frac{5}{6}e^{3t}t^3, \ \ t\geq 0 \blacksquare$$

Problem 44.9
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}] = e^{9(t-2)}h(t-2) = \begin{cases} 0, & 0\leq t<2 \\ e^{9(t-2)}, & t\geq 2 \end{cases} \blacksquare$$

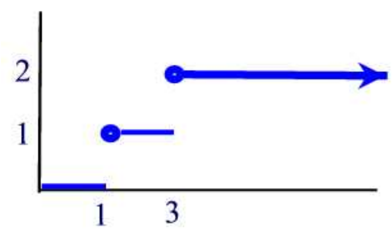
Problem 44.10
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}]$.

Solution.

$$\begin{aligned} \mathcal{L}^{-1}[\tfrac{e^{-3s}(2s+7)}{s^2+16}] &= 2\mathcal{L}^{-1}[\tfrac{e^{-3s}s}{s^2+16}]+\tfrac{7}{4}\mathcal{L}^{-1}[\tfrac{e^{-3s}}{4}s^2+16] \\ &= 2\cos 4(t-3)h(t-3)+\tfrac{7}{4}\sin 4(t-3), \ t\geq 0 \blacksquare \end{aligned}$$

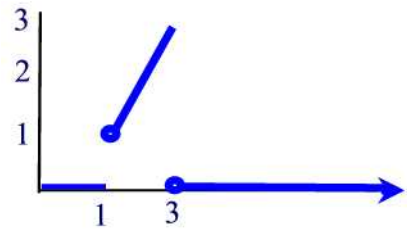
Problem 44.11
Graph the function $f(t) = h(t-1) + h(t-3)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that
$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$$
The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find
$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] + \mathcal{L}[h(t-3)] = \frac{e^{-s}}{s} + \frac{e^{-3s}}{s}, \ s > 0 \blacksquare$$



Problem 44.12
Graph the function $f(t) = t[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that
$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$
The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find
$$\begin{aligned} \mathcal{L}[f(t)] &= \mathcal{L}[(t-1)h(t-1) + h(t-1) - (t-3)h(t-3) - 3h(t-3)] \\ &= \mathcal{L}[(t-1)h(t-1)] + \mathcal{L}[h(t-1)] - \mathcal{L}[(t-3)h(t-3)] - 3\mathcal{L}[h(t-3)] \\ &= \tfrac{e^{-s}}{s^2} + \tfrac{e^{-s}}{s} - \tfrac{e^{3s}}{s^2} - \tfrac{e^{-3s}}{s}, \quad s > 1 \blacksquare \end{aligned}$$



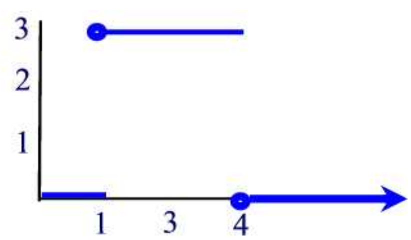
Problem 44.13
Graph the function $f(t) = 3[h(t-1) - h(t-4)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 3, & 1 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = 3\mathcal{L}[h(t-1)] - 3\mathcal{L}[h(t-4)] = \frac{3e^{-s}}{s} - \frac{3e^{-4s}}{s}, \ s > 0 \blacksquare$$



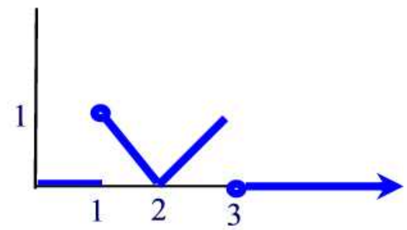
Problem 44.14
Graph the function $f(t) = |2-t|[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ |2-t|, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\begin{aligned}
\mathcal{L}[f(t)] &= \mathcal{L}[-(t-1)h(t-1) + h(t-1) + (t-2)h(t-2) + (t-2)h(t-2) - (t-3)h(t-3) - \\
&= -\mathcal{L}[(t-1)h(t-1)] + \mathcal{L}[h(t-1)] + 2\mathcal{L}[(t-2)h(t-2)] - \mathcal{L}[(t-3)h(t-3)] - \mathcal{L}[h(\\
&= -\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}, \ s > 1 \blacksquare
\end{aligned}$$



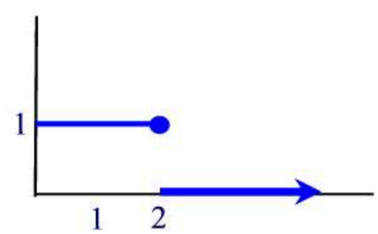
Problem 44.15
Graph the function $f(t) = h(2-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ 0, & t > 2 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = \int_0^2 e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^2 = \frac{1-e^{-2s}}{s}, \ s > 0 \blacksquare$$



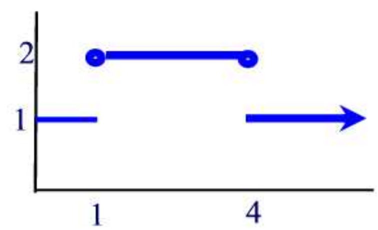
Problem 44.16
Graph the function $f(t) = h(t-1) + h(4-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

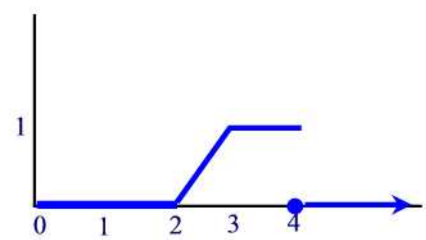
$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t \le 4 \\ 1, & t \ge 4 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] + \mathcal{L}[h(4-t)] = \frac{e^{-s}}{s} + \int_0^4 e^{-st}dt = \frac{1 + e^{-s} - e^{-4s}}{s}, \ s > 0 \blacksquare$$



Problem 44.17
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



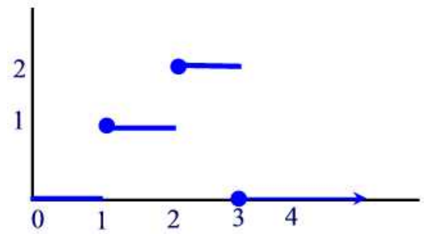
Solution.
From the graph we see that

$$\begin{aligned} f(t) &= (t-2)[h(t-2) - h(t-3)] + [h(t-3) - h(t-4)] \\ &= (t-2)h(t-2) - [(t-3)+1]h(t-3) + h(t-3) - h(t-4) \\ &= (t-2)h(t-2) - (t-3)h(t-3) - h(t-4) \end{aligned}$$

Thus,

$$\mathcal{L}[f(t)] = \mathcal{L}[(t-2)h(t-2)] - \mathcal{L}[(t-3)h(t-3)] - \mathcal{L}[h(t-4)] = \frac{e^{-2s} - e^{-3s}}{s^2} - \frac{e^{-4s}}{s}, \ s > 0 \blacksquare$$

Problem 44.18
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



Solution.
From the graph we see that

$$\begin{aligned} f(t) &= (t-1)[h(t-1)-h(t-2)]+(3-t)[h(t-2)-h(t-3)] \\ &= (t-1)h(t-1)-[(t-2)+1]h(t-2)+[-(t-2)+1]h(t-2)+(t-3)h(t-3) \\ &= (t-1)h(t-1)-2(t-2)h(t-2)+(t-3)h(t-3) \end{aligned}$$

Thus,

$$\mathcal{L}[f(t)] = \mathcal{L}[(t-1)h(t-1)]-2\mathcal{L}[(t-2)h(t-2)]+\mathcal{L}[(t-3)h(t-3)] = \frac{e^{-s}-2e^{-2s}+e^{-3s}}{s},\ s>0\ \blacksquare$$

Problem 44.19
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Solution.
Write

$$\frac{12}{(s-3)(s+1)} = \frac{A}{s-3}+\frac{B}{s+1}$$

Multiply both sides of this equation by $s-3$ and cancel common factors to obtain

$$\frac{12}{s+1} = A + \frac{B(s-3)}{s+1}.$$

Now, find A by setting $s=3$ to obtain $A=3$. Similarly, by multiplying both sides by $s+1$ and then setting $s=-1$ in the resulting equation leads to $B=-3$. Hence,

$$\frac{12}{(s-3)(s+1)} = 3\left(\frac{1}{s-3}-\frac{1}{s+1}\right)$$

Finally,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right] &= 3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]-3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]\\ &= 3e^{3t}-3e^{-t},\ t\geq 0 \blacksquare\end{aligned}$$

Problem 44.20
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Solution.
Write

$$\frac{24}{(s-3)(s+3)}=\frac{A}{s-3}+\frac{B}{s+3}$$

Multiply both sides of this equation by $s-3$ and cancel common factors to obtain

$$\frac{24}{s+3}=A+\frac{B(s-3)}{s+3}.$$

Now, find A by setting $s=3$ to obtain $A=4$. Similarly, by multiplying both sides by $s+3$ and then setting $s=-3$ in the resulting equation leads to $B=-4$. Hence,

$$\frac{24}{(s-3)(s+3)}=4\left(\frac{1}{s-3}-\frac{1}{s+3}\right)$$

Finally,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{(s-3)(s+3)}\right] &= 4\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s-3}\right]-4\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s+3}\right]\\ &= 4[e^{3(t-5)}-e^{-3(t-5)}]h(t-5),\ t\geq 0 \blacksquare\end{aligned}$$

Problem 44.21
Use Laplace transform technique to solve the initial value problem

$$y'+4y=g(t),\quad y(0)=2$$

where

$$g(t)=\begin{cases}0, & 0\leq t<1\\ 12, & 1\leq t<3\\ 0, & t\geq 3\end{cases}$$

Solution.
Note first that $g(t) = 12[h(t-1) - h(t-3)]$ so that

$$\mathcal{L}[g(t)] = 12\mathcal{L}[h(t-1)] - 12\mathcal{L}[h(t-3)] = \frac{12(e^{-s} - e^{-3s}}{s}, \ s > 0.$$

Now taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[g(t)].$$

But $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$. Letting $\mathcal{L}[y] = Y(s)$ we obtain

$$sY(s) - 2 + 4Y(s) = 12\frac{e^{-s} - e^{-3s}}{s}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{2}{s+4} + 12\frac{e^{-s} - e^{-3s}}{s(s+4)}.$$

But

$$\mathcal{L}^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}$$

and

$$\begin{aligned}
\mathcal{L}^{-1}\left[12\tfrac{e^{-s}-e^{-3s}}{s(s+4)}\right] &= 3\mathcal{L}^{-1}\left[(e^{-s} - e^{-3s})\left(\tfrac{1}{s} - \tfrac{1}{s+4}\right)\right] \\
&= 3\mathcal{L}^{-1}\left[\tfrac{e^{-s}}{s}\right] - 3\mathcal{L}^{-1}\left[\tfrac{e^{-3s}}{s}\right] - 3\mathcal{L}^{-1}\left[\tfrac{e^{-s}}{s+4}\right] + 3\mathcal{L}^{-1}\left[\tfrac{e^{-3s}}{s+4}\right] \\
&= 3h(t-1) - 3h(t-3) - 3e^{-4(t-1)}h(t-1) + 3e^{-4(t-3)}h(t-3)
\end{aligned}$$

Hence,

$$y(t) = 2e^{-4t} + 3[h(t-1) - h(t-3)] - 3[e^{-4(t-1)}h(t-1) - e^{-4(t-3)}h(t-3)], \ t \geq 0 \blacksquare$$

Problem 44.22
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution.
Taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y''] - 4\mathcal{L}[y] = \mathcal{L}[e^{3t}].$$

But $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2\mathcal{L}[y]$. Letting $\mathcal{L}[y] = Y(s)$ we obtain

$$s^2Y(s) - 4Y(s) = \frac{1}{s-3}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{1}{(s-3)(s-2)(s+2)}.$$

Using partial fraction decomposition

$$\frac{1}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s-2}$$

we find $A = \frac{1}{5},\ B = \frac{1}{20}$, and $C = -\frac{1}{4}$. Thus,

$$\begin{array}{rcl} y(t) & = & \mathcal{L}^{-1}[\frac{1}{(s-3)(s-2)(s+2)} = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\ & = & \frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t},\ t \geq 0 \blacksquare \end{array}$$

Problem 44.23
Obtain the Laplace transform of the function $\int_2^t f(\lambda)d\lambda$ in terms of $\mathcal{L}[f(t)] = F(s)$ given that $\int_0^2 f(\lambda)d\lambda = 3$.

