ADDITIONAL MATHEMATICS FORM 5 SBA

To determine the optimal angle, the sides of the trapezium should be bent to, in order to create the maximum carrying capacity.

AIM OF PROJECT

This project shows how realistic problems can be solved using and applying mathematical formulations.

A number of hardware and roofing companies are now offering consumers materials in a variety of styles. With specialised machinery, these sheets can be bent into any shape to create the stylish look the customer requires.

A major company in the city has new equipment capable of making guttering in the shape of an open top trapezium. The sheet metal used is 22 cm wide and bent such that the base is 14 cm.

This project relates the angle of the bent sides of the trapezium to its total area.

This project is targeted to evaluate the best angle to use to bend the sides of the trapezium. In order to maximise the carrying capacity of rain water run-off from the roof, the following must be investigated:

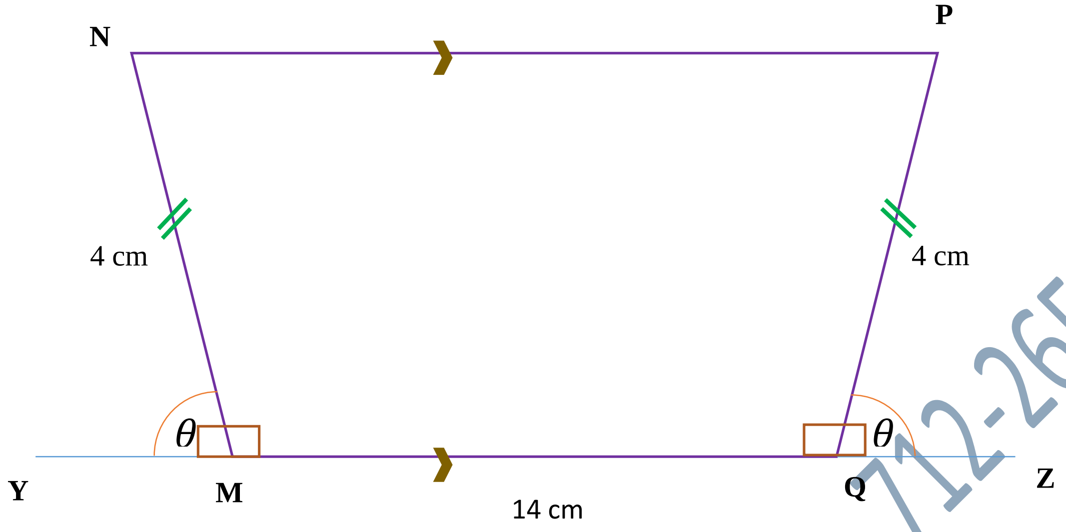
1. The relationship between the angle and the total area of the trapezium.
2. The size of the angle.
3. The maximum area that can be obtained.

This project uses two (2) mathematical method to solve this problem; a graphical method and an analytical method.

The variables manipulated are:

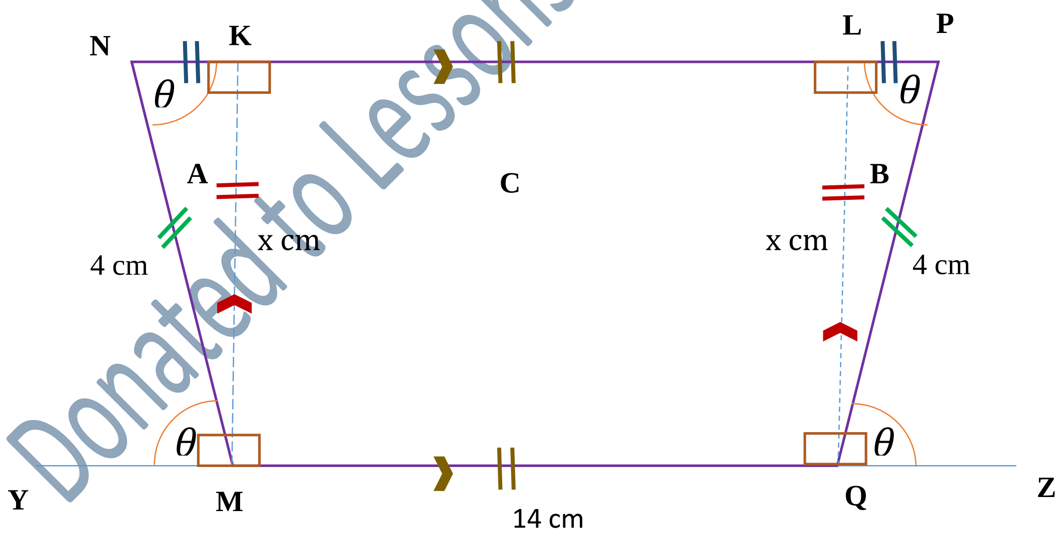
- X – Height of triangle or breadth of rectangle.
- A – Total area or area or trapezium.
- Ɵ - Angle the sides of the trapezium is bent.

