Achieving modulated oscillations by feedback control

Tian Ge,[1,2] Xiaoying Tian,[1] Jürgen Kurths,[3,4,5] Jianfeng Feng,[1,2] and Wei Lin[1,*]

[1]*School of Mathematical Sciences, Centre for Computational Systems Biology, Fudan University, Shanghai 200433, China, and Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, China*

[2]*Department of Computer Science, University of Warwick, Coventry CV4 7AL, United Kingdom*

[3]*Potsdam Institute for Climate Impact Research, D-14412 Potsdam, Germany*

[4]*Department of Physics, Humboldt University of Berlin, D-12489 Berlin, Germany*

[5]*Department of Control Theory, Nizhny Novgorod State University, Gagarin Avenue 23, 606950 Nizhny Novgorod, Russia*

(Received 22 November 2013; revised manuscript received 30 May 2014; published 14 August 2014)

In this paper, we develop an approach to achieve either frequency or amplitude modulation of an oscillator merely through feedback control. We present and implement a unified theory of our approach for any finite-dimensional continuous dynamical system that exhibits oscillatory behavior. The approach is illustrated not only for the normal forms of dynamical systems but also for representative biological models, such as the isolated and coupled FitzHugh-Nagumo model. We demonstrate the potential usefulness of our approach to uncover the mechanisms of frequency and amplitude modulations experimentally observed in a wide range of real systems.

DOI: 10.1103/PhysRevE.90.022909 PACS number(s): 05.45.−a, 02.30.Oz, 87.19.lr, 89.75.−k

I. INTRODUCTION

Oscillations are omnipresent in the real world, from mechanics to engineering, from optics to astro- and geophysics, and from economics to social science. In the human body, oscillators are particularly abundant and appear at many different levels, e.g., the oscillatory NFκB signaling pathway in the cell nucleus, the cell cycle, neural oscillations in the central nervous system, and circadian rhythms [1]. The advantages and functional roles of these oscillators have been under active investigation.

In telecommunications, information is encoded and transmitted in a carrier wave by varying the instantaneous frequency or strength of the wave, known as frequency modulation (FM) and amplitude modulation (AM), respectively [2]. Recent investigations indicate that FM and AM are not restricted to electronic applications but may be even essential mechanisms of information transmission in a wide range of biological systems. In particular, the period of the cell cycle oscillator ranges from about 10 min in rapidly dividing embryonic cells to tens of hours in somatic cells, while a variation in the amplitude of this oscillation seems neither necessary nor desirable [3]. This is a typical example of FM [4]. In the hippocampus or inferior temporal cortex in the brain, the amplitude of the theta waves (4–8 Hz) increases after learning, while the frequency is approximated invariant. This is a typical example of AM [5]. In spite of the increasing number of phenomenological findings, the common or unique mechanisms that underlie FM and AM in various systems including biological oscillators have remained largely unknown. Initial computational studies suggest that feedbacks play critical roles in generating FM. It is the negative-plus-positive feedback loop that makes some biological oscillators exhibit a widely tunable frequency with a near-constant amplitude [6].

It has been shown recently, in both deterministic and stochastic dynamical models, that feedbacks with or without time delays commonly act as controllers to suppress periodic or chaotic oscillations to equilibria or periodic orbits or to synchronize coupled oscillators to a common manifold [7]. However, to the best of our knowledge, few works aim at designing appropriate feedback control to achieve FM and AM in dynamical systems. In this paper, we develop an approach to achieve FM and AM in generic finite-dimensional continuous dynamical systems through feedback control.

II. FREQUENCY AND AMPLITUDE MODULATIONS IN NORMAL FORMS

To begin with, we note that any finite-dimensional continuous dynamical system that exhibits oscillatory behavior through the appearance of an Andronov-Hopf bifurcation can be reduced to a two-dimensional system topologically equivalent to

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}-\left(x_1^2+x_2^2\right)\begin{bmatrix}x_1\\x_2\end{bmatrix}, \qquad (1)$$

where α and β are real constants [8]. We therefore first investigate FM and AM and illustrate our approach in this normal form. System (1) undergoes a supercritical Andronov-Hopf bifurcation as α passes through zero, where a unique stable periodic orbit bifurcates from the equilibrium $\boldsymbol{x}=\boldsymbol{0}$. The limit cycle has the amplitude $r_0=\sqrt{\alpha}$ and the frequency $f_0=\beta/2\pi$, respectively. We now introduce to system (1) a linear feedback control as follows:

$$\begin{bmatrix}x_1\to x_1 & x_2\to x_1\\ x_1\to x_2 & x_2\to x_2\end{bmatrix}=\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}, \qquad (2)$$

where $x_i\to x_j$ denotes the feedback from x_i to x_j, and $i,j=1,2$, F_{ij}, and $i,j=1,2$, are the feedback gains. The goal is to design an appropriate linear feedback control by which the controlled system exhibits oscillatory behavior with a modulated amplitude or frequency. This can be easily achieved even if we require that the controlled system undergoes a supercritical Andronov-Hopf bifurcation at the same bifurcation point $\alpha^*=0$ as in the original system (1) and assume that both self-feedbacks vanish, i.e., $F_{11}=F_{22}=0$. Table I summarizes the designed feedback patterns and the resulting

*wlin@fudan.edu.cn

TABLE I. Frequency and amplitude modulations in the normal form of the Andronov-Hopf bifurcation.

	FM	AM
Feedback	$\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & c\beta/(c+\beta) \\ c & 0 \end{bmatrix}$
Frequency	$f_0 + c/2\pi$	f_0
Amplitude	r_0	$r_0\sqrt{\frac{2}{1+(1+c/\beta)^2}}$

amplitude and frequency of the controlled system, where we replace F_{21} by the parameter c for notational simplicity. The frequency of the system can be modulated in the range from zero to infinity with an invariant amplitude when a simple antisymmetric feedback is imposed. A positive c increases the frequency, while a negative c reduces it. The amplitude of the system is less tunable when the autoregulation is constrained at zero. The modulated amplitude is no larger than $\sqrt{2}r_0$, although the oscillation can be dramatically suppressed when c is sufficiently large.

III. FEEDBACK CONTROL THEORY

To see how the feedback control in Table I is analytically obtained, and, more generally, how FM and AM can be achieved in generic continuous dynamical systems with the constraint on self-feedbacks relaxed, we now present a sketch of our feedback control theory. Full details on the theoretical aspects of our approach are provided in the appendix.

We first note that the investigation of any finite-dimensional and continuous system that undergoes an Andronov-Hopf bifurcation can be reduced to the following generic two-dimensional polynomial system by the center manifold theorem [8] and a Taylor expansion at the equilibrium from which the limit cycle bifurcates,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} G_1(x_1,x_2;\alpha) \\ G_2(x_1,x_2;\alpha) \end{bmatrix} + O(\|\boldsymbol{x}\|^4), \tag{3}$$

where $G_\nu(x_1,x_2;\alpha) = \sum_{p+q\leqslant 3,p,q\geqslant 0} g^\nu_{pq}(\alpha)x_1^p x_2^q, \nu = 1,2$, are polynomials with degrees no larger than 3, p and q are integers, and α is a scalar parameter. We always, without loss of generality, set the equilibrium at the origin by a parameter-dependent coordinate transformation, which indicates that the constants in $G_\nu(x_1,x_2;\alpha), \nu = 1,2$, vanish. Now we analyze system (3) with the additional linear feedback control (2). The system can then be rewritten as follows:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(\alpha)\boldsymbol{x} + \boldsymbol{G}(\boldsymbol{x},\alpha) + O(\|\boldsymbol{x}\|^4), \tag{4}$$

where

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} g^1_{10} + F_{11} & g^1_{01} + F_{12} \\ g^2_{10} + F_{21} & g^2_{01} + F_{22} \end{bmatrix} \tag{5}$$

is the Jacobian matrix of the controlled system at the equilibrium $\boldsymbol{x} = \boldsymbol{0}$, and $\boldsymbol{G}(\boldsymbol{x};\alpha)$ summarizes all the quadratic and cubic terms.

The key condition for the appearance of an Andronov-Hopf bifurcation at the bifurcation point α^* is the existence of a pair of complex eigenvalues, $\lambda_\pm = \mu(\alpha) \pm i\omega(\alpha)$, on the imaginary axis for the Jacobian matrix $\boldsymbol{A}(\alpha)$ at α^*, which is equivalent to $\Delta(\alpha^*) < 0$ and $\mathrm{tr}[\boldsymbol{A}(\alpha^*)] = 0$, where $\Delta(\alpha)$ is the discriminant for the polynomial $\det[\lambda\boldsymbol{I} - \boldsymbol{A}(\alpha)]$. Once these two conditions are satisfied along with the transversality and nondegeneracy conditions, systems with and without linear feedback control can be transformed into their "normal forms" via invertible coordinate and parameter changes when the parameter α is sufficiently close to the bifurcation point α^* (see the appendix). In the complex domain, the normal form can be written as follows [8]:

$$\frac{dw}{dt} = \lambda w + \eta w^2\bar{w} + O(|w|^4), \tag{6}$$

where w is a complex variable, $\bar{w}$ is the conjugate of w, $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha)$, and $\eta = \eta(\alpha) := \chi(\alpha) + i\kappa(\alpha)$. If we further require that $\chi(\alpha) < 0$, system (4) undergoes a supercritical Andronov-Hopf bifurcation at $\alpha = \alpha^*$, i.e., a stable limit cycle bifurcates from the equilibrium when α passes through α^*. The amplitude of the invariant curve is approximately $r_{\boldsymbol{F}}(\alpha) = \sqrt{-\mu(\alpha)/\chi(\alpha)}$, and the frequency is approximately $f_{\boldsymbol{F}}(\alpha) = [\omega(\alpha) + \kappa r^2_{\boldsymbol{F}}]/2\pi$. Both the amplitude and the frequency are functions of α and depend on the feedback gains.

