THOMAS WHITHAM SIXTH FORM

Additional Mathematics

Revision Guide

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The book contains a number of worked examples covering the topics needed in the Additional Mathematics Specifications. This includes Calculus, Trigonometry, Geometry and the more advanced Algebra. Plus the topics required for Mechanics and statistics.

Algebra

- **Indices for all rational exponents**

$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $\left(a^m\right)^n = a^{mn}$, $a^0 = 1$,

$a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{n}} = \sqrt[n]{a}$, $a^{-n} = \dfrac{1}{a^n}$, $\dfrac{1}{a^{-n}} = a^n$, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Example $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ Example $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

Example $x^{\frac{2}{3}} = 4 \Rightarrow \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} \Rightarrow x = \left(\sqrt{4}\right)^3 = \underline{8}$

Example Simplify $2x^{-1} \times (3x)^2 \div 6x^3$

$$2x^{-1} \times (3x)^2 \div 6x^3 = \frac{2}{x} \times 9x^2 \times \frac{1}{6x^3} = \frac{18x^2}{6x^4} = \underline{\frac{3}{x^2}}$$

- **Quadratic equations** $ax^2 + bx + c = 0$

Examples (i) $4x^2 = 9$ (ii) $4x^2 = 9x$ (iii) $4x^2 = 9x - 2$

(i)	(ii)	(iii)
$x^2 = \frac{9}{4}$	$4x^2 - 9x = 0$	$4x^2 - 9x + 2 = 0$
	$x(4x-9) = 0$	$(4x-1)(x-2) = 0$
$\therefore x = \pm\frac{3}{2}$	$\therefore x = 0$ or $x = \frac{9}{4}$	$\therefore x = 2$ or $x = \frac{1}{4}$

Example Solve (i) $2x^2 + x - 5 = 0$ using the method of CTS

(ii) $5x^2 - 7x + 1 = 0$ using the formula.

(i) $2x^2 + x - 5 = 0$

$$x^2 + \frac{x}{2} = \frac{5}{2}$$

$$x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2 = \frac{5}{2} + \left(\frac{1}{4}\right)^2$$

$$\left(x + \tfrac{1}{4}\right)^2 = \tfrac{5}{2} + \tfrac{1}{16}$$

$$\left(x + \tfrac{1}{4}\right)^2 = \tfrac{41}{16}$$

$$x + \tfrac{1}{4} = \pm\sqrt{\tfrac{41}{16}}$$

$$x = -\tfrac{1}{4} \pm \sqrt{\tfrac{41}{16}}$$

$$\underline{\underline{x \approx 1.35, \quad -1.85}}$$

- **Simultaneous equations**

  Example   Solve simultaneously $2x+3y=8, \quad y=x^2-x+2$

  Here we substitute for *y* from the second equation into the first

  $2x+3y=8 \quad \therefore \; 2x+3\left(x^2-x+2\right)=8 \quad \therefore \; 3x^2-x-2=0$

  $\therefore \; (3x+2)(x-1)=0 \qquad \therefore \; x=-\frac{2}{3}, \; 1$

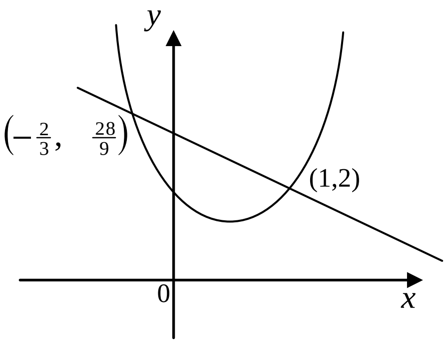
  when $x=-\frac{2}{3}$, $y=\frac{4}{9}+\frac{2}{3}+2=\frac{28}{9}$

  when $x=1$, $y=1-1+2=2$

  Solutions, $x=-\frac{2}{3}, \quad y=\frac{28}{9}$

  $x=1, \quad y=2$

  The geometrical interpretation here is that the straight line $2x+3y=8$ and the parabola $y=x^2-x+2$ intersect at points $\left(-\frac{2}{3}, \; \frac{28}{9}\right)$ $(1, \; 2)$



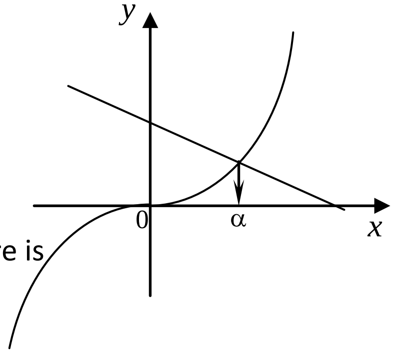
- **Intersection points of graphs to 'solve' equations**. There are many equations which can not be solved analytically. Approximate roots to equations can be found graphically if necessary.

  Example   What straight line drawn on the same axes as the graph of $y=x^3$ will give the real root of the equation $x^3+x-3=0$?

  $x^3+x-3=0 \;\Rightarrow\; x^3=3-x$

  $\therefore$ draw $y=3-x$

  As can be seen from the sketch there is only one real root α.



- **Expansions and factorisation –extensions**

Example

$$(2x-1)(x^2-x+3) = 2x^3-2x^2+6x$$
$$-x^2+x-3 \qquad \text{Expanding}$$
$$= \underline{2x^3-3x^2+7x-3}$$

Example $\quad x^3-9x = x(x^2-9) = \underline{x(x-3)(x+3)} \qquad$ Factorising

- **The remainder Theorem**

If the polynomial $p(x)$ be divided by $(ax+b)$ the remainder will be $p\left(-\frac{b}{a}\right)$

Example When $p(x)=2x^3-3x-5$ is divided by $2x+1$ the remainder is

$$p\left(-\tfrac{1}{2}\right) = -\tfrac{1}{4}+\tfrac{3}{2}-5 = \underline{\underline{-\tfrac{15}{4}}}$$

Example Find the remainder when $4x^3-3x^2+11x-2$ is divided by $x-1$.

$f(1)=4(1)^3-3(1)^2+11(1)-2=10 \qquad \therefore$ remainder is 10

- **The factor theorem** Following on from the last item

$p\left(-\frac{b}{a}\right)=0 \Rightarrow ax+b$ is a factor of $p(x)$

Example Show that $(x-2)$ is a factor of $6x^3-13x^2+x+2$ and hence solve the equation $6x^3-13x^2+x+2=0$

Let $p(x)=6x^3-13x^2+x+2$

$$p(2)=6(2)^3-13(2)^2+(2)+2=48-52+2+2=0$$

$\therefore$ $(x-2)$ is a factor of $p(x)$

$p(x)=(x-2)(6x^2-x-1)$ …by inspection

$$=(x-2)(3x+1)(2x-1)$$

$\therefore$ Solutions to the equation are $\underline{\underline{x=2,\ -\tfrac{1}{3},\ \tfrac{1}{2}}}$

- **Expansion of** $(a+b)^n$ where n is a ***positive*** integer.