MATHEMATICAL FOMULATION



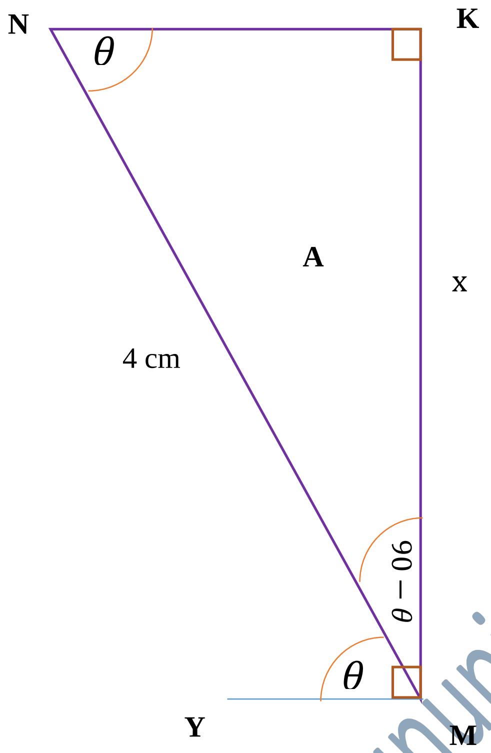
Above diagram NMQP, is a Trapezium made of a metal sheet 22 cm wide and bent such that the base is 14 cm, leaving 4 cm for each side to be bent.

The Trapezium is examined further to obtain additional information of it, by applying properties of a trapezium and mathematical formulae and expressions.



The Trapezium is divided into Three (3) Parts; MNK is Triangle A, MKLQ is Rectangle C and QLP is Triangle B. Triangle A and Triangle B are both right-angle triangles and are congruent. Rectangle C has sides 14cm and X. Angle YMN and Angle MNK are alternating angles and are therefore equal. In order to bend the sides to give the required trapezium above, the angle must be acute, which ranges from 0° to 90° (degrees).

Triangle A is examined in a clearer diagram:



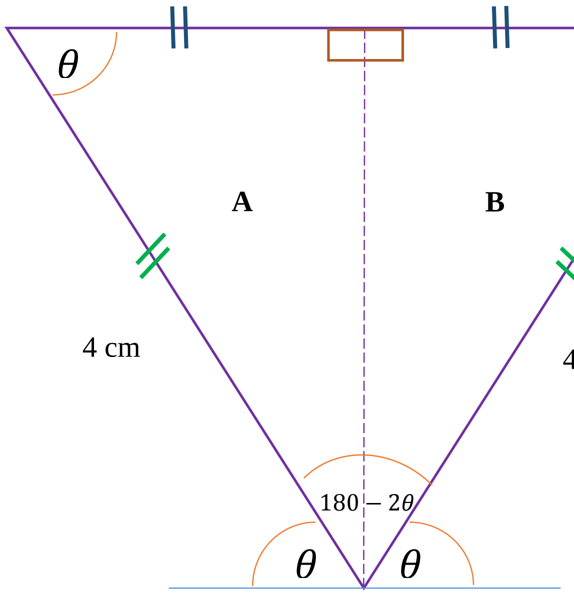
Examining Triangle A, there are two methods to solving for X, the height of triangle (breath of rectangle):

$$Either\ \sin\theta = \frac{Opposite}{Hypotenuse}\ or\ \cos(90-\theta) = \frac{Adjacent}{Hypotenuse}$$

$$Either\ \sin\theta = \frac{[x]}{4}\ or\ \cos(90-\theta) = \frac{[x]}{4}$$

$$Either\ [x] = 4\sin\theta\ \ or\ [x] = 4\cos(90-\ \theta)$$

Triangle A and Triangle B are combined and then examined:



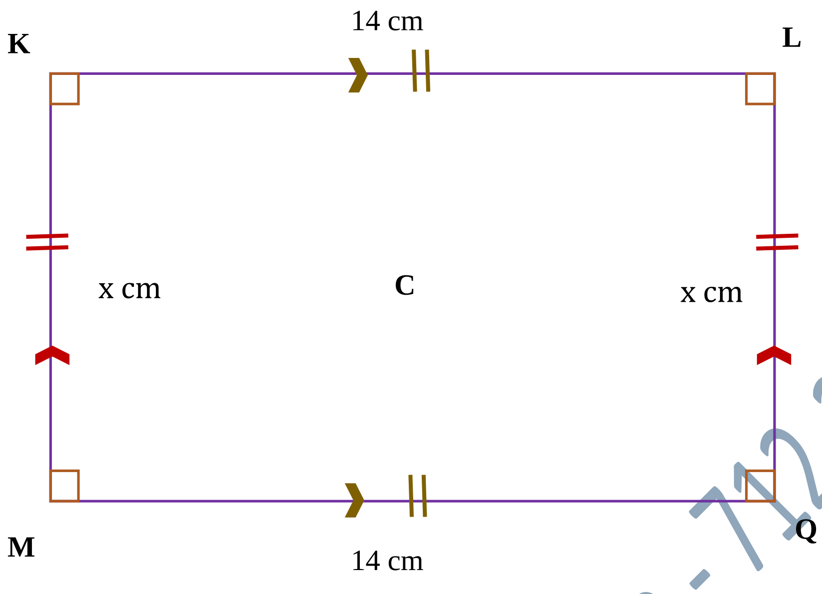
Examining combined Triangles A and B, the area of the combined Triangles A and B can be evaluated:

$$\text{Area of Triangle FORMULA } = \frac{[Side\ A] \text{ x } [Side\ C] \text{ x } [\sin\theta]}{2}$$

$$\text{Area of Combined Triangles A \& B } = \frac{[4] \text{ x } [4] \text{ x } [\sin(180 - 2\theta)]}{2}$$

$$\text{Area of Combined Triangles A \& B } = 8\sin(180 - 2\theta)$$

Rectangle C is examined:



By solving for X using Triangle A, the area of rectangle C can be evaluated:

Using Sine Angle Method:

$Area\ of\ Rectangle = [Length] \text{ x } [Breadth]$

$Area\ of\ Rectangle = [14] \text{ x } [4 \sin \theta]$

$Area\ of\ Rectangle = 56 \sin \theta$

Using Cosine Angle Method:

$Area\ of\ Rectangle = [Length] \text{ x } [Breadth]$

$Area\ of\ Rectangle = [14] \text{ x } [4 \cos(90 - \theta)]$

Apply Compound Angle Formula:

$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$

$Area\ of\ Rectangle = [14] \text{ x } [4\, \{\cos 90 \cos \theta + \sin 90 \sin \theta]$

$Area\ of\ Rectangle = [14] \text{ x } [4 \sin \theta]$

$Area\ of\ Rectangle = 56 \sin \theta$

End Result for using either method is the same:

$Area\ of\ Rectangle = 56 \sin \theta$

Knowing the areas Of Triangles A and B, and Rectangle C, the total area of the trapezium is found:

$$Total\ Area, [A] = [Area\ of\ Combined\ Triangles] + \ [\text{Area of Rectangle}]$$

$$[A] = [8 \sin(180 - 2\theta)] + \ [56 \sin \theta]$$

$$[A] = [8 \sin(180 - 2\theta) + 56 \sin \theta]$$

Apply Compound Angle Formula:

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$$

$$Total\ Area, [A] = [8\{[\sin 180\ \ \cos 2\theta] - \ [\sin 2\theta\ \ \cos 180]\} + 56 \sin \theta]$$

$$[A] = [8 \sin 2\theta + 56 \sin \theta]$$

Hence, the area of the trapezium, A cm^2, is given in terms of sine of the angle Ө, of the bent sides of the trapezium.

SOLUTION

Assumptions:

1. It is expected that the angle required to bend the sides of the trapezium is acute and therefore can only lie within the first quadrant, which ranges from 0° to 90° (degrees).
2. It is assumed that the trapezium can be divided into three (3) shapes, a rectangle at the centre and two (2) congruent triangles.
3. Consequently, finding the sum of these three (3) shapes is equivalent to the exact area of the trapezium.

