Abelian Regularization of Rings and Modules

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The project

The goal: This joint work, with Ivo Herzog, aims at obtaining for a noncommutative ring R, the **universal abelian regular** R-ring $R \to \hat{R}$; generalizing Olivier's construction of a universal commutative regular ring.

An axiomatization for the full subcategory $\hat{R}\text{-Mod} \subseteq R\text{-Mod}$ of modules over the universal abelian regular R-ring is also provided.

Finally, some topological spaces attached to the ring $\hat{R}$ are investigated.

Outline

1. Our context

2. Main Results

Some basic notions

Let R denote an associative ring with identity $1 \in R$.

Regular ring: An element $r \in R$ is **regular** if there exists a $y \in R$ such that $ryr = r$; then y is called a *generalized inverse* of r.
R is called **regular ring** if every element is regular.

Commutative reflexive inverse: A **reflexive inverse** of r is an element y such that

$$r = ryr, \qquad y = yry\,.$$

If y also commutes with r, the element is called a **commuting reflexive inverse** (CRI) of r.

Property The following are equivalent for an element $r \in R$:

1. r has a CRI in R;
2. there is an idempotent element $e \in R$ such that $rR = eR$ and $Rr = Re$;
3. there exist a direct sum decomposition $R_R = rR \oplus \text{r.ann}(r)$; and
4. For every left R-module ${}_RM$, $M = rM \oplus \mathrm{ann}_M(r)$;

Some basic notions

Abelian regular ring: A regular ring is said to be **abelian regular ring** if every element has a CRI. Equivalently, a regular ring is abelian if and only if every idempotent is central.

R-ring: A ring S is said to be an R-**ring** if there exists a ring morphism $f : R \to S$ of rings with domain R. An R-ring S can be thought as a left R-module ${}_RS$ via the action $rs = f(r)s$.

R-field: An R-ring $R \to \Delta$ is called R-**field** if Δ is a (not necessarily commutative) field.

Epic R-field: An R-field, which is generated, as a field, by the image of R is called **epic R-field**.

Olivier's construction

A new R-ring: Adjoin a CRI for every element of R. An R-ring is obtained by adjoining noncentral variables y_r, one for every $r \in R$,

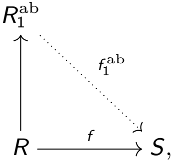
$$R \to R_1^{\mathrm{ab}} := R\{y_r \mid r \in R\}/I,$$

modulo the ideal $I = (ry_r r - r, y_r r y_r - y_r, ry_r - y_r r \mid r \in R)$ generated by the relations that ensure each $y_r + I = \overline{r}$ is a CRI of r.

If R is commutative, then R_1^{ab} is abelian regular: The R-ring $R \to R_1^{\mathrm{ab}}$ is universal with respect to the property that every $r \in R$ obtains a CRI.

Olivier's construction

A universal property: Every abelian regular R-ring $f : R \to S$ factors uniquely, through R_1^{ab},



An axiomatization: The full subcategory $R_1^{\mathrm{ab}}\text{-Mod} \subseteq R\text{-Mod}$ is axiomatizable, A left R-module $M \in R_1^{\mathrm{ab}}\text{-Mod}$ iff $\forall r \in R$

$$M \models \{\forall u\, \exists v, w\, [(u \doteq v + w) \wedge r|v \wedge rw \doteq 0]\} \wedge \forall u\, [(ru \doteq 0 \wedge r|u) \to u \doteq 0]$$

Non commutative version of Olivier's construction

This process can be iterated to obtain a denumerable sequence $R = R_0^{\mathrm{ab}} \longrightarrow R_1^{\mathrm{ab}} \longrightarrow R_2^{\mathrm{ab}} \longrightarrow \cdots$, of ring morphisms defined recursively by $R_{n+1}^{\mathrm{ab}} := (R_n)_1^{\mathrm{ab}}$.

The R-ring R^{ab}: Each of the compositions $R \to R_n^{\mathrm{ab}}$ is an epic R-ring and, therefore, so is the limit $R \to R^{\mathrm{ab}} := \lim_{\rightarrow} R_n^{\mathrm{ab}}$.