This has the same coefficients as the expansion of $(1+x)^n$. Each term will have degree *n*, powers of *a* descending from a^n

Example $\quad (1+x)^8 = 1+8x+28x^2+\ldots\ldots.$

$$(a+b)^8 = a^8+8a^7b+28a^6b^2+\ldots\ldots.$$

The general term will be ${}^nC_r \times a^{n-r}b^r$

Example

Use the binomial theorem to expand $(3+2x)^4$, simplifying each term of your expansion.

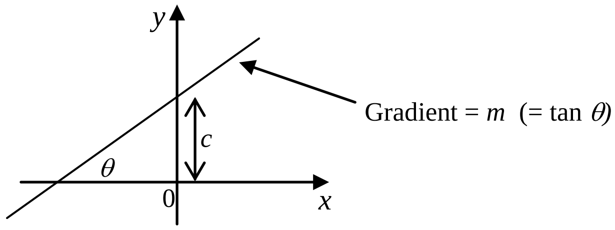
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\therefore\ (3+2x)^4 = (3)^4 + 4(3)^3(2x) + 6(3)^2(2x)^2 + 4(3)(2x)^3 + (2x)^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$

Geometry

- **<u>Gradient/ intercept form of a straight line</u>** Equation $y = mx + c$



- **<u>Distance between two points</u>**

Given A$(x_1,\ y_1)$ B$(x_2,\ y_2)$ then

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

- **<u>Gradient of a line through two points</u>**… $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ say

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- **<u>Equation of a line through (*x′*, *y′*) of gradient *m*</u>**

$$y - y' = m(x - x')$$

- **<u>Equation of a line through two points</u>**

Find the gradient using $m = \frac{y_2 - y_1}{x_2 - x_1}$ and use the formula as above.

- **<u>Parallel and perpendicular lines</u>**

Let two lines have gradients m_1 and m_2

Lines parallel $\Leftrightarrow m_1 = m_2$

Lines perpendicular $\Leftrightarrow m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

- **<u>Mid-point</u>** of line joining … $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ coordinates are

$$\left\{\tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(y_1 + y_2)\right\}$$

- **General form of a straight line**

  $ax+by+c=0.$ To find the gradient, rewrite in gradient/intercept form.

  Example Given points $A(-2, \quad 3)$ and $B(1, \quad -1)$ find

  (a) distance *AB*

  (b) the coordinates of the mid-point *M* of *AB*

  (c) the gradient of *AB*

  (d) the equation of the line through $C(5, \quad 2)$ parallel to *AB*

(a) $AB^2 = (1+2)^2 + (-1-3)^2 = 9+16 = 25$ $\therefore$ <u>*AB* = 5</u>

(b) $M\left(-\frac{1}{2}, \quad 1\right)$

(c) Gradient *AB* = $\dfrac{-1-3}{1+2} = -\dfrac{4}{3}$

(d) Point $(5, \quad 2)$ Gradient = $-\frac{4}{3}$

<u>Equation</u> $y-2 = -\frac{4}{3}(x-5)$

$3y-6 = -4x+20$

$3y+4x = 26$

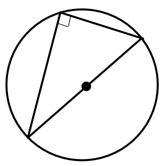
<u>Example</u> Find the gradient of the line $2x+3y=12$ and the equation of a perpendicular line through the point $(0, \quad -4)$

$2x+3y=12 \Rightarrow 3y=-2x+12 \quad \Rightarrow y=-\frac{2}{3}x+4$ $\therefore$ grad = $-\frac{2}{3}$

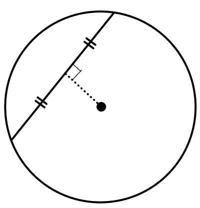
Gradient of perpendicular = $-\dfrac{1}{-\frac{2}{3}} = \frac{3}{2}$

<u>Equation</u> $y=\frac{3}{2}x-4$ $(y=mx+c)$

- **The Circle**



Angles in semicircle is 90°



Perpendicular to a chord from centre of circle bisects the chord.

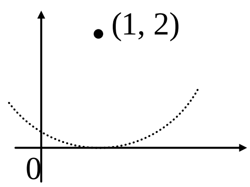
- **Centre, radius form of equation**

$$(x-a)^2+(y-b)^2=r^2$$

Centre (*a, b*) radius = *r*

Example Centre (2, -1) radius 3 equation $(x-2)^2+(y+1)^2=9$

Example Centre (1, 2) touching 0x Equation $(x-1)^2+(y-2)^2=4$



- **General form of equation**

$(x-a)^2+(y-b)^2=r^2$ Circle centre (a,b) with radius r

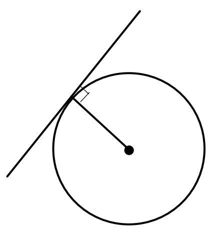
To find centre and radius, use the method of CTS to change into centre/radius form.

Example $x^2+y^2-2x+3y-3=0$

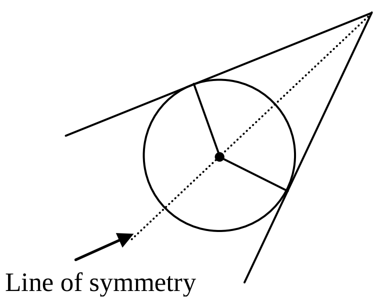
$$x^2+y^2-2x+3y-3=0$$
$$(x^2-2x)+(y^2+3y)=3$$
$$(x^2-2x+1)+\left(y^2+3y+\left(\tfrac{3}{2}\right)^2\right)=3+1+\left(\tfrac{3}{2}\right)^2$$
$$(x-1)^2+\left(y+\tfrac{3}{2}\right)^2=\tfrac{25}{4}$$

