AFOSR Scientific Report
AFOSR-67-2291

MILTON VAN DYKE

A SURVEY OF HIGHER-ORDER BOUNDARY-LAYER THEORY

SEPTEMBER
1967

Prepared for the Air Force Office of Scientific Research
Under Contract No. AF 49(638)-1274
Project No. 9781-01
Distribution of this document is unlimited.

SUDAAR
NO. 326

NOTICES

Qualified requesters may obtain additional copies from the Defense Documentation Center.

Research sponsored by Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force. Research monitored under the technical supervision of Donald L. Calvert, Major, AFOSR.

AFOSR No. 67-2291

Department of Aeronautics and Astronautics
Stanford University
Stanford, California

A SURVEY OF HIGHER-ORDER BOUNDARY-LAYER THEORY

by

Milton Van Dyke

SUDAAR No. 326

September 1967

Presented at AGARD Seminar on Numerical Methods for Viscous Flows
National Physical Laboratory
18-21 September 1967

Prepared for the Air Force Office of Scientific Research
Under Contract No. AF49(638)-1274
Project No. 9781-01

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

I. INTRODUCTION

If I undertake a survey of higher-order boundary-layer theory, at the present time it almost goes without saying that I am going to discuss only steady, plane or axisymmetric, laminar flows, and at most second-order theory. Laminar, because my knowledgeable colleagues assure me that turbulent boundary layers are not yet well enough understood that a sensible person would trouble himself with higher-order refinements. Second-order, because for gases on the one hand the Navier-Stokes equations are not valid to any higher order, and for liquids on the other hand the law of diminishing returns probably sets in after the second approximation — which itself extends the utility of Prandtl's theory down to Reynolds numbers of the order of ten. And steady and two-dimensional, because none of us has yet ventured further.

When we set out to improve upon boundary-layer theory in a systematic way, we naturally ask first what approximations were adopted by Prandtl in the classical theory. Consider first the simplest case of plane, steady, incompressible flow. The continuity equation and surface boundary conditions are left intact. Streamwise diffusion is neglected compared with transverse diffusion in the longitudinal momentum equation, the transverse pressure gradient is disregarded, and the distant boundary condition is replaced by the requirement that far out in the boundary layer the tangential velocity component approach the inviscid surface speed.

These three approximations introduce errors of relative order $R^{-1/2}$, where R is a representative Reynolds number. Hence if we count Prandtl's theory as the first approximation (some writers call it the "zeroth"!), second-order theory will add corrections of order $R^{-1/2}$, third-order theory terms of order R^{-1}, and so on.

II. DISPLACEMENT EFFECT

The neglect of streamwise diffusion actually causes only a third-order error. Likewise, for flat surfaces — plates and wedges — the normal pressure gradient exerts only a third-order effect. Hence for flat shapes the only second-order effect is the change in the outer tangential speed induced by the boundary layer itself. This is called the displacement effect.

This effect appears in the first discussion of higher-order boundary-layer theory that I know of, due to Prandtl himself. In volume three of Durand's "Aerodynamic Theory" he wrote, in discussing the flat plate

> The displacement of the stream-lines by the amount δ^* produces a slight alteration in the potential flow which was made the basis of the calculations. Instead of a simple parallel flow, the flow around a parabolic cylinder of thickness $2\delta^*$ should be introduced, which would slightly alter the pressure distribution. The above calculation would have to be repeated for this new pressure distribution and if necessary the process repeated on the basis of the new measure of displacement so obtained. Such calculations have so far not been performed; they would, in any case, make little difference in the regions where the calculations are usually applied in practice. They would however become necessary if the transition to smaller Reynolds number $u_o l/\nu$ were attempted.

To this we need only add that — according to thin-airfoil theory — it happens that a thin parabola induces no pressure change upon itself. Thus we see that there are no second-order corrections at all to the boundary layer on a semi-infinite plate.

For a finite flat plate, however, the displacement thickness is parabolic only back to the trailing edge, and then nearly constant in the wake. Consequently there is a small favorable pressure gradient induced upon the boundary layer. On this basis Kuo (1953) calculated the second-order effect, finding that the local skin friction is slightly increased everywhere. However, he made the mistake of integrating to find the total drag. This is not proper, because the boundary-layer approximation breaks down altogether in a small neighborhood of the leading edge; and, as I shall discuss later in more detail, this local deviation affects the drag to second order.

III. "ONE-AND-A-HALF-ORDER" THEORY

The displacement effect is often more difficult to calculate than any of the other second-order effects, because it alone is global in nature — the correction at any point depending upon the entire course of the boundary layer. Nevertheless, I have discussed it first because it is invariably present. (The only exception might arise if we contrived to apply suction to a porous wall, or to cool the wall in a compressible fluid, in just such a way that the displacement thickness was everywhere zero.)

Before discussing other second-order effects, I want to express the opinion that displacement effects deserve more attention than they have received. Indeed, very useful results can be obtained by stopping short of second-order boundary-layer theory, at what we might call "one-and-a-half-order theory" — that is, the classical boundary layer plus its flow due to displacement.

For example, chemical engineers have in the last few years disputed the old problem of viscous entry into a channel. This problem was first treated in 1934 by Schlichting, who applied boundary-layer theory to the walls, and assumed in between a uniform core that accelerates downstream. Obviously this assumption fails near the entry. Recently Wang and Longwell (1964) solved the full Navier-Stokes equations numerically for a cascade of plates at a Reynolds number of 150, based on channel width and upstream conditions. However, at such a large Reynolds number it is scarcely necessary to appeal to the full equations, or even to second-order boundary-layer theory. One need only calculate properly the flow due to displacement thickness.

Near the entry, the displacement thickness is a parabola for each plate, as indicated in Fig. 1. Hence the flow due to displacement thickness is just the potential flow past a cascade of parabolas, which can be found by elementary means (with due attention to indeterminate forms). Fig. 2 shows how well the result for the velocity profile across the entry plane agrees with the numerical solution of the full equations.

I have recently calculated also the flow induced by a variety of jets and plumes – results that have apparently never appeared in print, but are useful in understanding the flow pattern. For example, the well-known boundary-layer solution for an axisymmetric laminar jet yields, in cylindrical coordinates, the pattern of streamlines shown in Fig. 3, which appears in Prandtl's (1938) article. The flow far outside the jet happens, with this choice of coordinates, to be that appropriate to a jet issuing from an infinite plane wall. Suppose, however, that we are interested rather in a jet issuing from a long slender nozzle. Calculating the flow due to displacement shows that the outer stream surfaces are paraboloids of revolution. The composite solution shown in Fig. 4 is indistinguishable from the exact solution of the full Navier-Stokes equations sketched by Whitham on page 153 of Rosenhead's "Laminar Boundary Layers".

