Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Adaptive Model Predictive Control for Constrained Nonlinear Systems ⋆

Veronica Adetola * **Martin Guay** *

* *Dept. Chemical Engineering, Queen's University, Kingston Ontario Canada K7L 3N6 (e-mail: martin.guay@chee.queensu.ca).*

Abstract: A true adaptive nonlinear model predictive control (MPC) algorithm must address the issue of robustness to model uncertainty while the estimator is evolving. Unfortunately, this may not be achieved without introducing extra degree of conservativeness and/or computational complexity in the controller calculations. To attenuate this problem, we employ a finite time identifier and propose an adaptive predictive control structure that reduces to a nominal MPC problem when exact parameter estimates are obtained. The adaptive MPC is formulated in such a way that useful excitation is automatically injected into the closed loop system to decrease the identification period.

1. INTRODUCTION

Model predictive control (MPC) has attracted a great deal of interest from practitioners due to the relative ease constraints can be incorporated Qin and Badgwell (2003). Although it has been proven that a standard implementation of MPC using a nominal model of the system dynamics exhibit nominal robustness to sufficiently small disturbances Scokaert et al. (1997), such marginal robustness guarantee may be unacceptable in practical situations. Present uncertainties must be accounted for in the computation of the control law to achieve robust stability.

In general, most physical systems possess parametric uncertainties or unmeasurable parameters. Examples in chemical engineering include reaction rates, activation energies, fouling factors, and microbial growth rates. Since parametric uncertainty may degrade the performance of MPC, mechanisms to update the unknown or uncertain parameters are necessary in application. One possibility would be to use state measurements to update the model parameters off-line. A more attractive possibility is to apply adaptive extensions of MPC in which parameter estimation and control are performed online.

While some results are available for linear adaptive MPC (See Shouche et al. (1998) for example), very few adaptive MPC schemes have been developed for nonlinear systems Adetola and Guay (2004); Mayne and Michalska (1993). The result in Mayne and Michalska (1993), implements a certainty equivalence nominal-model MPC feedback to stabilize a parametric uncertain system subject to an input constraint. The result shows that there must exist a finite time such that an excitation condition is satisfied and thus parameter convergence is achieved. There is no mechanism to decrease the identification period in any way and moreover, it is only by assumption that the true system trajectory remains bounded during the identification phase. In Adetola and Guay (2004), an input-to-state stable control Lyapunov functions (iss-clf) is used to develop a MPC scheme that provides robust stabilization in the absence of parameter estimation algorithm, and ensures asymptotic stability of the closed loop with parameter adaptation. However, the work only deals with unconstrained nonlinear systems.

The design of adaptive nonlinear MPC schemes is very challenging because the "separation principle assumption" widely employed in linear control theory is not applicable to general class of nonlinear systems, in particular in the presence of constraints. A true adaptive nonlinear MPC algorithm must address the issue of robustness to model uncertainty while the estimator is evolving. Unfortunately, this may not be achieved without introducing extra degree of conservativeness and/or computational complexity in the controller calculations. The recent work DeHaan and Guay (2007) provided an adaptive robust MPC that deals with both state and input constraints within an adaptive framework. In the presented approach, a set valued description of the parametric uncertainty is adapted online to reduce the conservativeness of the solutions, especially with respect to the design of terminal penalty. The parameterization of the feedback MPC policy in terms of uncertainty set and the underlying min-max feedback MPC used in the study make the controller's computation very challenging. The result can be viewed as a conceptual result that focus on performance improvement rather than implementation.

In this paper, we assume that the uncertainties in the system are due to static nonlinearities, expressible in the form of unknown (constant) model parameters. As in DeHaan and Guay (2007), an adaptation mechanism is embedded within the frame work of a robust MPC to reduce the conservativeness of the MPC controller while retaining its stabilizing properties. The adaptive MPC is formulated in such a way that useful excitation is automatically injected into the closed loop system. In contrast to DeHaan and Guay (2007), simplicity is achieved by generating a parameter estimator for the unknown parameter vector rather than adapting a parameter uncertainty set directly. The

⋆ This work was supported in part by the Natural Sciences and Engineering Council of Canada.

identifier employed Adetola and Guay (2007) ensures that the degree of uncertainty is non-increasing at every time step. This means that the controller employs a process model which approaches that of the true system over time. Moreover, when an excitation condition is satisfied, the estimation routine recovers the true value of the uncertain parameters in a known finite time. Subsequently, the adaptive and robustness features of the MPC is eliminated and the complexity of the resultant controller reduces to that of nominal model predictive control.

The remainder of the paper is organized as follows. The problem description is given in section 2 while the finite time identifier employed is outlined in section 3. Our proposed robust adaptive MPC techniques are presented in sections 4 and 5. Simulation results are shown in section 6 and conclusions are given in section 7.

1.1 Nomenclature and Definitions

$\underline{\lambda}(M)$, $det(M)$ and $cond(M)$ denote the smallest eigenvalue, the determinant and the condition number of matrix M respectively. A continuous function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}_\infty$ if $\mu(0) = 0$, $\mu(.)$ is strictly increasing on $\mathbb{R}^+$ and is radially unbounded.