$\therefore$ Centre $\left(1,-\tfrac{3}{2}\right)$ radius = $\tfrac{5}{2}$

- **Tangents**



Angle between tangent and radius drawn to point of contact is 90°

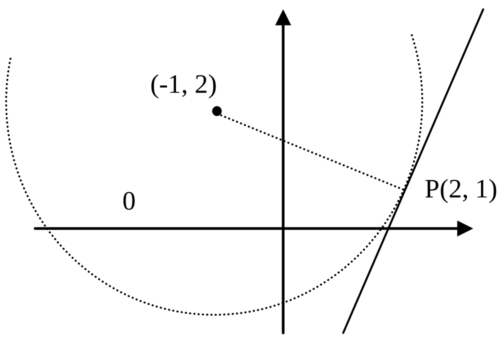


Tangents drawn from extended point

<u>Example</u> Find the equation of the tangent to the circle $x^2 + y^2 + 2x - 4y - 5 = 0$ at the point P(2, 1)

$$x^2 + y^2 + 2x - 4y - 5 = 0$$
$$(x^2 + 2x) + (y^2 - 4y) = 5$$
$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 5 + 1 + 4$$
$$(x+1)^2 + (y-2)^2 = 10$$

$\therefore$ Centre at $(-1,\ 2)$, radius $\sqrt{10}$



Gradient CP = $\dfrac{2-1}{-1-2} = -\dfrac{1}{3}$

$\therefore$ gradient of tangent at P = 3

<u>Equation</u>

$$y - 1 = 3(x - 2)$$
$$\underline{\underline{y = 3x - 5}}$$

Calculus

- **<u>Differentiation by rule</u>**

y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
ax^n	anx^{n-1}
ax	a
a	0
$u + v - w$	$\dfrac{du}{dx} + \dfrac{dv}{dx} - \dfrac{dw}{dx}$

<u>Examples</u>

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \tfrac{1}{2}x^{-\frac{1}{2}} = \underline{\frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left(\frac{4}{x}\right) = \frac{d}{dx}\left(4x^{-1}\right) = -4x^{-2} = \underline{-\frac{4}{x^2}}$$

$$\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{d}{dx}\left(\tfrac{1}{2}x\right) = \underline{\frac{1}{2}}$$

$$\frac{d}{dx}(10) = 0$$

$$\frac{d}{dx}\left(3x^2 - x - 5\right) = \underline{6x - 1}$$

- **<u>Vocabulary and more notation</u>**

$\dfrac{dy}{dx}$ is the derivative of *y* (with respect to *x*)

$\frac{dy}{dx}$ is the differential coefficient of *y* (with respect to *x*).

<u>Example</u> $y = x^3 - 4x^2 + 3x - 1$

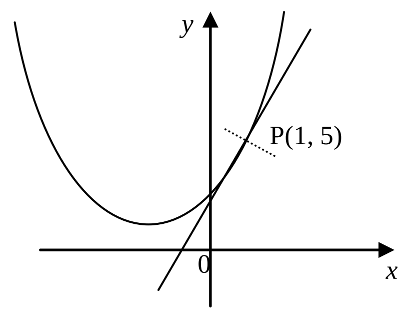
$$\frac{dy}{dx} = 3x^2 - 8x + 3$$

<u>Example</u> $f(x) = \frac{x^2 - 2}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}} = x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$

$$\therefore \quad f'(x) = \tfrac{3}{2}x^{\frac{1}{2}} - \left(-\tfrac{1}{2}\right)2x^{-\frac{3}{2}} = \tfrac{3}{2}x^{\frac{1}{2}} + \frac{1}{x^{\frac{3}{2}}} = \underline{\frac{3}{2}\sqrt{x} + \frac{1}{x\sqrt{x}}}$$

- **<u>The gradient of a curve</u>** at any point is given by the value of $\frac{dy}{dx}$ at that point.

<u>Example</u> Find the gradient at the point P(1, 5) on the graph of $y = x^2 + 2x + 2$. Hence find the equation of the tangent at P.



$y = x^2 + 2x + 2$

$\frac{dy}{dx} = 2x + 2$

$\therefore$ At P(1, 5) gradient = 4

<u>Tangent at P</u>

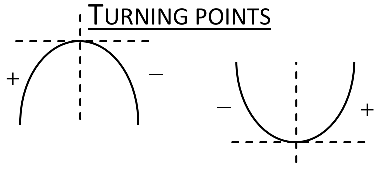
$y - 5 = 4(x - 1)$

$\therefore \underline{y = 4x + 1}$

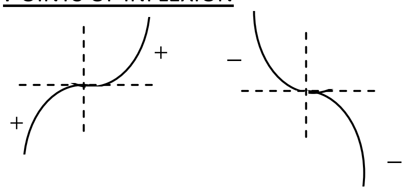
- **<u>Stationary points</u>** on the graph of a function are points where the gradient is zero.

<u>STATIONARY POINTS</u>

<u>TURNING POINTS</u>



<u>POINTS OF INFLEXION</u>



MAXIMUM POINT MINIMUM POINT TANGENT PASSING THROUGH THE CURVE

- **<u>To obtain coordinates of a SP</u>**. on the graph of $y = f(x)$

  (i) Put $f'(x) = 0$ and solve for *x*.

  (ii) If $x = a$ is a solution of (i) the SP will be $\{a, \quad f(a)\}$.

(iii) If $f''(a) > 0$ there will be a minimum point at $x = a$

If $f''(a) < 0$ there will be a maximum point at $x = a$

If $f''(a) = 0$ there could be max or min or inflexion so the second derivative rule fails. Investigate the gradient to the immediate left and right of the stationary point. (see the + and - signs on the diagrams in the previous section).

<u>Example</u> Find the stationary points on the graphs of

(i) $y = x^2 + 2x + 2$

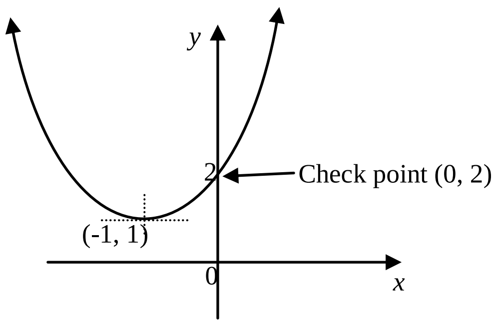
(ii) $y = x^3 - 3x + 2$

and sketch the graphs.

(i) Here we have a quadratic function, which will have a true max or min.