As an example, system (1) with feedback control (2) can be converted into the normal form (6), where

$$\lambda(\alpha) = \alpha + \frac{\mathrm{i}}{2}\sqrt{-\Delta}, \;\; \eta(\alpha) \equiv \frac{1}{2}\left[-1 + \frac{F_{21}+\beta}{F_{12}-\beta}\right] < 0,$$

as long as $\Delta(\alpha) \equiv (F_{11} - F_{22})^2 + 4(F_{12} - \beta)(F_{21} + \beta) < 0$, and $\mathrm{tr}[\boldsymbol{A}(0)] = F_{11} + F_{22} = 0$. The amplitude and frequency of system (1) with feedback control (2) are therefore $\sqrt{-\alpha/\eta}$ and $\sqrt{-\Delta}/4\pi$, respectively.

Having obtained the normal forms, it is then straightforward to impose further conditions on the amplitude or the frequency to achieve FM or AM. Specifically, for FM, the amplitude of the limit cycle is invariant with and without linear feedback control. This leads to the following:

$$r_0(\alpha) = r_{\boldsymbol{F}}(\alpha)|_{F_{ij}=0,i,j=1,2} = r_{\boldsymbol{F}}(\alpha). \tag{7}$$

By varying the feedback gains $F_{ij}, i,j = 1,2$, under condition (7), the amplitude of the oscillator is near constant, while the frequency is tunable. Analogously, for AM, an invariant frequency is required, giving

$$f_0(\alpha) = f_{\boldsymbol{F}}(\alpha)|_{F_{ij}=0,i,j=1,2} = f_{\boldsymbol{F}}(\alpha). \tag{8}$$

Varying the feedback gains F_{ij}, $i,j = 1,2$, under condition (8), the frequency of the oscillator will remain approximately invariant, while the amplitude can be modulated.

In real applications, optimal feedback control can be searched analytically or numerically in the four-dimensional space $\{F_{ij}\}_{i,j=1,2}$, which satisfies Eq. (7) or Eq. (8), while maximizing a physically or biologically inspired index $\mathcal{J}$ with certain regularizations on the feedback. For example, the index may be designed as

$$\mathcal{J} = \mathrm{argmax}_{\boldsymbol{F}}\{f_{\boldsymbol{F}}/f_0 - \rho\|\boldsymbol{F}\|_2\}, \tag{9}$$

where ρ is a regularization parameter and $\|\cdot\|_2$ is the 2-norm of a matrix. This index seeks a balance between maximizing the modulated frequency and the expense of the feedback control. Below we illustrate this approach in a representative biological model.

IV. APPLICATION TO THE ISOLATED FITZHUGH-NAGUMO MODEL

Consider the well-known FitzHugh-Nagumo (FHN) model [9],

$$\begin{aligned}\frac{dv}{dt} &= v(v-\theta)(1-v) - w + I := p(v)\\ \frac{dw}{dt} &= \epsilon(v - \gamma w),\end{aligned} \tag{10}$$

where v mimics the membrane potential of an excitable neuron, w is a recovery variable describing the activation of an outward current, the parameter θ determines the shape of the cubic parabola $v(v-\theta)(1-v)$, ϵ and γ are nonnegative parameters that describe the dynamics of the recovery variable w, and I is the injected current [9]. The FHN model may exhibit a variety of dynamics and can be viewed as a simplification of the Hodgkin-Huxley-type models. The intersection of the v-nullcline $p(v) = 0$ and the w-nullcline $w = v/\gamma$ is the equilibrium of system (10), denoted as (v_0, w_0). System (10) undergoes an Andronov-Hopf bifurcation at $\epsilon_0 = a_0/\gamma$ if the parameters satisfy $0 < a_0 < 1/\gamma$ or $1/\gamma < a_0 < 0$, where $a_0 = p'(v_0) = -3v_0^2 + 2(1+\theta)v_0 - \theta$. In addition, if we ensure that

$$\chi(\epsilon_0) = -\frac{1}{4}\left(\frac{3}{2} - \frac{b_0^2}{1/\gamma - a_0}\right) < 0, \tag{11}$$

where $b_0 = p''(v_0)/2 = -3v_0 + 1 + \theta$, the bifurcation is supercritical, i.e., the limit cycle that appears when $\epsilon < \epsilon_0$ is stable (see the appendix).

We introduce to (10) a linear feedback control as follows:

$$\begin{bmatrix} v \to v & w \to v \\ v \to w & w \to w \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} v - v_0 \\ w - w_0 \end{bmatrix}. \tag{12}$$

Figure 1 shows feasible feedback gains that achieve FM in the FHN model with an alteration of the amplitude no greater than 5%. Note that the frequency is widely tunable (approximately 20-fold). The color encodes the index defined in Eq. (9) with $\rho = 1/4$, i.e., a warmer color indicates a larger modulated frequency, penalized by the expense of the feedback control. The time traces of the original and the modulated systems that correspond to two sets of feedback gains are provided in the upper panel. It can be clearly seen that the frequency of the oscillator can either increase or decrease with a near-constant amplitude.

In real-world systems, not all the dynamical equations are accessible for manipulation. One advantage of our approach is that any constraint on the feedback gains can be easily imposed. For example, when the equation of the recovery variable w in the FHN model is not accessible [10], F_{21} and F_{22} can be fixed at zero, and the frequency of the system is still widely tunable with an appropriate feedback on the equation of the fast variable v only (see the bottom panel of Fig. 1).

Noise is omnipresent and inevitable in real world. To demonstrate the robustness of our approach with respect to noise perturbations, we consider the following FHN model

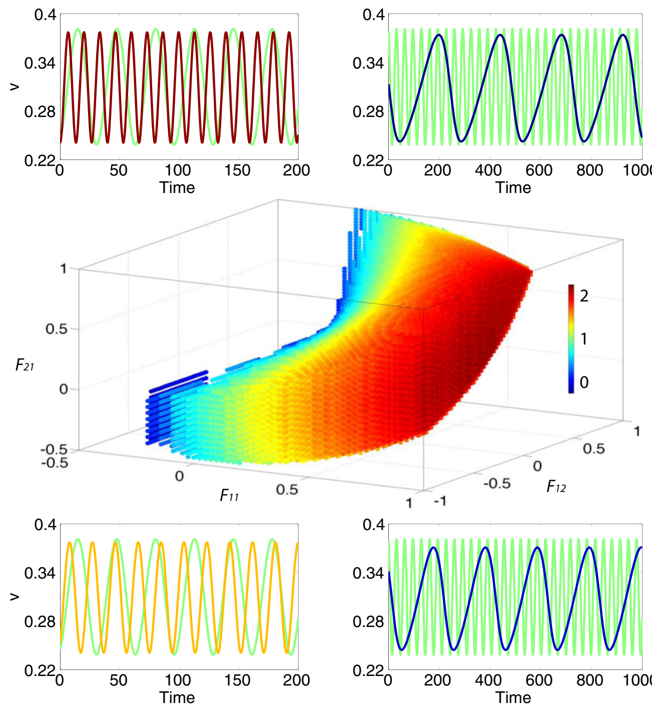

FIG. 1. (Color online) Frequency modulation of the FitzHugh-Nagumo model. The model parameters are selected as $\theta = 0.2$, $\gamma = 2.5$, and $I = 0.1$, which ensure that system (10) undergoes a supercritical Andronov-Hopf bifurcation at $\epsilon_0 = a_0/\gamma$. Feasible feedback gains F_{ij}, $i,j = 1,2$, to achieve FM with positive ϵ are numerically searched in the interval $[-1,1]$, with the alteration of the amplitude no greater than 5%. The color encodes the index defined in Eq. (9) with $\rho = 1/4$. The time traces of the original and the modulated systems that correspond to representative points on the surface are provided. The frequency of the system increases with the feedback gains $F_{11} = 1.000$, $F_{12} = -0.450$, $F_{21} = 0.375$, and $F_{22} = 0.900$ (upper left) and decreases with the feedback gains $F_{11} = -0.250$, $F_{12} = 0.950$, $F_{21} = -0.375$, and $F_{22} = 0.975$ (upper right). When the equation of the recovery variable w is not accessible for manipulation, i.e., F_{21} and F_{22} are constrained at zero, the frequency of the system can also be increased with feedback gains $F_{11} = 0.350$ and $F_{12} = -0.950$ (bottom left) and decreased with feedback gains $F_{11} = -0.250$ and $F_{12} = -0.850$ (bottom right).

