addition & subtraction: the learning and teaching trajectory

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addition & subtraction

- As soon as children can distinguish between differences in quantity, they begin to count and add.
- At first this addition is unitary, one by one, but then the counting becomes additive, skipping along by ever higher increments.
- Reflecting on these results reveal patterns that allow for faster and more efficient methods to be developed, and these methods form the basis for multiplication and division.

Competency with addition and subtraction is fundamental for the development of numeracy. In this session we look at the trajectory for learning about addition and subtraction.

addition & subtraction

- The additive world includes addition and subtraction.
- There are experiences, understanding and skills at every level of school for students (and teachers) to master

addition & subtraction

- For teachers, understanding the TRAJECTORY for learning about addition and subtraction is useful.
- Milestones, stepping stones, foundations, building blocks… it doesn't matter how you imagine children’s development in this area, what we DO know is that some skills are required before students can proceed to others.
- When we say “Our kids are having trouble with the subtraction algorithm” maybe they are really having trouble lower down on the ladder.

What is Arithmetic?

Numbers

Addition

Subtraction

Multiplication

Division

Fractions

Decimals

Place Value

the curriculum

By the end of	
Foundation	students make connections between number names, numerals and quantities up to 10. Students count to and from 20 and order small collections.
Year 1	students describe number sequences resulting from skip counting by 2s, 5s and 10s. They identify representations of one half. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies. They partition numbers using place value. They continue simple patterns involving numbers and objects.
Year 2	students recognise increasing and decreasing number sequences involving 2s, 3s and 5s. They represent multiplication and division by grouping into sets. Students identify the missing element in a number sequence. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies. They divide collections and shapes into halves, quarters and eighths. Students order shapes and objects using informal units.
Year 3	students recognise the connection between addition and subtraction and solve problems using efficient strategies for multiplication. They model and represent unit fractions. Students count to and from 10 000. They classify numbers as either odd or even. They recall addition and multiplication facts for single digit numbers. They continue number patterns involving addition and subtraction.
Year 4	students choose appropriate strategies for calculations involving multiplication and division. They recognise common equivalent fractions in familiar contexts and make connections between fraction and decimal notations up to two decimal places. They identify unknown quantities in number sentences. They describe number patterns resulting from multiplication. Students use the properties of odd and even numbers. They recall multiplication facts to 10 x 10 and related division facts. Students locate familiar fractions on a number line. They continue number sequences involving multiples of single digit numbers.
Year 5	students solve simple problems involving the four operations using a range of strategies. They check the reasonableness of answers using estimation and rounding. Students identify and describe factors and multiples. Students order decimals and unit fractions and locate them on number lines. They add and subtract fractions with the same denominator. Students continue patterns by adding and subtracting fractions and decimals. They find unknown quantities in number sentences.
Year 6	students recognise the properties of prime, composite, square and triangular numbers. They describe the use of integers in everyday contexts. They solve problems involving all four operations with whole numbers. Students connect fractions, decimals and percentages as different representations of the same number. They solve problems involving the addition and subtraction of related fractions. Students make connections between the powers of 10 and the multiplication and division of decimals. They describe rules used in sequences involving whole numbers, fractions and decimals. Students connect decimal representations to the metric system and choose appropriate units of measurement to perform a calculation. Students locate fractions and integers on a number line. They calculate a simple fraction of a quantity. They add, subtract and multiply decimals and divide decimals where the result is rational. Students calculate common percentage discounts on sale items. They write correct number sentences using brackets and order of operations.
Year 7	students solve problems involving the comparison, addition and subtraction of integers. They make the connections between whole numbers and index notation and the relationship between perfect squares and square roots. They solve problems involving percentages and all four operations with fractions and decimals. Students represent numbers using variables. They connect the laws and properties for numbers to algebra. They interpret simple linear representations and model authentic information. Students use fractions, decimals and percentages, and their equivalences. They express one quantity as a fraction or percentage of another. Students solve simple linear equations and evaluate algebraic expressions after numerical substitution.
Year 8	students solve everyday problems involving rates, ratios and percentages. They recognise index laws and apply them to whole numbers. They describe rational and irrational numbers. They make connections between expanding and factorising algebraic expressions. Students use efficient mental and written strategies to carry out the four operations with integers. They simplify a variety of algebraic expressions. They solve linear equations and graph linear relationships on the Cartesian plane.
Year 9	Students apply the index laws to numbers and express numbers in scientific notation. They expand binomial expressions. They find the distance between two points on the Cartesian plane and the gradient and midpoint of a line segment. They sketch linear and non-linear relations.
Year 10	students solve problems involving linear equations and inequalities. They make the connections between algebraic and graphical representations of relations. Students expand binomial expressions and factorise monic quadratic expressions. They find unknown values after substitution into formulas. They perform the four operations with simple algebraic fractions. Students solve simple quadratic equations and pairs of simultaneous equations.

some things for teachers to think about

some things for teachers to think about

The importance of an understanding of arithmetic to learning algebra

some things for teachers to think about

Key idea in the national curriculum:
Important for linking mathematics teaching and building mathematical understanding through primary and secondary years

some things for teachers to think about

ACARA on algebra

‘The curriculum must also provide access to future mathematics study. It is essential, for example, that all students have the opportunity to study algebra and geometry. The National Mathematics Advisory Panel (2008) argues that participation in algebra, for example, is connected to finishing high school; failing to graduate from high school is associated with under-participation in the workforce and high dependence on welfare. The study of algebra clearly lays the foundations not only for specialised mathematics study but also for vocational aspects of numeracy.’

some things for teachers to think about

The teaching of number and algebra are inextricably linked.

We cannot expect an improvement in student's learning of algebra until we succeed in building their understanding of arithmetic, i.e. knowledge of number and number operations, and mental computation techniques.

addition & subtraction

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- Milestones, stepping stones, foundations, building blocks… it doesn't matter how you imagine children’s development in this area, what we DO know is that some skills are required before students can proceed to others.

- When we say “Our kids are having trouble with the subtraction algorithm” maybe they are really having trouble lower down on the ladder.

things to watch for

The role of the equals sign

As teachers you need to set the example.

Only write the equals sign between quantities that are equal

Don't use the equals sign for "run on calculations"

For example, if solving "a boy has five marbles and then a friend gives him ten more, and then he loses two" then do NOT write

$$5 + 10 = 15 - 2 = 13$$

The role of vocabulary

As teachers you need to set the example.

NEVER say ‘do your sums’…

for *sum* implies addition ONLY!

the trajectory for addition and subtraction

- The Development of Additive Capacity

the trajectory for addition and subtraction

- Recognising differences in quantity (More or less)
- Subitising – (autonomic) recognition of quantities less than ten
- Unitary counting – one to one corresp., touch & count collections
- Skip Counting
- Place value
- Number facts with single digits to 10 - automatic and rapidly recalled
- Number facts with single digits to 18 - automatic and rapidly recalled
- Mental computation strategies for addition and subtraction
- Number buddies – I know 5 + 6 = … so I know 50 + 60 = … also 55 + 6 =
- Addition & subtraction by rules - algorithms as most efficient strategy
- Rules -Association, distribution, commutation Do they work for both?
- Addition and subtraction with fractions and decimals
- Contexts of addition: Geometry, clocks, probability
- Extending thinking into relational thinking – heading into algebra.
- **Senior:** Summation-Finite and Infinite Series
- Senior: Summation of Infinitesimals

Recognising differences in quantity (More or less)

Recognising differences in quantity (More or less)

Number with small people

Counting
Children's development of counting ability follows human development of number systems:

Counting by ones
Counting using groups and singles
Counting using five as a benchmark
Counting using tens and ones

You might design activities that force children to do this. My friend the Martian makes piles of five icy pole sticks and bundles them with a rubber band, can you count how many he has? etc.

Subitising – (autonomic) recognition of quantities less than ten

Number with small people

Subitizing
Subitizing is the ability to identify quickly the numbers in a set without counting.

For small children this is usually limited to two or three dots when the objects are randomly arranged. Children as young as two can do this, some say babies can do it too.

As numbers increase in magnitude older children and adults start to try to group them in some way or look for a pattern. There are some common and easy to recognise patterns. For example, dice patterns. Dice are an inexpensive and fun teaching tool, let your five year olds play with a handful and observe what they already know about number.