2. PROBLEM DESCRIPTION

The system considered is the following nonlinear parameter affine system

$$\dot{x} = f(x, u) + g(x, u)\theta \triangleq \mathcal{F}(x, u, \theta) \tag{1}$$

$\theta \in \mathbb{R}^p$ is the unknown parameter vector whose entries may represent physically meaningful unknown model parameters or could be associated with any finite set of universal basis functions. It is assumed that θ is uniquely identifiable and lie within an initially known compact set $\Theta^0 \triangleq B(\theta^0, z_\theta^0)$, a ball described by an initial nominal estimate θ^0 and associated error bound $z_\theta^0 = \sup_{s\in\Theta^0} \|s - \theta^0\|$. The mapping $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_\theta} \to \mathbb{R}^n$ is assumed to be locally Lipschitz with respect to its arguments and $\mathcal{F}(0, 0, \theta) = 0$. The state and the control input trajectories are assumed to be subject to pointwise constraints $x \in \mathbb{X} \in \mathbb{R}^n$ and $u \in \mathbb{U} \in \mathbb{R}^m$ respectively. The objective of the study is to (robustly) stabilize the plant by means of state feedback adaptive MPC. Optimality of the resulting trajectories are measured with respect to the accumulation of some stage cost $L(x, u) \geq 0$. The cost is assumed to be continuous, $L(0, 0) = 0$, and $L(x, u) \geq \mu_L(\|x, u\|)$, where μ_L is a $\mathcal{K}_\infty$ function.

3. FINITE TIME IDENTIFIER

In the following, we give a brief description of the identifier employed in the adaptive robust design framework. The finite time (FT) identification procedure assumes the state of the system $x(.)$ is accessible for measurement but do not require the measurement or computation of the velocity state vector $\dot{x}(.)$. The algorithm is independent of the control structure employed.

Let the state predictor for (1) be denoted as $\hat{x}$ and define

$$\dot{\hat{x}} = f(x, u) + g(x, u)\hat{\theta} + ke + w\dot{\hat{\theta}}, \tag{2}$$

where $\hat{\theta}$ is the parameter estimate vector, $k > 0$ a design constant or $(n \times n)$ matrix, $e = x - \hat{x}$ the prediction error and w the output of the filter

$$\dot{w} = g(x, u) - kw, \qquad w(t_0) = 0. \tag{3}$$

Denoting the parameter estimation error as $\tilde{\theta} = \theta - \hat{\theta}$, it follows from (1) and (2) that

$$\dot{e} = g(x, u)\tilde{\theta} - ke - w\dot{\hat{\theta}}. \tag{4}$$

Defining

$$\eta = e - w\tilde{\theta} \tag{5}$$

It follows from (3) and (4) that η can be generated from

$$\dot{\eta} = -k\eta, \qquad \eta(t_0) = e(t_0). \tag{6}$$

The adaptive compensator used in this work is given as

$$\dot{\hat{\theta}} = \text{Proj}\left\{\gamma w^T(e - \eta),\ \hat{\theta}\right\}, \quad \hat{\theta}(t_0) = \theta^0 \in \Theta^0 \tag{7}$$

where $\gamma = \gamma^T > 0$ and $\text{Proj}\{\phi,\ \hat{\theta}\}$ denotes a Lipschitz projection operator such that

$$-\text{Proj}\{\phi,\ \hat{\theta}\}^T\tilde{\theta} \leq -\phi^T\tilde{\theta}, \tag{8}$$

$$\hat{\theta}(t_0) \in \Theta^0 \Rightarrow \hat{\theta}(t) \in \Theta_\epsilon^0,\ \forall\, t \geq t_0. \tag{9}$$

where $\Theta_\epsilon^0 \triangleq B(0, \Pi + \epsilon)$, $\epsilon > 0$. More details on parameter projection can be found in Krstic et al. (1995).

Lemma 1. Let $Q \in \mathbb{R}^{p\times p}$ and $C \in \mathbb{R}^p$ be generated from the following dynamics:

$$\dot{Q} = w^T w, \qquad Q(t_0) = 0$$
$$\dot{C} = w^T(w\hat{\theta} + e - \eta), \qquad C(t_0) = 0$$

and define $\underline{\gamma} = \underline{\lambda}(\gamma)$, a Lyapunov function $V_{\tilde{\theta}} = \tilde{\theta}^T\tilde{\theta}$, excitation index $\mathcal{E}(t) = \underline{\lambda}\big(Q(t)\big)$, and contraction factor $0 < \alpha(t) = \frac{1}{1+\underline{\gamma}\mathcal{E}(t)} \leq 1$,

(1) the identifier (7) is such that the estimation error $\|\tilde{\theta}\| = \|\theta - \hat{\theta}\|$ is non-increasing and
$$V_{\tilde{\theta}}(t) \leq \alpha(t)V_{\tilde{\theta}}(t_0), \quad \text{for all} \quad t > t_0. \tag{10}$$
(2) suppose there exists a time t_c such that $Q(t_c)$ is invertible, then
$$\theta = Q(t)^{-1}C(t), \quad \text{for any} \quad t \geq t_c. \tag{11}$$

