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Spring 2018 Edition | **Volume I**

Samuel A. Broverman, Ph.D., ASA

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SOA Exam STAM Study Manual

Spring 2018 Edition | **Samuel A. Broverman, Ph.D., ASA**

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PRACTICE EXAMS

INTRODUCTORY COMMENTS

This study guide is designed to help in the preparation for the Society of Actuaries STAM Exam.

The first part of this manual consists of a summary of notes, illustrative examples and problem sets with detailed solutions. The second part consists of 5 practice exams.

The practice exams all have 35 questions. The level of difficulty of the practice exams has been designed to be similar to that of the past 3.5-hour exams. Some of the questions in the problem sets are taken from the relevant topics on SOA exams that have been released prior to 2009 but the practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official references. I have been, perhaps, more thorough than necessary on a couple of topics, such as maximum likelihood estimation and Bayesian credibility.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into 42 sections of varying lengths, with some suggested time frames for covering the material. There are almost 200 examples in the notes and over 900 exercises in the problem sets, all with detailed solutions. The 5 practice exams have 35 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Some of the in the problem sets that have come from previous SOA exams. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are almost 1300 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study guide.

I suggest that you work through the study guide by studying a section of notes and then attempting the exercises in the problem set that follows that section. The order of the sections of notes is the order that I recommend in covering the material, although the material on pricing and reserving in Sections 39 to 42 is independent of the other material on the exam. The order of topics in this manual is not the same as the order presented on the exam syllabus. About 80% or more of the material on the STAM Exam was on the former Exam C.

It has been my intention to make this study guide self-contained and comprehensive for the STAM Exam topics, however there are some exam topics for which the study notes are essentially summaries of concepts. For that material, I have attempted to summarize concepts as well, but it is best to refer to original reference material on all topics. At the time this study manual is being written, the STAM Exam Syllabus available on the SOA website contains a statement indicating that the study note titled "STAM-24-18 Supplement to Chapter 3 of *Intro to Ratemaking and Loss Reserving for Property and Casualty Insurance*. Fourth Edition" will be available at a later date. When that study note is available, ACTEX will release, if needed, a supplement to this study manual to cover the content of that study note.

While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations. In order for the review notes in this study guide to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study guide. The study guide begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance.

Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org .

If you have any questions, comments, criticisms or compliments regarding this study guide, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from www.actexmadriver.com .

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

Samuel A. Broverman January 2018
Department of Statistical Sciences www.sambroverman.com
University of Toronto E-mail: sam@utstat.toronto.edu or 2brove@rogers.com

NOTES

AND

PROBLEM SETS

SECTION 1 - PRELIMINARY REVIEW - PROBABILITY

Basic Probability, Conditional Probability and Independence

A significant part of the STAM Exam involves probability and statistical methods applied to various aspects of loss modeling and model estimation. A good background in probability and statistics is necessary to fully understand models and the modeling that is done. In this section of the study guide, we will review fundamental probability rules.

1.1 Basic Probability Concepts

Sample point and probability space

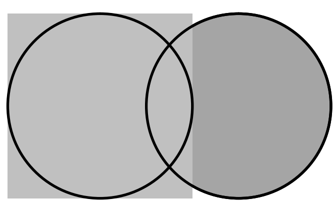
A sample point is the simple outcome of a random experiment. The probability space (also called sample space) is the collection of all possible sample points related to a specified experiment. When the experiment is performed, one of the sample points will be the outcome. An experiment could be observing the loss that occurs on an automobile insurance policy during the course of one year, or observing the number of claims arriving at an insurance office in one week. The probability space is the "full set" of possible outcomes of the experiment. In the case of the automobile insurance policy, it would be the range of possible loss amounts that could occur during the year, and in the case of the insurance office weekly number of claims, the probability space would be the set of integers $\{0, 1, 2, ...\}$.

Event

Any collection of sample points, or any subset of the probability space is referred to as an event. We say "event A has occurred" if the experimental outcome was one of the sample points in A.

Union of events A and B

$A \cup B$ denotes the union of events A and B, and consists of all sample points that are in either A or B.



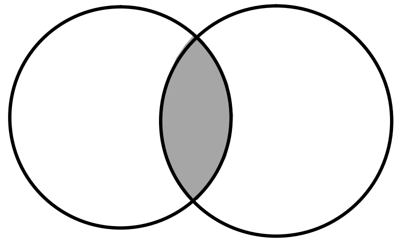
$A \cup B$

Union of events $A_1, A_2, ..., A_n$

$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ denotes the union of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are in at least one of the A_i's. This definition can be extended to the union of infinitely many events.

Intersection of events $A_1, A_2, \ldots, A_n$

$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ denotes the intersection of the events $A_1, A_2, \ldots, A_n$, and consists of all sample points that are simultaneously in all of the A_i's.



$A \cap B$

Mutually exclusive events $A_1, A_2, \ldots, A_n$

Two events are mutually exclusive if they have no sample points in common, or equivalently, if they have **empty intersection**. Events $A_1, A_2, \ldots, A_n$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$, where $\emptyset$ denotes the empty set with no sample points. Mutually exclusive events cannot occur simultaneously.

