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Invited Paper

ABOUT THE EQUIVALENT MODULUS OF ELASTICITY OF CABLES OF CABLE-STAYED BRIDGES

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Abstract. *The cables of cable-stayed bridges have a non-linear behaviour caused by their shape variation, when the stress conditions change. This is a most interesting theme, namely, the study of the effective cable modulus of elasticity on the exactness of the analysis, during the erection of the bridges.*
In the present paper, we examine the influence of the neglected component of the weight of the cable, which is parallel to the direction of the chord of the cable. We obtain an easy used and useful formula by means of which we can estimate the exactitude of the well-known Dischinger's formula and, also, Hajdin's and other's one. We, finally, draw useful diagrams.

1. INTRODUCTION

When the stress conditions change, the cables of a cable-stayed bridge demonstrate a non-linear behaviour, due to the change of their shape. The mostly used of the methods of analysis of such bridges, require linearity of the modulus of elasticity of the cables. That is the reason, which conducts us to the use of an equivalent modulus of elasticity.

The first who discussed this aspect of the problem was F. Dischinger [1], who, some years later, gave the expression of the equivalent tangent modulus of elasticity, by his well-known formula. Afterwards, H. Ernst [2] gave the equivalent secant modulus of elasticity.

There is a great number of papers on this topic that has prevailing been called Dischinger's formula [3], [4], [5], [6], [7], [8].

On the study of the equivalent modulus of elasticity, we, usually, accept two assumptions.

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The first one is the use of a parabolic shape for the inclined cable instead of its real form, which is the catenary (the influence of this assumption is, maximum, about 0,05%, for full-loaded cables).

The second is the neglect of the component of the weight of the cables, which is parallel to the chord of the cable.

In the present paper we examine the influence of this last assumption and we obtain an easily used formula, which contains terms that express the neglected component. That modulus, which is symbolized with $\breve{E}$, is used to study the change of the ratios $\frac{\breve{E}}{E}\%$ and $\frac{\breve{E}}{E_D}\%$, where E the known modulus of elasticity of the material of the cable and E_D Dishinger's effective cable modulus of elasticity. In this study the following quantities are used as parameters: A, the area of the cross-section of the cable, ℓ, the length of the cord of the cable and ρ, the angle between the cord of the cable and the horizontal axis.

From the diagrams that are obtained by the described method, we can estimate the influence of the above mentioned assumption and, also, determine the conditions under which we can use that assumption.

2. ANALYSIS

Notes:

1. The quantities a,b,s are known and therefore:

$$\left.\begin{aligned} \cos\varphi &= \frac{\alpha}{\ell} \\ \sin\varphi &= \frac{\beta}{\ell} \end{aligned}\right\} \tag{1}$$

are known.

2. We suppose that the equation of the catenary is expressed (with an acceptable accuracy for small values of the deformation w) by the following equation of a parabola of second order:

$$w = f\left[1-\left(\frac{x}{\delta}\right)^2\right] \quad \Bigg\} \tag{2}$$

which has been refered to the axes oxz of Fig. 1.

Then we can find that : $$\frac{dw}{dx} = -2f\frac{x}{\delta^2} \quad \Bigg\} \tag{3}$$

Putting:

$$\frac{T_i + T_k}{2} = T = (T_k + \delta\, g_x) \Bigg\} \tag{4}$$

We can write the equilibrium of half of the cable as shown in Fig. 2.

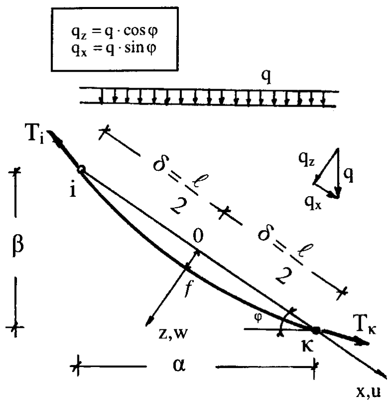

Fig. 1.

$$T.f - g_z \frac{\delta^2}{2} - \int_o^\delta g_x \cdot w(x) \cdot dx = 0 \text{ or}$$

$$f = \frac{1}{T}\left(g_z \frac{\delta^2}{2} + g_x f \int_o^\delta (1 - \frac{x^2}{\delta^2}) dx \right)$$