Graphical Method:

To obtain the best angle to give maximum carrying capacity of the guttering, a table will be created for values of angle ϴ using the domain 0° ≤ ϴ ≤ 90° (degrees) in intervals of 15° (degrees) and corresponding values of a will be calculated using the equation $[A] = [8\sin 2\theta + 56\sin\theta]$.

A graph of area of trapezium (A cm^2) VS the angle (ϴ° (degrees)) at which the sides of the trapezium are bent will then be plotted on graph paper and hence the best angle to give maximum area will be obtained.

Calculus Method:

To obtain the best angle to give maximum carrying capacity of the guttering, the area of the trapezium will be differentiated for the first derivative function $\frac{dA}{d\Theta}$. For stationary values (maximum or minimum), the first derivative function $\frac{dA}{d\Theta}$ will be equated to zero (0) since the turning points are the maximum or minimum value.

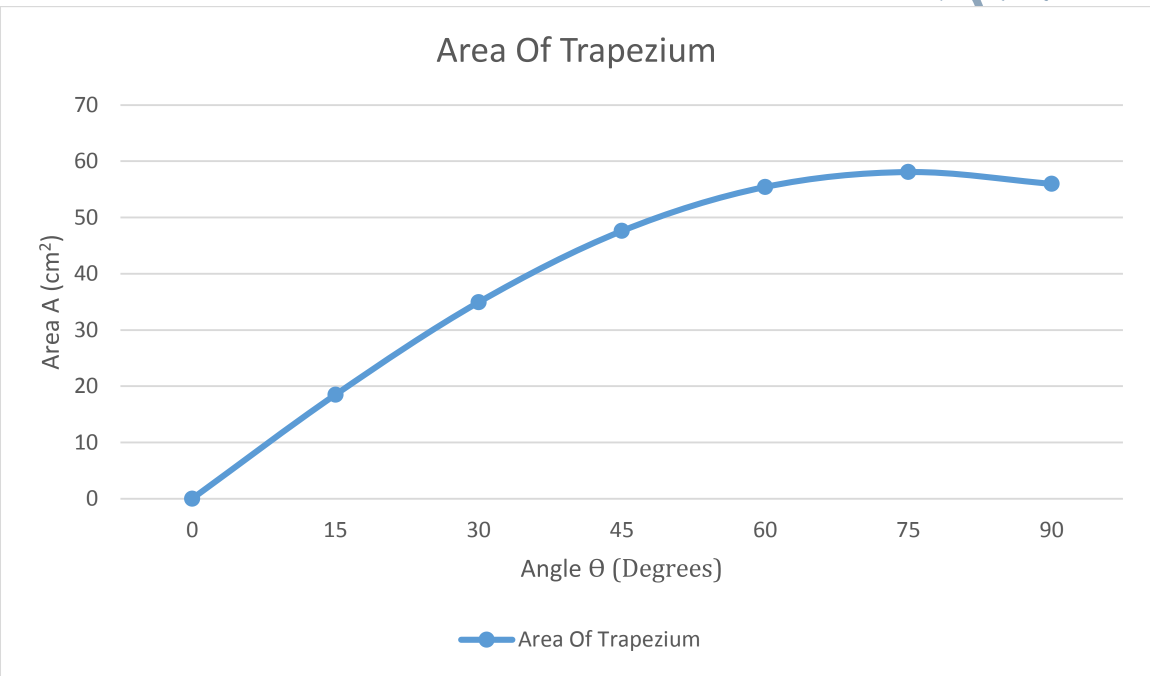
To determine whether the values obtained are maximum or minimum values, the second derivative $\frac{d^2A}{d\Theta^2}$ will then be found. A negative value of $\frac{d^2A}{d\Theta^2}$ will indicate that the area of the trapezium is a maximum value. A positive value of $\frac{d^2A}{d\Theta^2}$ will indicate that the area of the trapezium is a minimum value.

When the angle for the sides of the trapezium to be bent at is known, it will be substituted into the equation for A, the maximum area of the trapezium.

Graphical Method:

$$[A] = [8\sin 2\theta + 56\sin\theta]$$

Angle ϴ (Degrees)	Area A (cm^2)
0	0
15	18.49
30	34.93
45	47.60
60	55.43
75	58.09
90	56.00



From the graphical method, it is seen that the maximum value for the area A, of the trapezium, is 58.09 cm^2 and is obtained when the angle ϴ, the sides of the trapezium is bent at is 75° (degrees).

