A VERY SHORT INTRODUCTION TO DIFFERENTIAL FORMS AND RIEMANNIAN GEOMETRY

JUN LI

Abstract. This short notes (non-examinable) provide Math 433 students some connection between the surface theory and its generalization in higher dimensions, namely differential forms and Riemannian geometry. I tried best to make it self contained and short. Instead of being standard textbook style introducing too many abstract concepts, we will take a short cut to the main idea by making definition as simple as possible. Jump to Theorem 2.7 and Remarks 2.8, 2.9, and 2.10, you'll find something familiar.

Contents

1. Introduction

Many of us remember the following fact we mentioned in class: traveling with constant velocity along the great circle on the earth is actually not a constant move in $\mathbb{R}^3$, because the directions are changing. However, there should be a way to explain why it is reasonable for the traveler oneself thinking that he/she's doing a constant speed motion. This is why we define the covariant derivative (tangential derivative) which is different from Euclidean derivative, and define the great circle as the geodesic since the covariant derivative never changes along this curve. In higher dimension, we have similar concepts, generalizing the surface to a manifold, tangent plane to tangent space, 1st fundamental form to a Riemannian metric. Covariant derivatives and geodesics are defined similarly and many fundamental theorems and facts are still true in higher dimensions, e.g Gauss-Bonnet. Einstein adopted high-dimensional Riemannian geometry as the frame of his field equation (4) in General Relativity; while new phenomena appear as people explore many interesting questions, as discussed in Section 5.

A short reading guide: if you are interested in dimension 2 surface theory or $\mathbb{R}^3$, please take a look at Theorem 2.6, Remarks 4.6, and Question 5.4. If you are curious about what's going on in the higher dimension, then reading in a linear order will help.

Date: April 18, 2019.

2. Differential Forms in Euclidean Spaces

Many popular (standard) textbooks, such as do Carmo's *Riemannian Geometry* [CF92] and Guillemin-Pollack's *Differential Topology* [GP74] introduce forms by first playing with heavy algebraic/analytical facts of tensors. However, many people never get used to the whole mathematical theory of tensors and hence never really feel comfortable with the core part of differential form such as Stokes' Theorem, Hodge star operation, etc. Since our Math 433 students have such a diverse background, it's necessary to write some tensor free approach to differential forms. We start with a formal definition and computation rules, then directly move on to the key theorems.

Our approach is straightforward going into the core properties of differential forms, while the drawback is lack of intuition. Hence we also provide some geometry about differential forms in Remark 2.11. Also, tensors are important as they appear in physics, computer science, and statistics (e.g. nowadays they are popular in Learning theory). You can find a short introduction to tensors in Remark 4.7.

Here starts the introduction to differential forms. **We strongly encourage everyone who's interested to read [Arn89] Chapter 7 for more details.**

On $\mathbb{R}^n$, we start with the symbols $dx_1, \ldots, dx_n$(those are indeed basis on 1-forms), and define a "multiplication" operation (called **wedge product** or **exterior product**) on these symbols, denoted by $\wedge$, subject to the condition

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

Note that the anti-symmetry condition indeed means

$$dx_i \wedge dx_i = 0.$$

Also, if $I = \{i_1, \ldots, i_k\}$ is index set where we have repeated indices, then $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$. For example,

$$dx_1 \wedge dx_2 \wedge dx_1 = -dx_1 \wedge dx_1 \wedge dx_2 = -(dx_1 \wedge dx_1) \wedge dx_2 = 0.$$

Remark 2.1. *dx_i is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$. Here you can think it measures the length of a vector in the x_i direction (or the x_i component of the vector). And $dx_i \wedge dx_j$ measures the orientaed area of the parallelogram spanned by the vector in x_i and x_j direction. For general geometric meaning, see Remark 2.11.*

Another important property of the wedge operation is the linearity: let α, β, γ be arbitrary products of dx_i's, and c be any real number, then

$$\begin{aligned} (\alpha+\beta)\wedge\gamma &= \alpha\wedge\gamma+\beta\wedge\gamma \\ \alpha\wedge(\beta+\gamma) &= \alpha\wedge\beta+\alpha\wedge\gamma \\ (\alpha\wedge\beta)\wedge\gamma &= \alpha\wedge(\beta\wedge\gamma) \\ (c\alpha)\wedge\beta &= \alpha\wedge(c\beta)=c(\alpha\wedge\beta) \end{aligned} \tag{1}$$