From the last equation we have:

$$f = \frac{1}{T}\left(\frac{g_z \delta^2}{2} + g_x f \left[x - \frac{x^3}{3\delta^2} \right]_0^\delta \right) = \frac{g_z \delta^2}{2T} + \frac{g_x f}{T}\left(\delta - \frac{\delta}{3} \right) = \frac{g_z \delta^2}{2T} + \frac{2 g_x \delta}{3T} \cdot f$$

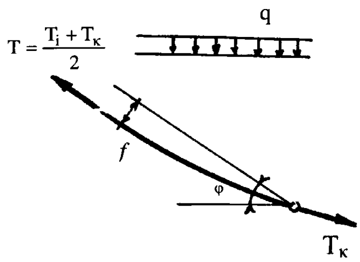

Fig. 2.

and finally we can write:

$$\left. f = \frac{3}{2} \cdot \frac{g_z \delta^2}{3T - 2 g_x \delta} \right\} \tag{5}$$

Substituting f from equation (5) in equation (3) we get:

$$\left. \frac{dw}{dx} = -2f \frac{x}{\delta^2} = -\frac{3 g_z \delta^2}{3T - 2 g_x \delta} \cdot \frac{x}{\delta^2} = -\frac{3 g_z x}{3T - 2 g_x \delta} \right\} \tag{6}$$

Now let us consider the infinitesimal element AB = ds of the cable ik at position x, loaded by its own weight and by the tensile forces T.

Under deformation line $\overline{AB}$ will be moved to the new position A'B'.

From Fig. 3 we can find:

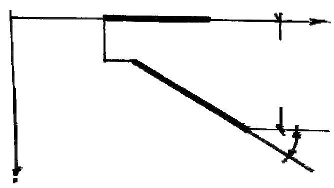

Fig. 3.

$$dx + u + du = u + (1 + \varepsilon)dx \cos\omega \text{ and}$$
$$w + dw - w = (1 + \varepsilon)dx \sin\omega$$

Finally we can write:

$$dx + du = (1+\varepsilon)dx \cos\omega$$
$$dw = (1+\varepsilon)dx \sin\omega$$

Extending cosω and sinω in form of a Maclaurin's series we have:

$$\left. \begin{aligned} \cos\omega &= 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \ldots \cong 1 - \frac{\omega^2}{2} \\ \sin\omega &= \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \ldots \cong \omega \end{aligned} \right\}$$

Then, we can change equs. (6) as follows:

$$\left. \begin{aligned} du + dx &= (1 + \varepsilon)dx(1 - \frac{\omega^2}{2}) = dx + \varepsilon dx - dx\frac{\omega^2}{2} - \varepsilon dx \frac{\omega^2}{2} \\ dw &= (1 + \varepsilon)dx.\omega = dx\omega + \varepsilon dx\omega \end{aligned} \right\}$$

Neglecting the infinitesimal values of the higher order terms $(\omega \cdot \varepsilon \cdot dx)$, $(\omega^2 \cdot \varepsilon \cdot dx)$ we get:

$$\left.\begin{aligned} du &= \varepsilon dx - dx\frac{\omega^2}{2} \\ dw &= \omega dx \end{aligned}\right\} \tag{7}$$

or:
$$\left.\varepsilon = \frac{du}{dx} + \frac{1}{2}\omega^2 = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\} \tag{8}$$

And then, for the tensile force we have:

$$T = A.\sigma = A.\varepsilon.E = A.E.\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

and finally:

$$\left.\frac{du}{dx} = \frac{T}{EA} - \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\} \tag{9}$$

Equation (9) because of equation (6) can be written as follows:

$$\frac{du}{dx} = \frac{T}{EA} - \frac{1}{2}\left(\frac{3g_z\delta}{3T - 2g_x\delta}\right)^2 \cdot \left(\frac{x}{\delta}\right)^2$$

The elongation of the cable is given by the following equation:

$$\left.\begin{aligned} \Delta\ell &= \int_{-\delta}^{\delta}\frac{du}{dx}dx = \int_{-\delta}^{\delta}\left\{\frac{T}{EA} - \frac{1}{2}\left(\frac{3g_z\delta}{3T - 2g_x\delta}\right)^2 \cdot \frac{x^2}{\delta^2}\right\}dx = \frac{2T\delta}{EA} - \frac{1}{2}\left(\frac{3g_z\delta}{3T - 2g_x\delta}\right)^2 \cdot \frac{1}{\delta^2} \cdot \left[\frac{x^3}{3}\right]_{-\delta}^{\delta} \\ &= \frac{2T\delta}{EA} - \frac{1}{2}\left(\frac{3g_z\delta}{3T - 2g_x\delta}\right)^2 \frac{2\delta}{3} = \frac{2T\delta}{EA} - \left(\frac{3g_z}{3T - 2g_x\delta}\right)^2 \frac{\delta^3}{3} \end{aligned}\right\} \tag{10}$$