Let me add one last remark on "one-and-a-half-order" theory. This is the province of Kaplun's (1954) optimal coordinates; and it seems to me imperative that we extend that remarkable idea – first to axisymmetric flows, and then if possible to three-dimensional and unsteady motions as well as to higher approximations.

IV. LONGITUDINAL CURVATURE

If the surface of a body is curved, rather than flat, centrifugal forces yield pressure changes across the boundary layer that exert a second-order effect. This effect of longitudinal curvature was first investigated by Tani in 1949. He studied the special case of a plate with curvature varying as the inverse square root of distance from the leading edge, because this admits a self-similar solution. He found a reduction of local skin friction due to convex curvature (in contrast to earlier Japanese work, based on a momentum integral, that suggested an increase).

The same problem was solved independently by Murphy in 1953, who found the same trend but a smaller coefficient. Tani thereupon realized that his treatment had been inconsistent, and in 1954 published a revised version with yet another value of the coefficient. Fig. 5 shows the subsequent history of this ridiculous comedy of errors, which seems to have been resolved only within the last year.

Of the investigators whose names appear here, Murphy, Cooke, Massey and Clayton, and Narasimha and Ojha have calculated the effects of longitudinal curvature for a more general class of flows. These are what I will call *completely self-similar* solutions, in the sense that the second-order correction is similar not only to itself but also to the first-order solution – which is a member of the Falkner-Skan family. With equal ease I have calculated several cases of what I may call *separately self-similar* solutions: the first- and second-order solutions are similar to themselves, but not to each other. These have the advantage that the curvature may be taken to be everywhere finite. It might be worthwhile to calculate a few more of these; whereas I believe the subject of completely self-similar flows is closed with the appearance of the definitive papers of Cooke and Narasimha and Ojha. (That of Murphy is invalidated by certain inconsistencies that have been pointed out by Massey and Clayton (1966), and the work of the latter is also open to some objection.)

Let me now make a possibly controversial comment on the range of applicability of these results. Murphy, Massey and Clayton, and Schultz-Grunow and Breuer assume – either explicitly or tacitly – that their solutions remain valid even when the wall curvature is so great that the radius is of the order of the boundary-layer thickness. I am sure that this is not true, and that – as I will discuss later – quite a different approximation must be adopted in that range. They therefore spend an unnecessary amount of labor in solving equations that are not split into first- and second-order components, and in presenting results for a range of curvature parameter. As Narasimha and Ojha point out, there is no justification for attaching any significance to the departure of their curves from the initial tangents.

V. TRANSVERSE CURVATURE

A second curvature effect arises when we extend our considerations to bodies of revolution. In the classical theory the boundary layer is negligibly thin compared with the local radius of the body; and this permits it to be related to an equivalent plane boundary layer by the Mangler transformation. However, on a very long slender body – a needle – the boundary layer may grow much thicker than the body even at high Reynolds number. We exclude this situation – which requires a fresh approach initiated by Glauert and Lighthill (1955) and Stewartson (1955) – and consider the effects of transverse curvature over a short body or the forward portions of a long one.*

Transverse curvature appears in its most nearly pure form in the boundary layer on a circular pipe. The internal flow was studied by Atkinson and Goldstein (cf. Goldstein 1938, p. 304), and the external flow by Seban and Bond (1959), with important corrections by Kelly (1954). The latter find the local skin friction to be increased over the flat-plate value by the factor

$$1 + 2.10 \sqrt{\frac{\nu}{Ua}} \sqrt{\frac{x}{a}} - 0.48 \frac{\nu}{Ua} \frac{x}{a} + \dots$$

where a is the radius of the pipe. In this form we recognize the second term as a second-order boundary-layer effect – proportional to the inverse square root of the Reynolds number – and the third term as a third-order effect. Whereas convex longitudinal curvature usually reduces the skin friction, convex transverse curvature appears always to increase it, as in this case.

Although longitudinal curvature is absent in this problem, the second-order displacement effect does not vanish for a tube as it does for a plate. Hence a correction for displacement must be added to the above result.

---

*We think of bodies that grow more slowly than a paraboloid. The situation is reversed for those that grow more rapidly, such as a cone, for which the ratio of boundary-layer thickness to body radius decreases downstream.

For the internal flow, Atkinson and Goldstein adopted Schlichting's idea of a uniform accelerated core; and this alters the coefficient of the second term. I intend to calculate properly the flow due to displacement thickness, in order to assess the error in these theories.

VI. EXTERNAL VORTICITY

The next second-order effect I want to mention was first recognized in supersonic problems, but can occur also in incompressible flows. Ferri and Libby (1954) pointed out that the boundary layer must be affected to some extent by the external vorticity generated by a curved bow shock wave. Li (1955) then proposed a simple incompressible model of this phenomenon that displays its essential features — a semi-infinite flat plate in a uniform shear flow. He at first omitted the pressure gradient that is induced by interaction of the external shear flow with the displacement thickness of the boundary layer, but corrected himself the following year (Li 1956). However, that correction was challenged by Glauert (1957) and others; and a lively and extended controversy arose. Finally, however, thanks to the careful analysis of Murray (1961) and the diplomatic intercession of Toomre and Rott (1964), the dispute has been resolved in a consensus of nearly all the participants.

VII. THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

We see that even in the simplest case of steady, plane, laminar, incompressible flow the development of higher-order boundary-layer theory has been marred by an unfortunate series of errors, misunderstandings, and controversies. The reason is simply that the insight of even a Prandtl begins to fail at about the second approximation. What one then wants is a rote procedure that can be applied automatically, without undue mental effort.

The required technique is the method of matched asymptotic (or "inner and outer") expansions. This useful method is in fact an outgrowth of

Prandtl's boundary-layer idea, as developed by Friedrichs (1953), Kaplun and Lagerstrom (1957), and others. I believe that this method can no longer be dismissed as an esoteric special technique, but should be part of the working equipment of every applied mathematician and theoretical engineer.

In the present subject, it is fair to assert that all the many errors and disputes have arisen from relying upon physical insight; and that not a single false step has been made by any of us who trusted rather to systematic application of the method of matched asymptotic expansions.

VIII. COMPRESSIBLE FLOW

Even though the emphasis of this meeting is on compressible flow, I have spoken so far only of incompressible motion, because it is simpler and exemplifies most of the essential features. Just as in the classical theory, dramatic compressibility effects are limited to the outer inviscid flow, and the boundary layer itself suffers changes only of detail, even into the hypersonic range. This point of view has recently been challenged by Weinbaum and Garvine (1966); but I think it would be charitable to say that they have misunderstood the asymptotic nature of boundary-layer theory, confusing it with the so-called "strong-interaction" theory, which is based upon quite a different double limit process.