Proof

(1) It follows from (5), (7), and (8) that
$$\dot{V}_{\tilde{\theta}} \leq -\underline{\gamma}\tilde{\theta}^T w^T w\tilde{\theta} \leq 0,$$
which guarantees non-increase of $\|\tilde{\theta}\|$. Also,
$$V_{\tilde{\theta}}(t) = V_{\tilde{\theta}}(t_0) + \int_{t_0}^t \dot{V}_{\tilde{\theta}} d\tau$$
$$\leq V_{\tilde{\theta}}(t_0) - \underline{\gamma}\underline{\lambda}\left(\int_{t_0}^t w(\tau)^T w(\tau) d\tau\right) \min_{\tau\in[t_0,t]} \|\tilde{\theta}(\tau)\|^2.$$
Hence
$$V_{\tilde{\theta}}(t) \leq V_{\tilde{\theta}}(t_0) - \underline{\gamma}\mathcal{E}(t)V_{\tilde{\theta}}(t) \tag{12}$$
Rearranging (12) concludes the proof of part (1).

(2) This can be easily shown by noting that
$$Q(t)\,\theta = \int_{t_0}^t w^T(\tau)w(\tau)\left[\hat{\theta}(\tau) + \tilde{\theta}(\tau)\right] d\tau. \tag{13}$$
Since $w\tilde{\theta} = e - \eta$, it follows from (13) that
$$\theta = Q(t)^{-1}\int_{t_0}^t \dot{C}(\tau)\, d\tau = Q(t)^{-1}C(t) \qquad \blacksquare$$

Let $\theta^c \triangleq Q(t_c)^{-1}C(t_c)$, the finite time (FT) identifier is given by

$$\hat{\theta}(t) = \begin{cases} \hat{\theta}(t), & \text{if } t < t_c \\ \theta^c, & \text{if } t \geq t_c. \end{cases} \tag{14}$$

Remark 2. The invertibility condition in theorem 1.2 is equivalent to the standard persistence of excitation condition required for parameter convergence in adaptive control. However, the superiority of the above design lies in the fact that we can actually compute the true parameter value at any time instant t_c the regressor matrix becomes positive definite and subsequently stop parameter adaptation.

4. ROBUST ADAPTIVE MPC - A MIN-MAX APPROACH

In this section, the concept of min-max robust MPC is employed to provide robustness for the MPC controller during the adaptation phase. The resulting optimization problem can either be solved in open-loop or closed-loop. In the presented approach, we choose the least conservative option by performing optimization with respect to closed loop strategies. As in typical feedback-MPC fashion, the controller chooses input u as a function of the current states. The formulation consists of maximizing a cost function with respect to θ and minimizing over feedback control policies κ. The uncertainty set is a ball of the form $\Theta \triangleq B(\hat{\theta}, z_\theta)$, described by a nominal estimate $\hat{\theta}$ and associated error bound $z_\theta \geq \|\theta - \hat{\theta}\|$, which is updated as

$$z_\theta(t) = \sqrt{\alpha(t)}z_\theta(t_0)$$

The receding horizon control law is defined by

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) \triangleq \kappa^*(0, x, \hat{\theta}, z_\theta) \tag{15a}$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot,\cdot,\cdot,\cdot)} J(x, \hat{\theta}, z_\theta, \kappa) \tag{15b}$$

where $J(x, \hat{\theta}, z_\theta, \kappa)$ is the (worst-case) cost associated with the optimal control problem:

$$J(x, \hat{\theta}, z_\theta, \kappa) \triangleq \max_{\theta \in B(\hat{\theta}, z_\theta)} \int_0^T L(\bar{x}, \bar{u})d\tau + W(\bar{x}(T), \tilde{\bar{\theta}}(T)) \tag{16a}$$

$$s.t.\ \forall \tau \in [0, T]$$

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}) + g(\bar{x}, \bar{u})\theta, \ \bar{x}(0) = x \tag{16b}$$

$$\dot{\bar{w}} = \beta \left(g^T(\bar{x}, \bar{u}) - k\bar{w}\right), \ \bar{w}(0) = w \tag{16c}$$

$$\dot{\hat{\bar{\theta}}} = \text{Proj}\left\{\gamma \bar{w}^T \bar{w}\tilde{\bar{\theta}}, \ \hat{\theta}\right\}, \ \tilde{\bar{\theta}} = \theta - \hat{\bar{\theta}}, \ \hat{\bar{\theta}}(0) = \hat{\theta} \tag{16d}$$

$$\bar{u}(\tau) \triangleq \kappa(\tau, \bar{x}(\tau), \hat{\bar{\theta}}(\tau)) \in \mathbb{U} \tag{16e}$$

$$\bar{x}(\tau) \in \mathbb{X}, \ \ \bar{x}(T) \in \mathbb{X}_f(\tilde{\bar{\theta}}(T)) \tag{16f}$$

In the above framework, $\beta \in \{0, 1\}$ is a design parameter. The bared variables denote the predicted values internal to the min-max nonlinear MPC controller. The effect of future parameter adaptation is accounted for, which results in less conservative worst-case predictions. Also, the conservativeness of the terminal cost is reduced by parameterizing both W and $\mathbb{X}_f$ as functions of $\tilde{\theta}_T$.