with a white noise added to the fast variable v:

$$\begin{aligned}\frac{dv}{dt} &= v(v-\theta)(1-v) - w + I + \sigma\xi,\\ \frac{dw}{dt} &= \epsilon(v - \gamma w),\end{aligned} \tag{13}$$

where σ is the standard deviation of the noise and ξ denotes a white noise with zero mean and unit variance. The top and middle panels of Fig. 2 show the time traces of the perturbed variable v at two different noise levels ($\sigma = 0.001$ on the left, and $\sigma = 0.01$ on the right). The model parameters and the feedback control are the same as in the bottom panel of Fig. 1. It can be seen that, in the presence of noise, the frequency of the system is still well controlled with the designed feedback. The bottom panels of Fig. 2 present the power spectrum density estimates, with normalized frequencies, of the original and

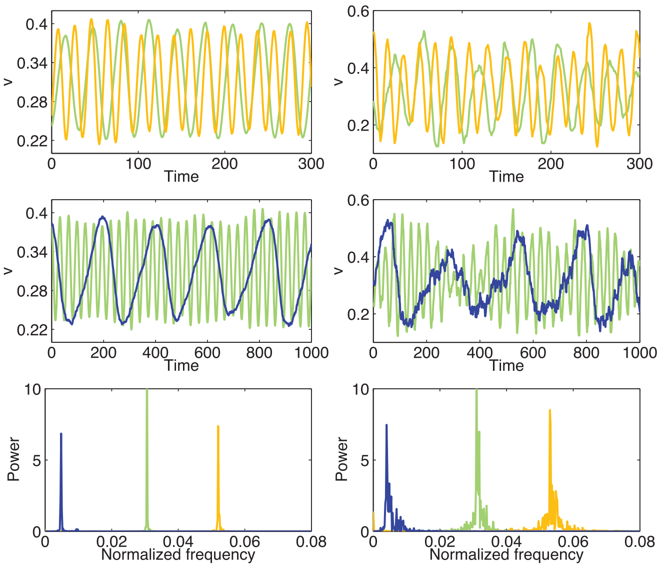

FIG. 2. (Color online) Frequency modulation of the noise perturbed FitzHugh-Nagumo model. The top and middle panels show the time traces of the fast variable v at two noise levels, $\sigma = 0.001$ on the left and $\sigma = 0.01$ on the right. The bottom panels present the power spectrum density estimates, with normalized frequencies, of the original and controlled systems. The model parameters and the feedback control are the same as in the bottom panel of Fig. 1.

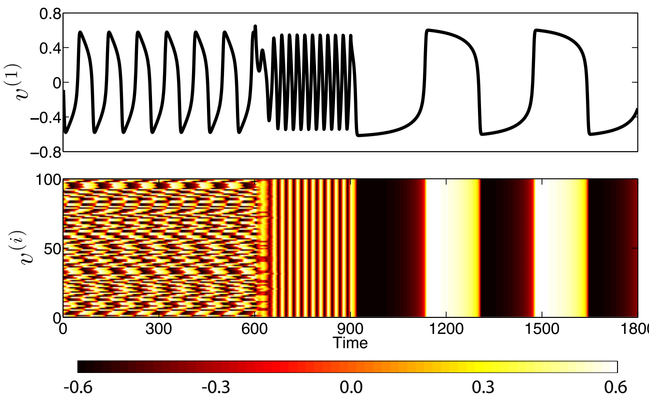

FIG. 3. (Color online) The dynamics of $N = 100$ coupled FitzHugh-Nagumo models with linear feedback control. The parameters are selected as $\theta = 0.2$, $\gamma = 2.5$, and $I = 0.112$. ϵ is chosen to ensure that each node exhibits oscillatory behavior. Upper panel: The time trace of a single neuron in the network. Lower panel: The dynamics of the 100 neurons in the network. Times 1–600: $\boldsymbol{Q} \equiv 0$, i.e., each node is an independent FHN neuron with randomly selected initial values. Times 601–900: When $F_{11} = 0.5000$, $F_{12} = -2.2024$, $F_{21} = 0.1072$, $F_{22} = -0.5000$, and $\boldsymbol{C}$ is selected as a compound symmetric matrix in which all diagonal terms are -0.1 and off-diagonal terms are $0.1/99$, synchronization in the network is achieved and the resulting synchronous oscillators have an increased frequency. Times 901–1800: When $F_{11} = -0.0400$, $F_{12} = 0.1762$, $F_{21} = -0.0071$, $F_{22} = 0.0040$, and $\boldsymbol{C}$ is selected as above, synchronization in the network is achieved and the resulting synchronous oscillators have an decreased frequency.

controlled systems, which further confirm that the frequency is widely tunable even under noise perturbations.

V. APPLICATION TO A COUPLED FITZHUGH-NAGUMO MODEL

In addition to FM and AM in isolated models, feedback control may play critical roles in networks with coupled oscillators, where FM or AM and synchronization may even be achieved simultaneously.

We illustrate this idea through a coupled FHN model. Specifically, we denote $\boldsymbol{x}(t) = [v(t), w(t)]^\top$ and summarize system (10) into $d\boldsymbol{x}/dt = \boldsymbol{H}(\boldsymbol{x};\epsilon)$, where the bifurcation parameter ϵ is assigned a proper value to generate oscillatory behavior. We now consider a complex dynamical network described by the following model:

$$\frac{d\boldsymbol{x}^{(i)}}{dt} = \boldsymbol{H}(\boldsymbol{x}^{(i)};\epsilon) + \boldsymbol{Q}^{(i)}\boldsymbol{X}, \qquad (14)$$

where $i = 1,2,\ldots,N$ is the node index, $\boldsymbol{X}(t)^\top = [\boldsymbol{x}^{(1)}(t)^\top, \ldots, \boldsymbol{x}^{(N)}(t)^\top]$ comprises the states of the nodes, and $\boldsymbol{Q}_{2N\times 2N}$ is a coefficient matrix that covers any type of linear feedback and coupling. Indeed, here we assume that it can be more specifically decomposed as $\boldsymbol{I}_{N\times N} \otimes \boldsymbol{F}_{2\times 2} + \boldsymbol{C}_{N\times N} \otimes \boldsymbol{I}_{2\times 2}$, where $\otimes$ denotes the Kronecker product, $\boldsymbol{I}_{2\times 2}$ is an identity matrix, $\boldsymbol{F}_{2\times 2}$ incorporates feedback gains, and $\boldsymbol{C}_{N\times N}$ is the coupling matrix satisfying each row sum to be 1. We note that $\boldsymbol{F}$ and $\boldsymbol{C}$ are two independent matrices. $\boldsymbol{F}$ is the feedback gain matrix applied to each two-dimensional system to achieve FM or AM, while $\boldsymbol{C}$ is a matrix coupling the N oscillators and can be appropriately selected to ensure synchronization in the network. $\boldsymbol{Q}^{(i)}$ in Eq. (14) represents two rows of $\boldsymbol{Q}$ that correspond to the i-th model.

To simultaneously achieve FM or AM and synchronization, we note that the variational equation of the complex network model on the synchronization manifold reads

$$\frac{d\boldsymbol{\delta}^{(i)}}{dt} = [D\boldsymbol{H}|_{\boldsymbol{x}^{(i)}} + \boldsymbol{F}]\boldsymbol{\delta}^{(i)} + \nu_i \boldsymbol{\delta}^{(i)}, \qquad (15)$$

where $D\boldsymbol{H}\big|_{\boldsymbol{x}^{(i)}}$ denotes the Jacobian matrix of $\boldsymbol{H}$ along the orbit $\{\boldsymbol{x}^{(i)}(t)\}$ and ν_i is the i-th eigenvalue of the coupling matrix $\boldsymbol{C}$. As each row sum of $\boldsymbol{C}$ equals 1, the matrix $\boldsymbol{C}$ is rank deficient. We therefore assume without loss of generality that $\nu_1 = 0$. The other ν_i's are selected to ensure $\|\boldsymbol{\delta}^{(i)}(n)\| \to \boldsymbol{0}$, as $n \to \infty$, for every i. This can be validated via numerically calculating the Lyapunov exponents of each variational equation transversal to the synchronization manifold.

The upper panel of Fig. 3 shows the time trace of a single neuron in the network, while the lower panel provides the dynamics of all the 100 neurons. In the first 600 time points, $\boldsymbol{Q}$ is set at zero, i.e., each node is an independent FHN neuron whose dynamics is governed by the self-dynamics function $\boldsymbol{H}$. All the oscillators have the same amplitude and frequency but different phases, since the initial values are randomly selected, producing a seemingly stochastic pattern. The next 300 time points show that when $\boldsymbol{Q}$ is appropriately designed, synchronization in the network can be achieved and the resulting synchronous oscillators have a dramatically increased frequency but an almost invariant amplitude. During the last 900 time points, $\boldsymbol{Q}$ is designed to decrease the frequency of the oscillators. This can be seen by noticing much wider stripe patterns compared to the time interval 601–900.

