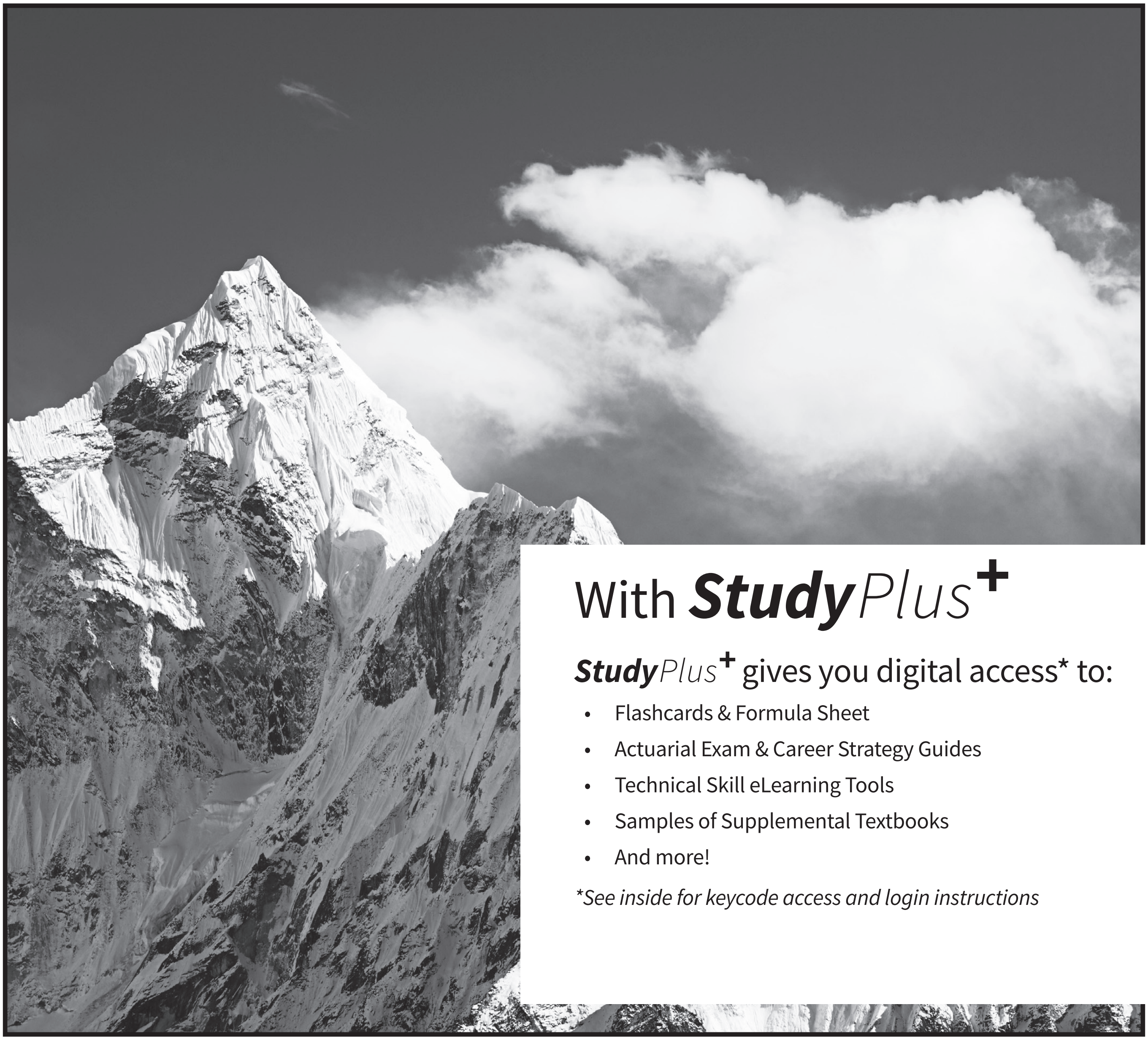
ACTEX

SOA Exam P Study Manual



With *Study*Plus+

***Study**Plus*+ gives you digital access* to:

- Flashcards & Formula Sheet
- Actuarial Exam & Career Strategy Guides
- Technical Skill eLearning Tools
- Samples of Supplemental Textbooks
- And more!

**See inside for keycode access and login instructions*

Spring 2018 Edition, Second Printing

Samuel A. Broverman, Ph.D., ASA

ACTEX Learning | Learn Today. Lead Tomorrow.

SOA Exam P Study Manual

Spring 2018 Edition, Second Printing

Samuel A. Broverman, Ph.D., ASA

ACTEX Learning
New Hartford, Connecticut

ACTEX Learning
Learn Today. Lead Tomorrow.

Actuarial & Financial Risk Resource Materials
Since 1972

ISBN: 978-1-63588-447-0

Printed in the United States of America.

YOUR OPINION IS IMPORTANT TO US

ACTEX is eager to provide you with helpful study material to assist you in gaining the necessary knowledge to become a successful actuary. In turn we would like your help in evaluating our manuals so we can help you meet that end. We invite you to provide us with a critique of this manual by sending this form to us at your convenience. We appreciate your time and value your input.

Publication:

ACTEX P Study Manual, Spring 2018 Edition, Second Printing

I found Actex by: (Check one)

☐ A Professor ☐ School/Internship Program ☐ Employer ☐ Friend ☐ Facebook/Twitter

In preparing for my exam I found this manual: (Check one)

☐ Very Good ☐ Good ☐ Satisfactory ☐ Unsatisfactory

I found the following helpful:

____

I found the following problems: (Please be specific as to area, i.e., section, specific item, and/or page number.)

____

To improve this manual I would:

____

Name: ____________________

Address: ____________________

Phone: ____________________ E-mail: ____________________

(Please provide this information in case clarification is needed.)

Send to: Stephen Camilli
ACTEX Learning
P.O. Box 715
New Hartford, CT 06057

Or visit our website at www.ActexMadRiver.com to complete the survey on-line. Click on the "Send Us Feedback" link to access the online version. You can also e-mail your comments to Support@ActexMadRiver.com.

NOT FOR PRINT: ADD KEYCODE
TEMPLATE PAGE HERE

NOT FOR PRINT: ADD KEYCODE TEMPLATE PAGE HERE

TABLE OF CONTENTS

SECTION 6 - FREQUENTLY USED DISCRETE DISTRIBUTIONS

SECTION 7 - FREQUENTLY USED CONTINUOUS DISTRIBUTIONS

SECTION 8 - JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS

SECTION 9 - TRANSFORMATIONS OF RANDOM VARIABLES

SECTION 10 - RISK MANAGEMENT CONCEPTS

INTRODUCTORY COMMENTS

This study guide is designed to help in the preparation for the Society of Actuaries Exam P. The study manual is divided into two main parts. The first part consists of a summary of notes and illustrative examples related to the material described in the exam catalog as well as a series of problem sets and detailed solutions related to each topic. Many of the examples and problems in the problem sets are taken from actual exams (and from the sample question list posted on the SOA website).

The second part of the study manual consists of ten practice exams, with detailed solutions, which are designed to cover the range of material that will be found on the exam. The questions on these practice exams are not from old Society exams and may be somewhat more challenging, on average, than questions from previous actual exams. Between the section of notes and the section with practice exams I have included the normal distribution table provided with the exam.

I have attempted to be thorough in the coverage of the topics upon which the exam is based. I have been, perhaps, more thorough than necessary on a couple of topics, particularly order statistics in Section 9 of the notes and some risk management topics in Section 10 of the notes.

Section 0 of the notes provides a brief review of a few important topics in calculus and algebra. This manual will be most effective, however, for those who have had courses in college calculus at least to the sophomore level and courses in probability to the sophomore or junior level.

If you are taking the Exam P for the first time, be aware that a most crucial aspect of the exam is the limited time given to take the exam (3 hours). It is important to be able to work very quickly and accurately. Continual drill on important concepts and formulas by working through many problems will be helpful. It is also very important to be disciplined enough while taking the exam so that an inordinate amount of time is not spent on any one question. If the formulas and reasoning that will be needed to solve a particular question are not clear within 2 or 3 minutes of starting the question, it should be abandoned (and returned to later if time permits). Using the exams in the second part of this study manual and simulating exam conditions will also help give you a feeling for the actual exam experience.

If you have any comments, criticisms or compliments regarding this study guide, please contact the publisher, ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you bring them to my attention. Any errors that are found will be posted in an errata file at the ACTEX website, www.actexmadriver.com .

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

Samuel A. Broverman
Department of Statistics
University of Toronto
www.sambroverman.com
E-mail: 2brove@rogers.com or sam@utstat.toronto.edu
www.brovermusic.com

April 2018

NOTES, EXAMPLES

AND PROBLEM SETS

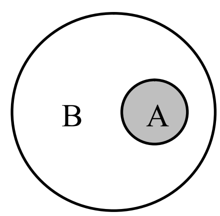
SECTION 0 - REVIEW OF ALGEBRA AND CALCULUS

In this introductory section, a few important concepts that are preliminary to probability topics will be reviewed. The concepts reviewed are set theory, graphing an inequality in two dimensions, properties of functions, differentiation, integration and geometric series. Students with a strong background in calculus who are familiar with these concepts can skip this section.