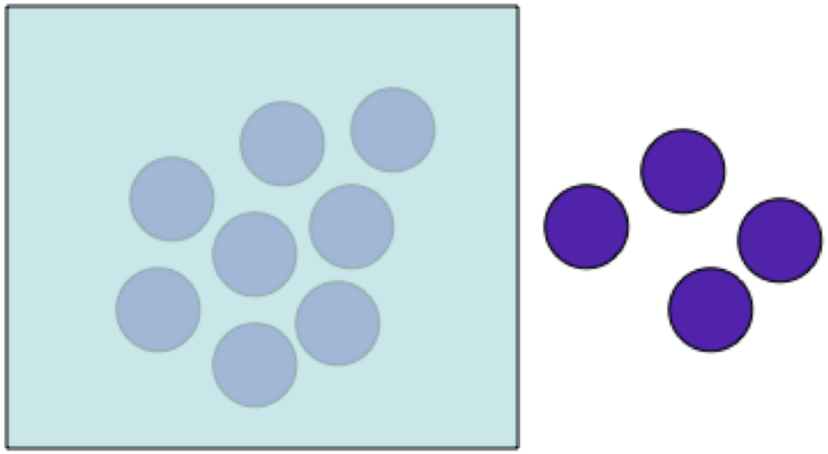
Unitary counting – one to one corresp., touch & count collections

Unitary counting – one to one corresp., touch & count collections

Cover up

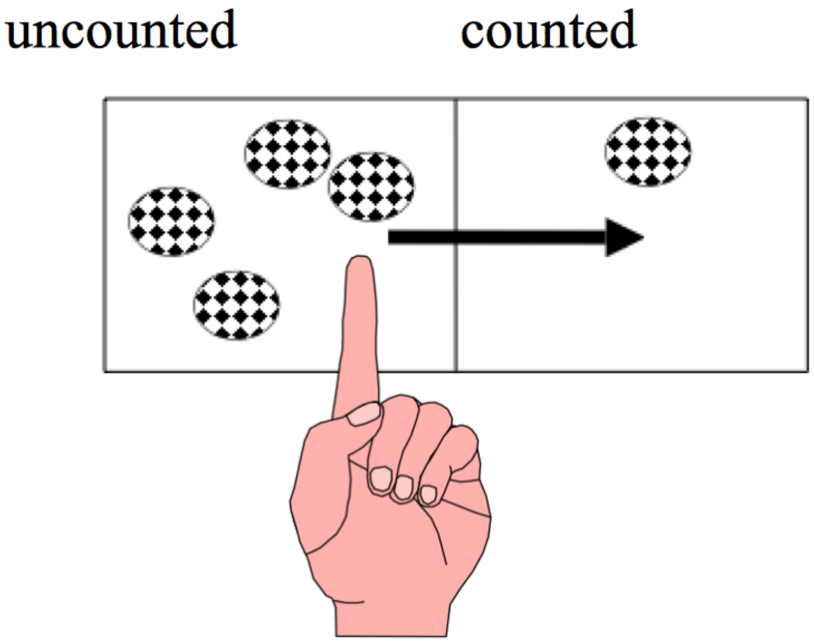
Encourages 'counting on'

Tell the child
"I have four buttons here and 8 more under the card."



Unitary counting – one to one corresp., touch & count collections

Counting mat



Skip Counting

Skip Counting

Number with small people

Whisper counting
Whisper Counting is developed later and is a precursor to skip counting. For example: "One, two, THREE, four, five, SIX, seven, eight, NINE. When Whisper Counting by three's, you say every third number loudly.
Gradually say the skip counted numbers louder and louder and the other numbers softer and softer. Increase the speed of counting until the child can count entirely in skip count mode (3, 6, 9, 12, 15...).
You can also clap (hop, ring a bell, etc.) instead of whispering as you move to skip counting.
Give children time to develop and feel successful with one number before moving to the next one.

Place value

Place value

- AMSI Modules on *Calculate*
- *Prep to 4: http://www.amsi.org.au/teacher_modules/Counting_and_place_valueK-4.html*
- *Year 4 to 7: http://www.amsi.org.au/teacher_modules/Using_place_value4-7.html*

Number facts with single digits to 10

- Automatic and rapidly recalled

Number facts with single digits to 18

- Needed for the subtraction algorithm
- Automatic and rapidly recalled

Number facts with single digits to 18

- Understand the relationship between addition/subtraction and multiplication/division and even square/square root
- Equivalence of