R^{ab} is abelian regular: if $r \in R^{\mathrm{ab}}$ is represented by some approximation $r_n \in R_n^{\mathrm{ab}}$, then the construction ensures that r_n obtains a CRI in R_{n+1}^{ab}.

First Result: abelian regularization

Theorem: Every ring R admits a universal abelian regular R-ring $R \to R^{\mathrm{ab}}$.

Corollary:
There is a bijection $P \mapsto R^{\mathrm{ab}}/P$ between the prime ideals of R^{ab} and the epic R-fields $R \to R^{\mathrm{ab}} \to R^{\mathrm{ab}}/P$. In particular, $R^{\mathrm{ab}} \neq 0$ iff there exists a nonzero epic R-field.

The notation $R \to \hat{\mathbf{R}} := R^{\mathrm{ab}}$ is used from now on.

Model theoretic context:

Let $\mathcal{L}(R) = (+, -, 0, r)_{r \in R}$ the language of left R-modules. For a pp formula $\rho(u, v)$ in two free variables and an R-module ${}_RM$, ρ defines the graph of a $\mathbb{Z}$-linear map $\rho : M \to M$,

$$M \models \forall u \exists! v \ \rho(u, v).$$

This pp definable function ρ is called a *definable scalar* on M.
The definable scalars on M form an R-ring $R \to R_M$.

First Result: abelian regularization

The lattice of pp definable subgroups: Denote by $\mathbb{L}(R,1)$, the lattice of pp formulae $\psi(u)$ in one variable. A morphism $f : R \to S$ of rings induces a morphism of languages $\mathcal{L}(f) : \mathcal{L}(R) \to \mathcal{L}(S)$, which induces the obvious morphism $\mathbb{L}(f,1) : \mathbb{L}(R,1) \to \mathbb{L}(S,1)$ of pp lattices.

The lattice $\mathbb{L}(\mathbf{R},\mathbf{1})_{\mathbf{M}}$: The pp definable subgroups $\psi(M) \subseteq M$ represent the elements of the quotient lattice $\mathbb{L}(R,1) \twoheadrightarrow \mathbb{L}(R,1)_M$, $\psi(u) \mapsto \psi(M)$, modulo the congruence given by equivalence relative to M, $\varphi(M) = \psi(M)$. The following diagram commutes:

$$\begin{array}{ccc} \mathbb{L}(R,1) & \longrightarrow & \mathbb{L}(R_M,1) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathbb{L}(R,1)_M & \rightarrowtail\!\!\!\!\twoheadrightarrow & \mathbb{L}(R_M,1)_M \end{array}$$

where the bottom horizontal arrow is an isomorphism.

Preliminary result

A coordinatized lattice: If R is a regular ring, then every pp formula $\psi(u)$ in one variable is equivalent to one of the form $e|u$ for some idempotent $e \in R$. The localization $\mathbb{L}(R,1) \to \mathbb{L}(R,1)_R$, $e|u \mapsto eR$ is an isomorphism.

Any complemented lattice, that is isomorphic to the lattice $\mathbb{L}(R,1)_R$ of principal right ideals of some regular ring R, is said to be **coordinatized** by R.

Proposition: Let R be an associative ring and M a left R-module for which $\mathbb{L}(R,1)_M$ is complemented. Then the vertical arrow in the diagram

$$\begin{array}{ccc} \mathbb{L}(R,1) & \longrightarrow & \mathbb{L}(R_M,1) \\ & \searrow & \downarrow \\ & & \mathbb{L}(R,1)_M \end{array}$$

is an isomorphism and $R \to R_M$ is a regular epic R-ring that coordinatizes $\mathbb{L}(R,1)_M$.

Main result

Abelian regular rings can also be characterized by the property that every element in the lattice $\mathbb{L}(R,1)_R$ of principal right ideals has a unique complement.

Theorem: The following are equivalent for a left R-module ${}_RM$:

1. $M \in \hat{R}\text{-Mod}$;
2. the R-ring $R \to R_M$ of definable scalars is abelian regular; and
3. every pp definable subgroup $\psi(M) \in \mathbb{L}(R,1)_M$ has a unique complement.

