A Theory of Credit Scoring and the Competitive Pricing of Default Risk*

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Abstract

We propose a theory of unsecured consumer credit where: (i) borrowers have the legal option to default; (ii) defaulters are not exogenously excluded from future borrowing; and (iii) there is free entry of lenders; and (iv) lenders cannot collude to punish defaulters. In our framework, limited credit or credit at higher interest rates following default arises from the lender's optimal response to limited information about the agent's type. The lender learns from an individual's borrowing and repayment behavior about his type and encapsulates his reputation for not defaulting in a credit score. We take the theory to data choosing the parameters of the model to match key data moments such as the overall delinquency rate. We use the model to quantify the value to having a good reputation in the credit market in a variety of ways, and also analyze the differential effects of static versus dynamic costs on credit market equilibria.

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1 Introduction

It is well known that lenders use credit scores to regulate the extension of consumer credit. People with high scores are offered credit on more favorable terms. People who default on their loans experience a decline in their scores and lose access to credit on favorable terms as a result. People who run up debt also experience a decline in their credit scores and have to pay higher interest rates on new loans. While credit scores play an important role in the allocation of consumer credit, credit scoring has not been adequately integrated into the theoretical literature on consumption smoothing and asset pricing. This paper attempts to remedy this gap.

We propose a theory of unsecured consumer credit where: (i) borrowers have the legal option to default; (ii) defaulters are not exogenously excluded from future borrowing; and (iii) there is free entry of lenders; and (iv) lenders cannot collude to punish defaulters. We use the framework to understand why households typically face limited credit or credit at higher interest rates following default, and how these terms evolve over time. We show that such outcomes arise from the lender's optimal response to limited information about the agent's type. The lender learns from an individual's borrowing and repayment behavior about his (or her) type and encapsulates his reputation for not defaulting in a credit score.

Beginning with the work of Athreya (2002), there has been a growing number of papers that have tried to understand bankruptcy data using quantitative, heterogeneous agent models (for example, Chatterjee et al. (2007); Livshits et al. (2007)). For simplicity, these models have assumed that an individual is exogenously excluded from future borrowing while a bankruptcy remains on his credit record. This exclusion restriction is often modeled as a Markov process and calibrated so that on average the household is excluded for 10 years, after which the Fair Credit Reporting Act requires that it be stricken from the household's record.[1]

While this exogenous exclusion restriction is broadly consistent with the empirical fact that following default households have a harder time obtaining credit, a fundamental question remains.[2]

[1] There is also a substantial literature (beginning with Kehoe and Levine (1993)) on endogenous incomplete markets with lack of commitment which assumes that a default triggers permanent exclusion from credit markets.

[2] Han et al. (2015) find that, although recent bankruptcy filers are not outright excluded from the unsecured credit market, their likelihood of receiving an offer is 6% lower than a non-filer's on average. Crucially, the terms of those offers are much more restrictive: for example, controlling for credit score and other borrower characteristics, having a recent bankruptcy on record is associated with a credit limit that is lower by $238 on average. Despite the extensive set of explanatory variables and flexible specifications, Table 3 in their paper documents that statistical models explain only a relatively small portion of the overall variation in contract terms. They state (p. 23) "Our low

Since a Chapter 7 filer is ineligible for a subsequent Chapter 7 discharge for 6 years (and at worst forced into a subsequent Chapter 13 repayment schedule), why don't we see more lending to those who declare bankruptcy? If lenders believe that the Chapter 7 bankruptcy signals something relatively permanent about the household's unobservable characteristics, then it may be optimal for lenders to limit future credit. But if the circumstances surrounding bankruptcy are temporary (like a transitory, adverse shock to expenses), those individuals who have just shed their previous obligations may be a good future credit risk. Competitive lenders use current repayment and bankruptcy status to try to infer an individual's future likelihood of default in order to correctly price loans. There is virtually no existing work embedding this inference problem into a quantitative, dynamic model.

This paper takes a step toward meeting the challenge of relaxing the assumption of exogenous exclusion from future borrowing following a default.[3] We know that if there was no cost or punishment for default, it would be impossible to support borrowing in equilibrium. Here we consider an environment with a continuum of infinitely-lived agents who at any point in time may be one of a finite number of types that affect their preferences. An agent's type is drawn independently from others and follows a persistent process. Importantly, a person's type is unobservable to the lender.[4] As in the discrete choice literature (McFadden (1973); Rust (1987)) the agent's choices are also subject to a purely transitory shock that enters his preferences.

These people interact with competitive financial intermediaries that can borrow in the international credit market at some fixed risk-free rate and make one-period loans to individuals at an interest rate that reflects that person's risk of default.[5] Because unobservable differences in preferences bear on the willingness of each type of agent to default, intermediaries must form some assessment of a person's type which is an input into his credit score. We model this assessment as a Bayesian inference problem; intermediaries use the recorded history of a person's actions in the

R-squareds are remarkable because the amount of information we use is similar to what a lender would have at its disposal in screening a consumer without a prior business relationship." Extending these findings beyond bankruptcy to insolvency, Albanesi and Nosal (2015) document that while bankruptcy is associated with lower access to credit than having a good record, individuals with a bankruptcy on record have greater access to credit than individuals who are insolvent (which may reflect the fact that Chapter 7 bankrupts cannot re-file for 8 years or Chapter 13 before 4 years.

[3]In Chatterjee et al. (2008) we use a related adverse selection model to show that credit can be supported even in a finite horizon model where trigger strategies cannot support credit.

[4]Ausubel (1999) documents adverse selection in the credit market both with respect to observable and unobservable household characteristics.

[5]Chatterjee et al. (2007) shows that there is not a big gain to relaxing the fixed risk-free rate assumption.

credit market to update their prior probability of his type and then charge an interest rate that is appropriate for that posterior. The fundamental inference problem for the lender is to assess whether a borrower or a defaulter is chronically "risky" or just experiencing a transitory shock. A rational expectations equilibrium requires that a lender's perceived probability of an agent's default must equal the objective probability implied by the agent's decision rule. Incorporating this equilibrium Bayesian credit scoring function into a dynamic incomplete markets model is the main technical challenge of our paper. The adoption of a discrete choice framework, which implies that all feasible actions are taken with positive probability, greatly simplifies the model by ruling out zeros in the denominator of the Bayesian posterior and speeds up the computation of an equilibrium.

This is possibly the simplest environment that can make sense of the observed connection between credit history and the terms of credit. Suppose it turns out that, in equilibrium, one type of person, say type g, has a lower probability of default than others. Then, under competition, the price of a discount bond (of any size) could be expected to be positively related to the probability of a person being of type g. Further, default will lower the *posterior* probability of being of type g because type g people default less frequently. This provides the basis for a theory of why people with high scores are offered credit on more favorable terms as in the data.

As in our predecessor paper Chatterjee et al. (2007), all one-period loans are viewed as discount bonds and the price of those bonds can depend on certain observable household characteristics. The price of a bond cannot, however, depend on unobservable characteristics like their preference type. Instead we assume that the bond depends on the agent's probability of repayment or credit score.[6] The probability of repayment depends on the posterior probability of a person being of a given type *conditional* on selling that particular sized bond. This is necessary because the different types will not have the same probability of default for any given sized bond and a person's asset choice is potentially informative about the person's type.[7] With this asset market structure, competition implies that the expected rate of return on each type of bond is equal to the (exogenous) risk-free rate. Our equilibrium concept is closest to a signaling game.

The paper is organized as follows. Section 2 describes our benchmark economy with private

[6] Livshits et al. (2015) document the increasing use of credit scorecards and provide a theory of why this might happen.

[7] Athreya et al. (2012) also consider a signaling model but assume anonymity so that past asset market choices encapsulated in a type score cannot be used as a prior when calculating posteriors associated with current asset market choices.

information. Section 3 describes the equilibrium problems faced by our agents. Section 4 describes our estimation procedure, and in particular includes a discussion on the import of the discrete choice shocks for our computational analysis and estimation in Section 4.2. Section 5 studies the properties of the benchmark model (5.1), and also compares this to a model in which there is no private information about the agent's persistent type (5.2). Section 6 presents additional quantitative analysis for the benchmark economy: specifically, we estimate the value to having a good reputation (6.1), and compare the static and dynamic costs of default (6.2). Section 7 concludes. All the details of the model's computation are contained in Appendix 8.

2 Environment

There is a unit measure of infinitely lived individuals. The persistent component of an individual's earnings, denoted $e_t \in \mathcal{E} = \{e_1, e_2, ..., e_E\} \subset \mathbb{R}_{++}$, is exogenously drawn from a stationary finite state Markov process $Q^e(e_{t+1}|e_t)$. The purely transitory component of an individual's earnings, denoted $z_t \in \mathcal{Z} = \{z_1, z_2, \ldots, z_Z\} \subset \mathbb{R}_{++}$, is exogenously drawn from a stationary probability distribution $H(z_t)$. All earnings draws are independent across individuals.

At time t individuals can borrow $a_{t+1} \in \mathcal{A}_{--} \subset \mathbb{R}_{--}$ or save $a_{t+1} \in \mathcal{A}_{+} \subset \mathbb{R}_{+}$ at discount price q_t determined in a competitive market with risk free net interest rate $r \geq 0$. We will assume that $\mathcal{A} = \mathcal{A}_{--} \cup \mathcal{A}_{+}$ is a finite set which includes 0. At the beginning of any period, if an agent holds debt (i.e. $a_t < 0$), he (or she) can choose whether or not to default $d_t \in \{0, 1\}$. If he defaults (i.e. $d_t = 1$), then he cannot borrow or save (i.e. $a_{t+1} = 0$), and his earnings become $(1 - \eta) \cdot (e_t + z_t)$ where the static costs of default $\eta \in [0, 1)$ (e.g. bankruptcy fees) are incurred in that period *only*. Let

$$\mathcal{Y} = \{(d_t, a_{t+1}) : (d_t, a_{t+1}) \in \{0\} \times \mathcal{A} \text{ or } (d_t, a_{t+1}) = (1, 0)\}$$

be the set of all possible asset choices and default. Let Y be the cardinality of the choice set $\mathcal{Y}$. An individual's action (d_t, a_{t+1}) along with their earnings, previous asset holdings, and prices will define their consumption $c_t^{(d_t, a_{t+1})}$.

Preferences are additively separable over time. In each period t, the individual orders consumption using a continuous, increasing, concave utility function $u(c_t) : \mathbb{R}_{++} \to \mathbb{R}$. Agents discount utility at rate $\beta_t \in \mathcal{B} = \{\beta_1, \beta_2, ..., \beta_B\} \subset [0, 1)$ exogenously drawn from a finite state Markov process $Q^\beta(\beta_{t+1}|\beta_t)$. These β_t shocks are drawn i.i.d. across agents. As in the discrete choice literature, for each action (d_t, a_{t+1}) agents draw an additive unobservable shock $\epsilon_t^{(d_t,a_{t+1})}$ from a Type 1 extreme value distribution $G(\cdot)$. Let the unobservable Y-dimensional action-specific utility shock vector $\epsilon_t = (\epsilon_t^{(d_t,a_{t+1})})_{(d_t,a_{t+1})\in\mathcal{Y}}$. These ϵ_t shocks are drawn i.i.d. across time and agents. The reward in period t from taking action (d_t, a_{t+1}) is given by $(1-\beta_t)u(c_t^{(d_t,a_{t+1})}) + \epsilon_t^{(d_t,a_{t+1})}$.

Intermediaries can observe individuals' earnings and asset market behavior (i.e. e_t, z_t, and (d_t, a_{t+1})), but cannot observe their preferences (i.e. ϵ_t and β_t). Since ϵ_t is i.i.d. over time and individuals, there is nothing to be learned about ϵ_{t+1} from knowledge of ϵ_t. However, since β_t is drawn from a persistent Markov process, there is something to be learned. We will call β_t an agent's type. In that case, the fraction of individuals of each type is given by the invariant distribution implied by Q^β. We denote the creditor's assessment of an individual's type at the beginning of period t before any actions are taken as $s_t = (s_t(\beta_1), ..., s_t(\beta_B)) \in \mathcal{S}$, a finite subset of $[0,1]^B$ with $\sum_{s_t\in\mathcal{S}} s_t = 1$.[8]

Given an individual's observable characteristics $\omega_t = (e_t, z_t, a_t, s_t)$ as well as their credit market actions (d_t, a_{t+1}), the creditor revises his assessment of an individual's type from s_t via Bayes' rule. We denote this update as $\psi_t^{(d_t,a_{t+1})}(\omega_t) \in [0,1]^B$. Finally, since the posterior ψ_t may not lie on the grid $\mathcal{S}$, it is assigned to one of the nearest two points randomly on $\mathcal{S}$. We denote the probability mass function implied by this random assignment rule by $Q^s(s_{t+1}|\psi_t)$.[9] Importantly, note that there is no further punishment to default except possibly loss of reputation since an individual's credit market behavior affects creditors' assessment of their unobservable type.

