ACTEX
CAS Exam MAS-I Study Manual

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Spring 2018 Edition | Volume I-III

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Preface

Exam MAS-I (*Modern Actuarial Statistics I*) is a new exam which will be offered for the first time in Spring 2018 on May 7, 2018 (Monday) by the Casualty Actuarial Society (CAS). It replaces its predecessor Exam S (*Statistics and Probabilistic Models*), which is a relatively short-lived exam offered only five times, from Fall 2015 to Fall 2017. Exam S, in turn, was developed from the old Exam LC (*Models for Life Contingencies*), Exam ST (*Models for Stochastic Processes and Statistics*), and Exam 3L (*Life Contingencies and Statistics*). The introduction of Exam MAS-I is in response to the discontinuation of Exam C/4 in July 2018, which the CAS sees as an opportunity to revamp the current Exam S with a heavier focus on contemporary statistical methods and the addition of statistical learning. The construction of MAS-I is a further step that the CAS has taken to enhance actuaries' statistical literacy, as the CAS finds a deep understanding of statistics to be more important for property and casualty actuaries than routine training in traditional actuarial subjects like life contingencies. You will considerably sharpen your statistics toolkit as a result of taking (and, in all likelihood, passing!) Exam MAS-I.

Syllabus

The syllabus of Exam MAS-I, available from `http://www.casact.org/admissions/syllabus/ExamMASI.pdf`, is extremely broad (but not necessarily deep) in scope, covering miscellaneous topics in applied probability, mathematical statistics, statistical modeling and time series analysis, many of which are new topics not tested in any SOA/CAS past exams. As a rough estimate, you need at least *three months* of intensive study to master the material in this exam[i]. The specific sections of the syllabus along with their approximate weights in the exam are shown below:

Section	Range of Weight
A. Probability Models (Stochastic Processes & Survival Models)	20–35%
B. Statistics	15–30%

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[i]It is true that one need not master every topic in order to pass this exam.

C. Extended Linear Models	30–50%
D. Time Series with Constant Variance	10–20%

Compared with the former Exam S, both Sections C and D have enjoyed a heavier weight, from 25–40% to 30–50% and from 5–10% to 10–20%, respectively. Sections A, B, and D are more or less taken intact from the syllabuses of Exam S, ST, LC, and 3L. As a result, you can find lots of relevant past exam questions on these two sections. Section C has experienced the biggest change, with a new textbook on statistical learning added.

Exam Format

Exam MAS-I is a four-hour multiple-choice exam. According to the list of MAS-I frequently asked questions (`http://www.casact.org/cms/files/New_CAS_Exams_MAS_I_and_II_FAQs_1.pdf`), the exam will consist of approximately 35 to 40 questions. Before the start of the exam, there will be a fifteen-minute reading period in which you can silently read the questions and check the exam booklet for missing or defective pages. However, writing will not be permitted during this time, neither will the use of calculators.

Given the similarity between Exam MAS-I and Exam S (whose syllabus is available from `http://www.casact.org/admissions/syllabus/ExamS.pdf`) in terms of their structure and topics, we may use Exam S as a rough proxy for Exam MAS-I. Each of the Exam S papers from Fall 2015 to Fall 2017 has 45 questions, categorized into the four sections as follows:

	Number of Questions				
Section	2015F	2016S	2016F	2017S	2017F
A. Probability Models	13	12	16	15	16
(Stochastic Processes & Survival Models)					
B. Statistics	17	15	14	13	14
C. Generalized Linear Models	11	15	11	14	12
D. Time Series with Constant Variance	4	3	4	3	3
Total	45	45	45	45	45

You can see that roughly the same number of exam questions was set on Sections A, B, and C, although Section C was proclaimed to be the most important section (perhaps even the examiners found it hard to set questions on this section!). To investigate whether such a distribution of exam questions is consistent with the distribution that the CAS announced in the exam syllabus, please try Practice Exam 2 Question 22 on page 1279. According to `http://www.casact.org/admissions/passmarks/examS.pdf`, the pass marks for Fall 2015,

Spring 2016, Fall 2016, and Spring 2017[ii] were **52.5**, **55.0**, **54.0**, and **50.50** respectively, which means that candidates needed to answer about **27 out of 45 questions** correctly to earn a pass (each question carries 2 points with the total score being $44 \times 2 = 88$ or $45 \times 2 = 90$).

Here are the characteristics of a typical CAS multiple-choice exam:

1. The questions are almost always arranged in the same order as the topics in the exam syllabus, so Question 1 is very likely a Poisson process question and Question 40 is a time series question. This implicitly gives you a hint as to which topic an exam question is testing.

2. A number of exam questions bear a striking resemblance to past CAS exam questions, sometimes even with the same numerical values. This attests to the importance of practicing numerous past exam problems, an abundance of which are discussed and solved in this study manual in full detail.

3. The scope of an exam can be narrow at times with several questions testing the same topic in much the same way. For example, Questions 43, 44, and 45 of the Fall 2016 Exam S all test time series forecasting for AR models, and Questions 21, 23, and 24 of the Fall 2017 Exam S all test the concepts of Type I and II errors.

4. Most answer choices are in the form of ranges, e.g.:

    A. Less than 1%

    B. At least 1%, but less than 2%

    C. At least 2%, but less than 3%

    D. At least 3%, but less than 4%

    E. At least 4%

    If your answer is much lower than the bound indicated by Answer A or much higher than that suggested by Answer E, do check your calculations. Chances are that you have made computational mistakes, but this is not definitely the case (sometimes the CAS examiners themselves made a mistake!).

    Note that unlike other multiple-choice exams you took before, a guessing adjustment will be in place in Exam MAS-I, so unless you can eliminate two or three of the answer choices, it will be wise of you not to answer questions which you are unsure of by pure guesswork.

What is Special about This Study Manual?

The objective of this study manual is to help you grasp the material in Exam MAS-I, which is a brand new exam, effectively and efficiently, and *pass it with considerable ease.* Here are some of the valuable features of this manual for achieving this all-important goal:

[ii]The pass mark for the Fall 2017 Exam S is not yet available when this manual goes in press.

- Each chapter and section starts by explicitly stating which learning outcomes and knowledge statements of the exam syllabus we are going to cover, to ensure that we are on track and hitting the right target.

- The knowledge statements of the syllabus are demystified by precise and concise expositions synthesized from the syllabus readings, helping you acquire a deep understanding of the subject matter.

- Formulas and results of utmost importance are $\boxed{\text{boxed}}$ for easy identification and memorization.

- While the focus of this study manual is on exam preparation, it is interspersed with the interpretation of various formulas that also fosters genuine understanding of the syllabus material and discourages rote learning. It is the author's belief and personal experience that a solid understanding of the subject is always conducive to achieving good exam results.

- Mnemonics and shortcuts are emphasized, so are highlights of important exam items and common mistakes committed by students.

- To succeed in any (actuarial) exam, the importance of practicing a wide variety of non-trivial problems to sharpen your understanding and to develop proficiency, as always, cannot be overemphasized. In this regard, there are no better learning resources than past Exam S and relevant SOA/CAS examination papers. The expositions of this manual are complemented by more than **550 in-text examples** and **750 end-of-chapter/section problems** (the harder ones are labeled as [**HARDER!**] or [**VERY HARD!!**]), which are original or taken from relevant past exams and/or required textbooks, all with step-by-step solutions and problem-solving remarks, to give you a sense of what you can expect to see in the real exam. All practice problems are phrased in a multiple-choice format to emulate the real exam environment. As you read this manual, skills are honed and confidence is built. As a general guide, you should *read through all of the in-text examples with particular attention paid to recent Exam S questions, and work out at least half of the end-of-chapter/section problems.*

- Three full-length practice exams updated for the MAS-I exam syllabus and designed to mimic the real exam conclude this study manual giving you a holistic review of the syllabus material.