Definition 2.2 (**Differential forms**)**.** *We define a k-**form** (or degree k-form) to be*

$$\sum_I f_I(x)dx_I,$$

where I is a index $I = \{i_1, \ldots, i_k\}$ of where$i_j \in \{1, 2, \ldots, n\}$ and dx_I means $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. And each dx_I has a smooth function f_I on $\mathbb{R}^n$ as its coefficient. Simply put, one can think the space of differential form a linear space spanned by

$$\{1; dx_1, dx_2, \cdots, dx_n; dx_i \wedge dx_j; \cdots; dx_1 \wedge \cdots \wedge dx_n\}$$

with smooth functions as their coefficients.

Question 2.3. *How many linearly independent basis of differential forms are there in $\mathbb{R}^n$?*

The answer is that for fixed degree k-forms, there are $\binom{n}{k}$ basis; while the total number is 2^n, simple combinatorics.

We are indeed very familiar with form on $\mathbb{R}^3$:

- 0-forms are smooth functions $F(x, y, z)$ on $\mathbb{R}^3$
- 1-forms can be uniquely written as $Pdx + Qdy + Rdz$, where P, Q and R are smooth functions and we denote dx, dy, dz for dx_1, dx_2, dx_3.
- 2-forms can be uniquely written as $Pdx \wedge dy + Qdz \wedge dx + Rdy \wedge dz$.
- 3-forms look like $F(x, y, z)dx \wedge dy \wedge dz$.
- Any k-form on $\mathbb{R}^3$ with $k > 3$ is zero, because there must be repeated indices in the index set.

You can check that on $\mathbb{R}^3$, there are 1,3,3,1 basis for degree $0, 1, 2, 3$ forms respectively, and $1 + 3 + 3 + 1 = 8 = 2^3$. Verifies the answer to Question 2.3.

Definition 2.4 (**Exterior Derivative**). *Let $\omega = \sum_I f_I dx^I$ be k-form, then we define the* exterior derivative

- *If $\omega = f$ is a 0-form:*
$$df(x) = \sum \frac{\partial f(x)}{\partial x^j} dx^j;$$
- *if ω is a k-form, $k \neq 0$,*
$$d\omega = \sum_I df_I(x) \wedge dx^I.$$

Here's one simple example $\mathbb{R}^2$, if $\omega = xydx + a^x dy$, then

$$\begin{aligned} d\omega &= ydx \wedge dx + xdy \wedge dx + ln(a)a^x dx \wedge dy + 0dy \wedge dy \\ &= (ln(a)a^x - x)dx \wedge dy. \end{aligned} \tag{2}$$

And we verified that in class that on $\mathbb{R}^3$, the exterior derivative

- acts on 0-forms (functions $F(x, y, z)$) are gradient $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$;
- acts on 1-forms $\omega = Pdx + Qdy + Rdz$ are curl, i.e. $d\omega = \nabla \times (P, Q, R)$;
- acts on 2-forms $\omega = Pdx \wedge dy + Qdz \wedge dx + Rdy \wedge dz$ are divergence, i.e. $d\omega = \nabla \cdot (P, Q, R) = (\frac{\partial R}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial z})dx \wedge dy \wedge dz$;
- acts on any k-form on $\mathbb{R}^3$ with $k \geq 3$ is zero, because higher degree forms vanishes.

As we mentioned, whatever in mathematics named a derivative(or deviation) will satisfy some **Leibnitz rule**:

Theorem 2.5. *If ω is a k-form and θ is an ℓ-form, then*
$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta).$$

Proof. By linearity, it suffices to check the basis, and we can assume $\omega = f_I dx^I$ and $\theta = g_J dx^J$.