Calculus Method:

$[A] = [8\sin 2\theta + 56\sin\theta]$

Differentiate:

$$\frac{dA}{d\theta} = [16\cos 2\theta + 56\cos\theta]$$

Maximum or Minimum Area occurs when gradient is zero: $\frac{dA}{d\theta} = 0$

$0 = [16\cos 2\theta + 56\cos\theta]$

Applying Trigonometry Identities:

$0 = [16\{(2\cos\theta)^2 - 1\} + 56\cos\theta]$

$0 = [32(\cos\theta)^2 - 16 + 56\cos\theta]$

Divide Throughout the Equation by Eight (8):

$0 = [4(\cos\theta)^2 - 2 + 7\cos\theta]$

Solve Quadratic By Applying Formula:

$$\cos\theta = \frac{-[b] \pm \sqrt{[\{b\}^2 - \{4\}\{a\}\{c\}]}}{[\{2\}\{a\}]}$$

$$\cos\theta = \frac{-[7] \pm \sqrt{[\{7\}^2 - \{4\}\{4\}\{-2\}]}}{[\{2\}\{4\}]}$$

$$\cos\theta = \frac{-7 \pm \sqrt{[49 + 32]}}{8}$$

$$\cos\theta = \frac{-7 \pm \sqrt{81}}{8}$$

$$\cos\theta = \frac{-7 \pm 9}{8}$$

$$Either \quad \cos\theta = \frac{1}{4} \quad or \quad \cos\theta = -2$$

$\cos\theta = -2$ Cannot be evaluated, therefore only solution to the problem is: $\cos\theta = \frac{1}{4}$

$$\cos\theta = \frac{1}{4}$$

$$\theta = \cos^{-1}\left(\frac{1}{4}\right)$$

This angle is in the first quadrant and fourth quadrant. However, the angle is acute and therefore can only be in the first quadrant:

$\theta = 75.52°$ (Degrees)

To determine whether it is a maximum or minimum value for the area of the trapezium, substitute angle ϴ into the second derivative:

Differentiate The First Derivative:

$$\frac{d^2A}{d\theta^2} = [-18\sin 2\theta - 56\sin\theta]$$

$$\frac{d^2A}{d\theta^2} = [-18\sin\{2(75.52)\} - 56\sin(75.52)]$$

$$\frac{d^2A}{d\theta^2} = [-18\sin(151.04) - 56\sin(75.52)]$$

$$\frac{d^2A}{d\theta^2} = [(-18)(0.48) - (56)(0.97)]$$

$$\frac{d^2A}{d\theta^2} = [-8.71 - 54.22]$$

$$\frac{d^2A}{d\theta^2} = -62.93$$

The angle obtained will give a maximum value for the area of the trapezium.

From the calculus method, it is seen that the best angle, the sides of the trapezium should bent at, is 75.52° (degrees).

Maximum Area of Trapezium:

$$[A] = [8\sin 2\theta + 56\sin\theta]$$

$$[A] = [8\sin 2(75.52) + 56\sin(75.52)]$$

$$[A] = [8\sin 151.04) + 56\sin(75.52)]$$

$$[A] = [(8)(0.48) + (56)(0.97)]$$

$$[A] = [3.87 + 54.22]$$

$$[A] = 58.09\ cm^2$$

From the calculus method and the angle obtained, the maximum area of the trapezium is 58.09 cm^2.

DICUSSION

It is noticeable that the values obtained for angle ϴ, at which the sides of the trapezium should be bent, differ slightly from the two (2) methods used. The graphical method gives a best approximated value (ϴ = 75° (degrees)) and is dependent on how precise the points were plotted and the curve drawn, whereas the calculus method provides a more precise value (ϴ = 75.52° (degrees)). Therefore, for these real-life problems, the calculus method should be used to attain a more accurate result, without the many errors associated with the graphical solutions.

CONCLUSION

Using the 22 cm wide metal sheet, the guttering in shape of a trapezium with a base of 14 cm, can be constructed to give a maximum carrying capacity of 58.09 cm^2 and this occurs when the angle at which the sides of the trapezium is bent, is 75.52° (degrees).

As observed, the value from the calculus method will be used.

SUGGESTIONS FOR FUTURE ANALYSIS

Further investigations involving the rates of inflow and outflow of the trapezium-shaped guttering can be carry out for future projects. In addition to this, guttering can be made in many other shapes such as a semi-circle or triangle which could have a better carrying capacity than a trapezium and therefore investigations pertaining to these shapes can be carried out for future projects.