Solution.
We have

$$\begin{array}{rcl} \mathcal{L}\left[\int_2^t f(\lambda)d\lambda\right] & = & \mathcal{L}\left[\int_0^t f(\lambda)d\lambda - \int_0^2 f(\lambda)d\lambda\right] \\ & = & \frac{F(s)}{s} - \mathcal{L}[3] \\ & = & \frac{F(s)}{s} - \frac{3}{s},\quad s > 0 \blacksquare \end{array}$$

45 The Laplace Transform and the Method of Partial Fractions

In Problems 45.1 - 45.4, give the form of the partial fraction expansion for $F(s)$. You need not evaluate the constants in the expansion. However, if the denominator has an irreducible quadratic expression then use the completing the square process to write it as the sum/difference of two squares.

Problem 45.1

$$F(s) = \frac{s^3 + 3s + 1}{(s-1)^3(s-2)^2}.$$

Solution.

$$F(s) = \frac{A_1}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_3}{s-1} + \frac{B_1}{(s-2)^2} + \frac{B_2}{s-2)} \blacksquare$$

Problem 45.2

$$F(s) = \frac{s^2 + 5s - 3}{(s^2+16)(s-2)}.$$

Solution.

$$F(s) = \frac{A_1 s + A_2}{s^2+16} + \frac{B_1}{s-2} \blacksquare$$

Problem 45.3

$$F(s) = \frac{s^3 - 1}{(s^2+1)^2(s+4)^2}.$$

Solution.

$$F(s) = \frac{A_1 s + A_2}{(s^2+1)^2} + \frac{A_3 s + A_4}{s^2+1} + \frac{B_1}{(s+4)^2} + \frac{B_2}{s+4} \blacksquare$$

Problem 45.4

$$F(s) = \frac{s^4 + 5s^2 + 2s - 9}{(s^2+8s+17)(s-2)^2}.$$

Solution.

$$F(s) = \frac{A_1}{(s-2)^2} + \frac{A_2}{s-2} + \frac{B_1s+B_2}{(s+4)^1+1} + \frac{B_3s+B_4}{(s+4)^2+1} \blacksquare$$

Problem 45.5
Find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right]$.

Solution.
Using Table $\mathcal{L}$ we find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right] = \frac{1}{2}e^{-t}t^2,\ t \ge 0$ ■

Problem 45.6
Find $\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-3s+2}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{2s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Multiplying both sides by $(s-1)(s-2)$ and simplifying to obtain

$$\begin{aligned} 2s-3 &= A(s-2) + B(s-1) \\ &= (A+B)s - 2A - B. \end{aligned}$$

Equating coefficients of like powers of s we obtain the system

$$\begin{cases} A+B = 2 \\ -2A - B = -3 \end{cases}$$

Solving this system by elimination we find $A = 1$ and $B = 1$. Now finding the inverse Laplace transform to obtain

$$\mathcal{L}^{-1}\left[\frac{2s-3}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^t + e^{2t},\ t \ge 0. \blacksquare$$

Problem 45.7
Find $\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s^3+s}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{4s^2+s+1}{s(s^2+1}=\frac{A}{s}+\frac{Bs+C}{s^2+1}.$$

Multiplying both sides by $s(s^2+1)$ and simplifying to obtain

$$\begin{array}{rcl} 4s^2+s+1 &=& A(s^2+1)+(Bs+C)s \\ &=& (A+B)s^2+Cs+A. \end{array}$$

Equating coefficients of like powers of s we obtain $A+B=4,\ C=1,\ A=1$. Thus, $B=3$. Now finding the inverse Laplace transform to obtain

$$\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s(s^2+1)}\right]=\mathcal{L}^{-1}\left[\frac{1}{s}\right]+3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]=1+3\cos t+\sin t,\ t\geq 0. \blacksquare$$

Problem 45.8
Find $\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{s^4+8s^2+16}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{s^2+6s+8}{(s^2+4)^2}=\frac{B_1s+C_1}{s^2+4}+\frac{B_2s+C_2}{s^2+4}.$$

Multiplying both sides by $(s^2+4)^2$ and simplifying to obtain

$$\begin{array}{rcl} s^2+6s+8 &=& (B_1s+C_1)(s^2+4)+B_2s+C_2 \\ &=& B_1s^3+C_1s^2+(4B_1+B_2)s+4C_1+C_2. \end{array}$$

Equating coefficients of like powers of s we obtain $B_1=0, C_1=1, B_2=6$, and $C_2=4$. Now finding the inverse Laplace transform to obtain

$$\begin{array}{rcl} \mathcal{L}^{-1}\left[\frac{s^2+6s+8}{(s^2+4)^2}\right] &=& \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]+6\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right]+4\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right] \\ &=& \frac{1}{2}\sin 2t+6\left(\frac{t}{4}\sin 2t\right)+4\left(\frac{1}{16}[\sin 2t-2t\cos 2t]\right) \\ &=& \frac{3}{2}t\sin 2t+\frac{3}{4}\sin 2t-\frac{1}{2}t\cos 2t,\ t\geq 0 \ \blacksquare \end{array}$$

Problem 45.9
Use Laplace transform to solve the initial value problem

$$y' + 2y = 26\sin 3t,\ \ y(0) = 3.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 26\mathcal{L}[\sin 3t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$sY(s) - y(0) + 2Y(s) = 26\left(\frac{3}{s^2+9}\right).$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{3}{s+2} + \frac{78}{(s+2)(s^2+9)}.$$

Using the partial fraction decomposition we can write

$$\frac{1}{s+2}s^2+9 = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}.$$

Multipliying both sides by $(s+2)(s^2+9)$ to obtain

$$\begin{array}{rcl} 1 &=& A(s^2+9) + (Bs+C)(s+2) \\ &=& (A+B)s^2 + (2B+C)s + 9A + 2C \end{array}$$

Equating coefficients of like powers of s we find $A+B=0,\ 2B+C=0$, and $9A+2C=1$. Solving this system we find $A=\frac{1}{13},\ B=-\frac{1}{13}$, and $C=\frac{2}{13}$. Thus,

$$Y(s) = \frac{9}{s+2} - 6\left(\frac{s}{s^2+9}\right) + 4\left(\frac{3}{s^2+9}\right).$$

Finally,

$$\begin{array}{rcl} y(t) &=& \mathcal{L}^{-1}[Y(s)] = 9\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] + 4\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] \\ &=& 9e^{-2t} - 6\cos 3t + 4\sin 3t,\ \ t \geq 0 \ \blacksquare \end{array}$$

Problem 45.10
Use Laplace transform to solve the initial value problem

$$y' + 2y = 4t,\ y(0) = 3.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 4\mathcal{L}[t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{3}{s+2} + \frac{4}{(s+2)s^2}.$$

Using the partial fraction decomposition we can write

$$\frac{1}{(s+2)s^2} = \frac{A}{s+2} + \frac{Bs+C}{s^2}.$$

Multipliying both sides by $(s+2)s^2$ to obtain

$$\begin{aligned} 1 &= As^2 + (Bs+C)(s+2) \\ &= (A+B)s^2 + (2B+C)s + 2C \end{aligned}$$

Equating coefficients of like powers of s we find $A+B=0,\ 2B+C=0$, and $2C=1$. Solving this system we find $A=4,\ B=-1$, and $C=2$. Thus,

$$Y(s) = \frac{4}{s+2} - \frac{1}{s} + 2\frac{1}{s^2}.$$

Finally,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] = 4\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \\ &= 4e^{-2t} - 1 + 2t,\ t \geq 0 \ \blacksquare \end{aligned}$$

Problem 45.11
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 6e^{-t},\ y(0) = 1,\ y'(0) = 2.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = 6\mathcal{L}[e^{-t}].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0))) + 2Y(s) = \frac{6}{s+1}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+5}{(s+1)(s+2)} + \frac{6}{(s+2)(s+1)^2} = \frac{s^2+6s+11}{(s+1)^2(s+2)}.$$

Using the partial fraction decomposition we can write

$$\frac{s^2+6s+11}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

Multipliying both sides by $(s+2)(s+1)^2$ to obtain

$$\begin{array}{rcl} s^2+6s+11 & = & A(s+1)^2 + B(s+1) + C \\ & = & As^2 + (2A+B)s + A + B + C \end{array}$$

Equating coefficients of like powers of s we find $A = 1,\ 2A + B = 6$, and $A + B + C = 11$. Solving this system we find $A = 3,\ B = -2$, and $C = 6$. Thus,

$$Y(s) = \frac{3}{s+2} - \frac{2}{s+1} + \frac{6}{(s+1)^2}.$$

Finally,

$$\begin{array}{rcl} y(t) & = & \mathcal{L}^{-1}[Y(s)] \quad = \quad 3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 6\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] \\ & = & 3e^{-2t} - 2e^{-t} + 6te^{-t} \blacksquare \end{array}$$

Problem 45.12
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos 2t,\ y(0) = 1,\ y'(0) = 1.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\cos 2t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+1}{s^2+4} + \frac{s}{(s^2+4)^2}.$$

Using Table $\mathcal{L}$ again we find

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\tfrac{s}{s^2+4}\right] + \tfrac{1}{2}\mathcal{L}^{-1}\left[\tfrac{2}{s^2+4}\right] + \mathcal{L}^{-1}\left[\tfrac{s}{(s^2+4)^2}\right] \\ &= \cos 2t + \tfrac{1}{2}\sin 2t + \tfrac{t}{4}\sin 2t,\ t \geq 0 \ \blacksquare \end{aligned}$$

Problem 45.13
Use Laplace transform to solve the initial value problem

$$y'' - 2y' + y = e^{2t},\ y(0) = 0,\ y'(0) = 0.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] - 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[e^{2t}].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + Y(s) = \frac{1}{s-2}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{1}{(s-1)^2(s-2)}.$$

Using the partial fraction decomposition, we can write

$$Y(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}.$$

Multipliying both sides by $(s-2)(s-1)^2$ to obtain

$$\begin{aligned} 1 &= A(s-1)(s-2)+B(s-2)+C(s-1)^2 \\ &= (A+C)s^2+(-3A+B-2C)s+2A-2B+C \end{aligned}$$

Equating coefficients of like powers of s we find $A+C=0,\ -3A+B-2C=0,$ and $2A-2B+C=1$. Solving this system we find $A=-1,\ B=-1,$ and $C=1$. Thus,

$$Y(s) = -\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{s-2}.$$

Finally,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] = -\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= -e^t - te^t + e^{2t},\ t \geq 0 \blacksquare \end{aligned}$$

Problem 45.14
Use Laplace transform to solve the initial value problem

$$y'' + 9y = g(t),\ y(0) = 1,\ y'(0) = 0$$

where

$$g(t) = \begin{cases} 6, & 0 \leq t < \pi \\ 0, & \pi \leq t < \infty \end{cases}$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 9\mathcal{L}[y] = \mathcal{L}[g(t)] = 6\mathcal{L}[h(t) - h(t-\pi)].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{6}{s} - 6\frac{e^{-\pi s}}{s}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+3}{s^2+9} + \frac{6}{s(s^2+9)}(1 - e^{-\pi s}).$$