Exhaustive events $B_1, B_2, \ldots, B_n$

If $B_1 \cup B_2 \cup \cdots \cup B_n = S$, the entire probability space, then the events $B_1, B_2, \ldots, B_n$ are referred to as exhaustive events.

Complement of event A

The complement of event A consists of all sample points in the probability space that are **not in A**. The complement is denoted $\overline{A}$, $\sim A$, A' or A^c and is equal to $\{x : x \notin A\}$. When the underlying random experiment is performed, to say that the complement of A has occurred is the same as saying that A has not occurred.

Subevent (or subset) A of event B

If event B contains all the sample points in event A, then A is a subevent of B, denoted $A \subset B$. The occurrence of event A implies that event B has occurred.

Partition of event A

Events $C_1, C_2, \ldots, C_n$ form a partition of event A if $A = \bigcup_{i=1}^{n} C_i$ and the C_i's are mutually exclusive.

DeMorgan's Laws

(i) $(A \cup B)' = A' \cap B'$, to say that $A \cup B$ has not occurred is to say that A has not occurred <u>and</u> B has not occurred ; this rule generalizes to any number of events;

$$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$$

(ii) $(A \cap B)' = A' \cup B'$, to say that $A \cap B$ has not occurred is to say that either A has not occurred <u>or</u> B has not occurred (or both have not occurred) ; this rule generalizes to any number of events, $\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$

Indicator function for event A

The function $I_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$ is the indicator function for event A, where x denotes a sample point. $I_A(x)$ is 1 if event A has occurred.

Some important rules concerning probability are given below.

(i) $P[S] = 1$ if S is the entire probability space (when the underlying experiment is performed, some outcome must occur with probability 1).

(ii) $P[\emptyset] = 0$ (the probability of no face turning up when we toss a die is 0).

(iii) If events $A_1, A_2, ..., A_n$ are mutually exclusive (also called disjoint) then

$$P[\bigcup_{i=1}^{n} A_i] = P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = \sum_{i=1}^{n} P[A_i] \quad (1.1)$$

This extends to infinitely many mutually exclusive events.

(iv) For any event A, $0 \leq P[A] \leq 1$

(v) If $A \subset B$ then $P[A] \leq P[B]$

(vi) For any events A, B and C, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ (1.2)

(vii) For any event A, $P[A'] = 1 - P[A]$ (1.3)

(viii) For any events A and B, $P[A] = P[A \cap B] + P[A \cap B']$ (1.4)

(ix) For exhaustive events $B_1, B_2, ..., B_n$, $P[\bigcup_{i=1}^{n} B_i] = 1$ (1.5)

If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive, they form a partition of the entire probability space, and for any event A,

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] = \sum_{i=1}^{n} P[A \cap B_i] \quad (1.6)$$

(x) The words "percentage" and "proportion" are used as alternatives to "probability". As an example, if we are told that the percentage or proportion of a group of people that are of a certain type is 20%, this is generally interpreted to mean that a randomly chosen person from the group has a 20% probability of being of that type. This is the "long-run frequency" interpretation of probability. As another example, suppose that we are tossing a fair die. In the long-run frequency interpretation of probability, to say that the probability of tossing a 1 is $\frac{1}{6}$ is the same as saying that if we repeatedly toss the die, the proportion of tosses that are 1's will approach $\frac{1}{6}$.

1.2 Conditional Probability and Independence of Events

Conditional probability arises throughout the STAM Exam material. It is important to be familiar and comfortable with the definitions and rules of conditional probability.

Conditional probability of event A given event B

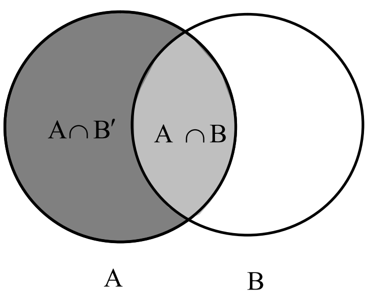
If $P(B) > 0$, then $\boldsymbol{P(A|B) = \frac{P(A\cap B)}{P(B)}}$ is the conditional probability that event A occurs given that event B has occurred. By rewriting the equation we get $P(A \cap B) = P(A|B) \cdot P(B)$.

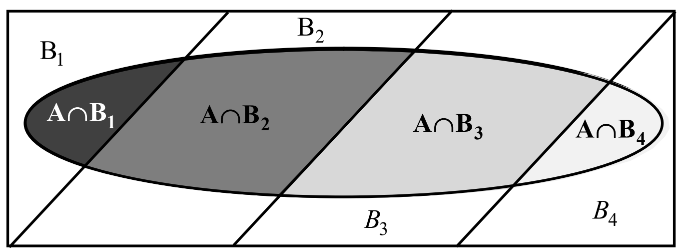
Partition of a Probability Space

Events $B_1, B_2, ..., B_n$ are said to form a partition of a probability space S if
(i) $B_1 \cup B_2 \cup \cdots \cup B_n = S$ and (ii) $B_i \cap B_j = \emptyset$ for any pair with $i \neq j$.

A partition is a disjoint collection of events which combines to be the full probability space.
A simple example of a partition is any event B and its complement B'.