$$6 \times 4 = 24$$

$$\Updownarrow$$

$$24 \div 4 = 6$$

$$12 + 8 = 20$$

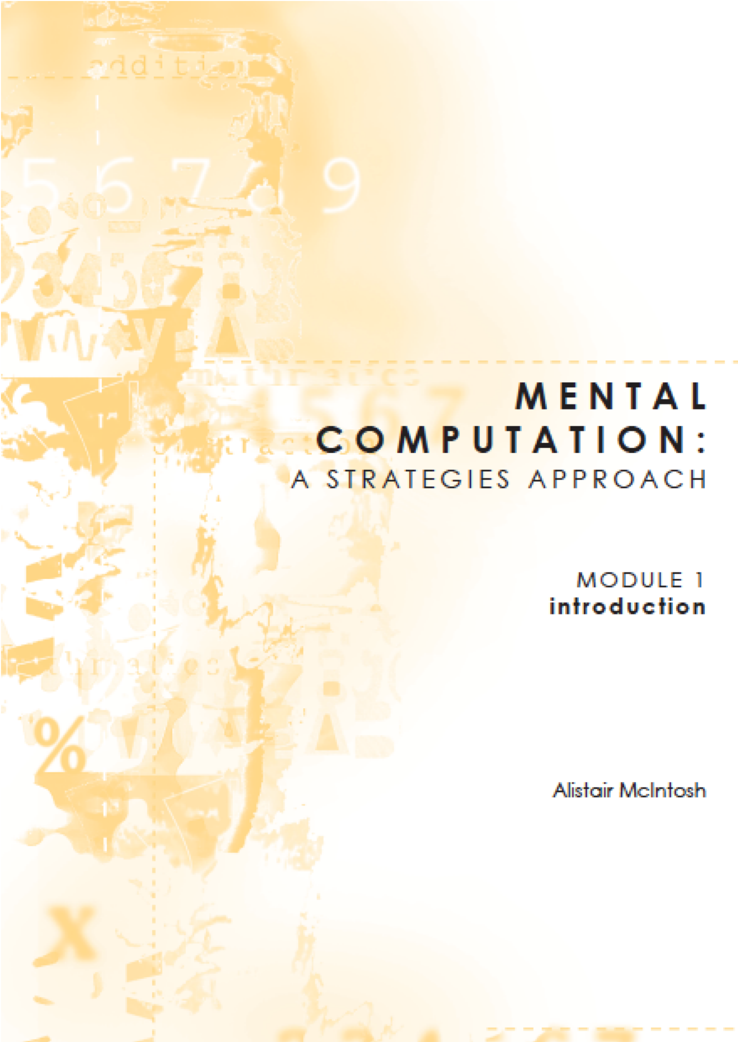
$$\Updownarrow$$

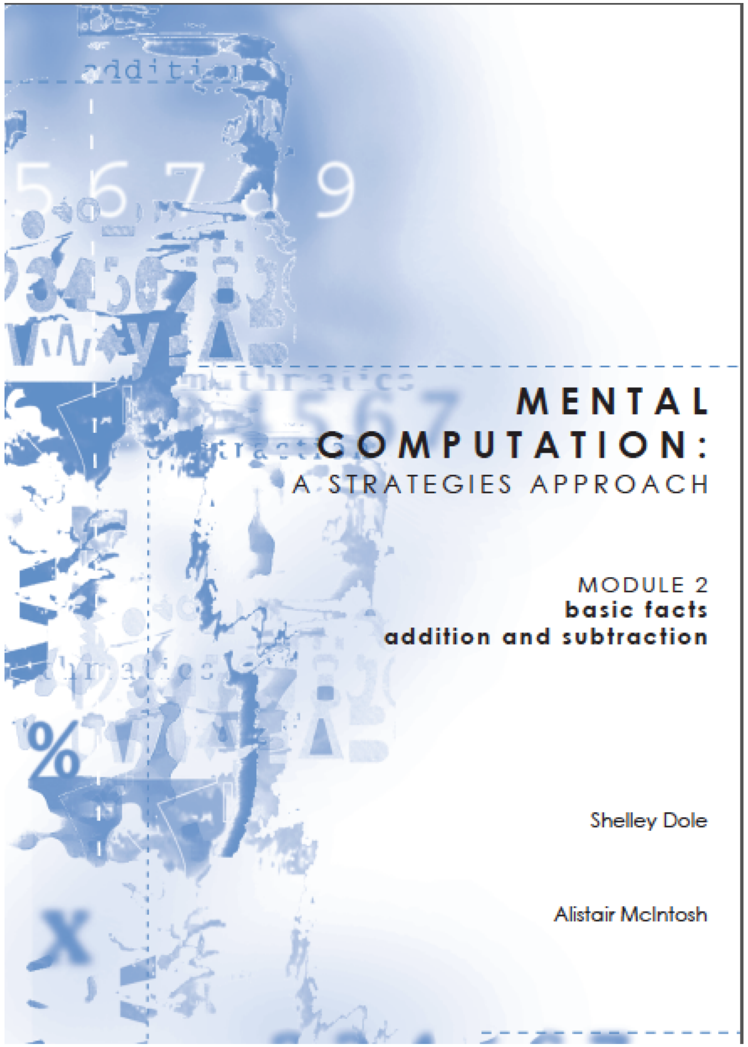
$$20 - 8 = 12$$

Mental computation strategies for addition and subtraction

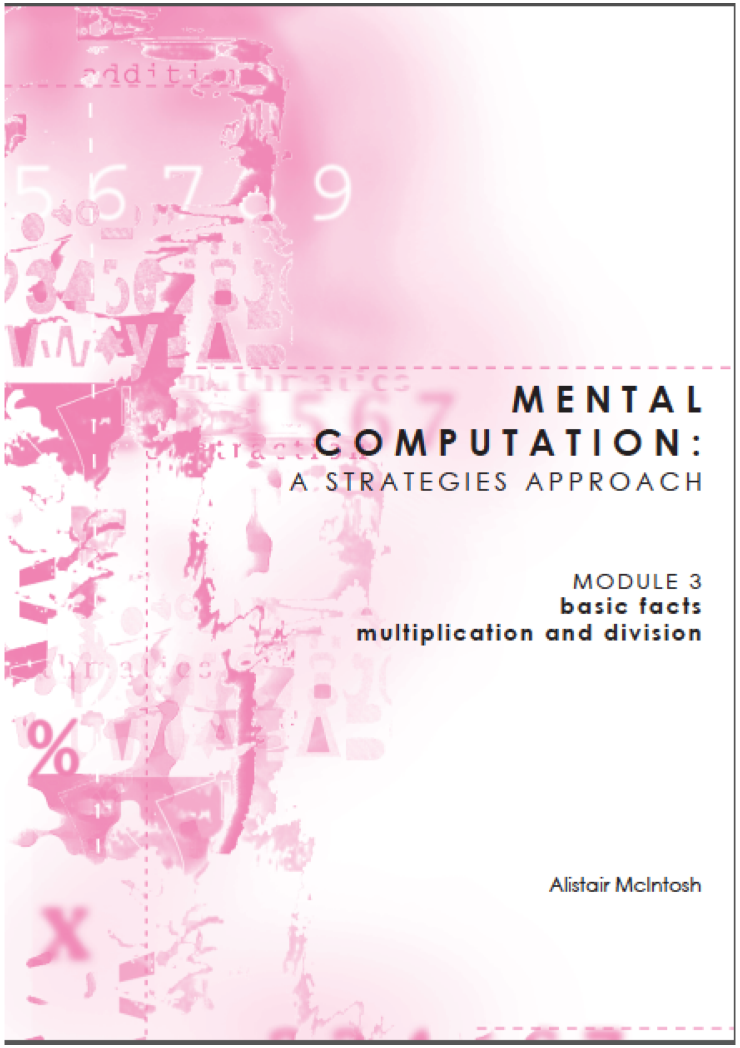
- Tasmanian modules on *Calculate*

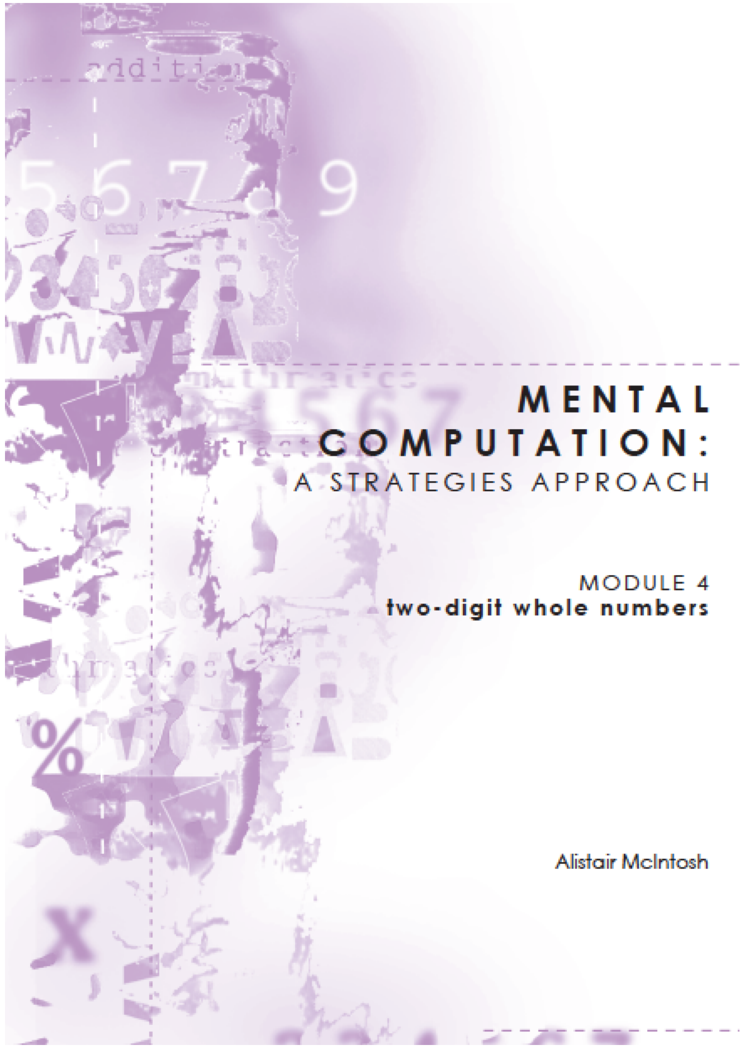
Mental computation strategies for addition and subtraction



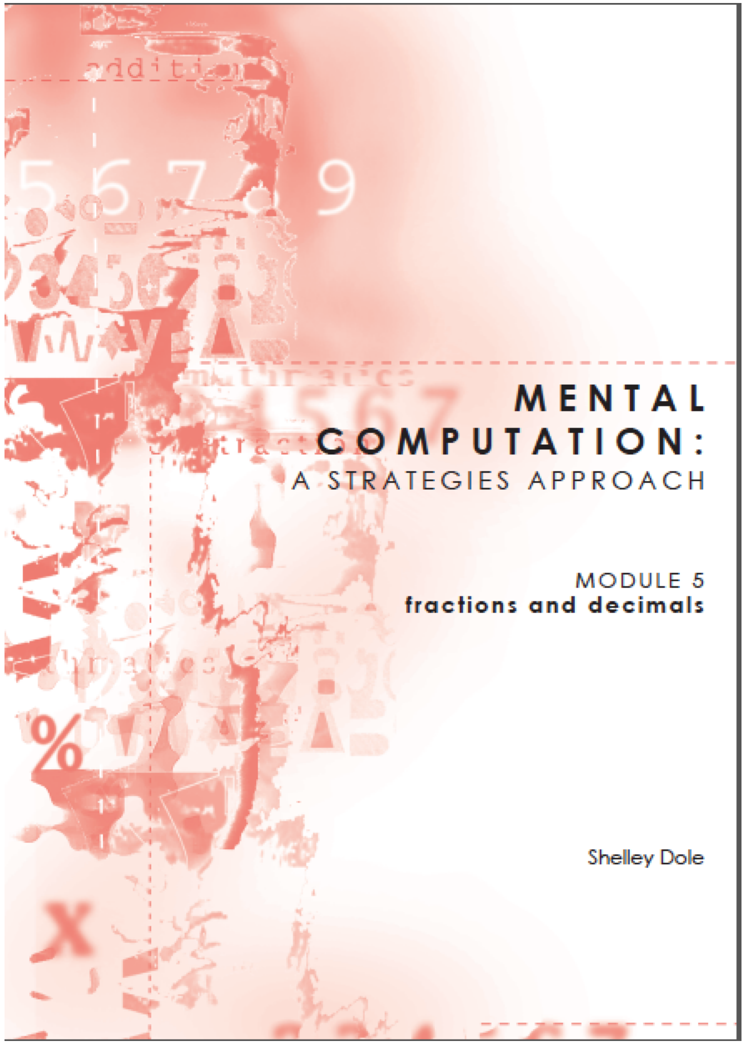


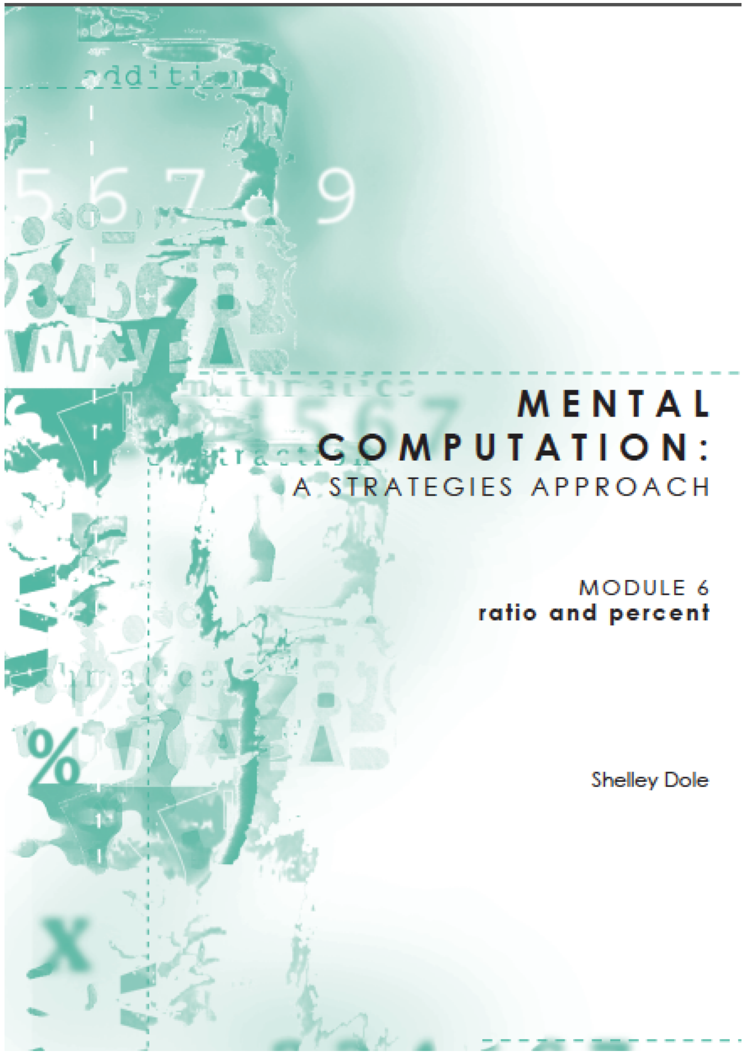
Mental computation strategies for addition and subtraction