As a result of this assessment, the prices faced by an individual in the credit market will also depend on his observable state and his credit market actions. Thus, we denote the price function by $q_t^{(0,d_{t+1})}(\omega_t)$. Note that in the absence of private information regarding type, the pricing function would be independent of z_t and a_t (as in fact is the case in Chatterjee et al. (2007)). Furthermore, there is dual dependence of prices on actions a_{t+1} and observables e_t as in Chatterjee et al. (2007). These values influence prices directly because they influence the likelihood of default next period on a loan *conditional* on type (as in standard debt and default models) and they do so indirectly by revealing information about the individual's current type (this is encoded in the update ψ).

[8] Of course the framework is rich enough to add more unobservables. For instance, if the persistent component of earnings are unobservable, then $s_t = (s_t(\beta_1, e_1), ..., s_t(\beta_B, e_E)) \in \mathcal{S} \subset [0,1]^{B\cdot E}$.

[9] We assume there is no insurance against this random assignment.

The timing in any given period is as follows:

1. Individuals begin period t with state vector (β_t, e_t, a_t, s_t).
2. Individuals receive a transitory shock z_t and an action-specific preference shock vector ϵ_t.
3. Given price $q_t^{(0,a_{t+1})}(\omega_t)$, agents choose $d_t \in \{0, 1\}$ if $a_t < 0$; if $d_t = 0$, they choose a_{t+1}.
4. Based on each individual's actions (d_t, a_{t+1}) and observable characteristics ω_t, intermediaries revise their assessments of an individual's type via Bayes' rule, updating s_t to ψ_t.
5. Beginning of next period realizations of β_{t+1} and e_{t+1} are drawn from the exogenous transition functions $Q^\beta(\cdot|\beta_t)$ and $Q^e(\cdot|e_t)$. The beginning of next period type score s_{t+1} is drawn from the probability mass function $Q^s(\cdot|\psi_t)$.

3 Equilibrium

3.1 Individuals' problem

Let a current value x_t be denoted x and next period's variable x_{t+1} be denoted x'. Denote the part of the state space observable to creditors by $\Omega = \{\mathcal{E} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{S}\}$ with typical element ω. After the realization of shocks, an individual begins the period in state $(\epsilon, \beta, \omega)$.

Each individual takes as given

- the price function $q^{(0,a')}(\omega) : \{\mathcal{Y} \backslash (1, 0)\} \times \Omega \to [0, 1/(1 + r)]$ [10]
- the type scoring function $\psi^{(d,a')}(\omega) : \mathcal{Y} \times \Omega \to [0, 1]^B$, which is a function that performs Bayesian updating of an individual's type based on all observables, including the current type score and the action taken.

For ease of notation, we will denote the vector-valued set of functions $\{q(\cdot), \psi(\cdot)\}$ by f.

Definition 1. Feasible Set: Given observable state ω and a set of equilibrium functions f the set of feasible actions is a finite set $\mathcal{F}(\omega|f) \subseteq \mathcal{Y}$ that contains all actions such that consumption,

[10] Note that we consider only the asset choice a' when $d = 0$ here since default in the model is assumed to be full: when $d = 1$, there is no repayment by the individual to the financial intermediary.

$c^{(d,a')}(\omega|q)$, satisfies:

$$c^{(d,a')}(\omega|q) = \begin{cases} e + z + a - q^{(0,a')}(\omega) \cdot a' > 0 & \text{for } d = 0, a' \neq 0 \\ (e + z) \cdot (1 - \eta) & \text{for } d = 1, a' = 0 \end{cases} \tag{1}$$

Given an individual's state and the functions f, the current-period return for an individual choosing any feasible action $(d, a') \in \mathcal{F}(\omega|f)$ is $u\left(c^{(d,a')}\right)$, where $u(\cdot)$ is continuous, strictly increasing and concave. Denote by $V(\epsilon, \beta, \omega|f) : \mathbb{R}^Y \times \mathcal{B} \times \Omega \to \mathbb{R}$ the value function of an individual in state $(\epsilon, \beta, \omega)$. Following the discrete choice literature, the shocks ϵ are drawn from $G(\epsilon)$ which is assumed to be a type 1 extreme value distribution with scale parameter $1/\alpha$. An individual's recursive decision problem is then given by

$$V(\epsilon, \beta, \omega|f) = \max_{(d,a') \in \mathcal{F}(\omega|f)} v^{(d,a')}(\beta, \omega|f) + \epsilon^{(d,a')} \tag{2}$$

where the conditional value function is given by

$$\begin{aligned} v^{(d,a')}(\beta, e, z, a, s|f) &= (1 - \beta) u\left(c^{(d,a')}\right) \\ &+ \beta \cdot \sum_{(\beta', e', z') \in \mathcal{B} \times \mathcal{E} \times \mathcal{Z}} Q^{\beta}(\beta'|\beta) Q^{e}(e'|e) H(z') Q^{s}(s'|\psi) W(\beta', e', z', a', s'|f) \end{aligned} \tag{3}$$

is the value associated with a specific action (d, a') and $W(\cdot)$ integrates the value function over transitory preference shocks: that is,

$$W(\beta, \omega|f) = \int V(\epsilon, \beta, \omega|f) dG(\epsilon). \tag{4}$$

Let $\sigma^{(d,a')}(\beta, \omega|f)$ be the probability that the individual in state (β, ω) chooses action $(d, a') \in \mathcal{F}(\omega|f)$. Given the form of the extreme value distribution, this probability has the following well-known form (see, for instance, Rust (1987)):

$$\sigma^{(d,a')}(\beta, \omega|f) = \frac{\exp\left\{\alpha \cdot v^{(d,a')}(\beta, \omega|f)\right\}}{\sum_{(\hat{d}, \hat{a}') \in \mathcal{F}(\omega|f)} \exp\left\{\alpha \cdot v^{(\hat{d}, \hat{a}')}(\beta, \omega|f)\right\}}. \tag{5}$$

And, given this expression,

$$W(\beta,\omega|f) = \frac{\gamma_{\mathrm{E}}}{\alpha} + \frac{1}{\alpha}\ln\left(\sum_{(d,a')\in\mathcal{F}(\omega|f)} \exp\left\{\alpha\cdot v^{(d,a')}(\beta,\omega|f)\right\}\right), \tag{6}$$

where $\gamma_{\mathrm{E}} = 0.56767\ldots$ is Euler's constant.

Theorem 1. *Given f, there exists a unique solution $W(f)$ to the individual's decision problem in (2) to (4). Furthermore, $W(f)$ is continuous in f.*

Proof. The proof relies on the Contraction Mapping Theorem. However, since $\epsilon^{(d,a')}$ can take any value on the real line, it is mathematically more convenient to seek a solution to (1), (3) and (6) (the extreme value errors do not appear in these). Define the operator $(T_f)(W) : \mathbb{R}^{B+|\Omega|} \to \mathbb{R}^{B+|\Omega|}$ as the map that takes in a $\mathbb{R}^{B+|\Omega|}$ vector of W values in (3) and returns a $\mathbb{R}^{B+|\Omega|}$ vector of W values via (6) using (1). We may easily verify that T_f satisfies Blackwell's sufficiency condition for a contraction map (with modulus β). Since $\mathbb{R}^{B+|\Omega|}$ is a complete metric space (with, say, the uniform metric $\rho(W,W') = \max_{1\leq i\leq B+|\Omega|}\|W_i - W_i'\|$), by Theorem 3.2 of Stokey and Lucas Jr. (1989), there exists a unique $W(f)$ satisfying $(T_f)(W) = W$.

To prove continuity of $W(f)$ we first show that the operator T_f is continuous in f. Let $f_n \in \mathbb{R}^{M+K}$ be a sequence converging to $\bar{f}$. Then, given the continuity of u with respect to c, the continuity of $c^{(d,a')}$ with respect to q for $(d,a') \in \mathcal{F}(\omega|f)$ and the continuity of Q^s with respect to ψ, it follows from (3) that $\lim_n T_{f_n} = T_{\bar{f}}$. Furthermore, since $\mathbb{R}^{M+K}$ is a Banach space, we may apply Theorem 4.3.6 in Hutson and Pym (1980) to conclude that $W(f)$ is continuous in f. □

Corollary 1. *For any $(d,a') \in \mathcal{F}(\omega|f)$, $v^{(d,a')}(\beta,\omega|f)$ and $\sigma^{(d,a')}(\beta,\omega|f)$ are continuous in f.*

3.2 Intermediaries' problem

Competitive intermediaries with deep pockets have access to an international credit market where they can borrow or lend at the risk-free interest rate $r \geq 0$. An intermediary also incurs a proportional cost $\iota \geq 0$ when making loans to individuals. Any given intermediary takes prices q and scoring function ψ (i.e. f) as given. The profit $\pi^{(0,a')}(\omega|f)$ on a financial contract of type $(0,a')$

made to individuals with observable characteristics ω is:

$$\pi^{(0,a')}(\omega|f) = \begin{cases} \frac{p^{(0,a')}(\omega|f)\cdot(-a')}{1+r+\iota} - q^{(0,a')}(\omega)\cdot(-a') & \text{if } a' < 0 \\ q^{(0,a')}\cdot a' - \frac{a'}{1+r} & \text{if } a' \geq 0 \end{cases} \tag{7}$$

where the probability of repayment on a financial contract of type $(0, a')$ made to individuals with observable characteristics ω is denoted $p^{(0,a')}(\omega|f) : (0 \times \mathcal{A}_{--}) \times \Omega \to [0, 1]$. Given perfect competition in financial intermediation and constant returns to scale in the lending technology, optimization by the intermediary implies that for each contract

$$q^{(0,a')}(\omega) = \begin{cases} \frac{p^{(0,a')}(\omega|f)}{1+r+\iota} & \text{if } a' < 0 \\ \frac{1}{1+r} & \text{if } a' \geq 0 \end{cases} \tag{8}$$

To assess an individual's probability $p^{(0,a')}(\omega|f)$ of repaying a debt *tomorrow* given their observable characteristics ω in order to price debt *today*, an intermediary must solve an inference problem since neither the persistent β nor the transitory ϵ are observable. This probability can be separated into two steps:

1. Assess the probability that an individual in state ω who takes action (d, a') today will be of unobservable type β' tomorrow via Bayes rule (the type scoring function $\psi_{\beta'}^{(d,a')}(\omega)$). Since the posterior ψ may not lie on the grid $\mathcal{S}$, for each β' randomly assign it to the nearest two points on $\mathcal{S}$. The endogenous transition function associated with this step is denoted $Q^s(s'|\psi)$.

2. For each possible future unobservable type β', compute the individual's probability of future repayment conditional on being that type and transitions over observable characteristics and then compute the weighted sum over future types to obtain p.

Starting with step 1, an individual's probability of being type $(\beta'_1, ..., \beta'_B)$ tomorrow is given by the Bayesian type scoring function $\psi^{(d,a')}(\omega) = \left(\psi_{\beta'_1}^{(d,a')}(\omega), ..., \psi_{\beta'_B}^{(d,a')}(\omega)\right)$. For each possible value of $\beta' \in \mathcal{B}$, the intermediary assigns probability

$$\psi_{\beta'}^{(d,a')}(\omega) = \sum_{\beta} Q^{\beta}(\beta'|\beta) \cdot \frac{\sigma^{(d,a')}(\beta, \omega|f) \cdot s(\beta)}{\sum_{\hat{\beta}} \left[\sigma^{(d,a')}(\hat{\beta}, \omega|f) \cdot s(\hat{\beta})\right]} \tag{9}$$

to an individual of type β with observable state ω and action (d, a') being of type β' tomorrow.[11] The first term in the sum in (9) is the probability of transitioning to the particular β' from a given β today. The numerator of the second term is the probability that an agent with a given β today and observable state ω chooses an action (d, a') today, weighted by the currently assessed probability that the agent in question actually has this β. Finally, the denominator of the second term is a scaling term to aggregate over all possible current values of β. Note that $\psi^{(\cdot)}(\cdot)$ is a B-vector-valued function as specified above and satisfies $\sum_{\beta'\in\mathcal{B}} \psi_{\beta'}^{(d,a')}(\omega) = 1$ for all $\omega \in \Omega$ and all (d, a') in the associated feasibility set $\mathcal{F}(\omega|f)$.