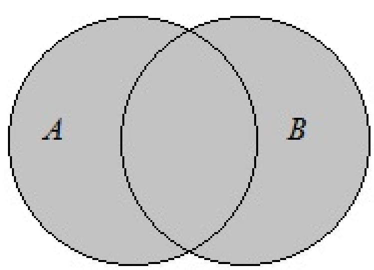
SET THEORY

A **set** is a collection of **elements**. The phrase **"$\boldsymbol{x}$ is an element of $\boldsymbol{A}$"** is denoted by $\boldsymbol{x \in A}$, and **"$\boldsymbol{x}$ is not an element of $\boldsymbol{A}$"** is denoted by $\boldsymbol{x \notin A}$.

Subset of a set: $\boldsymbol{A \subset B}$ means that each element of the set A is an element of the set B.
B may contain elements which are not in A, but A is totally contained within B. For instance, if A is the set of all odd, positive integers, and B is the set of all positive integers, then
$A = \{1\,,\,3\,,\,5\,,\,\ldots\}$ and $B = \{1\,,\,2\,,\,3\,,\,\ldots\}$.
The notation $A \subseteq B$ also denotes that A is a subset of B but that A may be equal to B.
For these two sets it is easy to see that $A \subset B$, since any member of A (any odd positive integer) is a member of B (is a positive integer). The Venn diagram below illustrates A as a subset of B.



Union of sets: $\boldsymbol{A \cup B}$ is the set of all elements in either A or B (or both).
$\boldsymbol{A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}}$



$A \cup B$

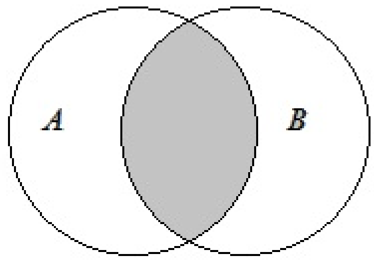
If A is the set of all positive even integers ($A = \{2\,,\,4\,,\,6\,,\,8\,,10\,,\,12\,,\,\ldots\}$) and B is the set of all positive integers which are multiples of 3 ($B = \{3\,,\,6\,,\,9\,,\,12\,,\,\ldots\}$) , then

$$A \cup B \;=\; \{2\,,\,3\,,\,4\,,\,6\,,\,8\,,\,9\,,\,10\,,\,12\,,\,\ldots\}$$

is the set of positive integers which are either multiples of 2 or are multiples of 3 (or both).

Intersection of sets: $\boldsymbol{A \cap B}$ is the set of all elements that are in both A and B.

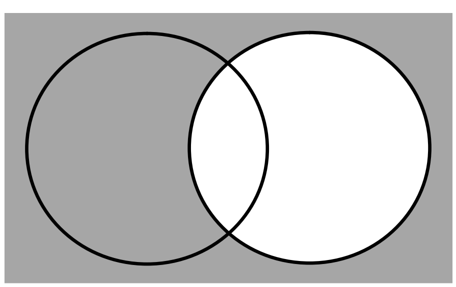
$\boldsymbol{A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}}$



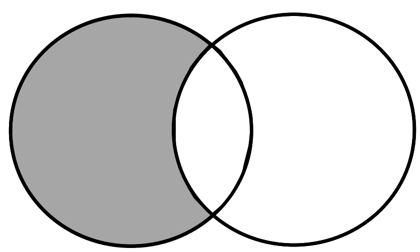
$A \cap B$

If A is the set of all positive even integers and B is the set of all positive integers which are a multiple of 3, then $A \cap B = \{6\ ,\ 12\ ,\ \ldots\}$ is the set of positive integers which are a multiple of 6. The elements of $A \cap B$ must satisfy the properties of **both** A and B. In this example, that means an element of $A \cap B$ must be a multiple of 2 and must also be a multiple of 3, and therefore must be a multiple of 6.

The complement of the set $\boldsymbol{B}$: The complement of B consists of all elements **not in $\boldsymbol{B}$**, and is denoted $\boldsymbol{B'}$, $\boldsymbol{\overline{B}}$ or $\boldsymbol{\sim B}$. $\boldsymbol{B' = \{x \,|\, x \notin B\}}$. When referring to the complement of a set, it is usually understood that there is some "full set", and the complement of B consists of the elements of the full set which are not in B. For instance, if B is the set of all positive even integers, and if the "full set" is the set of all positive integers, then B' consists of all positive odd integers. The set difference of "set A minus B" is $\boldsymbol{A - B = \{x \,|\, x \in A \textit{ and } x \notin B\}}$ and consists of all elements that are in A but not in B. Note that $\boldsymbol{A - B = A \cap B'}$. $A - B$ can also be described as the set that results when the intersection $A \cap B$ is removed from A.



$B' = \overline{B}$



$A - B = A \cap B'$

Example 0-1:

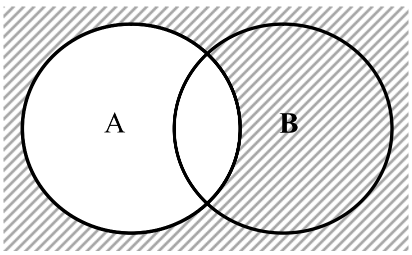
Verify the following set relationships (DeMorgan's Laws):

(i) $(A \cup B)' = A' \cap B'$ (the complement of the union of A and B is the intersection of the complements of A and B)

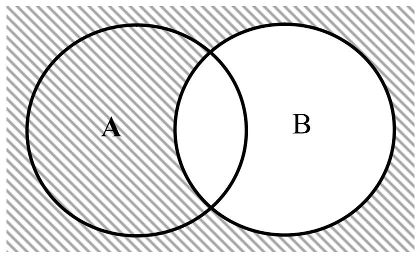
(ii) $(A \cap B)' = A' \cup B'$ (the complement of the intersection of A and B is the union of the complements of A and B)

Solution:

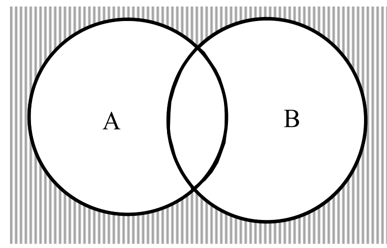
(i) Since the union of A and B consists of all points in either A **or** B, any point not in $A \cup B$ is in neither A nor B, and therefore must be in both the complement of A **and** the complement of B; this is the intersection of A' and B'. The reverse implication holds in a similar way; if a point is in the intersection of A' and B' then it is not in A **and** it is not in B, so it is not in $A \cup B$, and therefore it is in $(A \cup B)'$. Therefore, $(A \cup B)'$ and $A' \cap B'$ consist of the same collection of points, they are the same set.



A'

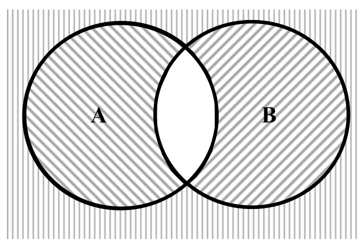


B'



$(A \cup B)' = A' \cap B'$

(ii) The solution is very similar to (i).



$(A \cap B)' = A' \cup B'$

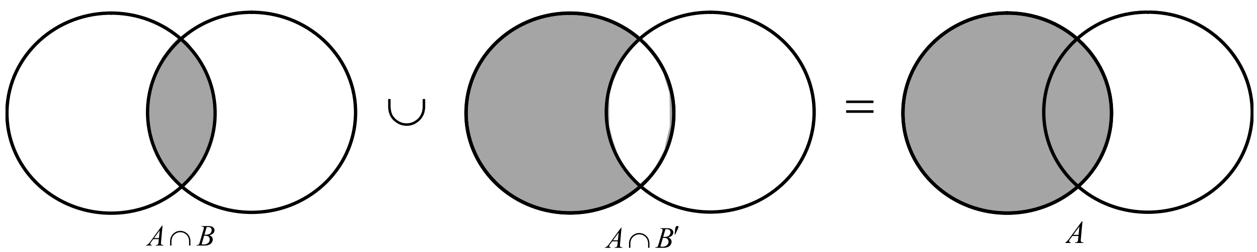
□

Empty set: The **empty set** is the set that contains no elements, and is denoted $\emptyset$. It is also referred to as the **null set**. Sets A and B are called **disjoint sets** if $A \cap B = \emptyset$ (they have no elements in common).

Relationships involving sets:

1. $A \cup B = B \cup A$; $A \cap B = B \cap A$; $A \cup A = A$; $A \cap A = A$
2. If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$ (this can be seen from the Venn diagram in the paragraph above describing subset)
3. For any set A, $\phi \subset A$ (the empty set is a subset of any other set A)
4. $A \cup \phi = A$; $A \cap \phi = \phi$; $A - \phi = A$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7. For any sets A and B, $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$
8. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

An important rule (that follows from point 4 above) is the following.
For any two sets A and B, we have $A = (A \cap B) \cup (A \cap B')$ **.**



Related to this is the property that if a finite set is made up of the union of disjoint sets, then the number of elements in the union is the sum of the numbers in each of the component sets.
For a finite set S, we define $n(S)$ to be the number of elements in S.
Two useful relationships for counting elements in a set are $n(A) = n(A \cap B) + n(A \cap B')$ (true since $A \cap B$ and $A \cap B'$ are disjoint), and
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ **(cancels the double counting of $A \cap B$).**

This rule can be extended to three sets,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) .$$

The main application of set algebra is in a probability context in which we use set algebra to describe events and combinations of events (this appears in the next section of this study guide). An understanding of set algebra and Venn diagram representations can be quite helpful in describing and finding event probabilities.