$y = x^2 + 2x + 2$

$\dfrac{dy}{dx} = 2x + 2$



$\therefore$ SP at $2x + 2 = 0$

i.e. at $x = -1$

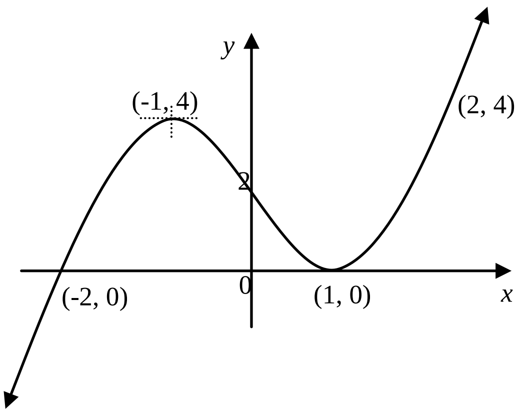
i.e. at $\underline{(-1, \quad 1)}$

$\dfrac{d^2y}{dx^2} = 2 > 0$

$\therefore$ SP is a minimum.

(ii) $y = x^3 - 3x + 2$

$\dfrac{dy}{dx} = 3x^2 - 3$



For SP $3x^2 - 3x = 0$

$\therefore \quad x^2 = 1$

$x = \pm 1$

$\therefore$ SPs at <u>(1, 0) (-1, 4)</u>

$\dfrac{d^2y}{dx^2} = 6x$

At (1, 0) $\dfrac{d^2y}{dx^2} = 6 > 0 \quad \therefore$ Min

At (-1, 4) $\frac{d^2y}{dx^2} = -6 < 0 \quad \therefore$ Max

Check points (0, 2) (2, 4) (-2, 0)

Note that the turning points are **<u>Local Max</u>** and **<u>Local Min</u>**

$$\frac{d^2y}{dx^2} = 6x$$

At (1, 0) $\frac{d^2y}{dx^2} = 6 > 0 \quad \therefore$ Min

At (-1, 4) $\frac{d^2y}{dx^2} = -6 < 0 \quad \therefore$ Max

Check points (0, 2) (2, 4) (-2, 0)

Note that the turning points are **<u>Local Max</u>** and **<u>Local Min</u>**

Integration

<u>Indefinite integrals</u>

Indefinite integration is the reverse of differentiation. Every indefinite integral must have an arbitrary constant

y	$\int y\,dx$	
x^n	$\frac{x^{n+1}}{n+1} + c \quad n \neq -1$	
ax^n	$\frac{ax^{n+1}}{n+1} + c$	Special cases worth remembering
x	$\frac{x^2}{2} + c$	
a	$ax + c$	
$u + v - w$	$\int u\,dx + \int v\,dx - \int w\,dx$	

$\int f(x)dx$ reads "the (indefinite) integral of $f(x)$ with respect to *x*"

$f(x)$ is called the integrand. dx is the differential of the integration and **<u>must never be omitted</u>**.

Example Find (i) $\int (3x+1)^2\,dx$ (ii) $\int \frac{x-1}{\sqrt{x}}\,dx$ (iii) $\int \frac{dx}{x^2}$

(i) $$\int (3x+1)^2\,dx = \int (9x^2 + 6x + 1)dx = \frac{9x^3}{3} + \frac{6x^2}{2} + x + c$$
$$= \underline{\underline{3x^3 + 3x^2 + x + c}}$$

(ii) $$\int \frac{x-1}{\sqrt{x}}\,dx = \int\left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \underline{\underline{\tfrac{2}{3}x\sqrt{x} - 2\sqrt{x} + c}}$$

(iii) Not a misprint! $$\int \frac{dx}{x^2} = \int \frac{1}{x^2}.dx = \int x^{-2}dx = \frac{x^{-1}}{-1} + c$$

$$= \underline{\underline{-\frac{1}{x} + c}}$$

- **Definite integrals**

If $I = \int f(x)dx = F(x) + c$, then the definite integral $\int_a^b f(x)dx$ is the difference in the value of *I* when $x = b$ and $x = a$.

i.e. $\int_a^b f(x)dx = F(b) - F(a)$ no constant!

The limits of the definite integral are *a* (lower limit) and *b* (upper limit).

Note the use of square brackets. $[F(x)]_a^b = F(b) - F(a)$

Example Evaluate $\int_1^3 \left(x + \frac{1}{x^3}\right)dx$

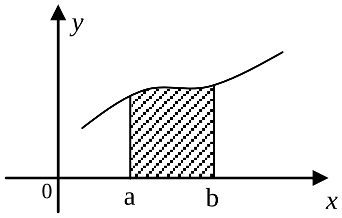
$$\int_1^3 \left(x + \frac{1}{x^3}\right)dx = \int_1^3 \left(x + x^{-3}\right)dx = \left[\frac{x^2}{2} + \frac{x^{-2}}{(-2)}\right]_1^3$$

$$= \left[\frac{x^2}{2} - \frac{1}{2x^2}\right]_1^3 = \left(\frac{9}{2} - \frac{1}{18}\right) - \left(\frac{1}{2} - \frac{1}{2}\right) = \underline{\underline{\frac{40}{9}}}$$

- **Area on a graph as a definite integral**

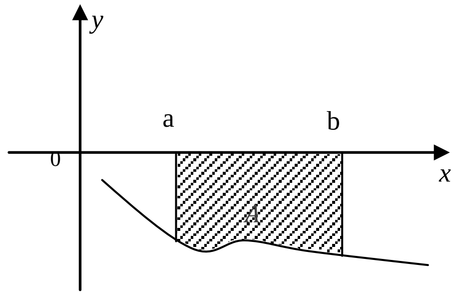
(i)

$$A = \int_a^b y\,dx$$



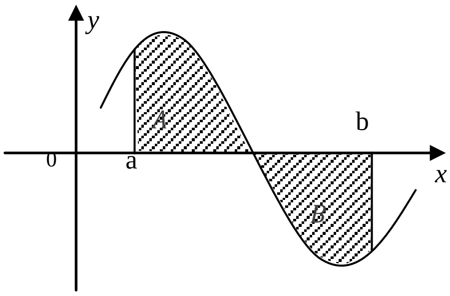
$$\int_a^b y\,dx = -A$$

i.e. the value of the definite integral will be negative if y is negative for $a \le x \le b$



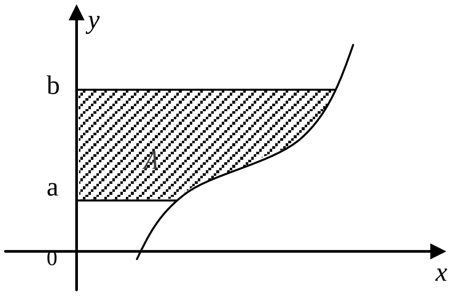
(iii)

$$\int_a^b y\,dx = A - B$$



(iv)