If A is any event in probability space S and $\{B_1, B_2, ..., B_n\}$ is a partition of probability space S, then
$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$.
A special case of this rule is $P(A) = P(A \cap B) + P(A \cap B')$ for any two events A and B.





Bayes rule and Bayes Theorem

For any events A and B with $P(A) > 0$, $\boldsymbol{P(B|A) = \frac{P(A|B)\times P(B)}{P(A)}}$ (1.7)

If $B_1, B_2, ..., B_n$ form a partition of the entire sample space S, then

$$P(B_j|A) = \frac{P(A|B_j)\times P(B_j)}{\sum_{i=1}^{n} P(A|B_i)\times P(B_i)} \quad \text{for each } j = 1, 2, ..., n \tag{1.8}$$

The values of $P(B_j)$ are called prior probabilities, and the value of $P(B_j|A)$ is called a posterior probability. Variations on this rule are very important in Bayesian credibility.

Independent events A and B

If events A and B satisfy the relationship $\boldsymbol{P(A \cap B) = P(A) \times P(B)}$, then the events are said to be independent or stochastically independent or statistically independent. The independence of (non-empty) events A and B is equivalent to $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

Mutually independent events $A_1, A_2, ..., A_n$

The events are mutually independent if

(i) for any A_i and A_j, $P(A_i \cap A_j) = P(A_i) \times P(A_j)$, and
(ii) for any A_i, A_j and A_k, $P(A_i \cap A_j \cap A_k) = P(A_i) \times P(A_j) \times P(A_k)$,
and so on for any subcollection of the events, including all events:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n) = \prod_{i=1}^{n} P(A_i) \tag{1.9}$$

Here are some rules concerning conditional probability and independence. These can be verified in a fairly straightforward way from the definitions given above.

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B (1.10)

(ii) $P(A \cap B) = P(B|A) \times P(A) = P(A|B) \times P(B)$ for any events A and B (1.11)

(iii) If $B_1, B_2, ..., B_n$ form a partition of the sample space S, then for any event A

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) \times P(B_i) \tag{1.12}$$

As a special case, for any events A and B, we have

$$P(A) = P(A \cap B) + P(A \cap B') = P(A|B) \times P(B) + P(A|B') \times P(B') \tag{1.13}$$

(iv) If $P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

(v) $P(A') = 1 - P(A)$ and $P(A'|B) = 1 - P(A|B)$ (1.14)

(vi) if $A \subset B$ then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$, and $P(B|A) = 1$

(vii) if A and B are independent events then A' and B are independent events, A and B' are independent events, and A' and B' are independent events

(viii) since $P(\emptyset) = P(\emptyset \cap A) = 0 = P(\emptyset) \cdot P(A)$ for any event A, it follows that $\emptyset$ is independent of any event A

Example 1-1:
Suppose a fair six-sided die is tossed. We define the following events:

$A =$ "the number tossed is ≤ 3" $= \{1, 2, 3\}$, $B =$ "the number tossed is even" $= \{2, 4, 6\}$
$C =$ "the number tossed is a 1 or a 2" $= \{1, 2\}$
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1, 6\}$

The conditional probability of A given B is

$$P(A|B) = \frac{P(\{1,2,3\} \cap \{2,4,6\})}{P(\{2,4,6\})} = \frac{P(\{2\})}{P(\{2,4,6\})} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Events A and B are not independent, since $\frac{1}{6} = P(A \cap B) \neq P(A) \times P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,
or alternatively, events A and B are not independent since $P(A|B) \neq P(A)$.

$P(A|C) = 1 \neq \frac{1}{2} = P(A)$, so that A and C are not independent.
$P(B|C) = \frac{1}{2} = P(B)$, so that B and C are independent

(alternatively, $P(B \cap C) = P(\{2\}) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(B) \cdot P(C)$).
It is not difficult to check that both A and B are independent of D. □

IMPORTANT NOTE: The following manipulation of event probabilities arises from time to time:
$\boldsymbol{P(A) = P(A|B) \cdot P(B) + P(A|B') \times P(B')}$. If we know the conditional probabilities for event A given some other event B and its complement B', and if we know the (unconditional) probability of event B, then we can find the probability of event A. One of the important aspects of applying this relationship is the determination of the appropriate events A and B.

Example 1-2:
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. An Urn is chosen at random, and a ball is randomly selected from that Urn. Find the probability that the ball chosen is white.

Solution:
Let B be the event that Urn I is chosen and B' is the event that Urn II is chosen. The implicit assumption is that both Urns are equally likely to be chosen (this is the meaning of "an Urn is chosen at random"). Therefore, $P(B) = \frac{1}{2}$ and $P(B') = \frac{1}{2}$. Let A be the event that the ball chosen in white. If we know that Urn I was chosen, then there is $\frac{1}{2}$ probability of choosing a white ball (2 white out of 4 balls, it is assumed that each ball has the same chance of being chosen); this can be described as $P(A|B) = \frac{1}{2}$.