Example 0-2:

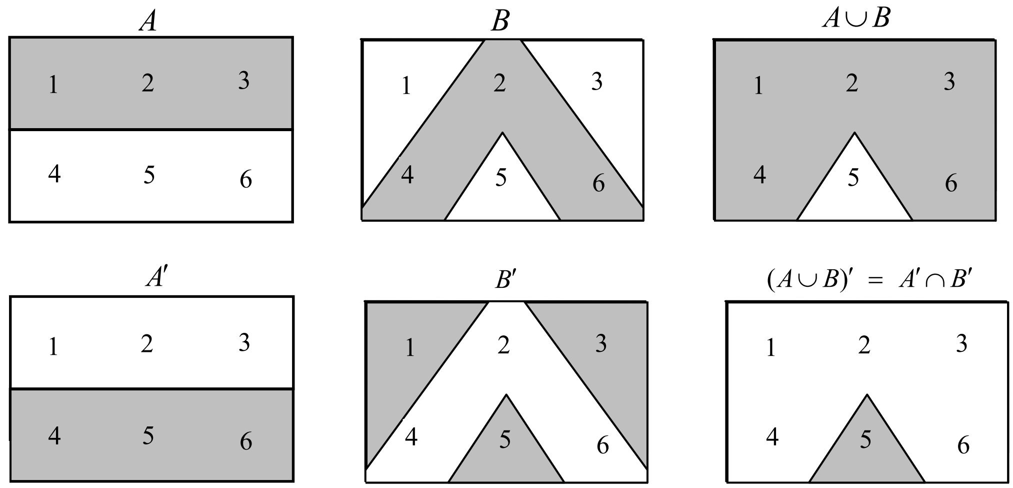
Suppose that the "total set" S consists of the possible outcomes that can occur when tossing a six-faced die. Then $S = \{1, 2, 3, 4, 5, 6\}$. We define the following subsets of S:

$A = \{1, 2, 3\}$ (a number less than 4 is tossed) ,
$B = \{2, 4, 6\}$ (an even number is tossed) ,
$C = \{4\}$ (a 4 is tossed) .

Then $A \cup B = \{1, 2, 3, 4, 6\}$; $A \cap B = \{2\}$;
A and C are disjoint since $A \cap C = \emptyset$; $C \subset B$;
$A' = \{4, 5, 6\}$ (complement of A); $B' = \{1, 3, 5\}$; $A \cup B = \{1, 2, 3, 4, 6\}$; and
$(A \cup B)' = \{5\} = A' \cap B'$ (this illustrates one of DeMorgan's Laws).

This is illustrated in the following Venn diagrams with sets identified by shaded regions.

Example 0-2 continued:



□

Venn diagrams can sometimes be useful when analyzing the combinations of intersections and unions of sets and the numbers of elements in various. The following examples illustrates this.

Example 0-3:

A heart disease researcher has gathered data on 40,000 people who have suffered heart attacks. The researcher identifies three variables associated with heart attack victims:

A - smoker, B - heavy drinker , C - sedentary lifestyle .

The following data on the 40,000 victims has been gathered:

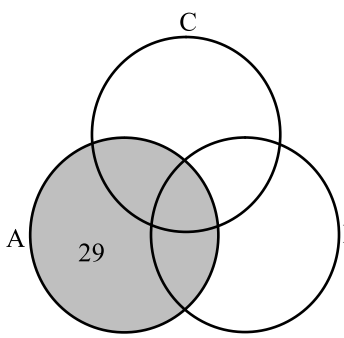
29,000 were smokers ; 25,000 were heavy drinker ; 30,000 had a sedentary lifestyle ;
22,000 were both smokers and heavy drinkers ;
24,000 were both smokers and had a sedentary lifestyle ;
20,000 were both heavy drinkers and had a sedentary lifestyle ; and
20,000 were smokers, and heavy drinkers and had a sedentary lifestyle.

Determine how many victims were:

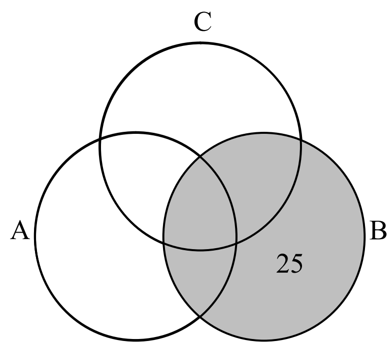
(i) neither smokers, nor heavy drinkers, nor had a sedentary lifestyle;
(ii) smokers but not heavy drinkers;
(iii) smokers but not heavy drinkers and did not have a sedentary lifestyle?
(iv) either smokers or heavy drinkers (or both) but did not have a sedentary lifestyle?

Solution:

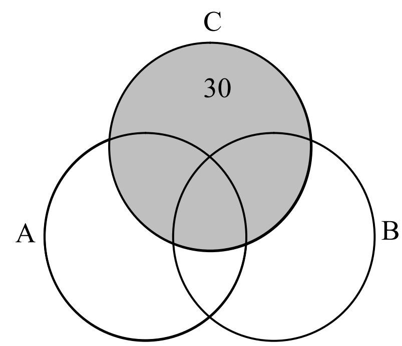
It is convenient to represent the data in Venn diagram form. For a subset S, $n(S)$ denotes the number of elements in that set (in thousands). The given information can be summarized in Venn diagram form as follows:



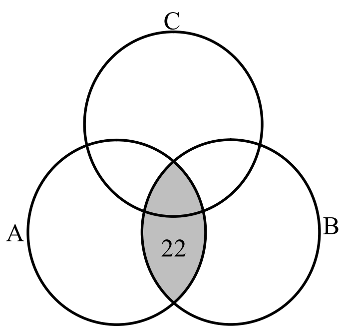
$n(A) = 29{,}000$ (smoker)



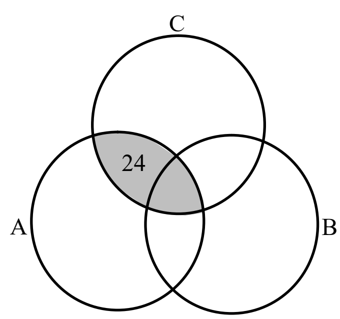
$n(B) = 25,000$ (heavy drinker)



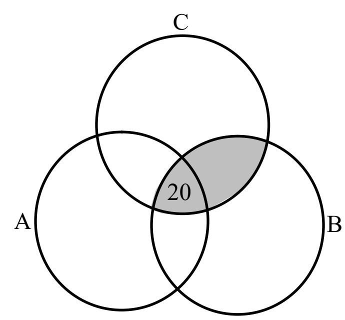
$n(C) = 30,000$ (sedentary lifestyle)



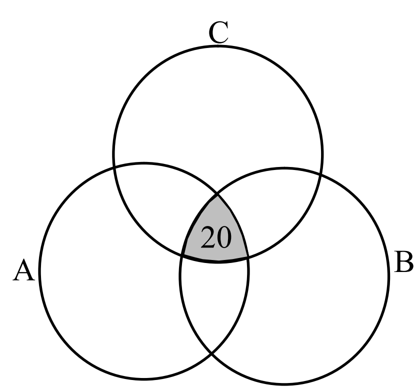
$n(A \cap B) = 22,000$
(smoker and heavy drinker)



$n(A \cap C) = 24,000$
(smoker and sedentary lifestyle)



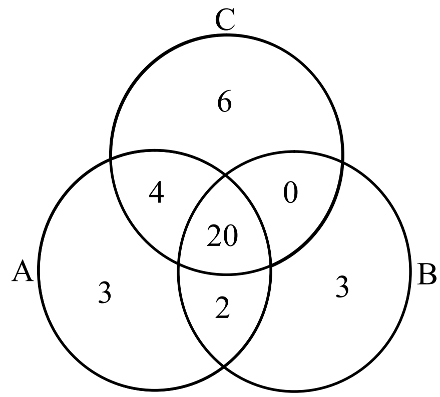
$n(B \cap C) = 20,000$
(heavy drinker and sedentary lifestyle)



$(A \cap B \cap C) = 20,000$
(smoker and heavy drinker and sedentary lifestyle)

Example 0-3 continued:

Working from the inside outward in the Venn diagrams, we can identify the number within each minimal subset of all of the intersections:



A typical calculation to fill in this diagram is as follows. We are given $n(A \cap B \cap C) = 20,000$ and $n(A \cap B) = 22,000$; we use the relationship

$$22,000 = n(A \cap B) = n(A \cap B \cap C) + n(A \cap B \cap C') = 20,000 + n(A \cap B \cap C')$$

to get $n(A \cap B \cap C') = 2,000$ (this shows that the 22,000 victims in $A \cap B$ who are both smokers and heavy drinker can be subdivided into those who also have a sedentary lifestyle $n(A \cap B \cap C) = 20,000$, and those who do not have a sedentary lifestyle, $n(A \cap B \cap C')$, the other $= 2,000$). Other entries are found in a similar way. From the diagram we can gain additional insight into other combinations of subsets. For instance, 6,000 of the victims have a sedentary lifestyle, but are neither smokers nor heavy drinkers; this is the entry "6", which in set notation is $n(A' \cap B' \cap C) = 6,000$. Also, the number of victims who were both heavy drinkers and had a sedentary lifestyle but were not smokers is 0.