Using the partial fraction decomposition, we can write

$$\frac{6}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}.$$

Multipliying both sides by $s(s^2+9)$ to obtain

$$\begin{aligned} 6 &= A(s^2+9)+(Bs+C)s \\ &= (A+B)s^2+Cs+9A \end{aligned}$$

Equating coefficients of like powers of s we find $A+B=0,\ C=0$, and $9A=6$. Solving this system we find $A=\frac{2}{3},\ B=-\frac{2}{3}$, and $C=0$. Thus,

$$Y(s)=\frac{s}{s^2+9}+\frac{3}{s^2+9}+(1-e^{-\pi s})\left(\frac{2}{3}\frac{1}{s}-\frac{2}{3}\frac{s}{s^2+9}\right).$$

Finally,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)]=\cos 3t+\sin 3t+\tfrac{2}{3}(1-\cos 3t)-\tfrac{2}{3}(1-\cos 3(t-\pi))h(t-\pi) \\ &= \cos 3t+\sin 3t+\tfrac{2}{3}(1-\cos 3t)-\tfrac{2}{3}(1+\cos 3t)h(t-\pi),\ t\geq 0 \ \blacksquare \end{aligned}$$

Problem 45.15
Determine the constants α,β,y_0, and y_0' so that $Y(s)=\frac{2s-1}{s^2+s+2}$ is the Laplace transform of the solution to the initial value problem

$$y''+\alpha y'+\beta y=0,\ y(0)=y_0,\ y'(0)=y_0'.$$

Solution.
Taking the Laplace transform of both sides we find

$$s^2Y(s)-sy_0-y_0'+\alpha sY(s)-\alpha y_0+\beta Y(s)=0.$$

Solving for $Y(s)$ we find

$$Y(s)=\frac{sy_0+(y_0'+\alpha y_0)}{s^2+\alpha s+\beta}=\frac{2s-1}{s^2+s+2}.$$

By identification we find $\alpha=1,\ \beta=2,\ y_0=2$, and $y_0'=-3$ $\blacksquare$

Problem 45.16
Determine the constants α,β,y_0, and y_0' so that $Y(s)=\frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y''+\alpha y'+\beta y=0,\ y(0)=y_0,\ y'(0)=y_0'.$$

Solution.
Taking the Laplace transform of both sides we find

$$s^2Y(s) - sy_0 - y_0' + \alpha sY(s) - \alpha y_0 + \beta Y(s) = 0.$$

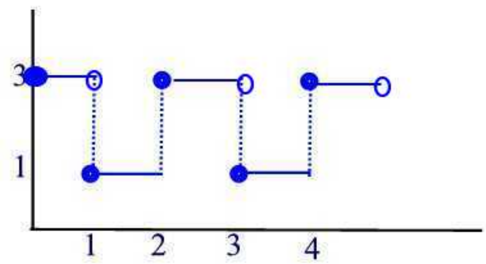
Solving for $Y(s)$ we find

$$Y(s) = \frac{sy_0 + (y_0' + \alpha y_0)}{s^2 + \alpha s + \beta} = \frac{s}{s^2 + 2s + 1}.$$

By identification we find $\alpha = 2,\ \beta = 1,\ y_0 = 1$, and $y_0' = -2$ ■

47 Laplace Transforms of Periodic Functions

Problem 47.1
Find the Laplace transform of the periodic function whose graph is shown.



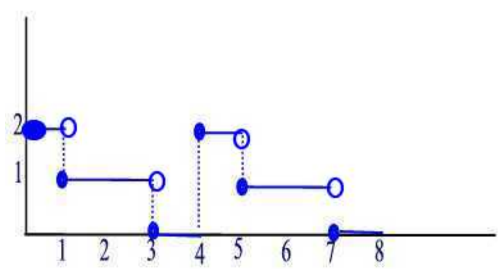
Solution.
The function is of period $T = 2$. Thus,

$$3\int_0^1 e^{-st}dt + \int_1^2 e^{-st}dt = \left[-\frac{3}{s}e^{-st}\right]_0^1 - \left[\frac{e^{-st}}{s}\right]_1^2 = \frac{1}{s}(3 - 2e^{-s} - e^{-2s}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{3 - 2e^{-s} - e^{-2s}}{s(1 - e^{-2s}} \blacksquare$$

Problem 47.2
Find the Laplace transform of the periodic function whose graph is shown.



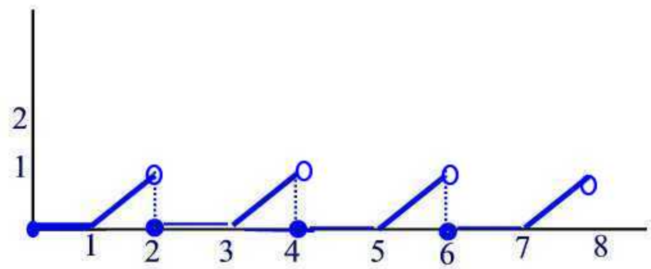
Solution.
The function is of period $T = 4$. Thus,

$$2\int_0^1 e^{-st}dt + \int_1^3 e^{-st}dt = \left[-\frac{2}{s}e^{-st}\right]_0^1 - \left[\frac{e^{-st}}{s}\right]_1^3 = \frac{1}{s}(2 - e^{-s} - e^{-3s}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{2 - e^{-s} - e^{-3s}}{s(1 - e^{-4s}} \blacksquare$$

Problem 47.3
Find the Laplace transform of the periodic function whose graph is shown.



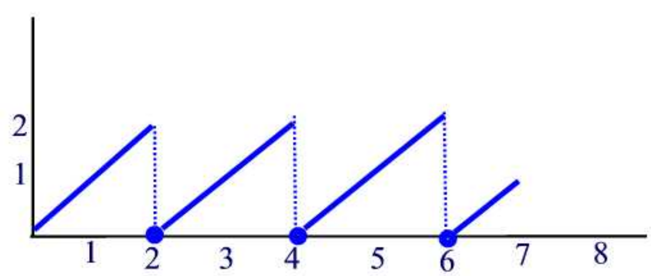
Solution.
The function is of period $T = 2$. Thus,

$$\int_1^2 (t-1)e^{-st}dt = \int_0^1 ue^{(1+u)s}du = \left[-\frac{e^{-s}}{s^2}(su+1)e^{-su}\right]_0^1 = -\frac{e^{-s}}{s^2}[(s+1)e^{-s}-1].$$

Hence,

$$\mathcal{L}[f(t)] = \frac{e^{-s}}{s^2(1 - e^{-2s}}[1 - (s+1)e^{-s}] \blacksquare$$

Problem 47.4
Find the Laplace transform of the periodic function whose graph is shown.



Solution.
The function is of period $T = 2$. Thus,

$$\int_0^2 te^{-st}dt = \left[-\frac{1}{s^2}(st+1)e^{-st}\right]_0^2 = -\frac{1}{s^2}[(2s+1)e^{-2s} - 1].$$

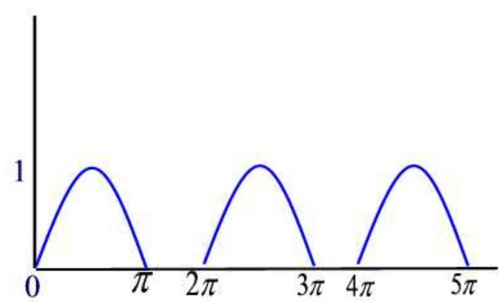
Hence,

$$\mathcal{L}[f(t)] = \frac{1}{s^2(1-e^{-2s}}[1-(2s+1)e^{-2s}] \blacksquare$$

Problem 47.5
State the period of the function $f(t)$ and find its Laplace transform where

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t),\ t \ge 0.$$

Solution.
The graph of $f(t)$ is shown below.



The function $f(t)$ is of period $T = 2\pi$. The Laplace transform of $f(t)$ is

$$\mathcal{L}[f(t)] = \frac{\int_0^\pi \sin te^{-st}dt}{1-e^{-2\pi s}}$$

Using integration by parts twice we find

$$\int \sin te^{-st}dt = -\frac{e^{-st}}{1+s^2}(\cos t + s\sin t)$$

Thus,

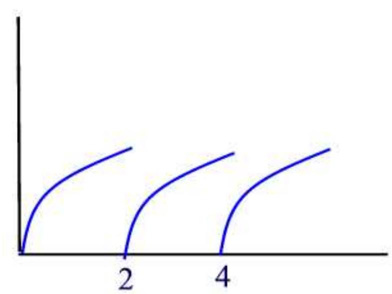
$$\begin{array}{lll}\int_0^\pi \sin t e^{-st}dt & = & \left[-\frac{e^{-st}}{1+s^2}(\cos t + s\sin t)\right]_0^\pi \\ & = & \frac{e^{-\pi s}}{1+s^2} + \frac{1}{1+s^2} \\ & = & \frac{1+e^{-\pi s}}{1+s^2}\end{array}$$

Hence,

$$\mathcal{L}[f(t)] = \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-2\pi s})} \blacksquare$$

Problem 47.6
State the period of the function $f(t) = 1 - e^{-t},\ 0 \le t < 2, \quad f(t+2) = f(t)$, and find its Laplace transform.

Solution.
The graph of $f(t)$ is shown below



The function is periodic of period $T = 2$. Its Laplace transform is

$$\mathcal{L}[f(t)] = \frac{\int_0^2 (1-e^{-t})e^{-st}dt}{1-e^{-2s}}.$$

But

$$\int_0^2 (1-e^{-t})e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[\frac{e^{(s+1)t}}{s+1}\right]_0^2 = \frac{1}{s}(1-e^{-2s}) - \frac{1}{s+1}(1-e^{-2(s+1)}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1-e^{-2(s+1)}}{(s+1)(1-e^{-2s}} \blacksquare$$

Problem 47.7
Using Example 44.3 find

$$\mathcal{L}^{-1}\left[\frac{s^2-s}{s^3}+\frac{e^{-s}}{s(1-e^{-s})}\right].$$

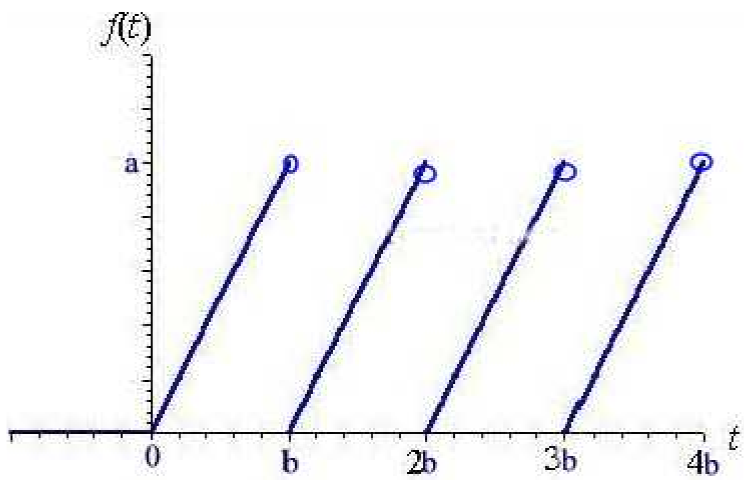
Solution.
Note first that

$$\frac{s^2-s}{s^3}+\frac{e^{-s}}{s(1-e^{-s})}=\frac{1}{s}-\left(\frac{1}{s^2}-\frac{se^{-s}}{s^2(1-e^{-s})}\right).$$

Using Example 44.3, we find

$$f(t)=1-g(t)$$

where $g(t)$ is the sawtooth function shown below



Problem 47.8
An object having mass m is initially at rest on a frictionless horizontal surface. At time $t=0$, a periodic force is applied horizontally to the object, causing it to move in the positive x-direction. The force, in newtons, is given by

$$f(t)=\begin{cases} f_0, & 0\le t\le \frac{T}{2}\\ 0, & \frac{T}{2}<t<T \end{cases} \qquad f(t+T)=f(t),\ t\ge 0.$$

The initial value problem for the horizontal position, $x(t)$, of the object is

$$mx''(t) = f(t), \quad x(0) = x'(0) = 0.$$

(a) Use Laplace transforms to determine the velocity, $v(t) = x'(t)$, and the position, $x(t)$, of the object.
(b) Let $m = 1\ kg,\ f_0 = 1\ N$, and $T = 1\ sec$. What is the velocity, v, and position, x, of the object at $t = 1.25$ sec?