The four second-order effects that I have discussed so far — displacement, longitudinal curvature, transverse curvature, and external vorticity — persist for compressible motion. In my own work, I found it convenient to subdivide the effect of external vorticity into that of entropy gradient and of stagnation enthalpy gradient, the latter being absent for the usual isoenergetic flows of aerodynamics. To these are added two new phenomena associated with the boundary conditions at the surface: the effects of slip and of temperature jump. It should be emphasized, however, as pointed out by Rott and Lenard (1962), that this classification is not unique, and a considerable part of the controversy in this subject has arisen only because of different ways of dividing among displacement, curvature, and external vorticity.

In contrast to the incompressible theory, the more complicated compressible second-order boundary-layer theory has been developed with a minimum of error. In particular, the comprehensive analyses of myself (Van Dyke 1961), Maslen (1962), and Lenard (1962) — developed independently in about the same year — seem to have withstood the test of time.

IX. APPLICATIONS IN COMPRESSIBLE FLOW

The first applications of the theory for compressible flow were again to stagnation points and leading edges, for which self-similar solutions exist (Van Dyke 1961, Maslen 1962, Fannelöp and Flügge-Lotz 1965, Davis and Flügge-Lotz 1964a). Later, numerical integration of the first- and second-order equations was undertaken (Devan 1964), the most comprehensive results — until this meeting — being those of Fannelöp and Flügge-Lotz (1966) for plane flow past a circular cylinder and a plate with semi-circular leading edge, and of Davis and Flügge-Lotz (1964b) for axisymmetric flow past a paraboloid, sphere, and hyperboloid.

The most troublesome component of these calculations is the flow due to displacement thickness. In principle, we should perturb the basic inviscid blunt-body solution. However, we have all resorted to the strategem of approximating the body plus the displacement thickness by a magnified and shifted replica of itself; and this does not seem to have introduced serious errors.

These results show that the various second-order contributions may vary widely in sign and magnitude, depending upon body shape, surface temperature, and other parameters in the problem. The same is true of the resultant, which is often smaller than any of its components. Consequently it is important to calculate all second-order effects if any significance is to be attached to the result.

Experimental confirmation is perhaps still not conclusive. A few years ago the situation seemed to be that experiments carried out in New York agreed with the rather large effects predicted by several partial theories developed in the same state, and experiments in California tended to agree with the smaller effects predicted by theories developed there.

I am not sure how much this situation has been clarified; but I hope to learn more about it at this meeting.

X. SEPARATION

One point of special interest at this meeting is the light that second-order theory can shed on laminar separation. Until recently, most of us believed that the classical boundary-layer theory breaks down shortly before the skin friction vanishes – as in Howarth's (1938) solution for a linearly decreasing surface speed. As usual in perturbation theories, we might expect this failure of the first approximation to be confirmed by compounded singularities in higher approximations.

Two bits of evidence suggest that this does happen. In their completely self-similar solutions for incompressible flow, Narasimha and Ojha have observed that the second-order coefficient of skin friction due to longitudinal curvature seems to be rising rapidly as the critical value of the Falkner-Skan parameter is approached. Again, in their full second-order solution for a sphere at Mach number 10, Davis and Flügge-Lotz (1964b) found the effects of longitudinal curvature becoming large as the skin friction fell.

Our ideas on separation have, however, been overturned by the recent discovery of Catherall and Mangler (1966) that the classical boundary-layer solution will proceed smoothly through zero skin friction if it is permitted the slightest freedom to choose the local pressure distribution so as to avoid catastrophe. It would be interesting to re-examine the second-order theory in the light of this remarkable turn of events.

XI. CORNERS AND EDGES

I have already expressed the opinion that higher-order boundary-layer theory will break down long before the surface curvature becomes so great that the radius of curvature is comparable with the boundary-layer thickness – and this would be true of transverse as well as longitudinal curvature. I believe that the proper way of treating such problems has been pointed out by Brailovskaya (1965) and Neiland and Sychev (1966). They consider plane flow past a corner that is slightly rounded, with a radius of the order of the local boundary-layer thickness. Classical boundary-layer theory holds as a first approximation ahead of the corner and again behind it. In the immediate neighborhood of the corner, however, the small viscous forces are insignificant compared with the pressure and inertial forces, so the flow is governed locally by the Euler equations of rotational inviscid flow. This local solution matches the boundary layers upstream and downstream in the sense of the method of matched asymptotic expansions. Finally, because the local inviscid solution violates the no-slip condition, a thin sub-boundary layer must be added close to the wall.

Neiland and Sychev consider only rounded corners in order to avoid having to deal with the full Navier-Stokes equations. For if the corner is sharp, a local solution of the full equations is evidently required. However, it will have a certain simple and universal character, and may therefore be worth working out numerically.

One case that seems to be well in hand is incompressible flow near a cusped leading edge. The local problem is the standard one of viscous flow past a semi-infinite flat plate. Ten years ago Imai (1957) showed, by ingenious use of global momentum balance, that although the Prandtl-Blasius boundary-layer solution breaks down near the leading edge, it can be used to find the second term in the integrated skin friction (which from the crude point of view of the boundary-layer approximation appears as a concentrated force at the leading edge). More recently, Davis (1967) has solved the problem in detail using the semi-numerical method of series truncation; and his solution agrees so well with both the global result of Imai and a much-neglected analysis of Dean (1954) that we can accept it with confidence. This local correction can be applied to Kuo's solution for the finite flat plate, to the cascade of flate plates, to the solutions of Atkinson and Goldstein and Seban and Bond for the circular pipe, and so on. In supersonic and hypersonic flow the problem is more difficult, and has not yet been satisfactorily solved.

Perhaps the simplest case of flow over a sharp corner is a cusped trailing edge in an incompressible stream, because there is no question of separation. If we consider, for example, the standard problem of the finite flate plate, we see that the Prandtl-Blasius solution applies over most of the surface, and the wake solution of Goldstein (1930) and Tollmien (1931) almost everywhere behind it. The boundary-layer solution fails in a circular neighborhood of the leading edge whose radius is of the order of R^{-1} times the length of the plate, R being the Reynolds number based on length; and here we can use the results of Imai, Dean, and Davis. At the trailing edge the thickness of the boundary layer is of order $R^{-1/2}$, and the theory of Neiland and Sychev would suggest that a local Euler solution is required in a neighborhood of that size. However, in this simple case that correction vanishes. I have convinced myself that a correction is then required only in a smaller neighborhood of the trailing edge, whose radius is of the order of $R^{-3/4}$ times the length of the plate. (This corresponds to Neiland and Sychev's sub-boundary layer.) The full Navier-Stokes equations must be solved there; and we hope to carry this out by series truncation.

Without making the detailed calculations, we can see that this trailing-edge correction will contribute to the integrated skin friction a term of order $R^{-5/4}$. Thus for cusp-ended shapes we must reconsider our numbering scheme. What we have heretofore called second-order theory adds a correction of relative order $R^{-1/2}$, and third-order theory a term of order R^{-1}; but now we see that the trailing-edge correction supplies a "two-and-one-half-order" term in $R^{-3/4}$, and so on (whatever that may mean!).