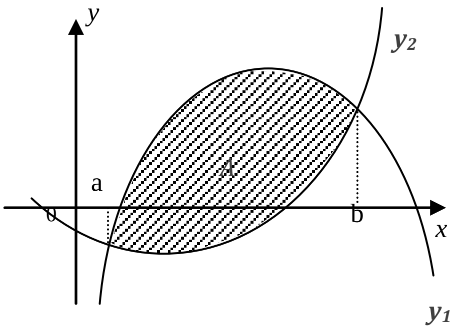
$$\int_a^b x\,dy = A$$



(v)

$$\int_a^b (y_1 - y_2)\,dx = A$$

NB $y_1 > y_2$ for $a < x < b$

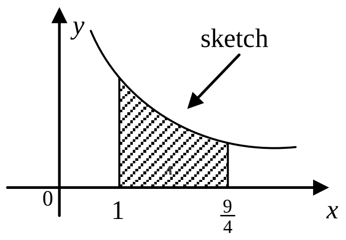


NB (i) most certainly will be tested, and (iv) could be.

(ii) and (iii) most unlikely to be included.

(iv) most likely area between a line and a curve.

<u>Example</u> Find the area enclosed between the graph of $y = 1 + \dfrac{1}{\sqrt{x}}$ the *x*-axis and the ordinates at *x* = 1 and $x = \frac{9}{4}$



$$A=\int_1^{\frac{9}{4}}\left(1+\frac{1}{\sqrt{x}}\right)dx=\int_1^{\frac{9}{4}}\left(1+x^{-\frac{1}{2}}\right)dx$$

$$=\left[x+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^{\frac{9}{4}}=\left[x+2\sqrt{x}\right]_1^{\frac{9}{4}}$$

$$=\left(\frac{9}{4}+\cancel{2}\times\frac{3}{\cancel{2}}\right)-(1+2)=\underline{\underline{\frac{9}{4}}}$$

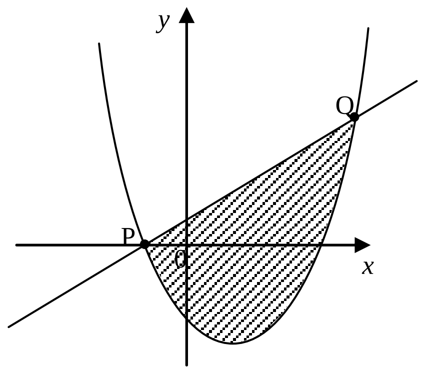
<u>Example</u> The diagram shows the sketch of graph of $y=x^2-2x-3$ and $y=x+1$. Find the x coordinates of the points of intersection P and Q of the graphs.
Calculate the shaded area.

For P, Q

$$\left.\begin{array}{r} y=x^2-2x-3 \\ y=x+1 \end{array}\right\} x^2-2x-3=x+1$$

$$\therefore \quad \begin{array}{l} x^2-3x-4=0 \\ (x+1)(x-4)=0 \end{array}$$

$$\therefore \quad \underline{\underline{x=-1,\ \ 4}}$$



Shaded area = $\int (y_1-y_2)dx$

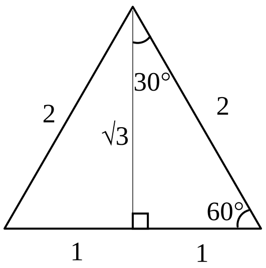
$$=\int_{-1}^{4}\{(x+1)-(x^2-2x-3)\}dx=\int_{-1}^{4}(3x+4-x^2)dx$$

$$=\left[\frac{3x^2}{2}+4x-\frac{x^3}{3}\right]_{-1}^{4}=\left(24+16-\frac{64}{3}\right)-\left(\frac{3}{2}-4+\frac{1}{3}\right)$$

$$=\underline{\underline{21\tfrac{1}{6}}}$$

Trigonometry

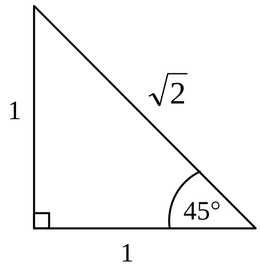
- **Trig ratios for 30°, 60°, 45°**



$\sin 30 = \cos 60 = \frac{1}{2}$

$\sin 60 = \cos 30 = \frac{\sqrt{3}}{2}$

$\tan 60 = \sqrt{3} \quad \tan 30 = \frac{1}{\sqrt{3}}$



$\sin 45 = \cos 45 = \frac{1}{\sqrt{2}}$

$\tan 45 = 1$

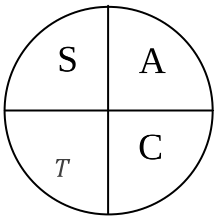
- **Trig ratios for all angles** NB the CAST DIAGRAM

For the sign of a trig ratio

All positive in first quadrant

Sine (only) in second quadrant

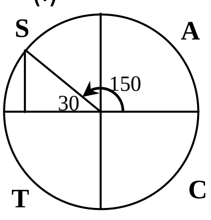
Etc…



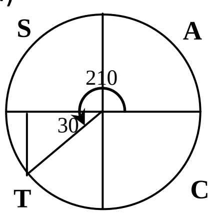
Example Without using a calculator find

(i) $\cos 150^\circ$ (ii) $\tan 210^\circ$ (iii) $\sin(-240^\circ)$

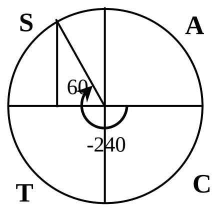
(i)



(ii)



(iii)



$\cos 150^\circ = -\cos 30$

$= -\frac{\sqrt{3}}{2}$

$\tan 210^\circ = \tan 30$

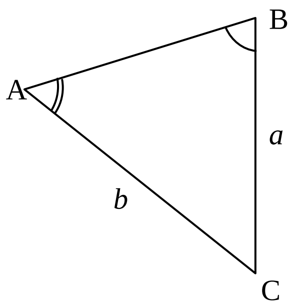
$= \frac{1}{\sqrt{3}}$

$\sin(-240^\circ) = \sin 60$

$= \frac{\sqrt{3}}{2}$

- **Trig of Scalene triangles**

<u>Sine rule</u>

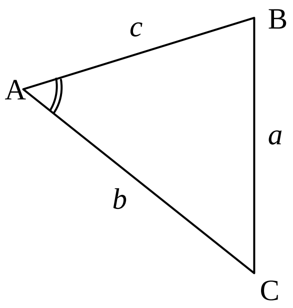


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \left(\frac{c}{\sin C}\right)$$

Given AAS use it to find a second side

Given SSA use it to find a second angle (but take care to choose the angle size appropriately –it could be acute or obtuse).