Solution.
(a) Taking Laplace transform of both sides we find $ms^2X(s) = \frac{f_0\int_0^{\frac{T}{2}} e^{-st}dt}{1-e^{-sT}} = \frac{f_0}{s}\left(\frac{1-e^{-s\frac{T}{2}}}{1-e^{-sT}}\right)$. Solving for $X(s)$ we find

$$X(s) = \frac{f_0}{m} \cdot \frac{1}{s^3} \cdot \frac{1}{1+e^{-s\frac{T}{2}}}.$$

Also,

$$V(s) = \mathcal{L}[v(t)] = sX(s) = \frac{f_0}{m} \cdot \frac{1}{s^2} \cdot \frac{1}{1+e^{-s\frac{T}{2}}} = \frac{1}{m} \cdot \frac{1}{s} \cdot \frac{1}{s(1+e^{-s\frac{T}{2}})}.$$

Hence, by Example 44.1 and Table $\mathcal{L}$ we can write

$$v(t) = \frac{1}{m}\int_0^t f(u)du.$$

Since $X(s) = \frac{1}{m}\frac{1}{s^2}\frac{f_0}{s(1+e^{-s\frac{T}{2}})} = \frac{1}{m}\mathcal{L}[t]\mathcal{L}[f(t)] = \frac{1}{m}\mathcal{L}[t*f(t)]$ we have

$$x(t) = \frac{1}{m}(t*f(t)) = \frac{1}{m}\int_0^t (t-u)f(u)du.$$

(b) We have $x(1.25) = \int_0^{\frac{1}{2}}(\frac{5}{4}-u)du + \int_1^{\frac{5}{4}}(\frac{5}{4}-w)dw = \frac{17}{32}$ meters and $v(1.25) = \int_0^{\frac{5}{4}} f(u)du = \int_0^{\frac{1}{2}} dt + \int_1^{\frac{5}{4}} dt = \frac{3}{4}$ m/sec ■

Problem 47.9
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that the transfer function of this system is given by $\Phi(s) = \frac{1}{2s^2+5s+2}$.
(a) What are the constants $a,\ b$, and c?
(b) If $f(t) = e^{-t}$, determine $F(s),\ Y(s)$, and $y(t)$.

Solution.
(a) Taking the Laplace transform of both sides we find $as^2Y(s) + bsY(s) + cY(s) = F(s)$ or

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{as^2 + bs + c} = \frac{1}{2s^2 + 5s + 2}.$$

By identification we find $a = 2,\ b = 5$, and $c = 2$.
(b) If $f(t) = e^{-t}$ then $F(s) = \mathcal{L}[e^{-t}] = \frac{1}{s+1}$. Thus,

$$Y(s) = \Phi(s)F(s) = \frac{1}{(s+1)(2s^2 + 5s + 2)}.$$

Using partial fraction decomposition

$$\frac{1}{(s+1)(2s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{2s+1} + \frac{C}{s+2}$$

Multiplying both sides by $s + 1$ and setting $s = -1$ we find $A = -1$. Next, multiplying both sides by $2s+1$ and setting $s = -\frac{1}{2}$ we find $B = \frac{4}{3}$. Similarly, multiplying both sides by $s + 2$ and setting $s = -2$ we find $C = \frac{1}{3}$. Thus,

$$\begin{aligned} y(t) &= -\mathcal{L}^{-1}\left[\tfrac{1}{s+1}\right] + \tfrac{2}{3}\mathcal{L}^{-1}\left[\tfrac{1}{s+\frac{1}{2}}\right] + \tfrac{1}{3}\mathcal{L}^{-1}\left[\tfrac{1}{s+2}\right] \\ &= -e^{-t} + \tfrac{2}{3}e^{-\frac{t}{2}} + \tfrac{1}{3}e^{-2t} \ \blacksquare \end{aligned}$$

Problem 47.10
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that an input $f(t) = t$, when applied to the above system produces the output $y(t) = 2(e^{-t} - 1) + t(e^{-t} + 1), \quad t \geq 0$.
(a) What is the system transfer function?
(b) What will be the output if the Heaviside unit step function $f(t) = h(t)$ is applied to the system?

Solution.
(a) Since $f(t) = t$, we have $F(s) = \frac{1}{s^2}$. Aslo, $Y(s) = \mathcal{L}[y(t)] = \mathcal{L}[2e^{-t} - 2 + te^{-t} + t] = \frac{2}{s+2} - \frac{2}{s} + \frac{1}{(s+1)^2} + \frac{1}{s^2} = \frac{1}{s^2(s+1)^2}$. But $\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{(s+1)^2}$.

(b) If $f(t) = h(t)$ then $F(s) = \frac{1}{s}$ and $Y(s) = \Phi(s)F(s) = \frac{1}{s(s+1)^2}$. Using partial fraction decomposition we find

$$\begin{array}{rcl} \frac{1}{s(s+1)^2} & = & \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \\ 1 & = & A(s+1)^2 + Bs(s+1) + Cs \\ 1 & = & (A+B)s^2 + (2A+B+C)s + A \end{array}$$

Equating coefficients of like powers of s we find $A = 1, \quad B = -1$, and $C = -1$. Therefore,

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

and

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 1 - e^{-t} - te^{-t} \blacksquare$$

Problem 47.11
Consider the initial value problem

$$y'' + y' + y = f(t), \quad y(0) = y'(0) = 0,$$

where

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

Solution.
(a) Taking the Laplace transform of both sides we find

$$s^2Y(s) + sY(s) + Y(s) = F(s)$$

so that

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{s^2+s+1}.$$

(b) But

$$\begin{array}{rcl} \int_0^2 f(t)e^{-st}dt & = & \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt \\ & = & \left[\frac{e^{-st}}{-s}\right]_0^1 - \left[\frac{e^{-st}}{-s}\right]_1^2 \\ & = & \frac{1}{s}(1-e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) \\ & = & \frac{(1-e^{-s})^2}{s} \end{array}$$

Hence,

$$F(s) = \frac{(1-e^{-s})^2}{s(1-e^{-2s})} = \frac{(1-e^{-s})}{s(1+e^{-s})}$$

and

$$Y(s) = \Phi(s)F(s) = \frac{(1-e^{-s})}{s(1+e^{-s})(s^2+s+1)} \blacksquare$$

Problem 47.12
Consider the initial value problem

$$y''' - 4y = e^t + t, \quad y(0) = y'(0) = y''(0) = 0.$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

Solution.
(a) Taking Laplace transform of both sides we find

$$s^3Y(s) - 4Y(s) = F(s).$$

Thus,

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{s^3-4}.$$

(b) We have

$$F(s) = \mathcal{L}[e^t + t] = \frac{1}{s-1} + \frac{1}{s^2} = \frac{s^2+s-1}{(s-1)s^2}.$$

Hence,

$$Y(s) = \frac{s^2+s-1}{s^2(s-1)(s^3-4)} \blacksquare$$

Problem 47.13
Consider the initial value problem

$$y'' + by' + cy = h(t), \quad y(0) = y_0, \ y'(0) = y'_0, \ t > 0.$$

Suppose that $\mathcal{L}[y(t)] = Y(s) = \frac{s^2+2s+1}{s^3+3s^2+2s}$. Determine the constants $b, \ c, \ y_0$, and y'_0.

Solution.
Take the Laplace transform of both sides to obtain

$$s^2Y(s) - sy_0 - y_0' + bsY(s) - by_0 + cY(s) = \frac{1}{s}.$$

Solve to find

$$\begin{aligned} Y(s) &= \tfrac{1}{s^3+bs^2+cs)} + \tfrac{sy_0+y_0'+by_0}{s^2+bs+c} \\ &= \tfrac{s^2y_0+s(y_0'+by_0)+1}{s^3+bs^2+cs} \\ &= \tfrac{s^2+2s+1}{s^3+3s^2+2s}. \end{aligned}$$

By comparison we find $b = 3,\ c = 2,\ y_0 = 1$, and $y_0' + by_0 = 2$ or $y_0' = -1$ ■

47 Convolution Integrals

Problem 47.1
Consider the functions $f(t) = g(t) = h(t),\ t \geq 0$ where $h(t)$ is the Heaviside unit step function. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Solution.
(a) We have

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t h(t-s)h(s)ds = \int_0^t ds = t$$

(b) Since $F(s) = G(s) = \mathcal{L}[h(t)] = \frac{1}{s}$, $(f*g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[\frac{1}{s^2}] = t$ ■

Problem 47.2
Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t},\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Solution.
(a) We have

$$\begin{array}{lll} (f * g)(t) & = & \int_0^t f(t-s)g(s)ds = \int_0^t e^{(t-s)}e^{-2s}ds \\ & = & e^t \int_0^t e^{-3s}ds = \left[\frac{e^{(t-3s)}}{-3}\right]_0^t \\ & = & \frac{e^t - e^{-2t}}{3} \end{array}$$

(b) Since $F(s) = \mathcal{L}[e^t] = \frac{1}{s-1}$ and $G(s) = \mathcal{L}[e^{-2t}] = \frac{1}{s+2}$ we have $(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[\frac{1}{(s-1)(s-2)}]$. Using partial fractions decomposition we find

$$\frac{1}{(s-1)(s+2)} = \frac{1}{3}(\frac{1}{s-1} - \frac{1}{s+2}).$$

Thus,

$$(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \frac{1}{3}(\mathcal{L}^{-1}[\frac{1}{s-1}] - \mathcal{L}^{-1}[\frac{1}{s+2}] = \frac{e^t - e^{-2t}}{3} \ ■$$

Problem 47.3
Consider the functions $f(t) = \sin t$ and $g(t) = \cos t,\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Solution.
(a) Using the trigonometric identity $2\sin p \cos q = \sin(p+q) + \sin(p-q)$ we find that $2\sin(t-s)\cos s = \sin t + \sin(t-2s)$. Hence,

$$\begin{array}{rcl} (f*g)(t) &=& \int_0^t f(t-s)g(s)ds = \int_0^t \sin(t-s)\cos s ds \\ &=& \frac{1}{2}[\int_0^t \sin t ds + \int_0^t \sin(t-2s)ds] \\ &=& \frac{t\sin t}{2} + \frac{1}{4}\int_{-t}^t \sin u du \\ &=& \frac{t\sin t}{2} \end{array}$$

(b) Since $F(s) = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$ and $G(s) = \mathcal{L}[\cos t] = \frac{s}{s^2+1}$ we have

$$(f*g)(t) = \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[\frac{s}{(s^2+1)^2}] = \frac{t}{2}\sin t \ \blacksquare$$

Problem 47.4
Use Laplace transform to compute the convolution $\mathbf{P} * \mathbf{y}$, where $|bfP(t) = \left[\begin{array}{cc} h(t) & e^t \\ 0 & t \end{array}\right]$ and $\mathbf{y}(t) = \left[\begin{array}{c} h(t) \\ e^{-t} \end{array}\right]$.