Remark 3. The fact that the terminal penalty is parameterized as a function of $\tilde{\theta}$ ensures that the algorithm will seek to reduce the parameter error in the process of optimizing the cost function J. However, to further improve the quality of excitation in the closed loop and thereby achieve parameter convergence in a minimum time, the PE condition can be *explicitly* incorporated in the min-max optimization problem by selecting terminal function of the form

$$W(\bar{x}_T, \tilde{\theta}_T) = W_1(\bar{x}_T, \tilde{\theta}_T) + \alpha(T)V_{\tilde{\theta}}(0) \tag{17}$$

where $\alpha(T) = \frac{1}{1+\gamma\underline{\mathcal{E}}(T)}$ as given in lemma 1

4.1 Implementation Algorithm

Algorithm 1. The MPC algorithm performs as follows: At sampling instant t_i

(1) Measure the current state of the plant x
(2) Obtain the current value of matrices w, Q
(3) If $det(Q) = 0$ or $cond(Q)$ is not satisfactory

   If $z_\theta(t_i) \leq z_\theta(t_{i-1}) - \|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\|$

   $$\hat{\theta} = \hat{\theta}(t_i), \quad z_\theta = z_\theta(t_i), \quad \beta = 1$$

   Else

   $$\hat{\theta} = \hat{\theta}(t_{i-1}), \quad z_\theta = z_\theta(t_{i-1}), \quad \beta = 1$$

   End

   Elseif $det(Q) > 0$ and $cond(Q)$ is satisfactory,

   $$\hat{\theta} = \theta^c \triangleq Q^{-1}C, \quad z_\theta = 0, \quad \beta = 0$$

   End
(4) Solve the optimization problem (15) and apply the resulting feedback control law to the plant until the next sampling instant
(5) Increment $i = i + 1$. If $z_\theta > 0$, repeat the procedure from step 1 for the next sampling instant. Otherwise, repeat only steps 1 and 4 for the next sampling instant.

Implementing the adaptive MPC controller according to algorithm 1 guarantees that the uncertainty ball $\Theta \triangleq B(\hat{\theta}, z_\theta)$ is contained in the previous one, that is, $\Theta(t_i) \subseteq \Theta(t_{i-1})$. Hence, a successive reduction in the computational requirement of (15) is ensured. Moreover, when the parameter estimate θ^c becomes available, the uncertainty set $B(.,.)$ reduces to a single point with $\tilde{\theta} = 0$ and the predictive robust control structure becomes that of a nominal MPC:

$$u = \kappa_{mpc}(x) \triangleq \kappa^*(0, x) \tag{18a}$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot,\cdot)} J(x, \kappa) \triangleq \int_0^T L(\bar{x}, \bar{u})d\tau + W(\bar{x}(T)) \tag{18b}$$

$$s.t.\ \forall \tau \in [0, T]$$

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}) + g(\bar{x}, \bar{u})\theta^c, \ \bar{x}(0) = x \tag{18c}$$

$$\bar{u}(\tau) \triangleq \kappa(\tau, \bar{x}(\tau)) \in \mathbb{U}, \ \bar{x}(\tau) \in \mathbb{X}, \ \bar{x}(T) \in \mathbb{X}_f \tag{18d}$$

In the remainder of this section, we drop constraints (16e) and (16f) by using the convention that if some of the constraints is not satisfied, then the value of J is $+\infty$.

4.2 Closed loop Robust Stability

Robust stability is guaranteed if predicted state at terminal time belong to a robustly invariant set for all possible uncertainties. Let $\tilde{\Theta}^0 = \{\tilde{\theta} \mid \|\tilde{\theta}\| \leq z_\theta(t_0)\}$, sufficient conditions for the robust MPC (15,16) to guarantee stabilization of the origin is outlined below:

Criterion 4. The terminal penalty function $W : \mathbb{X}_f \times \tilde{\Theta}^0 \to [0, +\infty]$ and the terminal constraint function $\mathbb{X}_f : \tilde{\Theta}^0 \to \mathbb{X}$ are such that for each $(\theta, \hat{\theta}, \tilde{\theta}) \in (\Theta^0 \times \Theta^0_\epsilon \times \tilde{\Theta}^0)$, there exists a feedback $k_f(., \hat{\theta}) : \mathbb{X}_f \to \mathbb{U}$ satisfying

(1) $0 \in \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}$, $\mathbb{X}_f(\tilde{\theta})$ closed
(2) $k_f(x, \hat{\theta}) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f(\tilde{\theta})$
(3) $W(x, \tilde{\theta})$ is continuous with respect to $x \in \mathbb{R}^n$
(4) $\mathbb{X}_f(\tilde{\theta})$ is strongly positively invariant under $k_f(x, \hat{\theta})$
(5) $W(x(t+\delta), \tilde{\theta}(t)) - W(x(t), \tilde{\theta}(t)) \leq - \int_t^{t+\delta} L(x, k_f(x, \hat{\theta}))d\tau, \ \forall x \in \mathbb{X}_f(\tilde{\theta})$.