The Condition (2) can be used to axiomatize the elementary class $\hat{R}\text{-Mod} \subseteq R\text{-Mod}$.

Corollary: A module $M \in \hat{R}\text{-Mod}$ iff for every definable scalar $\rho(u,v) \in R_M$,

$$M \models \{\forall u \, \exists v, w \, [(u \doteq v + w) \wedge \exists u' \; \rho(u',v) \wedge \rho(w,0)]\} \wedge$$

$$\wedge \forall u \, [(\rho(u,0) \wedge \exists v \rho(v,u)) \to u \doteq 0].$$

Other comments

Further axiomatization: A nicer system of axioms could be given if we could find $\forall\varphi$ an explicit form for a pp formula $\varphi^{\perp}$ itha defines in M the unique element of $\varphi(M)$ in $\mathbb{L}(R,1)_M$.

The axiom schema would then be of the form $\varphi(M) \oplus \varphi^{\perp}(M) = M$.

A possible way: Given a pp formula $\varphi(u)$, the task therefore is to find a pp formula $\varphi^{\perp}(u)$ such that for every epic R-field Δ, $\varphi^{\perp}(\Delta) = \Delta$ if and only if $\varphi(\Delta) = 0$.

This is possible when the ring R is commutative.

Some topological spaces

The Cohn spectrum: Let $\mathrm{Spec}(R)$ denote the Cohn spectrum of a ring R. The points of $\mathrm{Spec}(R)$ are the epic R-fields $R \to \Delta$, with a basis of quasi-compact open subsets given by

$$\mathcal{O}(A) := \{\Delta \mid A \text{ is invertible in } \Delta\},$$

as A ranges over the square matrices with entries in R.

Clopen sets: If R is abelian regular, then the Cohn spectrum $\mathrm{Spec}(R)$ is a totally disconnected compact space with a clopen basis given by

$$\mathcal{O}(e) = \{\mathcal{P} \mid e \notin \mathcal{P}\},$$

where e ranges over the idempotent elements in R and $\mathcal{P}$ over the maximal ideals of R.

The constructible Cohn spectrum

We can introduce the patch topology on it. This space is the **constructible Cohn spectrum** of R, denoted by $\widehat{\mathrm{Spec}}(R)$; an open basis is given by the boolean combinations of quasi-compact open subsets of $\mathrm{Spec}(R)$.

Theorem: The universal abelian regular R-ring $R \to \hat{R}$ induces a homeomorphism $\mathrm{Spec}(\hat{R}) \to \widehat{\mathrm{Spec}}(R)$, $\mathcal{P} \mapsto \hat{R}/\mathcal{P}$, of constructible Cohn spectra.

The Ziegler spectrum $\mathrm{Zg}(R)$ of a ring R is the space whose points are given by indecomposable pure injective left R-modules, with a basis of open subsets:

$$\mathcal{O}(\varphi/\psi) := \{U \in \mathrm{Zg}(R) \mid \varphi(U)/\psi(U) \neq 0\},$$

as $\psi \leq \varphi$ range over $\mathbb{L}(R, 1)$. The quasi-compact open subsets of this topology have the form $\mathcal{O}(\varphi/\psi)$, as $\psi \leq \varphi$ range over the various $\mathbb{L}(R, n)$, $n \geq 1$.

The Ziegler Spectrum

Endosimple modules: A module ${}_R U$ is **endosimple** if it is simple as a module over its endomorphism ring $\mathrm{End}_R U$.

An example: Every epic R-field $R \to \Delta$ becomes, by restriction of scalars, an indecomposable endosimple left R-module

$\mathrm{Zg}_1(R)$ denotes the subspace of endosimple points of $\mathrm{Zg}(R)$ and forms a closed subset.