VI. APPLICATION TO THREE-DIMENSIONAL SYSTEMS

Our proposed approach to achieve FM and AM is also applicable to higher-dimensional systems, which resorts to the computation of the center manifold, followed by our generic feedback control theory in two-dimensional systems presented above. As an illustrative example, we consider the following three-dimensional dynamical system:

$$\begin{aligned}\frac{dx_1}{dt} &= (\eta-1)x_1 - x_2 + x_1x_3,\\ \frac{dx_2}{dt} &= x_1 + (\eta-1)x_2 + x_2x_3,\\ \frac{dx_3}{dt} &= \eta x_3 - \left(x_1^2+x_2^2+x_3^2\right),\end{aligned} \tag{16}$$

where η is a parameter. The system has an equilibrium at $(0,0,\eta)$, which can be set at the origin via the coordinate transformation $x_1 = y_1, x_2 = y_2$, and $x_3 = y_3 + \eta$. The transformed system takes the form

$$\begin{aligned}\frac{dy_1}{dt} &= (2\eta-1)y_1 - y_2 + y_1y_3,\\ \frac{dy_2}{dt} &= y_1 + (2\eta-1)y_2 + y_2y_3,\\ \frac{dy_3}{dt} &= -\eta y_3 - \left(y_1^2+y_2^2+y_3^2\right),\end{aligned} \tag{17}$$

and has the following Jacobian matrix $\boldsymbol{A}(\eta)$ evaluated at the equilibrium:

$$\boldsymbol{A}(\eta) = \begin{bmatrix} 2\eta-1 & -1 & 0\\ 1 & 2\eta-1 & 0\\ 0 & 0 & -\eta\end{bmatrix}. \tag{18}$$

The three eigenvalues of $\boldsymbol{A}(\eta)$ are $\lambda_{1,2}(\eta) = 2\eta - 1 \pm \mathrm{i}$, and $\lambda_3(\eta) = -\eta$. In particular, when $\eta = 1/2$, the equilibrium is not hyperbolic, and the two conjugate eigenvalues $\lambda_{1,2}$ have zero real part. By the center manifold theory [8], for each fixed η with $|\eta - 1/2|$ sufficiently small, there is a locally defined smooth two-dimensional invariant center manifold $\mathcal{W}^c_\eta$, which is tangent at the origin to the generalized eigenspace of $\boldsymbol{A}(\eta)$ corresponding to $\lambda_{1,2}(\eta)$. This center manifold $\mathcal{W}^c_\eta$ can be locally represented as a graph of a smooth function,

$$\mathcal{W}^c_\eta = \{(y_1,y_2,y_3) : y_3 = h(y_1,y_2)\}, \tag{19}$$

and, due to the tangent property of $\mathcal{W}^c_\eta$, we write $y_3 = h(y_1,y_2) = h_{20}y_1^2 + h_{11}y_1y_2 + h_{02}y_2^2 + o(\|\boldsymbol{y}\|^2)$, where h_{20}, h_{11}, and h_{02} are functions of η. By noticing the fact that

$$\frac{dy_3}{dt} = \frac{\partial h(y_1,y_2)}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial h(y_1,y_2)}{\partial y_2}\frac{dy_2}{dt}, \tag{20}$$

we find $h_{20} = h_{02} = 1/(2-5\eta)$ and $h_{11} = 0$ by expanding both sides of Eq. (20) and comparing the coefficients of quadratic terms. By the reduction principle [8], for sufficiently small $|\eta - 1/2|$, system (17) is locally topologically equivalent near the origin to the system,

$$\begin{aligned}\frac{dy_1}{dt} &= (2\eta-1)y_1 - y_2 - \frac{1}{5\eta-2}y_1\left(y_1^2+y_2^2\right),\\ \frac{dy_2}{dt} &= y_1 + (2\eta-1)y_2 - \frac{1}{5\eta-2}y_2\left(y_1^2+y_2^2\right),\\ \frac{dy_3}{dt} &= -\eta y_3.\end{aligned} \tag{21}$$

Note that the equation for y_3 is uncoupled with y_1 and y_2 and has an exponentially decaying solution when η is close to 1/2. Therefore, the stability of system (21) is determined by the equations for y_1 and y_2, i.e., the restriction of the system to its center manifold. Having reduced the system to two dimensions, the standard normal form theory of the Andronov-Hopf bifurcation then follows. In particular, if we introduce a complex variable $z = y_1 + \mathrm{i}y_2$ and transform the system to its polar form by using the representation $z = \rho e^{\mathrm{i}\phi}$, we obtain

$$\begin{aligned}\frac{d\rho}{dt} &= (2\eta-1)\rho - \frac{1}{5\eta-2}\rho^3 + o(\rho^3)\\ \frac{d\phi}{dt} &= 1.\end{aligned} \tag{22}$$

Since $5\eta - 2 > 0$ for sufficiently small $|\eta - 1/2|$, the system undergoes a supercritical bifurcation at $\eta = 1/2$, and a stable limit cycle bifurcates from the equilibrium when $\eta > 1/2$, with a radius approximately $\sqrt{(2\eta-1)(5\eta-2)}$ and a constant frequency $1/2\pi$. The upper panel of Fig. 4 shows an orbit of system (16) with $\eta = 1/2 + 10^{-3}$ and the initial values $(0.01,0.01,1.5)$. Consistent with the theory, x_3 approaches the center manifold quickly, on which the orbit converges to the limit cycle.

Higher-dimensional systems offer many more possibilities for the design of feedback control than two-dimensional systems. Here we consider a simple case for illustration. We introduce to the first two components of system (16) an antisymmetric feedback as follows:

$$\begin{bmatrix} x_1\to x_1 & x_2\to x_1\\ x_1\to x_2 & x_2\to x_2\end{bmatrix} = \begin{bmatrix}0 & -\beta\\ \beta & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \tag{23}$$

and to the third component a self-feedback,

$$x_3 \to x_3 = \gamma(x_3 - \eta), \tag{24}$$

where β and γ are feedback gains. Repeating the computation that gives Eq. (22), we get the normal form of the controlled system restricted to the center manifold for proper values of β and γ,

$$\begin{aligned}\frac{d\rho}{dt} &= (2\eta-1)\rho - \frac{1}{5\eta-2-\gamma}\rho^3 + o(\rho^3)\\ \frac{d\phi}{dt} &= 1+\beta.\end{aligned} \tag{25}$$

The controlled system then has the same bifurcation point $\eta = 1/2$ as the original system and has a stable limit cycle with a radius approximately $\sqrt{(2\eta-1)(5\eta-2-\gamma)}$, and a frequency $(1+\beta)/2\pi$, when η is slightly off the critical point. The middle and bottom panels of Fig. 4 show that the frequency

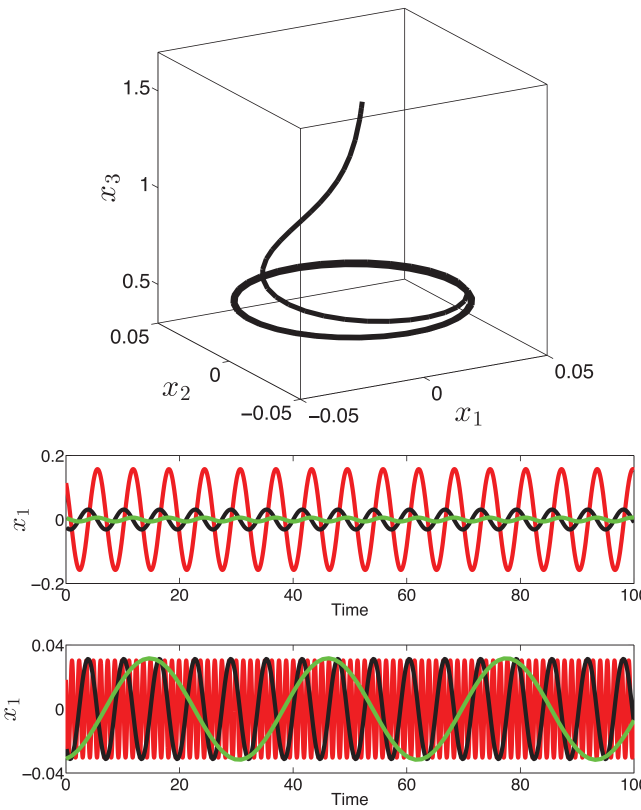

FIG. 4. (Color online) Frequency and amplitude modulations of an illustrative three-dimensional system. Upper panel: An orbit of system (16) with $\eta = 1/2 + 10^{-3}$ and the initial values (0.01,0.01,1.5). Middle panel: Amplitude modulation of system (16) with the feedback specified in Eq. (24). When $\eta = 1/2 + 10^{-3}$, the amplitude of the original system (black) increases with $\gamma = -12$ (red) and decreases with $\gamma = 0.48$ (green). Bottom panel: Frequency modulation of system (16) with the feedback specified in Eq. (23). When $\eta = 1/2 + 10^{-3}$, the frequency of the original system (black) increases with $\beta = 4$ (red) and decreases with $\beta = -0.8$ (green).

and amplitude of system (16) can be dramatically modulated when β and γ are appropriately selected.