$$\begin{aligned} \omega \wedge \theta &= f_I(x) g_J(x) dx^I \wedge dx^J \\ d(\omega \wedge \theta) &= \sum_j \partial_j (f_I(x) g_J(x)) dx^j \wedge dx^I \wedge dx^J \end{aligned}$$

$$\begin{aligned}
&= \sum_j (\partial_j f_I(x)) g_J(x) dx^j \wedge dx^I \wedge dx^J \\
&\quad + \sum_j f_I(x) \partial_j g_J(x) dx^j \wedge dx^I \wedge dx^J \\
&= \sum_j (\partial_j f_I(x)) dx^j \wedge dx^I \wedge g_J(x) dx^J \\
&\quad + (-1)^k \sum_j f_I(x) dx^I \wedge \partial_j g_J(x) dx^j \wedge dx^J \\
&= (d\omega) \wedge \theta + (-1)^k \omega \wedge d\theta. \qquad (3)
\end{aligned}$$

□

The following is called "Poincarè Lemma." Even through this is called a Lemma, it is indeed the most important fact about exterior derivative. This is the generalization that **partial derivatives are commute i.e.** $\frac{\partial^2}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u}$:

Theorem 2.6 (**Poincarè Lemma**).
$d(d\omega) = 0$. (Simple denote $d^2 = 0$.)

Proof. It's easy for you to check that on $\mathbb{R}^3$,

$$(\nabla\times)\nabla = 0,$$

$$(\nabla\cdot)(\nabla\times) = 0.$$

The reason is simply $\frac{\partial^2}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u}$.

In higher dimension, this can be verified by (a similar) direction computation, and it is true for the same reason. □

Theorem 2.7 (**Stokes' Theorem**). *Let M be a compact oriented n-dimensional manifold-with-boundary ∂M, and let ω be an $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

There's indeed some subtle orientation issue on the boundary ∂M, and in the class, I'm being a little bit sloppy on this point, due to the limited time. You can check it by figuring out the following Remarks (where we indeed did in class) 2.8, 2.9, 2.10 by yourself:

Remark 2.8. *Suppose that I is a compact connected oriented 1-manifold-with-boundary. (Simply put, think about a closed interval $[a, b]$.) The Stokes' Theorem on I is the* **Fundamental Theorem of Calculus.**

Remark 2.9. *Suppose that R is a bounded region in $\mathbb{R}^2$. Then the Stokes' Theorem on R is the* **Green's Theorem.**

Remark 2.10. *Suppose that D is a bounded domain in $\mathbb{R}^3$ with boundary surface $S = \partial E$. Then the Stokes' Theorem on E is the* **Divergence Theorem.**

Remark 2.11. *By doing the above exercises, we may have realized that the geometry meaning of a differential form is an oriented volume element. When doing integration over a manifold of the correct dimension, the integration gives the measurement of the manifold in terms if the oriented volume element.*

3. Smooth Manifolds (you may skip detais of this part when first reading)

Everyone in our class has some feeling about topological spaces. But mathematics treatment needs to deal with it in full generality. We refer to Hitchin's note page 10 for the notion of a "topological structure," "Hausdorff" and "continuity, homeomorphism." Once we know these concepts, we have the following, which can be found in any standard book on smooth manifolds, for example, Guillemin and Pollack [GP74]:

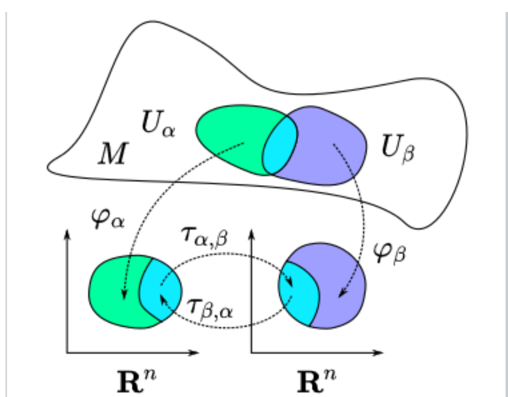
In mathematics, a manifold is a topological space that locally resembles Euclidean space near each point.

Definition 3.1 (n-dim **Topological Manifold**). *An* ***n-dimensional topological manifold*** $\mathcal{N}$*, is a Hausdorff topological space* $(\mathcal{N}, \tau_{\mathcal{N}})$ *which is locally homeomorphic to the Euclidean space* $\mathbb{R}^n$*. This means that any point* $P \in \mathcal{N}$ *is contained in some neighborhood* $V_P \subseteq \mathcal{N}$*, homeomorphic to a domain* $U = \phi(V_P) \subseteq \mathbb{R}^n$ *of the Euclidean space.*

Thus when $\mathcal{N}$ is an n-dim. topological manifold, we can find in $\mathcal{N}$ a system of open sets V_i numbered by finitely (or infinitely) many indices i and a system of homeomorphisms $\phi_i : V_i \to \phi(V_i) \equiv U_i \subseteq \mathbb{R}^n$ of the open sets V_i on the open domains U_i. The system of the open sets $\{V_i\}$ must cover the space $\mathcal{N}$ i.e. $\mathcal{N} = \bigcup_i V_i$ and the domains U_i may, in general, intersect each other.

