About Numbers

How these Basic Tools Appeared and Evolved in Diverse Cultures

By **Allen Klinger,** Ph.D., *New York Iota '57*

MANY BIRDS AND insects possess a "number sense."[1] "If … a bird's nest contains four eggs, one may be safely taken; but if two are removed, the bird becomes aware of the fact and generally deserts."[2] The fact that many forms of life "sense" number or symmetry may connect to historic evolution of quantity in different human societies. We begin with the distinction between cardinal (counting) numbers and ordinal ones (that show position as in 1st or 2nd).

Humans sought greater understanding of numerical properties to better deal with daily occupations. One means of representing numbers uses the fingers. The matching principle involved in indicating a finger for each object is called one-to-one correspondence. The importance of man's ten fingers in numeration is indicated by the fact that "… those savages who have not reached the stage of finger counting are almost completely deprived of all perception of number."[1]

AUTHOR'S NOTE

The original version of this article is on the web at *http://web.cs.ucla.edu/~klinger/number.pdf*

It was written when I was a freshman. The humanities course had an assignment to write a paper on anthropology. The instructor approved the topic "number in early man."

At a reunion in 1997, I met a classmate from 1954, who remembered my paper from the same year. As a pack rat, somehow I found the original.

The Bent offers this to encourage exploring topics outside the engineering curriculum.

Key words: History and Civilization, Number Words and Base, Calculation.

First Attempts

The cardinal concept implied in matching was the basis of the first attempts to keep a record of number. Pebbles or sticks were cast into a pile, notches cut in wood or knots made in cord. The early origin of written number records is shown by pieces of carved bone. Making a notch on a bone with a carved head of some particular animal at the end, is almost recording a word message like "taken, one more such animal."


Franz Boas

Etymology indicates how recently our ancestors made use of these simple methods of recording number. Tally comes from the French tailler, to cut, which was derived from the Latin talea, a stick or cutting. Similarly, calculate/calculus come from the Latin word for pebble— calculus.

Representation of quantity by the principle of one-to-one correspondence was undoubtedly accompanied, and perhaps preceded, by creation of number-words. These can be divided into two main categories: those that arose before the concept of number unrelated to concrete objects, and those that arose after it.

An extreme instance of the development of number-words before the abstract concept of number is that of the Tsimshian language of a tribe in British Columbia. This language has seven distinct classes of number-words: for flat objects and animals, round objects and time, long objects and trees, men, canoes, measurements and, as a recent development, for counting when no definite object is referred to.[1] This was shown in studies by German-American anthropologist Franz Boas.[3] In a like manner, the English language has many words for types of collections (set, flock, herd, lot, bunch, etc.) but the more general collection and aggregate are of foreign origin.

Another peculiarity of English, the many words for two things, led to the comment, "It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number two."[4]

Some number-words developed as an abstraction of a general quality of quantity. That is the case of number word *five*. It had the original meaning *hand* in many languages: pantcha was five in Sanskrit—modern Persian uses pentcha for hand; five in Russian is piat—piast is Russian for the outstretched hand.

Abstract Nature

Many societies found it difficult to develop number-words besides "one," "two," and "many."

The idea of number as opposed to the number of a specific set of things was not completely comprehended in early societies even when there was but a single word to express a particular quantity. The collection word for five brought forth the image of a hand (or another group of five objects). This image was compared with the set of things whose number was to be determined.

The difficulty in grasping the idea of the abstract nature of number may lie in its being a logical, rather than numerical, concept. Some experiments with children

RIGHT: The Maya numbering system of dots and dashes is shown in this page of a pre-Columbian Maya book of the 11th or 12th century. It is believed to be a copy of an original text of some three or four hundred years earlier, and is the oldest book written in the Americas known to historians. It is located at the Saxon State Library in Dresden, Germany.

Jean Piaget

performed by Swiss psychologist Jean Piaget seem to indicate this.[5]

When a child is asked to lay out a number of blue chips equal to the number of red chips that are in a row spaced at uniform intervals, his reaction will depend on his age. If the child is five or younger, he will lay out a row of chips of the same length as the model row but will disregard spacing (placing them very close to each other) indicating that he believes the number of them is the same if the lengths of the rows are equal. At the age of six, the child will match his chips to those on the table. But if the length of one row is increased (by changing the spacing) he believes that row has more than the other. A third stage is reached by the time the child is six-and-a-half to seven: he realizes that the number of chips is independent of their spacing. "In short, children must grasp the principle of conservation of quantity before they can develop the concept of number."