In a similar way, if Urn II is chosen, then $P(A|B') = \frac{3}{5}$ (3 white out of 5 balls). We can now apply the relationship described prior to this example. $P(A \cap B) = P(A|B) \times P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$, and

$P(A \cap B') = P(A|B') \times P(B') = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) = \frac{3}{10}$. Finally,
$P(A) = P(A \cap B) + P(A \cap B') = \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$.

The order of calculations can be summarized in the following table

	A
B	**1.** $P(A \cap B) = P(A|B) \times P(B)$
B'	**2.** $P(A \cap B') = P(A|B') \times P(B')$
	3. $P(A) = P(A \cap B) + P(A \cap B')$

□

Example 1-3:
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. One ball is chosen at random from Urn I and transferred to Urn II, and then a ball is chosen at random from Urn II. The ball chosen from Urn II is observed to be white. Find the probability that the ball transferred from Urn I to Urn II was white.

Solution:
Let B denote the event that the ball transferred from Urn I to Urn II was white and let A denote the event that the ball chosen from Urn II is white. We are asked to find $P(B|A)$.

From the simple nature of the situation (and the usual assumption of uniformity in such a situation, meaning all balls are equally likely to be chosen from Urn I in the first step), we have $P(B) = \frac{1}{2}$ (2 of the 4 balls in Urn I are white), and by implication, it follows that $P[B'] = \frac{1}{2}$.

If the ball transferred is white, then Urn II has 4 white and 2 black balls, and the probability of choosing a white ball out of Urn II is $\frac{2}{3}$; this is $P(A|B) = \frac{2}{3}$.

If the ball transferred is black, then Urn II has 3 white and 3 black balls, and the probability of choosing a white ball out of Urn II is $\frac{1}{2}$; this is $P(A|B') = \frac{1}{2}$.

All of the information needed has been identified. We do calculations in the following order:

1. $P[A \cap B] = P[A|B] \times P[B] = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$
2. $P[A \cap B'] = P[A|B'] \times P[B'] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
3. $P[A] = P[A \cap B] + P[A \cap B'] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
4. $P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{1/3}{7/12} = \frac{4}{7}$ □

Example 1-4:
Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

Solution:
The event " a 6 is thrown" is denoted by "6"

$$P[\text{die 1}|\text{"6"}] = \frac{P[(\text{die 1}) \cap (\text{"6"})]}{P[\text{"6"}]} = \frac{P[\text{"6"}|\text{die 1}] \times P[\text{die 1}]}{P[\text{"6"}]} = \frac{p \cdot \frac{1}{3}}{P[\text{"6"}]}$$

But

$$P[\text{"6"}] = P[(\text{"6"}) \cap (\text{die 1})] + P[(\text{"6"}) \cap (\text{die 2})] + P[(\text{"6"}) \cap (\text{die 3})]$$
$$= P[\text{"6"}|\text{die 1}] \times P[\text{die 1}] + P[\text{"6"}|\text{die 2}] \times P[\text{die 2}] + P[\text{"6"}|\text{die 3}] \times P[\text{die 3}]$$
$$= p \times \tfrac{1}{3} + q \times \tfrac{1}{3} + r \times \tfrac{1}{3} = \frac{p+q+r}{3} \rightarrow P[\text{die 1}|\text{"6"}] = \frac{p \times \frac{1}{3}}{P[\text{"6"}]} = \frac{p \times \frac{1}{3}}{(p+q+r) \cdot \frac{1}{3}} = \frac{p}{p+q+r}$$ □

SECTION 1 PROBLEM SET
Preliminary Review - Probability

1. A survey of 1000 people determines that 80% like walking and 60% like biking, and all like at least one of the two activities. How many people in the survey like biking but not walking?

   A) 0 B) .1 C) .2 D) .3 E) .4

2. A life insurer classifies insurance applicants according to the following attributes:
   M - the applicant is male
   H - the applicant is a homeowner
   Out of a large number of applicants the insurer has identified the following information:
   40% of applicants are male, 40% of applicants are homeowners and
   20% of applicants are female homeowners.

   Find the percentage of applicants who are male and do not own a home.

   A) .1 B) .2 C) .3 D) .4 E) .5

3. Let A, B, C and D be events such that $B = A'$, $C \cap D = \emptyset$, and
   $P[A] = \frac{1}{4}$, $P[B] = \frac{3}{4}$, $P[C|A] = \frac{1}{2}$, $P[C|B] = \frac{3}{4}$, $P[D|A] = \frac{1}{4}$, $P[D|B] = \frac{1}{8}$

   Calculate $P[C \cup D]$.

   A) $\frac{5}{32}$ B) $\frac{1}{4}$ C) $\frac{27}{32}$ D) $\frac{3}{4}$ E) 1

4. You are given that $P[A] = .5$ and $P[A \cup B] = .7$.
   Actuary 1 assumes that A and B are independent and calculates $P[B]$ based on that assumption.
   Actuary 2 assumes that A and B mutually exclusive and calculates $P[B]$ based on that assumption.
   Find the absolute difference between the two calculations.

   A) 0 B) .05 C) .10 D) .15 E) .20

5. A test for a disease correctly diagnoses a diseased person as having the disease with probability .85. The test incorrectly diagnoses someone without the disease as having the disease with a probability of .10. If 1% of the people in a population have the disease, what is the chance that a person from this population who tests positive for the disease actually has the disease?