Solution.
We have

$$\begin{array}{rcl} (\mathbf{P}*\mathbf{y})(t) &=& \mathcal{L}^{-1}\left\{\left[\begin{array}{cc} \frac{1}{s} & \frac{1}{s-1} \\ 0 & \frac{1}{s^2} \end{array}\right]\left[\begin{array}{c} \frac{1}{s} \\ \frac{1}{s+1} \end{array}\right]\right\} \\ &=& \mathcal{L}^{-1}\left\{\left[\begin{array}{c} \frac{1}{s} \quad \frac{1}{s^2} + \frac{1}{(s-1)(s+1)} \\ \frac{1}{s^2(s+1)} \end{array}\right]\right\} \end{array}$$

But

$$\frac{1}{(s-1)(s+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

and

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

Hence,

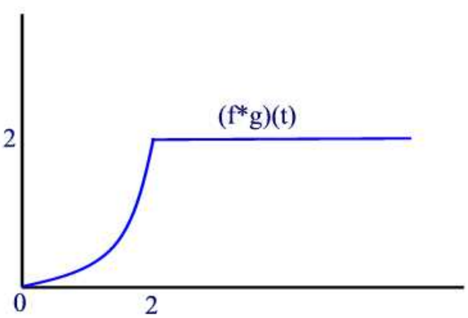
$$(\mathbf{P} * \mathbf{y})(t) = \left[\begin{array}{c} t + \frac{e^t}{2} - \frac{e^{-t}}{2} \\ t - 1 + e^{-t} \end{array}\right] \blacksquare$$

Problem 47.5
Compute and graph $f * g$ where $f(t) = h(t)$ and $g(t) = t[h(t) - h(t-2)]$.

Solution.
Since $f(t) = h(t)$, $F(s) = \frac{1}{s}$. Similarly, since $g(t) = th(t) - th(t-2) = th(t) - (t-2)h(t-2) - 2h(t-2)$, $G(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$. Thus, $F(s)G(s) = \frac{1}{s^3} - \frac{e^{-2s}}{s^3} - \frac{2e^{-2s}}{s^2}$. It follows that

$$(f * g)(t) = \frac{t^2}{2} - \frac{(t-2)^2}{2}h(t-2) - 2(t-2)h(t-2).$$

The graph of $(f * g)(t)$ is given below $\blacksquare$

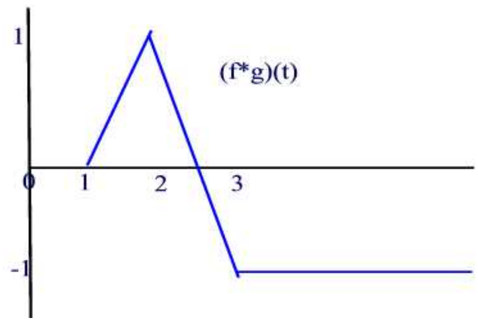


Problem 47.6
Compute and graph $f * g$ where $f(t) = h(t) - h(t-1)$ and $g(t) = h(t-1) - 2h(t-2)]$.

Solution.
Since $f(t) = h(t) - h(t-1)$, $F(s) = \frac{1}{s} - \frac{e^{-s}}{s}$. Similarly, since $g(t) = h(t-1) - 2h(t-2)$, $G(s) = \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s}$. Thus, $F(s)G(s) = \frac{e^{-s} - 3e^{-2s} + 2e^{-3s}}{s^2}$. It follows that

$$(f * g)(t) = (t-1)h(t-1) - 3(t-2)h(t-2) + 2(t-3)h(t-3).$$

The graph of $(f * g)(t)$ is given below $\blacksquare$



Problem 47.7
Compute $t * t * t$.

Solution.
By the convolution theorem we have $\mathcal{L}[t*t*t] = (\mathcal{L}[t])^3 = \left(\frac{1}{s^2}\right)^3 = \frac{1}{s^6}$. Hence, $t * t * t = \mathcal{L}^{-1}\left[\frac{1}{s^6}\right] = \frac{t^5}{5!} = \frac{t^5}{120}$ ■

Problem 47.8
Compute $h(t) * e^{-t} * e^{-2t}$.

Solution.
By the convolution theorem we have $\mathcal{L}[h(t)*e^{-t}*e^{-2t}] = \mathcal{L}[h(t)]\mathcal{L}[e^{-t}]\mathcal{L}[e^{-2t}] = \frac{1}{s} \cdot \frac{1}{s+1} \cdot \frac{1}{s+2}$. Using the partial fractions decomposition we can write

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2}.$$

Hence,

$$h(t) * e^{-t} * e^{-2t} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \blacksquare$$

Problem 47.9
Compute $t * e^{-t} * e^t$.

Solution.
By the convolution theorem we have $\mathcal{L}[t * e^{-t} * e^t] = \mathcal{L}[t]\mathcal{L}[e^{-t}]\mathcal{L}[e^t] = \frac{1}{s^2} \cdot \frac{1}{s+1} \cdot \frac{1}{s-1}$. Using the partial fractions decomposition we can write

$$\frac{1}{s^2(s+1)(s-1)} = -\frac{1}{s^2} - \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1}.$$

Hence,

$$t * e^{-t} * e^t = -t + \frac{e^t}{2} - \frac{e^{-t}}{2} \blacksquare$$

Problem 47.10

Suppose it is known that $\overbrace{h(t) * h(t) * \cdots * h(t)}^{n \ functions} = Ct^8$. Determine the constants C and the poisitive integer n.

Solution.

We know that $\mathcal{L}[\overbrace{h(t) * h(t) * \cdots * h(t)}^{n \ functions}] = (\mathcal{L}[h(t)])^n = \frac{1}{s^n}$ so that $\mathcal{L}^{-1}[\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!} = Ct^8$. It follows that $n = 9$ and $C = \frac{1}{8!}$ $\blacksquare$

Problem 47.11

Use Laplace transform to solve for $y(t)$:

$$\int_0^t \sin{(t-\lambda)} y(\lambda) d\lambda = t^2.$$

Solution.

Note that the given equation reduces to $\sin t * y(t) = t^2$. Taking Laplace transform of both sides we find $\frac{Y(s)}{s^2+1} = \frac{2}{s^3}$. This implies $Y(s) = \frac{2(s^2+1)}{s^3} = \frac{2}{s} + \frac{2}{s^3}$. Hence, $y(t) = \mathcal{L}^{-1}[\frac{2}{s} + \frac{2}{s^3}] = 2 + t^2$ $\blacksquare$

Problem 47.12

Use Laplace transform to solve for $y(t)$:

$$y(t) - \int_0^t e^{(t-\lambda)} y(\lambda) d\lambda = t.$$

Solution.

Note that the given equation reduces to $e^t * y(t) = y(t) - t$. Taking Laplace transform of both sides we find $\frac{Y(s)}{s-1} = Y(s) - \frac{1}{s^2}$. Solving for $Y(s)$ we find $Y(s) = \frac{s-1}{s^2(s-2)}$. Using partial fractions decomposition we can write

$$\frac{s-1}{s^2(s-2)} = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{(s-2)}.$$

Hence,

$$y(t) = -\frac{1}{4} + \frac{t}{2} + \frac{1}{4}e^{2t} \blacksquare$$

Problem 47.13
Use Laplace transform to solve for $y(t)$:

$$t * y(t) = t^2(1 - e^{-t}).$$

Solution.
Taking Laplace transform of both sides we find $\frac{Y(s)}{s^2} = \frac{2}{s^3} - \frac{2}{(s+1)^3}$. This implies $Y(s) = \frac{2}{s} - \frac{2s^2}{(s+1)^3}$. Using partial fractions decomposition we can write

$$\frac{s^2}{(s+1)^3} = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}.$$

Hence,

$$y(t) = 2 - 2(e^{-t} - 2te^{-t} + \frac{t^2}{2}e^{-t} = 2\left(1 - (1 - 2t + \frac{t^2}{2})e^{-t}\right) \blacksquare$$

Problem 47.14
Use Laplace transform to solve for $y(t)$:

$$\mathbf{y}' = h(t) * \mathbf{y}, \quad \mathbf{y}(0) = \left[\begin{array}{c} 1 \\ 2 \end{array}\right].$$

Solution.
Taking Laplace transform of both sides we find $s\mathbf{Y} - \mathbf{y}(0) = \frac{1}{s}\mathbf{Y}$. Solving for $\mathbf{Y}$ we find

$$\mathbf{Y}(s) = \frac{s}{s^2 - 1}\left[\begin{array}{c} 1 \\ 2 \end{array}\right] = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right)\left[\begin{array}{c} 1 \\ 2 \end{array}\right].$$

Hence,

$$\mathbf{y}(t) = \frac{1}{2}(e^t + e^{-t})\left[\begin{array}{c} 1 \\ 2 \end{array}\right] = \cosh t\left[\begin{array}{c} 1 \\ 2 \end{array}\right] \blacksquare$$

Problem 47.15
Solve the following initial value problem.

$$y' - y = \int_0^t (t - \lambda)e^{\lambda}d\lambda, \quad y(0) = -1.$$

Solution.
Note that $y' - y = t * e^t$. Taking Lalplace transform of both sides we find $sY - (-1) - Y = \frac{1}{s^2} \cdot \frac{1}{s-1}$. This implies that $Y(s) = -\frac{1}{s-1} + \frac{1}{s^2(s-1)^2}$. Using partial fractions decomposition we can write

$$\frac{1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}.$$

Thus,

$$Y(s) = -\frac{1}{s-1} + \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{1}{(s-1)^2}.$$

Finally,

$$y(t) = 2 + t - 3e^t + te^t \blacksquare$$

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49 The Dirac Delta Function and Impulse Response

Problem 49.1
Evaluate

(a) $\int_0^3 (1+e^{-t})\delta(t-2)dt.$
(b) $\int_{-2}^1 (1+e^{-t})\delta(t-2)dt.$
(c) $\int_{-1}^2 \begin{bmatrix} \cos 2t \\ te^{-t} \end{bmatrix} \delta(t)dt.$
(d) $\int_{-1}^2 (e^{2t}+t) \begin{bmatrix} \delta(t+2) \\ \delta(t-1) \\ \delta(t-3) \end{bmatrix} dt.$

Solution.
(a) $\int_0^3 (1+e^{-t})\delta(t-2)dt = 1+e^{-2}.$
(b) $\int_{-2}^1 (1+e^{-t})\delta(t-2)dt = 0$ since 2 lies outside the integration interval.
(c)

$$\begin{aligned}
\int_{-1}^2 \begin{bmatrix} \cos 2t \\ te^{-t} \end{bmatrix} \delta(t)dt &= \begin{bmatrix} \int_{-1}^2 \cos 2t\delta(t)dt \\ \int_{-1}^2 te^{-t}\delta(t)dt \end{bmatrix} \\
&= \begin{bmatrix} \cos 0 \\ 0 \times t^0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
\int_{-1}^2 (e^{2t}+t) \begin{bmatrix} \delta(t+2) \\ \delta(t-1) \\ \delta(t-3) \end{bmatrix} dt \\
&= \begin{bmatrix} \int_{-1}^2 (e^{2t}+t)\delta(t+2)dt \\ \int_{-1}^2 (e^{2t}+t)\delta(t-1)dt \\ \int_{-1}^2 (e^{2t}+t)\delta(t-3)dt \end{bmatrix} \\
&= \begin{bmatrix} e^{-4}-2 \\ e^2+1 \\ 0 \end{bmatrix} \blacksquare
\end{aligned}$$

Problem 49.2
Let $f(t)$ be a function defined and continuous on $0 \le t < \infty$. Determine

$$(f * \delta)(t) = \int_0^t f(t-s)\delta(s)ds.$$

Solution.
Let $g(s) = f(t-s)$. Then

$$\begin{array}{rcl} (f*\delta)(t) & = & \int_0^t f(t-s)\delta(s)ds = \int_0^t g(s)\delta(s)ds \\ & = & g(0) = f(t) \blacksquare \end{array}$$

Problem 49.3
Determine a value of the constant t_0 such that $\int_0^1 \sin^2[\pi(t-t_0)]\delta(t-\frac{1}{2})dt = \frac{3}{4}$.