Since the posterior ψ may not lie on the grid $\mathcal{S}$, we randomly assign it to one of the two nearest s' on $\mathcal{S}$. For each possible value of $\beta' \in \mathcal{B}$, we find two adjacent grid points $s'_i(\beta')$ and $s'_j(\beta')$ such that $s'_i(\beta') \leq \psi_{\beta'}^{(d,a')}(\omega) \leq s'_j(\beta')$, and assign probability $\chi(\beta'|\psi)$ to $s'_i(\beta')$ and $1 - \chi(\beta'|\psi)$ to $s'_j(\beta')$ where

$$\chi(\beta'|\psi) = \frac{s'_j(\beta') - \psi_{\beta'}^{(d,a')}(\omega)}{s'_j(\beta') - s'_i(\beta')}. \tag{10}$$

For all s' such that $s'(\beta') \in \{s'_i(\beta'), s'_j(\beta')\}$ for all $\beta' \in \mathcal{B}$, the probability of being assigned type score s' in period $t+1$ is equal to

$$Q^s(s'|\psi) = \prod_{s'(\beta')=s'_i(\beta')} \chi(\beta'|\psi) \cdot \prod_{s'(\beta')=s'_j(\beta')} (1 - \chi(\beta'|\psi)), \tag{11}$$

where $\chi(\beta'|\psi)$ is given by (10). For all other s', we have $Q^s(s'|\psi) = 0$. Note that we assume in (11) that the likelihood of being assigned to a given $s'(\beta') \in \{s'_i(\beta'), s'_j(\beta')\}$ is independent across β'.

Given observable state ω, the type scoring s' in (11) given s and ψ from (9), we obtain the probability of repayment the intermediary uses for pricing debt (i.e. for $a' < 0$) via:

$$\begin{aligned} p^{(0,a')}(\omega|f) \quad = \quad & \sum_{s'\in\mathcal{S}} Q^s(s'|\psi) \cdot \Bigg[\sum_{(e',z')\in\mathcal{E}\times\mathcal{Z}} Q^e(e'|e) \cdot H(z') \\ & \cdot \sum_{\beta'\in\mathcal{B}} s'(\beta') \left(1 - \sigma^{(1,0)}(\beta', \omega'|f)\right) \Bigg]. \end{aligned} \tag{12}$$

[11] It is worth noting a critical distinction here. The type score s and s', which is updated according the $\psi(\cdot)$ function, reflects an individual's *assessed* probability of being a given type. The *objective* probability of being a given type in the current period (unconditionally) is given by the stationary distribution implied by $Q^\beta(\cdot)$; conditional on today's type, the objective probability of being a given type β' tomorrow is given by $Q^\beta(\beta'|\beta)$.

The first term captures the discrete transition over type scores across periods; the second term captures the exogenous transition over future earnings (conditional on the persistent component of today's earnings); and the final term is an agent's probability of repayment tomorrow conditional on type and the observable future state. Equation (12) can be separated in the following way: the bottom line is the probability of repayment in period $t+1$ conditional on the entire observable state, while the the top line captures the probability of transitioning to that observable state in $t+1$ conditional on date t observables.

Remark 1. Note that equations (10) through (12) have implicit dependence on the full observable state of the individual, ω, through the $\psi(\cdot)$ function in equation (9). This dependence is suppressed for notational convenience, however, since these state variables only matter in how they impact the assessment of an individual's type, which is governed entirely by (9). An important exception is equation (12), in which must incorporate e directly in order to account for earnings transitions.

3.3 Evolution

The probability that an individual in state (β, e, z, a, s) transits to state (β', e', z', a', s') given a set of functions f is:

$$T^*(\beta', \omega'|\beta, \omega; f) = \sigma^{(d,a')}(\beta, \omega|f) \cdot Q^s(s'|\psi^{(d,a')}(\omega)) \cdot Q^\beta(\beta'|\beta) \cdot Q^e(e'|e) \cdot H(z') \tag{13}$$

where the first term reflects the probability that an individual in state (β, e, z, a, s) chooses the asset position a', and the remaining terms govern the transitions – both endogenous and exogenous – over the remaining state variables.

Let $\mu(\beta, e, z, a, s|f)$ be the measure of individuals in state (β, e, z, a, s) today for a given set of equilibrium functions f. Then, the cross-sectional distribution evolves according to

$$\mu'(\beta', \omega'|f) = \sum_{(\beta,\omega)\in\mathcal{B}\times\Omega} T^*(\beta', \omega'|\beta, \omega; f) \cdot \mu(\beta, \omega|f). \tag{14}$$

An invariant distribution in this model is a fixed point $\overline{\mu}(\cdot)$ of (14).

Lemma 1. *There exists a unique invariant distribution $\overline{\mu}$.*

Proof. (Sketch) We will use Theorem 11.2 in Stokey and Lucas Jr. (1989) to establish this result.

To connect to that theorem, let $S = \mathcal{B} \times \Omega$, with $M = |S|$ so that $S = \{s_1, ..., s_M\}$ is a finite set. Further, let the transition matrix Π in their theorem correspond to T^* in (14). We need to establish that there exists a state s_j such that the following holds: for every $i \in \{1, ..., M\}$, there exists $n \geq 1$ such that $\pi_{i,j}^{(n)} > 0$. Clearly in (13), the exogenous transitions Q^β and Q^e from any β and e to β' and e' take a finite number of steps as assumed in the environment section of the paper and the exogenous probability distribution H has full support. Next, note that $Q^s(s'|\psi)$ in (13) has full support since ψ depends on $Q^\beta(\beta'|\beta)$ in (9). That is, the fact that any individual's type may switch exogenously implies that the posterior assign some weight to an individual of being each type after a finite number of periods. Finally, for any f, since $a' = 0$ in $\sigma^{(d,a')}(\beta, \omega|f)$ in (13) is chosen with some probability due to the extreme value shocks for any a (default $(1, 0)$ is always feasible for $a < 0$ and $a' = 0$ is always feasible for any $a \geq 0$). Thus there exists an s_j with $a' = 0$ for every s_i such that $\pi_{i,j}^{(n)} > 0$. $\square$

Remark 2. Note that although the invariant distribution is critical for computing cross-sectional moments used to map the model to the data in later sections of the paper, none of the other equilibrium objects (i.e. the set of functions f, the value function $V(\cdot)$ or the decision rule $\sigma^{(\cdot)}(\cdot)$) take $\mu(\cdot)$ as an argument. This simplifies the model and eases the computational burden, but is not necessary. Other specifications in which knowledge of the distribution is required are possible, but we do not consider these in the benchmark case.

3.4 Equilibrium definition

We can now give the definition of a stationary recursive competitive equilibrium.

Definition 2. Stationary recursive competitive equilibrium: A stationary recursive competitive equilibrium is a vector-valued pricing function q^*, a vector-valued type scoring function ψ^*, a vector-valued quantal response function σ^*, and a steady state distribution $\overline{\mu}^*$ such that:

- $\sigma^{(d,a')*}(\beta, \omega|f^*)$ satisfies (5) for all $(\beta, \omega) \in \mathcal{B} \times \Omega$ and $(d, a') \in \mathcal{F}(\omega|f^*)$,
- $q^{(0,a')*}(\omega)$ satisfies (8) for all $\omega \in \Omega$ and $(d, a') \in \mathcal{F}(\omega|f^*)$ with $p^{(0,a')*}(\omega|f^*)$ satisfying (12) for all $\omega \in \Omega$ and $(d, a' < 0) \in \mathcal{F}(\omega|f^*)$,
- $\psi_{\beta'}^{(d,a')*}(\omega)$ satisfies (9) for all $(\beta', \omega) \in \mathcal{B} \times \Omega$ and $(d, a') \in \mathcal{F}(\omega|f^*)$, and

- $\overline{\mu}^*(\beta, \omega | f^*)$ solves (14) for T^* in (13).

3.5 Existence of Equilibrium

Theorem 2. *There exists a stationary recursive competitive equilibrium.*

Proof. (Sketch) Let $\mathcal{G} = \{((d, a'), \beta, \omega) : (d, a') \in \mathcal{F}(\omega|f), \beta \in \mathcal{B}, \omega \in \Omega\} \subset \mathcal{Y} \times B \times \Omega$. Define the cardinalities $M = |\mathcal{G}|$ and $K = |\mathcal{A}_{--} \times \Omega|$.

1. Let f be the vector composed by stacking $q \in [0, 1]^K$ and $\psi \in [0, 1]^M$ so $f \in [0, 1]^{K+M}$.

2. Let
$$W = W(f) : [0, 1]^{K+M} \to \mathbb{R}^{B+|\Omega|}$$
be the solution established in Theorem 1.

3. Given W, use (3) to construct function $v^{(d,a')}(\beta, \omega|f) : \mathcal{G} \to \mathbb{R}$ and then construct the vector-valued function
$$v = J_1(W) : \mathbb{R}^{B+|\Omega|} \to \mathbb{R}^M$$

4. Given v, construct the function $\sigma^{(d,a')}(\beta, \omega|f) : \mathcal{G} \to (0, 1)$ using the mapping in (5) and then construct the vector-valued function
$$\sigma = J_2(v) : \mathbb{R}^M \to (0,\ 1)^M.$$

5. Given σ and ψ, use the mapping in (12) to construct the function $p^{(0,a')}(\omega|f) : \mathcal{A}_{--} \times \Omega \to [0, 1]$ and then construct the vector-valued function
$$p = J_3(\sigma, \psi) : (0,\ 1)^M \times [0, 1]^M \to [0, 1]^{|\mathcal{A}_{--} \times \Omega|}.$$

6. Given p, use the mapping in (8) to construct the function $q_{\text{new}}^{(0,a')}(\omega) : \{\mathcal{Y} \backslash (1, 0)\} \times \Omega \to [0, 1]$ and given σ, use the mapping in (9) to construct the function $\psi_{\text{new}}^{(d,a')}(\omega) : \mathcal{Y} \times \Omega \to [0, 1]^B$. Then construct the $K + M$ vector

$$f_{\text{new}} = (q_{\text{new}}, \psi_{\text{new}}) = J_4(p, \sigma) : [0,1]^{|\mathcal{A}_{--} \times \Omega|} \times [0,1]^M \to [0,1]^{K+M}.$$

7. Finally, denote by $J(f) : [0,\,1]^{K+M} \to [0,\,1]^{K+M}$ the composite mapping $J_4 \circ J_3 \circ J_2 \circ J_1 \circ W$. By Theorem 1 $W(f)$ is continuous. By inspection, the functions J_i, $i \in \{1,2,3,4\}$ are also continuous. Hence J is a continuous self-map.

8. Since $[0,\,1]^K$ is a compact and convex subset of $\mathbb{R}^K$, the existence of $f^* = J(f^*)$ is guaranteed by the Brouwer's FPT.

9. Given $f^* = (q^*, \psi^*)$, the existence of a unique $\overline{\mu}^*$ follows from Lemma 1.

□

4 Estimation

In this section, we describe our calibration of the model, and particularly describe the role of the extreme value preference shocks. All of our quantitative work assumes a period length of one year.

4.1 Parameters and moments

We calibrate the earnings process outside the model using estimates from Floden and Lindé (2001), Table 4. In particular, the variance of the log of the transitory component of earnings reported in Floden and Lindé (2001) is 0.0421. We approximate this process by a three-point uniform distribution on support $\mathcal{Z} = \{-z, 0, z\}$, where $z = \sqrt{\frac{3}{2} \cdot 0.0421} = 0.18$. The persistent component of earnings is an AR(1) in logs, with autocorrelation of 0.9136 and innovation variance of 0.0426. We approximate this process by a 3-state Markov process using the method developed by Adda and Cooper (2003). The resulting support, $\mathcal{E}$, and transitions, $Q^e(e'|e)$, are given in Table 1b.[12]

Aside from the earnings process, there are 9 parameters that must be chosen. In addition to these, we must also set appropriate grids for the remaining state variables, (β, a, s), which are relevant for an individual's decision problem. Details on parameters are contained in Tables 1a; similar information on grid is contained in the Appendix, Table 4.