VII. DISCUSSION AND FUTURE DIRECTIONS

In this paper, we have presented an approach to design feedback control and modulate the frequency or amplitude of oscillators modeled by generic continuous dynamical systems. In complex network models with coupled oscillators, FM or AM and synchronization can even be achieved simultaneously. The proposed method, when combined with the computation of center manifold, is applicable to FM and AM in higher-dimensional dynamical systems as well.

We note that the normal form theory is accurate when the bifurcation parameter is sufficiently close to the critical point. This somewhat indicates that the above-proposed approach is only suitable for analyzing the behavior of the system near the equilibrium or fixed point, whereas natural systems are believed to operate far from steady states [11]. However, we found that in many cases the feedback control designed near the bifurcation point still can achieve AM and FM when the bifurcation parameter varies in a wide region. For example, unlike the dynamics presented in Figs. 1 and 2 where the bifurcation parameter is near the critical point ($\epsilon = \epsilon^* - 10^{-3}$), in the FHN network model presented above, ϵ is set at $\epsilon^* - 0.1$. The system exhibits a much richer dynamics than a regular oscillation only, and the effects of FM are still clearly observed. Hence, our strategies for achieving FM or AM are applicable to oscillations even far from steady states. Having said this, strategies based on global bifurcation theory are also expected for rigorous analysis of the systems not close to the equilibrium or fixed point.

The approach for FM and AM is not restricted to continuous dynamical systems and can be easily applied to discrete dynamical systems that exhibit oscillatory behavior via the appearance of a Neimark-Sacker bifurcation, the counterpart of the Andronov-Hopf bifurcation. Our approach may also be extended to engineering oscillators with nonlinear feedbacks or time delays. Our investigations may invite systematic studies on oscillator control theory and contribute to a general principle to engineer biological oscillators and has the potential to provide a theoretical support to the mechanisms of FM and AM postulated in the literature.

ACKNOWLEDGMENTS

This work was supported by the NNSF of China (Grants No. 11322111 and No. 61273014) and by Talents Programs (Grants No. 10SG02, No. 11QH1400200, and No. NCET-11-0109).

APPENDIX

In this appendix, we provide details of the analytical approach to achieve frequency and amplitude modulations in generic continuous dynamical systems and expatiate on the computational details when applying the theory to the normal form of the Andronov-Hopf bifurcation and the FitzHugh-Nagumo model.

1. Frequency and amplitude modulations in continuous dynamical systems

We note that the investigation of any finite-dimensional system that undergoes an Andronov-Hopf bifurcation can be reduced to the following generic two-dimensional polynomial system by the central manifold theorem [8] and a Taylor expansion at the equilibrium from which the limit cycle bifurcates,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} G_1(x_1,x_2;\gamma) \\ G_2(x_1,x_2;\gamma) \end{bmatrix} + O(\|\boldsymbol{x}\|^4), \tag{A1}$$

where $G_\nu(x_1,x_2;\gamma) = \sum_{p+q\leqslant 3,p,q\geqslant 0} g_{pq}^{\nu}(\gamma) x_1^p x_2^q,\ \ \nu = 1,2$, are polynomials with degrees no larger than 3, p and q are integers, and γ is a scalar parameter. Suppose that the system undergoes a supercritical Andronov-Hopf bifurcation. Then the system has a unique equilibrium when the parameter γ is sufficiently close to the bifurcation point. Therefore, we always set the equilibrium at the origin by a parameter-dependent coordinate transformation. This indicates that the constants in $G_\nu(x_1,x_2;\gamma)$, $\nu = 1,2$, vanish. Now we consider system (A1) with an additional linear feedback control,

$$\begin{bmatrix} x_1 \to x_1 & x_2 \to x_1 \\ x_1 \to x_1 & x_2 \to x_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $x_i \to x_j$ denotes the feedback from x_i to x_j and $i,j = 1,2$, F_{ij}, $i,j = 1,2$, are the feedback gains. The system can then be reorganized as follows:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(\gamma)\boldsymbol{x} + \boldsymbol{G}(\boldsymbol{x},\gamma) + O(\|\boldsymbol{x}\|^4), \tag{A2}$$

where

$$\boldsymbol{A}(\gamma) = \begin{bmatrix} g_{10}^1 + F_{11} & g_{01}^1 + F_{12} \\ g_{10}^2 + F_{21} & g_{01}^2 + F_{22} \end{bmatrix}$$

is the Jacobian matrix of the controlled system at the equilibrium $\boldsymbol{x} = \boldsymbol{0}$,

$$\boldsymbol{G}(\boldsymbol{x},\gamma) = \boldsymbol{G}_q(\boldsymbol{x},\gamma) + \boldsymbol{G}_c(\boldsymbol{x},\gamma).$$

Here $\boldsymbol{G}_q$ denotes all the quadratic terms as follows:

$$\boldsymbol{G}_q(\boldsymbol{x},\gamma) = \begin{bmatrix} g_{20}^1 x_1^2 + g_{11}^1 x_1 x_2 + g_{02}^1 x_2^2 \\ g_{20}^2 x_1^2 + g_{11}^2 x_1 x_2 + g_{02}^2 x_2^2 \end{bmatrix},$$

$\boldsymbol{G}_c$ summarizes all the cubic terms,

$$\boldsymbol{G}_c(\boldsymbol{x},\gamma) = \begin{bmatrix} g_{30}^1 x_1^3 + g_{21}^1 x_1^2 x_2 + g_{12}^1 x_1 x_2^2 + g_{03}^1 x_2^3 \\ g_{30}^2 x_1^3 + g_{21}^2 x_1^2 x_2 + g_{12}^2 x_1 x_2^2 + g_{03}^2 x_2^3 \end{bmatrix},$$

and $g_{pq}^\nu = g_{pq}^\nu(\gamma)$, $1 \leqslant p+q \leqslant 3$, $p,q \geqslant 0$, $\nu = 1,2$, are all smooth functions of γ.

The goal is to design an appropriate linear feedback control by which the controlled system undergoes a supercritical Andronov-Hopf bifurcation at the bifurcation point γ^* and additionally exhibits oscillatory behavior with a desired amplitude or frequency.

The key condition for the appearance of an Andronov-Hopf bifurcation is the existence of a pair of complex eigenvalues on the imaginary axis for the Jacobian matrix $\boldsymbol{A}(\gamma)$ at the bifurcation point $\gamma = \gamma^*$, which is equivalent to $\Delta(\gamma^*) < 0$ and $\mathrm{tr}[\boldsymbol{A}(\gamma^*)] = 0$, where $\Delta(\gamma)$ is the discriminant for the polynomial $\det[\lambda \boldsymbol{I} - \boldsymbol{A}(\gamma)]$. Therefore, we have

$$\Delta(\gamma^*) = \left[g_{10}^1(\gamma^*) - g_{01}^2(\gamma^*) + F_{11} - F_{22}\right]^2 + 4\left[g_{01}^1(\gamma^*) + F_{12}\right]\left[g_{10}^2(\gamma^*) + F_{21}\right] < 0 \tag{A3}$$

and

$$\mathrm{tr}[\boldsymbol{A}(\gamma^*)] = g_{10}^1(\gamma^*) + g_{01}^2(\gamma^*) + F_{11} + F_{22} = 0. \tag{A4}$$

Lemma 1. Suppose that $g_{pq}^\nu = g_{pq}^\nu(\gamma), 1 \leqslant p+q \leqslant 3, p,q \geqslant 0, \nu = 1,2$, satisfy the transversality and nondegeneracy conditions at $\gamma = \gamma^*$. Furthermore, conditions Eq. (A3) and Eq. (A4) hold. Then system (A2) undergoes an Andronov-Hopf bifurcation at $\gamma = \gamma^*$.

Suppose that the conditions in Lemma 1 hold. Then system (A2) can be transformed into its normal form via invertible coordinate and parameter changes. Specifically, Eq. (A3) and Eq. (A4) ensure that, for all sufficiently small $|\gamma - \gamma^*|$, the Jacobian matrix $\boldsymbol{A}(\gamma)$ has two eigenvalues as follows:

$$\lambda_\pm(\gamma) = \mu(\gamma) \pm \mathrm{i}\omega(\gamma),$$

where $\mu(\gamma) = \frac{1}{2}[g_{10}^1(\gamma) + g_{01}^2(\gamma) + F_{11} + F_{22}]$ with $\mu(\gamma^*) = 0$ and $\omega(\gamma) = \frac{1}{2}\sqrt{-\Delta(\gamma)}$. Hence, there exists an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A}(\gamma) = \boldsymbol{P}\boldsymbol{J}(\gamma)\boldsymbol{P}^{-1},$$

where $\boldsymbol{J}(\gamma) = [\begin{smallmatrix} \mu & -\omega \\ \omega & \mu \end{smallmatrix}]$ is the Jordan normal form of $\boldsymbol{A}(\gamma)$. Denote $\xi = \xi(\gamma) = 2[g_{01}^1 + F_{12}]$ and $\zeta = \zeta(\gamma) = g_{01}^2 - g_{10}^1 + F_{22} - F_{11}$. $\boldsymbol{P}$ can then be selected as

$$\boldsymbol{P} = \frac{1}{\xi}\begin{bmatrix} \xi & 0 \\ \zeta & -\sqrt{-\Delta} \end{bmatrix}.$$

By a coordinate transformation $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$, system (A2) can be transformed into

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{J}(\gamma)\boldsymbol{y} + \boldsymbol{K}(\boldsymbol{y},\gamma) + O(\|\boldsymbol{y}\|^4), \tag{A5}$$

where $\boldsymbol{K}(\boldsymbol{y},\gamma) = \boldsymbol{P}^{-1}\boldsymbol{G}(\boldsymbol{P}\boldsymbol{y},\gamma) = \boldsymbol{K}_q(\boldsymbol{y},\gamma) + \boldsymbol{K}_c(\boldsymbol{y},\gamma)$, with $\boldsymbol{K}_q(\boldsymbol{y},\gamma)$ and $\boldsymbol{K}_c(\boldsymbol{y},\gamma)$ denoting the quadratic and cubic terms of $\boldsymbol{K}(\boldsymbol{y},\gamma)$, respectively. Note that this transformation leaves the first component of the coordinate unchanged.