Mental computation strategies for addition and subtraction





Two ways to think about subtraction

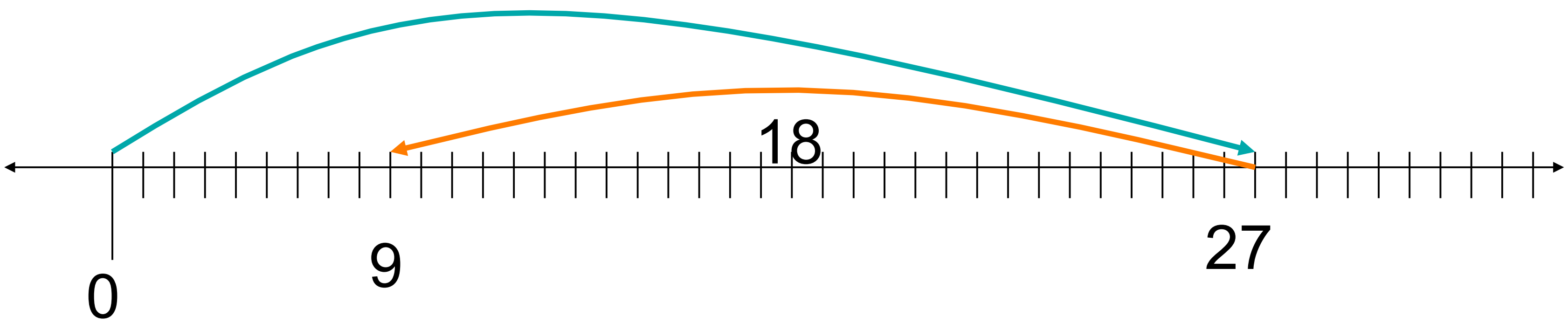
Two ways of thinking about subtraction

1. taking away

For example, 27 – 18

On the number line

“27 take away 18 is …”



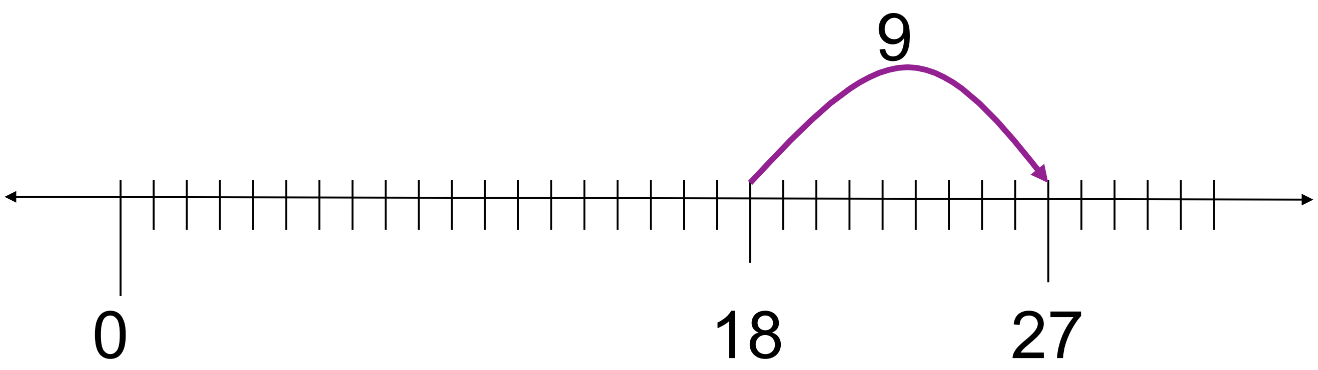
Two ways to think about subtraction

Two ways of thinking about subtraction

2. adding on

On the number line

"What do I add to 18 to get to 27?"



Number buddies

- I know 5 + 6 = … so I know 50 + 60 = …

also 55 + 6 =

Addition & subtraction by rules - algorithms as most efficient strategy

Addition & subtraction by rules - algorithms as most efficient strategy

- AMSI Modules

- Prep to 4:
  http://www.amsi.org.au/teacher_modules/Addition_and_subtraction.html

- 4 to 7:
  http://www.amsi.org.au/teacher_modules/addition_of_whole_numbers.html

- and 4 to 7:
  http://www.amsi.org.au/teacher_modules/Subtraction_of_whole_numbers.html

Subtraction

Two different algorithms:

Borrow and pay back

Equal addition

Renaming

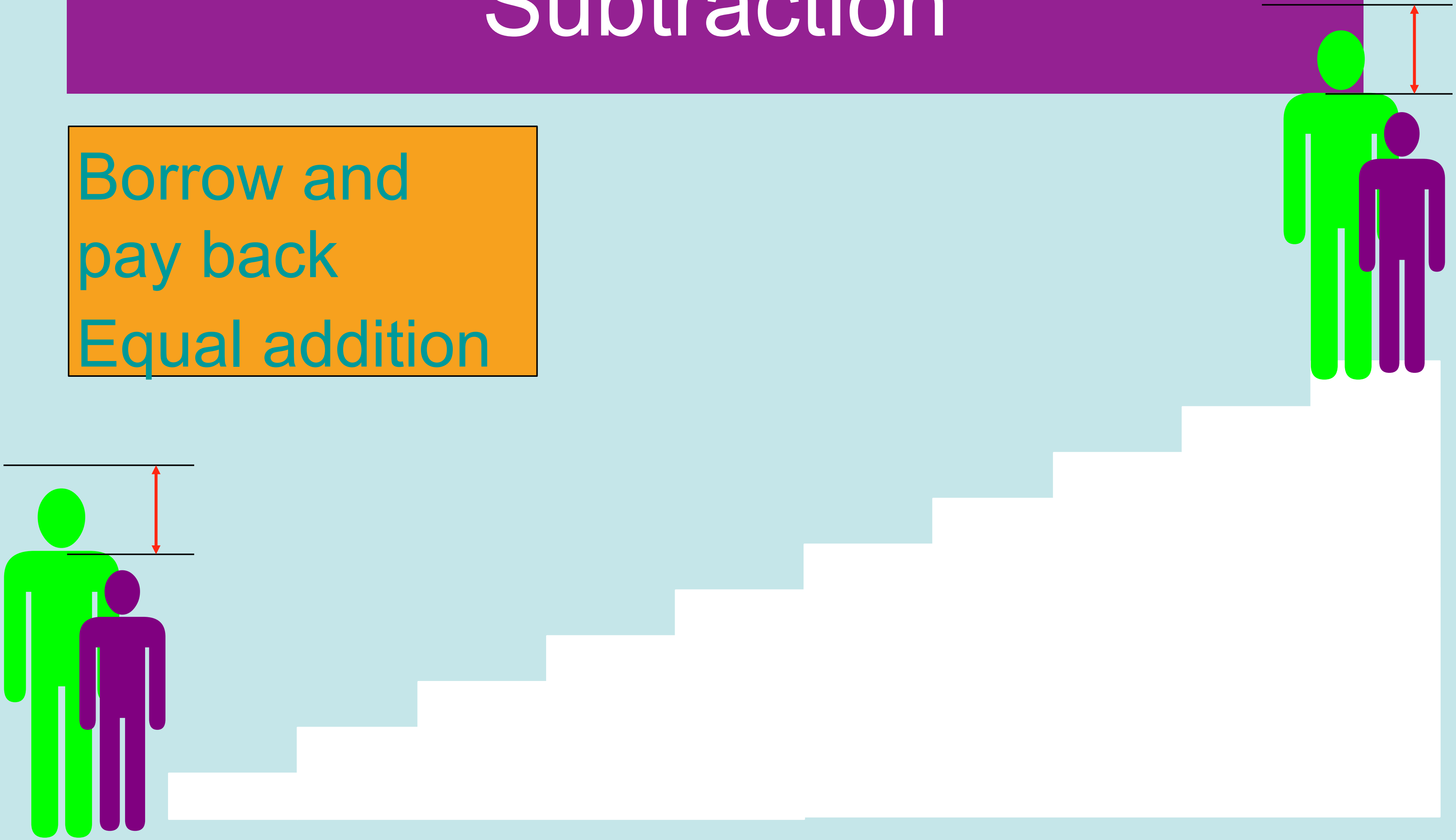
Decomposition

Trading

Subtraction

Borrow and pay back

Equal addition



Subtraction

Borrow and pay back

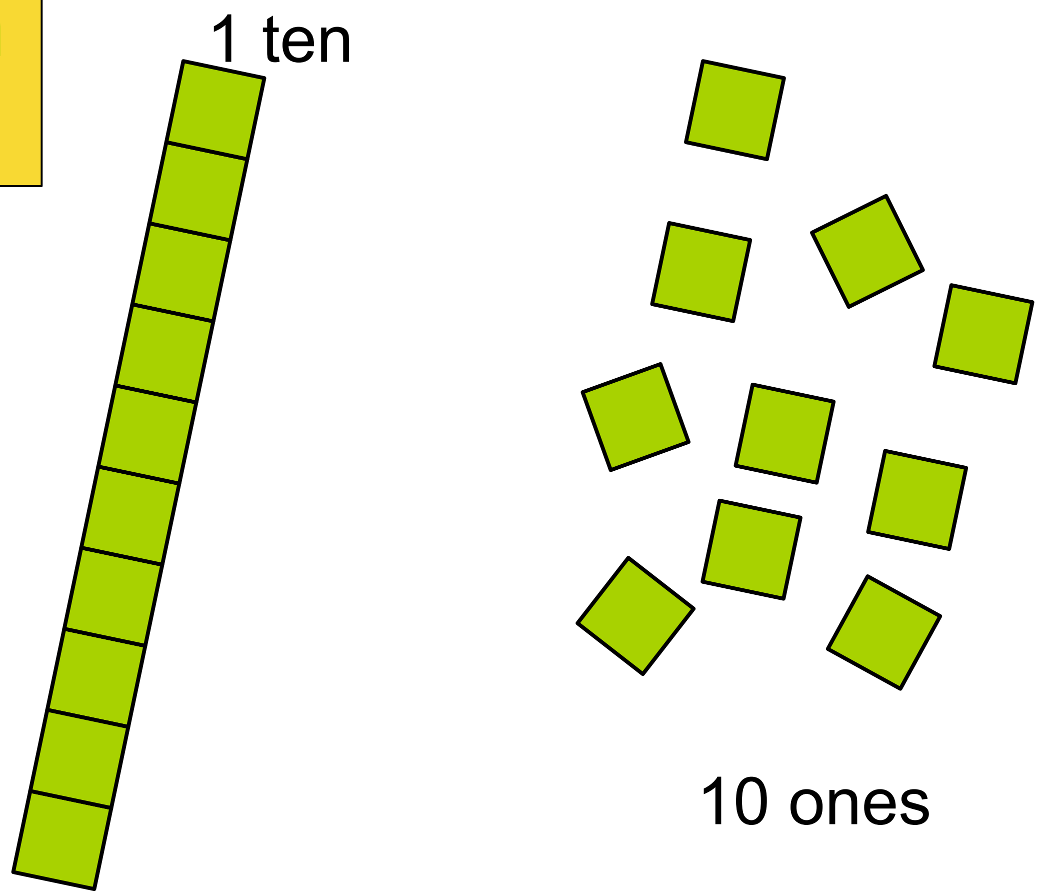
Equal addition

$$\begin{array}{r} 4\ {}^{1}2 \\ -\ 2_{1}\ 9 \\ \hline 1\ \ 3 \\ \hline \end{array}$$

$$\begin{aligned} &(40 + 2) - (20 + 9) \\ &= (40 + 10 + 2) - (20 + 10 + 9) \\ &= (40 - 30) + (12 - 9) \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

Subtraction

Renaming
Decomposition
Trading



Subtraction

Renaming

Decomposition

Trading

$$\begin{array}{r} \overset{3}{\cancel{4}}\;{}^{1}2 \\ -\;\;2\;\;9 \\ \hline 1\;\;3 \\ \hline \end{array}$$

$$\begin{aligned} &(40 + 2) - (20 + 9) \\ &= (30 + 12) - (20 + 9) \\ &= (30 - 20) + (12 - 9) \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

Subtraction

Now, calculate 33 – 19 using both methods.

Borrow and pay back
Equal addition

Renaming
Decomposition
Trading

Rules - Association, distribution, commutation

- Do they work for both addition and subtraction?

Rules - Association, distribution, commutation

As well as being commutative, addition is associative, meaning that for all numbers a, b and c:

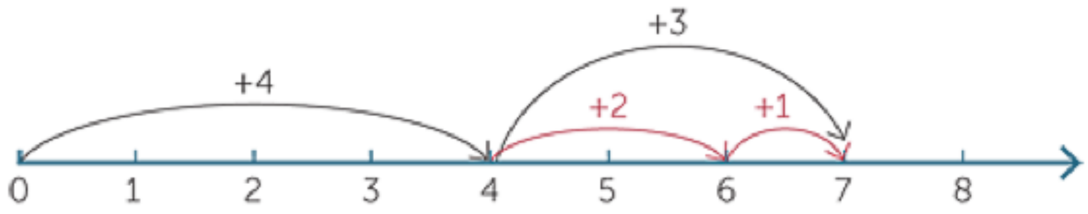
$$a + (b + c) = (a + b) + c$$

Because of the associative law we have

$4 + (2 + 1) = (4 + 2) + 1$

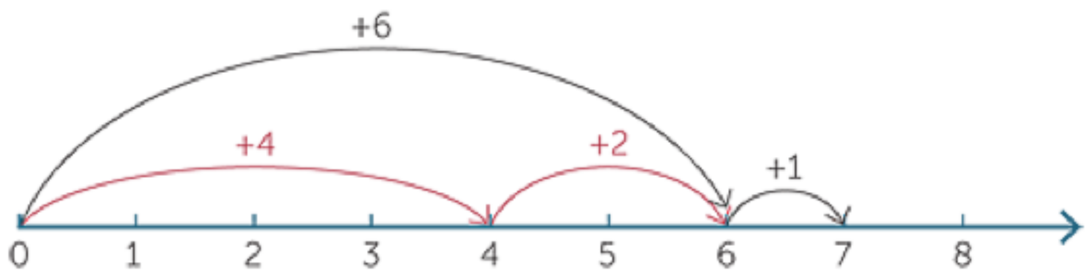
Rules - Association, distribution, commutation

4 + (2 + 1) corresponds to



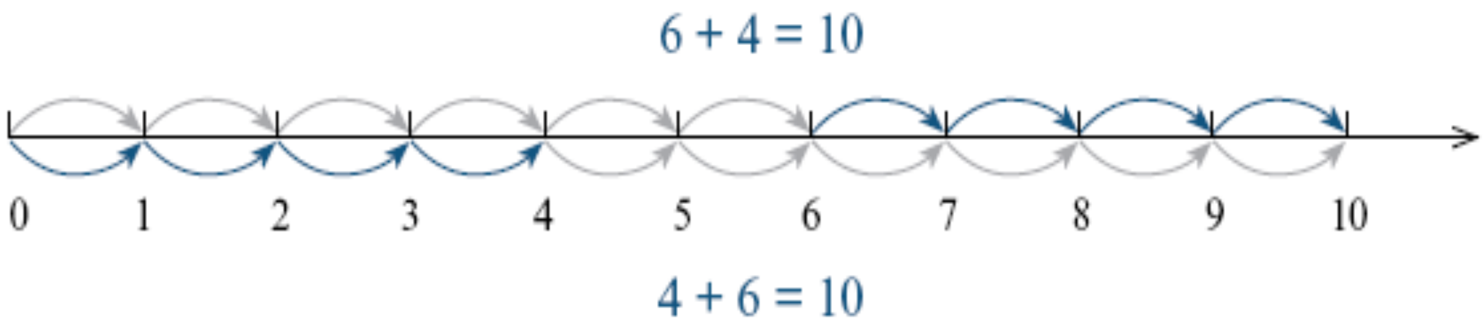
whereas

(4 + 2) + 1 corresponds to



Rules - Association, distribution, commutation

The order in which we perform addition does not matter. For example, $6 + 4 = 4 + 6$. This can be shown on the number line.



This property is called the **commutative law for addition.**

Rules - Association, distribution, commutation

We can produce a long list of arithmetic statements such as

$$4 + 3 = 3 + 4$$

$$2 + 6 = 6 + 2$$

$$8 + 3 = 3 + 8$$

Each is an example of the commutative property of addition.

Rules - Association, distribution, commutation

One algebraic statement defines the commutative law

$$a + b = b + a$$

where ***a*** and ***b*** are whole numbers.

Rules - Association, distribution, commutation

Any order principle of addition

$$13 + 25 + 45 + 27$$

Rules - Association, distribution, commutation

Students … use number properties in combination to facilitate computations

For example, 7 + 10 + 13 = 10 + 7 + 13 = 10 + 20)

We could also say:

12 + 28 = 10 + 2 + 28 = 10 + 30

Where we have moved 2 from the first to the second number (in an additive way)

Later students will need to do this:

12 x 25 = 3 x 4 x 25 = 3 x 100

Where we have moved 4 from the first to the second number (in an multiplicative way- “factor”)

Rules - Association, distribution, commutation

- Why is it important to think about arithmetic in this way?
- Does it apply only to addition?

Rules - Association, distribution, commutation

Any order principle of addition

$3x + 4y + 7x + 11y$

Rules - Association, distribution, commutation

Any order principle of multiplication

$$25 \times 7 \times 4 \times 3$$

Rules - Association, distribution, commutation

Any order principle of multiplication

$3xy^2 \times 4x2y^3$

Rules - Association, distribution, commutation

<u>Using the distributive law to multiply two-digit by one digit numbers mentally</u>

This is one of those truly multiplicative things. If we want to know what 34×8 is, splitting it into more manageable sums will help.

We can do 30×8 (that's 240 we used place value to extend our tables) and 4×8 (we know that's 32 from our tables) in our heads, then we add the two answers. $240 + 32$ is 272.

We distribute multiplication OVER addition.