Definition 3.2 (**local chart**). *If* $\mathcal{N}$ *is a topological manifold, any pair* (V, ϕ) *will be called a* ***local chart****, where* V *is an open subset of* $\mathcal{N}$ *and* $\phi : V \to U \equiv \phi(V) \subseteq \mathbb{R}^n$ *an homeomorphism onto an open domain* U *of* $\mathbb{R}^n$

Suppose that $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two charts for a manifold M such that $U_\alpha \cap U_\beta$ is non-empty. The transition map $\tau_{\alpha,\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is the map defined by $\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$.



Two charts on a manifold, and their respective **transition map**

Figure 1. transition map, picture taken from Wikipedia

Definition 3.3 (**smooth manifold**). *If $\mathcal{N}$ is a topological manifold, if there exists a atlas $(U_\alpha, \varphi_\alpha)$ where all the transition maps are diffeomorphism between $\mathbb{R}^n$, then $\mathcal{N}$ with charts $(U_\alpha, \varphi_\alpha)$ is called a smooth manifold.*

And we have the following (slight different phrased but equivalent as we talked in class) theorem:

Theorem 3.4 (**partition of unity**). *On a smooth manifold $\mathcal{N}$ with charts $(U_\alpha, \varphi_\alpha)$ there exist a smooth partition of unity. Here a partition of unity on $\mathcal{N}$ is a collection g_i ($i \in I$ where I is a possibly uncountable index set)of smooth real valued functions on X such that (1) $0 \le g_i \le 1$ for each i; (2) every $x \in \mathcal{N}$ has a neighborhood U such that $U \cap \{g_i \neq 0) = \emptyset$ for all but finitely many g_i; (3) for each $x \in \mathcal{N}$*

$$\sum_i g_i(x) = 1.$$

Remark 3.5. *A partition of unity can be used to patch together objects defined locally. For instance, we can extend the differential forms in $\mathbb{R}^n$, a metric of $\mathbb{R}^n$, or a vector field to any smooth manifold.*

4. Riemannian Geometry in Arbitrary (finite) Dimensions

We indeed covered basic Riemannian Geometry in dimension 2, where we can regard surface as a dim-2 manifold and the 1st fundamental form as a Riemannian metric. Now we assume people know a bit basics of a smooth manifold of dimension n, denoted by M^n for details, see section 3. Throughout the section, we use the coordinate-dependent approach to define everything. The advantage is this way is the simplest and mostly same as the the surface theory, and the drawback is a bit loss of generality, see Remark 4.8.

Through out this section the setting is that there's a local chart $\mathbb{R}^n \to N_p \subset M$, where $p \in N_p$ is an open neighborhood of any point $p \in M_n$. Then we choose orthonormal coordinates in $\mathbb{R}^n$ as $(x_1, \cdots x_n)$, where the unit tangent vector fields on the x_i curve in M^n is denoted as $\vec{X}_i$. Then the tangent space T_pM^nat p is spanned by the vectors $\{\vec{X}_1, \cdots, \vec{X}_n\}$ at point p. Note that there's coordinate-free approach of defining the tangent space, but it's too complicated to introduce here. We refer to do Carmo's *Riemannian Geometry* [CF92] for details.