Solution.
We have

$$\begin{array}{rcl} \int_0^1 \sin^2[\pi(t-t_0)]\delta(t-\frac{1}{2})dt & = & \frac{3}{4} \\ \sin^2\left[\pi\left((\frac{1}{2}-t_0\right)\right] & = & \frac{3}{4} \\ \sin\left[\pi\left((\frac{1}{2}-t_0\right)\right] & = & \pm\frac{\sqrt{3}}{2} \end{array}$$

Thus, a possible value is when $\pi\left((\frac{1}{2}-t_0\right) = \frac{\pi}{3}$. Solving for t_0 we find $t_0 = \frac{1}{6}$ $\blacksquare$

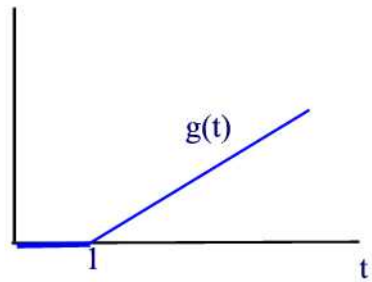
Problem 49.4
If $\int_1^5 t^n\delta(t-2)dt = 8$, what is the exponent n?

Solution.
We have $\int_1^5 t^n\delta(t-2)dt = 2^n = 8$. Thus, $n = 3$ $\blacksquare$

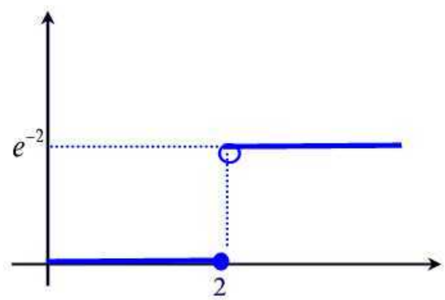
Problem 49.5
Sketch the graph of the function $g(t)$ which is defined by $g(t) = \int_0^t\int_0^s \delta(u-1)duds,\ 0 \le t < \infty$.

Solution.
Note first that $\int_0^s \delta(u-1)du = 1$ if $s > 1$ and 0 otherwise. Hence,

$$g(t) = \left\{ \begin{array}{cc} 0, & \text{if } t \le 1 \\ \int_1^t h(s-1)ds = t-1, & \text{if } t > 1 \end{array} \right. \blacksquare$$



Problem 49.6
The graph of the function $g(t) = \int_0^t e^{\alpha t}\delta(t - t_0)dt,\ 0 \le t < \infty$ is shown. Determine the constants α and t_0.



Solution.
Note that

$$g(t) = \begin{cases} 0, & 0 \le t \le t_0 \\ e^{\alpha t_0}, & t_0 < t < \infty \end{cases}$$

It follows that $t_0 = 2$ and $\alpha = -1$ ■

Problem 49.7
(a) Use the method of integarting factor to solve the initial value problem $y' - y = h(t),\ y(0) = 0$.
(b) Use the Laplace transform to solve the initial value problem $\phi' - \phi = \delta(t),\ \phi(0) = 0$.
(c) Evaluate the convolution $\phi * h(t)$ and compare the resulting function with the solution obtained in part(a).

Solution.
(a) Using the method of integrating factor we find, for $t \geq 0$,

$$\begin{aligned} y' - y &= h(t) \\ (e^{-t}y)' &= e^{-t} \\ e^{-t}y &= -e^{-t} + C \\ y &= -1 + Ce^t \\ y &= -1 + e^t \end{aligned}$$

(b) Taking Laplace of both sides we find $s\mathbf{\Phi} - \mathbf{\Phi} = 1$ or $\mathbf{\Phi}(s) = \frac{1}{s-1}$. Thus, $\phi(t) = e^t$.
(c) We have

$$(\phi * h)(t) = \int_0^t e^{(t-s)}h(s)ds = \int_0^t e^{(t-s)}ds = -1 + e^t \blacksquare$$

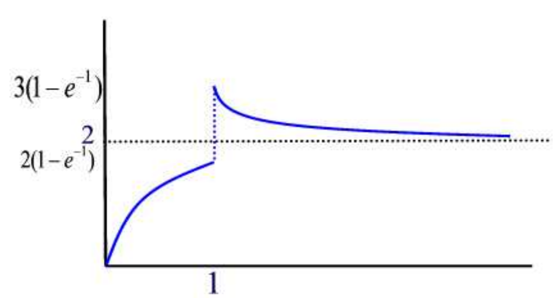
Problem 49.8
Solve the initial value problem

$$y' + y = 2 + \delta(t-1), \quad y(0) = 0, \ 0 \leq t \leq 6.$$

Graph the solution on the indicated interval.

Solution.
Taking Laplace of both sides to obtain $sY + Y = \frac{2}{s} + e^{-s}$. Thus, $Y(s) = \frac{2}{s(s+1)} + \frac{e^{-s}}{s+1} = \frac{2}{s} - \frac{2}{s+1} + \frac{e^{-s}}{s+1}$. Hence,

$$y(t) = \begin{cases} 2 - 2e^{-t}, & t < 1 \\ 2 + (e-2)e^{-t}, & t \geq 1 \end{cases} \blacksquare$$



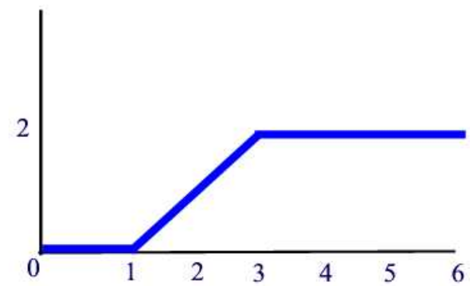
Problem 49.9
Solve the initial value problem

$$y'' = \delta(t-1) - \delta(t-3), \quad y(0) = 0, \; y'(0) = 0, \; 0 \le t \le 6.$$

Graph the solution on the indicated interval.

Solution. Taking Laplace of both sides to obtain $s^2Y = e^{-s} - e^{-3s}$. Thus, $Y(s) = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}$. Hence,

$$y(t) = (t-1)h(t-1) - (t-3)h(t-3).$$



Problem 49.10
Solve the initial value problem

$$y'' - 2y' = \delta(t-1), \quad y(0) = 1, \; y'(0) = 0, \; 0 \le t \le 2.$$

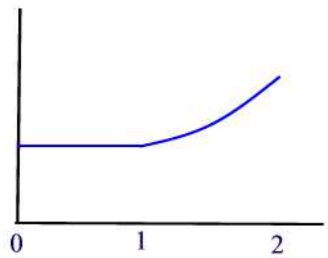
Graph the solution on the indicated interval.

Solution.
Taking Laplace transform of both sides and using the initial conditions we find

$$s^2Y - s - 2(sY - 1) = e^{-s}.$$

Solving for s we find $Y(s) = \frac{1}{s} + \frac{e^{-s}}{s(s-2)} = \frac{1}{s} - \frac{e^{-s}}{2s} + \frac{e^{-s}}{s-2}$. Hence,

$$y(t) = 1 - \frac{1}{2}h(t-1) + \frac{1}{2}e^{2(t-1)}h(t-1) \blacksquare$$

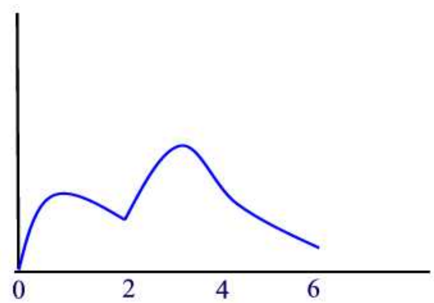


Problem 49.11
Solve the initial value problem

$$y'' + 2y' + y = \delta(t-2), \quad y(0) = 0,\ y'(0) = 1,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

Solution.
Taking Laplace transform of both sides to obtain $s^2Y - 1 + 2sY + Y = e^{-2s}$. Solving for $Y(s)$ we find $Y(s) = \frac{1}{(s+1)^2} + \frac{e^{-2s}}{(s+1)^2}$. Therefore, $y(t) = te^{-t} + (t-2)e^{-(t-2)}h(t-2)$ ■



49 Solving Systems of Differential Equations Using Laplace Transform

Problem 49.1
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \frac{d}{dt}\left[\begin{array}{c} e^{-t}\cos 2t \\ 0 \\ t+e^t \end{array}\right]$$

Solution.

$$\begin{aligned}
\mathcal{L}[\mathbf{y}(t)] &= \mathcal{L}\left\{\left[\begin{array}{c} -e^{-t}\cos 2t + 2e^{-t}\sin 2t \\ 0 \\ 1+e^t \end{array}\right]\right\} \\
&= \left[\begin{array}{c} -\frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4} \\ 0 \\ \frac{1}{s}+\frac{1}{s-1} \end{array}\right] \\
&= \left[\begin{array}{c} \frac{3-s}{(s+1)^2+4} \\ 0 \\ \frac{1}{s}+\frac{1}{s-1} \end{array}\right] \blacksquare
\end{aligned}$$

Problem 49.2
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \int_0^t \left[\begin{array}{c} 1 \\ u \\ e^{-u} \end{array}\right] du$$

Solution.

$$\begin{aligned}
\mathcal{L}[\mathbf{y}(t)] &= \mathcal{L}\left\{\left[\begin{array}{c} t \\ \frac{t^2}{2} \\ -e^{-t}+1 \end{array}\right]\right\} \\
&= \left[\begin{array}{c} \frac{1}{s^2} \\ \frac{1}{s^3} \\ \frac{1}{s}-\frac{1}{s+1} \end{array}\right] \blacksquare
\end{aligned}$$

Problem 49.3
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{2}{s^2+2s+2} \\ \frac{1}{s^2+s} \end{bmatrix}$$

Solution.
We have

$$\begin{array}{rcl} \mathcal{L}^{-1}[\frac{1}{s}] & = & 1 \\ \mathcal{L}^{-1}[\frac{2}{s^2+2s+2}] & = & \mathcal{L}^{-1}[\frac{2}{(s+1)^2+1^2}] = 2e^{-t}\sin t \\ \mathcal{L}^{-1}[\frac{1}{s^2+s}] & = & \mathcal{L}^{-1}[\frac{1}{s} - \frac{1}{s+1}] = 1 - e^{-t} \end{array}$$

Thus,

$$\mathcal{L}^{-1}[\mathbf{Y}(s)] = \begin{bmatrix} 1 \\ 2e^{-t}\sin t \\ 1 - e^{-t} \end{bmatrix} \blacksquare$$

Problem 49.4
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{L}[t^3] \\ \mathcal{L}[e^{2t}] \\ \mathcal{L}[\sin t] \end{bmatrix}$$

Solution.
We have

$$\begin{array}{rcl} \mathbf{Y}(s) & = & \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{L}[t^3] \\ \mathcal{L}[e^{2t}] \\ \mathcal{L}[\sin t] \end{bmatrix} \\ & = & \mathcal{L}\left\{ \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} t^3 \\ e^{2t} \\ \sin t \end{bmatrix} \right\} \\ & = & \mathcal{L}\left\{ \begin{bmatrix} t^3 - e^{2t} + 2\sin t \\ 2t^3 + 3\sin t \\ t^3 - 2e^{2t} + \sin t \end{bmatrix} \right\} \end{array}$$

Thus,

$$\mathcal{L}^{-1}[\mathbf{Y}(s)] = \begin{bmatrix} t^3 - e^{2t} + 2\sin t \\ 2t^3 + 3\sin t \\ t^3 - 2e^{2t} + \sin t \end{bmatrix} \blacksquare$$

Problem 49.5
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$s\mathbf{Y} = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} 0 & \frac{1}{s} \end{bmatrix}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\mathbf{Y}(s) = \begin{bmatrix} -\frac{4}{s^2(s-1)} \\ \frac{s-5}{s^2(s-1)} \end{bmatrix}.$$