[12] Observe, in particular, that this earnings parameterization satisfies our earlier assumption that $\min \mathcal{E} + \min \mathcal{Z} + \min \mathcal{A} > 0$. Therefore, all debt choices are feasible for all types of agents in all states.

We divide the set of parameters into two groups: those we calibrate, and those we choose outside the model. The calibrated parameters include: (i) the extreme value scale parameter, α; (ii/iii) the two switching probabilities for each β type, $Q^{\beta}(\beta'_L|\beta_H)$ and $Q^{\beta}(\beta'_H|\beta_L)$; (iv) the discount factor for the low type, β_L; and (v) the exogenous default cost, η.[13] We use a two-state β-type process because this allows us to collapse the generally vector-valued state function $\psi(\cdot)$ into a scalar, easing computation.[14] Subsection 4.2 below discusses the scale parameter in detail. Intuitively, the nature of the discount factor type process is important in the model. If the two types were very different (i.e. $\beta_H - \beta_L$ large), type scores should have only a small impact on prices because agents' fundamentally preferred actions would be different enough that the intermediary can easily assess agents' true type based on their behavior over time. Put differently, the benefit of acting like the other type will be small relative to the cost of taking these actions, and so the types will separate more.[15] Similarly, higher type persistence (i.e. lower $Q^{\beta}(\beta'_L|\beta_H)$ and $Q^{\beta}(\beta'_H|\beta_L)$) increases the benefit to the intermediary of inferring correctly an agent's type, since this assessment is likely to be persistently correct. Finally, the exogenous default cost η is used to mitigate the extent of default in the model, and to study the relative efficacy of static and dynamic punishments in mitigating default (Section 6.2).

The parameters we select rather than calibrate are: (i) the discount factor of the high β type, β_H; (ii), the coefficient of relative risk aversion, ν; (iii) the risk-free rate r; and (iv) the cost of intermediation, ι. The high β is standard for models with a period length of one year. The coefficient of relative risk aversion, which indexes the curvature of the flow utility function $u(c) = c^{1-\nu}/(1-\nu)$, is standard in the macro literature. A risk-free rate of 3% is consistent with the observed average for a one-year time horizon, and intermediation costs of 1% roughly reflect fixed costs in operating a bank.

We calibrate these parameters to a set of five moments drawn from Chatterjee et al. (2007) and Chatterjee and Eyigungor (2015). These moments, presented in Table 2 below, are: (i) the economy-wide default rate; (ii) the average interest rate paid in the economy; (iii) median assets to median income; (iv) the fraction of households in debt; and (v) the aggregate debt to earnings

[13]The η parameter reflects static costs of default (filing costs, legal fees, etc.), which have been shown to play a key role in the filing decision by Albanesi and Nosal (2015).

[14]That is, we now have $\psi_{\beta'_L}(\omega) = 1 - \psi_{\beta'_H}(\omega)$.

[15]Note that in our discrete choice framework, *perfect* separation is not possible because all feasible actions are chosen with positive probability by all types.

Parameter	Notation	Value
Calibrated		
Low type discount factor	β_L	0.89
Low β to high β transition probability	$Q^\beta(\beta'_H\|\beta_L)$	0.05
High β to low β transition probability	$Q^\beta(\beta'_L\|\beta_H)$	0.11
Exogenous default cost	η	9.8%
Extreme value scale parameter	α	183.3
Selected		
High type discount factor	β_H	0.97
Coefficient of relative risk aversion	ν	3
Risk-free rate	r	3.0%
Intermediation cost	ι	1.0%
Earnings	$e, z, Q^e(\cdot\|\cdot), H(\cdot)$	See Table 1b

(a) **Model parameters**

Persistent		e'		
$Q^e(e'\|e)$		$e'_1 = 0.575$	$e'_2 = 1.000$	$e'_3 = 1.740$
	$e_1 = 0.575$	0.818	0.178	0.004
e	$e_2 = 1.000$	0.178	0.643	0.178
	$e_3 = 1.740$	0.004	0.178	0.818

Transitory				
level	z	$z_1 = -0.18$	$z_2 = 0$	$z_3 = 0.18$
probability	$H(z)$	1/3	1/3	1/3

(b) **Detail: persistent component of earnings**

Table 1: **Parameterization**

ratio. For details on the computation of these moments within the model, please see Appendix 8.3. We have chosen these targets for two reasons. First, since the fact that they are standard in the literature allows for simple comparison across studies. Second, they provide convenient metrics for the size and riskiness of unsecured credit markets without constraining our model to directly match key facts about credit scores and prices, allowing these moments to serve as validation for our model.

For a given set of parameters, we can compute the model analogs of the moments presented in Table 2, applying Simulated Method of Moments (SMM).[16] We require that the aggregate line of the benchmark model match the data as closely as possible: the other figures in the table are

[16] Let M^D be the 5-vector of data moments, and let $M(x)$ be the analogous vector of model moments implied by

	Default rate (%)	Average interest rate (%)	Med. net worth to med. income	Fraction HH in debt (%)	Average debt to income (%)
Data					
aggregate	0.54%	11.35%	1.28	6.73%	0.67%
Benchmark					
aggregate	0.53	9.98	2.13	8.24	0.64
β_H	0.39	10.06	2.80	5.24	0.44
β_L	0.61	9.92	1.76	10.22	0.77
Full information					
aggregate	0.45	11.61	2.20	7.98	0.61
β_H	0.42	12.94	2.92	5.02	0.45
β_L	0.50	10.77	1.83	9.86	0.72

Table 2: **Model moments: data, benchmark, and full information**

presented for discussion and comparison. The benchmark model delivers a tight fit to the data in the aggregate, with the notable exception of median net worth to median income.[17]

4.2 Scale parameter and the impact of extreme value preference shocks

Relative to standard work in macroeconomics and finance, one of the key modifications in our model is the inclusion of the additive, action-specific preference shocks. Although these extreme value shocks are assumed to be mean zero, we have allowed the variance to be general in formulating the model through the scale parameter α.[18] How does behavior in the model change with this variance? Following Train (2002), a simple calculation shows that $\partial\sigma^{(d,a')}(\beta,\omega)/\partial\alpha$ takes the sign of

$$\sum_{(\tilde{d},\tilde{a}')\in\mathcal{F}(\omega|f)} \left[v^{(d,a')}(\beta,\omega) - v^{(\tilde{d},\tilde{a}')}(\beta,\omega)\right] \cdot \exp\left\{\alpha \cdot \left(v^{(d,a')}(\beta,\omega) + v^{(\tilde{d},\tilde{a}')}(\beta,\omega)\right)\right\}$$

the set of parameters $x = (\alpha, Q^\beta(\beta_L'|\beta_H), Q^\beta(\beta_H'|\beta_L), \beta_L, \eta)$. The estimation problem, then, is simply

$$\hat{x} = \arg\min_x (M^D - M(x))' W (M^D - M(x)), \tag{15}$$

where W is an appropriately chosen positive semi-definite weighting matrix. In our computations, we set $W = I_5$. Because each computation of our model is quite costly and the equilibrium objects are highly nonlinear, we use the derivative-free, least squares minimization routine developed in Zhang et al. (2010).

[17] This is to be expected, since ours is a net worth model in which agents cannot simultaneously hold both assets and debt.

[18] This amounts to setting the location parameter of the extreme value distribution equal to 0. Since our model contains no outside option, and only the *difference* between the shocks associated with each pair of actions matters for the determination of choice probabilities, this normalization has no impact on behavior in the model.

Furthermore, examining equation (5) reveals that

$$\arg \max_{(d,a')\in\mathcal{F}(\omega|f)} \sigma^{(d,a')}(\beta,\omega) = \arg \max_{(d,a')\in\mathcal{F}(\omega|f)} v^{(d,a')}(\beta,\omega),$$

so that the action which delivers the highest total utility *before* the extreme value shock is chosen with the highest probability. Combining these two pieces of information, we see that as α increases, the probability of choosing the action with the highest conditional value increases relative to all other feasible actions. Put differently, as α increases, (i) more and more weight is placed on the modal action, and (ii) the mean action converges to the modal action. For actions that are "suboptimal" in the sense that they deliver lower conditional value than the modal action, the change in weight depends on the difference in conditional value, weighted by the total value of these actions. This can have a meaningful impact on the mean action taken, if not the mode, which can effect prices and type scores significantly.

Figure 1 demonstrates the impact of changing α on decisions in the model. The top two panels present the *modal* decisions for each β across all a for $e = 1, z = 0$ in the full information case of the model, discussed in Section 5.2 below.[19] That is, we plot the a' level that corresponds to $\arg\max_{(d,a')} \sigma^{(d,a')}(\beta, 1, 0, a)$. Note that the bottom of the asset grid is -0.25, and we represent the default decision $(d, a') = (1, 0)$ by $a' = -1.25$ on the graph. The bottom two panels depict the *mean* decision $E(a') = \sum_{a'\in\mathcal{Y}} a' \cdot \sigma^{(d,a')}(\beta, 1, 0, a)$ for the same subset of the state space. The left panels are for a high value of the scale parameter (low dispersion), $\alpha = 300$; the right panels are for a lower value of $\alpha = 10$ (high dispersion).

Immediately, we see that there are two key changes to observed decisions in the model resulting from changing α. First, as the bottom two panels reveal, the mean action is much closer to the modal action for the higher value of α; this graphically confirms the intuition discussed above, since the modal action is chosen with higher probability. Second, we can observe that the modal action itself is not invariant to α. It is crucial to note that this is an *equilibrium property* of our model, whereas the first point would remain true even if we solved only the decision problem in partial equilibrium, taking the set of equilibrium functions f as given.

Consider the example of the high β type default decision (the solid black line in Figure 1). With

[19] We use the full information case simply to reduce the dimensionality of the state space to depict in figures.

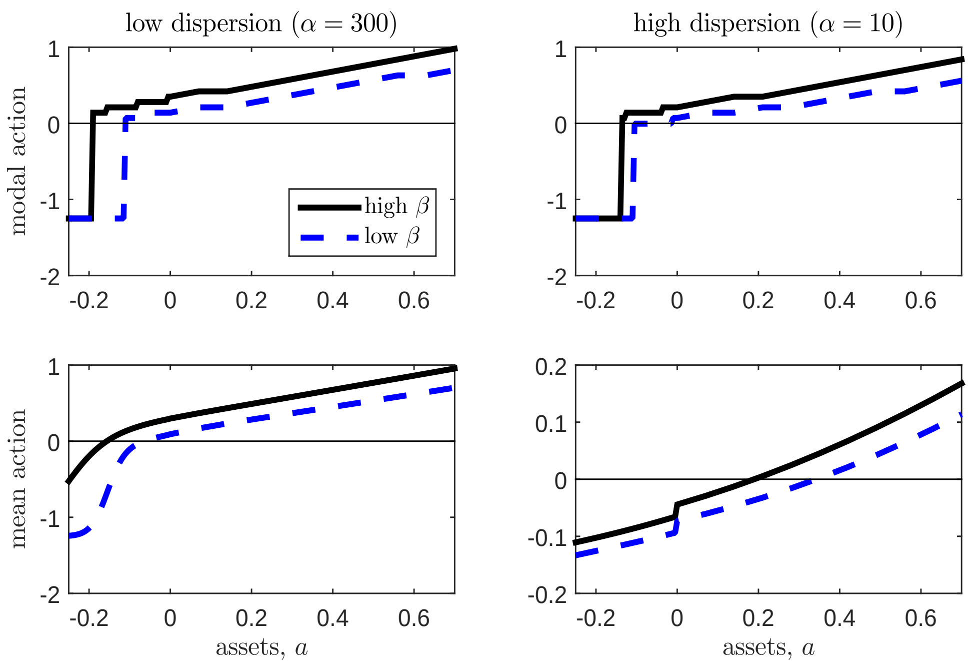


Figure 1: **Impact of extreme value preference shocks**

$\alpha = 300$, the modal action is default for the high β type only at the very bottom of the asset grid. When $\alpha = 10$, though, she defaults (in the mode) for every asset position lower than about -0.11. How might this come about? Since higher α translates to lower weight placed on actions that yield less than the conditional value, the agent rarely defaults on debts in the model with high α. However, as the agent goes deeper into debt, the benefit of defaulting relative to other feasible actions increases as the required repayment increases, and so default begins to look more attractive: this effect is present in both cases in Figure 1. Only for the lower α, though, is the weight placed on the suboptimal default action sufficiently high to actually impact prices. This price effect makes defaulting relatively more attractive than choosing other (still feasible) debt levels, and so defaulting becomes the modal action for sufficiently large debts.