New to the Spring 2018 Edition

- Old SOA/CAS exam questions before 2000 are added as appropriate. Despite their seniority, these old exam questions, which are not easily available nowadays, do illustrate some less commonly tested concepts and are of considerable value.

- All of the 45 questions from the very recent Fall 2017 Exam S are inserted into this manual and carefully discussed and solved. Variants of some of these exam questions are developed.

- A number of sections have been substantially revised, partly in response to the recent Exam S papers, e.g. Section 1.2 on the properties of the exponential distribution, Section 10.6 on model diagnostics, Section 12.2 on the method of scoring. A number of new examples and end-of-chapter/section problems have been added. *In Bayesian parlance, we learn from experience as life moves on!*

- All known typographical errors have been fixed.

Exam Tables

In the exam, you will be supplied with a variety of tables, including:

- *Standard normal distribution table* (used throughout this study manual)

  You will need this table for values of the standard normal distribution function or standard normal quantiles, when you work with normally distributed random variables or perform normal approximation.

- *Illustrative Life Table* (used mostly in Chapter 4 of this study manual)

  You will need this when you are told that mortality of the underlying population follows the Illustrative Life Table.

- *A table of distributions for a number of common continuous and discrete distributions and the formulas for their moments and other probabilistic quantities* (used throughout Parts I and II of this study manual)

  This big table provides a great deal of information about some common as well as non-common distributions (e.g. inverse exponential, inverse Gaussian, Pareto, Burr, etc.). When an exam question centers on these distributions and quantities such as their means or variances are needed, consult this table.

- *Quantiles of t-distribution, F-distribution*[iii], *chi-square distribution* (used in Chapters 8, 10, 12 and 13 of this study manual)

  These quantiles will be of use when you perform parametric hypothesis tests.

You should download these tables from `http://www.casact.org/admissions/syllabus/MASI_Tables.pdf` right away, print out a copy and learn how to locate the relevant entries in these tables because they will be intensively used during your study as well as in the exam.

[iii]The *F*-table has a duplicate page.

Acknowledgment

I would like to thank my colleagues, Professor Elias S. W. Shiu and Dr. Michelle A. Larson, at the University of Iowa for sharing with me many pre-2000 SOA/CAS exam papers. These hard-earned old exam papers have proved invaluable in illustrating a number of less commonly tested exam topics. Thanks are also due to Mr. Zhaofeng Tang, doctoral student in actuarial science at the University of Iowa, for his professional assistance in the production of some of the graphs in this study manual.

Errata

While we go to great lengths to polish and proofread this manual, some mistakes will inevitably go unnoticed. The author wishes to apologize in advance for any errors, typographical or otherwise, and would greatly appreciate it if you could bring them to his attention by *sending any errors* you identify or *any criticisms or compliments about this manual to* `ambrose-lo@uiowa.edu` *and c.c.* `support@actexmadriver.com`. The author will try his best to respond to any inquiries within 48 hours and an ongoing errata list will be maintained online at `https://sites.google.com/site/ambroseloyp/publications/cas-exam-mas-I-study-manual`. More importantly, *students who report errors will quality for a quarterly drawing for a $100 in-store credit.*

<div align="right">

Ambrose Lo
January 2018
Iowa City, IA

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About the Author

Professor Ambrose Lo was born, raised, and educated in Hong Kong. He earned his B.S. in Actuarial Science (first class honors) and Ph.D. in Actuarial Science from The University of Hong Kong in 2010 and 2014 respectively. He joined the Department of Statistics and Actuarial Science at The University of Iowa in August 2014 as an Assistant Professor in Actuarial Science. He is a Fellow of the Society of Actuaries (FSA) and a Chartered Enterprise Risk Analyst (CERA). His research interests lie in dependence structures, quantitative risk management as well as optimal (re)insurance. His research papers have been published in top-tier actuarial journals, such as *Insurance: Mathematics and Economics*, *Scandinavian Actuarial Journal* and *ASTIN Bulletin: The Journal of the International Actuarial Association.*

Besides dedicating himself to actuarial research, Ambrose attaches equal importance to teaching, through which he nurtures the next generation of actuaries and serves the actuarial profession. He has taught courses on financial derivatives, mathematical finance, life contingencies, credibility theory, advanced probability theory, and regression analysis. His emphasis in teaching is always placed on thorough understanding of the subject matter complemented by concrete problem-solving skills. As a result of his exceptional teaching performance, Ambrose has won numerous teaching awards ever since he was a graduate student (see `http://www.scifac.hku.hk/news/comm/award-excellence-teaching-assistant-2011-12-2` for example).

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Part I

Probability Models
(Stochastic Processes and
Survival Models)

Chapter 1

Poisson Processes

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LEARNING OBJECTIVES

1. Understand and apply the properties of Poisson processes:
   - For increments in the homogeneous case
   - For interval times in the homogeneous case
   - For increments in the non-homogeneous case
   - Resulting from special types of events in the Poisson process
   - Resulting from sums of independent Poisson processes

   Range of weight: 0-5 percent

2. For any Poisson process and the inter-arrival and waiting distributions associated with the Poisson process, calculate:
   - Expected values
   - Variances
   - Probabilities

   Range of weight: 0-5 percent

3. For a compound Poisson process, calculate moments associated with the value of the process at a given time.

   Range of weight: 0-5 percent

4. Apply the Poisson Process concepts to calculate the hazard function and related survival model concepts.
   - Relationship between hazard rate, probability density function and cumulative distribution function
   - Effect of memoryless nature of Poisson distribution on survival time estimation

   Range of weight: 2-8 percent

</div>

Chapter overview: As a prospective P&C actuary, you would be interested in monitoring the number of insurance claims an insurance company receives as time goes by and how these claims can be appropriately analyzed by means of sound statistical analysis. In Exam MAS-I, we shall learn one way of modeling the flow of insurance claims – the Poisson process.

This part of the syllabus has two required readings:

(1) *A study note by J.W. Daniel*

The study note is precise and concise, introducing main results mostly without proof and supplementing its exposition with a few simple examples. It is suitable for a first-time introduction to Poisson processes.

(2) *The book entitled Introduction to Probability Models by S.M. Ross.*

This is a textbook used by a number of college courses on elementary applied probability. It balances rigor and intuition, and presents the theory of Poisson processes at a level that is much deeper than that in the study note by Daniel. In particular, it treats the conditional distribution of the arrival times as well as the interplay between two independent Poisson processes. A conspicuous feature of this book is its large number of sophisticated examples and exercises which require a large amount of ingenuity and cannot be done in a reasonable exam setting. This study manual improves the practicality of the book and rephrases these otherwise intractable examples in an exam tone.

You can download this book (Eleventh Edition) "legally" from *ScienceDirect* via the following link, chapter by chapter, if your university has subscribed to it:

`http://www.sciencedirect.com/science/book/9780124079489`

You should do so because the book has a number of good exercises which will be solved in full in this study manual (the questions cannot be reproduced here because of copyright issues). The exercises had been the theme of some past SOA/CAS examination questions, so you should not despise these exercises as irrelevant and useless.