Taking into account the following facts:

$$x_1^2 = y_1^2,$$
$$x_1 x_2 = \frac{y_1}{\xi}[\zeta y_1 - \sqrt{-\Delta} y_2],$$
$$x_2^2 = \frac{1}{\xi^2}[\zeta^2 y_1^2 - \Delta y_2^2 - 2\sqrt{-\Delta}\zeta y_1 y_2],$$

and after intensive calculation, we obtain the coefficients in $\boldsymbol{K}_q(\boldsymbol{y},\gamma)$ and $\boldsymbol{K}_c(\boldsymbol{y},\gamma)$,

$$\boldsymbol{K}_q(\boldsymbol{y},\gamma) = \begin{bmatrix} k_{20}^1 y_1^2 + k_{11}^1 y_1 y_2 + k_{02}^1 y_2^2 \\ k_{20}^2 y_1^2 + k_{11}^2 y_1 y_2 + k_{02}^2 y_2^2 \end{bmatrix},$$

where $k_{pq}^\nu = k_{pq}^\nu(\gamma)$, $p+q = 2$, $p,q \geqslant 0$, $\nu = 1,2$, are smooth functions of γ,

$$k_{20}^1 = g_{20}^1 + \frac{\zeta}{\xi} g_{11}^1 + \frac{\zeta^2}{\xi^2} g_{02}^1,$$

$$k_{11}^1 = -\frac{\sqrt{-\Delta}}{\xi}\left[g_{11}^1 + \frac{2\zeta}{\xi} g_{02}^1\right], \quad k_{02}^1 = -\frac{\Delta}{\xi^2} g_{02}^1,$$

$$k_{20}^2 = \frac{1}{\sqrt{-\Delta}}\left\{\zeta g_{20}^1 - \xi g_{20}^2\right\} + \frac{1}{\sqrt{-\Delta}}\left\{\frac{\zeta}{\xi}\left[\zeta g_{11}^1 - \xi g_{11}^2\right] + \frac{\zeta^2}{\xi^2}\left[\zeta g_{02}^1 - \xi g_{02}^2\right]\right\},$$

$$k_{11}^2 = -\frac{1}{\xi}\left\{\zeta g_{11}^1 - \xi g_{11}^2 + \frac{2\zeta}{\xi}\left[\zeta g_{02}^1 - \xi g_{02}^2\right]\right\},$$

$$k_{02}^2 = \frac{\sqrt{-\Delta}}{\xi^2}\left[\zeta g_{02}^1 - \xi g_{02}^2\right]. \tag{A6}$$

$$\boldsymbol{K}_c(\boldsymbol{y},\gamma) = \begin{bmatrix} k_{30}^1 y_1^3 + k_{21}^1 y_1^2 y_2 + k_{12}^1 y_1 y_2^2 + k_{03}^1 y_2^3 \\ k_{30}^2 y_1^3 + k_{21}^2 y_1^2 y_2 + k_{12}^2 y_1 y_2^2 + k_{03}^2 y_2^3 \end{bmatrix},$$

where $k^{\nu}_{pq} = k^{\nu}_{pq}(\gamma)$, $p+q=3$, $p,q \geqslant 0$, $\nu = 1,2$, are smooth functions of γ,

$$\begin{aligned}
k^1_{30} &= g^1_{30} + \frac{\zeta}{\xi} g^1_{21} + \frac{\zeta^2}{\xi^2} g^1_{12} + \frac{\zeta^3}{\xi^3} g^1_{03},\\
k^1_{21} &= -\frac{\sqrt{-\Delta}}{\xi}\left[g^1_{21} + \frac{2\zeta}{\xi} g^1_{12} + \frac{3\zeta^2}{\xi^2} g^1_{03}\right],\\
k^1_{12} &= -\frac{\Delta}{\xi^2}\left[g^1_{12} + \frac{3\zeta}{\xi} g^1_{03}\right], \quad k^1_{03} = \frac{\Delta\sqrt{-\Delta}}{\xi^3} g^1_{03},\\
k^2_{30} &= \frac{1}{\sqrt{-\Delta}}\left\{\zeta g^1_{30} - \xi g^2_{30} + \frac{\zeta}{\xi}\left[\zeta g^1_{21} - \xi g^2_{21}\right]\right\}\\
&\quad + \frac{1}{\sqrt{-\Delta}}\left\{\frac{\zeta^2}{\xi^2}\left[\zeta g^1_{12} - \xi g^2_{12}\right] + \frac{\zeta^3}{\xi^3}\left[\zeta g^1_{03} - \xi g^2_{03}\right]\right\},\\
k^2_{21} &= -\frac{1}{\xi}\left\{\zeta g^1_{21} - \xi g^2_{21}\right\}\\
&\quad - \frac{1}{\xi}\left\{\frac{2\zeta}{\xi}\left[\zeta g^1_{12} - \xi g^2_{12}\right] + \frac{3\zeta^2}{\xi^2}\left[\zeta g^1_{03} - \xi g^2_{03}\right]\right\},\\
k^2_{12} &= \frac{\sqrt{-\Delta}}{\xi^2}\left\{\zeta g^1_{12} - \xi g^2_{12} + \frac{3\zeta}{\xi}\left[\zeta g^1_{03} - \xi g^2_{03}\right]\right\},\\
k^2_{03} &= \frac{\Delta}{\xi^3}\left[\zeta g^1_{03} - \xi g^2_{03}\right].
\end{aligned} \tag{A7}$$

Since $\boldsymbol{K}(\boldsymbol{y},\gamma)$ satisfies that $\boldsymbol{K}(\boldsymbol{0},\gamma) = \boldsymbol{0}$ for all sufficiently small $|\gamma - \gamma^*|$, by introducing a complex variable $z = y_1 + iy_2$, Eq. (A5) can be transformed into the following complex form:

$$\begin{aligned}
\frac{dz}{dt} &= \lambda z + \frac{l_{20}}{2} z^2 + l_{11} z\bar{z} + \frac{l_{02}}{2}\bar{z}^2\\
&\quad + \frac{l_{30}}{6} z^3 + \frac{l_{21}}{2} z^2\bar{z} + \frac{l_{12}}{2} z\bar{z}^2 + \frac{l_{03}}{6}\bar{z}^3 + O(|z|^4),
\end{aligned}$$

where $\lambda = \lambda(\gamma) = \mu(\gamma) + i\omega(\gamma)$, $l_{pq} = l_{pq}(\gamma)$, $p+q = 2,3$, $p,q \geqslant 0$, are smooth functions of γ.

Specifically,

$$\begin{aligned}
l_{20} &= \frac{1}{2}\left[\left(k^1_{20} + k^2_{11} - k^1_{02}\right) + i\left(k^2_{20} - k^1_{11} - k^2_{02}\right)\right],\\
l_{11} &= \frac{1}{2}\left[\left(k^1_{20} + k^1_{02}\right) + i\left(k^2_{20} + k^2_{02}\right)\right],\\
l_{02} &= \frac{1}{2}\left[\left(k^1_{20} - k^2_{11} - k^1_{02}\right) + i\left(k^2_{20} + k^1_{11} - k^2_{02}\right)\right],\\
l_{21} &= \frac{1}{4}\left(3k^1_{30} + k^1_{12} + k^2_{21} + 3k^2_{03}\right)\\
&\quad + \frac{i}{4}\left(3k^2_{30} - k^1_{21} + k^2_{12} - 3k^1_{03}\right),
\end{aligned}$$

where $k^{\nu}_{pq} = k^{\nu}_{pq}(\gamma)$ are specified in Eq. (A6) and Eq. (A7).