Using partial fractions decomposition we find

$$-\frac{4}{s^2(s-1)} = \frac{4}{s^2} + \frac{4}{s} - \frac{4}{s-1}$$

and

$$\frac{s-5}{s^2(s-1)} = \frac{5}{s^2} + \frac{4}{s} - \frac{4}{s-1}.$$

Hence,

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{4}{s^2} + \frac{4}{s} - \frac{4}{s-1} \\ \frac{5}{s^2} + \frac{4}{s} - \frac{4}{s-1} \end{bmatrix}.$$

Finally,

$$\mathbf{y}(t) = \begin{bmatrix} 4t + 4 - 4e^t \\ 5t + 4 - 4e^t \end{bmatrix} \blacksquare$$

Problem 49.6
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$s\mathbf{Y} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}\mathbf{Y}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\begin{aligned} \mathbf{Y}(s) &= \tfrac{1}{(s-1)(s-2)}\begin{bmatrix} s+2 & -4 \\ 3 & s-5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \tfrac{1}{(s-1)(s-2)}\begin{bmatrix} 3s-2 \\ 2s-1 \end{bmatrix} \end{aligned}$$

Using partial fractions decomposition we find

$$\frac{3s-2}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{4}{s-2}$$

and

$$\frac{2s-1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{3}{s-2}.$$

Hence,

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{-1}{s-1} + \frac{4}{s-2} \\ \frac{-1}{s-1} + \frac{3}{s-2} \end{bmatrix}.$$

Finally,

$$\mathbf{y}(t) = \begin{bmatrix} -e^t + 4e^{2t} \\ -e^t + 3e^{2t} \end{bmatrix} \blacksquare$$

Problem 49.7
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 3e^t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$\begin{aligned} s\mathbf{Y} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} 0 \\ \frac{3}{s-1} \end{bmatrix} \\ \begin{bmatrix} s-1 & -4 \\ 1 & s-1 \end{bmatrix}\mathbf{Y} &= \begin{bmatrix} 3 \\ \frac{3}{s-1} \end{bmatrix} \end{aligned}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\begin{aligned}\mathbf{Y}(s) &= \frac{1}{(s-1)^2+4}\begin{bmatrix} s-1 & 4 \\ -1 & s-1 \end{bmatrix}\begin{bmatrix} 3 \\ \frac{3}{s-1} \end{bmatrix} \\ &= \frac{1}{(s-1)^2+4}\begin{bmatrix} 3(s-1)+\frac{12}{s-1} \\ 0 \end{bmatrix}\end{aligned}$$

Using partial fractions decomposition we find

$$\frac{3(s-1)+\frac{12}{s-1}}{(s-1)^2+4} = 3\frac{s-1}{(s-1)^2+4} + \frac{12}{(s-1)[(s-1)^2+4]}.$$

But

$$\frac{12}{(s-1)[(s-1)^2+4]} = \frac{3}{s-1} - 3\frac{s-1}{(s-1)^2+4}.$$

Hence,

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{3}{s-1} \\ 0 \end{bmatrix}.$$

Finally,

$$\mathbf{y}(t) = \begin{bmatrix} 3e^t \\ 0 \end{bmatrix} \blacksquare$$

Problem 49.8
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{y}'(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$\begin{aligned} s^2\mathbf{Y} - s\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{Y} \\ \begin{bmatrix} s^2+3 & 2 \\ -4 & s^2-3 \end{bmatrix}\mathbf{Y} &= \begin{bmatrix} s \\ 1 \end{bmatrix} \end{aligned}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\begin{aligned}\mathbf{Y}(s) &= \tfrac{1}{s^4-1}\left[\begin{array}{cc} s^2-3 & -2 \\ 4 & s^2+3\end{array}\right]\left[\begin{array}{c} s \\ 1\end{array}\right] \\ &= \tfrac{1}{s^4-1}\left[\begin{array}{c} s^3-3s-2 \\ s^2+4s+3\end{array}\right]\end{aligned}$$

Using partial fractions decomposition we find

$$\frac{s^3-3s-2}{s^4-1} = -\frac{1}{s-1} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

and

$$\frac{s^2+4s+3}{(s-1)(s+1)(s^2+1)} = \frac{s+3}{(s-1)(s^2+1)} = \frac{2}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}.$$

Hence,

$$\mathbf{Y}(s) = \left[\begin{array}{c} -\frac{1}{s-1} + 2\frac{s}{s^2+1} + \frac{1}{s^2+1} \\ \frac{2}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\end{array}\right].$$

Finally,

$$\mathbf{y}(t) = \left[\begin{array}{c} -e^t + 2\cos t + \sin t \\ 2e^t - \cos t - \sin t\end{array}\right] \blacksquare$$

Problem 49.9
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \left[\begin{array}{cc} 1 & -1 \\ 1 & -1\end{array}\right]\mathbf{y} + \left[\begin{array}{c} 2 \\ 1\end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 0 \\ 1\end{array}\right], \ \mathbf{y}'(0) = \left[\begin{array}{c} 0 \\ 0\end{array}\right]$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$\begin{aligned} s^2\mathbf{Y} - s\left[\begin{array}{c} 0 \\ 1\end{array}\right] &= \left[\begin{array}{cc} 1 & -1 \\ 1 & -1\end{array}\right]\mathbf{Y} + \left[\begin{array}{c} \frac{2}{s} \\ \frac{1}{s}\end{array}\right] \\ \left[\begin{array}{cc} s^2-1 & 1 \\ -1 & s^2+1\end{array}\right]\mathbf{Y} &= \left[\begin{array}{c} \frac{2}{s} \\ \frac{1}{s}+s\end{array}\right]\end{aligned}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\begin{aligned}\mathbf{Y}(s) &= \frac{1}{s^4}\left[\begin{array}{cc} s^2+1 & -1 \\ 1 & s^2-1 \end{array}\right]\left[\begin{array}{c} \frac{2}{s} \\ \frac{1}{s}+s \end{array}\right] \\ &= \left[\begin{array}{c} \frac{1}{s^3}+\frac{1}{s^5} \\ \frac{1}{s}+\frac{1}{s^5} \end{array}\right]\end{aligned}$$

$$\mathbf{y}(t) = \left[\begin{array}{c} \frac{t^2}{2}+\frac{t^3}{4!} \\ 1+\frac{t^4}{4!} \end{array}\right] \blacksquare$$

Problem 49.10
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} e^t \\ 1 \\ -2t \end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

Solution.
Taking Laplace of both sides and using the initial condition we find

$$\begin{aligned}s\mathbf{Y} &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]\mathbf{Y} + \left[\begin{array}{c} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{array}\right] \\ \left[\begin{array}{ccc} s-1 & 0 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s-2 \end{array}\right]\mathbf{Y} &= \left[\begin{array}{c} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{array}\right]\end{aligned}$$

Solving this matrix equation for $\mathbf{Y}$ we find

$$\begin{aligned}\mathbf{Y}(s) &= \left[\begin{array}{ccc} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s-2)} \\ 0 & 0 & \frac{1}{s-2} \end{array}\right]\left[\begin{array}{c} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{array}\right] \\ &= \left[\begin{array}{c} \frac{1}{(s-1)^2} \\ \frac{s(s-2)-2}{s^2(s+1)(s-2)} \\ \frac{-2}{s^2(s-2)} \end{array}\right]\end{aligned}$$

Using partial fractions decomposition we find

$$\frac{s(s-2)-2}{s^2(s+1)(s-2)} = \frac{1}{s^2} + \frac{1}{2s} - \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s-2}$$

and

$$\frac{-2}{s^2(s-2)} = \frac{1}{s^2} + \frac{1}{2s} - \frac{1}{2}\cdot\frac{1}{s-2}.$$

Hence,

$$\mathbf{Y}(s) = \left[\begin{array}{c} \frac{1}{(s-1)^2} \\ \frac{1}{s^2} + \frac{1}{2s} - \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s-2} \\ \frac{1}{s^2} + \frac{1}{2s} - \frac{1}{2}\cdot\frac{1}{s-2} \end{array}\right].$$

Finally,

$$\mathbf{y}(t) = \left[\begin{array}{c} te^t \\ t + \frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{2t} \\ t + \frac{1}{2} - \frac{1}{2}e^{2t} \end{array}\right] \blacksquare$$

Problem 49.11

The Laplace transform was applied to the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y},\ \mathbf{y}(0) = \mathbf{y}_0$, where $\mathbf{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right]$, $\mathbf{A}$ is a 2×2 constant matrix, and $\mathbf{y}_0 = \left[\begin{array}{c} y_{1,0} \\ y_{2,0} \end{array}\right]$. The following transform domain solution was obtained

$$\mathcal{L}[\mathbf{y}(t)] = \mathbf{Y}(s) = \frac{1}{s^2-9s+18}\left[\begin{array}{cc} s-2 & -1 \\ 4 & s-7 \end{array}\right]\left[\begin{array}{c} y_{1,0} \\ y_{2,0} \end{array}\right].$$

(a) what are the eigenvalues of $\mathbf{A}$?
(b) Find $\mathbf{A}$.

Solution.

(a) $det(s\mathbf{I} - \mathbf{A}) = s^2 - 9s + 18 = (s-3)(s-6) = 0$. Hence, the eigenvalues of $\mathbf{A}$ are $r_1 = 3$ and $r_2 = 6$.
(b) Taking Laplace transform of both sides of the differential equation we find

$$\begin{array}{rcl} s\mathbf{Y} - \mathbf{y}_0 & = & \mathbf{A}\mathbf{Y} \\ \mathbf{Y} & = & (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}_0 \end{array}$$

Letting $s = 0$ we find

$$\mathbf{Y}(0) = -\mathbf{A}^{-1} = \frac{1}{18}\left[\begin{array}{cc} -2 & -1 \\ 4 & -7 \end{array}\right]$$

Hence,

$$\mathbf{A}^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix}$$

and

$$det(\mathbf{A}^{-1}) = \frac{1}{18}.$$

It follows that

$$\mathbf{A} = (\mathbf{A}^{-1})^{-1} = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix} \blacksquare$$

50 Numerical Methods for Solving First Order Linear Systems: Euler's Method

In Problems 50.1 - 50.3 answer the following questions:
(a) Solve the differential equation analytically using the appropriate method of solution.
(b) Write the Euler's iterates: $y_{k+1} = y_k + hf(t_k, y_k)$.
(c) Using step size $h = 0.1$, compute the Euler approximations $y_k,\ k = 1, 2, 3$ at times $t_k = a + kh$.
(d) For $k = 1, 2, 3$ compute the error $y(t_k) - y_k$ where $y(t_k)$ is the exact value of y at t_k.

Problem 50.1

$$y' = 2t - 1, \quad y(1) = 0.$$

Solution.
(a) $y = t^2 - t$
(b) $y_{k+1} = y_k + h(2t_k - 1)$
(c)

$$\begin{aligned} y_1 =& y_0 + 0.1(2t_0 - 1) = 0 + 0.1(2(1) - 1) = 0.1 \\ y_2 =& y_1 + 0.1(2t_1 - 1) = 0.1 + 0.1(2(1.1) - 1) = 0.22 \\ y_3 =& y_2 + 0.1(2t_2 - 1) = 0.22 + 0.1(2(1.2) - 1) = 0.36 \end{aligned}$$

(d)

$$\begin{aligned} y_1^{\text{err}} =& 0.11 - 0.1 = 0.01 \\ y_2^{\text{err}} =& 0.24 - 0.22 = 0.02 \\ y_3^{\text{err}} =& 0.39 - 0.36 = 0.03 \ \blacksquare \end{aligned}$$

Problem 50.2

$$y' = -ty, \quad y(0) = 1.$$

Solution.
(a) Using the method of integrating factor we find $y = Ce^{-\frac{t^2}{2}}$. Since $y(0) = 1$, we find $c = 1$ so that $y = e^{-\frac{t^2}{2}}$.