Remembering that $\delta = \frac{\ell}{2}$ for a loading on the bridge that gives tensile force T, we get:

$$\left.\frac{\Delta\ell}{\ell} = \frac{T}{EA} - \frac{1}{24}\left(\frac{3g_z\ell}{3T - g_x\ell}\right)^2\right\} \tag{11}$$

Analogically, for a loading on the bridge that gives tensile force $\overline{T}$ we will have:

$$\left.\frac{\overline{\Delta\ell}}{\ell} = \frac{\overline{T}}{EA} - \frac{1}{24}\left(\frac{3g_z\ell}{3\overline{T} - g_x\ell}\right)^2\right\} \tag{12}$$

Then, the deformation of an equivalent system of a chord that would be elongated by a tensile force ΔT is:

$$\left.\Delta\tilde{\varepsilon} = \frac{\Delta\overline{\ell}}{\ell} - \frac{\Delta\ell}{\ell} = \frac{\overline{T} - T}{EA} - \frac{1}{24}\left\{\left(\frac{3g_z\ell}{3\overline{T} - g_x\ell}\right)^2 - \left(\frac{3g_z\ell}{3T - g_x\ell}\right)^2\right\}\right\} \tag{13}$$

But we know that: $$\Delta\tilde{\varepsilon} = \frac{\Delta\sigma}{\overline{E}} = \frac{1}{\overline{E}} \cdot \frac{\overline{T} - T}{A} \tag{14}$$

Taking into account equations (13) and (14) we can write:

$$\frac{1}{\overline{E}A} = \frac{1}{EA} - \frac{1}{\overline{T} - T} \cdot \frac{1}{24}\left[\left(\frac{3g_z\ell}{3\overline{T} - g_x\ell}\right)^2 - \left(\frac{3g_z\ell}{3T - g_x\ell}\right)^2\right] \Rightarrow$$

$$\frac{1}{\overline{E}A} = \frac{1}{EA} + \frac{1}{\overline{T} - T} \cdot \frac{1}{24}\left[\frac{(3\overline{T} - g_x\ell)^2 - (3T - g_x\ell)^2}{(3\overline{T} - g_x\ell)^2(3T - g_x\ell)^2}\right](3g_z\ell)^2$$

Or finally :

$$\frac{1}{\overline{E}A} = \frac{1}{EA} + \frac{1}{24} \cdot \frac{(3g_z\ell)^2}{\overline{T} - T} \frac{3(\overline{T} - T)(3\overline{T} + 3T - 2g_x\ell)}{[(g_x\ell)^2 - (3T + 3\overline{T})\, 3g_x\ell + 3T.3\overline{T}]^2}$$

$$= \frac{1}{EA}\left[1 + \frac{EA(g_z\ell)^2}{24} \cdot \frac{\left(\overline{T} + T - \frac{2}{3}g_x\ell\right)}{\left[T\overline{T} - \frac{g_x\ell}{3}(T + \overline{T}) + \left(\frac{g_x\ell}{3}\right)^2\right]^2}\right]$$

And finally we obtain the following expression for the equivalent modulus of elasticity of the cable:

$$\overline{E}A = \frac{EA}{1 + \frac{EA(g_z\ell)^2}{24} \frac{\left(\overline{T} + T - \frac{2}{3}g_x\ell\right)}{\left[T\overline{T} - \frac{g_x\ell}{3}\left(T + \overline{T}\right) + \left(\frac{g_x\ell}{3}\right)^2\right]^2}} \tag{15}$$

Putting $g_x = 0$ we obtain the formula of paper [8]:

$$\left.\overline{E}A = \frac{EA}{1 + \frac{1}{24}\left(\frac{g_z\ell}{T.\overline{T}}\right)^2 EA(T + \overline{T})}\right\} \tag{16}$$

And for $T = \overline{T}$ we obtain Dischinger's formula:

$$\left.\overline{E}A = \frac{EA}{1 + \frac{1}{12}\left(\frac{g_z\ell}{T^2}\right)^2 EAT}\right\} \tag{17}$$

3. Numerical results and discussion

For an illustrated example, we use the new Cable Bridge in Poland, which has been designed by the first of the authors. In table 1, the characteristics of the 14 cables of different lengths, inclinations and areas are shown. In table 2 is shown the error (on per cent), because of the neglect of the inclined component of the weight of the cables.

