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A Sufficient Condition for the Circle of the 6 Points to Become Euler's Circle

In Ion Patrascu, Florentin Smarandache: "Complements to Classic Topics of Circles Geometry". Brussels (Belgium): Pons Editions, 2016

In this article, we prove the theorem relative to the **circle of the 6 points** and, requiring on this circle to have three other remarkable triangle's points, we obtain the **circle of 9 points** (the Euler's Circle).

1st Definition.

It is called cevian of a triangle the line that passes through the vertex of a triangle and an opposite side's point, other than the two left vertices.

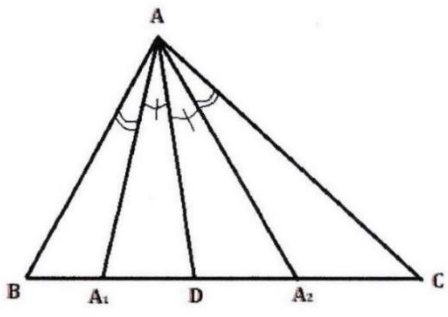

Figure 1.

1st Remark.

The name *cevian* was given in honor of the italian geometrician Giovanni Ceva (1647 - 1734).

2nd Definition.

Two cevians of the same triangle's vertex are called *isogonal cevians* if they create congruent angles with triangle's bisector taken through the same vertex.

2nd Remark.

In the *Figure 1* we represented the bisector AD and the isogonal cevians AA_1 and AA_2. The following relations take place:

$\widehat{A_1AD} \equiv \widehat{A_2AD}$;

$\widehat{BAA_1} \equiv \widehat{CAA_2}$.

1st Proposition.

In a triangle, the height and the radius of the circumscribed circle corresponding to a vertex are isogonal cevians.

Proof.

Let ABC an acute-angled triangle with the height AA' and the radius AO (see *Figure 2*).

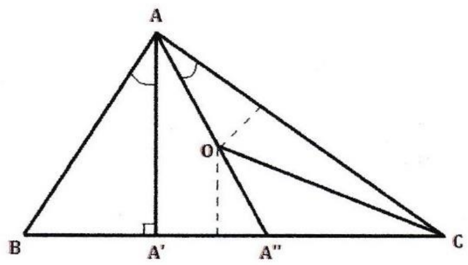

Figure 2.

The angle AOC is a central angle, and the angle ABC is an inscribed angle, so $\widehat{AOC} = 2\widehat{ABC}$. It follows that $\widehat{AOC} = 90^0 - \hat{B}$.

On the other hand, $\widehat{BAA'} = 90^0 - \hat{B}$, so AA' and AO are isogonal cevians.

The theorem can be analogously proved for the obtuse triangle.

3^rd^ Remark.

One can easily prove that in a triangle, if AO is circumscribed circle's radius, its isogonal cevian is the triangle's height from vertex A.

3rd Definition.

Two points P_1, P_2 in the plane of triangle ABC are called *isogonals* (isogonal conjugated) if the cevians' pairs (AP_1, AP_2), (BP_1, BP_2), (CP_1, CP_2), are composed of isogonal cevians.

4th Remark.

In a triangle, the orthocenter and circumscribed circle's center are isogonal points.

1st Theorem.

(The 6 points circle)

If P_1 and P_2 are two isogonal points in the triangle ABC, and A_1, B_1, C_1 respectively A_2, B_2, C_2 are their projections on the sides BC, CA and AB of the triangle, then the points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical.

Proof.

The mediator of segment $[A_1A_2]$ passes through the middle P of segment $[P_1, P_2]$ because the trapezoid $P_1A_1A_2P_2$ is rectangular and the mediator of $[A_1A_2]$ contains its middle line, therefore (see *Figure 3*), we have: $PA_1 = PA_2$ (1). Analogously, it follows that the mediators of segments $[B_1B_2]$ and $[C_1C_2]$ pass through P, so $PB_1 = PB_2$ (2) and $PC_1 = PC_2$ (3). We denote by A_3 and A_4 respectively the middles of segments $[AP_1]$ and $[AP_2]$. We prove that the triangles PA_3C_1 and B_2A_4P are congruent. Indeed, $PA_3 = \frac{1}{2}AP_2$ (middle line), and $B_2A_4 = \frac{1}{2}AP_2$, because it is a median in the rectangular triangle P_2B_2A, so $PA_3 = B_2A_4$; analogously, we obtain that $A_4P = A_3C_1$.