<u>Cosine rule</u>



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

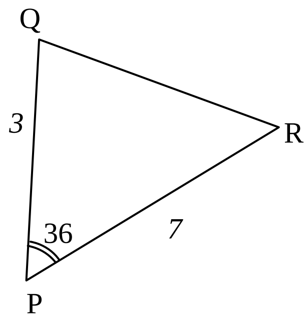
Both formulae with two more sets.

Given SAS use it to find the third side

Given SSS use it to find an angle (no possible ambiguity here).

<u>Example</u> Triangle *PQR* has *PR* = 3cm, *QR* = 7cm and $Q\hat{P}R = 36^\circ$

Find (i) QR using the cosine rule and then (ii) $P\hat{Q}R$ using the sine rule.



(i) $QR^2 = 9 + 49 - 42\cos 36 = 24.021...$

$QR = 4.901.. \approx \underline{\underline{4.90}}$

(ii) $\dfrac{7}{\sin PQR} = \dfrac{4.901..}{\sin 36}$

$\sin PQR = \dfrac{7\sin 36}{4.901..} = 0.8394...$

$\therefore\ PQR = 57.086..$ or $PQR = 122.914..$

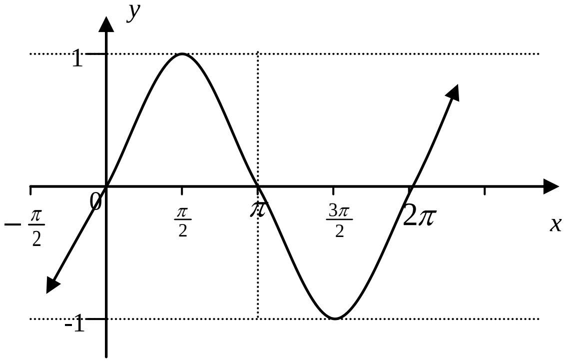
It can't be 57.08.. since R would be 86.92.. and would be the largest angle in the triangle, but R faces the smallest side so is the smallest angle. Hence $\underline{PQR = 122.91}$

- **Area "$\frac{1}{2}ab\sin C$" rule** given SAS

Area of triangle = $\frac{1}{2}ab\sin C$

- **<u>Graphs of trig functions</u> (all periodic)**

1. Graph of $y = \sin x$

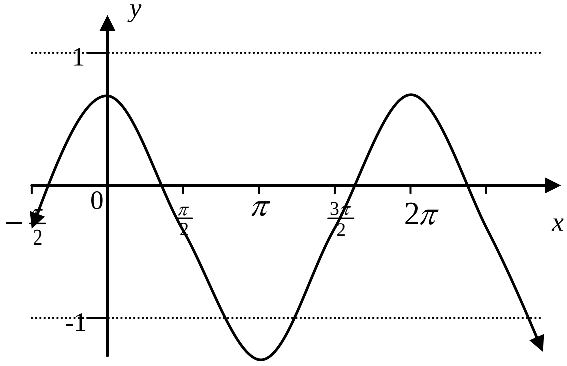


Period 2π

$\sin(2\pi + x) = \sin x$

$|\sin x| \leq 1$

2. Graph of $y = \cos x$

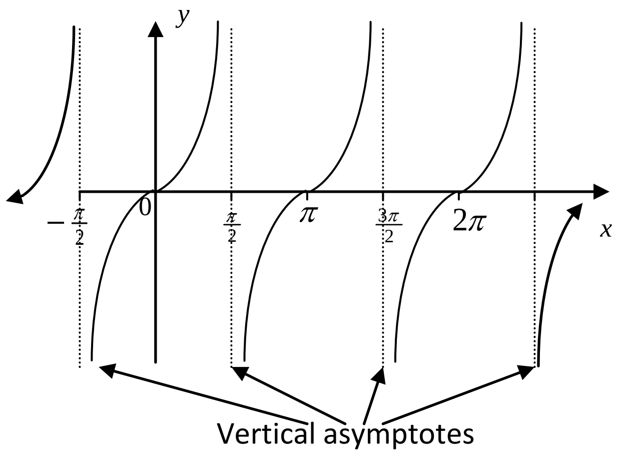


Period 2π

$\cos(2\pi + x) = \cos x$

$|\cos x| \leq 1$

3. Graph of $y = \tan x$



Period π

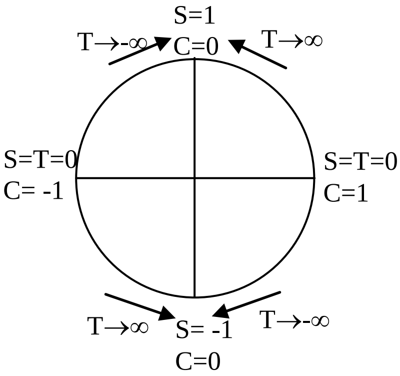
$\tan(\pi + x) = \tan x$

Vertical asymptotes at

$x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$, etc

Boundary values of trig ratios

Verify these from graphs



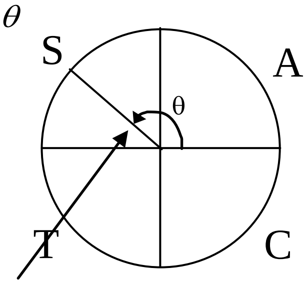
Two important trig identities

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \qquad\qquad \sin^2\theta + \cos^2\theta = 1$$

<u>Example</u> Given θ is obtuse and $\sin\theta = \frac{8}{17}$ find the values of $\cos\theta$ and $\tan\theta$.

$\sin^2\theta + \cos^2\theta = 1$

$$\therefore\ \cos^2\theta = 1 - \sin^2\theta$$
$$= 1 - \tfrac{64}{289}$$
$$= \tfrac{225}{289}$$
$$\therefore\ \cos\theta = -\tfrac{15}{17}$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \therefore\ \tan\theta = \frac{\frac{8}{17}}{-\frac{15}{17}} = -\tfrac{8}{15}$$

<u>NB</u> Learn how to rearrange the identities

$$\sin\theta = \cos\theta\tan\theta \qquad \cos\theta = \frac{\sin\theta}{\tan\theta}$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad \sin^2\theta = 1 - \cos^2\theta$$

Trig equations

<u>Trig equations</u> Remember that from your calculator $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ give the ***principal value*** (p.v.)