The Daniel study note has been on the syllabuses of Exams 3, 3L and ST, whereas *Introduction to Probability Models* just entered the syllabus of Exam S in Fall 2015. As a result of the addition of the latter reading, we expect more complex exam questions on Poisson processes in Exam MAS-I. In total, expect about 4 questions on the material of the entire chapter.

1.1 Fundamental Properties

> **KNOWLEDGE STATEMENTS**
>
> 1a. Poisson process
>
> 1b. Non-homogeneous Poisson process
>
> **OPTIONAL SYLLABUS READING(S)**
>
> - Ross, Subsections 5.3.1 and 5.3.2
> - Daniel, Section 1.1 and Subsection 1.4.1

Definition. By definition, a *Poisson process* $\{N(t),\, t \geq 0\}$ with *rate function* (also known as *intensity function*) $\lambda(\cdot)$ [i] is a stochastic process, namely, a collection of random variables indexed by time t (in an appropriate unit, e.g. minute, hour, month, year, etc.), satisfying the following properties:

1. *(Counting)* $N(0) = 0$, $N(t)$ is non-decreasing in t and takes non-negative integer values only.

   Interpretation: $N(t)$ counts the number of claims which are submitted on or before time t. Thus $N(0)$ is 0 (we assume that should be no claims before the insurance company starts its business), $N(t)$ cannot decrease in time and must be integer-valued.

2. *(Distribution of increments are Poisson random variables)* For $s < t$, the increment $N(t) - N(s)$, which counts the number of events in the interval $(s, t]$, is a Poisson random variable with mean $\Lambda = \int_s^t \lambda(y) \, dy$.

   Interpretation: Increments of a Poisson process, as its name suggests, are Poisson random variables with mean computed by integrating the rate function over the same interval. In this regard, we can see that the rate function of a Poisson process completely specifies the distribution of each increment.

3. *(Increments are independent)* If $(s_1, t_1]$ and $(s_2, t_2]$ are non-overlapping intervals, then $N(t_1) - N(s_1)$ and $N(t_2) - N(s_2)$ are independent random variables.

   Interpretation: This is the most amazing property of a Poisson process. Its increments not only follow Poisson distribution, but also are independent on disjoint intervals (e.g. $(0, 1)$ and $(2, 5)$ are disjoint intervals, so are $(3, 4]$ and $(4, 5]$). This means that, in this model, the frequency of claims you received last month has nothing to do with the frequency this month.

---

[i]The study note by Daniel simply writes a Poisson process as N in short. While this is a perfectly correct way of writing, some students may confuse that with a Poisson random variable N. Also, here we write $\lambda(\cdot)$ with a parenthesis containing the argument of the function instead of just λ to emphasize that $\lambda(\cdot)$ is a function.

In the context of insurance applications, we interpret $N(t)$ as the number of claims that occur on or before time t. The same interpretation can easily carry over to more general contexts where we are interested in counting a particular type of event, e.g. the number of customers that enter a store, the number of cars passing through an intersection, the number of lucky candidates passing Exam MAS-I, etc.

Homogeneous Poisson processes. A Poisson process whose rate function is constant, say $\lambda(t) = \lambda$ for all $t \geq 0$, is called a *homogeneous* Poisson process. In addition to having independent increments, a homogeneous Poisson process also possesses *stationary increments*, meaning that the distribution of $N(t+s) - N(s)$ depends only on the length of the interval, which is t in this case, but not on s.

Probability calculations.   The second and third properties of a Poisson process allow us to calculate many probabilistic quantities, such as the probability of a certain number of events, as well as the expected and variance of the number of events in a particular time interval. These two properties will be intensively used in exam questions. The following string of past exam questions serves as excellent illustrations.

RECALL

Just in case you forgot:

1. The probability mass function of a Poisson random variable X with parameter λ (a scalar, not a function) is given by

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

   The mean and variance of X are both equal to λ.

2. If $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with respective means $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $X_1 + X_2 + \cdots + X_n$ is also a Poisson random variable with a mean of $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. In other words, the sum of independent Poisson random variables is also a Poisson random variable whose mean is the sum of the individual Poisson means.

Example 1.1.1. (SOA/CAS Exam P/1 Sample Question 173: Warm-up question) In a given region, the number of tornadoes in a one-week period is modeled by a Poisson distribution with mean 2. The numbers of tornadoes in different weeks are mutually independent.

  Calculate the probability that fewer than four tornadoes occur in a three-week period.

   A. 0.13

   B. 0.15

   C. 0.29

   D. 0.43

   E. 0.86

Ambrose's comments: This is not even a Poisson process question. It simply reminds you of how probabilities for a Poisson random variable are typically calculated.

Solution. We are interested in $\Pr(N_1 + N_2 + N_3 < 4)$, where N_i is the number of tornadoes in the ith week for $i = 1, 2, 3$. As $N_1 + N_2 + N_3$ is also a Poisson random variable with a mean of $3(2) = 6$, we have

$$
\begin{aligned}
\Pr(N_1 + N_2 + N_3 < 4) &= \sum_{i=0}^{3} \Pr(N_1 + N_2 + N_3 = i) \\
&= e^{-6}\left(1 + 6 + \frac{6^2}{2!} + \frac{6^3}{3!}\right) \\
&= \boxed{0.1512}. \quad \textbf{(Answer: B)}
\end{aligned}
$$

$\square$

Example 1.1.2. (CAS Exam 3L Fall 2013 Question 9: Probability $-$ I) You are given that claim counts follow a non-homogeneous Poisson Process with $\lambda(t) = 30t^2 + t^3$.
   Calculate the probability of at least two claims between time 0.2 and 0.3.

   A. Less than 1%

   B. At least 1%, but less than 2%

   C. At least 2%, but less than 3%

   D. At least 3%, but less than 4%

   E. At least 4%

Solution. The number of claims between time 0.2 and 0.3 is a Poisson random variable with parameter

$$
\int_{0.2}^{0.3} (30t^2 + t^3)\, \mathrm{d}t = 10t^3 + \frac{t^4}{4}\bigg|_{0.2}^{0.3} = 0.191625.
$$

Hence the probability of at least two claims between time 0.2 and 0.3 is the complement of the probability of having 0 or 1 claim:

$$1 - \Pr(0 \text{ claim}) - \Pr(1 \text{ claim}) = 1 - e^{-0.191625}(1 + 0.191625) = \boxed{0.0162}. \quad \textbf{(Answer: B)}$$

□

Example 1.1.3. (CAS Exam 3L Spring 2010 Question 12: Probability – II)
Downloads of a song on a musician's Web site follow a heterogeneous Poisson process with the following Poisson rate function:

$$\lambda(t) = e^{-0.25t}$$

Calculate the probability that there will be more than two downloads of this song between times $t = 1$ and $t = 5$.