Table 1
(cable characteristics)

Cable No	Length (m)	Inclination (deg)	G.S. Area (cm^2)	Weight (kN/m)	g_Z (kN/m)	g_X (kN/m)	50% of Full Tension (kN)	Full Tension (kN)
87	135.30	27.51	256.31	2.26	2.00	1.04	8060.5	16121
88	119.12	28.18	276.32	2.43	2.14	1.15	8437.5	16875
89	102.96	29.05	230.14	2.03	1.77	0.98	7859.0	15718
90	86.83	30.26	229.37	2.02	1.74	1.02	7782.0	15564
91	70.75	32.01	280.17	2.47	2.09	1.31	9524.5	19049
92	54.79	34.78	168.56	1.48	1.22	0.85	5726.0	11452
93	39.05	39.81	153.94	1.35	1.04	0.87	5262.5	10525
94	187.00	19.53	267.08	2.35	2.22	0.79	7205.5	14411
95	163.72	20.10	237.83	2.09	1.97	0.72	8119.0	16238
96	140.45	20.85	228.60	2.01	1.88	0.72	7778.5	15557
97	117.22	21.91	224.75	1.98	1.83	0.74	7611.0	15222
98	94.05	23.50	228.60	2.01	1.84	0.80	7760.0	15520
99	71.00	26.11	163.94	1.44	1.30	0.64	5544.5	11089
100	48.23	31.22	157.79	1.39	1.19	0.72	5359.0	10718

Table 2
(numerical results)

Cable	Full Tension		50% of Full Tension	
No	A	B	A	B
87	-2.897 %	-0.0441 %	-7.235 %	-0.1359 %
88	-2.254 %	-0.0306 %	-5.721 %	-0.0972 %
89	-1.223 %	-0.0133 %	-3.143 %	-0.0427 %
90	-0.955 %	-0.0096 %	-2.445 %	-0.0303 %
91	-0.619 %	-0.0054 %	-1.589 %	-0.0169 %
92	-0.350 %	-0.0025 %	-0.901 %	-0.0080 %
93	-0.134 %	-0.0007 %	-0.349 %	-0.0023 %
94	-8.695 %	-0.1489 %	-20.042 %	-0.4286 %
95	-3.522 %	-0.0434 %	-8.734 %	-0.1342 %
96	-2.765 %	-0.0313 %	-6.900 %	-0.0963 %
97	-2.064 %	-0.0211 %	-5.178 %	-0.0650 %
98	-1.257 %	-0.0109 %	-3.200 %	-0.0340 %
99	-0.712 %	-0.0052 %	-1.826 %	-0.0163 %
100	-0.276 %	-0.0015 %	-0.716 %	-0.0050 %

$$\text{A: } \frac{(EA)_{gx} - (EA)}{(EA)}\,\%\text{ , B: } \frac{(EA)_{gx} - (EA)_{Hajdin}}{(EA)_{Hajdin}}\,\%$$

In this last table we compare the values we got from the proposed formula for the equivalent modulus E to these of the well-known tangential E (columns A) and, also, to these which are given by Hajdin's formula (columns B), for full tension of the cables and for 50% of the full tension.

We notice that the error of the second comparison is small and that Hajdin's formula is acceptable for its exactitude.

Finally, two indicative diagrams for the cable 94 and 95 are shown.

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EKVIVALENTNI MODUL ELASTIČNOSTI KABLOVA KOD VISEĆIH MOSTOVA

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Kablovi kod visećih mostova imaju nelinearno ponašanje izazvano promenom njihovog oblika tokom naprezanja. Proučavanje ekvivalentnog modula elsatičnosti kablova je jako zanimljiva tema jer direktno utiče na tačnost analize tokom proračuna a zatim i na samo ponašanje izvedenog mosta.

U radu je ispitivan uticaj zanemarivanja komponente sopstvene težine kablova koja je paralelna sa njihovom osom. Izvedena je korisna formula koja je jednostavna za primenu a pomoću koje se može proceniti tačnost kako dobro poznate Dišingerove formule tako i Hajdinove i ostalih. Na kraju, u radu su dati i korisni dijagrami.