$\mathcal{O}(\varphi/\psi)$**:** the quasi-compact open subsets of $\mathrm{Zg}_1(R)$ are also closed: if $\psi \leq \varphi$ in $\mathbb{L}(R,1)$ and $\Delta \in \mathcal{O}(\varphi/\psi)$, then $\varphi(\Delta) = \Delta$ and $\psi(\Delta) = 0$, and so in $Zg_1(R)$,

$$\mathcal{O}(\varphi/\psi)^c = \mathcal{O}(u \doteq u/\varphi(u)) \,\cup\, \mathcal{O}(\psi(u)/u \doteq 0)$$

is also open. So, $\mathrm{Zg}_1(R)$ is equipped by patch topology.

Relation between Cohn and Ziegler Spectrum

If R is abelian regular, then the points of the Ziegler spectrum are given by the endosimple modules $R/\mathcal{P}$, as ranges over the prime, i.e., maximal, ideals.

Proposition: If R is abelian regular, then $\mathrm{Zg}(R) = \mathrm{Zg}_1(R)$ and the function $\mathrm{ann} : \mathrm{Zg}(R) \to \mathrm{Spec}(R)$, $\Delta \mapsto \mathrm{ann}(\Delta)$, is a homeomorphism.

Theorem: The universal abelian regular R-ring $R \to \hat{R}$ induces a homeomorphism

$$\widehat{\mathrm{Spec}}(R) = \mathrm{Zg}(\hat{R}) \to \mathrm{Zg}_1(R) \subseteq \mathrm{Zg}(R)$$

from the constructible Cohn spectrum of R to the closed subset of endosimple points in the Ziegler spectrum.

The étale bundle of definable scalars

Our goal: We show how to present an abelian regular ring R as the ring of global sections of a suitable sheaf over the constructible Cohn Spectrum.

The topological space $\mathrm{Zg}^*(R)$: Consider the **Zariski topology** $\mathrm{Zg}^*(R)$, introduced as a dual topology on the Ziegler spectrum $\mathrm{Zg}(R)$ whose basic open subsets are the complements $\mathcal{O}(\varphi/\psi)^c$, $\psi \leq \varphi \in \mathbb{L}(R, n)$, of the quasi-compact open subsets of $\mathrm{Zg}(R)$.

If R is abelian regular, then $\mathrm{Zg}^*(R) = \mathrm{Zg}(R)$.

Topological bundle: Let $\rho(u, v)$ be a pp formula in two variables. Then ρ defines a scalar on every point in the Zariski open subset

$$\mathcal{O}_{\mathrm{Zar}}(\rho(u, v)) := \mathcal{O}(u \doteq u/\exists v\ \rho(u, v))^c \ \cap\ \mathcal{O}(\rho(0, v)/v \doteq 0)^c$$

of $\mathrm{Zg}^*(R)$. Thus $U \in \mathcal{O}_{\mathrm{Zar}}(\rho(u, v))$ if and only if $\rho(U) \in R_U$.

The étale bundle of definable scalars

Topological bundle: Let $\mathbf{Bun}(R) := \dot{\bigcup} \{R_U \mid U \in \mathrm{Zg}^*(R)\}$ be the disjoint union of the R_U and define $p : \mathrm{Bun(R)} \to \mathrm{Zg}^*(\mathrm{R})$ to be a **topological bundle**, that is, a function whose fiber $p^{-1}\{U\} = R_U$. There is a commutative diagram

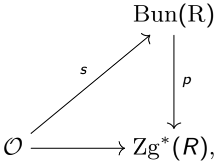
$$\begin{array}{ccc} & & \mathrm{Bun(R)} \\ & \overset{\mathrm{Ev}(\rho)}{\nearrow} & \downarrow p \\ \mathcal{O}_{\mathrm{Zar}}(\rho(u,v)) & \longrightarrow & \mathrm{Zg}^*(R), \end{array}$$

where $\mathbf{Ev}(\rho)(U) := \rho(U) \in R_U$.

Proposition: The topological bundle $p : \mathrm{Bun(R)} \to \mathrm{Zg}^*(\mathrm{R})$ is an **étale bundle**, with a subbasis of open subsets for $\mathrm{Bun(R)}$ given by the images $\mathrm{Im}\ \mathrm{Ev}(\rho)$, $\rho(u,v) \in \mathbb{L}(R,2)$, and preimages $p^{-1}(\mathcal{O})$, as $\mathcal{O}$ ranges over a basis for $\mathrm{Zg}^*(R)$.