   A. Less than 29%

   B. At least 29%, but less than 30%

   C. At least 30%, but less than 31%

   D. At least 31%, but less than 32%

   E. At least 32%

Solution. Because $N(5) - N(1)$ is a Poisson random variable with parameter

$$\int_1^5 \lambda(t)\,\mathrm{d}t = \int_1^5 e^{-0.25t}\,\mathrm{d}t = \frac{e^{-0.25(1)} - e^{-0.25(5)}}{0.25} = 1.969184,$$

the required probability equals

$$
\begin{aligned}
&\Pr(N(5) - N(1) > 2)\\
={}& 1 - \Pr(N(5) - N(1) = 0) - \Pr(N(5) - N(1) = 1) - \Pr(N(5) - N(1) = 2)\\
={}& 1 - e^{-1.969184}\left(1 + 1.969184 + \frac{1.969184^2}{2}\right)\\
={}& \boxed{0.3150}. \quad \textbf{(Answer: D)}
\end{aligned}
$$

□

Example 1.1.4. (CAS Exam S Spring 2016 Question 3: Probability − III) You are given:

- The number of claims, $N(t)$, follows a Poisson process with intensity:

$$\lambda(t) = \frac{1}{2}t, \quad 0 < t < 5$$
$$\lambda(t) = \frac{1}{4}t, \quad t \geq 5$$

- By time $t = 4$, 15 claims have occurred.

Calculate the probability that exactly 16 claims will have occurred by time $t = 6$.

    A. Less than 0.075

    B. At least 0.075, but less than 0.125

    C. At least 0.125, but less than 0.175

    D. At least 0.175, but less than 0.225

    E. At least 0.225

Solution. The number of claims between $t = 4$ and $t = 6$ is a Poisson random variable with mean

$$\int_4^5 \frac{1}{2}t \, dt + \int_5^6 \frac{1}{4}t \, dt = \frac{5^2 - 4^2}{2(2)} + \frac{6^2 - 5^2}{4(2)} = 3.625.$$

The probability of having exactly one($= 16 - 15$) claim between $t = 4$ and $t = 6$ is

$$3.625e^{-3.625} = \boxed{0.0966}. \quad \textbf{(Answer: B)}$$

$\square$

Remark. More formally, the probability we seek is

$$\Pr(N(6) = 16 | N(4) = 15) = \Pr(N(6) - N(4) = 1 | N(4) = 15)$$
$$= \Pr(N(6) - N(4) = 1)$$

due to the property of independent increments.

Example 1.1.5. (CAS Exam 3L Spring 2012 Question 9: Expected value) Claims reported for a group of policies follow a non-homogeneous Poisson process with

rate function:

$$\lambda(t) = 100/(1+t)^3, \text{ where } t \text{ is the time (in years) after January 1, 2011.}$$

Calculate the expected number of claims reported after January 1, 2011 for this group of policies.

    A. Less than 45

    B. At least 45, but less than 55

    C. At least 55, but less than 65

    D. At least 65, but less than 75

    E. At least 75

Solution. We are interested in $N(\infty) = \lim_{t\to\infty} N(t)$, which is a Poisson random variable with mean

$$\int_0^\infty \lambda(t)\,dt = \int_0^\infty \frac{100}{(1+t)^3}\,dt = 100\left[-\frac{1}{2(1+t)^2}\right]_0^\infty = \boxed{50}. \quad \textbf{(Answer: B)}$$

$\square$

Example 1.1.6. (CAS Exam 3L Spring 2013 Question 9: Variance) You are given the following:

- An actuary takes a vacation where he will not have access to email for eight days.

- While he is away, emails arrive in the actuary's inbox following a non-homogeneous Poisson process where

$$\lambda(t) = 8t - t^2 \text{ for } 0 \le t \le 8. \, (t \text{ is in days})$$

Calculate the variance of the number of emails received by the actuary during this trip.

    A. Less than 60

    B. At least 60, but less than 70

    C. At least 70, but less than 80

    D. At least 80, but less than 90

    E. At least 90

Solution. The trip of the actuary lasts for 8 days, during which the number of emails is a Poisson random variable with variance (same as the Poisson parameter)

$$\int_0^8 (8t - t^2)\,\mathrm{d}t = \left[4t^2 - \frac{t^3}{3}\right]_0^8 = \boxed{85.3333}. \quad \textbf{(Answer: D)}$$

□

Example 1.1.7. (CAS Exam ST Fall 2015 Question 1: Calculation of homogeneous Poisson intensity) For two Poisson processes, N_1 and N_2, you are given:

- N_1 has intensity function $\lambda_1(t) = \begin{cases} 2t & \text{for } 0 < t \le 1 \\ t^3 & \text{for } t > 1 \end{cases}$

- N_2 is a homogeneous Poisson process.

- $\mathrm{Var}[N_1(3)] = 4\mathrm{Var}[N_2(3)]$

Calculate the intensity of N_2 at $t = 3$.

   A. Less than 1

   B. At least 1, but less than 3

   C. At least 3, but less than 5

   D. At least 5, but less than 7

   E. At least 7

Solution. Note that $N_1(3)$ has a mean and variance equal to

$$\int_0^3 \lambda_1(t)\,\mathrm{d}t = \int_0^1 2t\,\mathrm{d}t + \int_1^3 t^3\,\mathrm{d}t = [t^2]_0^1 + \left[\frac{t^4}{4}\right]_1^3 = 1 + \frac{3^4 - 1^4}{4} = 21,$$

while $N_2(3)$ has a mean and variance equal to $3\lambda_2$, where λ_2 is the constant intensity of N_2. As $\mathrm{Var}[N_1(3)] = 4\mathrm{Var}[N_2(3)]$, we have $21 = 4(3\lambda_2)$, so $\lambda_2 = 21/12 = \boxed{1.75}$. **(Answer: B)**

□

Probabilities involving overlapping intervals. A harder exam question may ask that you determine probabilities for increments on overlapping intervals. The key step to calculate these probabilities lies in rewriting the events in terms of increments on *non-overlapping* intervals, which are independent according to the definition of a Poisson process.

Example 1.1.8. (Probability for overlapping increments I) The number of calls received in a telephone exchange follow a homogeneous Poisson process with a rate of 30 per hour.

Calculate the probability that there are exactly 2 calls in the first ten minutes and exactly 5 calls in the first twenty minutes.

  A. Less than 0.01

  B. At least 0.01, but less than 0.02

  C. At least 0.02, but less than 0.03

  D. At least 0.03, but less than 0.04

  E. At least 0.04

Solution. When time is measured in hours, the required probability is

$$\Pr(N(1/6) = 2, N(1/3) = 5) = \Pr(N(1/6) = 2, N(1/3) - N(1/6) = 3),$$

which can be factored, because of independence, into

$$
\begin{aligned}
\Pr(N(1/6) = 2)\Pr(N(1/3) - N(1/6) = 3) &= \frac{e^{-30/6}(30/6)^2}{2!} \times \frac{e^{-30/6}(30/6)^3}{3!} \\
&= \boxed{0.0118}. \quad \textbf{(Answer: B)}
\end{aligned}
$$

□

Example 1.1.9. (Probability for overlapping increments II) Customers arrive at a post office in accordance with a Poisson process with a rate of 5 per hour. The post office opens at 9:00 am.

Calculate the probability that only one customer arrives before 9:20 am and ten customers arrive before 11:20 am.