5 Model Properties

In this section, we present the key properties of the benchmark model presented in Section 3 and calibrated in Section 4. In order to understand the role of private information in the model, we also explore a version of the model in which types are directly observable in Section 5.2.

5.1 Benchmark equilibrium

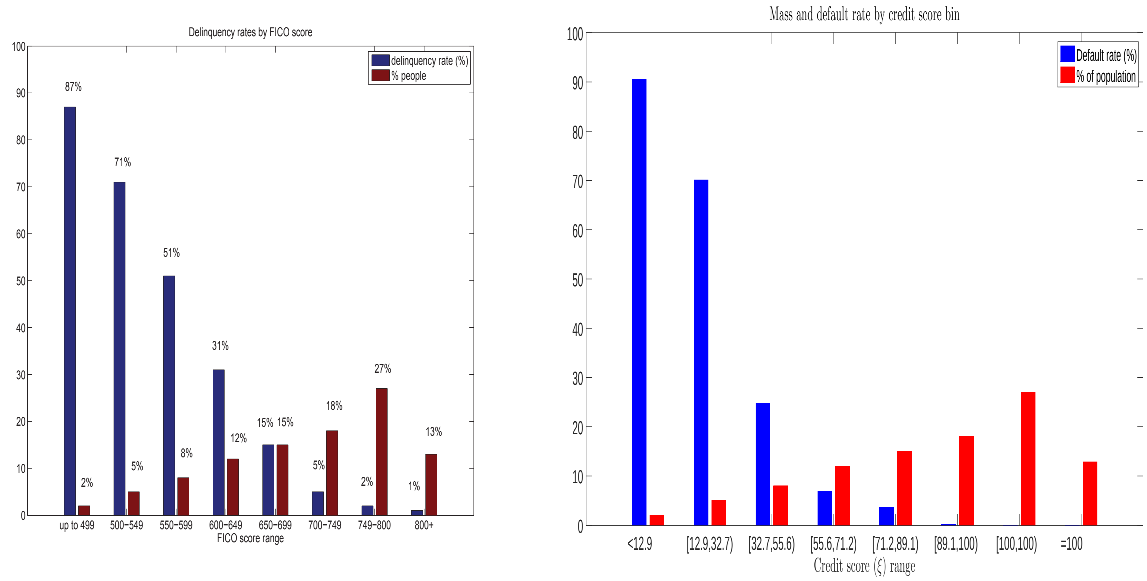
It is critical to understand the behavior of the different types in the model. Table 2, reveals that high β types: (i) default at a much lower rate (0.39%, compared to 0.61% for low β types); (ii) face higher interest rates on average (10.06% vs. 9.92%); (iii) hold more assets and take on debt less frequently (high median net worth to median income, half as high a fraction in debt as low types); and (iv) take on smaller debts when they do go into debt. At first glance, all these properties appear consistent with intuition, except for the interest rates. How can agents who default less often face higher interest rates on average in an environment with perfectly competitive, risk neutral pricing? The answer lies in a selection effect: for a given debt level choice, high β types tend to face more favorable terms, but they tend to choose deeper debt levels whose increased risk demands a higher interest rate. Note that these debts are larger only in the absolute sense, since higher β types tend to take out less debt relative to their income (last column).

While these core moments help underscore the differences in behavior between the two types in the benchmark model, it is difficult to tease out the critical effects of reputation by considering only the stationary equilibrium of the benchmark economy. Therefore, in the remaining sections of the paper, we use the estimated model to compute moments, conduct simulations, and run counterfactual experiments which can more directly address this question.

5.1.1 Credit scoring: data vs. model

In the introduction, we motivated our analysis of credit scoring by appealing to the stark trend of delinquency profiles by credit score subgroup in Figure 2. Since we have not calibrated our model to anything to do with credit scores directly, a good test of the model is to compute credit scores, divide the population into ranges, and compute the default rate within each credit score range in order to compare the profiles of each credit score subgroup in our model to those in the real world. The result of this analysis is presented in Figure 2 below.

In order to construct Figure 2, we must first compute the analog of a credit score in our model. While the technical details of this procedure are contained in Appendix 8.3, it is worthwhile to motivate the definition and outline the computational procedure here. In the real world, consumers have a credit score which, in principle, summarizes all the knowable, relevant information about



(a) Data (b) Model

Figure 2: Distribution of credit scores and default

the consumer which affects their probability of repayment. In general, the consumer then takes on credit at a rate jointly determined by (i) this credit score and (ii) the size of the loan they wish to take out. In our model, there is a vector of prices, one for each possible action the agent can take, which also depends on all the observable information about the agent, ω. Since these prices are action-specific, the appropriate construction of a credit score must integrate out over possible actions a given agent in observable state ω can take using knowledge of (i) the equilibrium choice probabilities $\sigma(\cdot)$ and (ii) the stationary distribution of agents in the population, $\mu(\cdot)$. Performing this calculation gives the average probability of default in the next period *before* a given action is chosen. An analogous procedure can give the probability of default in the next n periods.

Upon examining Figure 2, we immediately see that the distribution of default rates over the different credit score brackets in our model *very* closely resembles those in the real world presented in Figure 2. Figure 2 is constructed by taking the population fractions within each credit score bracket as given in order to define the relevant score thresholds, and then computing the default rate between these thresholds. We see that default is very common (about 90%) for the lowest credit scores, and becomes less and less common as credit score increases, until it disappears at the top end of the distribution. One interesting divergence between the two figures is that our model produces more default on the low end of the credit score range than what we observe in the real world, and

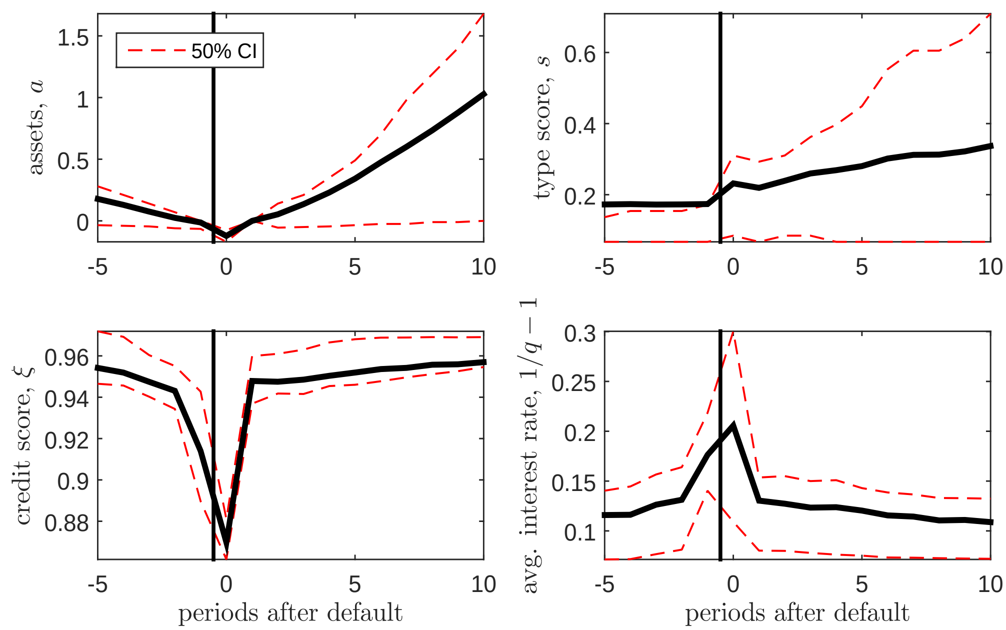


Figure 3: **Key state variables**

less default at the high end of the range. This is directly attributable to the fact that, in our model, the high end of the credit score range is occupied exclusively by individuals who have $a > 0$. Since our model features only a *net worth* savings decision – i.e. agents cannot simultaneously have a stock of assets ($a > 0$) and take on debts ($a < 0$) – these agents by construction cannot default.[20]

5.1.2 What happens before and after default?

In the introduction, we highlighted evidence that individuals tend to face significantly less favorable terms in the wake of defaulting. Is this the case in our model? If so, why? In this section, we use the estimated model to simulate a panel of individuals over a large number of periods in order to isolate trends in key state variables and pricing terms leading up to and following default events.[21]

Each panel of Figure 3 depicts a relevant state or pricing variable of an individual who chooses to default in the period indexed as 0. The solid black lines represent the mean, and the dotted red

[20] This is an important difference between the environment of our model and the one in the real world, where having positive assets (say, a savings account) and debts (say, a mortgage) are very common. The reasons for this are outside the scope of the current study.

[21] Specifically, we construct a pseudo-panel with $N = 5,000$ individuals for $T = 1,000$ periods. Then, we drop the first 100 periods, and collect all the default events with 5 periods preceding and 10 periods trailing. All summary statistics are then computed off of this sub-sample with even weighting for all observations. Note that since we simulate from the stationary distribution of the model, a default in period t is the same as a default in period t'.

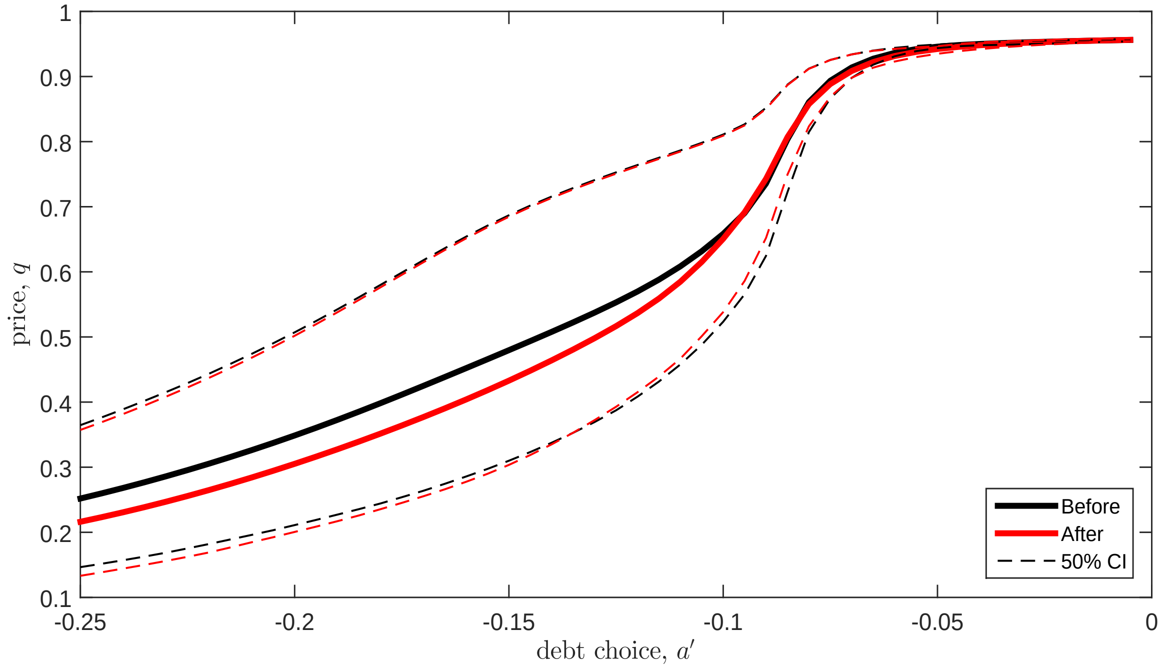


Figure 4: **Price menu for debt before and after default**

lines are the inter-quartile range. It is most natural to read this figure from left to right and top to bottom. First, in the top left we see that agents naturally tend to decline into debt, default on a relatively large debt, then immediately begin to save out of debt. Interestingly, the dispersion in wealth post-default is quite wide. Next, in the top right, we see that agents tend to be assigned very low type scores (i.e. high probability of having $\beta = \beta_L$) while declining into debt. Then, as they begin to save out of debt in the wake of their default, their reputation recovers, and they are assigned higher and higher probabilities of having a high β. Third, agents credit scores (bottom left) begin to decline as they accumulate debt, and then plummet in the period in which they default. This is consistent with the findings of Musto (2004). Then, as agents shed debt, they immediately begin to sharply improve their credit score. It follows logically that the first order effect of a default in our model is actually positive on net because of debt forgiveness: even if your reputation declines, shedding debt lowers your subsequent default risk. Finally, the bottom right panel plots the average (net) interest rate agents face in taking out debt before and after default. As the asset position and reputation weaken in the lead-up to a default, the typical terms of credit worsen in kind. This trend then spikes in the period after default (i.e. it would be very costly to go immediately back into debt after defaulting), and then settles back to normal in subsequent periods.