We can now find the requested numbers.

(i) The number of victims who had at least one of the three specified conditions is $n(A \cup B \cup C)$, which, from the diagram can be calculated from the disjoint components:

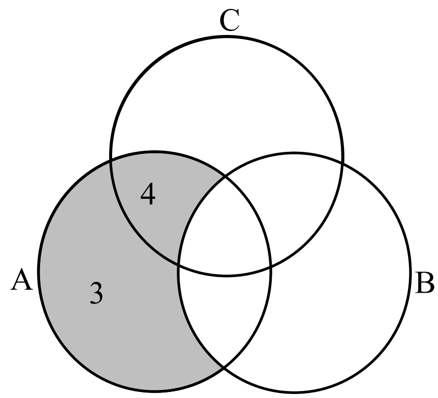
$$n(A \cup B \cup C) = 20,000 + 2,000 + 4,000 + 0 + 3,000 + 3,000 + 6,000 = 38,000.$$

The "total set" in this example is the set of all 40,000 victims. Therefore, there were 2,000 heart attack victims who had none of the three specified conditions; this is the complement of $n(A \cup B \cup C)$.

Algebraically, we have used the extension of one of DeMorgan's laws to the case of three sets, "none of A or B or C" $= (A \cup B \cup C)' = A' \cap B' \cap C' =$ "not A" and "not B" and "not C" .

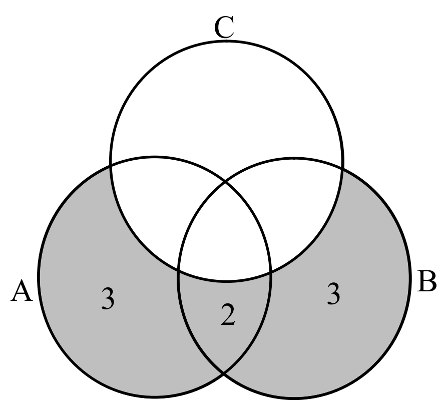
Example 0-3 continued:

(ii) The number of victims who were smokers but not heavy drinkers is $n(A \cap B') = 3,000 + 4,000$. This can be seen from the following Venn diagram



(iii) The number of victims who were smokers but not heavy drinkers and did not have a sedentary lifestyle is $n(A \cap B' \cap C') = 3,000$ (part of the group in (ii)).

(iv) The number of victims who were either smokers or heavy drinkers (or both) but did not have a sedentary lifestyle is $n[(A \cup B) \cap C']$. This is illustrated in the following Venn diagram.



$n[(A \cup B) \cap C'] = 3,000 + 2,000 + 3,000 = 8,000$. □

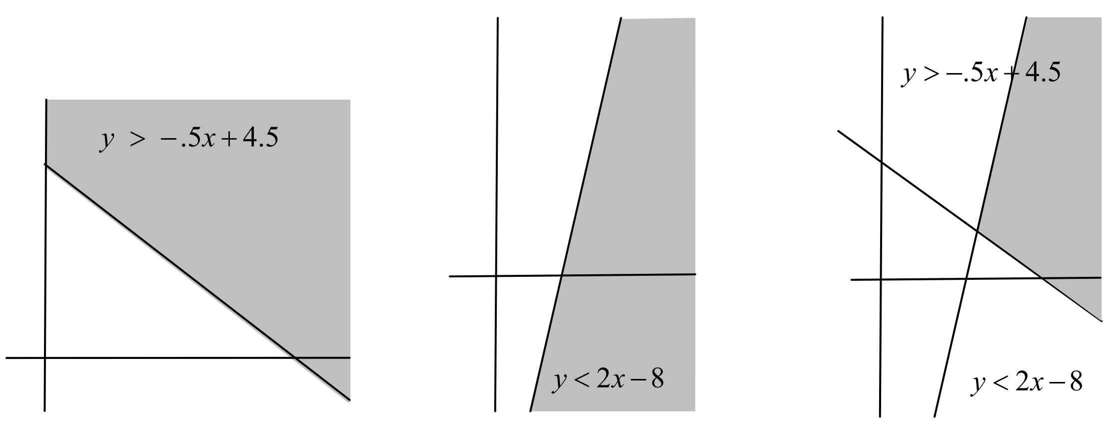
GRAPHING AN INEQUALITY IN TWO DIMENSIONS

The joint distribution of a pair of random variables X and Y is sometimes defined over a two dimensional region which is described in terms of linear inequalities involving x and y. The region represented by the inequality $y > ax + b$ is the region above the line $y = ax + b$ (and $y < ax + b$ is the region below the line).

Example 0-4: Using the lines $y = -\frac{1}{2}x + \frac{9}{2}$ and $y = 2x - 8$, find the region in the x-y plane that satisfies both of the inequalities $y > -\frac{1}{2}x + \frac{9}{2}$ and $y < 2x - 8$.

Solution:
We graph each of the straight lines, and then determine which side of the line is represented by the inequality. The first graph below is the graph of the line $y = -\frac{1}{2}x + \frac{9}{2}$, along with the shaded region, which is the region $y > -\frac{1}{2}x + \frac{9}{2}$, consisting of all points "above" that line. The second graph below is the graph of the line $y = 2x - 8$, along with the shaded region, which is the region $y < 2x - 8$, consisting of all points "below" that line.

The third graph is the intersection (first region and second region) of the two regions. Although the boundary lines of the regions in the graphs are solid lines, the ineqaulities are strict inequalities.



□

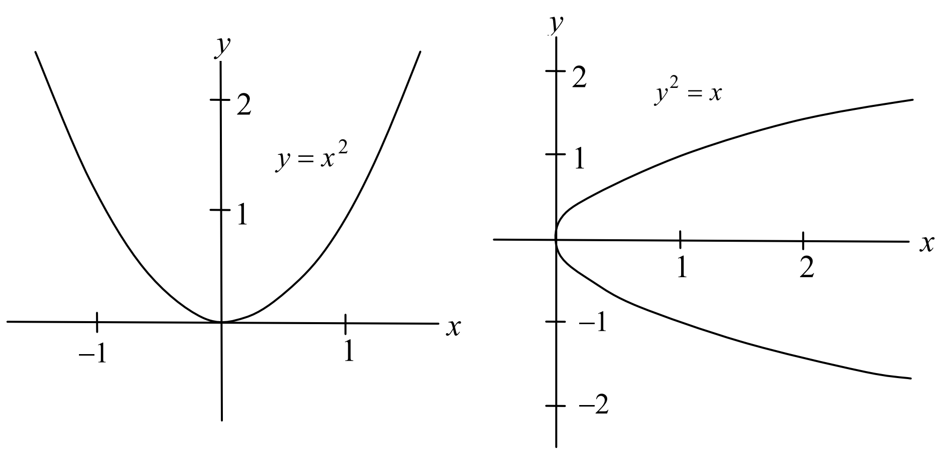
PROPERTIES OF FUNCTIONS

Definition of a function f: A function $f(x)$ is defined on a subset (or the entire set) of real numbers. For each x, the function defines a number $f(x)$. **The domain** of the function f is the set of x-values for which the function is defined. The **range of f** is the set of all $f(x)$ values that can occur for x's in the domain. Functions can be defined in a more general way, but we will be concerned only with real valued functions of real numbers. Any relationship between two real variables (say x and y) can be represented by its graph in the (x, y)-plane. If the function $y = f(x)$ is graphed, then for any x in the domain of f, the vertical line at x will intersect the graph of the function at exactly one point; this can also be described by saying that for each value of x there is (at most) one related value of y.

Example 0-5:

(i) $y = x^2$ defines a function since for each x there is exactly one value x^2. The domain of the function is all real numbers (each real number has a square). The range of the function is all real numbers ≥ 0, since for any real x, the square is $x^2 \geq 0$.

(ii) $y^2 = x$ does not define a function since if $x > 0$, there are two values of y for which $y^2 = x$. These two values are $\pm\sqrt{x}$. This is illustrated in the graphs below



□

Functions defined piecewise:

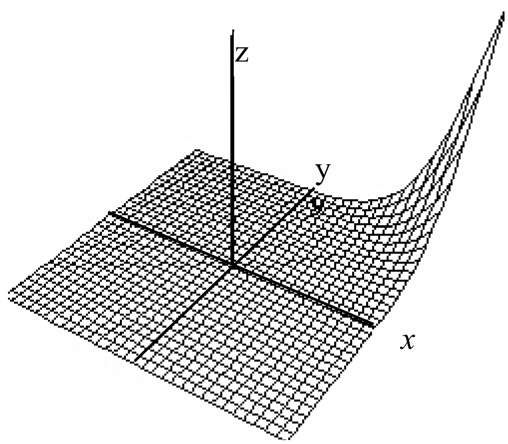
A function that is defined in different ways on separate intervals is called a **piecewise defined function**. The absolute value function is an example of a piecewise defined function:

$$|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}.$$

Multivariate function: A function of more than one variable is called a multivariate function.