   A) .0085 B) .0791 C) .1075 D) .1500 E) .9000

6. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl 1. Find the probability that bowl 1 still has 5 black and 5 white balls.

   A) $\frac{2}{3}$ B) $\frac{3}{5}$ C) $\frac{6}{11}$ D) $\frac{1}{2}$ E) $\frac{6}{13}$

7. People passing by a city intersection are asked for the month in which they were born. It is assumed that the population is uniformly divided by birth month, so that any randomly passing person has an equally likely chance of being born in any particular month. Find the minimum number of people needed so that the probability that no two people have the same birth month is less than .5.

A) 2 B) 3 C) 4 D) 5 E) 6

8. In a T-maze, a laboratory rat is given the choice of going to the left and getting food or going to the right and receiving a mild electric shock. Assume that before any conditioning (in trial number 1) rats are equally likely to go the left or to the right. After having received food on a particular trial, the probability of going to the left and right become .6 and .4, respectively on the following trial. However, after receiving a shock on a particular trial, the probabilities of going to the left and right on the next trial are .8 and .2, respectively. What is the probability that the animal will turn left on trial number 2?

A) .1 B) .3 C) .5 D) .7 E) .9

9. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change his choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that he gets the door with the prize?

A) 0 B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

10. A supplier of a testing device for a type of component claims that the device is highly reliable, with $P[A|B] = P[A'|B'] = .95$, where

$A =$ device indicates component is faulty, and
$B =$ component is faulty .

You plan to use the testing device on a large batch of components of which 5% are faulty. Find the probability that the component is faulty given that the testing device indicates that the component is faulty .

A) 0 B) .05 C) .15 D) .25 E) .50

11. An insurer classifies flood hazard based on geographical areas, with hazard categorized as low, medium and high. The probability of a flood occurring in a year in each of the three areas is

Area Hazard	low	medium	high
Prob. of Flood	.001	.02	.25

The insurer's portfolio of policies consists of a large number of policies with 80% low hazard policies, 18% medium hazard policies and 2% high hazard policies. Suppose that a policy had a flood claim during a year. Find the probability that it is a high hazard policy.

A) .50 B) .53 C) .56 D) .59 E) .62

12. One of the questions asked by an insurer on an application to purchase a life insurance policy is whether or not the applicant is a smoker. The insurer knows that the proportion of smokers in the general population is .30, and assumes that this represents the proportion of applicants who are smokers. The insurer has also obtained information regarding the honesty of applicants:

40% of applicants that are smokers say that they are non-smokers on their applications,
none of the applicants who are non-smokers lie on their applications.

What proportion of applicants who say they are non-smokers are actually non-smokers?

A) 0 B) $\frac{6}{41}$ C) $\frac{12}{41}$ D) $\frac{35}{41}$ E) 1

13. When sent a questionnaire, 50% of the recipients respond immediately. Of those who do not respond immediately, 40% respond when sent a follow-up letter. If the questionnaire is sent to 4 persons and a follow-up letter is sent to any of the 4 who do not respond immediately, what is the probability that at least 3 never respond?

A) $(.3)^4 + 4(.3)^3(.7)$
B) $4(.3)^3(.7)$
C) $(.1)^4 + 4(.1)^3(.9)$
D) $.4(.3)(.7)^3 + (.7)^4$
E) $(.9)^4 + 4(.9)^3(.1)$

14. A fair coin is tossed. If a head occurs, 1 fair die is rolled; if a tail occurs, 2 fair dice are rolled. If Y is the total on the die or dice, then $P[Y = 6] =$

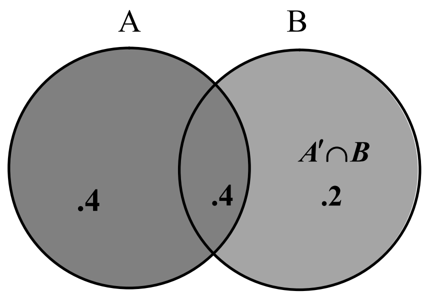
A) $\frac{1}{9}$ B) $\frac{5}{36}$ C) $\frac{11}{72}$ D) $\frac{1}{6}$ E) $\frac{11}{36}$

15. In Canada's national 6-49 lottery, a ticket has 6 numbers each from 1 to 49, with no repeats. Find the probability of matching exactly 4 of the 6 winning numbers if the winning numbers are all randomly chosen.