Rules - Association, distribution, commutation

This also works for two digit × two digit multiplication, but you have to break it down more and it is a bit trickier for students:

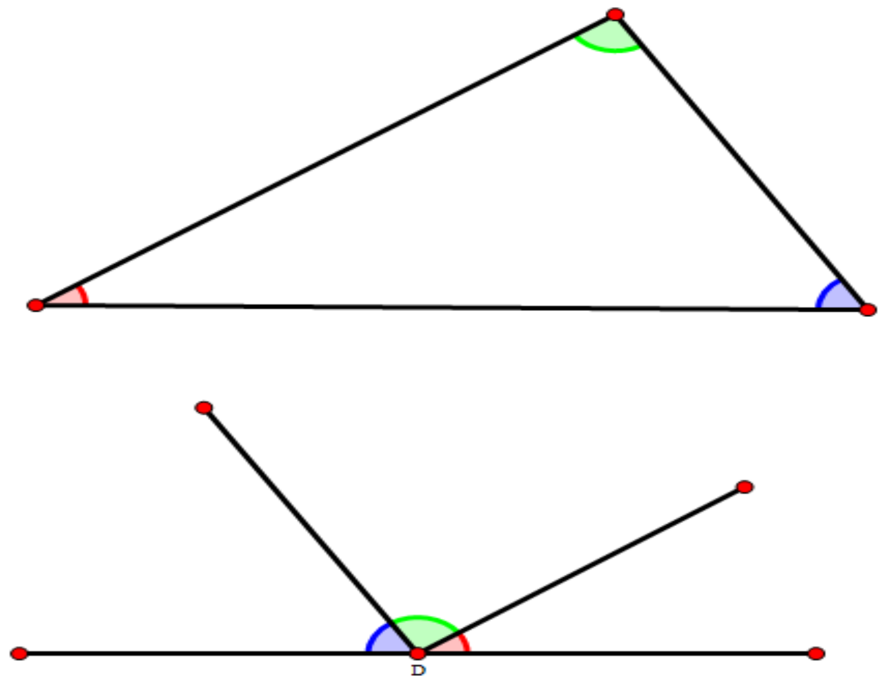
$$
\begin{aligned}
34 \times 18 &= (30 \times 10) + (30 \times 8) + (4 \times 10) + (4 \times 8) \\
&= 300 + 240 + 40 + 32 \\
&= 612
\end{aligned}
$$

Addition and subtraction with fractions and decimals

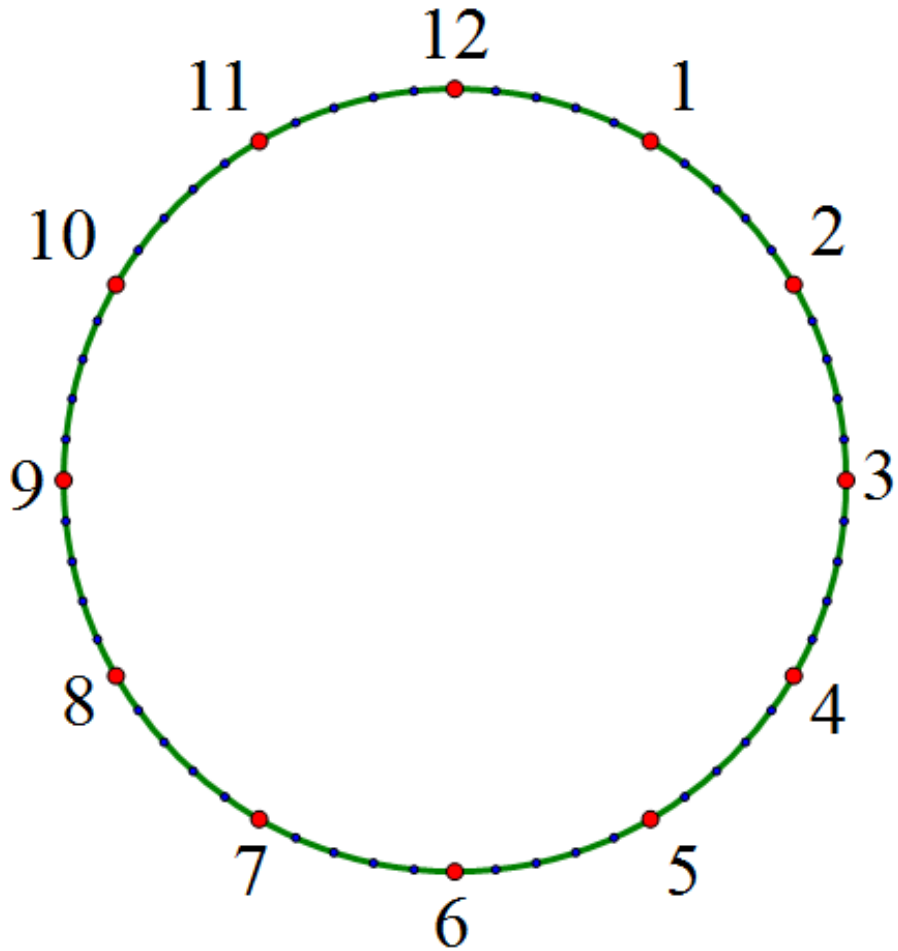
Contexts of addition:
Geometry, clocks, probability

Angle Arithmetic

And what about the angle sum of a polygon?



What Patterns can you make by repeated addition of one number?

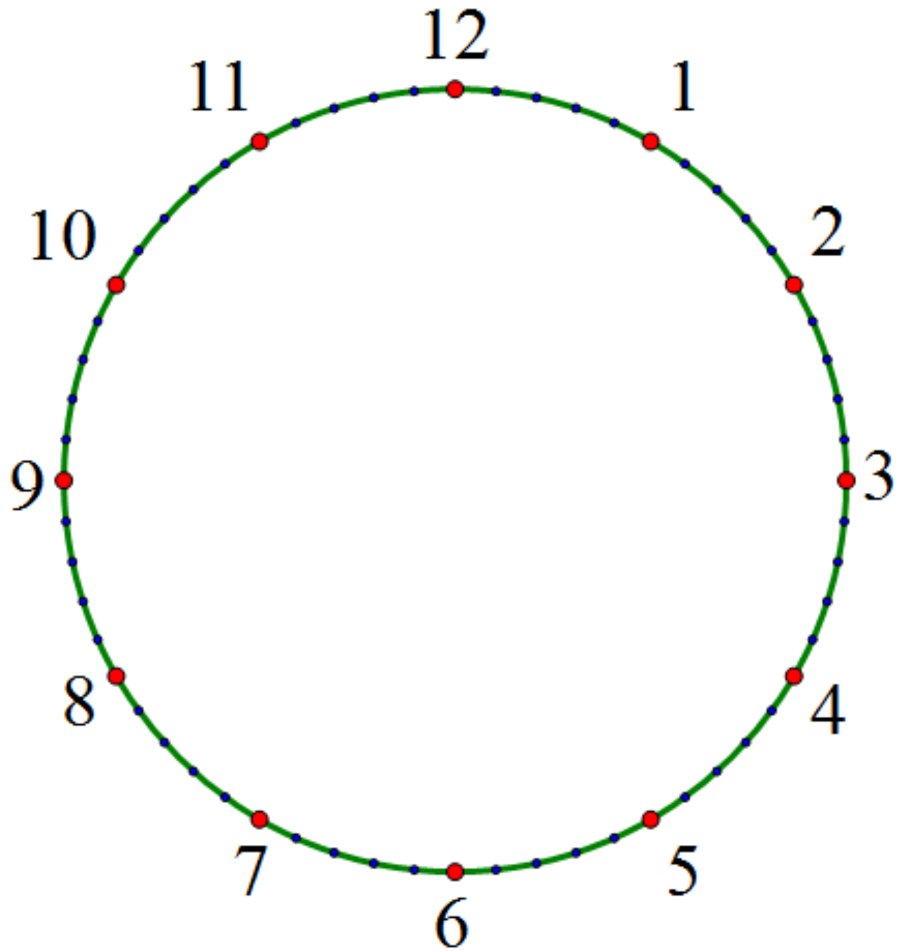


Clock Arithmetic

Sometimes Addition and subtraction don't always end in increasingly "larger numbers" going to infinity.

Consider what we do with time.

If it is 5 O'Clock now, and we add 10 hours, what time will the clock say?



Probability Arithmetic

Or the sum of probabilities?

Extending thinking into relational thinking – heading into algebra

Extending thinking into relational thinking – heading into algebra

Students need to be familiar with

- arithmetic
- patterns in arithmetic
- relationships between numbers and operations

Extending thinking into relational thinking – heading into algebra

- The teaching of number and algebra are inextricably linked.

- We cannot expect an improvement in student's learning of algebra until we succeed in building their understanding of arithmetic, i.e. knowledge of number and number operations, and mental computation techniques.

Leading back to arithmetic

$$345\,6789 \times 42 + 345\,6789 \times 58$$

Relational thinking: scenario

With thanks to Dr Max Stephens, University of Melbourne

The teacher wrote an open number sentence:

$$7 + 6 = \square + 5$$

and asked children to find the missing number and to say how they found it

Here are four different responses:

$$7 + 6 = \square + 5$$

Luke: 7 + 6 = 13 + 5

Teacher: *Luke, what number did you put in the box?*

Luke: *Thirteen*

Teacher: *How did you decide?*

Luke: *7 and 6 are 13*

Teacher: *What about the 5?*

Luke: *It doesn't matter. The answer to 7 + 6 is 13*

Teacher: *What is the 5 doing then?*

Luke: *It's just there*.