  A. Less than 0.01

  B. At least 0.01, but less than 0.02

  C. At least 0.02, but less than 0.03

  D. At least 0.03, but less than 0.04

  E. At least 0.04

Solution. The required probability is

$$
\begin{aligned}
\Pr\left(N(1/3) = 1, N(7/3) = 10\right) &= \Pr\left(N(1/3) = 1, N(7/3) - N(1/3) = 9\right) \\
&= \Pr\left(N(1/3) = 1\right)\Pr\left(N(7/3) - N(1/3) = 9\right) \\
&= e^{-5/3}\left(\frac{5}{3}\right) \times e^{-5(2)}\frac{[5(2)]^9}{9!} \\
&= \boxed{0.0394}. \quad \textbf{(Answer: D)}
\end{aligned}
$$

$\square$

Conditional distribution of $N(t)$ given $N(s)$ for $s \leq t$. Suppose that we know the value of the Poisson process at one time point s with $N(s) = m$, and we wish to study the probabilistic behavior of the Poisson process at a later time point t with $s \leq t$. Then $N(t)$ turns out to be a *translated Poisson* random variable in the sense that it has the same distribution as the sum of a Poisson random variable and a constant. To see this, let's write $N(t)$ as

$$
N(t) = [N(t) - N(s)] + N(s).
$$

The second term $N(s)$ is known to be m, while the first term, owing to the property of independent increments of a Poisson process, is a Poisson random variable, say M, whose distribution does not depend on the value of m. Therefore, we have the distributional representation

$$
\boxed{[N(t)|N(s) = m] \overset{d}{=} M + m, \quad s \leq t,}
$$

where "$\overset{d}{=}$" means equality in distribution. This result allows us to answer questions about many probabilistic quantities associated with $N(t)$ when the value of $N(s)$ is given.

Example 1.1.10. (CAS Exam S Fall 2017 Question 3: Conditional probability) You are given:

- A Poisson process N has a rate function: $\lambda(t) = 3t^2$

- You've already observed 50 events by time $t = 2.1$.

Calculate the conditional probability, $\Pr[N(3) = 68 \mid N(2.1) = 50]$.

    A. Less than 5%

    B. At least 5%, but less than 10%

    C. At least 10%, but less than 15%

    D. At least 15%, but less than 20%

E. At least 20%

Solution. The conditional probability can be determined as

$$\Pr[N(3) = 68 \mid N(2.1) = 50] = \Pr[N(3) - N(2.1) = 18 \mid N(2.1) = 50]$$
$$= \Pr[N(3) - N(2.1) = 18],$$

where $N(3) - N(2.1)$ is a Poisson random variable with mean $\int_{2.1}^{3} 3t^2 \, \mathrm{d}t = 3^3 - 2.1^3 = 17.739$. The final answer is

$$\frac{\mathrm{e}^{-17.739} 17.739^{18}}{18!} = \boxed{0.0934}. \quad \textbf{(Answer: B)}$$

□

Example 1.1.11. (CAS Exam 3L Fall 2010 Question 11: Conditional variance)
You are given the following information:

- A Poisson process N has a rate function $\lambda(t) = 3t^2$.

- You have observed 50 events by time $t = 2.1$.

Calculate $\mathrm{Var}[N(3) \mid N(2.1) = 50]$.

    A. Less than 10

    B. At least 10, but less than 20

    C. At least 20, but less than 30

    D. At least 30, but less than 40

    E. At least 40

Solution. Conditional on $N(2.1) = 50$, $N(3)$ has the same distribution as $M + 50$, where M is a Poisson random variable with mean and variance

$$\int_{2.1}^{3} \lambda(t) \, \mathrm{d}t = \int_{2.1}^{3} 3t^2 \, \mathrm{d}t = t^3 \Big|_{2.1}^{3} = 17.739.$$

Hence

$$\mathrm{Var}[N(3) \mid N(2.1) = 50] = \mathrm{Var}(M + 50) = \mathrm{Var}(M) = \boxed{17.739} \quad \textbf{(Answer: B)}.$$

□

Normal approximation. For more cumbersome probabilities such as $\Pr(N(t) > c)$ with c being a large number, exact calculations can be tedious and normal approximation may be used. That is, we approximate $N(t)$ by a normal random variable with the same mean and variance, and instead calculate the probability of the same event for this normal random variable using the standard normal distribution table you have in the exam. Because the distribution of $N(t)$ is discrete, and a continuous distribution (i.e., normal) is used to approximate this discrete distribution, a continuity correction should be made.

Recall - Continuity correction

Let X be a random variable taking values in the set of integers $\{0, \pm 1, \pm 2, \ldots\}$ and N is a normal random variable having the same mean and variance as X. The following shows how various probabilities are approximated using the normal approximation *with* continuity correction: (c is an integer)

Probability of Interest		Approximant
$\Pr(X \leq c)$	$\approx$	$\Pr(N \leq c + 0.5)$
$\Pr(X < c)$	$\approx$	$\Pr(N \leq c - 0.5)$
$\Pr(X \geq c)$	$\approx$	$\Pr(N \geq c - 0.5)$
$\Pr(X > c)$	$\approx$	$\Pr(N > c + 0.5)$

In the second column, it does not matter whether we take sharp or strict inequalities because N is a continuous random variable. In other words, we may replace "$\leq$" by "$<$" and "$\geq$" by "$>$".

Throughout this study manual, we denote the distribution function of the standard normal distribution by Φ.

Example 1.1.12. (CAS Exam 3L Spring 2008 Question 11: Normal approximation) A customer service call center operates from 9:00 AM to 5:00 PM. The number of calls received by the call center follows a Poisson process whose rate function varies according to the time of day, as follows:

Time of Day	Call Rate (per hour)
9:00 AM to 12:00 PM	30
12:00 PM to 1 :00 PM	10
1:00 PM to 3:00 PM	25
3:00 PM to 5:00 PM	30

Using a normal approximation, what is the probability that the number of calls received from 9:00AM to 1:00PM exceeds the number of calls received from 1:00PM to 5:00PM?

A. Less than 10%

B. At least 10%, but less than 20%

C. At least 20%, but less than 30%

D. At least 30%, but less than 40%

E. At least 40%

Solution. The number of calls received from 9:00AM to 1:00PM is a Poisson random variable N^1 with parameter $30(3) + 10(1) = 100$, while the number of calls received from 1:00PM to 5:00PM is a Poisson random variable N^2 with parameter $25(2) + 30(2) = 110$. Because N^1 and N^2 are independent,

$$\mathrm{E}[N^1 - N^2] = \mathrm{E}[N^1] - \mathrm{E}[N^2] = 100 - 110 = -10,$$

and

$$\mathrm{Var}(N^1 - N^2) = \mathrm{Var}(N^1) + \mathrm{Var}(N^2) = 100 + 110 = 210.$$

Using the normal approximation with continuity correction, we have

$$\Pr(N^1 > N^2) = \Pr(N^1 - N^2 > 0) \approx \Pr\left(\underbrace{\mathrm{N}(-10, 210)}_{\substack{\text{a normal r.v. with mean} \\ -10 \text{ and variance } 210}} > 0.5 \right),$$

which, upon standardization, equals

$$\Pr\left(\mathrm{N}(0, 1) > \frac{0.5 - (-10)}{\sqrt{210}} \right) = 1 - \Phi(0.72) = 1 - 0.7642 = \boxed{0.2358}. \quad \textbf{(Answer: C)}$$

$\square$

Remark. If you do not use continuity correction, you will calculate

$$
\begin{aligned}
\Pr(N^1 > N^2) &\approx \Pr\left(\mathrm{N}(-10, 210) > 0 \right) \\
&= 1 - \Phi\left(\frac{0 - (-10)}{\sqrt{210}} \right) = 1 - \underbrace{\Phi(0.69)}_{0.7549} = 0.2451,
\end{aligned}
$$

in which case you will also end up with Answer C.