A) .00095 B) .00097 C) .00099 D) .00101 E) .00103

SECTION 1 PROBLEM SET SOLUTIONS

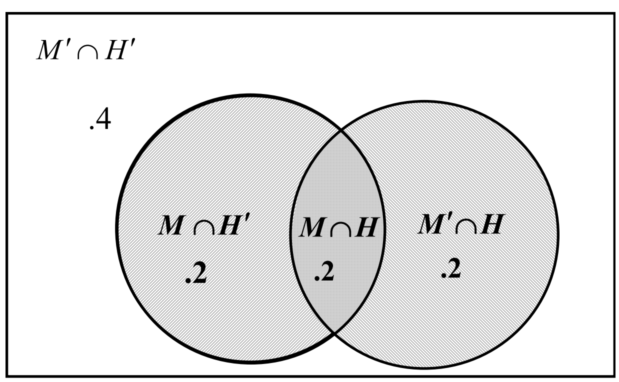
1. Let $A =$ "like walking" and $B =$ "like biking". We use the interpretation that "percentage" and "proportion" are taken to mean "probability".
We are given $P(A) = .8$, $P(B) = .6$ and $P(A \cup B) = 1$.
From the diagram below we can see that since $A \cup B = A \cup (B \cap A')$ we have
$P(A \cup B) = P(A) + P(A' \cap B) \rightarrow P(A' \cap B) = .2$ is the proportion of people who like biking but (and) not walking . In a similar way we get $P(A \cap B') = .4$



An algebraic approach is the following. Using the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get $1 = .8 + .6 - P(A \cap B) \rightarrow P(A \cap B) = .4$. Then, using the rule
$P(B) = P(B \cap A) + P(B \cap A')$, we get $P(B \cap A') = .6 - .4 = .2$. Answer: C

2. $P[M] = .4$, $P[M'] = .6$, $P[H] = .4$, $P[H'] = .6$, $P[M' \cap H] = .2$,
We wish to find $P[M \cap H']$. From probability rules, we have

$.6 = P[H'] = P[M' \cap H'] + P[M \cap H']$, and
$.6 = P[M'] = P[M' \cap H] + P[M' \cap H'] = .2 + P[M' \cap H']$.
Thus, $P[M' \cap H'] = .4$ and then $P[M \cap H'] = .2$. The following diagram identifies the component probabilities.



The calculations above can also be summarized in the following table. The events across the top of the table categorize individuals as male (M) or female (M'), and the events down the left side of the table categorize individuals as homeowners (H) or non-homeowners (H').

	$P(M) = .4$, given		$P(M') = 1 - .4 = .6$
$P(H) = .4$ given $\Downarrow$	$P(M \cap H)$ $= P(H) - P(M' \cap H) = .4 - .2 = .2$	$\Leftarrow$	$P(M' \cap H) = .2$, given
$P(H') = 1 - .4 = .6$	$P(M \cap H') = P(M) - P(M \cap H) = .4 - .2 = .2$		Answer: B

3. Since C and D have empty intersection, $P[C \cup D] = P[C] + P[D]$.

Also, since A and B are "exhaustive" events (since they are complementary events, their union is the entire sample space, with a combined probability of

$$P[A \cup B] = P[A] + P[B] = 1 \text{).}$$

We use the rule $P[C] = P[C \cap A] + P[C \cap A']$, and the rule $P[C|A] = \frac{P[A \cap C]}{P[A]}$ to get

$P[C] = P[C|A] \times P[A] + P[C|A'] \times P[A'] = \frac{1}{2} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} = \frac{11}{16}$ and
$P[D] = P[D|A] \times P[A] + P[D|A'] \times P[A'] = \frac{1}{4} \times \frac{1}{4} + \frac{1}{8} \times \frac{3}{4} = \frac{5}{32}$.

Then, $P[C \cup D] = P[C] + P[D] = \frac{27}{32}$. Answer: C.

4. Actuary 1: Since A and B are independent, so are A' and B'.
$P[A' \cap B'] = 1 - P[A \cup B] = .3$.

But $.3 = P[A' \cap B'] = P[A'] \cdot P[B'] = (.5)P[B'] \to P[B'] = .6 \to P[B] = .4$.
Actuary 2: $.7 = P[A \cup B] = P[A] + P[B] = .5 + P[B] \to P[B] = .2$.
Absolute difference is $|.4 - .2| = .2$. Answer: E

5. We define the following events: D - a person has the disease,
TP - a person tests positive for the disease. We are given $P[TP|D] = .85$ and
$P[TP|D'] = .10$ and $P[D] = .01$. We wish to find $P[D|TP]$.
Using the formulation for conditional probability we have $P[D|TP] = \frac{P[D \cap TP]}{P[TP]}$.
But $P[D \cap TP] = P[TP|D] \times P[D] = (.85)(.01) = .0085$, and
$P[D' \cap TP] = P[TP|D'] \times P[D'] = (.10)(.99) = .099$. Then,
$P[TP] = P[D \cap TP] + P[D' \cap TP] = .1075 \to P[D|TP] = \frac{.0085}{.1075} = .0791$.

The following table summarizes the calculations.

$$\begin{array}{lcl}
P[D] = .01 \text{ , given} & \Rightarrow & P[D'] = 1 - P[D] = .99 \\
\quad\Downarrow & & \quad\Downarrow \\
P[D \cap TP] & & P[D' \cap TP] \\
= P[TP|D] \times P[D] = .0085 & & = P[TP|D'] \times P[D'] = .099
\end{array}$$

$$\Downarrow$$

$$P[TP] = P[D \cap TP] + P[D' \cap TP] = .1075$$

$$\Downarrow$$

$$P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{.0085}{.1075} = .0791.$$

Answer: B

6. Let C be the event that bowl 1 has 5 black balls after the exchange.
Let B_1 be the event that the ball chosen from bowl 1 is black, and let B_2 be the event that the ball chosen from bowl 2 is black.