Figure 4 builds on this last panel of Figure 3. Rather than simply showing the average interest rate from *chosen* actions, which includes the effect of transitory preference shocks, this figure depicts the entire price menu an agent faces on average before (black) and after (red) default. This allows us to see not only the selected interest rates, but also – critically – the pricing terms which drive those decisions. Most notable is the fact that terms worsen in the wake of default most severely for higher levels of debt. This completes the logical circle which extends back to the top two panels of Figure 3: as prices for larger debt levels worsen in the wake of default, agents are disincentivized from going deep into debt and tend to save.

5.2 Equilibrium with observable types

In order to understand the effect of private information in our model, we consider a case where β is observable while ϵ remains unobservable as in standard discrete choice models like Rust (1987). In this case, the relevant state for an individual is simply $\omega^{FI} = (\beta, e, z, a)$. Each component is directly observable, obviating type scoring.[22] That is, intermediaries need not *assess* types since they can actually *see* them. Therefore, the set of equilibrium functions is simply $f^{FI} = q(\cdot)$, since type scores – $\psi(\cdot)$ – are no longer necessary. This case offers a desirable comparison, since it isolates the effects of pricing without being clouded by the inference problem introduced in the full model.

We first compare the model's performance to our targeted set of moments. The results of this comparison are presented in Table 2. Note that we do not recalibrate the model for the full information case: rather, we simply compute the full information model for the parameters given in Table 1. Relative to the benchmark case, in the full information model: (i) agents default less frequently (0.45%, compared to 0.53% in the benchmark); (ii) average interest rates are much higher (by about 2 percentage points); (iii) median assets are a slightly higher fraction of income (2.20 vs. 2.13), while slightly fewer households take on debt (7.98% vs 8.24%); and (iv) the size of debt choices relative to income are about the same.

The best way to understand how private information influences behavior in the model is to compare the differences in the behavior between types under the benchmark case and the full information case. Table 2 shows that default is much less dispersed across types under full information,

[22] Note that $\omega = (\omega^{FI}, s)$, since the only part of the state space that changes in the full information model is that s is dropped.

suggesting that when high β types do not stand to suffer a reputational loss (their true β is known), the incentive not to default is lowered moderately. Conversely, low β types default less (0.50% vs 0.61%) because they obtain on average less debt (debt to income of 0.72% vs. 0.77%), and take on debt less frequently (9.86% vs. 10.22%). Average interest rates rise across the board in the full information model.

Finally, we complete the discussion of the full information model's properties by presenting the full price schedules for all debt choices in Figure 5. The black lines are for the "best" types in the full information and benchmark models: in the full information model (dotted) these are simply high β types, while in the benchmark model (solid) these are the agents with a current type score at the highest possible level, $s = \overline{s}$. Likewise, the blue lines are for the worst types: low β in the full information case, $s = \underline{s}$ in the benchmark. Much like in the pre- and post-default image of Figure 4, prices worsen more and more for worse types (assessed or actual) for deeper and deeper choices of a'. Perhaps most interesting, though is the fact that price dispersion is much greater in the full information model than in the benchmark. That is, the solid lines in the figure are much closer than the dashed. This suggests that the presence of private information in the benchmark model limits the extent to which intermediaries can effectively price discriminate based on type. Furthermore, prices are "less extreme" in the benchmark model in the sense that agents with good reputation face less favorable terms than those known to have high β in the full information case; likewise, agents with bad reputation face more favorable terms than those known to have low β.

5.2.1 Welfare analysis

How much more consumption per period must an agent receive in the benchmark economy to be indifferent with the full information economy? In order to answer this question, we can define a "consumption equivalent" measure. Given benchmark and full information expected values, $W(\beta, \omega)$ and $W^{FI}(\omega^{FI})$, respectively, we can compute the measure

$$\lambda(\beta, \omega) = \left[\frac{W^{FI}(\omega^{FI})}{W(\beta, \omega)} \right]^{\frac{1}{1-\nu}}, \tag{16}$$

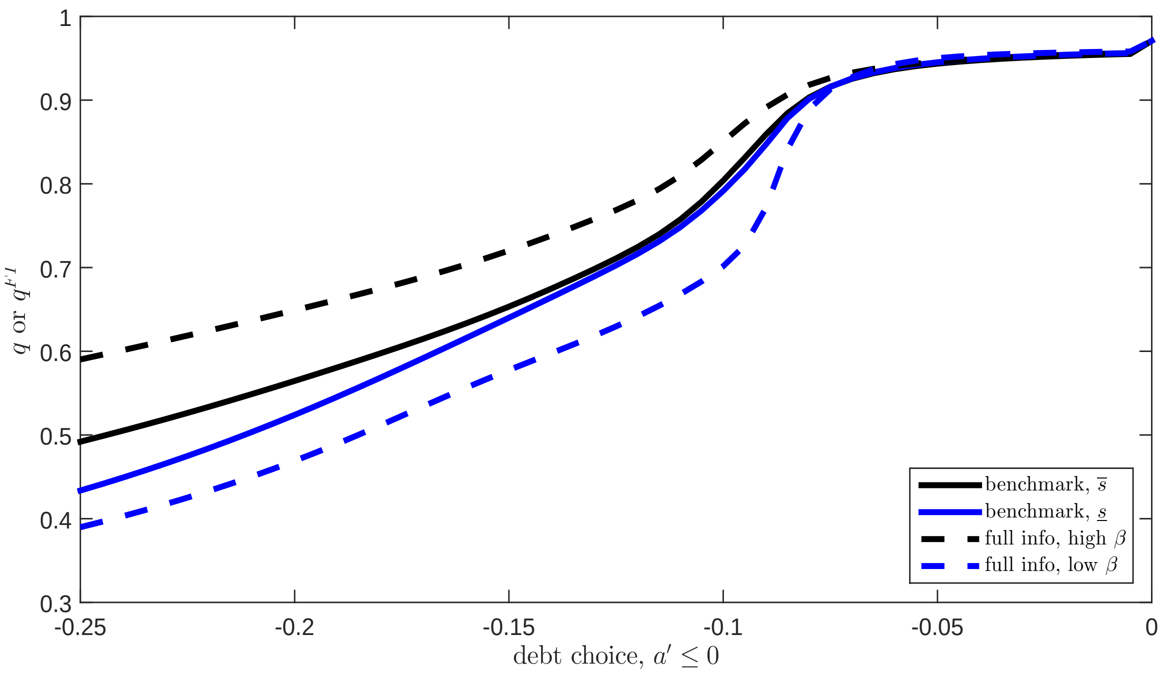


Figure 5: **Prices of debt by reputation (type) in benchmark (full information) model**

where ν is the coefficient of relative risk aversion.[23] Note that a positive value of λ implies that agents are better off in the full information economy than in the benchmark economy with private information.

The results of computing this measure over the stationary distribution and for different subsets of the population are depicted below in Table 6a.[24] Immediately, we see that agents much prefer the full information economy on the whole, with an average value of 0.038% for λ. High β agents prefer the full information setup more than low β agents (0.063% vs. 0.021%) because they stand to benefit from the reputation boost they'd receive if their types were revealed. Notably, low β types with bad reputation who are in debt actually *prefer* the benchmark economy with private information. This is because these agents represent the small subset who both use the credit market and stand to gain a better reputation when information is removed from the economy. Finally, in Figure 6b we plot the average value of λ across the asset space and the entire range of type scores,

[23]It is important for this analysis to integrate out the transitory preference shocks. Therefore, we use $W(\cdot)$ instead of $V(\cdot)$ in the analysis. Expression (16) for λ is obtained by solving

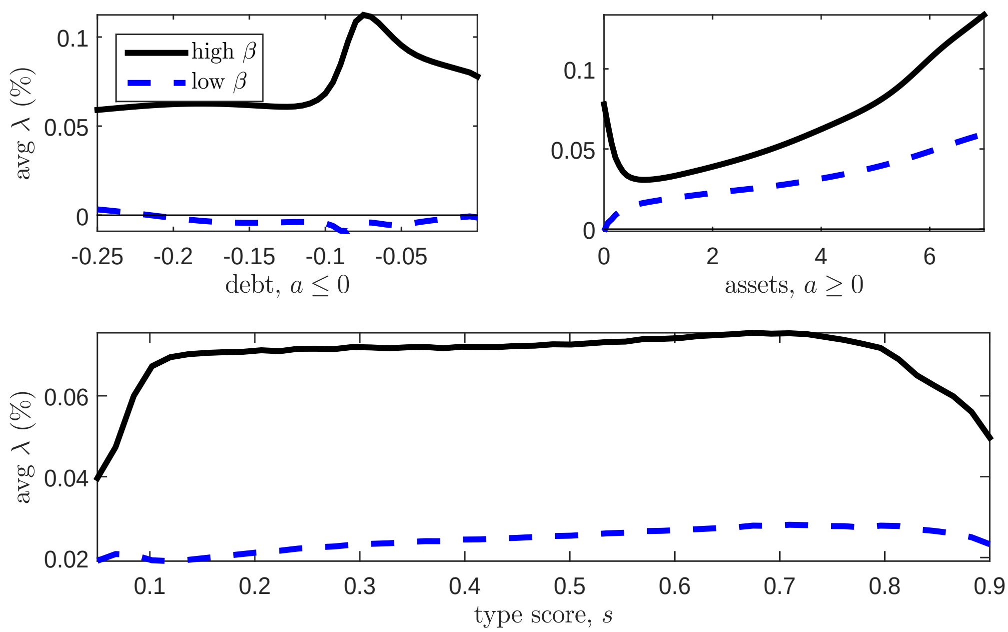
$$W^{FI}(\omega^{FI}) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta_t^t u\left(c_t(\beta,\omega) \cdot (1+\lambda(\beta,\omega)\right)\right],$$

where $c_t(\beta, \omega)$ is the consumption implied by agents' optimal policies in the benchmark model.

[24]Throughout, all aggregation is done with respect to the stationary distribution of the model, $\mu(\cdot)$.

$\overline{\lambda}$ (%)	**aggregate**	**in debt** $(a<0)$ **total**	**worst rep.**	**best rep.**	**saving** $(a \geq 0)$ **total**	**worst rep.**	**best rep.**
aggregate	0.038	0.016	0.020	0.048	0.040	0.895	0.049
β_H	0.063	0.052	0.076	0.165	0.062	1.729	0.000
β_L	0.021	0.014	-0.003	0.029	0.024	0.845	0.000

(a) **By sub-population**



(b) **Across assets and type scores**

Figure 6: **Welfare analysis, benchmark to full information**

corroborating the evidence in the table above.

6 Further Quantitative Analysis

6.1 Measuring the value of reputation

In order to measure reputation, we define for each state (β, ω) a number $\tau(\beta, \omega)$ such that for all type scores $s \geq \underline{s}$, where $\underline{s}$ is the lowest possible type score (equal to the probability of transitioning from β_L to β_H),

$$W(\beta, e, z, a, s) = W(\beta, e, z, a + \tau(\beta, e, z, a, s), \underline{s}). \tag{17}$$

In this sense, $\tau(\beta, \omega)$ measures how much an agent in state (β, ω) would be willing to give up in terms of assets in the current period in order to avoid being re-assigned *today* to the lowest possible type score. We can gain insight into the relative size of reputation by integrating these τ values over the stationary distribution, i.e. by looking at

$$\overline{\tau} = \sum_{\beta,\omega} \tau(\beta, \omega) \cdot \mu(\beta, \omega), \tag{18}$$

or by looking at how $\tau(\cdot)$ across possible states.[25] In our model, integrated over the stationary distribution, we obtain a value of $\overline{\tau}$; for different subsets of the population, the results are presented in Table 7a and Figure 7b.