Example 0-6:

$z = f(x, y) = e^{x+y}$ is a function of two variables, the domain is the entire 2-dimensional plane (the set $\{(x, y) |\ x, y$ are both real numbers$\}$) , and the range is the set of strictly positive real numbers. The function could be graphed in 3-dimensional x-y-z space. The domain would be the (horizontal) x-y plane, and the range would be the (vertical) z-dimension.
The 3-dimensional graph is shown below.



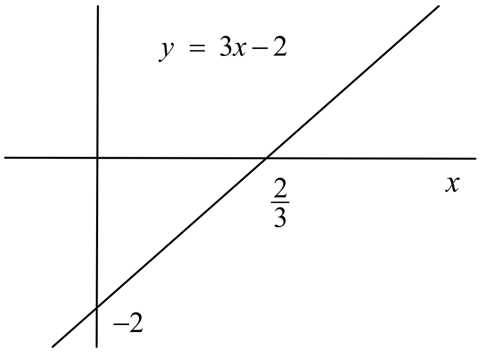
□

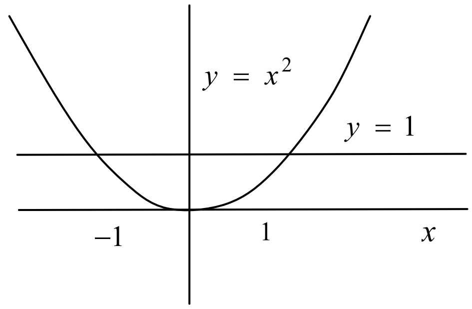
The concept of the inverse of a function is important when formulating the distribution of a transformed random variable. A preliminary concept related to the inverse of a function is that of a one-to-one function.

One-to-one function: The function f is called a one-to-one if the equation $f(x) = y$ has at most one solution x for each y (or equivalently, different x-values result in different $f(x)$ values). If a graph is drawn of a one-to-one function, any horizontal line crosses the graph in at most one place.

Example 0-7:

The function $f(x) = 3x - 2$ is one-to-one, since for each value of y, the relation $y = 3x - 2$ has exactly one solution for x in terms of y; $x = \frac{y+2}{3}$. The function $g(x) = x^2$ with the whole set of real numbers as its domain is not one-to-one, since for each $y > 0$, there are two solutions for x in terms of y for the relation $y = x^2$ (those two solutions are $x = \sqrt{y}$ and $x = -\sqrt{y}$; note that if we restrict the domain of $g(x) = x^2$ to the positive real numbers, it becomes a one-to-one function). The graphs are below.





□

Inverse of function f: The inverse of the function f is denoted f^{-1}. The inverse exists only if f is one-to-one, in which case, $f^{-1}(y) = x$ is the (unique) value of x which satisfies $f(x) = y$ (finding the inverse of $y = f(x)$ means that we solve for x in terms of y, $x = f^{-1}(y)$). For instance, for the function $y = 2x^3 = f(x)$, if $x = 1$ then $y = f(1) = 2(1^3) = 2$, so that $1 = f^{-1}(2) = (2/2)^{1/3}$. For the example just considered, the inverse function applied to $y = 2$ is the value of x for which $f(x) = 2$, or equivalently, $2x^3 = 2$, from which we get $x = 1$.

Example 0-8:

(i) The inverse of the function $y = 5x - 1 = f(x)$ is the function $x = \frac{y+1}{5} = f^{-1}(y)$ (we solve for x in terms of y).

(ii) Given the function $y = x^2 = f(x)$, solving for x in terms of y results in $x = \pm\sqrt{y}$, so there are two possible values of x for each value of y; this function does not have an inverse. However, if the function is defined to be $y = x^2 = f(x)$ **for $x \geq 0$ only**, then $x = +\sqrt{y} = f^{-1}(y)$ would be the inverse function, since f is one-to-one on its domain which consists of non-negative numbers. □

Quadratic functions and equations:

A quadratic function is of the form $p(x) = ax^2 + bx + c$.

The roots of the quadratic equation $ax^2 + bx + c = 0$ are $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

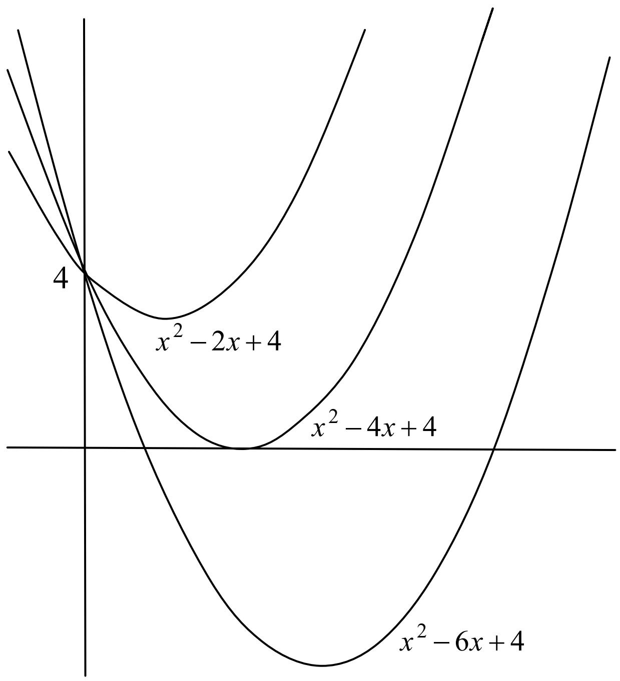
The quadratic equation has:

(i) distinct real roots if $b^2 - 4ac > 0$,

(ii) distinct complex roots if $b^2 - 4ac < 0$, and

(iii) equal real roots if $b^2 - 4ac = 0$.

Example 0-9:
The quadratic equation $x^2 - 6x + 4 = 0$ has two distinct real solutions: $x = 3 \pm \sqrt{5}$. The quadratic equation $x^2 - 4x + 4 = 0$ has both roots equal: $x = 2$.
The quadratic equation $x^2 - 2x + 4 = 0$ has two distinct complex roots: $x = 1 \pm i\sqrt{3}$.



□

Exponential and logarithmic functions: Exponential functions are of the form $f(x) = b^x$, where $b > 0,\ b \neq 1$, and the inverse of this function is denoted $log_b(y)$.
Thus $y = b^x \Leftrightarrow log_b(y) = x$. The log function with base e is the **natural logarithm**, $log_e(y) = ln\,y$
Some important properties of these functions are:

$b^0 = 1$	$log_b(1) = 0$
$domain(f) = \mathbb{R} = range(f^{-1})$	$range(f) = (0, +\infty) = domain(f^{-1})$
$b^{log_b(y)} = y$ for $y > 0$	$log_b(b^x) = x$ for all x
$b^x = e^{x \cdot ln\,b}$	$log_b(y) = \frac{ln\,y}{ln\,b}$
$(b^x)^y = b^{xy}$	$log_b(y^k) = k \cdot log_b(y)$
$b^x b^y = b^{x+y}$	$log_b(yz) = log_b(y) + log_b(z)$
$b^x / b^y = b^{x-y}$	$log_b(y/z) = log_b(y) - log_b(z)$

For the function e^x, we have $e^{ln\,y} = y$ for an $y > 0$, and for the natural log function, we have $ln\,e^x = x$ for any real number x.

LIMITS AND CONTINUITY

Intuitive definition of limit: The expression $\lim_{x \to c} f(x) = L$ means that as x gets close to (approaches) the number c, the value of $f(x)$ gets close to L.

Example 0-10:

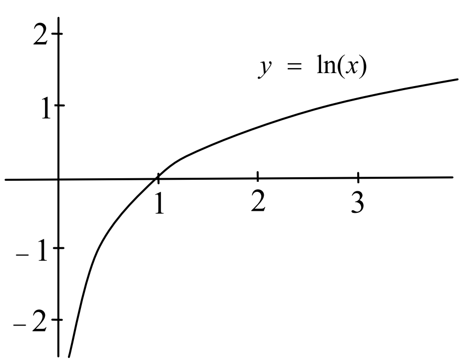
$\lim_{x \to 1}(x+3) = 4$, $\lim_{x \to +\infty} e^{-x} = 0$ and $\lim_{x \to 1} \frac{x^2+2x-3}{x-1} = \lim_{x \to 1}(x+3) = 4$ (for this last limit, note that $\frac{x^2+2x-3}{x-1} = \frac{(x+3)(x-1)}{x-1} = x+3$ if $x \neq 1$, but in taking this limit we are only concerned with what happens "near" $x = 1$, that fact that $\frac{x^2+2x-3}{x-1} = \frac{0}{0}$ at $x = 1$ does not mean that the limit does not exist; it means that the function does not exist at the point $x = 1$). □

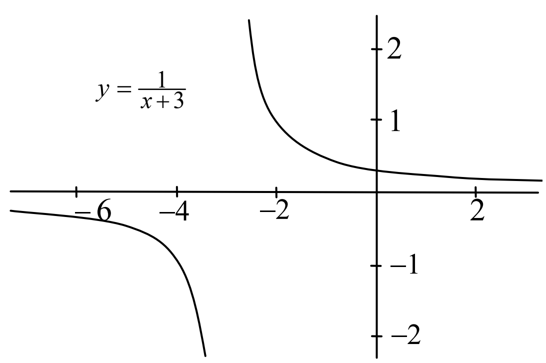
Continuity: The function f is continuous at the point $x = c$ if there is no "break" or "hole" in the graph of $y = f(x)$, or equivalently, if $\boldsymbol{\lim_{x \to c} f(x) = f(c)}$. In Example 0-10 above, the third function is not continuous at $x = 1$ because $f(1) = \frac{0}{0}$ is not defined. Another reason for a discontinuity in $f(x)$ occurring at $x = c$ is that the limit of $f(x)$ is different from the left than it is from the right.