The situation is different again when we encounter large or infinite <u>transverse</u> curvature. The simplest example is perhaps the incompressible flow along a corner, which has been re-examined recently by Rubin (1966) from the point of view of matched asymptotic expansions. In this case the crucial problem to be solved in the immediate vicinity of the corner involves equations simpler than the full Navier-Stokes equations, but more complicated than the conventional boundary-layer equations. The only attempt at solving this problem numerically was made by Pearson (1957) ten years ago in his unpublished Cambridge University thesis. If this correction is applied to the flow inside a rectangular channel, we see that it contributes a term of relative order $R^{-1/2}$ to the drag — and is therefore to be included with the second-order displacement effect discussed earlier

The flow near the outside corner on such a channel is more complicated, as is indicated by Stewartson's (1961) study of the quarter-infinite plate. And again the corresponding supersonic problems are still more difficult.

XII. SINGULAR OUTER FLOWS

Finally, I want to discuss an intriguing new field of application for higher-order boundary-layer theory. This is motion in which the basic inviscid flow is singular at the surface of the body. This situation has arisen recently in various branches of high-speed aerodynamics, of which I will mention four:

1. Perhaps the simplest case to understant is the inviscid stagnation region of a blunt body in hypersonic flight through a completely transparent radiating gas. A particle of fluid on the stagnation streamline requires an infinite time to reach the stagnation point, and so — because there is no re-absorption — radiates away all its energy. Hence the inviscid surface streamline is at absolute zero temperature. In a model of this phenomenon studied by Burggraf (1966) the velocity and enthalpy both vanish as regative fractional powers of the logarithm of the distance from the wall.
2. Similarly, for an inviscid stagnation point in a chemically-reacting gas, the temperature and degree of dissociation (as well as the other thermodynamic properties) approach some equilibrium values at the stagnation point; and the normal gradients are zero for sufficiently fast reactions, finite for a particular intermediate rate, and infinite for slower reactions (Fig. 6). This behavior has been discussed by Conti and myself (1966).

3. Hayes (1964) has studied the rotational inviscid flow near a three-dimensional stagnation point. He finds that in all but very special cases the solution is non-analytic and the vorticity infinite at the wall, the stagnation streamline being tangent to the surface.
4. In hypersonic small-disturbance theory the self-similar solutions associated with strong power-law bow shock waves are singular at the surface of the body. This case is the subject of a paper to be presented at this meeting by Lee and Cheng, entitled "Higher-order approximation in the theory of hypersonic boundary layers on slender bodies".

If we now consider applying boundary-layer theory to any of these problems, questions arise that force us to re-examine the basis of Prandtl's classical theory. Should we still use the inviscid surface speed as the tangential velocity at the outer edge of the boundary layer, even though the gradient is infinite? Is the boundary-layer thickness still of order $R^{-1/2}$? Do higher approximations proceed in the usual way? Is a single boundary layer sufficient, or are intermediate transition layers required?

Conti and I have concluded that – at least in the first two cases, of stagnation points in radiating or reacting flows – the situation is in general as follows: Classical boundary-layer theory remains valid even though the inviscid surface gradients are infinite. That is, the boundary-layer thickness is still of order $R^{-1/2}$, and the inviscid surface speed is approached at the outer edge. However, important differences appear in higher approximations. Rather than being smaller by a full inverse half power of Reynolds number, the second-order correction follows close on the heels of the classical solution. In the case of slow chemical reactions, it may differ by only a very small negative power of Reynolds number, so that several or even a great many higher-order terms intervene before the conventional second-order correction. And in the case of radiation, successive terms differ from one another only by fractional powers of the logarithm of the Reynolds number, so that an infinite number of terms intervene.

These conclusions differ somewhat from those of Burggraf. The reason is that he considers only the degenerate case of vanishing surface temperature. Then the surface boundary conditions on both velocity and temperature are satisfied by the inviscid solution, so that no conventional boundary layer is required. The first correction is what would ordinarily be the second-order term; and it is a consequence of the non-linearity of the governing equations that the boundary layer then has a thickness that is slightly greater than usual, by an amount smaller than any power of the Reynolds number.

Our conclusions appear also to differ from those reached by Lee and Cheng in their study of power-law bodies in hypersonic flow, for they invoke a third region that serves to join the inviscid flow to the boundary layer. From my point of view, this meeting will be a success if we can clarify our thoughts on this fascinating new branch of the subject.

REFERENCES

BRAILOVSKAYA, I. YU., "A method for calculating flows with strong viscous interaction", (Russian) Doklady Akad. Nauk SSR 162, 36-40, 1965.

BURGGRAF, O. R., "Asymptotic solution for the viscous radiating shock layer", AIAA J. 4, 1725-1734, 1966.

CATHERALL, D. and MANGLER, K. W., "The integration of the two-dimensional laminar boundary-layer equations past the point of vanishing skin friction", J. Fluid Mech. 26, 163-182, 1966.

CONTI, R. J. and VAN DYKE, M., "Nonequilibrium flow near a symmetrical plane stagnation point", Bull. Amer. Phys. Soc. (2) 12, Abstract M6, 1967.

DAVIS, R. T., "Laminar incompressible flow past a semi-infinite flat plate", J. Fluid Mech. 27, 691-704, 1967.

DAVIS, R. T. and FLÜGGE-LOTZ, I., "The laminar compressible boundary-layer in the stagnation-point region of an axisymmetric blunt body including the second-order effect of vorticity interaction", Int. J. Heat Mass Transfer 7, 341-370, 1964a.

DAVIS, R. T. and FLÜGGE-LOTZ, I., "Second-order boundary-layer effects in hypersonic flow past axisymmetric blunt bodies", J. Fluid Mech. 20, 593-623, 1964b.

DEAN, W. R., "On the steady motion of viscous liquid past a flat plate", Mathematika 1, 143-156, 1954.

COOKE, J. C., "Similar second order incompressible laminar boundary layers", RAE Tech. Rept. No. 65044, 1965.

DEVAN, L., "Second order incompressible laminar boundary layer development on a two-dimensional semi-infinite body", Ph.D. diss., Univ. Calif., Los Angeles, 1964.

FANNELÖP, T. K. and FLÜGGE-LOTZ, I., "Two-dimensional hypersonic stagnation flow at low Reynolds numbers", Z. Flugwiss. 13, 282-29[illegible], 1965.

FANNELÖP, T. K. and FLÜGGE-LOTZ, I., "Viscous hypersonic flow over simple blunt bodies; comparison of a second-order theory with experimental results," J. Mécanique 5, 69-100, 1966.

FERRI, A. and LIBBY, P. A., "Note on an interaction between the boundary layer and the inviscid flow", J. Aero. Sci. 21, 130, 1954.

FRIEDRICHS, K. O., "Special Topics in Fluid Dynamics", pp. 120-130, New York Univ., 1953.