Looking at this table, we see immediately that the average household requires compensation in order to accept the lowest possible reputation. As expected, this compensation tends to be higher for high types than for low types (0.020% vs. 0.011%). Moreover, the value of τ tends to be much higher (9.2 times on average) for agents in debt, for whom reputation immediately and tangibly affects consumption through the equilibrium debt pricing schedule. This effect is particularly severe for agents in debt who have the best possible reputation: for example, low β types who are incorrectly assessed with $s = \overline{s}$ require compensation of 0.847% of median earnings to have their assessed type reset to the lowest level. On the other hand, agents who save require very little compensation to have their reputations lowered. Since saving always occurs at the risk-free rate in the model, they both receive no immediately adverse price impacts and tend to recover their reputations quickly by saving. Figure 7b fleshes out these ideas, plotting τ for each type over various slices of the state space. Most notable in this figure is the top left panel, which isolates agents currently in debt. Agents very deeply in debt can require up to 3.5% of median earnings as compensation to have their reputation set to $\underline{s}$.

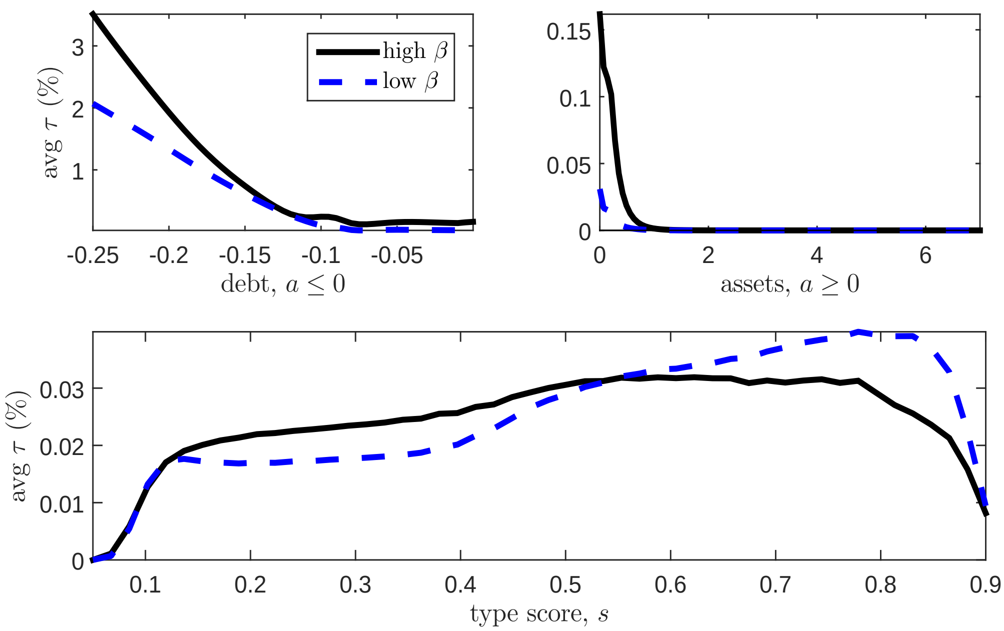
6.2 Dynamic and static costs of default

Examining the quantitative impact of a significant bankruptcy reform that went into effect in 2005 in the United States, Albanesi and Nosal (2015) find that increased costs of bankruptcy filings

[25] Note that $\tau(\cdot)$ has the property that $\tau(\beta, e, z, a, \underline{s}) = 0$, since agents already in the lowest score can't be made to pay to avoid it.

$\overline{\tau}$ (%)	aggregate	in debt $(a < 0)$			saving $(a \geq 0)$		
		total	worst rep.	best rep.	total	worst rep.	best rep.
aggregate	0.015	0.139	0.000	0.613	0.004	0.000	0.006
β_H	0.020	0.252	0.000	0.586	0.007	0.000	0.006
β_L	0.011	0.101	0.000	0.847	0.002	0.000	0.006

(a) **By sub-population**



(b) **Across assets and type scores**

Figure 7: **Reputation analysis, asset value of avoiding bad reputation**

deter agents from filing bankruptcy and exacerbate their financial constraints. Motivated by their analysis, in this section we examine how changes in the static cost of default, η, impact the dynamic value of reputation, τ. To this end, Table 3 contains two sets of results: the first with the benchmark calibrated value of $\eta = 9.8\%$, and the second with no static income loss, $\eta = 0$. For the second model, we maintain all other parameters at the values indicated in Table 1.

The first two moments in Table 3, which (loosely) measure the riskiness of the credit market, increase each by a factor of about 5 when we do away with the income loss associated with default. The last three moments, which (again, loosely) measure the size of the credit market in the model economy, change significantly less. In particular, the fraction of households in debt decreases from 8.24% to 6.69%, although the size of the average debt relative to income increases from 0.64% to

Moment	Data	Model	
		$\eta = 9.8\%$	$\eta = 0$
Targets			
Default rate (%)	0.54	0.53	2.63
Average interest rate (%)	11.35	9.98	57.73
Median net worth to median income	1.28	2.13	2.20
Fraction of households in debt	6.73	8.24	6.69
Average debt to income ratio	0.67	0.64	0.82
Other			
$\overline{\tau}$ (%)	-	0.02	0.21

Table 3: **Dynamic and static costs of default**

0.82%. Most interestingly, though, the value of τ increases by a factor of 10.5 when we go from the model with positive η to the one with $\eta = 0$. This suggests that having a good reputation becomes much, much more value the lower is the static incentive of agents to repay.

7 Conclusion

In this paper, we presented a model of unsecured consumer credit with an endogenous default decision and no exogenous exclusion from credit markets in the event of a default. In an environment in which households are subject to both persistent and transitory preference shocks, lenders have to solve an inference problem in order to properly assess agents' type. Given these assessments, lenders price debt in order to break even in expectation given the households' endogenously determined default risk, given perfect competition. In equilibrium, assessed default risk must be consistent with the actual choices of borrowers.

From a methodological perspective, we contribute to the extant literature in two primary ways. First, we scrap the assumption of exogenous stochastic exclusion from credit markets following a default event, and instead endogenize the dynamic punishment for default through prices. To do so, we must solve for the optimal pricing function on the lender side, which necessarily incorporates dynamic assessments of an individual's underlying type. In our model, this type captures the propensity to default. In executing this first contribution, we deliver our second contribution: the adoption of techniques from the discrete choice literature. Given the necessity of Bayesian updating for assessing the probability of an agent being of a given type, this inclusion has the desirable property that all feasible actions are chosen with positive probability by all agents, imposing good behavior on the type scoring function. Additionally, these techniques ease computation and therefore estimation of the model.

Given the calibrated version of the model described above, we explore the model's main properties and run a series of quantitative experiments designed to assess: (i) the effect of private information; (ii) the value of reputation; and (iii) the efficacy of static versus dynamic punishments in deterring default and sustaining credit in an environment with limited commitment. We find that agents would on average need to have their consumption increased by about 0.03% per period in the benchmark economy in order to be indifferent between this economy and one with full information about type. Furthermore, agents would require a non-trivial amount of compensation – about 0.015% of median earnings – in order to "lose their reputation," or be assigned to the lowest possible type score. Finally, we find evidence of significant substitutability between static and dynamic punishment for default. In the benchmark model with a relatively high income loss in the event of a default, the value of maintaining a good reputation is relatively low: that is, most of the deterrence from default is achieved by the static component of the punishment. In a world with no income loss from default, however, the value of maintaining a good reputation is significantly higher. This suggests that acting in a way which signals a good underlying type is more important the less important is the static deterrent.

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8 Appendix

8.1 Computational algorithm

In this section, we describe the algorithm used to compute the benchmark model presented in this paper. Note that the model is calibrated by using the procedure below to solve the model for a given set of parameters, and then updating these parameters to minimize the distance between the model moments and the data moments.

1. Set parameters and tolerances for convergence and create grids for (β, e, z, a, s).[26] Denote the length of these grids by $n_\beta, n_e, n_z, n_a,$ and n_s respectively. The parameters should include the exogenous transition matrices $Q^\beta(\beta'|\beta)$ and $Q^e(e'|e)$. Further, note that the variable s lies in the range $[Q^\beta(\beta'_L|\beta_H), 1 - Q^\beta(\beta'_H|\beta_L)]$.

2. Initialize the following equilibrium objects with sensible initial conditions:

   (a) $W(\beta, \omega) = 0$ for all β, ω.

   (b) $\psi^{(d,a')}(\omega) = \mu = \frac{Q^\beta(\beta'_H|\beta_L)}{Q^\beta(\beta'_H|\beta_L)+Q^\beta(\beta'_L|\beta_H)}$ for all $\omega, (d, a')$, so that there is no information initially on types other than the stationary distribution implied by $Q^\beta(\cdot)$. Note that this implies that, initially,

   $$\begin{aligned} s'_i &= \max\{s \in \mathcal{S} | s \leq \mu\} \\ s'_j &= \max\{s \in \mathcal{S} | s \geq \mu\} \\ \omega &= \frac{s'_j - \frac{Q^\beta(H|L)}{Q^\beta(H|L)+Q^\beta(L|H)}}{s'_j - s'_i} \end{aligned}$$

   (c) $p^{(0,a')}(\omega) = 1$ for all a', ω.

   (d) $q(a', p) = \frac{1}{1+r}$ for all a', so that initial pricing is consistent with the assumption on p.

   (e) $\mu(\beta, e, z, a, s) = \frac{1}{n_\beta \cdot n_e \cdot n_z \cdot n_a \cdot n_s}$ for all β, e, z, a, s so that the initial guess of the distribution of agents is uniform over the state space.

3. Taking the current guess of the equilibrium functions $f_0 = \{q_0, \psi_0\}$ as given, enter the equilibrium computation loop:

   (a) Solve for the expected value function $W_1(\cdot|f_0)$ taking as given $W_0(\cdot|f_0)$:

      i. Assess budget feasibility, finding the set of feasible actions $\mathcal{F}(\omega|f_0)$.

      ii. For each value of β, ω, compute the conditional value associated with each action $(d, a') \in \mathcal{F}(\omega|f_0)$, $v_1^{(d,a')}(\beta, \omega|f_0)$, according to (3), with $W(\cdot) = W_0(\cdot)$.

[26]Though the algorithm is presented here with the separated e and z components of the earnings process for consistency with the text, the code condenses these two states into one. As long as both are observable, this simplification is completely without loss of generality.

iii. Having looped over all feasible actions, aggregate the conditional values $v_1^{(d,a')}(\cdot|f_0)$ into the new expected value function $W_1(\cdot|f_0)$ according to (2).

iv. Assess value function convergence in terms of the sup norm metric,

$$dist = \sup_{\beta,\omega} |W_1(\beta,\omega|f_0) - W_0(\beta,\omega|f_0)|$$

If $dist < tol$, go to 3.b; if $dist \geq tol$, set $W_0(\cdot|f_0) = W_1(\cdot|f_0)$ and go back to 3.a.ii.

(b) Compute the decision probabilities $\sigma_1(\cdot|f_0)$ implied by $W_1(\cdot|f_0)$ according to (5).

(c) Given the decision probabilities $\sigma_1(\cdot|f_0)$, compute the new set of equilibrium functions, $f_1 = \{q_1, \psi_1\}$:

i. Compute $\psi_1(\cdot)$ according to (9).

ii. Compute $q_1(\cdot)$ according to (8).

A. Note that this involves, as preliminary steps, computing $p_1(\cdot)$ according to (12), $\chi(\beta'|\psi)$ and $Q^s(s'|\psi)$ for all the values of ψ found in step 3.c.i. This is performed simply iteration by iteration via (10) and (11).

(d) Assess equilibrium function convergence in terms of the sup norm metric:

$$dist = \max \{\sup |\psi_1 - \psi_0|, \sup |q_1 - q_0|\}$$

If $dist < tol$, proceed to step 4; otherwise, set $f_0 = f_1$ and go back to the beginning of step 3.