Example 0-11:

(i) If $f(x) = ln\, x$ and $c = 0$ then f is discontinuous at $c = 0$ since the function $ln\, x$ is not defined at the point $x = 0$ (this would also be the case for the function $f(x) = \frac{1}{x+3}$ and $c = -3$).

(ii) If $f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$, then $f(x)$ is discontinuous at $x = 0$ since even though $f(0)$ is defined, $\lim_{x \to 0} f(x) \neq f(0)$ ($\lim_{x \to 0} f(x)$ doesn't exist).





□

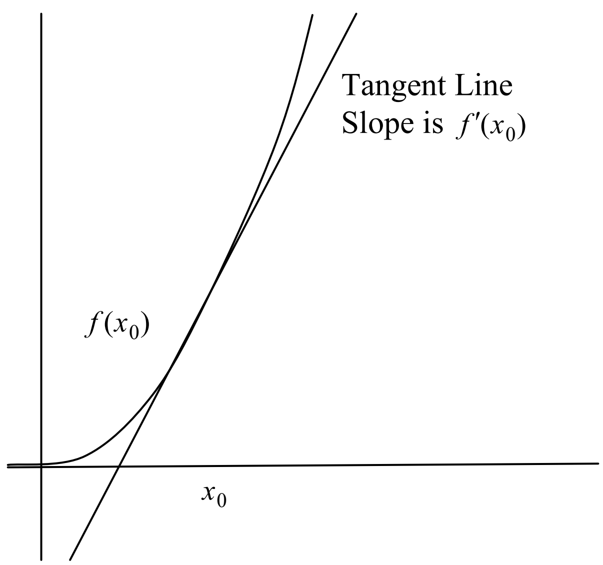
DIFFERENTATION

Geometric interpretation of derivative: The derivative of the function $f(x)$ at the point $x = x_0$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$. The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$.

This is also referred to as the derivative of f with respect to x at the point $x = x_0$.

The algebraic definition of $f'(x_0)$ is

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$



The second derivative of f at x_0 is the derivative of $f'(x)$ at the point x_0. It is denoted $f''(x_0)$ or $f^{(2)}(x_0)$ or $\left.\frac{d^2f}{dx^2}\right|_{x=x_0}$. The n-th order derivative of f at x_0 (n repeated applications of differentiation) is denoted $f^{(n)}(x_0) = \left.\frac{d^nf}{dx^n}\right|_{x=x_0}$.

The derivative as a rate of change: Perhaps the most important interpretation of the derivative $f'(x_0)$ is as the **"instantaneous" rate at which the function is increasing or decreasing as x increases (if $f' > 0$, the graph of $y = f(x)$ is rising, with the tangent line to the graph having positive slope, and if $f' < 0$, the graph of $y = f(x)$ is falling)**, and if $f'(x_0) = 0$ then the tangent line at that point is horizontal (has slope 0). This interpretation is the one most commonly used when analyzing physical, economic or financial processes.

The following is a summary of some important differentiation rules.

Rules of differentiation:

	$f(x)$	$f'(x)$
	c (a constant)	0
Power rule -	cx^n $(n \in \mathbb{R})$	cnx^{n-1}
	$g(x) + h(x)$	$g'(x) + h'(x)$
Product rule -	$g(x) \cdot h(x)$	$g'(x) \cdot h(x) + g(x) \cdot h'(x)$
	$u(x)v(x)w(x)$	$u'vw + uv'w + uvw'$
Quotient rule -	$\frac{g(x)}{h(x)}$	$\frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$
Chain rule -	$g(h(x))$	$g'(h(x)) \cdot h'(x)$
	$e^{g(x)}$	$g'(x) \cdot e^{g(x)}$
	$ln(g(x))$	$\frac{g'(x)}{g(x)}$
	a^x $(a > 0)$	$a^x \, ln\, a$
	e^x	e^x
	$ln\, x$	$\frac{1}{x}$
	$log_b x$	$\frac{1}{x\, lnb}$
	$sin\, x$	$cos\, x$
	$cos\, x$	$-\, sin\, x$

Example 0-12:

What is the derivative of $f(x) = 4x(x^2+1)^3$?

Solution:

We apply the product rule and chain rule:

$$f(x) = g(x) \cdot h(x)\,,$$

where

$$g(x) = 4x\ ,\ h(x) = (x^2+1)^3\ ,\ g'(x) = 4\ ,\ h'(x) = 3(x^2+1)^2 \cdot 2x\ .$$

$$f'(x) = 4x \cdot 3(x^2+1)^2 \cdot 2x + 4(x^2+1)^3 = 4(x^2+1)^2(7x^2+1)\ .$$

Notice that $h(x) = (x^2+1)^3 = [w(x)]^3 = h(w(x))$, where $h(w) = w^3$ and $w(x) = x^2+1$.

The chain rule tells us that $h'(x) = h'(w) \cdot w'(x) = 3w^2 \cdot (2x) = 3(x^2+1)^2 \cdot (2x)$. □

L'Hospital's rules for calculating limits: A limit of the form $\lim_{x\to c}\frac{f(x)}{g(x)}$ is said to be in indeterminate form if both the numerator and denominator go to 0, or if both the numerator and denominator go to $\pm\infty$. L'Hospital's rules are:

1. **IF** $\begin{cases} \text{(i) } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f'(c) \text{ exists, } \textbf{and} \\ \text{(iii) } g'(c) \text{ exists and is } \neq 0 \end{cases}$ **THEN** $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$

2. **IF** $\begin{cases} \text{(i) } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f \text{ and } g \text{ are differentiable near } c, \textbf{ and} \\ \text{(iii) } \lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ exists} \end{cases}$ **THEN** $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

In 1 or 2, the conditions $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ can be replaced by the conditions $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, and the point c can be replaced by $\pm\infty$ with the conclusions remaining valid.

Example 0-13: Find $\lim_{x\to 2}\frac{3^{x/2}-3}{3^x-9}$.

Solution:

The limits in both the numerator and denominator are 0, so we apply l'Hospital's rule. $\frac{d}{dx}3^x = 3^x \ln 3$, and $\frac{d}{dx}3^{x/2} = 3^{x/2}\cdot\frac{1}{2}\ln 3$, so that $\lim_{x\to 2}\frac{3^{x/2}-3}{3^x-9} = \lim_{x\to 2}\frac{3^{x/2}\cdot\frac{1}{2}\ln 3}{3^x \ln 3} = \frac{1}{6}$. This limit can also be found by factoring the denominator into $3^x - 9 = (3^{x/2}-3)(3^{x/2}+3)$, and then canceling out the factor $3^{x/2}-3$ in the numerator and denominator. □

Differentiation of functions of several variables - partial differentiation:

Given the function $f(x,y)$, a function of two variables, the partial derivative of f with respect to x at the point (x_0, y_0) is found by differentiating f with respect to x and regarding the variable y as constant - then substitute in the values $x = x_0$ and $y = y_0$. The partial derivative of f with respect to x is usually denoted $\frac{\partial f}{\partial x}$. The partial derivative with respect to y is defined in a similar way: "Higher order" partial derivatives can be defined - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$; and "mixed partial" derivatives can be defined (the order of partial differentiation does not usually matter) -
$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$.

Example 0-14:

If $f(x,y) = x^y$ for $x, y > 0$ then find $\left.\frac{\partial f}{\partial x}\right|_{(4,\frac{1}{2})}$ and $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(4,\frac{1}{2})}$.

Solution:

$\frac{\partial f}{\partial x} = yx^{y-1}\Big|_{(4,\frac{1}{2})} = (\frac{1}{2})(4)^{-1/2} = \frac{1}{4}$, and

$\frac{\partial f}{\partial y} = x^y(\ln x)$ and $\frac{\partial^2 f}{\partial y^2} = x^y(\ln x)^2\Big|_{(4,\frac{1}{2})} = 4^{1/2}(\ln 4)^2 = 2(\ln 4)^2$. □

INTEGRATION

Geometric interpretation of the "definite integral" - the area under the curve:

Given a function $f(x)$ on the interval $[a, b]$, the definite integral of $f(x)$ over the interval is denoted $\int_a^b f(x)\,dx$, and is equal to the "signed" area between the graph of the function and the x-axis from $x = a$ to $x = b$. Signed area is positive when $f(x) > 0$ and is negative when $f(x) < 0$. What is meant by signed area here is the area from the interval(s) where $f(x)$ is positive minus the area from the intervals where $f(x)$ is negative.