[HARDER!] Conditional distribution of $N(s)$ given $N(t)$ with $s \leq t$. We have learned that conditional on $N(s)$, the distribution of $N(t)$, where $0 \leq s \leq t$, is that of a translated Poisson distribution. What about the conditional distribution of $N(s)$ given

$N(t)$? To answer this question, we consider, for $k = 0, 1, \ldots, n$,

$$
\begin{aligned}
\Pr\left(N(s) = k | N(t) = n\right) &= \frac{\Pr\left(N(s) = k, N(t) = n\right)}{\Pr\left(N(t) = n\right)} \\
&= \frac{\Pr\left(N(s) = k, N(t) - N(s) = n - k\right)}{\Pr\left(N(t) = n\right)}.
\end{aligned}
$$

Because a (homogeneous or non-homogeneous) Poisson process possesses independent increments, the preceding probability can be further written as

$$
\begin{aligned}
\Pr\left(N(s) = k | N(t) = n\right) &= \frac{\Pr\left(N(s) = k\right)\Pr\left(N(t) - N(s) = n - k\right)}{\Pr\left(N(t) = n\right)} \\
&= \frac{e^{-m(s)}[m(s)]^k / k! \times e^{-[m(t) - m(s)]}[m(t) - m(s)]^{n-k} / (n-k)!}{e^{-m(t)}[m(t)]^n / n!} \\
&= \frac{n!}{k!(n-k)!}\left(\frac{m(s)}{m(t)}\right)^k\left(1 - \frac{m(s)}{m(t)}\right)^{n-k} \\
&= \binom{n}{k}\left(\frac{m(s)}{m(t)}\right)^k\left(1 - \frac{m(s)}{m(t)}\right)^{n-k}.
\end{aligned}
$$

In other words, given $N(t) = n$, $N(s)$ is a binomial random variable with parameters n and $m(s)/m(t)$. In particular, for a homogeneous Poisson process with rate λ, i.e., $m(t) = \lambda t$ for $t \geq 0$, then

$$
\boxed{N(s) | N(t) = n \quad \sim \quad \text{Binomial}\left(n, \frac{s}{t}\right),}
$$

which is free of λ.

Example 1.1.13. (CAS Exam ST Spring 2016 Question 1: Probability for $N(s)$ given $N(t)$ with $s \leq t$) You are given that $N(t)$ follows the Poisson process with rate $\lambda = 2$.

Calculate $\Pr[N(2) = 3 | N(5) = 7]$.

A. Less than 0.25

B. At least 0.25, but less than 0.35

C. At least 0.35, but less than 0.45

D. At least 0.45, but less than 0.55

E. At least 0.55

Solution. Conditional on $N(5) = 7$, $N(2)$ is a binomial random variable with parameters 7 and 2/5, so

$$
\Pr[N(2) = 3 | N(5) = 7] = \binom{7}{3}\left(\frac{2}{5}\right)^3\left(1 - \frac{2}{5}\right)^4 = \boxed{0.290304}. \quad \textbf{(Answer: B)}
$$

$\square$

Remark. We do not need the value of λ to get the answer.

Problems

Problem 1.1.1. (SOA Exam P Sample Question 280: Conditional Poisson mean)
The number of burglaries occurring on Burlington Street during a one-year period is Poisson distributed with mean 1.

   Calculate the expected number of burglaries on Burlington Street in a one-year period, given that there are at least two burglaries.

   A. 0.63

   B. 2.39

   C. 2.54

   D. 3.00

   E. 3.78

Solution. Let N be the number of burglaries on Burlington Street in a specified one-year period. Given that there are at least two burglaries, the expected value of N is

$$
\begin{aligned}
\mathrm{E}[N|N \geq 2] &= \frac{\sum_{n=2}^{\infty} n \Pr(N = n)}{\Pr(N \geq 2)} \\
&= \frac{\mathrm{E}[N] - 1 \times \Pr(N = 1)}{1 - \Pr(N = 0) - \Pr(N = 1)} \\
&= \frac{1 - \mathrm{e}^{-1}}{1 - \mathrm{e}^{-1} - \mathrm{e}^{-1}} \\
&= \boxed{2.3922}. \quad \textbf{(Answer: B)}
\end{aligned}
$$

□

Problem 1.1.2. (CAS Exam 3 Fall 2006 Question 26: True-of-false questions)
Which of the following is/are true?

   1. A counting process is said to possess independent increments if the number of events that occur between time s and t is independent of the number of events that occur between time s and $t + u$ for all $u > 0$.

   2. All Poisson processes have stationary and independent increments.

   3. The assumption of stationary and independent increments is essentially equivalent to asserting that at any point in time the process probabilistically restarts itself.

A. 1 only

B. 2 only

C. 3 only

D. 1 and 2 only

E. 2 and 3 only

Solution. Only 3. is correct. (**Answer: C**)

1. This would be true if the second s is changed to t.

2. A non-homogeneous Poisson process does not have stationary increments in general.

□

Problem 1.1.3. (CAS Exam 3 Fall 2006 Question 28: Piecewise linear intensity function) Customers arrive to buy lemonade according to a Poisson distribution with $\lambda(t)$, where t is time in hours, as follows:

$$\lambda(t) = \begin{cases} 2 + 6t & 0 \le t \le 3 \\ 20 & 3 < t \le 4 \\ 36 - 4t & 4 < t \le 8 \end{cases}$$

At 9:00 a.m., t is 0.

Calculate the number of customers expected to arrive between 10:00 a.m. and 2:00 p.m.

A. Less than 63

B. At least 63, but less than 65

C. At least 65, but less than 67

D. At least 67, but less than 69

E. At least 69

Solution. The expected number of customers arriving between 10:00 a.m. ($t = 1$) and 2:00 p.m. ($t = 5$) is

$$
\begin{aligned}
\int_1^5 \lambda(t)\,dt &= \int_1^3 (2 + 6t)\,dt + \int_3^4 20\,dt + \int_4^5 (36 - 4t)\,dt \\
&= [2t + 3t^2]_1^3 + 20 + [36t - 2t^2]_4^5 \\
&= \boxed{66}. \quad (\textbf{Answer: C})
\end{aligned}
$$

□

Remark. Because the intensity function is piecewise linear, integrating it is the same as calculating the areas of trapeziums.

Problem 1.1.4. (SOA Course 3 Fall 2004 Question 26: Linear rate function)
Customers arrive at a store at a Poisson rate that increases linearly from 6 per hour at 1:00 p.m. to 9 per hour at 2:00 p.m.

Calculate the probability that exactly 2 customers arrive between 1:00 p.m. and 2:00 p.m.

    A. 0.016

    B. 0.018

    C. 0.020

    D. 0.022

    E. 0.024

Solution. Let 1:00 p.m. be time 0 and measure time in hours. The rate function is given by

$$\lambda(t) = 6 + 3t, \quad t \geq 0.$$

You can check that $\lambda(0) = 6$ and $\lambda(1) = 9$. The number of customers that arrive between 1:00 p.m. and 2:00 p.m. is a Poisson random variable with a mean of $\int_0^1 \lambda(t)\, dt = 6 + 3/2 = 7.5$. The probability of having 2 customers in the same period is

$$\frac{e^{-7.5}(7.5)^2}{2!} = \boxed{0.0156}. \quad \textbf{(Answer: A)}$$

$\square$

Problem 1.1.5. (CAS Exam 3L Fall 2008 Question 1: Expected value) The number of accidents on a highway from 3:00 PM to 7:00 PM follows a nonhomogeneous Poisson process with rate function

$$\lambda = 4 - (t-2)^2, \text{ where } t \text{ is the number of hours since 3:00 PM.}$$

How many more accidents are expected from 4:00 PM to 5:00 PM than from 3:00 PM to 4:00PM?