GLAUERT, M. B., "The boundary layer in simple shear flow past a flat plate", J. Aero. Sci. 24, 848-849, 1957.

GLAUERT, M. B. and LIGHTHILL, M. J., "The axisymmetric boundary layer on a long thin cylinder", Proc. Roy. Soc. A 230, 188-203, 1955.

GOLDSTEIN, S., "Concerning some solutions of the boundary-layer equations in hydrodynamics," Proc. Camb. Phil. Soc. 26, 1-30, 1930.

GOLDSTEIN, S., "Modern Developments in Fluid Dynamics", Clarendon Press, Oxford, 1938.

HAYASHI, N., "Effects of surface curvature on laminar boundary-layer flows", AIAA J. 1, 914-915, 1963.

HAYES, W. D., "Rotational stagnation point flow", J. Fluid Mech. 19, 366-374, 1964.

HOWARTH, L., "On the solution of the laminar boundary layer equations", Proc. Roy. Soc. A 164, 547-579, 1938.

IMAI, I., "Second approximation to the laminar boundary-layer flow over a flat plate", J. Aero. Sci. 24, 155-156, 1957.

KAPLUN, S., "The role of coordinate systems in boundary-layer theory", ZAMP 5, 111-135, 1954.

KAPLUN, S. and LAGERSTROM, P. A., "Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers", J. Math. Mech. 6, 585-593, 1957.

KELLY, H. R., "A note on the laminar boundary layer on a circular cylinder in axial incompressible flow", J. Aero. Sci. 21, 634.

KUO, Y. H., "On the flow of an incompressible viscous fluid past a flat plate at moderate Reynolds numbers", J. Math. and Phys. 32, 83-101, 1953.

LI, T. Y., "Simple shear flow past a flat plate in an incompressible fluid of small viscosity", J. Aero. Sci. 22, 651-652, 1955.

LI, T. Y., "Effects of free stream vorticity on the behavior of a viscous boundary layer", J. Aero. Sci. 23, 1128-1129, 1956.

MASLEN, S. H., "Second-order effects in laminar boundary layers", AIAA J. 1, 33-40, 1963.

MASSEY, B. S. and CLAYTON, B. R., "Laminar boundary layers and their separation from curved surfaces", Trans. A.S.M.E., J. Basic Engg. 87D, 483, 1965.

MASSEY, B. S. and CLAYTON, B. R.,"Comment on 'Extensions of the Falkner-Skan similar solutions to flows with surface curvature'", AIAA J. 4, 1149, 1966.

MURPHY, J. S., "Some effects of surface curvature on laminar boundary-layer flow", J. Aero. Sci. 20, 338-334, 1953.

MURPHY, J. S., "Extensions of the Falkner-Skan similar solutions of the boundary-layer equations to flows with surface curvature", Douglas Aircraft Co. Rept. ES 40487, 1961.

MURPHY, J. S., "Extensions of the Falkner-Skan similar solutions to flow with surface curvature", AIAA J. 3, 2043-2049, 1965.

NARASIMHA, R. and OJHA, S. K., "Effect of longitudinal surface curvature on boundary layers", J. Fluid Mech. 29, 187-199, 1967.

NEILAND, V. YA. and SYCHEV, V. V., "Asymptotic solutions of the Navier-Stokes equations in regions with large local perturbations", (Russian), Mekh. Zhid. i Gaza 43-49, 1966, English transl.: Grumman Research Dept. Transl. TR-40, 1967.

ROTT, N. and LENARD, M., "The effect of slip, particularly for highly cooled walls", J. Aero. Sci. 29, 591-595, 1962.

RUBIN, S. G., "Incompressible flow along a corner", J. Fluid Mech. 26, 97-110, 1966.

SCHLICHTING, H., "Laminare Kanaleinlaufströmung" ZAMM 14, 368, 1934.

SCHULTZ-GRUNOW, F. and BREUER, W., "Laminar boundary layers on cambered walls", In Basic Developments in Fluid Dynamics, (ed. M. Holt), New York and London: Acadamic Press, 1965.

SEBAN, R. A. and BOND, R., "Skin-friction and heat-transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow", J. Aero. Sci. 18, 671-675, 1951.

STEWARTSON, K., "The asymptotic boundary layer on a circular cylinder in axial incompressible flow", Quart. Appl. Math. 13, 113-122, 1955.

TANI, I., "Effect of curvature on the velocity distribution in the boundary layer", (Japanese), J. Japan Soc. Mech. Engrs. 52, 476-477, 1949.

TANI, I., "Effect of curvature on the velocity distribution in the boundary layer (revised)", (Japanese), J. Japan Soc. Mech. Engrs. 57, 596-598, 1954.

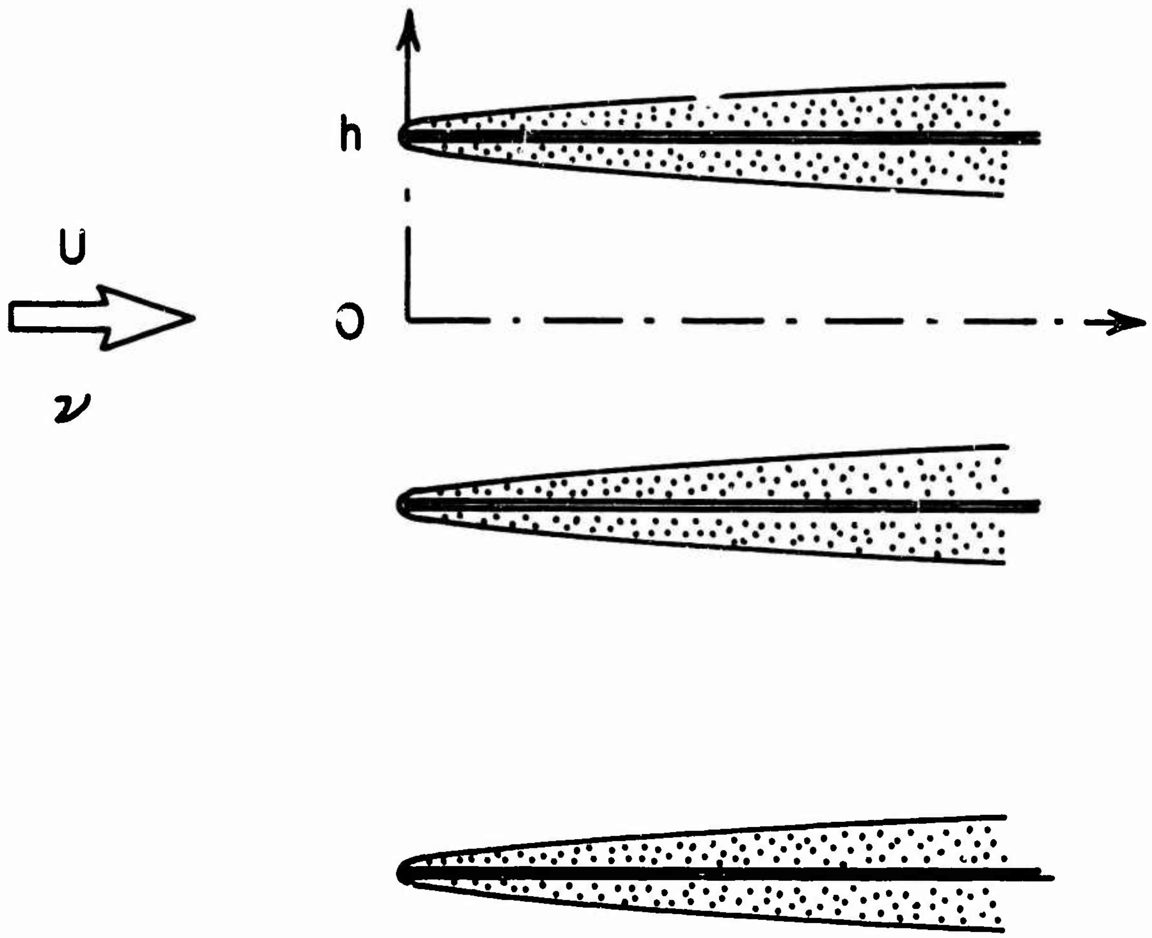
TOOMRE, A. and ROTT, N., "On the pressure induced by the boundary layer on a flat plate in shear flow", J. Fluid Mech. 19, 1-10, 1964.