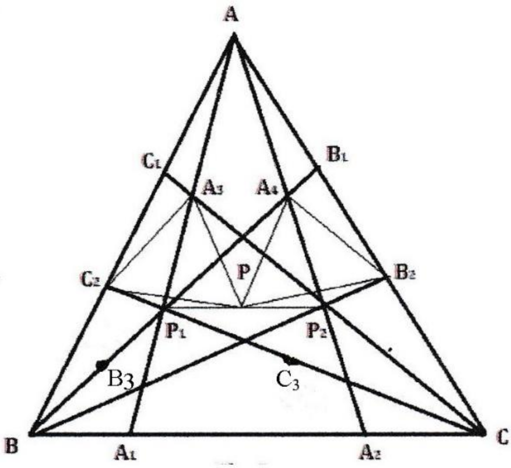

Figure 3.

We have that:

$$\widehat{PA_3C_1} = \widehat{PA_3P_1} + \widehat{P_1A_3C_1} = \widehat{P_1AP_2} + 2\widehat{P_1AC_1} =$$
$$= \hat{A} + \widehat{P_1AB};$$
$$\widehat{B_2A_4P} = \widehat{B_2A_4P_2} + \widehat{PA_4P_2} = \widehat{P_1AP_2} + 2\widehat{P_2AB_2} =$$
$$= \hat{A} + \widehat{P_2AC}.$$

But $\widehat{P_1AB} = \widehat{P_2AC}$, because the cevians AP_1 and AP_2 are isogonal and therefore $\widehat{PA_3C_1} = \widehat{B_2A_4P}$. Since $\Delta PA_3C_1 = \Delta B_2A_4P$, it follows that $PB_2 = PC_1$ (4).

Repeating the previous reasoning for triangles PB_3C_1 and A_2B_4P, where B_3 and B_4 are respectively the middles of segments (BP_1) and (BP_2), we find that they are congruent and it follows that $PC_1 = PA_2$ (5).

The relations (1) - (5) lead to $PA_1 = PA_2 = PB_1 = PB_2 = PC_1 = PC_2$, which shows that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are located on a circle centered in P, the middle of the segment determined by isogonal points given by P_1 and P_2.

4th Definition.

It is called the 9 points circle or Euler's circle of the triangle ABC the circle that contains the middles of triangle's sides, the triangle heights' feet and the middles of the segments determined by the orthocenter with triangle's vertex.

2nd Proposition.

If P_1, P_2 are isogonal points in the triangle ABC and if on the circle of their corresponding 6 points there also are the middles of the segments (AP_1), (BP_1), (CP_1), then the 6 points circle coincides with the Euler's circle of the triangle ABC.

1st Proof.

We keep notations from *Figure 3*; we proved that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are on the 6 points circle of the triangle ABC, having its center in P, the middle of segment $[P_1P_2]$.

If on this circle are also situated the middles A_3, B_3, C_3 of segments (AP_1), (BP_1), (CP_1), then we have $PA_3 = PB_3 = PC_3$.

We determined that PA_3 is middle line in the triangle P_1P_2A, therefore $PA_3 = \frac{1}{2}AP_2$, analogously $PB_3 = \frac{1}{2}BP_2$ and $PC_3 = \frac{1}{2}CP_2$, and we obtain that $P_2A = P_2B = P_2C$, consequently P is the center of the circle circumscribed to the triangle ABC, so $P_2 = O$.

Because P_1 is the isogonal of O, it follows that $P_1 = H$, therefore the circle of 6 points of the isogonal points O and H is the circle of 9 points.

2nd ***Proof.***

Because A_3B_3 is middle line in the triangle P_1AB, it follows that

$$\sphericalangle P_1AB \equiv \sphericalangle P_1A_2B_3. \quad (1)$$

Also, A_3C_3 is middle line in the triangle P_1AC, and A_3C_3 is middle line in the triangle P_1AP_2, therefore we get

$$\sphericalangle PA_3C_3 \equiv \sphericalangle P_2AC. \quad (2)$$

The relations (1), (2) and the fact that AP_1 and AP_2 are isogonal cevians lead to:

$$\sphericalangle P_1A_2B_3 \equiv PA_3C_3. \quad (3)$$

The point P is the center of the circle circumscribed to $A_3B_3C_3$; then, from (3) and from isogonal cevians' properties, one of which is circumscribed circle radius, it follows that in the triangle $A_3B_3C_3$ the line P_1A_3 is a height, as $B_3C_3 \parallel BC$, we get that P_1A is a height in the triangle ABC and, therefore, P_1 will be the orthocenter of the triangle

ABC , and P_2 will be the center of the circle circumscribed to the triangle ABC.

5th Remark.

Of those shown, we get that the center of the circle of 9 points is the middle of the line determined by triangle's orthocenter and by the circumscribed circle's center, and the fact that Euler's circle radius is half the radius of the circumscribed circle.

References.

[1] Roger A. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publications, 2007.

[2] Cătălin Barbu: *Teoreme fundamentale din geometria triunghiului* [Fundamental theorems of triangle's geometry]. Bacău: Editura Unique, 2008.