    A. Less than 0.75

    B. At least 0.75, but less than 1.25

    C. At least 1.25, but less than 1.75

    D. At least 1.75, but less than 2.25

E. At least 2.25

Solution.   • The expected number of accidents from 3:00 PM to 4:00PM is

$$\int_0^1 [4 - (t-2)^2]\,\mathrm{d}t = \left[4t - \frac{(t-2)^3}{3}\right]_0^1 = \frac{5}{3}.$$

• The expected number of accidents from 4:00 PM to 5:00 PM is

$$\int_1^2 [4 - (t-2)^2]\,\mathrm{d}t = \left[4t - \frac{(t-2)^3}{3}\right]_1^2 = \frac{11}{3}.$$

The difference is $\boxed{2}$. **(Answer: D)**   □

Problem 1.1.6. (CAS Exam 3L Fall 2008 Question 2: Probability, homogeneous)
You are given the following:

• Hurricanes occur at a Poisson rate of $1/4$ per week during the hurricane season.

• The hurricane season lasts for exactly 15 weeks.

Prior to the next hurricane season, a weather forecaster makes the statement, "There will be
at least three and no more than five hurricanes in the upcoming hurricane season."
   Calculate the probability that this statement will be correct.

A. Less than 54%

B. At least 54%, but less than 56%

C. At least 56%, but less than 58%

D. At least 58%, but less than 60%

E. At least 60%

Solution. Note that $N(15)$, the number of hurricanes during the 15-week hurricane season, is
a Poisson random variable with a mean of $15/4 = 3.75$. The probability that the statement
will be correct is

$$\Pr\left(3 \le N(15) \le 5\right) = \mathrm{e}^{-3.75}\left(\frac{3.75^3}{3!} + \frac{3.75^4}{4!} + \frac{3.75^5}{5!}\right) = \boxed{0.5458}. \quad \textbf{(Answer: B)}$$

□

Problem 1.1.7. (CAS Exam 3L Spring 2008 Question 10: Probability, non-homogeneous) Car accidents follow a Poisson process, as described below:

• On Monday and Friday, the expected number of accidents per day is 3.

- On Tuesday, Wednesday, and Thursday, the expected number of accidents per day is 4.

- On Saturday and Sunday, the expected number of accidents per day is 1.

Calculate the probability that exactly 18 accidents occur in a week.

A. Less than .06

B. At least .06 but less than .07

C. At least .07 but less than .08

D. At least .08 but less than .09

E. At least .09

Solution. The total number of accidents in a week is a Poisson random variable with a mean of $3(2) + 4(3) + 1(2) = 20$, so the probability of having exactly 18 accidents in a week is

$$\frac{\mathrm{e}^{-20}20^{18}}{18!} = \boxed{0.0844}. \quad (\textbf{Answer: D})$$

<div align="right">□</div>

Problem 1.1.8. (CAS Exam 3 Spring 2006 Question 33: Probability, non-homogeneous)
While on vacation, an actuarial student sets out to photograph a Jackalope and a Snipe, two animals common to the local area. A tourist information booth informs the student that daily sightings of Jackalopes and Snipes follow independent Poisson processes with intensity parameters:

$$\lambda_J(t) = \frac{t^{1/3}}{5} \qquad \text{for Jackalopes}$$
$$\lambda_S(t) = \frac{t^{1/2}}{10} \qquad \text{for Snipes}$$

where: $0 \le t \le 24$ and t is the number of hours past midnight
    If the student takes photographs between 1 pm and 5 pm, calculate the probability that he will take at least 1 photograph of each animal.

A. Less than 0.45

B. At least 0.45, but less than 0.60

C. At least 0.60, but less than 0.75

D. At least 0.75, but less than 0.90

E. At least 0.90

Solution. The number of Jackalopes and Snipes between 1 pm and 5 pm are Poisson random variables with respective means

$$\frac{1}{5} \int_{13}^{17} t^{1/3} \, dt = \frac{3}{4(5)} (17^{4/3} - 13^{4/3}) = 1.971665$$

and

$$\frac{1}{10} \int_{13}^{17} t^{1/2} \, dt = \frac{2}{3(10)} (17^{3/2} - 13^{3/2}) = 1.548042.$$

Because the two Poisson processes are independent (note: here we are not using the property of independent increments),

$$\begin{aligned}
& \Pr \left(N_J(17) - N_J(13) \geq 1, N_S(17) - N_S(13) \geq 1 \right) \\
= \ & \Pr \left(N_J(17) - N_J(13) \geq 1 \right) \Pr \left(N_S(17) - N_S(13) \geq 1 \right) \\
= \ & \left[1 - \Pr \left(N_J(17) - N_J(13) = 0 \right) \right] \left[1 - \Pr \left(N_S(17) - N_S(13) = 0 \right) \right] \\
= \ & (1 - e^{-1.971665})(1 - e^{-1.548042}) \\
= \ & \boxed{0.6777}. \quad \textbf{(Answer: C)}
\end{aligned}$$

$\square$

Problem 1.1.9. (CAS Exam 3 Fall 2005 Question 26: Probability, non-homogeneous)
The number of reindeer injuries on December 24 follows a Poisson process with intensity function:
$$\lambda(t) = (t/12)^{1/2} \qquad 0 \leq t \leq 24, \text{ where } t \text{ is measured in hours}$$

Calculate the probability that no reindeer will be injured during the last hour of the day.

A. Less than 30%

B. At least 30%, but less than 40%

C. At least 40%, but less than 50%

D. At least 50%, but less than 60%

E. At least 60%

Solution. We need

$$\begin{aligned}
\Pr \left(N(24) - N(23) = 0 \right) &= \exp \left[-\int_{23}^{24} (t/12)^{1/2} \, dt \right] \\
&= \exp \left[-\frac{2}{3(12)^{1/2}} (24^{3/2} - 23^{3/2}) \right] \\
&= \boxed{0.24675}. \quad \textbf{(Answer: A)}
\end{aligned}$$

$\square$

Problem 1.1.10. [HARDER!] (Rate function mimics the normal density function)
You are given that claim counts follow a non-homogeneous Poisson process with intensity
function $\lambda(t) = e^{-t^2/4}$.

Calculate the probability of at least two claims between time 1 and time 2.

  A. Less than 0.10

  B. At least 0.10, but less than 0.15

  C. At least 0.15, but less than 0.20

  D. At least 0.20, but less than 0.25

  E. At least 0.25

Solution. The number of claims between time 1 and time 2 is a Poisson random variable
with mean
$$\int_1^2 \lambda(t)\,dt = \int_1^2 e^{-t^2/4}\,dt.$$
Note that the integrand resembles the density function of a normal distribution with mean
0 and variance 2, except that the normalizing constant $1/\sqrt{2\pi(2)}$ is missing. Hence

$$
\begin{aligned}
\int_1^2 e^{-t^2/4}\,dt &= \sqrt{2\pi(2)} \int_1^2 \frac{1}{\sqrt{2\pi(2)}} e^{-t^2/4}\,dt \\
&= 2\sqrt{\pi}\,\Pr\left(1 < N(0,2) < 2\right) \\
&= 2\sqrt{\pi}\left[\Phi\left(\frac{2}{\sqrt{2}}\right) - \Phi\left(\frac{1}{\sqrt{2}}\right)\right] \\
&= 2\sqrt{\pi}[\Phi(1.41) - \Phi(0.71)] \\
&= 2\sqrt{\pi}(0.9207 - 0.7611) \\
&= 0.565767.
\end{aligned}
$$

Finally, the required probability is

$$
\begin{aligned}
\Pr\left(N(2) - N(1) \geq 2\right) &= 1 - \Pr\left(N(2) - N(1) \leq 1\right) \\
&= 1 - e^{-0.565767}(1 + 0.565767) \\
&= \boxed{0.1108}. \quad \textbf{(Answer: B)}
\end{aligned}
$$

$\square$

Problem 1.1.11. (Probability for $N(s)$ given $N(t)$ for $s \leq t$) Customers arrive at a
post office in accordance with a Poisson process with a rate of 5 per hour. The post office
opens at 9:00 am.