Event C is the disjoint union of $B_1 \cap B_2$ and $B_1' \cap B_2'$ (black-black or white-white picks), so that $P[C] = P[B_1 \cap B_2] + P[B_1' \cap B_2']$.
The black-black combination has probability $(\frac{6}{11})(\frac{1}{2})$, since there is a $\frac{5}{10}$ chance of picking black from bowl 1, and then (with 6 black in bowl 2, which now has 11 balls) $\frac{6}{11}$ is the probability of picking black from bowl 2. This is

$$P[B_1 \cap B_2] = P[B_2|B_1] \times P[B_1] = (\tfrac{6}{11})(\tfrac{1}{2}).$$

In a similar way, the white-white combination has probability $(\frac{6}{11})(\frac{1}{2})$.

Then $P[C] = (\frac{6}{11})(\frac{1}{2}) + (\frac{6}{11})(\frac{1}{2}) = \frac{6}{11}$. Answer: C

7. $A_2 =$ event that second person has different birth month from the first.
$P(A_2) = \frac{11}{12} = .9167$.
$A_3 =$ event that third person has different birth month from first and second.

Then, the probability that all three have different birthdays is
$P[A_3 \cap A_2] = P[A_3|A_2] \cdot P(A_2) = (\frac{10}{12})(\frac{11}{12}) = .7639$.
$A_4 =$ event that fourth person has different birth month from first three.

Then, the probability that all four have different birthdays is
$$\begin{aligned} P[A_4 \cap A_3 \cap A_2] &= P[A_4|A_3 \cap A_2] \cdot P[A_3 \cap A_2] \\ &= P[A_4|A_3 \cap A_2] \cdot P[A_3|A_2] \cdot P(A_2) = (\tfrac{9}{12})(\tfrac{10}{12})(\tfrac{11}{12}) = .5729. \end{aligned}$$
$A_5 =$ event that fifth person has different birth month from first four.

Then, the probability that all five have different birthdays is
$$\begin{aligned} P[A_5 \cap A_4 \cap A_3 \cap A_2] &= P[A_5|A_4 \cap A_3 \cap A_2] \times P[A_4 \cap A_3 \cap A_2] \\ &= P[A_5|A_4 \cap A_3 \cap A_2] \times P[A_4|A_3 \cap A_2] \times P[A_3|A_2] \times P(A_2) \\ &= (\tfrac{8}{12})(\tfrac{9}{12})(\tfrac{10}{12})(\tfrac{11}{12}) = .3819 . \end{aligned}$$
Answer: D

8. $L1 =$ turn left on trial 1, $R1 =$ turn right on trial 1, $L2 =$ turn left on trial 2 .

We are given that $P[L1] = P[R1] = .5$.
$P[L2] = P[L2 \cap L1] + P[L2 \cap R1]$ since $L1, R1$ form a partition .
$P[L2|L1] = .6$ (if the rat turns left on trial 1 then it gets food and has a .6 chance of turning left on trial 2). Then $P[L2 \cap L1] = P[L2|L1] \times P[L1] = (.6)(.5) = .3$.

In a similar way, $P[L2 \cap R1] = P[L2|R1] \times P[R1] = (.8)(.5) = .4$.

Then, $P[L2] = .3 + .4 = .7$. Answer: D

9. We define the events $A =$ prize door is chosen after contestant switches doors , $B =$ prize door is initial one chosen by contestant. Then $P[B] = \frac{1}{3}$, since each door is equally likely to hold the prize initially. To find $P[A]$ we use the Law of Total Probability.

$$P[A] = P[A|B] \times P[B] + P[A|B'] \times P[B'] = (0)(\tfrac{1}{3}) + (1)(\tfrac{2}{3}) = \tfrac{2}{3}$$

If the prize door is initially chosen, then after switching, the door chosen is not the prize door, so that $P[A|B] = 0$. If the prize door is not initially chosen, then since the host shows the other non-prize door, after switching the contestant definitely has the prize door, so that $P[A|B'] = 1$.
Answer: E

10. We are given $P[B] = .05$. We can calculate entries in the following table in the order indicated.

	A	A'
B $P[B] = .05$ (given)	$P[A\|B] = .95$ (given) **1.** $P[A \cap B] = P[A\|B] \times P[B] = .0475$	$P[A'\|B'] = .95$ (given)
B' $P[B']$ $= 1 - P[B]$ $= .95$	**3.** $P[A \cap B']$ $= P[B'] - P[A' \cap B']$ $= .95 - .9025 = .0475$	**2.** $P[A' \cap B']$ $= P[A'\|B'] \times P[B']$ $= .95^2 = .9025$
	4. $P[A] = P[A \cap B] + P[A \cap B'] = .095$ **5.** $P[B\|A] = \frac{P[B \cap A]}{P[A]} = \frac{.0475}{.095} = .5$	Answer: E

11. This is a classical Bayesian probability situation. Let C denote the event that a flood claim occurred. We wish to find $P(H|C)$.

We can summarize the information in the following table, with the order of calculations indicated.