Numeration would be a cumbersome process if it were necessary to form model collections and special words for every number. Fortunately, that is not the case because of the principle of forming large numbers by combining smaller ones. The last simple number is known as the *base* or *radix* of its number system. Our system is decimal: all numbers after ten are compounded of the first ten numerals (eleven and twelve are compound numbers whose forms are unrecognizable because they are Anglo-Saxon in origin.)

Primitive Language

Among the bases in use are two, three, four, five, six, eight, ten, twelve, twenty, and sixty. Of these, the most common today are five, ten and twenty.

The decimal base mechanism follows from the literal meaning of eleven in a primitive language: a man and one on the hand of another man. Of the non-decimal number systems, a typical binary system is that of a tribe of the Torres Straits between Australia and New Guinea:

(1) urapun
(2) okosa
(3) okosa-urapun
(4) okosa-okosa
(5) okosa-okosa-urapun
(6) okosa-okosa-okosa

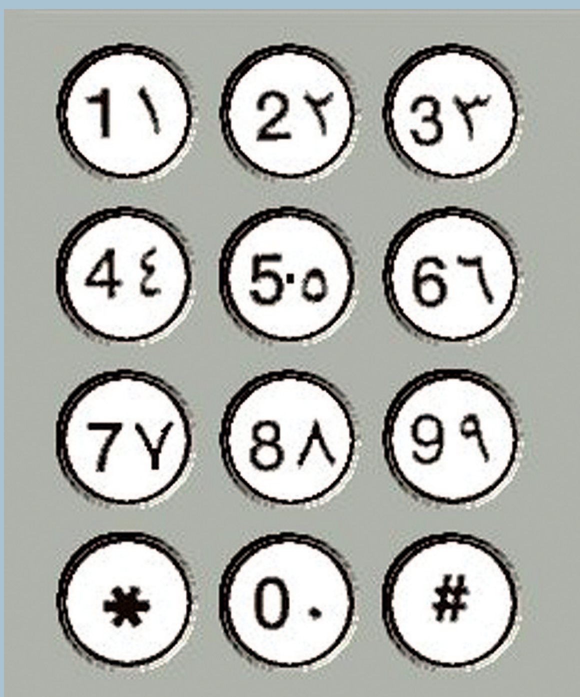
A typical quinary system is that of the Api language of the New Hebrides:

(1) tai
(2) lua
(3) tolu
(4) vari
(5) luna (hand)
(6) o tai (other one)
(7) o lua (other two)
(8) o tolu (other three)
(9) o vari (other four)
(10) luna luna (two hands)

Quinary, decimal and vigesimal number systems arose from finger counting, the "whole man" being regarded, respectively, as a hand, both hands, or the hands and feet. Binary systems are probably based on the symmetry of man's body; a Brazilian tribe developed a tertiary number system because they counted on the joints of the fingers; the quaternary system arose in California because of the religious significance of the four quarters of the sky. A system using eight basic numbers arose because counting was performed on the spaces between fingers not on the fingers themselves.

Apparently the simplest system, the binary, was adopted initially and discarded in favor of a system of higher radix in many parts of the world. The need for a system using a higher base arises from the difficulty of expressing large numbers in the binary system. The decimal and quinary systems have developed and been rapidly adopted in regions that previously used simpler systems of numeration. The existence of remnants of other number bases in our decimal system is readily ap-

YBC 7289 is an Old Babylonian clay tablet (circa 1800–1600 BCE) from the Yale Babylonian Collection. An approximately 8-cm diameter hand tablet, it appears to be a practice school exercise undertaken by a novice scribe. But, mathematically speaking, it is one of the most fascinating extant clay tablets because it contains not only a constructed illustration of a geometric square with intersecting diagonals, but also, in its text, a numerical estimate of 2√ correct to three sexagesimal or six decimal places. The value is read from the uppermost horizontal inscription and demonstrates the greatest known computational accuracy obtained anywhere in the ancient world. It is believed that the tablet's author copied the results from an existing table of values and did not compute them himself. The contents of this tablet were first translated and transcribed by Otto Neugebauer and Abraham Sachs in their 1945 book, *Mathematical Cuneiform Texts* (New Haven, CT: American Oriental Society). Photos: Bill Casselman, *www.math.ubc.ca/~cass/euclid/ybc/ybc*



Modern-day Arab telephone keypad with two forms of Arabic numerals: Western Arabic/European numerals on the left and Eastern Arabic numerals on the right

parent: having sixty minutes in an hour, dozen and gross, and the Biblical "three score and ten" are among the many examples.