In addition to criterion (4), the $\tilde{\theta}$ dependence of W and $\mathbb{X}_f$ is required to satisfy the following:

Criterion 5. For any $\tilde{\theta}_1, \ \tilde{\theta}_2 \in \tilde{\Theta}^0$ *s.t.* $\|\tilde{\theta}_1\| \leq \|\tilde{\theta}_2\|$,

(1) $W(x, \tilde{\theta}_1) \leq W(x, \tilde{\theta}_2), \ \forall x \in \mathbb{X}_f(\tilde{\theta}_2)$
(2) $\mathbb{X}_f(\tilde{\theta}_1) \supseteq \mathbb{X}_f(\tilde{\theta}_2)$

Note that criterion (4) require only the existence, not knowledge, of $k_f(x, \hat{\theta})$ and the stability condition requires the terminal penalty function $W(x, \tilde{\theta})$ to be a robust-CLF on the domain $\mathbb{X}_f(\tilde{\theta})$. Criterion (5) requires W to decrease and the domain $\mathbb{X}_f$ to enlarge with decreased parametric uncertainty as expected.

Theorem 6. Let $X_0(\Theta^0)$ denote the set of initial states with uncertainty Θ^0 for which (16) has a solution. Assuming criteria 4 and 5 are satisfied, then the closed loop system state x, given by (1,14,15), originating from any $x_0 \in X_0$ feasibly approaches the origin as $t \to +\infty$.

Proof. The stability of the closed loop system is established by proving strict decrease of the optimal cost $J^*(x, \hat{\theta}, z_\theta) \triangleq J(x, \hat{\theta}, z_\theta, \kappa^*)$. Let the trajectories x^p, $\hat{\theta}^p$, $\tilde{\theta}^p$, z^p_θ and control u^p correspond to any worst case minimizing solution of $J^*(x, \hat{\theta}, z_\theta)$. If $x^p_{[0,T]}$ were extended to $\tau \in [0, T+\delta]$ by implementing the feedback $u(\tau) = k_f(x^p(\tau), \hat{\theta}^p(\tau))$ on $\tau \in [T, t+\delta]$, then criterion 5.5 guarantees the inequality

$$\int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p))d\tau + W(x^p_{T+\delta}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) \leq 0 \tag{19}$$

where in (19) and in the remainder of the proof, $x^p_\sigma \triangleq x^p(\sigma)$, $\tilde{\theta}^p_\sigma \triangleq \tilde{\theta}^p(\sigma)$, for $\sigma = T, T+\delta$.

The optimal cost $J^*(x, \hat{\theta}, z_\theta)$

$$\begin{aligned}
&= \int_0^T L(x^p, u^p)d\tau + W(x^p_T, \tilde{\theta}^p_T) \\
&\geq \int_0^T L(x^p, u^p)d\tau + W(x^p_T, \tilde{\theta}^p_T) + \int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p))d\tau \\
&\quad + W(x^p_{T+\delta}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) && (20) \\
&\geq \int_0^\delta L(x^p, u^p)d\tau + \int_\delta^T L(x^p, u^p)d\tau \\
&\quad + \int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p))d\tau + W(x^p_{T+\delta}, \tilde{\theta}^p_{T+\delta}) && (21) \\
&\geq \int_0^\delta L(x^p, u^p)d\tau + J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) && (22)
\end{aligned}$$

Hence, it follows from (22) that

$$J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) - J^*(x, \hat{\theta}, z_\theta) \leq - \int_0^\delta L(x^p, u^p)d\tau$$

■

Remark 7. In the above proof,

- (20) is obtained using inequality (19)
- (21) follows from criterion 5.1 and the fact that $\|\tilde{\theta}\|$ is non-increasing
- (22) follows by noting that the last 3 terms in (21) is a (potentially) suboptimal cost on the interval $[\delta, T+\delta]$ starting from the point $(x^p(\delta), \hat{\theta}^p(\delta))$ with associated uncertainty set $B(\hat{\theta}^p(\delta), z^p_\theta(\delta))$.

5. ROBUST ADAPTIVE MPC - A LIPSHITZ BASED APPROACH

Due to the computational complexity associated with (feedback) min-max optimization problem for non-linear systems, it is (sometimes) more practical to use a more conservative but computationally efficient methods.

In this section, we present a Lipschitz based method whereby the nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction and parameter estimation error bounds are determined based on the Lipschitz continuity of the model. A knowledge of appropriate Lipschitz bounds for the x-dependence of the dynamics $f(x, u)$ and $g(x, u)$, and for the penalty functions $L(x, u)$ and $W(x, \theta)$ are assumed as follows:

Assumption 8. A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^+$, $j \in \{f, g, L\}$ and $\mathcal{L}_W : \mathbb{X} \times \Theta \to \mathbb{R}^+$ are known which satisfy

$$\begin{aligned}
&\mathcal{L}^o_j(\mathbb{X}, u) \geq \min \Big\{ \mathcal{L}_j \ \Big| \\
&\qquad \sup_{x_1, x_2 \in \mathbb{X}} \Big(\|j(x_1, u) - j(x_2, u)\| - \mathcal{L}_j \|x_1 - x_2\| \Big) \leq 0 \Big\}, \\
&\mathcal{L}^o_W(\mathbb{X}, \theta) \geq \min \Big\{ \mathcal{L}_W \ \Big| \\
&\qquad \sup_{x_1, x_2 \in \mathbb{X}} \Big(W(x_1, \theta) - W(x_2, \theta) - \mathcal{L}_W \|x_1 - x_2\| \Big) \leq 0 \Big\}
\end{aligned}$$

where for $j \equiv g$ is interpreted as an induced norm.