Integration is related to the antiderivative of a function. Given a function $f(x)$, an antiderivative of $f(x)$ is any function $F(x)$ which satisfies the relationship $F'(x) = f(x)$. According to the Fundamental Theorem of Calculus, the definite integral for $f(x)$ can be found by first finding $F(x)$, an antiderivative of $f(x)$. The basic relationships relating integration and differentiation are:

(i) If $F'(x) = f(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

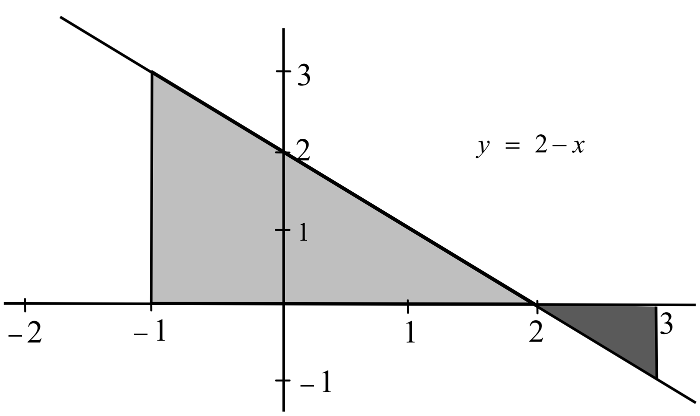
(ii) If $G(x) = \int_a^x g(t)\,dt$, then $G'(x) = g(x)$

Example 0-15:

Find the definite integral of the function $f(x) = 2 - x$ on the interval $[-1, 3]$.

Solution:

The graph of the function is given below. It is clear that $f(x) > 0$ for $x < 2$, and $f(x) < 0$ for $x > 2$. An antiderivative for $f(x)$ is $F(x) = 2x - \frac{x^2}{2}$. The definite integral will be $\int_{-1}^{3}(2 - x)\,dx = F(3) - F(-1) = (6 - \frac{3^2}{2}) - (-2 - \frac{(-1)^2}{2}) = 4$. Note that the area between the graph and the x-axis from $x = -1$ to $x = 2$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$, and the signed area between the graph and the x-axis from $x = 2$ to $x = 3$ is $-\frac{1}{2}(1)(1) = -\frac{1}{2}$. The total signed area is $\frac{9}{2} - \frac{1}{2} = 4$.



□

Antiderivatives of some frequently used functions:

$f(x)$	$\int f(x)dx$ (antiderivative)
$g(x)+h(x)$	$\int g(x)dx + \int h(x)dx + c$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}$	$ln\, x + c$
e^x	$e^x + c$
$a^x \ (a>0)$	$\frac{a^x}{ln\, a} + c$
xe^{ax}	$\frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + c$.
$sin\, x$	$-cos\, x + c$
$cos\, x$	$sin\, x + c$

Integration of f on $[a,b]$ when f is not defined at a or b, or when a or b is $\pm\infty$:

Integration over an infinite interval (an "improper integral") is defined by taking limits:
$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$, with a similar definition applying to $\int_{-\infty}^b f(x)\,dx$,
and $\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to+\infty}\int_{-a}^a f(x)\,dx$.

If f is not defined at $x=a$ (also called an improper integral), or if f is discontinuous at $x=a$, then
$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$.

A similar definition applies if f is not defined at $x=b$, or if f is discontinuous at $x=b$.

If $f(x)$ has a discontinuity at the point $x=c$ in the interior of $[a,b]$, then
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 0-16:

(a) $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-1/2}dx = \lim_{c\to 0^+}\left[2x^{1/2}\Big|_{x=c}^{x=1}\right] = \lim_{c\to 0^+}[2-2\sqrt{c}] = 2$,

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{c\to\infty}\int_1^c x^{-1/2}dx = \lim_{c\to\infty}\left[2x^{1/2}\Big|_{x=1}^{x=c}\right] = \lim_{c\to\infty}[2\sqrt{c}-2] = +\infty$.

(b) $\int_1^{+\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_{x=1}^{x=b}\right] = \lim_{b\to\infty}[-\frac{1}{b}-(-1)] = 1$

(c) $\int_{-\infty}^1 \frac{1}{x^2}\,dx$. Note that $\frac{1}{x^2}$ has a discontinuity at $x=0$, so that
$\int_{-\infty}^1 \frac{1}{x^2}\,dx = \int_{-\infty}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$. The second integral is
$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}[-1+\frac{1}{a}] = +\infty$, thus, the second improper integral
does not exist (when $\lim_{\to}\int$ is infinite or does not exist, the integral is said to "diverge"). □

A few other useful integration rules are:

(i) for integer $n \geq 0$ and real number $c > 0$ $\int_0^\infty x^n e^{-cx}\,dx = \frac{n!}{c^{n+1}}$.

(ii) if $G(x) = \int_a^{h(x)} f(u)\,du$, then $G'(x) = f[h(x)] \cdot h'(x)$,

(iii) if $G(x) = \int_x^b f(u)\,du$, then $G'(x) = -f(x)$,

(iv) if $G(x) = \int_{g(x)}^b f(u)\,du$, then $G'(x) = -f[g(x)] \cdot g'(x)$,

(v) if $G(x) = \int_{g(x)}^{h(x)} f(u)\,du$, then $G'(x) = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$.

Double integral: Given a continuous function of two variables, $f(x, y)$ on the rectangular region bounded by $x = a$, $x = b$, $y = c$ and $y = d$, it is possible to define the definite integral of f over the region. It can be expressed in one of two equivalent ways:

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

The interpretation of the first expression is $\int_a^b [\int_c^d f(x, y)\,dy]\,dx$, in which the "inside integral" is $\int_c^d f(x, y)\,dy$, and it is calculated assuming that the value of x is constant (it is an integral with respect to the variable y). When this definite "inside integral" has been calculated, it will be a function of x alone, which can then be integrated with respect to x from $x = a$ to $x = b$. The second equivalent expression has a similar interpretation; $\int_a^b f(x, y)\,dx$ is calculated assuming that y is constant; this results in a function of y alone which is then integrated with respect to y from $y = c$ to $y = d$. Double integration arises in the context of finding probabilities for a joint distribution of continuous random variables.

Example 0-17:

Find $\int_0^1 \int_1^2 \frac{x^2}{y}\,dy\,dx$.

Solution:

First we assume that x is constant and find $\int_1^2 \frac{x^2}{y}\,dy = x^2(\ln y)\Big|_{y=1}^{y=2} = x^2(\ln 2)$. Then we find

$$\int_0^1 [x^2(\ln 2)]\,dx = (\ln 2) \cdot \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{\ln 2}{3} .$$

We can also write the integral as $\int_1^2 \int_0^1 \frac{x^2}{y}\,dx\,dy$, and first find

$$\int_0^1 \frac{x^2}{y}\,dx = \frac{x^3}{3y}\Big|_{x=0}^{x=1} = \frac{1}{3y} . \text{ Then, } \int_1^2 \frac{1}{3y}\,dy = \frac{1}{3}(\ln y)\Big|_{y=1}^{y=2} = \frac{1}{3}(\ln 2) .$$ □

For double integration over the rectangular two-dimensional region $a \leq x \leq b$, $c \leq y \leq d$, as the expression $\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$ indicates, it is possible to calculate the double integral by integrating with respect to the variables in either order (y first and x second for the integral on the left, and x first and y second for the integral on the right of the " $=$ " sign).

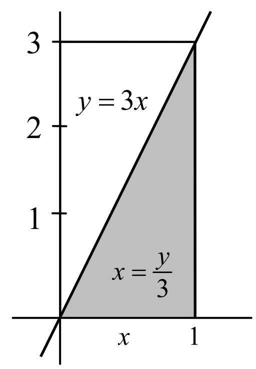
Formulations of probabilities and expectations for continuous joint distributions sometimes involve integrals over a non-rectangular two-dimensional region. It will still be possible to arrange the integral for integration in either order ($dy\,dx$ or $dx\,dy$), but care must be taken in setting up the limits of integration. If the limits of integration are properly specified, then the double integral will be the same whichever order of integration is used. Note also that in some situations, it may be more efficient to formulate the integration in one order than in the other.

Example 0-18:
Which of the following integrals is equal to $\int_0^1 \int_0^{3x} f(x,y)\,dy\,dx$
for every function for which the integral exists?

A) $\int_0^3 \int_0^{y/3} f(x,y)\,dx\,dy$ B) $\int_0^1 \int_{3x}^3 f(x,y)\,dx\,dy$ C) $\int_0^3 \int_{3y}^1 f(x,y)\,dx\,dy$

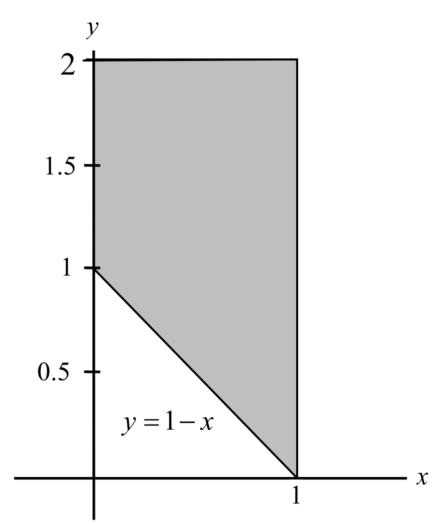
D) $\int_0^1 \int_0^{x/3} f(x,y)\,dx\,dy$ E) $\int_0^3 \int_{y/3}^1 f(x,y)\,dx\,dy$

Solution:
The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $0 \le y \le 3x$. Writing $y = 3x$ as $x = \frac{y}{3}$, we see that the inequalities translate into $0 \le y \le 3$, and $\frac{y}{3} \le x \le 1$. Answer: E □



Example 0-19: The function $f(x,y)$ is to be integrated over the two-dimensional region defined by the following constraints: $0 \le x \le 1$ and $1-x \le y \le 2$. Formulate the double integration in the $dy\,dx$ order and then in the $dx\,dy$ order.