Lemma 2. [8]. The equation

$$\begin{aligned}
\frac{dz}{dt} &= \lambda z + \frac{l_{20}}{2} z^2 + l_{11} z\bar{z} + \frac{l_{02}}{2}\bar{z}^2\\
&\quad + \frac{l_{30}}{6} z^3 + \frac{l_{21}}{2} z^2\bar{z} + \frac{l_{12}}{2} z\bar{z}^2 + \frac{l_{03}}{6}\bar{z}^3, + O(|z|^4),
\end{aligned}$$

where $\lambda = \lambda(\gamma) = \mu(\gamma) + i\omega(\gamma)$, $\mu(\gamma^*) = 0$, $\omega(\gamma^*) > 0$, and $l_{pq} = l_{pq}(\gamma)$ can be transformed by an invertible parameter-dependent change of complex coordinate, which is smoothly dependent on the parameter,

$$\begin{aligned}
z &= w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2}\bar{w}^2\\
&\quad + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2\bar{w} + \frac{h_{12}}{2} w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3,
\end{aligned}$$

for all sufficiently small $|\gamma - \gamma^*|$, into a map with only one cubic term (the *resonant term*),

$$\frac{dw}{dt} = \lambda w + \eta w^2\bar{w} + O(|w|^4),$$

where

$$\eta = \eta(\gamma) = \frac{l_{20} l_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|l_{11}|^2}{\lambda} + \frac{|l_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{l_{21}}{2}.$$

Let $\chi(\gamma) = \mathrm{Re}[\eta(\gamma)]$. Then the coefficient $\chi(\gamma^*)$ determines the stability of the invariant orbit in system (A2) which undergoes an Andronov-Hopf bifurcation.

Having obtained the normal forms, the stability as well as the amplitude and frequency of the invariant orbit can be extracted, as summarized in the following lemma.

Lemma 3. Suppose that the conditions Eq. (A3), Eq. (A4), and $\chi(\gamma^*) < 0$ are satisfied, along with the nondegenerate condition $\ell(\gamma^*) \neq 0$ and the transversality condition $\mu'(\gamma^*) \neq 0$. Then system (A2) undergoes a supercritical Andronov-Hopf bifurcation at $\gamma = \gamma^*$. A stable closed invariant orbit bifurcates from the equilibrium when γ passes through γ^*. The direction of this bifurcation is determined by the sign of $\mu'(\gamma^*)$. Specifically, the invariant orbit appears for $\gamma > \gamma^*$ if $\mu'(\gamma^*) > 0$ and for $\gamma < \gamma^*$ if $\mu'(\gamma^*) < 0$. The radius of the invariant orbit is approximately $r_{\boldsymbol{F}}(\gamma) = \sqrt{-\mu(\gamma)/\chi(\gamma)}$, and the frequency is approximately $f_{\boldsymbol{F}}(\gamma) = \omega(\gamma)/2\pi$.

To modulate the frequency of the oscillator with an invariant amplitude, all the conditions in Lemma 3 are required to ensure the appearance of a supercritical Andronov-Hopf bifurcation. Moreover, the amplitude of the stable closed invariant orbit that bifurcates from the equilibrium is invariant with and without linear feedback control. This leads to the following:

$$r_{\boldsymbol{F}}(\gamma)|_{F_{ij}=0,i,j=1,2} = r_{\boldsymbol{F}}(\gamma). \tag{A8}$$

By varying the feedback gains F_{ij}, $i,j = 1,2$, under Eq. (A8), the amplitude of the stable closed invariant orbit is approximately invariant while the frequency is tunable.

Similarly, to modulate the amplitude of the oscillator with an invariant frequency, it is required, in addition to all the conditions in Lemma 3, that

$$f_{\boldsymbol{F}}(\gamma)|_{F_{ij}=0,i,j=1,2} = f_{\boldsymbol{F}}(\gamma). \tag{A9}$$

By varying the feedback gains F_{ij}, $i,j = 1,2$, under Eq. (A9), the frequency of the stable closed invariant orbit remains approximately invariant while the amplitude can be modulated.

In real applications, conditions (A8) and (A9) can either be calculated analytically or validated numerically.

2. Computational details of the normal form

We consider the normal form of the Andronov-Hopf bifurcation in Eq. (1), with the linear feedback control (2).

Following the notations in the above section, the controlled system can then be reorganized as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(\alpha)\boldsymbol{x} + \boldsymbol{G}(\boldsymbol{x},\alpha), \tag{A10}$$

where

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} \alpha + F_{11} & -\beta + F_{12} \\ \beta + F_{21} & \alpha + F_{22} \end{bmatrix}$$

and

$$\boldsymbol{G}(\boldsymbol{x},\alpha) = \boldsymbol{G}_q(\boldsymbol{x},\alpha) + \boldsymbol{G}_c(\boldsymbol{x},\alpha),$$

with $g^1_{30} = g^1_{12} = g^2_{21} = g^2_{03} = -1$, and all the other coefficients $g^{\nu}_{pq} = g^{\nu}_{pq}(\gamma)$, $p + q = 2,3$, $p,q \geqslant 0$, $\nu = 1,2$, vanish. To ensure the existence of an Andronov-Hopf bifurcation at the same bifurcation point as the original system, it is required that the Jacobi matrix $\boldsymbol{A}(\alpha)$ has a pair of complex eigenvalues on the imaginary axis at the critical point $\alpha^* = 0$, giving

$$\Delta(\alpha) = \Delta(\alpha^*) \equiv (F_{11} - F_{22})^2 + 4(F_{12} - \beta)(F_{21} + \beta) < 0$$

and

$$\mathrm{tr}[\boldsymbol{A}(\alpha^*)] \equiv F_{11} + F_{22} = 0.$$

Suppose that the above two conditions are satisfied, and $|\alpha|$ is sufficiently close to zero. Then $\boldsymbol{A}(\alpha)$ has two eigenvalues,

$$\lambda_{\pm}(\alpha) = \mu(\alpha) \pm i\omega(\alpha),$$

where $\mu(\alpha) = \alpha$ and $\omega(\alpha) = \frac{1}{2}\sqrt{-\Delta(\alpha)}$. Let $\xi = 2(F_{12} - \beta)$, $\zeta = F_{22} - F_{11}$, and

$$\boldsymbol{P} = \frac{1}{\xi}\begin{bmatrix} \xi & 0 \\ \zeta & -\sqrt{-\Delta} \end{bmatrix}.$$

By a coordinate transformation $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$, system (A10) can be transformed into

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{J}(\alpha)\boldsymbol{y} + \boldsymbol{K}(\boldsymbol{y},\alpha), \tag{A11}$$

where $\boldsymbol{J}(\alpha) = [\begin{smallmatrix} \alpha & -\omega \\ \omega & \alpha \end{smallmatrix}]$, $\boldsymbol{K}(\boldsymbol{y},\alpha) = \boldsymbol{P}^{-1}\boldsymbol{G}(\boldsymbol{P}\boldsymbol{y},\alpha) = \boldsymbol{K}_q(\boldsymbol{y},\alpha) + \boldsymbol{K}_c(\boldsymbol{y},\alpha)$, with

$$k^1_{20} = k^1_{11} = k^1_{02} = k^2_{20} = k^2_{11} = k^2_{02} = 0$$

and

$$k^1_{30} = -1 - \frac{\zeta^2}{\xi^2}, \quad k^1_{21} = \frac{2\zeta\sqrt{-\Delta}}{\xi^2}, \quad k^1_{12} = \frac{\Delta}{\xi^2}, \quad k^1_{03} = 0,$$

$$k^2_{30} = 0, \quad k^2_{21} = -1 - \frac{\zeta^2}{\xi^2}, \quad k^2_{12} = \frac{2\zeta\sqrt{-\Delta}}{\xi^2}, \quad k^2_{03} = \frac{\Delta}{\xi^2}.$$

By introducing a complex variable $z = y_1 + iy_2$, system (A11) can be transformed into the following complex form:

$$\frac{dz}{dt} = \lambda z + \frac{l_{20}}{2}z^2 + l_{11}z\bar{z} + \frac{l_{02}}{2}\bar{z}^2 + \frac{l_{30}}{6}z^3 + \frac{l_{21}}{2}z^2\bar{z} + \frac{l_{12}}{2}z\bar{z}^2 + \frac{l_{03}}{6}\bar{z}^2, \tag{A12}$$

where $\lambda = \lambda(\alpha) = \alpha + i\omega(\alpha)$,

$$l_{20} = l_{11} = l_{02} = 0, \; l_{21} = -1 - \frac{\zeta^2}{\xi^2} + \frac{\Delta}{\xi^2} < 0.$$

Finally, system (A12) can be transformed by an invertible parameter-dependent change of complex coordinate, which is smoothly dependent on the parameter, for sufficiently small $|\alpha|$, into its normal form with only one cubic term (the resonant term),

$$\frac{dw}{dt} = \lambda w + \eta w^2\bar{w},$$

where

$$\eta = \eta(\alpha) = \frac{l_{21}}{2} \equiv \frac{1}{2}\left[-1 + \frac{F_{21} + \beta}{F_{12} - \beta}\right] < 0.$$

Since $\chi(\alpha) = \mathrm{Re}[\eta(\alpha)] = \eta(\alpha) < 0$, the controlled system (A10) undergoes a supercritical Andronov-Hopf bifurcation at $\alpha^* = 0$. The radius of the limit cycle is

$$r_{\boldsymbol{F}}(\alpha) = \sqrt{-\frac{\alpha}{\eta}},$$

and the frequency of the invariant orbit is

$$\frac{\omega}{2\pi} = \frac{1}{4\pi}\sqrt{-\Delta} = \frac{1}{4\pi}\sqrt{-(F_{11} - F_{22})^2 - 4(F_{12} - \beta)(F_{21} + \beta)}.$$