Examination of the number base from a mathematical standpoint yields several criteria for a satisfactory radix: it should be sufficiently large to express large figures concisely, but small enough to lessen the words to be memorized to a reasonable amount. The next quality is how factorable the base should be. Mathematicians are divided on this point. Some favor a prime number so as to eliminate the ambiguity that arises in the expression of fractions (3/25 represents itself, 6/50, 12/100, etc.). This type of radix makes all decimals non-terminating (e.g. 1/3 = 0.3333 …) and is impractical from the standpoint of everyday use.

Other mathematicians favor a base that is evenly divisible by many numbers. The radix fulfilling the other requirements and that of factorability best is twelve. A duodecimal system would employ two additional symbols (e.g. X for ten and L for eleven) and would represent 12 by 10, 144 by 100, etc. The many factors of twelve simplify calculations by making quotients "come out even" more frequently and facilitate mental calculations: if a number ends in 0 in the decimal system we know it is divisible by 2 and 5; if it ends in 0 in the duodecimal system we know that it is divisible by 2, 3, 4, and 6. At any rate, ten would not be used as the mathematically chosen radix since it is neither prime nor highly factorable.

The universal need for describing quantity led to representation by one-to-one correspondence, development of number words, and a method for limiting the amount of words needed to express large figures. An important contribution to this was the development of a group of symbols.

Written numeration is probably as old as private property. There is little doubt that it originated in man's desire to keep a record of his flocks and other goods. Archeological researchers trace such records to times immemorial, and they are found in the caves of prehistoric man in Europe, Africa, and Asia. Numeration is at least as old as written language, and there is evidence that it preceded this. Perhaps, even, the recording of numbers suggested the recording of sounds.[1]

First Written Numbers

The first written numbers that were not solely tally marks probably occurred in Egypt about 3,400 B.C.[6] The Egyptian numerals employed the tally principle, as did most early systems, and were mainly straight lines. The distinction between simple tallying and the Egyptian numeration is made because the Egyptians used special symbols to signify large quantities. The Egyptians had basic symbols for one, ten, one hundred, and other powers of ten. The value of a number was the sum of the values of the symbols comprising it, the value of a symbol being repeated the number of times that the symbol was repeated.

Another system was the cuneiform symbols developed in Mesopotamia about 3,000 B.C. The system used by the Babylonians was both sexagesimal and decimal. Ordinary computation was carried on in the decimal system and astronomical work in the sexegesimal. The Babylonians used the wedge-like imprint that the stylus made in the clay as one, sixty, three hundred sixty, and, in general, sixty raised to any power, the meaning of the symbol being derived from the context. Similarly, the symbol for ten served as ten times sixty raised to any power.

As a later development, the Babylonians attempted to eliminate the indefinite representation that their system produced. In lists where we make the entry *'0'*, as an alternative to leaving blank, the Babylonians in some cases make use of the sign *ul*, meaning 'not,' 'nothing.' The Babylonian "zero" was a punctuation mark like our dash. About 400 B.C., the Babylonian mathematician Naburianu realized the significance of ul and used it as we use our zero. This development was lost to the world because it was in the sexagesimal system which was incomprehensible to the neighboring peoples who used decimal systems.[2]

The Greeks and Hebrews used fundamentally different numeration systems. Both were derived from the Phoenician; they followed an ordinal scheme of written numbers. Each of these civilizations used alphabetic elements—letters—to represent numbers. The Greeks originally represented numerals by the initial letter of the number word, but adopted the Phoenician convention of using the letters of the alphabet in their natural succession to represent numbers. The first nine letters of the Greek alphabet were, successively, symbols for one through nine, the next nine represented ten and its multiples through nine hundred (the Greeks added three letters to their alphabet in order to have twenty-seven symbols).

Return to Cardinal

The Roman numeral system was a return to the cardinal principle, and was similar to the Egyptian. Two interesting theories have been proposed to explain the derivation of the Roman symbols for five and ten. The first considers the symbol for ten as two fives. This theory is based on the fact that the Roman "V" is heavier on the left side. Consequently, the five is regarded as a hand. The idea is that the left side of the symbol represents the four fingers and the right side the thumb. The second theory is based on the tallying origin of cardinal systems of numeration: "X" is considered an abbreviation for "||||||||| over written by a /," and "V" is understood as one half of "X."