The étale bundle of definable scalars

The sheaf Def: The sheaf Def of sections associated to the étale bundle $p : \mathrm{Bun(R)} \to \mathrm{Zg}^*(\mathrm{R})$ assigns to an open subset $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ the R-ring $R \to \mathrm{Def}(\mathcal{O})$ of continuous maps $s : \mathcal{O} \to \mathrm{Bun(R)}$ for which the diagram



commutes, where the horizontal arrow is the inclusion morphism.

Definable sections: A section $s \in \mathrm{Def}(\mathcal{O})$ is **definable** if there is pp formula $\rho(u, v)$ such that $\mathcal{O} \subseteq \mathcal{O}_{\mathrm{Zar}}(\rho)$ and $s = \mathrm{Ev}(\rho)|_{\mathcal{O}}$. Prest defined the notion of a **presheaf-on-a-basis** of definable scalars, which assigns to a basic open subset $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ the R-ring of definable sections on $\mathcal{O}$. The sheaf Def on $\mathrm{Zg}^*(R)$ is the sheafification of this presheaf.

The étale bundle of definable scalars

Pullback of étale bundles If $f : R \to S$ is an epic R-ring, then the induced homeomorphic embedding $\mathrm{Zg}(f) : \mathrm{Zg}(S) \to \mathrm{Zg}(R)$ is also continuous with respect to the Zariski topology $\mathrm{Zg}(f) = \mathrm{Zg}^*(f) : \mathrm{Zg}^*(S) \to \mathrm{Zg}^*(R)$. The action of every element $s \in S$ on a left S-module ${}_SM$ is a definable scalar over R. So if $U \in \mathrm{Zg}^*(S)$ is an indecomposable pure injective, then $S_U = R_U$ and the obvious morphism $\mathrm{Bun(f)} : \mathrm{Bun(S)} \to \mathrm{Bun(R)}$ of étale bundles given by

$$\begin{array}{ccc} \mathrm{Bun(S)} & \xrightarrow{\mathrm{Bun(f)}} & \mathrm{Bun(R)} \\ \downarrow p_S & & \downarrow p_R \\ \mathrm{Zg}^*(S) & \xrightarrow{\mathrm{Zg}^*(f)} & \mathrm{Zg}^*(R), \end{array}$$

where $p_S = p_R|_{\mathrm{Bun(S)}}$, is a pullback diagram.

The étale bundle of definable scalars

The correspondence between sheaves and topological bundles, implies that the sheaf of locally definable scalars on $\mathrm{Zg}^*(S)$ is given by the pullback along $\mathrm{Zg}^*(f)$ of the sheaf of locally definable scalars on $\mathrm{Zg}^*(R)$.

Coming back to our regular R-ring $R \to \hat{R}$, we can prove as follows.

Theorem: Let R be a ring with universal abelian regular R-ring $R \to \hat{R}$. The sheaf $\mathrm{Def}(\hat{R})$ of locally definable scalars over the constructible Cohn spectrum $\widehat{\mathrm{Spec}}(R) = \mathrm{Zg}^*(\hat{R})$ is obtained by the image sheaf $\mathrm{Def}(R)$ along the homeomorphic embedding $\mathrm{Zg}^*(\hat{R}) \to \mathrm{Zg}^*(R)$.

Further Investigations

Ring with involution Olivier's construction can still be generalized to obtain $(\hat{R}, \hat{*})$, the universal $*$-regular $(R, *)$-ring over a noncommutative ring $(R, *)$ with involution.

The construction mimics Olivier's construction with the **Moore-Penrose inverse** replacing the role of the commuting reflexive inverse in Olivier's construction.

It is shown that $(\hat{R}, \hat{*})$ coordinatizes the universal quantum logic of $(R, *)$, defined to be the lattice $\mathbb{L}(R, 1)$ modulo the least congruence for which the involution designates an orthogonal complement. This congruence is generated by the Laws of Contradiction and Excluded Middle, so that the R-modules that arise from the universal $*$-regularization are axiomatized by these laws.