4. Compute the stationary distribution associated with the equilibrium behavior and the equilibrium functions computed in step 3.

(a) For each state β, ω, compute $\mu_1(\beta, \omega)$ according to (14), with $\mu(\cdot) = \mu_0(\cdot)$ and given the set of equilibrium functions solved for above.

(b) Assess convergence based on the sup norm metric $dist = \sup |\mu_1(\cdot) - \mu_0(\cdot)|$. If $dist < tol$, go to step 5; otherwise, set $\mu_0 = \mu_1$ and go back to 4.a.

5. Compute moments.

Variable	Notation	No. points	Range / Values	Notes
Discount factor	β	2	$\{0.97, 0.89\}$	2-point support makes Bayesian functions scalar-valued.
Earnings (persistent)	e	3	$\{0.58, 1.00, 1.74\}$	See Section 4.
Earnings (transitory)	z	3	$\{-0.18, 0.00, 0.18\}$	See Section 4.
Assets	a	150	$[-0.25, 8.00]$	50 points in neg. region, 100 in pos. Density close to 0 is critical.
Type score	s	50	$[0.05, 0.89]$	bounded below by low β to high β transition, above by high to low.

Table 4: **Grids used in computational analysis**

8.2 Grids

Table 4 presents the key grid used in the computational analysis. Note in particular that the asset and type score grids are quite dense in order to insure convergence, while in contrast the earnings and type grids are coarse in order to ease the computational burden and simplify the analysis.

8.3 Model moment definitions

8.3.1 Default rate

The default rate is computed as the total fraction of the population who defaults within a given period. The probability of a given state is given by $\mu(\cdot)$, and the probability of default given a state is $\sigma^{(1,0)}(\cdot)$, and so the *aggregate* default rate is

$$\text{aggregate default rate } = \sum_{\beta,\omega} \sigma^{(1,0)}(\beta,\omega) \cdot \mu(\beta,\omega). \tag{19}$$

By type, we have $\sum_{\omega} \sigma^{(1,0)}(\beta,\omega) \cdot \mu(\beta,\omega) / \sum_{\hat{\omega}} \mu(\beta,\hat{\omega})$.

8.3.2 Fraction in debt

This is simply the fraction of of the population with $a < 0$ in a given period, and so

$$\text{fraction in debt } = \sum_{\omega | a<0} \mu(\beta,\omega). \tag{20}$$

By type, the analogous figure is $\sum_{\omega|a<0} \mu(\beta,\omega) / \sum_{\hat{\omega}} \mu(\beta,\hat{\omega})$.

8.3.3 Median net worth to median income

This is simply the ratio of the relevant medians, computed with respect to the stationary distribution.

8.3.4 Debt to income

Income in the model is given by the sum of earnings (persistent and transitory) and net interest on assets. That is, income $= e + z + (1/q(a', p) - 1) \cdot a$. Therefore, debt to income is computed as the weighted average of the ratio of assets, a, to income conditional on a being negative:

$$\text{debt to earnings} = \sum_{\beta,\omega|a<0} \frac{a}{e + z + (1/q - 1) \cdot a} \cdot \frac{\mu(\beta, \omega)}{\sum_{\hat{\beta},\hat{\omega}|\hat{a}<0} \mu(\hat{\beta}, \hat{\omega})}, \tag{21}$$

8.3.5 Average interest rate

The average interest paid (or received) by the agents in the economy is the weighted average of the interest rates paid, $1/q - 1$, over the stationary distribution and decision probabilities.

$$\text{average interest rate} = \sum_{\omega,a'} \left(\frac{1}{q} - 1\right) \cdot \sum_{\beta} \sigma^{(0,a')}(\beta, \omega) \cdot \frac{\mu(\beta, \omega)}{\sum_{\hat{\beta}} \mu(\hat{\beta}, \omega)} \tag{22}$$

8.3.6 Credit scores

The goal of this section is to map the key equilibrium objects of the model into credit scores which reflect the key features of credit scores that we observe in the real world. It is important to note, however, that these credit scores are secondary moments in our model, and not key drivers of the pricing of debt. This is because a credit score must necessarily aggregate over *possible future actions*, as will be made clear below.

The basic idea of a credit score is to measure an agent's probability of a default event within a certain period of time, given today's observables. We can compute these probabilities for windows of $n = 1, ..., N$ periods ahead, where N is an arbitrary finite number greater than or equal to 1. In this sense, we can compute an N-vector of credit scores based on the observable state ω, $\xi(\omega)$, such that

$$\xi(\omega) = (\xi_1(\omega), ..., \xi_N(\omega)),$$

where $\xi_n(\omega)$ represents the probability of a default event within n periods.

How can we compute these scores? The $p(\cdot)$ function computes the probability of repayment next period on a given action for a given state today; $\sigma(\cdot)$ indicates the probability of each of these actions, and we can weight out the unobservable parts of the state relevant for $\sigma(\cdot)$ (i.e. β) using the stationary distribution $\mu(\cdot)$. Let's begin with the 1-period credit score:

$$\xi_1(\omega) = \sum_{(d,a')\in\mathcal{Y}} \left[p^{(0,a')}(\omega) \cdot \sum_{\beta\in\mathcal{B}} \sigma^{(d,a')}(\beta, \omega) \cdot \frac{\mu(\beta, \omega)}{\sum_{\hat{\beta}\in\mathcal{B}} \mu(\hat{\beta}, \omega)} \right]$$

The first term captures the probability that an agent in state ω today (period t) choosing a' will default tomorrow (period $t + n = t + 1$). The second two terms capture the probability that an agent in state ω will choose action a': $\sigma^{(d,a')}(\beta, \omega)$ is the probability that an agent with full state (β, ω) will choose a', and $\frac{\mu(\beta,\omega)}{\sum_{\hat{\beta}} \mu(\hat{\beta},\omega)}$ is the share of β-types in the sub-population of agents in state ω. Multiplying these terms and summing over β gives the desired action weight. To ease notation in what follows, define the *observable* decision rule $\tilde{\sigma}(\cdot)$ by

$$\tilde{\sigma}^{(d,a')}(\omega) = \sum_{\beta\in\mathcal{B}} \sigma^{(d,a')}(\beta, \omega) \cdot \frac{\mu(\beta, \omega)}{\sum_{\hat{\beta}\in\mathcal{B}} \mu(\hat{\beta}, \omega)}$$

so that

$$\xi_1(\omega) = \sum_{(d,a')\in\mathcal{Y}} p\left(a', \psi^{(d,a')}(\omega), e\right) \cdot \tilde{\sigma}^{(d,a')}(\omega). \tag{23}$$

This definition is particularly useful given the stationarity of the distribution $\mu(\cdot)$.

We can perform an analogous procedure for subsequent periods, and there turns out to be a nice recursive formulation of the n-period score. In words, $\xi_1(\omega)$ represents the assessed probability that an agent in state

(ω) in period t repays his debt (whatever that turns out to be) in period $t+1$; in probability notation, $\xi_1(\omega) = \Pr(\text{repay in } t+1|\omega \text{ in } t)$, and likewise $\xi_n(\omega) = \Pr(\text{repay in } t+n|\omega \text{ in } t)$.

Starting with $n = 2$, we have

$$\begin{aligned}\xi_2(\omega) &= \Pr(\text{repay in } t+2|\omega \text{ in } t)\\ &= \sum_{\omega'}\left[\Pr(\text{repay in } t+2|\omega' \text{ in } t+1)\cdot\Pr(\omega' \text{ in } t+1|\omega \text{ in } t)\right]\\ &= \sum_{\omega'}\xi_1(\omega')\cdot\Pr(\omega' \text{ in } t+1|\omega \text{ in } t).\end{aligned}$$

As the expression above makes clear, once we have computed the one period ahead score across all states, all that remains is to compute the conditional probability of transitioning states, which is given by $\Pr(\omega' \text{ in } t+1|\omega \text{ in } t) \equiv Q(\omega'|\omega)$, where

$$Q(\omega'|\omega) = Q^e(e'|e)\cdot H(z')\cdot\tilde{\sigma}^{(d,a')}(\omega)\cdot Q^s(s'|\psi^{(d,a')}(\omega)). \tag{24}$$

We now sum to obtain:

$$\xi_2(\omega) = \sum_{\omega,\omega'}\xi_1(\omega')\cdot Q(\omega'|\omega).$$

Repeating this procedure, we find that

$$\xi_{n+1}(\omega) = \sum_{\omega,\omega'}\xi_n(\omega')\cdot Q(\omega'|\omega), \tag{25}$$

for all $n = 1, ..., N-1$, and so we have computed the entire range of credit scores across possible time horizons.

8.3.7 Credit score transitions

Individuals naturally move through the range of possible credit scores over time: indeed, this is precisely what we observe in the Equifax data. The goal of this section is to map the transition matrix of agents through credit scores. That is, for each $n = 1, ..., N$, we would like to find a transition matrix $Q_n^{\xi}(\xi_n'|\xi_n)$ that gives the probability of transiting from credit score ξ_n today to credit score ξ_n' tomorrow. Once we have computed this, we can aggregate into a transition matrix $Q_n^X(X_n'|X_n)$, which gives the probability of transitioning from a given credit score *group* $X_n = \{\xi_n^1, ..., \xi_n^J\}$ today to a different group $X_n' = \{\xi_n^{1\prime}, ..., \xi_n^{J\prime}\}$ tomorrow.

Every possible state $\omega \in \Omega$ maps into a credit score (N-vector) $\xi = \xi(\omega)$; therefore, the transition over credit scores is governed entirely by the transition over states, $\omega \to \omega'$. However, the mapping is not one-to-one (for example, multiple states can have certain repayment for a credit score of $\xi_1 = 1$), and so the score transitions must sum over states that have the same credit score.

Let ξ_n and ξ_n' be arbitrary n-period credit scores for today and tomorrow respectively. The probability of transitioning from $\xi_n \to \xi_n'$ is the probability of transitioning from any state ω today such that $\xi_n = \xi_n(\omega)$ to any state (ω') tomorrow such that $\xi_n' = \xi_n'(\omega')$. Define

$$\mathcal{X}(\xi_n) = \{\omega \in \Omega|\xi_n(\omega) = \xi_n\} \tag{26}$$

to be the set of all states that map into a given credit score. Then, the probability of transitioning from ξ_n to ξ_n' is simply the double sum[27]

$$Q_n^{\xi}(\xi_n'|\xi_n) = \sum_{\omega\in\mathcal{X}(\xi_n)}\sum_{\omega'\in\mathcal{X}(\xi_n')}Q(\omega'|\omega)\cdot\frac{\sum_{\beta}\mu(\beta,\omega)}{\sum_{\beta,\omega\in\mathcal{X}(\xi_n)}\mu(\beta,\omega)}, \tag{27}$$

[27]NB: Though we use discrete grids to compute the model, this sum notation is not technically mathematically correct because s is, in principle, a continuous variable.

where $Q(\omega'|\omega)$ is given by (24) and the second term weights the relative size of each particular state within the group of states that yield the given current score.

Having defined the score-by-score transitions in the preceding section, we can now extend the procedure to "score brackets" or transitions over ranges of credit scores. Define a score bracket X_n to be a collection of J credit scores $\{\xi_n^1, ..., \xi_n^J\}$.[28] Then, we can define the set of all states that map into a credit score in bracket X_n to be

$$\mathcal{X}(X_n) = \bigcup_{j=1}^{J} \mathcal{X}(\xi_n^j), \tag{28}$$

where $\mathcal{X}(\xi_n)$ is given by (26). Then, the transition from score bracket $X_n \to X_n'$ is simply

$$Q_n^X(X_n'|X_n) = \sum_{\omega \in \mathcal{X}(X_n)} \sum_{\omega' \in \mathcal{X}(X_n')} Q(\omega'|\omega) \cdot \frac{\sum_{\beta} \mu(\beta, \omega)}{\sum_{\beta, \omega \in \mathcal{Z}(Z_n)} \mu(\beta, \omega)}, \tag{29}$$

where again $Q(\omega'|\omega)$ is given by (24) and the second term weights the relative size of each particular state within the group of states that yield scores within the given current score range.

[28] Again, here we run into the mathematical detail of continuous vs. discrete variables: in principle, the space of credit scores could be continuous, and then this procedure we outline would not be exhausted. Computationally, though, it is not an issue.