While these number systems were developed in the Old World, the Mayan Indians of the Yucatan Peninsula developed a vigesimal system which employed place notation. The Mayans wrote numbers in vertical columns, using only three symbols: . (one), _ (five) and (a version of an oval) (zero). Symbols on the lowest level were units, the next higher level, multiples of twenty, and successively, multiples of 360 and 7,200. One could try to represent these four examples 11, 60, 47, and 723 following Mayan three-symbol notation.

As a simple, cheap and reliable device, the Russian abacus remained in use in shops and markets throughout the former Soviet Union. Its use was taught in most schools until the 1990s.

It is interesting to note the pebble and stick form that the Mayan system takes. The use of "stick" or straight line numerals was extremely common. The Egyptian and Roman symbols were vertical lines but horizontal forms were used in the Far East. Our "2" and "3" are derived from the cursive forms of the corresponding horizontal line symbols, "z" (two horizontal lines, one on top of the other, joined by an oblique line) and three horizontal lines stacked together.

Each of the systems of written numbers that we have discussed are different from our system in one major respect: they were used to record figures but were not used directly in calculations. A primitive calculating instrument, the abacus, was used in one form or another in almost every civilization. The abacus was a method of representing numbers in a positional notation. Reduced to its basic elements, the abacus consisted of a set of parallel columns. The columns successively represented units, tens, hundreds, etc. Counters were placed on the columns (or marks made on columns in the dust) to indicate how many units, tens, or hundreds there were in a number. How would the number 13,204 be represented on an abacus?

The abacus was used in calculations by adding figures to their appropriate column and transferring when the marks in a column exceeded nine. Addition and subtraction as performed on the abacus can be easily visualized, but the processes of multiplication and division are too involved to be discussed here. A basic method for multiplication, "duplation," was used in Egypt at an early period (c. 3,400 B.C.) and was extremely important in Europe, surviving until the Renaissance period. Using our notation, multiplication of 32 by 17 using duplation was accomplished in this manner:

1 x 32	=	32		
2 x 32	=	64		
4 x 32	=	128	(4 x 32 = 2 x 64)	
				[1 + 2 + 4 = 7]
8 x 32	=	256		
2 x 32	=	64		
				[7 + 8 + 2 = 17]
17 x 32	=	32 + 64 + 128 + 256 + 64 = 544		

Division was accomplished by the analogous "mediation." These processes were taught in universities, one institution excelling in their course in mediation, another in duplation. The product of thousands of years of civilization was an inflexible number system "so crude as to make progress well-nigh impossible" and "a calculating device so limited in scope that even elementary calculations called for the services of an expert."[8]

Pierre-Simon, Marquis de Laplace

In India the use of a cardinal system of numbers made the abacus a fundamental tool. It was, however, difficult to record counting-board operations: the figure could represent 32, 302, 320, and so on. In order to make an unambiguous record of a series of calculations it became customary to place a mark called *sunya* (meaning empty or blank but having no connotations of void or nothingness) to indicate an empty place on the abacus. The symbol used was 0, o, or . and was eventually accepted as a number itself. It was placed at the bottom of the ordinal series 1, 2, 3, … and was adopted by the Arabs along with the rest of the Indian numerals.

Greatly Simplified

The Arabs, of course, introduced their numbers to Europe. The Hindu-Arabic symbols greatly simplified all calculations; their effectiveness is illustrated by multiplication. The example that required the services of an expert in duplation is within the capability of most children because of the new numerals. The method used in multiplication involves separating one of the numbers to be multiplied into units and tens, multiplying each of these quantities by the second number and adding the partial products:

The French mathematician Pierre-Simon, Marquis de Laplace, commented on the Hindu-Arabic numerals by saying:

"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."[7]

Our modern unambiguous place notation was developed to serve practical needs. It was originated in three places: Mesopotamia, India, and the Yucatan Peninsula. The last step, recognition of zero as a number by the public, took place only in India.

We have seen the development of cardinal and ordinal number concepts, the creation of the number word, the grouping of numerals by using a base, the use of simple symbols, the discovery of the principle of place notation, the development of a symbol for the absence of a digit, and the recognition of zero as a number with which computations may be made. Each step was important in producing our arithmetic, a fundamental tool.

NOTES

This paper has a few of the citations from 1954: for more, please see the online pdf. In asking a reader to respond to questions it presents issues about number representations that appear in the original work. Any that seem unclear can be resolved by consulting the scanned original work at:
http://web.cs.ucla.edu/~klinger/number.pdf

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