(b) $y_{k+1} = y_k - h(t_k y_k)$
(c)

$$\begin{aligned} y_1 =& y_0 - 0.1(t_0 y_0) = 1 - 0.1(0 \times 1) = 1 \\ y_2 =& y_1 - 0.1(t_1 y_1) = 1 - 0.1(0.1)(1) = 0.99 \\ y_3 =& y_2 - 0.1(t_2 y_2) = 0.99 - 0.1(0.2 \times 0.99) = 0.9702 \end{aligned}$$

(d)

$$\begin{aligned} y_1^{\text{err}} =& 0.99501 - 1 = -0.00499 \\ y_2^{\text{err}} =& 0.98109 - 0.99 = -0.00891 \\ y_3^{\text{err}} =& 0.95599 - 0.9702 = -0.01421 \blacksquare \end{aligned}$$

Problem 50.3

$$y' = y^2, \quad y(0) = 1.$$

Solution.
(a) Using the method of integrating factor we find $-\frac{1}{y} = t + C$. Since $y(0) = 1$, we find $c = -1$ so that $y = \frac{1}{1-t}$.
(b) $y_{k+1} = y_k + hy_k^2$
(c)

$$\begin{aligned} y_1 =& y_0 + 0.1y_0^2 = 1 + 0.1(1^2) = 1.1 \\ y_2 =& y_1 + 0.1y_1^2 = 1.1 + 0.1(1.1)^2 = 1.221 \\ y_3 =& y_2 + 0.1y_2^2 = 1.221 + 0.1(1.221)^2 = 1.370084 \end{aligned}$$

(d)

$$\begin{aligned} y_1^{\text{err}} =& 1.1111 - 1.1 = 0.0111 \\ y_2^{\text{err}} =& 1.25 - 1.221 = 0.029 \\ y_3^{\text{err}} =& 1.4286 - 1.370084 = 0.058516 \blacksquare \end{aligned}$$

In Problems 50.4 - 50.6 answer the following questions:
(a) Write the Euler's method algorithm in explicit form. Specify the starting values t_0 and $\mathbf{y}_0$.
(b) Give a formula for the kth $t-$value, t_k. What is the range of the index k if we choose $h = 0.01$?
(c) Use a calculator to carry out two steps of Euler's method, finding $\mathbf{y}_1$ and $\mathbf{y}_2$.

Problem 50.4

$$\mathbf{y}' = \begin{bmatrix} -t^2 & t \\ 2-t & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}, \ \mathbf{y}(1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad 1 \le t \le 4.$$

Solution.
(a)

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h\left(\begin{bmatrix} -t_k^2 & t_k \\ 2-t_k & 0 \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} 1 \\ t_k \end{bmatrix}\right)$$

where $t_0 = 1$ and

$$\mathbf{y}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(b) We have $t_k = 1 + kh$. Now, $h = \frac{b-a}{N} \to 0.01 = \frac{3}{N} \to N = 300$. Thus, the range of the index k is $0 \le k \le 300$.
(c) We have

$$\begin{aligned}
\mathbf{y}_1 =& \mathbf{y}_0 + h\left(\begin{bmatrix} -t_0^2 & t_0 \\ 2-t_0 & 0 \end{bmatrix} \mathbf{y}_0 + \begin{bmatrix} 1 \\ t_k \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0.01\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} \\
\mathbf{y}_2 =& \mathbf{y}_1 + h\left(\begin{bmatrix} -t_1^2 & t_1 \\ 2-t_1 & 0 \end{bmatrix} \mathbf{y}_1 + \begin{bmatrix} 1 \\ t_1 \end{bmatrix}\right) = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + 0.01\left(\begin{bmatrix} -(1.01)^2 & 1.01 \\ 0.99 & 0 \end{bmatrix}\begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.01 \end{bmatrix}\right.
\end{aligned}$$

Problem 50.5

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 2 \\ t \end{bmatrix}, \ \mathbf{y}(-1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad -1 \le t \le 0.$$

Solution.
(a)

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h\left(\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} 0 \\ 2 \\ t_k \end{bmatrix}\right)$$

where $t_0 = 1$ and

$$\mathbf{y}_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b) We have $t_k = -1 + kh$. Now, $h = \frac{b-a}{N} \to 0.01 = \frac{1}{N} \to N = 100$. Thus, the range of the index k is $0 \le k \le 100$.
(c) We have

$$\begin{aligned}
\mathbf{y}_1 =& \mathbf{y}_0 + h\left(\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_0 + \begin{bmatrix} 0 \\ 2 \\ t_0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0.01\left(\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & \\ 2 & -1 \end{bmatrix}\right) = \Bigg[\\
\mathbf{y}_2 =& \mathbf{y}_1 + h\left(\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_1 + \begin{bmatrix} 0 \\ 2 \\ t_1 \end{bmatrix}\right) = \begin{bmatrix} 1.02 \\ 0.06 \\ 1.00 \end{bmatrix} + 0.01\left(\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1.02 \\ 0.06 \\ 1.00 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -0.99 \end{bmatrix}\right)
\end{aligned}$$

Problem 50.6

$$\mathbf{y}' = \begin{bmatrix} \frac{1}{t} & \sin t \\ 1-t & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}, \ \mathbf{y}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1 \le t \le 6.$$

Solution.
(a)

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h\left(\begin{bmatrix} \frac{1}{t_k} & \sin t_k \\ 1-t_k & 1 \end{bmatrix}\mathbf{y}_k + \begin{bmatrix} 0 \\ t_k^2 \end{bmatrix}\right)$$

where $t_0 = 1$ and

$$\mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) We have $t_k = 1 + kh$. Now, $h = \frac{b-a}{N} \to 0.01 = \frac{5}{N} \to N = 500$. Thus, the range of the index k is $0 \le k \le 500$.
(c) We have

$$\begin{aligned}
\mathbf{y}_1 =& \mathbf{y}_0 + h\left(\begin{bmatrix} \frac{1}{t_0} & \sin t_0 \\ 1-t_0 & 1 \end{bmatrix}\mathbf{y}_0 + \begin{bmatrix} 0 \\ t_0^2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.01\left(\begin{bmatrix} 1 & \sin 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.0 \end{bmatrix} \\
\mathbf{y}_2 =& \mathbf{y}_1 + h\left(\begin{bmatrix} \frac{1}{t_1} & \sin t_1 \\ 1-t_1 & 1 \end{bmatrix}\mathbf{y}_1 + \begin{bmatrix} 0 \\ t_1^2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + 0.01\left(\begin{bmatrix} \frac{1}{1.01} & \sin 1.01 \\ 0.01 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + \Big[\ (1
\end{aligned}$$

In Problems 50.7 - 50.8 answer the following questions.
(a) Rewrite the given initial value problem as an equivalent initial value problem for a first order system, using the substitution $z_1 = y, z_2 = y', z_3 = y'', \cdots$.
(b) Write the Euler's method algorithm $\mathbf{z}_{k+1} = \mathbf{z}_k + h[P(t_k)\mathbf{z}_k + \mathbf{g}(t_k)]$, in explicit form. Specify the starting values t_0 and $\mathbf{z}_0$.

(c) Using a calculator with step size $h = 0.01$, carry out two steps of Euler's method, finding $\mathbf{z}_1$ and $\mathbf{z}_2$ What are the corresponding numerical approximations to the solution $\mathbf{y}(t)$ at times $t = 0.01$ and $t = 0.02$?

Problem 50.7

$$y'' + y' + t^2 y = 2, \quad y(1) = 1, \quad y'(1) = 1.$$

Solution.
(a) Let $z_1 = y$ and $z_2 = y'$ so that $z_1' = z_2$ and $z_2' = y''$. Thus,

$$\begin{aligned} z_1' =& z_2 \\ z_2' =& y'' = 2 - t^2 y - y' = 2 - t^2 z_1 - z_1' \\ =& - z_2 - t^2 z_1 + 2 \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{Z}'(t) = \left[\begin{array}{c} z_1 \\ z_2 \end{array} \right] =& \left[\begin{array}{c} z_2 \\ -t^2 z_1 - z_2 + 2 \end{array} \right] \\ =& \left[\begin{array}{cc} 0 & 1 \\ -t^2 & -1 \end{array} \right] \left[\begin{array}{c} z_1 \\ z_2 \end{array} \right] + \left[\begin{array}{c} 0 \\ 2 \end{array} \right] \end{aligned}$$

with

$$\mathbf{Z}(1) = \left[\begin{array}{c} 1 \\ 1 \end{array} \right].$$

(b) We have

$$\mathbf{Z}_{k+1} = \mathbf{Z}_k + h \left(\left[\begin{array}{cc} 0 & 1 \\ -t_k^2 & -1 \end{array} \right] \mathbf{Z}_k + \left[\begin{array}{c} 0 \\ 2 \end{array} \right] \right)$$

with $t_0 = 1$ and

$$\mathbf{Z}_0 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$

(c) We have

$$\begin{aligned}
\mathbf{Z}_1 =& \mathbf{Z}_0 + h\left(\begin{bmatrix} 0 & 1 \\ -t_0^2 & -1 \end{bmatrix}\mathbf{Z}_0 + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\\
=& \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.01\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\\
=& \begin{bmatrix} 1.01 \\ 1 \end{bmatrix}\\
\mathbf{Z}_2 =& \mathbf{Z}_1 + h\left(\begin{bmatrix} 0 & 1 \\ -t_1^2 & -1 \end{bmatrix}\mathbf{Z}_1 + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\\
=& \begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + 0.01\left(\begin{bmatrix} 0 & 1 \\ -(1.01)^2 & -1 \end{bmatrix}\begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\\
=& \begin{bmatrix} 1.02 \\ 0.99969 \end{bmatrix} \blacksquare
\end{aligned}$$

Problem 50.8

$$y''' + 2y' + ty = t + 1, \quad y(0) = 1,\ y'(0) = -1,\ y''(0) = 0.$$

Solution.
(a) Let $z_1 = y, z_2 = y'$ and $z_3 = y''$. Then

$$\begin{aligned}
z_1' =& z_2\\
z_2' =& y'' = z_3\\
z_3' =& y''' = t + 1 - ty - 2y' = t + 1 - tz_1 - 2z_2
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{Z}'(t) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} =& \begin{bmatrix} z_2 \\ -t^2 z_1 - z_2 + 2 \end{bmatrix}\\
=& \begin{bmatrix} 0 & 1 \\ -t^2 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\end{aligned}$$

with

$$\mathbf{Z}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) We have

$$\mathbf{Z}_{k+1} = \mathbf{Z}_k + h\left(\begin{bmatrix} 0 & 1 \\ -t_k^2 & -1 \end{bmatrix}\mathbf{Z}_k + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

with $t_0 = 1$ and

$$\mathbf{Z}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) We have

$$\begin{aligned}
\mathbf{Z}_1 =& \mathbf{Z}_0 + h\left(\begin{bmatrix} 0 & 1 \\ -t_0^2 & -1 \end{bmatrix}\mathbf{Z}_0 + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\
=& \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.01\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\
=& \begin{bmatrix} 1.01 \\ 1 \end{bmatrix} \\
\mathbf{Z}_2 =& \mathbf{Z}_1 + h\left(\begin{bmatrix} 0 & 1 \\ -t_1^2 & -1 \end{bmatrix}\mathbf{Z}_1 + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\
=& \begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + 0.01\left(\begin{bmatrix} 0 & 1 \\ -(1.01)^2 & -1 \end{bmatrix}\begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\
=& \begin{bmatrix} 1.02 \\ 0.99969 \end{bmatrix} \blacksquare
\end{aligned}$$