$7 + 6 = \square + 5$

Cameron: $7 + 6 = 18 + 5$

Teacher: *Cameron, what number did you put in the box?*

Cameron: *Eighteen*

Teacher: *How did you decide?*

Cameron: *7 and 6 are 13 and 5 more is 18*

Teacher: *Does 7 plus 6 equal to 18 plus 5?*

Cameron: *7 + 6 is 13 and 5 more is 18*

$7 + 6 = \square + 5$

Fiona: $7 + 6 = 8 + 5$

Teacher: *Fiona, what number did you put in the box?*

Fiona: *Eight*

Teacher: *How did you decide?*

Fiona: *7 and 6 gives 13 and I then thought what number goes with 5 to give 13.*

7 + 6 is 13 and 5 + 8 is 13

$7 + 6 = \square + 5$

Chris: $7 + 6 = 8 + 5$

Teacher: *Chris, what number did you put in the box?*

Chris: *Eight*

Teacher: *How did you decide?*

Chris: (Points to the numbers)

$7 + 6 = \square + 5$

5 is one less than 6, so you need a number that is one more than 7 to go in the $\square$ so it all balances.

$7 + 6 = \square + 5$

Luke and Cameron

Each child interpreted the number sentence and the equal sign in their own way

Luke and Cameron made reasonable attempts to deal with an unfamiliar problem

They appear to think that the equal sign is always followed by an answer

The equal sign is being used in a quite different way in these expressions

$7 + 6 = \square + 5$

Fiona and Chris

Fiona and Chris see the equal sign as expressing an equivalence between the numbers represented on both sides.

They can accept that the equal sign is not always followed by an ***answer.***

They are both comfortable in having number sentences with different forms.

Children need to meet number sentences with different *forms* and with different *operations.*

$7 + 6 = \square + 5$

Fiona's method

Fiona added both numbers on the left side, and then looked for a number to place in the box that would give the same total. Perhaps she thought:

$7 + 6 = \square + 5$

$7 + 6 = 13$, so

$13 = \square + 5$, so

what do I have to add to 5 to get 13. Yes 8.

She sees that the results of the calculations on both sides have to be the same.

$7 + 6 = \square + 5$

Chris' method

Chris' method is subtly different.

> *5 is one less than 6, so you need a number that is one more than 7 to go in the □ so it all balances.*

She looks at the relation between the two addition expressions on either side of the equal sign, not just at the answers of the two calculations.

What would you ask to see if Chris is a confident user of this kind of thinking?

It is possible that Fiona can think this way too – we don't know for sure – but it is ***unlikely*** that Luke or Cameron can think in this way without further teaching.

$7 + 6 = \square + 5$

Relational thinking

The ability to identify and use relations within number sentences such as:

$7 + 6 = \square + 5$, or

$27 + \square = 30 + 5$

is very important for students in the primary years:

- to think about the structure of arithmetic relations, and
- to use that knowledge as a bridge to algebra

Another problem for students

Loretta has written the following number sentence

$34 + 29 = 33 + 30$

She did not have to add up the numbers to know this. Why?

Two students' responses to $34 + 29 = 33 + 30$

One Year 6 student said, "Loretta just knows that they both add up to 63".

A Year 5 student said: "Loretta can do this because she did it in her head".

Neither student could explain why Loretta **did not need** to add the numbers in order to know that she was correct without using an explanation based on computation.

Two other students’ responses to $34 + 29 = 33 + 30$

One Year 5 student drew the following:

$+1$

$$34 + 29 = 33 + 30$$

-1

Referring to the 29, the student wrote, “It increases by 1 to give 30, so 34 has to decrease by 1 to give 33”.

Two other students' responses to $34 + 29 = 33 + 30$

A second Year 5 student inserted an arrow:

$$34 + 29 = 33 + 30$$

No number was attached to the arrow, but the student wrote: "If one unit moves from the 30, the other number becomes 34"

Teaching implications

Introducing young children to relational thinking is not an easy task when teachers’ vision has for so long been restricted to thinking of arithmetic as calculation.

In the primary school, this means attending to the structure of arithmetic operations.

Without these experiences, many students fail to understand these structures necessary for a successful transition to algebra.

Moving beyond addition

Teachers could introduce number sentences involving subtraction (difference) such as:

$41 - 15 = 43 - \square$

$104 - 45 = \square - 46$

Use "difference" rather than "subtraction"

Represent this difference on a number line

Payoffs for the Primary Years

How would you calculate?

$$\begin{array}{r} 3000 \\ -\underline{1563} \\ \\ \underline{} \end{array}$$

Payoffs for the Primary Years

Can relational thinking help us to find alternatives?

$$\begin{array}{r} 3000 \\ -\underline{1563} \\ \underline{} \end{array}$$

Some students find algorithms for subtraction (e.g. method of decomposition) difficult and time consuming

Payoffs for the Primary Years

Given 3000 – 1563

Increase both numbers by 37 gives

3037 – 1600

This can be calculated more easily!! 1437

(Why did we choose 37? Because it makes the second number 1600 and easier to subtract)

Payoffs for the Primary Years

We can also change the first number:

Given 3000 – 1563

Decrease both numbers by 1: 2999 – 1562

$$\begin{array}{r} 2999 \\ -\ \underline{1562} \\ 1437 \end{array}$$

This can be calculated more easily!! 1437

Relational thinking

- Is a powerful way of drawing attention to some fundamental structures of arithmetic

- Two key ideas are:

  – equivalence of expressions, and

  – compensation, including knowing the direction in which compensation takes place

- These ideas also provide a foundation for algebraic thinking

Comparing Addition and Subtraction (inverse operations)

Addition

Changes to 1st number	Changes to 2nd number	Sum
+5	- 5	same
-3.5	+3.5	same
- $^{1}/_{4}$	+ $^{1}/_{4}$	same

Subtraction

Changes to 1st number	Changes to 2nd number	Difference
+5	+5	same
-3.5	-3.5	same
+ $^{1}/_{4}$	+ $^{1}/_{4}$	same

Comparing Multiplication and Division (inverse operations)

Multiplication

Changes to 1st number	Changes to 2nd number	Product
x 5	÷ 5	same
÷ 25	x 25	same
÷ 10	x 10	same

Division

Changes to 1st number	Changes to 2nd number	Quotient
x 5	x 5	same
÷ 25	÷ 25	same
x 10	x 10	same

Summary of 4 operations

Operation in number sentence	Keep this the same	Direction of adjustments* to the original numbers	Operations involved in these adjustments
Addition	Sum	Opposite directions	Additive (+/-)
Subtraction	Difference	Same direction	Additive (+/-)
Multiplication	Product	Opposite directions	Multiplicative (x/÷)
Division	Quotient	Same direction	Multiplicative (x/÷)

*** Note that the numbers involved in these adjustments can be any type of number (eg whole numbers, fractions, decimals)**

Extending thinking into relational thinking – heading into algebra

We can keep the sum or the difference the same by making particular adjustments to each number using the operations of addition and subtraction.

- To keep the sum the same, one number is increased by a certain amount and the other number is decreased by that same amount.
- To keep the difference the same, both numbers are increased (or decreased) by the same amount.

In each case, the amount of increase or decrease can be any type of number (whole, fraction, decimal)

In each case, the type of increase or decrease is additive (which means involves only addition or subtraction)

Senior: Summation-Finite and Infinite Series

Achilles and the Tortoise

- Achilles and a tortoise are having a race over 100 metres.
- To be fair, Achilles gives the tortoise a 50 metre head start.
- In the time that it takes Achilles to cover half the original distance between himself and the tortoise, the tortoise has moved to a new position.

Infinite Sums

- Add the following: 1 + ½ + 1/3 + ¼ + …
- 8 + 4 + 2 + 1 + ….
- ½ + ¼ + 1/8 + …

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Senior: Summation of Infinitesimals

To think about

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$\pi = 6\left(\frac{1}{2} + \frac{1}{2\cdot3\cdot2^3} + \frac{1\cdot3}{2\cdot4\cdot5\cdot2^5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot2^7} + \cdots\right)$$

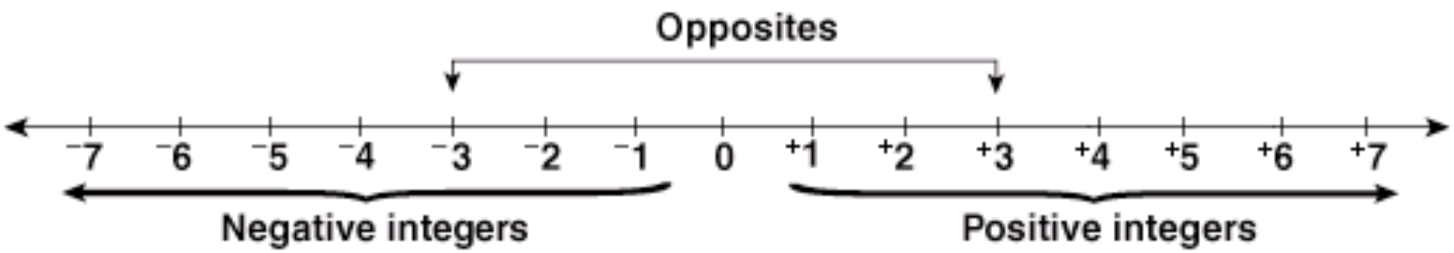
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

the integers

The Integers

A Definition

Integers are the set of whole numbers and their opposites.



The Integers

The Australian Curriculum

Year 6

1. Integers

Read, represent, write, interpret and order positive and negative numbers.