Solution: The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $1-x \le y \le 2$. The integral can be formulated in the $dy\,dx$ order as $\int_0^1 \int_{1-x}^2 f(x.y)\,dy\,dx$; for each x, the integral in the vertical direction starts on the line $y = 1-x$ and continues to the upper boundary $y = 2$. To use the $dx\,dy$ order, we must split the integral into two double integrals; $\int_0^1 \int_{1-y}^1 f(x,y)\,dx\,dy$ to cover the triangular area below $y = 1$, and $\int_1^2 \int_0^1 f(x,y)$ $dx\,dy$ to cover the square area above $y = 1$. □



There are a few integration techniques that are useful to know. The integrations that arise on Exam P are usually straightforward, but knowing a few additional techniques of integration are sometimes useful in simplifying an integral in an efficient way.

The Method of Substitution: Substitution is a basic technique of integration that is used to rewrite the integral in a standard form for which the antiderivative is well known. In general, to find $\int f(x)\,dx$ we may make the substitution $u = g(x)$ for an "appropriate" function $g(x)$.
We then define the "differential" du to be $du = g'(x)\,dx$, and we try to rewrite $\int f(x)\,dx$ as an integral with respect to the variable u.

For example, to find $\int (x^3-1)^{4/3}x^2dx$, we let $u = x^3 - 1$, so that $du = 3x^2dx$, or equivalently, $\frac{1}{3}\cdot du = x^2dx$; then the integral can be written as $\int u^{4/3}\cdot\frac{1}{3}\,du$, which has antiderivative $\int u^{4/3}\cdot\frac{1}{3}\,du = \frac{1}{3}\cdot\int u^{4/3}du = \frac{1}{3}\cdot\frac{u^{7/3}}{7/3} = \frac{1}{7}u^{7/3}(+c)$.
We can then write the antiderivative in terms of the original variable x -
$\int (x^3-1)^{4/3}x^2dx = \frac{1}{7}u^{7/3} = \frac{1}{7}(x^3-1)^{7/3}$.

The main point to note in applying the substitution technique is that the choice of $u = g(x)$ should result in an antiderivative which is easier to find than was the original antiderivative.

Example 0-20:
Find $\int_0^1 x\sqrt{1-x^2}dx$.

Solution:
Let $u = 1 - x^2$ Then $du = -2x\,dx$, so that $-\frac{1}{2}\cdot du = x\,dx$, and the antiderivative can be written as $\int u^{1/2}\cdot(-\frac{1}{2})du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}$.
The definite integral is then $\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_{x=0}^{x=1} = -0-(-\frac{1}{3}) = \frac{1}{3}$.
Note that once the appropriate substitution has been made, the definite integral may be calculated in terms of the variable u: $u(0) = 1$ and $u(1) = 0$ -

$$\int_0^1 x\sqrt{1-x^2}dx = \int_{u(0)=1}^{u(1)=0} u^{1/2}\cdot(-\frac{1}{2})du = -\frac{1}{3}u^{3/2}\Big|_{u=1}^{u=0} = -0-(-\frac{1}{3}) = \frac{1}{3}\ .\ \square$$

Integration by parts:

This technique of integration is based upon the product rule
$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$. This can be rewritten as
$f(x) \cdot g'(x) = \frac{d}{dx}[f(x) \cdot g(x)] - f'(x) \cdot g(x)$, which means that the antiderivative of $f(x) \cdot g'(x)$ can be written as $\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx$.

This technique is useful if $f'(x) \cdot g(x)$ has an easier antiderivative to find than $f(x) \cdot g'(x)$. Given an integral, it may not be immediately apparent how to define $f(x)$ and $g(x)$ so that the integration by parts technique applies and results in a simplification. It may be necessary to apply integration by parts more than once to simplify an integral.

Example 0-21:

Find $\int xe^{ax}\, dx$, where a is a constant.

Solution:

If we define $f(x) = x$ and $g(x) = \frac{e^{ax}}{a}$, then $g'(x) = e^{ax}$, and
$\int xe^{ax}\, dx = \int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$.
Since $f'(x) = 1$, it follows that $\int f'(x)g(x)\, dx = \int \frac{e^{ax}}{a} dx = \frac{e^{ax}}{a^2}$, and therefore
$\boldsymbol{\int xe^{ax}\, dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + c}$ **.**

An alternative to integration by parts is the following approach:
$\boldsymbol{\frac{d}{da}\int e^{ax}\, dx = \int xe^{ax}\, dx}$ and $\boldsymbol{\frac{d}{da}\int e^{ax}\, dx = \frac{d}{da}\,\frac{e^{ax}}{a} = \frac{axe^{ax} - e^{ax}}{a^2}}$
so it follows that $\boldsymbol{\int xe^{ax}\, dx = \frac{axe^{ax} - e^{ax}}{a^2} = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}}$ **.**

This integral has appeared a number of times on the exam, usually with $a < 0$ (it is valid for any $a \neq 0$) and it is important to be familiar with it. □

An alternative way to apply integration by parts is via "Tabular Integration". Suppose that we wish to find the integral $\int u(x) \cdot v(x)\, dx$. We create two columns, a column of the successive derivatives of $u(x)$ and a column of the successive antiderivatives of $v(x)$. In order for the method to work efficiently, we try to choose $u(x)$ to be a polynomial which will eventually have a derivative of 0. The integral in Example 0-21 will be used to illustrate tabular integration. We make the following choices for $u(x)$ and $v(x)$:
$u(x) = x$, $v(x) = e^{ax}$. The two columns are

Row	Derivatives of $u(x) = x$	Antiderivatives of $v(x) = e^{ax}$
0	x	e^{ax}
1	1	e^{ax}/a
2	0	e^{ax}/a^2

We pair up entries in each row of the "Derivatives of $u(x)$" column with the following row of the "Antiderivatives of $v(x)$" column and multiply the pairs, alternative " $+$ " and " $-$ " for each pair, and add them up. In this example, we pair x (Row 0 of "Derivatives") with e^{ax}/a (Row 1 of "Antiderivatives"), then we pair 1 (Row 1 of "Derivatives") with e^{ax}/a^2 (Row 2 of "Antiderivatives") and apply " $-$ " to this pair. For this example, we can stop at this point, since all higher order derivatives of $u(x)$ are 0 in Row 2 and higher. The integral $\int xe^{ax}\,dx$ is $x \cdot e^{ax}/a - 1 \cdot e^{ax}/a^2 = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$, as in Example 0-21.

NOTE: An extension of Example 0-21 shows that for integer $n \geq 0$ and $c > 0$
$\int_0^{\infty} x^n\, e^{-cx}\, dx = \frac{n!}{c^{n+1}}$. This is another useful identity for the exam.

GEOMETRIC AND ARITHMETIC PROGRESSIONS

Geometric progression: $a,\, ar,\, ar^2,\, ar^3,\, \ldots,$ sum of the first n terms is
$a + ar + ar^2 + \cdots + ar^{n-1} = a[1 + r + r^2 + \cdots + r^{n-1}] = a \cdot \frac{r^n - 1}{r-1} = a \cdot \frac{1-r^n}{1-r}$,
if $-1 < r < 1$ then the infinite series can be summed, $a + ar + ar^2 + \cdots = \frac{a}{1-r}$

Arithmetic progression: $a,\, a+d,\, a+2d,\, a+3d,\, \ldots,$
sum of the first n terms of the series is $na + d \cdot \frac{n(n-1)}{2}$,
a special case is the sum of the first n integers - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Example 0-22:
A product sold 10,000 units last week, but sales are forecast to decrease 2% per week if no advertising campaign is implemented. If an advertising campaign is implemented immediately, the sales will decrease by 1% of the previous week's sales but there will be 200 new sales for the week (starting with this week). Under this model, calculate the number of sales for the 10-th week, 100-th week and 1000-th week of the advertising campaign (last week is week 0, this week is week 1 of the campaign).

Solution:
Week 1 sales: $(.99)(10{,}000) + 200$,
Week 2 sales: $(.99)[(.99)(10{,}000) + 200] + 200 = (.99)^2(10{,}000) + (200)[1 + .99]$
Week 3 sales: $(.99)[(.99)^2(10{,}000) + (200)[1 + .99]] + 200$
$= (.99)^3(10{,}000) + (200)[1 + .99 + .99^2]$
$\vdots$
Week 10 sales: $(.99)^{10}(10{,}000) + (200)[1 + .99 + .99^2 + \cdots + .99^9]$
$= (.99)^{10}(10{,}000) + (200)[\frac{1-.99^{10}}{1-.99}] = 10{,}956.2.$
Week 100 sales: $(.99)^{100}(10{,}000) + (200)[\frac{1-.99^{100}}{1-.99}] = 16{,}339.7.$
Week 1000 sales: $(.99)^{1000}(10{,}000) + (200)[\frac{1-.99^{1000}}{1-.99}] = 19{,}999.6.$ □