Suppose that both self-feedbacks vanish, i.e., $F_{11} = F_{22} = 0$. To modulate the frequency of the oscillator with an invariant amplitude, we require that

$$\sqrt{\alpha} = r_0 = r_{\boldsymbol{F}}(\alpha)|_{F_{ij}=0,i,j=1,2} = r_{\boldsymbol{F}}(\alpha) = \sqrt{-\frac{\alpha}{\eta}},$$

giving $F_{12} = -F_{21}$. That is, any feedback control that takes the form $[\begin{smallmatrix} 0 & -c \\ c & 0 \end{smallmatrix}]$ can achieve FM, with a modulated frequency

$$f_{\boldsymbol{F}} = (\beta + c)/2\pi = f_0 + c/2\pi.$$

To modulate the amplitude of the oscillator with an invariant frequency, we require that

$$\frac{\beta}{2\pi} = f_0 = f_{\boldsymbol{F}}(\alpha)|_{F_{ij}=0,i,j=1,2} = f_{\boldsymbol{F}}(\alpha) = \frac{1}{4\pi}\sqrt{-\Delta} = \frac{1}{2\pi}\sqrt{(\beta - F_{12})(\beta + F_{21})},$$

giving $F_{12} = \beta F_{21}/(\beta + F_{21})$. That is, any feedback control that takes the form $[\begin{smallmatrix} 0 & c\beta/(c+\beta) \\ c & 0 \end{smallmatrix}]$ can achieve AM, with a modulated amplitude

$$r_{\boldsymbol{F}} = \sqrt{\frac{2\alpha}{1 + (1 + c/\beta)^2}} = r_0\sqrt{\frac{2}{1 + (1 + c/\beta)^2}}.$$

3. Computational details of the FitzHugh-Nagumo model

Consider the FHN model described by the continuous dynamical system (10). The Jacobi matrix of system (10) at the equilibrium (v_0,w_0) is

$$A_0 = \begin{bmatrix} a_0 & -1 \\ \epsilon & -\gamma\epsilon \end{bmatrix},$$

where $a_0 = p'(v_0) = -3v_0^2 + 2(1+\theta)v_0 - \theta$. The eigenvalues of A_0 are

$$\lambda_\pm = \mu(\epsilon) \pm i\omega(\epsilon) = \frac{a_0-\gamma\epsilon}{2} \pm\sqrt{\left(\frac{a_0-\gamma\epsilon}{2}\right)^2 + 2\frac{a_0-\gamma\epsilon}{2}\left(\frac{1}{\gamma}-a_0\right) - a_0\left(\frac{1}{\gamma}-a_0\right)}.$$

Therefore, the FitzHugh-Nagumo model undergoes an Andronov-Hopf bifurcation at $\epsilon_0 = a_0/\gamma$, if the parameters satisfy $0 < a_0 < 1/\gamma$ or $1/\gamma < a_0 < 0$. At the bifurcation point, the Jacobi matrix A_0 has a pair of complex eigenvalues, $\pm i\sqrt{a_0(1/\gamma - a_0)}$, on the imaginary axis. Following the procedures described above, system (10) can be transformed into the following complex form:

$$\frac{dz}{dt} = \lambda z + \frac{b_0}{4}\left(1 + i\frac{\mu-a_0}{\omega}\right)(z+\bar{z})^2 - \frac{1}{8}\left(1 + i\frac{\mu-a_0}{\omega}\right)(z+\bar{z})^3,$$

where $\lambda = \lambda(\epsilon) = \mu(\epsilon) + i\omega(\epsilon)$, $b_0 = p''(v_0)/2 = -3v_0 + 1 + \theta$, and it can finally be converted into its normal form,

$$\frac{dw}{dt} = \lambda w + \eta w^2 \bar{w},$$

where

$$\eta(\epsilon) = \frac{b_0^2}{8}\left\{\frac{1}{\mu^2+\omega^2}\left[5\mu - 2(\mu-a_0) - \mu\left(\frac{\mu-a_0}{\omega}\right)^2\right]\right\} + \frac{b_0^2}{8}\left\{\frac{\mu}{\mu^2+9\omega^2}\left[1+\left(\frac{\mu-a_0}{\omega}\right)^2\right]\right\} - \frac{3}{8}$$
$$- i\frac{b_0^2}{8}\omega\left\{\frac{1}{\mu^2+\omega^2}\left[1 - 6\frac{\mu}{\omega}\frac{\mu-a_0}{\omega} + \left(\frac{\mu-a_0}{\omega}\right)^2\right]\right\}$$
$$- i\frac{b_0^2}{8}\omega\left\{\frac{3}{\mu^2+9\omega^2}\left[1+\left(\frac{\mu-a_0}{\omega}\right)^2\right]\right\} - i\frac{3}{8}\frac{\mu-a_0}{\omega},$$

and, in particular,

$$\mathrm{Re}[\eta(\epsilon_0)] = -\frac{1}{4}\left(\frac{3}{2} - \frac{b_0^2}{1/\gamma - a_0}\right).$$

Therefore, if we ensure that $\chi(\epsilon_0) = \mathrm{Re}[\eta(\epsilon_0)] < 0$, the Andronov-Hopf bifurcation of the FitzHugh-Nagumo model is supercritical, i.e., the limit cycle that appears when $\epsilon < \epsilon_0$ is stable.

We now introduce to (10) the linear feedback control (12). Let $x = [v - v_0, w - w_0]^\top$. The controlled FHN model can be reorganized as follows:

$$\frac{dx}{dt} = A(\epsilon)x + G(x,\epsilon),$$

where

$$A(\epsilon) = \begin{bmatrix} a_0 + F_{11} & -1 + F_{12} \\ \epsilon + F_{21} & -\gamma\epsilon + F_{22} \end{bmatrix}$$

and

$$G(x,\epsilon) = G_q(x,\epsilon) + G_c(x,\epsilon),$$

with $g_{20}^1 = b_0$, $g_{30}^1 = -1$, and all the other coefficients $g_{pq}^\nu = g_{pq}^\nu(\gamma)$, $p+q = 2,3$, $p,q \geqslant 0$, $\nu = 1,2$, vanish. Therefore, we have obtained all the coefficients $g_{pq}^\nu = g_{pq}^\nu(\gamma)$, $1 \leqslant p+q \leqslant 3$, $p,q \geqslant 0$, $\nu = 1,2$, for this polynomial system. We can verify all the conditions required for FM and AM following the results established in this Appendix.

[1] L. Ashall *et al.*, Science **324**, 242 (2009); C. Gerard and A. Goldbeter, Proc. Natl. Acad. Sci. USA **106**, 21643 (2009); A. Goldbeter *et al.*, FEBS Lett. **586**, 2955 (2012).

[2] J. Proakis and M. Salehi, *Digital Communications* (McGraw-Hill, New York, 2007).

[3] J. R. Pomerening *et al.*, Cell **122**, 565 (2005).

[4] L. Cai *et al.*, Nature **455**, 485 (2008).

[5] A. B. Tort *et al.*, Proc. Natl. Acad. Sci. USA **106**, 20942 (2009); K. M. Kendrick *et al.*, BMC Neurosci. **12**, 55 (2011).

[6] T. Y. Tsai *et al.*, Science **321**, 126 (2008); A. Goldbeter, Nature **420**, 238 (2002).

[7] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); K. Pyragas, Phys. Lett. A **170**, 421 (1992); **206**, 323 (1995); Philos. Trans. R. Soc. London, Ser. A **364**, 2309 (2006); W. Lin, H. Ma, J. Feng, and G. Chen, Phys. Rev. E **82**, 046214 (2010); J. Lu *et al.*, IEEE Trans. Neural Netw. **23**, 285 (2012); Y. Guo *et al.*, New J. Phys. **14**, 083022 (2012); G. Yan, J. Ren, Y.-C. Lai, C. H. Lai, and B. Li, Phys. Rev. Lett. **108**, 218703 (2012); W. Lin *et al.*, Europhy. Lett. **102**, 20003 (2013); H. Ma, W. Lin, and Y.-C. Lai, Phys. Rev. E **87**, 050901(R) (2013); J. Sun and A. E. Motter, Phys. Rev. Lett. **110**, 208701 (2013); A. Koseska, E. Volkov, and J. Kurths, *ibid.* **111**, 024103 (2013); P. Menck *et al.*, Nat. Phys. **9**, 89 (2013); Z. Yuan *et al.*, Nat. Comm. **4**, 2447 (2013).

[8] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Spring-Verlag, New York, 1998).

[9] R. FitzHugh, Biophys. J. **1**, 445 (1961); J. Nagumo *et al.*, Proc. IRE **50**, 2061 (1962); C. Rocsoreanu *et al.*, *The FitzHugh-Nagumo Model: Bifurcation and Dynamics* (Kluwer Academic, Boston, 2000).

[10] Although the recovery variable w cannot be experimentally measured in real applications, it can be estimated through the fast variable v by several online estimation methods, such as the adaptive synchronization techniques recently developed in the literature. Therefore, the feedback that uses the signal of w can still be realized in our approach.

[11] K. S. McCann, Nature **405**, 228 (2000); D. L. DeAngelis and J. C. Waterhouse, Ecol. Monogr. **57**, 1 (1987).