VAN DYKE, M., "Second-order compressible boundary-layer theory with application to blunt bodies in hypersonic flow", Stanford Univ. SUDAER Rept. No. 112, 1961. Hypersonic Flow Research (ed. F. R. Riddell), pp. 37-76, New York, Academic Press.

WANG, Y. L. and LONGWELL, P. A., "Laminar flow in the inlet section of parallel plates", A.I.Ch.E.J. 10, 323-329, 1964.

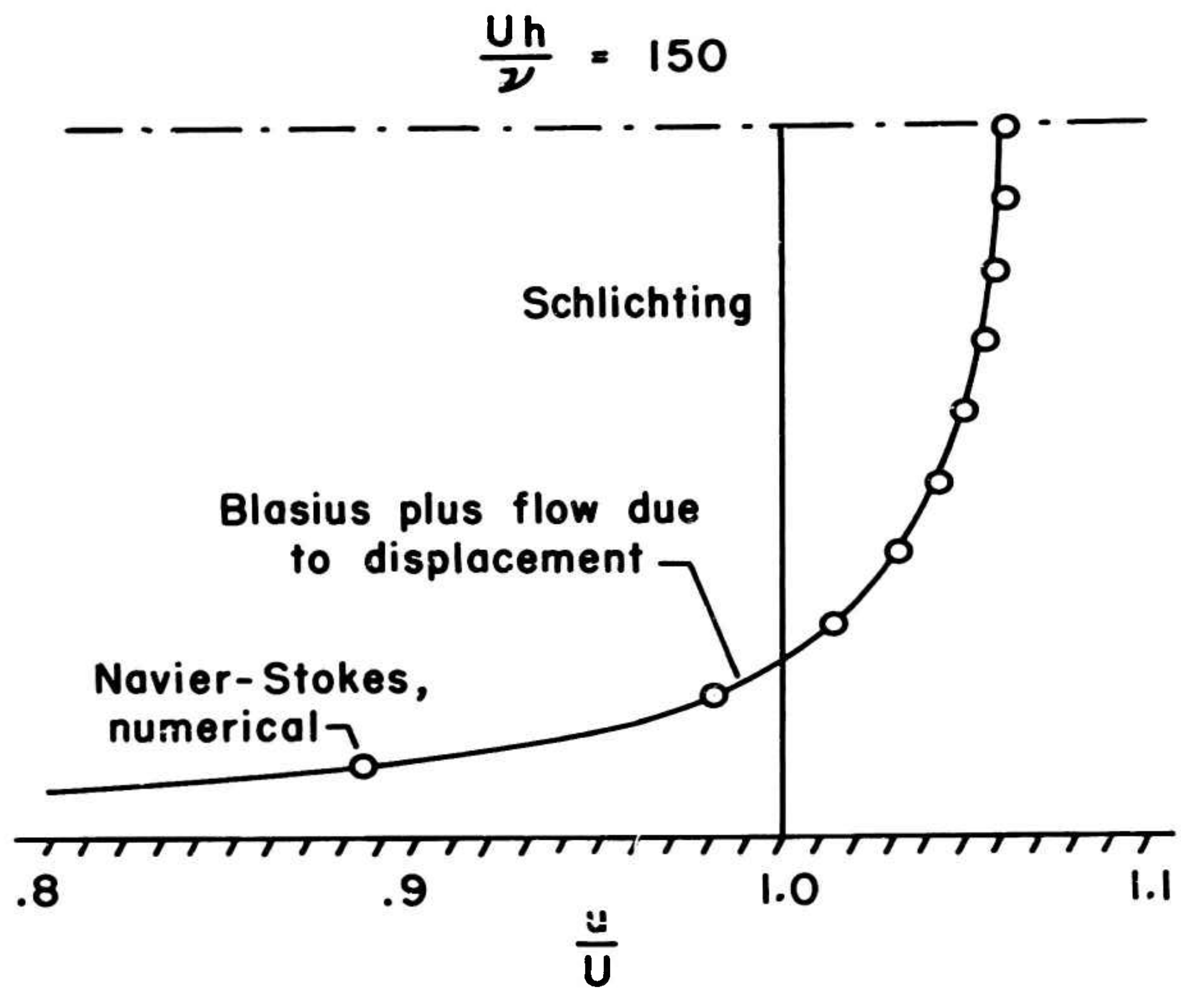
WEINBAUM, S. and GARVINE, R. W., "The pressure field in compressible boundary layer theory – a critical re-examination and treatment", General Electric Space Sci. Lab. Rept. No. R66SD13, 1966.

YEN, K. T. and TOBA, K., "A theory of the two-dimensional boundary layer over a curved surface", J. Aero. Sci. 28, 877-884, 1961.