Note that the functions $\mathcal{L}^o_f$, $\mathcal{L}^o_g$, $\mathcal{L}^o_L$ in assumption 8 can be parameterized in terms of u.

5.1 Prediction of State and Parameter Estimation Errors Bounds

State Prediction: Consider the actual system

$$\dot{x} = f + g\theta, \tag{24}$$

and the nominal model

$$\dot{\bar{x}} = \bar{f} + \bar{g}\bar{\theta}, \tag{25}$$

where $f \triangleq f(x, \bar{u})$, $g \triangleq g(x, \bar{u})$, $\bar{f} \triangleq f(\bar{x}, \bar{u})$, $\bar{g} \triangleq g(\bar{x}, \bar{u})$ and $\bar{\theta}$ is a constant estimate of θ, it can be seen that

$$\|\dot{x} - \dot{\bar{x}}\| = \|f - \bar{f}\| + \|g\theta - \bar{g}\theta\| + \|\bar{g}\theta - \bar{g}\bar{\theta}\|.$$

Defining $\bar{z}_x \triangleq \max_{\theta\in\Theta} \|x-\bar{x}\|$ and $\Pi_\theta = \|\hat{\theta}\|+z_\theta$, it follows that

$$\dot{\bar{z}}_x = (\mathcal{L}_f^o + \mathcal{L}_g^o \Pi_\theta)\bar{z}_x + \|\bar{g}\| z_\theta, \quad \bar{z}_x(t_0) = 0 \tag{26}$$

provides a bound on the worst-case deviation of the nominal state trajectory from the solution of the actual system.

Parameter Estimation Error Prediction: Using a nominal model prediction, it is impossible to predict the actual future behavior of the parameter estimation error as was possible in the min-max framework. However, based upon the excitation of the nominal prediction, one can generate a lower bound on the excitation index $\mathcal{E}(.)$ and hence an upper bound on the future parameter estimation error.
To this end, let $\zeta = w^T w - \bar{w}^T \bar{w}$, it follows that

$$\dot{\zeta} = -2k\zeta + w^T g - \bar{w}^T \bar{g} + g^T w - \bar{g}^T \bar{w}$$

By adding and subtracting some extra terms, we have

$$\dot{\zeta} = -2k\zeta + \bar{w}^T(g-\bar{g}) + (g-\bar{g})^T\bar{w} + (w-\bar{w})^T\bar{g} + \bar{g}^T(w-\bar{w}) + (w-\bar{w})^T(g-\bar{g}) + (g-\bar{g})^T(w-\bar{w})$$

Thus, an upper bound $\bar{z}_\zeta \geq \|\zeta\|$ can be obtained from

$$\dot{\bar{z}}_\zeta = 2\left\{k\bar{z}_\zeta + \|\bar{w}\|\mathcal{L}_g^o\bar{z}_x + \|\bar{g}\|\bar{z}_w + \mathcal{L}_g^o\bar{z}_x\bar{z}_w\right\}, \quad \bar{z}_\zeta(t_0) = 0. \tag{27}$$

where $\bar{z}_w \geq \|w - \bar{w}\|$ is generated from

$$\dot{\bar{z}}_w = k\bar{z}_w + \mathcal{L}_g^o\bar{z}_x, \quad \bar{z}_w(t_0) = 0. \tag{28}$$

Lemma 9. (Bhatia (1997), Corollary III.2.6)
Let A and B be real-symmetric matrices of order n with eigenvalues $\lambda^{\downarrow}(.) = \lambda_1, \cdots, \lambda_n$ arranged in decreasing order. Then

$$\max_j \mid \lambda^{\downarrow}(A) - \lambda^{\downarrow}(B) \mid \leq \|A - B\|. \tag{29}$$

Using lemma (9), we have that

$$\mid \underline{\lambda}(w^T w) - \underline{\lambda}(\bar{w}^T\bar{w}) \mid \leq \bar{z}_\zeta, \tag{30}$$

which implies that

$$\underline{\lambda}\left(\int_{t_0}^{\tau} w(\sigma)^T w(\sigma) d\sigma\right) \geq \bar{\mathcal{E}}(\tau) \tag{31}$$

where $\bar{\mathcal{E}}(\tau) \triangleq \underline{\lambda}\left(\int_{t_0}^{\tau} \bar{w}(\sigma)^T\bar{w}(\sigma)d\sigma\right) - \bar{z}_\zeta(\tau - t_0)$

So, the parameter estimation error bound $\bar{z}_{\theta\zeta} \geq \|\theta - \hat{\theta}\|$ is given as

$$\bar{z}_{\theta\zeta}(\tau) \triangleq \sqrt{\alpha_\zeta(\tau)}\, \|\tilde{\theta}(t_0)\| \tag{32}$$

$$\alpha_\zeta(\tau) = \begin{cases} \bar{\alpha}(\tau) \triangleq \dfrac{1}{1+\underline{\gamma}\bar{\mathcal{E}}(\tau)}, & \text{if } 0 < \bar{\alpha}(\tau) < 1 \\ 1, & \text{otherwise.} \end{cases} \tag{33}$$