<u>Example</u> Solve the equations

(i) $\tan\theta = -1.5$ for $0^\circ < \theta < 360^\circ$

(ii) $\sin 2\theta = 0.5$ for $-180^\circ < \theta < 180^\circ$

(iii) $2\cos^2\theta = 1 - \sin\theta$ for $0^\circ < \theta < 360^\circ$

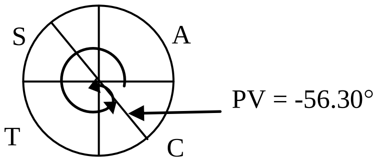
(iv) $2\sin^2\theta = \sin\theta\cos\theta$ for $0^\circ < \theta < 360^\circ$

(v) $\sin(\theta - 80) = \dfrac{\sqrt{3}}{2}$ for $-180^\circ < \theta < 180^\circ$

(i)

$\tan\theta = -1.5$

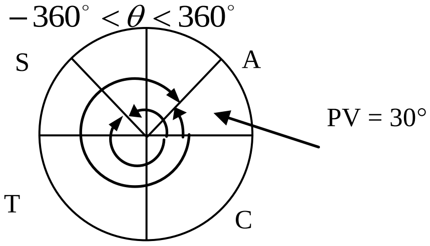
$\theta \approx 124^\circ,\ \ 304^\circ$



(ii) $\sin 2\theta = 0.5$ .....first solve for 2θ for $-360^\circ < \theta < 360^\circ$

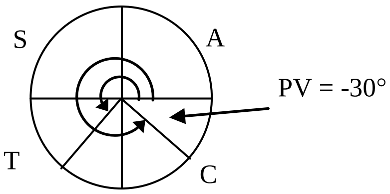
$2\theta = 30,\ 150;\ -210,\ -330$

$\therefore \theta = 15^\circ,\ 75^\circ;\ -105^\circ,\ -165^\circ$



(iii) (In this example, use $\cos^2\theta = 1 - \sin^2\theta$)

$$2\cos^2\theta = 1 - \sin\theta$$
$$2\left(1 - \sin^2\theta\right) = 1 - \sin\theta$$
$$2 - 2\sin^2\theta = 1 - \sin\theta$$
$$2\sin^2\theta - \sin\theta - 1 = 0$$
$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$



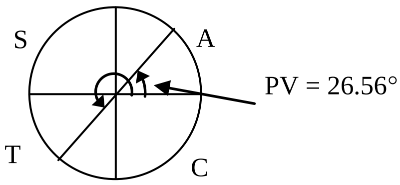
$\therefore$ $\sin\theta = 1$, $\theta = 90^\circ$ or $\sin\theta = -\frac{1}{2}$, $\theta = 210^\circ,\ \ 330^\circ$

$\therefore\ \ \theta = 90^\circ,\ \ 210^\circ,\ \ 330^\circ$

(iv) Don't cancel out $\sin\theta$. Bring to LHS and factorise

$$2\sin^2\theta = \sin\theta\cos\theta$$
$$2\sin^2\theta - \sin\theta\cos\theta = 0$$
$$\sin\theta\left(2\sin\theta - \cos\theta\right) = 0$$

$\therefore\ \ \sin\theta = 0$ or $2\sin\theta = \cos\theta$



$\theta = 0^\circ,\ \ 180^\circ$

$$\frac{\sin\theta}{\cos\theta} = \frac{1}{2}$$

$$\tan\theta = \frac{1}{2}$$

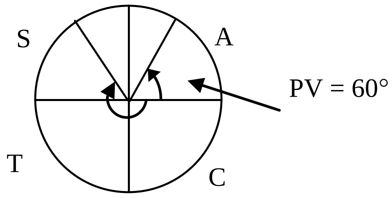
$$\theta \approx 27^\circ,\ \ 207^\circ$$

$\therefore\ \ \theta = 0^\circ,\ \ 180^\circ,\ \ 27^\circ,\ \ 207^\circ$

(v) $\sin(\theta - 80) = \frac{\sqrt{3}}{2}$ solve first for $-260^\circ < \theta < 100^\circ$

$(\theta - 80) = 60^\circ, \ -240^\circ$

$\therefore \underline{\theta = 140^\circ, \ -160^\circ}$



In the next example, angles are in radians. The radian sign ^{c} is sometimes omitted, but is implied when the interval contains π.

Example Solve the following equations

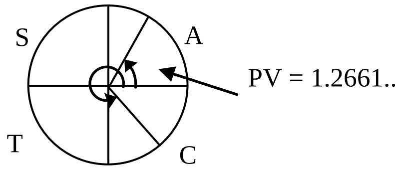
(i) $\cos x = 0.3$ for $0 < x < 2\pi$, answers correct to 2d.p.

(ii) $\tan\frac{x}{2} = \sqrt{3}$ for $-2\pi < x < 2\pi$, answers in exact form

(i) $\cos x = 0.3$ ……put calculator into RAD mode.

$x = 1.266..., \ 2\pi - 1.266...$

$\underline{x \approx 1.27, \ 5.02}$



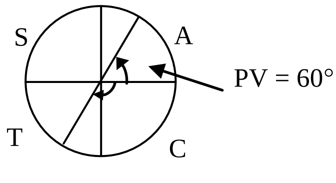
(ii) In exact terms means in terms of π. The implication is that the angles will be exact form in degrees. So, work in degrees first and then convert to radians.

$\tan\frac{x}{2} = \sqrt{3}$ … solve first for $-\pi < x < \pi$

$\frac{x}{2} = 60^\circ, -120^\circ$

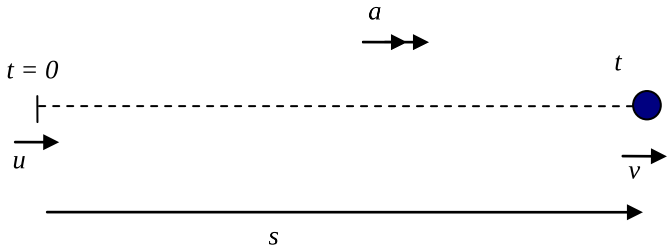
$\therefore x = 120^\circ, -240^\circ$

$\therefore \underline{x = \frac{2\pi}{3}, \ -\frac{4\pi}{3}}$



Mechanics

1. <u>Rectilinear motion with constant acceleration</u>



Remember that *s* is a displacement, is directed, and should be shown with <u>**one**</u> arrow head.