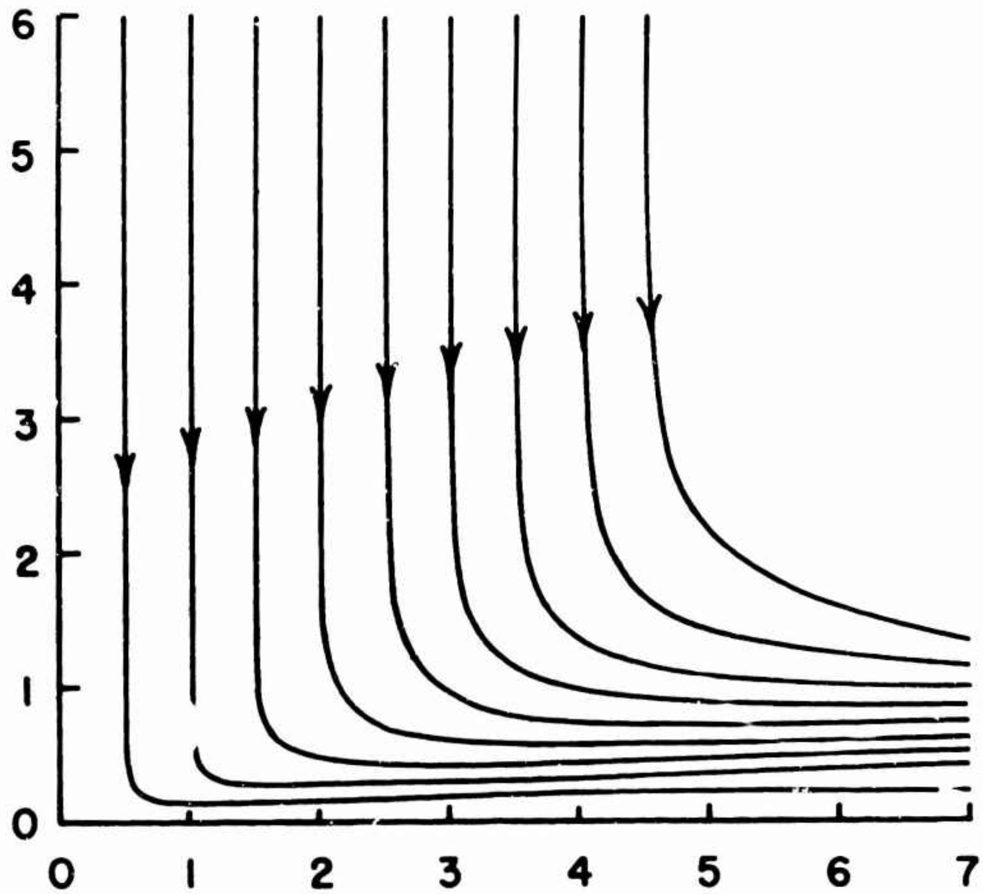


BOUNDARY LAYERS ON CASCADE

FIGURE 1.