5.2 Lipschitz Based Finite Horizon optimal Control Problem

The model predictive feedback is defined as

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) = u^*(0) \tag{34a}$$

$$u^*(.) \triangleq \arg\min_{\bar{u}_{[0,T]}} J(x, \hat{\theta}, z_\theta, \bar{u}) \tag{34b}$$

where $J(x, \hat{\theta}, z_\theta, \bar{u})$ is given by the optimal control problem:

$$J(x, \hat{\theta}, z_\theta, \bar{u}) = \int_0^T L'(\bar{x}, \bar{u}, \bar{z}_x)d\tau + W'(\bar{x}, \bar{z}_x, \bar{z}_\theta) \mid_T \tag{35a}$$

s.t. $\forall\tau \in [0, T]$

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}) + g(\bar{x}, \bar{u})\bar{\theta}, \quad \bar{\theta} = \hat{\theta} \ \ \bar{x}(0) = x \tag{35b}$$

$$\dot{\bar{z}}_x = \varphi(\mathcal{L}_f^o + \mathcal{L}_g^o\Pi)\bar{z}_x + \|\bar{g}\|z_\theta, \quad \bar{z}_x(0) = 0 \tag{35c}$$

$$\bar{X}(\tau) \triangleq B(\bar{x}(\tau), \bar{z}_x(\tau)) \subseteq \mathbb{X}, \quad \bar{u}(\tau) \in \mathbb{U} \tag{35d}$$

$$\bar{X}(T) \subseteq \mathbb{X}_f(\bar{z}_\theta(T)) \tag{35e}$$

In the proposed formulation, $\varphi \in \{0, 1\}$ is a design parameter. The uncertainty radius z_θ in (35c) and $\bar{\theta}$ in (35b) are held constant over the prediction horizon. However, the fact that they are updated at every sampling instant enlarges the terminal domain and hence reduces the conservativeness of the robust MPC that would otherwise have been designed based on a large initial uncertainty $z_\theta(t_0)$.

For simplicity, $\bar{z}_\theta(T)$ which appears in the terminal expressions of (35a) and (35e) can be selected as z_θ. However, using $\bar{z}_\theta(T) \triangleq \sqrt{\alpha_\zeta(T)}\, z_\theta$ can further improve the overall performance by reducing the conservatism of the terminal penalty (i.e. the robust-CLF estimate of the remaining cost-to-go).

The objective function requires the minimization of the worst case cost since we explicitly incorporate $\bar{z}_x$ in to the penalties as follows:

$$L'(\bar{x}, \bar{u}, \bar{z}_x) = L(\bar{x}, \bar{u}) + \mathcal{L}_L^o\bar{z}_x \tag{36}$$

$$W'(\bar{x}, \bar{z}_x, \bar{z}_\theta) = W(\bar{x}, \bar{z}_\theta) + \mathcal{L}_W^o\bar{z}_x \tag{37}$$

5.3 Implementation and Result

The Lipshitz based MPC is implemented according to algorithm 1. Implementing the algorithm ensures that the size of the uncertainty cone $B(\bar{x}(\tau), \bar{z}_x(\tau))$ around the nominal state trajectory reduces as z_θ shrinks and the problem becomes that of a nominal MPC (18) when exact parameter estimate vector θ^c is obtained.

Theorem 10. Let $X_0'(\Theta^0)$ denote the set of initial states for which (34) has a solution. Assuming criteria 4 and 5 are satisfied, then the origin of the closed loop system given by (1,14,34) is feasibly asymptotically stabilized from any $x_0 \in X_0'$.

Stability proof is performed similar to that of theorems 6.

6. SIMULATION EXAMPLE

An adaptive extremum seeking control problem is used to demonstrate the applicability of the proposed adaptive MPC scheme. In this type of problem, the desired target is the operating set-point that optimizes an uncertain cost function. Any set-point chosen *a priori* will be suboptimal and can be improved by using some sort of adaptation and perturbation to search for the optimal operating condition in real time.
We consider two parallel isothermal stirred-tank reactors DeHaan and Guay (2005) in which reagent A forms product B and waste-product C

$$A \xrightarrow{1} B, \qquad 2A \xrightarrow{2} C$$

Reaction kinetic constants are only nominally known. Material balances for the reactors give

$$\frac{dA_i}{dt} = A^{in}\frac{F_i^{in}}{V_i} - A_i\frac{F_i^{out}}{V_i} - k_{i1}A_i - k_{i2}A_i^2,$$
$$\frac{dB_i}{dt} = -B_i\frac{F_i^{out}}{V_i} + k_{i1}A_i,$$
$$\frac{dC_i}{dt} = -C_i\frac{F_i^{out}}{V_i} + k_{i2}A_i^2,$$

where A_i, B_i, C_i denote concentrations in reactor i. The inlet flows F_i^{in} are the control inputs, while the outlet flows F_i^{out} are governed by PI controllers which regulate reactor volume to V_i^0. Denoting $x_p = [A_1,\ A_2]^T$, and $\theta = [k_{11}, k_{12}, k_{21}, k_{22}]^T$, the net expense of operating the process at steady state is given by

$$p(x_p,\ s,\ \theta) = \sum_{i=1}^{2}[(p_{i1}s_i + P_A - P_B)k_{i1}A_iV_i^0 + (p_{i2}s_i + 2P_A)k_{i2}A_i^2V_i^0], \tag{38}$$

where P_A, P_B denote component prices, p_{ij} is the net operating cost of reaction j in reactor i and measured disturbances s_1 and s_2 reflect changes in the operating cost (utilities, etc,) of each reactor.