The Integers

The Australian Curriculum

Year 6

Elaborations

- whole numbers can be positive and negative and continue indefinitely

- investigate everyday situations to understand integers

- using number lines to position and order positive and negative integers

- solve everyday additive problems without developing formal rules

The Integers

The Australian Curriculum

Year 7

2. Integers

Order, add and subtract integers fluently and identify patterns for multiplication and division using ICT.

The Integers

The Australian Curriculum

Year 7

Elaborations

- understanding the number system extends beyond positive numbers and that rules for operations on integers can be developed from patterns

- becoming fluent with operations with integers and identifying patterns

- interpreting the use of integers in real world contexts e.g. temperature, sea level and money

- recognising and using the language of integers accurately

The Integers

The Australian Curriculum

Year 7

Elaborations

- comparing and ordering integers using strategies such as locating on a number line

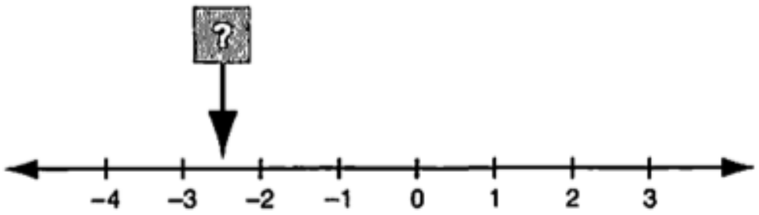
- using a range of strategies for adding and subtracting integers, including number lines and virtual manipulatives

- investigating multiplication of positive and negative integers, using a range of strategies including patterning, multiplication as repeated addition and identifying the process for division as the inverse of multiplication.

The Integers

Year 7 & 9 2009 NAPLAN Question

7



Write your answer in the box.

The arrow points to a position on the number line.

What number is at this position?

The Integers

Year 9 2009 NAPLAN Question

20 Two numbers added together equal 1.
The two numbers multiplied together equal –30.

Write your answers in the boxes.

What are the two numbers? [ ] and [ ]

The Integers

Year 9 2009 NAPLAN Question

22 When $m = 2$ and $n = -2$, what is the value of $m^2 + n^2$?

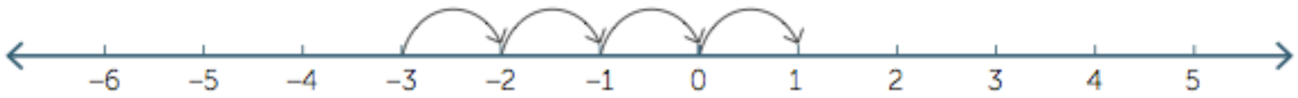
$m^2 + n^2 =$ $\boxed{}$

The Integers

Addition and subtraction of positive integers

Adding a positive integer

When you add a positive integer, you move to the right along the number line.



For example, to calculate –3 + 4, start at –3 and move 4 steps to the right, so –3 + 4 = 1

Money also provides a useful model. For example: the statement –3 + 4 = 1 can be modelled as:

I had a debt of \$3, but I earned \$4. I now have \$1.

The Integers

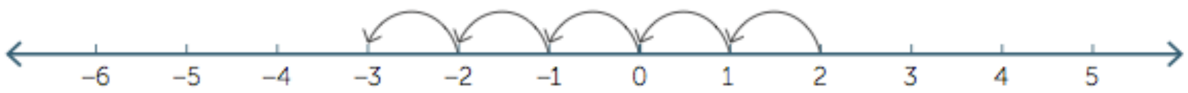
Addition and subtraction of positive integers

Subtracting a positive integer

We will start by thinking of subtraction as '**taking away**'.

When you subtract a positive integer, move to the left along the number line.

For example, to calculate 2 – 5: start at 2 and move 5 steps to the left. We see that 2 – 5 = –3.



This subtraction can also be understood in terms of money.

I had \$2 and I spent \$5. I now have a debt of \$3.

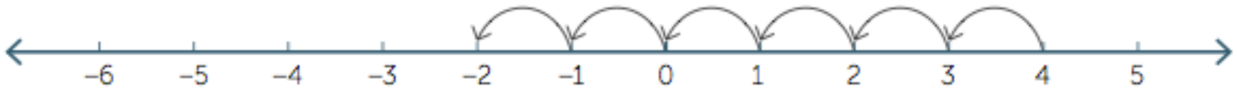
The Integers

Addition and subtraction of a negative integer

Adding a negative integer

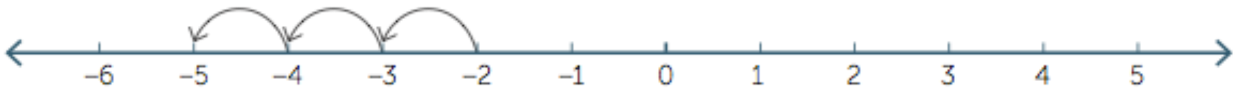
Adding a negative integer to another integer means that you take a certain number of steps to the left on the number line.

The result of the addition 4 + (– 6) is the number you get by moving 6 steps to the left, beginning at 4.



$4 + (-6) = -2$

You can also start at a negative number. The result of the addition –2 + (– 3) is the number you get by moving 3 steps to the left, beginning at –2.



Notice that –2 – 3 is also equal to –5.

The Integers

Addition and subtraction of a negative integer

Subtracting a negative integer

We have already seen that adding -2 means taking 2 steps to the left.
For example, $7 + (-2) = 5$.

We want subtracting -2 to be the reverse process of adding -2.

So to subtract -2, we will take two steps to the right. For example, $7 - (-2) = 9$.

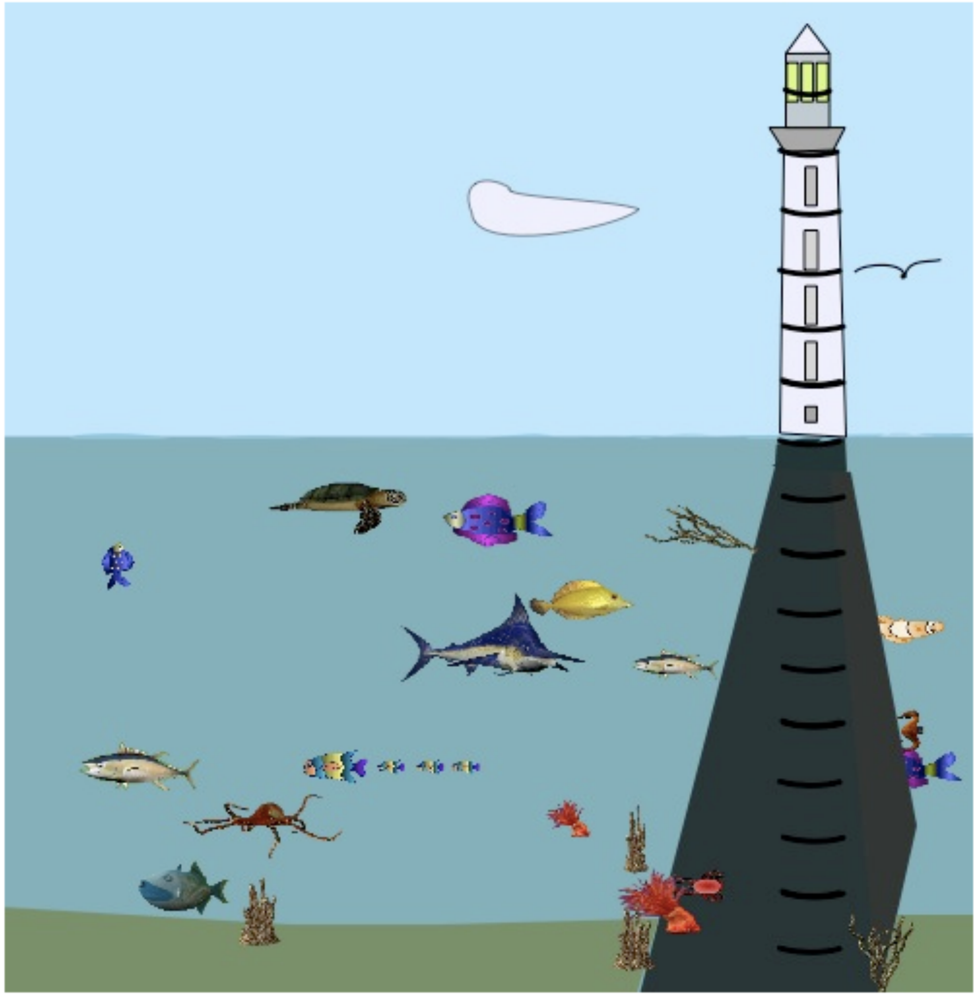
There is a very simple way to state this rule:

To subtract a negative number, add its opposite.

For example, $7 - (-2) = 7 + 2$

The Integers

Using the number line to order and compare



http://nrich.maths.org/public/viewer.php?obj_id=5929&part=index

The Integers

Using the number line to order and compare integers

- Walking the number line

- Dice games

- Walk the Plank

- The number ladder

-

The Integers

Integer grids - addition

+		6		10
	-1			
	7	11		
-7			3	
				2

The Integers

Integer grids - subtraction

-		8	-5	
2	5			2
	-6			
-1		-9		
			6	

The Integers

Magic Squares

Use the integers -10, -8, -6, -4, -2, 0, 2, 4, & 6 to complete the Magic Square.

	-2	

The Integers

Magic Squares

The Magic Number = -6

4	-10	0
-6	-2	2
-4	6	-8