	L , $P(L) = .8$ (given)	M , $P(M) = .18$ (given)	H , $P(H) = .02$ (given)
C	$P(C\|L) = .001$ (given)	$P(C\|M) = .02$ (given)	$P(C\|H) = .25$ (given)
	1. $P(C \cap L)$ $= P(C\|L) \times P(L)$ $= .0008$	**2.** $P(C \cap M)$ $= P(C\|M) \times P(M)$ $= .0036$	**3.** $P(C \cap H)$ $= P(C\|H) \times P(H)$ $= .005$

4. $P(C) = P(C \cap L) + P(C \cap M) + P(C \cap H) = .0094$

5. $P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{.005}{.0094} = .532$ Answer: B

12. We identify the following events:

S - the applicant is a smoker, NS - the applicant is a non-smoker $= S'$
DS - the applicant declares to be a smoker on the application
DN - the applicant declares to be non-smoker on the application $= DS'$.

The information we are given is $P[S] = .3$, $P[NS] = .7$, $P[DN|S] = .4$, $P[DS|NS] = 0$. We wish to find $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]}$.

We calculate $.4 = P[DN|S] = \frac{P[DN \cap S]}{P[S]} = \frac{P[DN \cap S]}{.3} \rightarrow P[DN \cap S] = .12$,
and $0 = P[DS|NS] = \frac{P[DS \cap NS]}{P[NS]} = \frac{P[DS \cap NS]}{.7} \rightarrow P[DS \cap NS] = 0$.

Using the rule $P[A] = P[A \cap B] + P[A \cap B']$, and noting that $DS = DN'$ and $S = NS'$ we have

$P[DS \cap S] = P[S] - P[DN \cap S] = .3 - .12 = .18$, and
$P[DN \cap NS] = P[NS] - P[DS \cap NS] = .7 - 0 = .7$, and
$P[DN] = P[DN \cap NS] + P[DN \cap S] = .7 + .12 = .82$.

Then, $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$.
These calculations can be summarized in the order indicated in the following table.

$P(S)$, .3 given	$\Rightarrow$	**1.** $P(NS) = 1 - P(S) = .7$ $\Downarrow$
6. $DS \Leftarrow$ $P(DS)$ $= P(DS \cap S)$ $+ P(DS \cap NS)$ $= .18 + 0 = .18$	**5.** $P(DS \cap S)$ $= P(S) - P(DN \cap S)$ $= .3 - .12 = .18$	**2.** $P(DS|NS) = 0$, given $P(DS \cap NS)$ $= P(DS|NS) \times P(NS)$ $= (0)(.7) = 0$
$\Downarrow$	$\Uparrow$	
7. DN $P(DN)$ $= 1 - P(DS)$ $= 1 - .18$ $= .82$	**4.** $P(DN|S) = .4$ given $P(DN \cap S)$ $= P(DN|S) \cdot P(S)$ $= (.4)(.3) = .12$	**3.** $P(DN \cap NS) =$ $= P(NS) - P(DS \cap NS)$ $= .7 - 0 = .7$

Then,
8. $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$ Answer: D

13. The probability that an individual will not respond to either the questionnaire or the follow-up letter is $(.5)(.6) = .3$. The probability that all 4 will not respond to either the questionnaire or the follow-up letter is $(.3)^4$.

$$\begin{aligned} P[\text{3 don't respond}] &= P[\text{1 response on 1st round, no additional responses on 2nd round}] \\ &\quad + P[\text{no responses on 1st round, 1 response on 2nd round}] \\ &= 4[(.5)^4(.6)^3] + 4[(.5)^4(.6)^3(.4)] = 4(.3)^3(.7)\ .\ \text{Then,} \end{aligned}$$

$P[\text{at least 3 don't respond}] = (.3)^4 + 4(.3)^3(.7)$. Answer: A

14. If 1 fair die is rolled, the probability of rolling a 6 is $\frac{1}{6}$, and if 2 fair dice are rolled, the probability of rolling a 6 is $\frac{5}{36}$ (of the 36 possible rolls from a pair of dice, the rolls 1-5, 2-4, 3-3, 4-2 and 5-1 result in a total of 6), Since the coin is fair, the probability of rolling a head or tail is .5. Thus, the probability that $Y = 6$ is $(.5)(\frac{1}{6}) + (.5)(\frac{5}{36}) = \frac{11}{72}$.

Answer: C

15. Suppose you have bought a lottery ticket. There are $\binom{6}{4} = 15$ ways of picking 4 numbers from the 6 numbers on your ticket. Suppose we look at one of those subsets of 4 numbers from your ticket. In order for the winning ticket number to match exactly those 4 of your 6 numbers, the other 2 winning ticket numbers must come from the 43 numbers between 1 and 49 that are not numbers on your ticket. There are $\binom{43}{2} = \frac{43\times 42}{2\times 1} = 903$ ways of doing that, and since there are 15 subsets of 4 numbers on your ticket, there are $15 \times 903 = 13,545$ ways in which the winning ticket numbers match exactly 3 of your ticket numbers. Since there are a total of 13,983,816 ways of picking 6 out of 49 numbers, your chance of matching exactly 4 of the winning numbers is $\frac{13{,}545}{13{,}983{,}816} = .00096862$. Answer: B