Definition 4.1 (**Metric**). *At every point p we define an bilinear form $< -, - >$ on T_pM^n and denote $g_{ij} =< \vec{X}_i, \vec{X}_j >$. Then at every point p we have a $n \times n$ symmetric matrix. If the matrix $(g_{ij})_{n\times n}$ is smooth on M^n, which means every entry g_{ij} as a function on M^n is smooth, then $(g_{ij})_{n\times n}$ is called a (smooth) metric on M^n. When the matrix $(g_{ij})_{n\times n}$ is positive definite, it is called a* **Riemannian metric.** *We also denote $(g^{ij})_{n\times n}$ as the inverse of the matrix $(g_{ij})_{n\times n}$. i.e.*

$$(g^{ij})_{n\times n} = (g_{ij})^{-1}_{n\times n}.$$

On a surface, $(g^{ij})_{2\times 2} = I$, namely,

$$g_{11} = E, g_{12} = F, g_{21} = F, g_{22} = G.$$

A metric is a smooth 2-tensor field on a manifold. Simply put, a 2-tensor is a matrix, where input 2 vectors output a scalar. A 2-tensor field on M^n just means on every point there's a 2-tensor, namely a matrix $(g_{ij})_{n\times n}$. Strictly speaking, we defined a $(0, 2)$-tensor field, see Remark 4.8.

As we did in the surface theory, we need a notion of derivative in the curved space, such that the derivative of any tangential vector field is again a tangential vector field. Throughout this section we follow Einstein summation convention if any upper index appears to be the same as a lower index, that means we are taking sum on that index. For example, $a^i b_i c_k := \sum_{i=1}^n a^i b_i c_k$, where i takes value through 1 to n and k is a fixed index.

Definition 4.2 (**Covariant Derivative**). *We again take the coordinate-depend approach, and only define Covariant Derivative along coordinate lines $\{x_i\}$ or vector fields $\vec{X}_j$. For any i, j the covariant derivative of $\vec{X}_j$ along the $\{x_i\}$ curve is defined as*

$$\nabla_i \vec{X}_j := \Gamma_{ij}^k \vec{X}_k = \sum_{k=1}^n \Gamma_{ij}^k \vec{X}_k.$$

Here Γ_{ij}^k is the Christofell symbol, given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

The above definition may looks complicated, however, we have already seen them several times in class. On a surface, let $i = 1, j = 2$,

$$\nabla_u \vec{X}_v = \Gamma_{12}^1 \vec{X}_1 + \Gamma_{12}^2 \vec{X}_2.$$

And for the Christofell symbol, let $i = 1, j = 2, k = 1$, check the following agrees with the book expression:

$$\Gamma_{12}^1 = \frac{1}{2}(g^{11}(F_u + E_v - F_u) + g^{12}(G_u + F_v - F_v)) = \frac{GE_v - FG_u}{2(EG - F^2)}.$$

Note: The name "Covariant" comes from physics, means that the derivative behaves well (covariant) under change of coordinate. We also have the notion of contravariant, where we will not go into details and anyone who's interested can look at Wikipedia or talk to me in person.

A bit different (but closely related, see Remark 4.6), we have many different curvatures in higher dimensions, where we address 3 of them here: **Riemann curvature tensor, Ricci curvature tensor, and the scalar curvature**.

Definition 4.3 (**Riemann curvature tensor**). *We again take the coordinate-depend approach, and only define Riemann curvature tensor for coordinate vector fields $\vec{X}_j$. Riemann curvature tensor R_{ijk}^m is a 4-tensor (strictly speaking, a (0,4)-tensor), namely, for any i, j, k, m the R_{ijk}^m is a linear operation taking 4 tangent vectors $\vec{X}_i, \vec{X}_j, \vec{X}_k, \vec{X}_m$ as input and a scalar R_{ijk}^m as output. Here's the details: firstly it measure how much the tangential derivatives fail to commute (compare this with Theorem 2.6):*

$$R(\vec{X}_i, \vec{X}_j)\vec{X}_k := (\nabla_j \nabla_i - \nabla_i \nabla_j)\vec{X}_k.$$

*Note this is a tangential derivative, which should be a linear combination of the basis of tangent vectors, and we define R_{ijk}^m to be the coefficient of $\vec{X}_m$ in this linear combination (***we saw this in class when proving Gauss Theorema Egregium and Gauss-Bonnet):**

$$(\nabla_j \nabla_i - \nabla_i \nabla_j)\vec{X}_k = R_{ijk}^m \vec{X}_m.$$

Also, we have another 4-tensor R_{ijkl}, defined by

$$R_{ijkl} = R(\vec{X}_i, \vec{X}_j)\vec{X}_k \cdot \vec{X}_l = R_{ijk}^m g_{ml},$$

which is very useful.