The control objective is to steer the system to optimal operating conditions that optimize the economic steady state cost function (38) subject to the uncertainty $0.01 \leq k_{ij} \leq 0.2$, the state constraint $0 \leq x_{pi} \leq 3.0$, and the control input constraint $0.05 \leq u_i \leq 0.15$ (for $i = 1, 2$ and $j = 1, 2$).

The system was simulated subject to a ramping economic disturbance in s_2 for $t \in [2, 6]$. The extremum seeking trajectory $x_p(t, s, \hat{\theta})$ was generated online via a Lyapunov based set-point update law DeHaan and Guay (2005). At every sampling instant, the current value of the $\hat{\theta}$-dependent optimal setpoint is passed to the adaptive MPC for implementation. For this simulation, we used the proposed Lipschitz based adaptive MPC procedure. Fig. 1(a) shows that the cost function converges to the unknown optimal $p^*(x_p^*, \theta)$, while Fig. 1(b) shows that the state x_p converges to the optimum value. The parameter estimates converge to their true values as shown in Fig. 1(c-d) and the constrained control inputs, Fig. 1(e), are implementable.

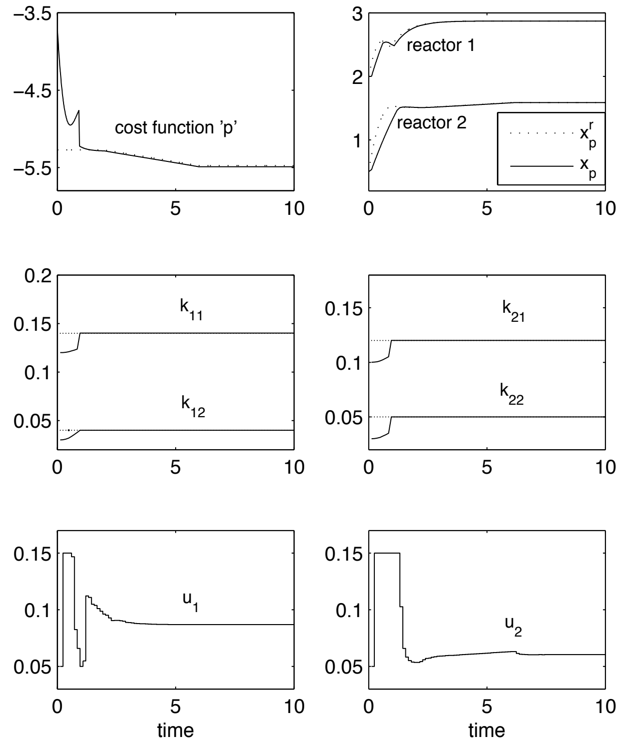

Fig. 1. Simulated system trajectories: (a) cost function, (b) reference set-point and state, (c, d) unknown parameters and estimates for reactor 1 and 2 respectively, (e, f) control inputs

7. CONCLUSIONS

This paper provides an adaptive MPC design technique for constrained nonlinear systems with parametric uncertainties. Robustly stabilizing MPC schemes are developed to ensure robustness to parameter estimation error during the adaptation phase. The true unknown parameter values are reconstructed when the given invertibility condition is satisfied and the computational requirement/conservativeness of the adaptive MPC reduces to that of a nominal MPC. Extension of the approach to systems with both parametric uncertainties and time varying disturbances is underway.

REFERENCES

V. Adetola and M. Guay. Finite-time parameter estimation in adaptive control of nonlinear systems. In *Proc. of American Control Conference*, July 2007.

V. Adetola and M. Guay. Adaptive receding horizon control of nonlinear systems. In *Proc. of IFAC Symposium on Nonlinear Control systems*, pages 1055–1060, Stuttgart, 2004.

Rajendra Bhatia. *Matrix Analysis*. Graduate Texts in Mathematics. Springer-Verlag New York, Inc, 1997.

D. DeHaan and M. Guay. Extremum seeking control of state constrained nonlinear systems. *Automatica*, 41(9): 1567–1574, 2005.

Darryl DeHaan and Martin Guay. Adaptive robust mpc: A minimally-conservative approach. In *Proc. of American Control Conference*, July 2007.

M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley and Sons Inc, Toronto, 1995.

D. Q. Mayne and H. Michalska. Adaptive receding horizon control for constrained nonlinear systems. In *In Proc. of IEEE Conference on Decision and Control*, pages 1286–1291, San Antonio, Texas, 1993.

S. Joe Qin and Thomas A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11:733–764, 2003.

P.O.M. Scokaert, J.B. Rawlings, and E.S. Meadows. Discrete-time stability with perturbations: Application to model predictive control. *Automatica*, 33(3):463–470, 1997.

M. Shouche, H. Genceli, P. Vuthandam, and M. Nikolaou. Simultaneous constrained model predictive control and identification of DARX processes. *Automatica*, 34(12): 1521–1530, 1998.