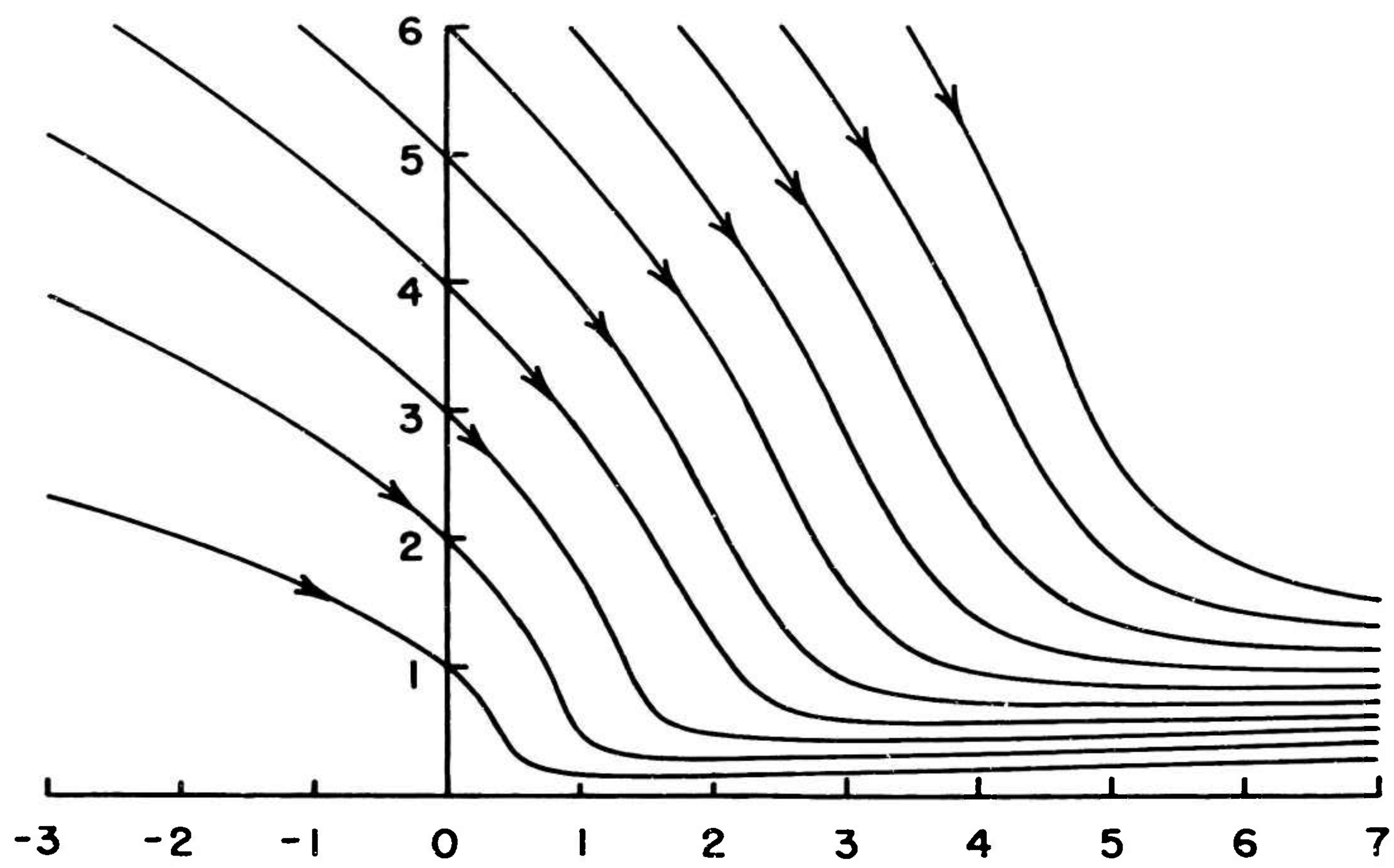


ENTRY PROFILE



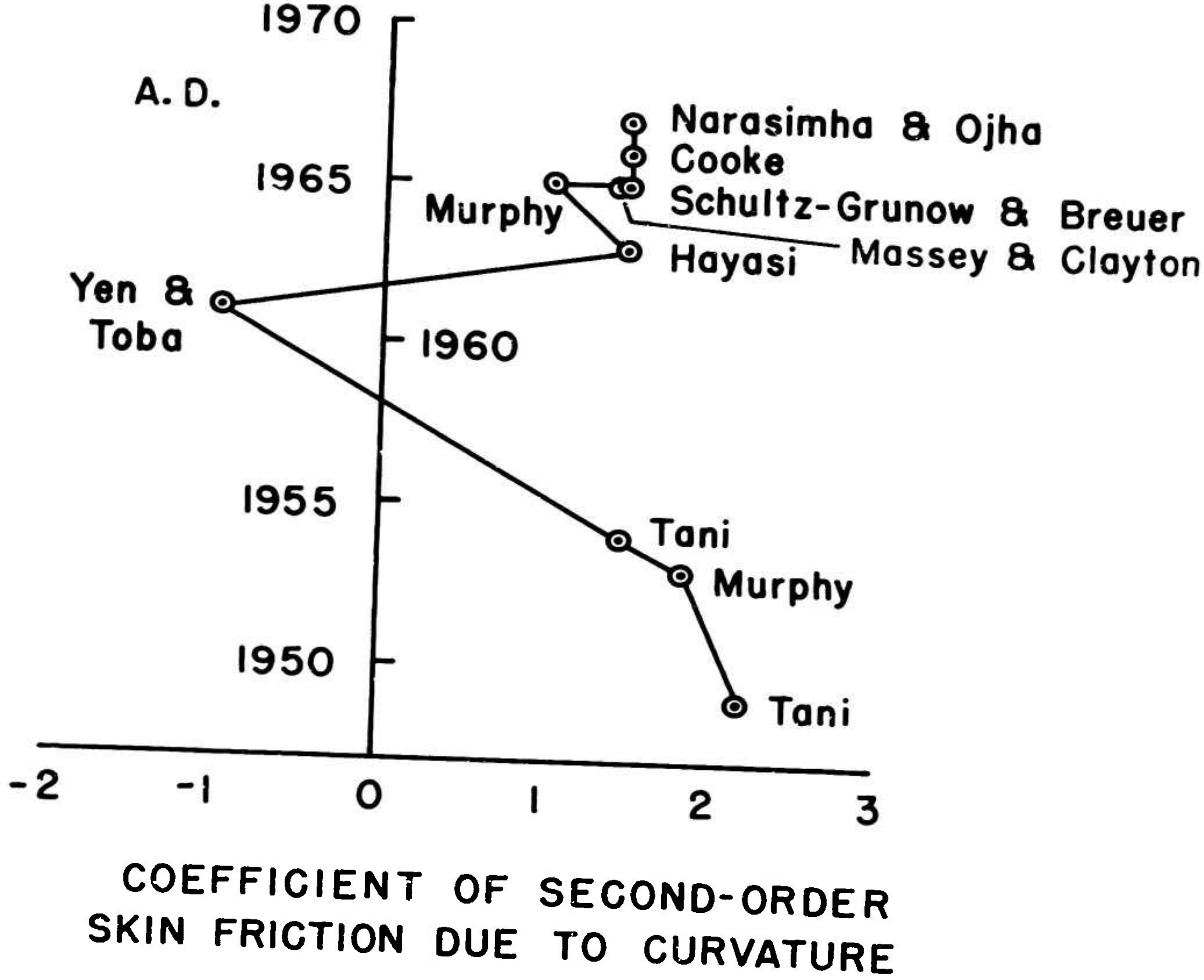
BOUNDARY-LAYER SOLUTION FOR AXISYMMETRIC LAMINAR JET IN CYLINDRICAL COORDINATES

FIGURE 3.

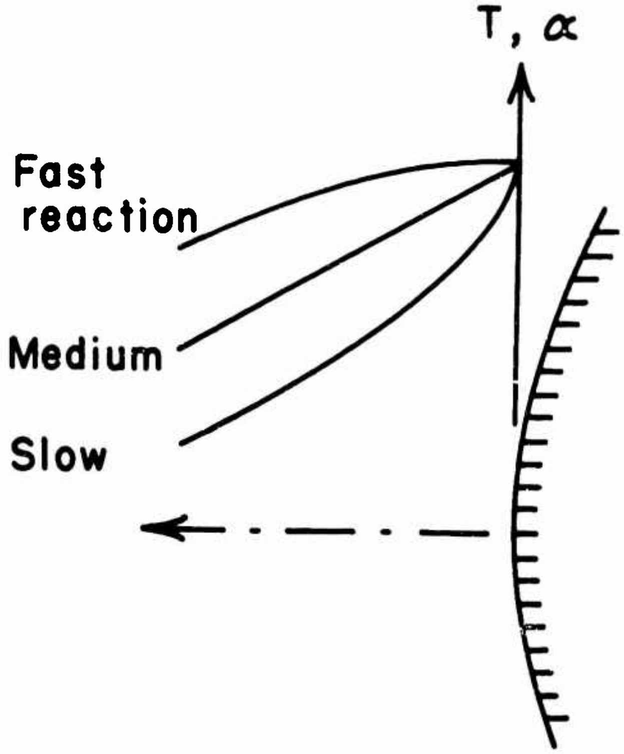


BOUNDARY LAYER PLUS DISPLACEMENT FLOW FOR
AXISYMMETRIC LAMINAR JET FROM NOZZLE

FIGURE



1970
A. D.
1965
1960
1955
1950
Narasimha & Ojha
Cooke
Schultz-Grunow & Breuer
Massey & Clayton
Murphy
Hayasi
Yen & Toba
Tani
Murphy
Tani
-2
-1
0
1
2
3
COEFFICIENT OF SECOND-ORDER SKIN FRICTION DUE TO CURVATURE



NONANALYTIC OUTER FLOW
NEAR STAGNATION POINT

FIGURE 6.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Stanford University Department of Aeronautics and Astronautics Stanford, California 94305	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

A SURVEY OF HIGHER-ORDER BOUNDARY-LAYER THEORY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

Milton Van Dyke

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1967	26	49

8a. CONTRACT OR GRANT NO. AF 49(638)1274 b. PROJECT NO. 9781-01 c. 61445014 d. 681307	9a. ORIGINATOR'S REPORT NUMBER(S) SUDAAR 326 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFOSR 67-2291

10. DISTRIBUTION STATEMENT

1. Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES Tech-Other	12. SPONSORING MILITARY ACTIVITY AF Office of Scientific Research (SREM) 1400 Wilson Boulevard Arlington, Virginia 22209

13. ABSTRACT

A survey of higher-order boundary-layer theory is made. The solutions to a number of cases of steady, plane or axisymmetry, laminar flows are reviewed where one or more of the higher order effects such as, displacement, longitudinal curvature, transverse curvature, and external vorticity are considered. The method of matched asymptotic expansion is offered as a technique which can be applied to such problems without having to relay upon physical insight. The more complicated compressible second-order boundary theory in contrast with the incompressible case is seen developed with minimum of error. A new field of application for higher-order boundary-layer thoery is discussed. This is motion in which the basic inviscid flow is singular at the surface of the body. This situation has arisen recently in various branches of high-speed aerodynamics.

DD FORM 1473

UNCLASSIFIED
Security Classification

BEST AVAILABLE COPY

Higher-order approximation						
Boundary-layer thoery						
Aerodynamics						