Note that the R_{ijk}^m and R_{ijkl} are anti-symmetric on the index pair (i, j) or (k, l). This can be easily checked using our HW5 question 4:

$$\frac{d}{dt} < \vec{v(t)}, \vec{w(t)} > = < \nabla_\gamma v(t), \vec{w(t)} > + < \vec{v(t)}, \nabla_\gamma w(t) > .$$

Definition 4.4 (Ricci curvature tensor). *Ricci curvature tensor R_{ik} is a 2-tensor (strictly speaking, a (0,2)-tensor), which is the matrix trace of R_{ijkl} over indices j, l. Namely,*

$$R_{ik} := R_{ijkl} g^{jl} = R_{ijk}^j.$$

Note that taking the trace of a 4-tensor we get a 2-tensor(a matrix), and if we further take the trace of a 2-tensor, we will get a 0-tensor(a scalar). Indeed the scalar curvature is obtained in this way:

Definition 4.5 (Scalar curvature). *The scalar curvature R is a scalar (0-tensor), which is the trace of the Ricci curvature tensor R^{ij} Namely,*

$$R := R_{ij} g^{jl}.$$

Remark 4.6. *Suppose we have a surface S with first fundamental form I_S, we indeed covered the above curvatures in class:*

- *Riemann curvature tensor: By the anti-symmetry, the only non-zero tensor on S is R_{1212}. Indeed we saw this in Hitchin's proof of Gauss' Theorema Egregium, where we amount to prove that*

$$R_{1212} = K det I_S,$$

  where K is the Gaussian curvature of the surface S.
- *Ricci curvature tensor: there are 4 of them and uniformly we have*

$$R_{ij} = K g_{ij}.$$

  More explicitly, if we write the intrinsic expression of R_{ij} using definition on the left-hand side, and the right-hand side is one of EK, FK, GK, this is the Gauss equation.
- *Scalar curvature is exactly twice of the Gaussian curvature, namely,*

$$R = 2K.$$

Remark 4.7. *This is indeed linear algebra (where they call multi-linear algebra). We know if we have some linear space V, the collection of linear functions on V form a linear space, which is isomorphic to V itself, called the dual space, and denoted by V^*. In general, a (p, q)-tensor T is a linear function*

$$T : \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \to \mathbb{R}.$$

Now come back to the question why we have (0,4) or (0,2) curvature tensors: our tangent space $T_p M^n$ is the above linear space V, its dual is called the cotangent space denoted by $T_p^* M^n$ or simply V^*. Our Riemannian curvature tensor takes 4 tangent vectors(which lives in $T_p M^n$) linearly to a real number, and hence it is a (0,4)-tensor. Similarly, Ricci is a (0,2)-tensor.

Remark 4.8.
- *For any tangential vector fields $\vec{X}, \vec{Y}, \vec{Z}$, people in general define*
$$R(\vec{X},\vec{Y})\vec{Z} := (\nabla_{\vec{X}}\nabla_{\vec{Y}} - \nabla_{\vec{Y}}\nabla_{\vec{X}})\vec{Z} - \nabla_{[\vec{X},\vec{Y}]}\vec{Z}.$$
Here $[,]$ is the Lie bracket defined as
$$[\sum u^i\vec{X}_i, \sum v^j\vec{X}_j] = \sum u^i\partial_i(v^j)\vec{X}_j - \sum v^j\partial_j(u^i)\vec{X}_i.$$
It measures how much the non-commutivity of two vector fields $\sum u^i\vec{X}_i$ and $\sum v^j\vec{X}_j$. And in our case the Lie brackets for coordinate vector fields are zero.
- *Different people use different sign conventions for Riemannian curvature tensor. You might see some definition being the negative of our definition. But more than 90% literature uses our sign convention.*

(Sub-)Riemannian Geometry (including Minkowski geometry) becomes the core part of modern mathematics after the 1900s, one important reason is the
Einstein field equations for general relativity:

$$R_{ij} - \frac{1}{2}g_{ij}R + g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij}. \tag{4}$$

The left side is about the geometry (R_{ij} is the Ricci, R is the scalar curvature, and g_{ij} is the Minkowski metric meaning that its signature is $1, 1, \cdots, -1$ since time coordinate is different from the space coordinate) of the universe and the right side is the physics of the universe. Here's what other terms mean: G is Newton's gravitational constant, c is the the speed of light, Λ is the cosmological constant, T_{ij} is the stress-energy tensor (which measures the matter/energy of spacetime).