Ten customers have arrived before 11:00 am.

Calculate the probability that only two customers have arrived before 9:30 am.

  A. Less than 0.15

B. At least 0.15, but less than 0.20

C. At least 0.20, but less than 0.25

D. At least 0.25, but less than 0.30

E. At least 0.30

Solution. Note that $N(0.5)|N(2) = 10$ has a binomial distribution with parameters 10 and $0.5/2 = 0.25$. The conditional probability that $N(0.5) = 2$ equals

$$\binom{10}{2}0.25^2(1 - 0.25)^8 = \boxed{0.2816}. \quad \textbf{(Answer: D)}$$

$\square$

Problem 1.1.12. [HARDER!] (Mean of a conditional sandwiched Poisson process value) You are given that $\{N(t)\}$ is a Poisson process with rate $\lambda = 2$.

   Calculate the expected value of $N(3)$, conditional on $N(2) = 3$ and $N(5) = 10$.

A. Less than 5

B. At least 5, but less than 6

C. At least 6, but less than 7

D. At least 7, but less than 8

E. At least 8

Solution. We are interested in the distribution of the value of a Poisson process at a particular time point, given the process values at an *earlier* time as well as a *later* time. To reduce this two-condition setting to the one-condition setting involving $N(s)$ given $N(t)$ for $s \leq t$, we consider the *translated Poisson process* $\{N^2(t)\}_{t\geq 0}$ defined by $N^2(t) := N(2+t) - N(2)$. This translated process looks at the original Poisson process $\{N(t)\}$ from time 2 (hence the superscript "2") onward, but with values translated downward by $N(2)$ units. It is easy to conceive (and can be rigorously shown) that $\{N^2(t)\}_{t\geq 0}$ is indeed a Poisson process (see Exercise 5.35 of Ross). Moreover, because of independent increments, $\{N^2(t)\}_{t\geq 0}$ is independent of $N(2)$. In terms of the translated Poisson process, $N(3)$ can be written as the telescoping sum

$$N(3) = [N(3) - N(2)] + N(2) = N^2(1) + 3.$$

As $\{N(2) = 3, N(5) = 10\} = \{N(2) = 3, N^2(3) = 7\}$, the conditional distribution of $N_2(1)$ is

$$N^2(1) \mid [\underbrace{N(2) = 3}_{\text{get rid of this}}, N^2(3) = 7] \quad \sim \quad N^2(1) \mid N^2(3) = 7 \quad \sim \quad \text{Bin}(7, 1/3).$$

Finally, the conditional expected value of $N(3)$ is

$$
\begin{aligned}
\mathrm{E}[N(3)|N(2) = 3, N(5) = 10] &= 3 + \mathrm{E}[N^2(1)|N(2) = 3, N^2(3) = 7] \\
&= 3 + \frac{7}{3} \\
&= \boxed{5.3333}. \quad \textbf{(Answer: B)}
\end{aligned}
$$

$\square$

Remark. In general, for $t_1 \leq s \leq t_2$ and a non-homogeneous Poisson process with mean value function $m(\cdot)$, the distribution of $N(s)$ conditional on $N(t_1) = A$ and $N(t_2) = B$ is

$$
A + \mathrm{Bin}\left(B - A, \frac{m(s) - m(t_1)}{m(t_2) - m(t_1)}\right).
$$

The conditional expected value is

$$
A + (B - A)\left[\frac{m(s) - m(t_1)}{m(t_2) - m(t_1)}\right] = \left[\frac{m(t_2) - m(s)}{m(t_2) - m(t_1)}\right] A + \left[\frac{m(s) - m(t_1)}{m(t_2) - m(t_1)}\right] B,
$$

which is a weighted average of A and B.

Part V

Practice Examinations

Prelude. It is now time to test your understanding of the whole syllabus by working out three comprehensive practice exams. Each of these three exams has 40 multiple-choice questions, sorted and distributed in line with the distribution of questions in the MAS-I exam syllabus. You should time yourself and work on each exam in a simulated exam environment. Detailed solutions are provided and the relevant chapters and sections in this manual identified following each practice exam. The difficulty of these three practice exams is slightly higher than that of a typical CAS exam, so if you do well in these three exams (say, you answer more than 30 out of 40 questions correctly in each exam), you should be able to pass Exam MAS-I with ease!

An abridged version of the CAS exam instructions is reproduced overleaf for your information.

Practice Examination 1

1. People arrive at a train station in accordance with a Poisson process with rate λ.

   At time 0, the train station is empty. At time 10, the bus departs.

   Five people get on the train when the bus departs.

   Calculate the expected amount of waiting time of the person who first arrives at the bus stop.

   A. Less than 2.5

   B. At least 2.5, but less than 5.0

   C. At least 5.0, but less than 7.5

   D. At least 7.5, but less than 10.0

   E. There is not enough information to determine the answer.

2. You are given the following information about the arrival of vehicles:

   - Taxis arrivals follow the Poisson process with rate $\lambda = 1$ per 10 minutes.
   - Bus arrivals follow the Poisson process with rate $\lambda = 4$ per 30 minutes.
   - Streetcar arrivals follow the Poisson process with rate $\lambda = 2$ per hour.

   Calculate the probability that the second vehicle arrives within 10 minutes.

   A. Less than 0.5

   B. At least 0.5, but less than 0.6

   C. At least 0.6, but less than 0.7

   D. At least 0.7, but less than 0.8

   E. At least 0.8

Exam Continued On Next Page

Solutions to Practice Examination 1

Answer Key

Question	Answer		Question	Answer
1	D		21	D
2	D		22	D
3	B		23	A
4	A		24	B
5	A		25	D
6	B		26	B
7	D		27	C
8	B		28	D
9	D		29	A
10	C		30	C
11	E		31	B
12	E		32	E
13	E		33	B
14	D		34	A
15	C		35	E
16	B		36	B
17	C		37	B
18	E		38	D
19	E		39	C
20	D		40	D

1. **(Conditional arrival time)**

   Solution. Given that $N(10) = 5$, the conditional expected value of $S_1(= T_1)$, the arrival time of the first person arriving at the train station, is given by

   $$\mathrm{E}[S_1|N(10)] = 5] = \frac{1}{5+1}(10) = \frac{5}{3}.$$

   As the bus is scheduled to depart at time 10, the conditional expected waiting time is

   $$10 - \mathrm{E}[S_1|N(10)] = 5] = \frac{25}{3} = \boxed{8.3333}. \quad \textbf{(Answer: D)}$$

   $\square$

   Remark. The value of the Poisson rate λ is not needed to solve this problem.

2. **(Modified from CAS Exam ST Fall 2014 Question 1: Probability associated with waiting time)**