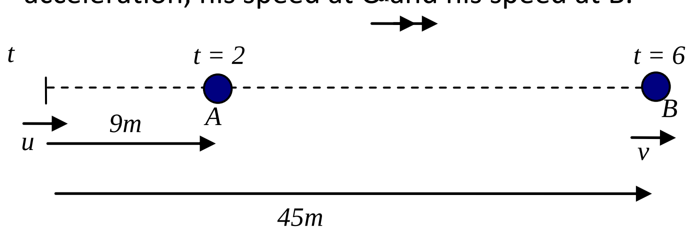
$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{1}{2}(u + v)t$$

<u>Example</u> A cyclist moves along a straight line passing through points O, A and B with constant acceleration. 2 seconds after passing O he is at A where OA = 9m and after a further 4 seconds he is at B where AB = 36m. Find his constant acceleration, his speed at O and his speed at B.



$\underline{s = ut + \frac{1}{2}at^2}$

O – A $\quad 9 = 2u + 2a$

O – B $\quad 45 = 6u + 18a \qquad \Rightarrow \; 18 = 2a$

$$\therefore \; \underline{\underline{a = 1.5m/s^2}} \quad , \quad \underline{\underline{u = 3m/s}}$$

$\underline{v = u + at}$

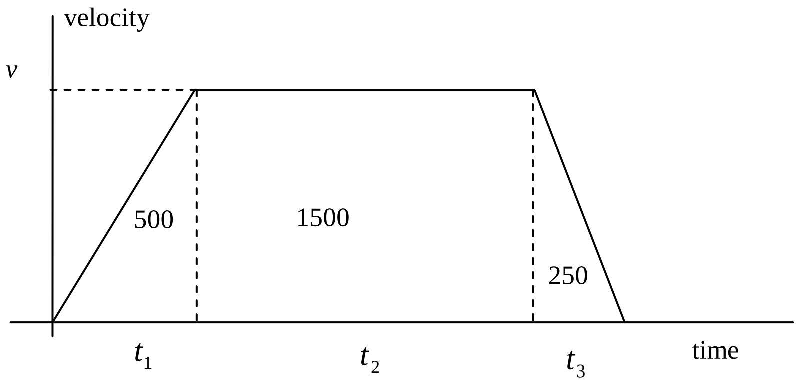
O – B $\quad v = 3 + 1.5 \times 6 = \underline{\underline{12m/s}}$

<u>Properties of the velocity/time graph</u>

Gradient = acceleration

Area under graph = distance travelled

Example A train starting from rest is uniformly accelerated during the first ½ km of its run, maintains its acquired speed for the next 1½ km, and is then brought to rest with uniform retardation in the last ¼ km. The time for the whole journey is 5 minutes. Find the acceleration in the first part of the run and the retardation in the final stage.



Area $\frac{1}{2}t_1v = 500$ $\therefore$ $t_1v = 1000$

$t_2v = 1500$

$\frac{1}{2}t_3v = 250$ $\therefore$ $t_3v = 500$

$t_1v + t_2v + t_3v = 3000$

$\therefore$ $v(t_1 + t_2 + t_3) = 3000$

$v \times 300 = 3000$

$v \times 300 = 3000$

$v = 10m/s$

$\therefore$ $t_1 = 100$ and $t_3 = 50$

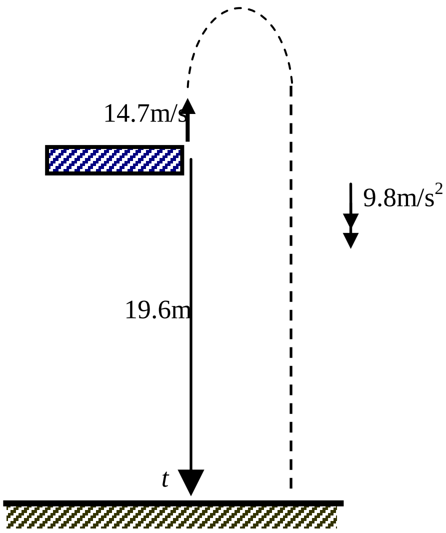
Gradient 1st stage acceleration = $\frac{10}{100} = \frac{1}{10}m/s^2$

3rd stage deceleration = $\frac{10}{50} = \frac{1}{5}m/s^2$

2. Vertical motion under gravity

The assumption here will be that the motion is unrestricted, so that the constant acceleration formulae can be used with *g* usually given as having magnitude 9.8 m/s^2

Example A ball is thrown vertically upwards with speed 14.7 m/s from a platform 19.6m above level ground. Find the time for the ball to reach the ground (g = 9.8 m/s^2) State any assumption that you make.



Notice here how the displacement is shown from the platform downwards and NOT from ground upwards. This is very important.

$s = ut + \frac{1}{2}at^2$ $\qquad\qquad 19.6 = -14.7t + \frac{1}{2} \times 9.8t^2$

$\therefore \ t^2 - 3t - 4 = 0$

$\therefore \ (t+1)(t-3) = 0$

$\therefore \ t = 3s$

Assume no air resistance during motion.

Probability Distribution –Binomial

The binomial random variable X is where the probability of success remains constant from trial to trial –say ‘p’. Therefore the probability of failure is ‘q’ where $q = 1 - p$. When the trial is repeated for n trials then a probability can be found using the following formula.

$$P(X = r) = {}^nC_r \times p^r \times q^{n-r} \quad \text{where } r = 1,2,3,4, \ldots . n$$

Example

When a drawing pin is thrown onto a table, the probability that it will fall ‘point upwards’ is 0.2. Ten drawing pins in a packet are thrown onto a table, work out the probability that

(i) All land point down

(ii) Exactly three land point up

(iii) Less than two land point up.

$p = 0.2, q = 0.8$

(i) P(all land point down) = $0.80^{10} = 0.107$

(ii) P(exactly three land point up) = $P(X = 3) = {}^{10}C_3 \times 0.2^3 \times 0.8^7$

$$= 120 \times 0.2^3 \times 0.8^7 = 0.201$$

(iii) P(less than two land point up) = $P(X < 2) = P(X = 0) + P(X = 1)$

$$= 0.107 + 10 \times 0.2 \times 0.8^9$$
$$= 0.375$$