This is a system of $n \times n$ equations, it is a partial differential equation of the metric g_{ij}, since the left-hand side is all about g_{ij} and their derivatives. The simplest case is the vacuum with the cosmological constant $\lambda = 0$, these equations become $R_{ij} = 0$, and manifolds with zero Ricci tensor is called a Ricci-flat manifold. You may hear the name Calabi-Yau manifolds, and these are the most important examples of Ricci-flat manifolds

Also, the topology of the Ricci-flat manifold or more generally, of a manifold with given curvature condition, is a very important and interesting problem in modern mathematics, see section 5.

5. Problems in Geometry/Topology and the Summary

Here are some key questions in the theory of topological/smooth manifold theory, the answer to which is one of the great mathematical achievements in the 20th century. We only give a list of question and known answers, for details see two nicely written article by John Milnor [Mil15; Mil11]:

Question 5.1 (**Are smooth structures Unique?**)**.** *For a given topological manifold $\mathcal{N}$, is smooth structure unique on $\mathcal{N}$? How many different smooth structures can we find there?*

People really address the question for the Euclidean space $\mathbb{R}^n$ and the sphere S^n. The reason is if there are more than one smooth structures $\mathbb{R}^n$, then any non-compact n-dim manifold will have different smooth structures, simply by changing an open subset(same as $\mathbb{R}^n$); and similarly if there are more than one smooth structures $\mathbb{R}^n$, then any compact n-dim manifold will have different smooth structures, simply by doing connect sum with S^n.

The answer for $\mathbb{R}^n$ is as follows:

- $n = 1, 2, 3$, there's a unique smooth structure on $\mathbb{R}^n$, this is essentially known to Poincaré when he invented Topology around 1900.
- $\mathbb{R}^4$ is extremely hard, Donaldson-Freedman in 1982 proved that there's more than one smooth structure. Later, Taubes 1987 showed that there are uncountably many smooth structures on $\mathbb{R}^4$. Donaldson won 1986 Fields medal for this work.
- For $\mathbb{R}^n \geq 5$, Stallings 1961 showed that there's a unique smooth structure.

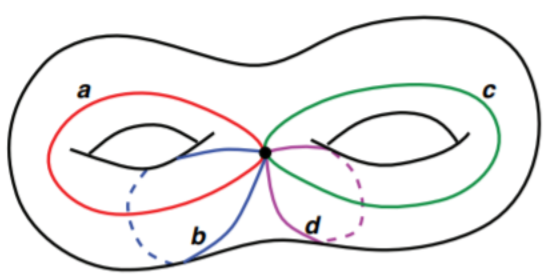
The answer for S^n is as follows:

- $n = 1, 2$, there's a unique smooth structure on S^n, this is also known to Poincaré when he invented Topology around 1900.
- S^3 is solved recently, by Perelman's 2002 work, see the next question.
- For S^4, this question is still open, see the next question.
- For $S^n, n \geq 5$, John Milnor in 1956 discovered exotic smooth structures (He won the 1962 Fields medal for this work). In the next 20 years people fully understood smooth structures on high dimensional spheres, they form a finite group.

Question 5.2 (**How do we characterize a sphere, topologically and smoothly**). *Poincaré posed the following question, which he initially thought been proved by himself but later found his proof being wrong: suppose $M^n, n \geq 2$ is a $n-$dimensional compact manifold, and every simple closed loop on M^n is contractible (this is precise only in dimension $n = 3$, the precise assumption is " M^n is homotopic to the the sphere S^n"), then M^n is equivalent to the $n-$dimensional sphere S^n.*

*People call this **Poincaré Conjecture**, and there are two versions:*
•*Topological: M^n is homeomorphic to the $n-$dimensional sphere S^n.*
•*Smooth: M^n is diffeomorphic to the $n-$dimensional sphere S^n.*

Here is a bit note of a contractible loop: we know on S^2 any simple closed loop is contractible, meaning that it can be continuously deformed to a point. And the following picture shows that on a genus 2 surface, there are many non-contractible loops (a,b,c,d in the picture):



The answer for the Topological Poincaré Conjecture Yes:

- $n = 2$, Poincaré observed this around 1900, and this is why he proposes this conjecture.
- S^3 is solved by Perelman's 2002 work, using Ricci flow, which is a PDE about the Ricci curvature as defined in Definition 4.4. He was awarded the 2006 Fields medal but later on, sadly, he declined it and left math.
- For S^4, this question is done by Freedman in 1980s, where he won the 1986 Fields medal for this work.
- For $S^n, n \geq 5$, Smale (he attended college and grad school here at Michigan) in 1961 and 1962 proved this result using a very elegant method, which was quite surprising because people thought higher dimensional questions are harder. He won the 1966 Fields medal for this work.

The answer to the Smooth Poincaré Conjecture(there's a bit overlap with smooth structures on S^n):

- $n = 2$, there's a unique smooth structure on S^2, this is also known to Poincaré and you can try to prove this by yourself
- S^3 is solved also by Perelman's 2002 work, he indeed proved Thurston's geometrization program, which can be thought as a 3-d analog of the classification of compact surfaces we learned.
- For S^4, this question is still open, it is considered as one of the most difficult math questions today. People even don't know how to guess the answer. Indeed there are some possible exotic S^4's given by the surgery called "Gluck twists," but nobody knows how to tell whether or not they are standard, because of the lack of smooth invariants in dim=4.
- For $S^n, n \geq 5$, essentially, smooth Poincaré is "almost" true since there're only finitely many different smooth structures (on a compact manifold of dim≥ 5).

Question 5.3 (**Are manifolds subsets of Euclidean space?**). *For a given smooth manifold M^n, or a Riemannian manifold (M^n, g) where g is a Riemannian metric, people ask whether it can be smoothly or even better, isometrically embedded into some Euclidean space?*

The answer to the embedding question is **Yes** for both settings:

- The smooth embedding: **Whitney (1936) embedding theorem** states that any smooth real n-dimensional manifold (required also to be Hausdorff and second-countable) can be smoothly embedded in the real $2n$-space ($\mathbb{R}^{2n}$), if $n > 0$. Note that $2n$ is the upper bound, and in our class, we know that a compact orientable surface can be smoothly embedded into $\mathbb{R}^3$, which is better than this bound.
- The Riemannian setting(note that this is much stronger than the smooth setting): **Nash (1956) embedding theorem**: if M^n is a given n-dimensional Riemannian manifold (analytic or of class $C^k, 3 \leq k \leq \infty$), then there exists a number N (with $N \leq n(3n + 11)/2$ if M^n is a compact manifold, or $N \leq n(n + 1)(3n + 11)/2$ if M^n is a non-compact manifold), such that M^n isometrically embedded into $\mathbb{R}^N$. Yes, this is the Nobel laureate John Nash who is famous for Game Theory. But the mathematics community considered this theorem as Nash's most important achievement.

Let's end with the last and most important question for our 433 class:

Question 5.4 (**Why manifold?**). *The answer to the previous question tells us that manifold is just some subsets of some high dimensional* $\mathbb{R}^n$*, why don't we just study Euclidean spaces instead of manifolds?*

The answer is the following: theoretically yes, we can and we are studying subsets of some very high dimensional Euclidean spaces. However, practically we have to deal with the manifold because we live in a universe where we can only measure from inside. This means what we know is just some metric on a manifold. Then it is curial to develop some theory which tells us information of the manifold only from the metric, i.e. the study of intrinsic property which only depends on the metric. And this is the main idea of our course, and also of Riemannian Geometry.

References

[Arn89] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. 3rd ed. Graduate texts in Mathematics 60. 1989.

[CF92] Manfredo P. do Carmo and Francis Flaherty. *Riemannian geometry*. 1st ed. Mathematics. Theory and Applications. Birkhauser, 1992. ISBN: 9780817634902,0817634908,3764334908.

[GP74] Victor Guillemin and Alan Pollack. *Differential Topology*. 1974. ISBN: 0132126052,9780132126052.

[Mil11] John Milnor. "Differential Topology Forty-six Years Later". In: *Bulletin of AMS* (2011).

[Mil15] John Milnor. "Topology through the centuries: Low dimensional manifolds". In: *Bulletin of AMS* (2015).

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: `lijungeo@umich.edu`