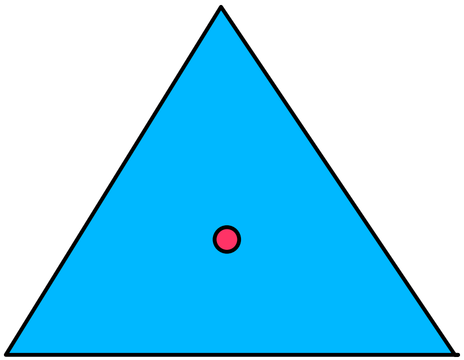
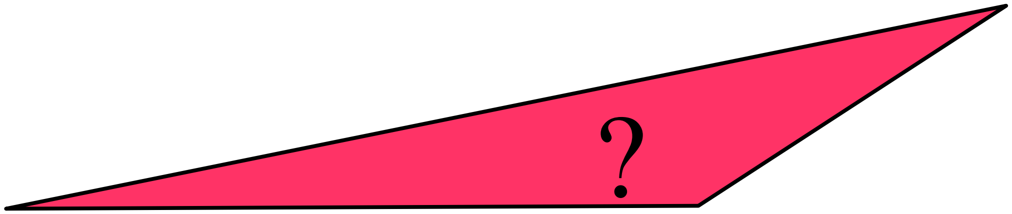
Advanced Euclidean Geometry

What is the center of a triangle?

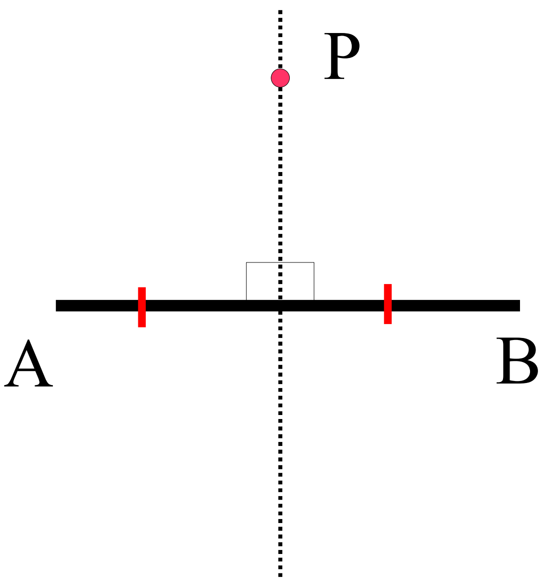


But what if the triangle is not equilateral?

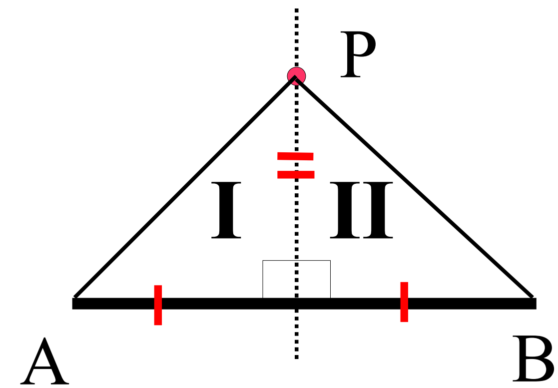


Circumcenter

Equally far from the vertices?

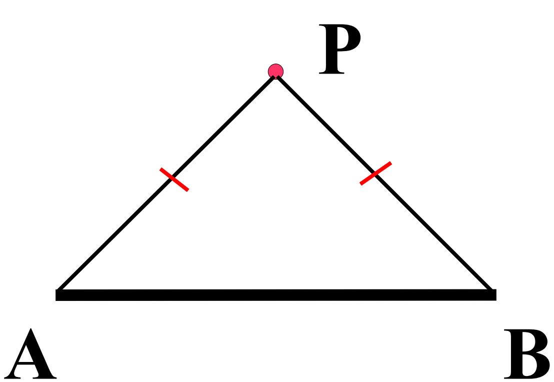


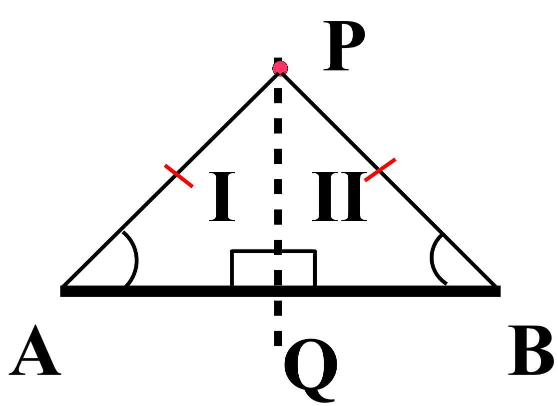
Points are on the perpendicular bisector of a line segment iff they are equally far from the endpoints.



Δ I ≅ Δ II (SAS)

PA = PB



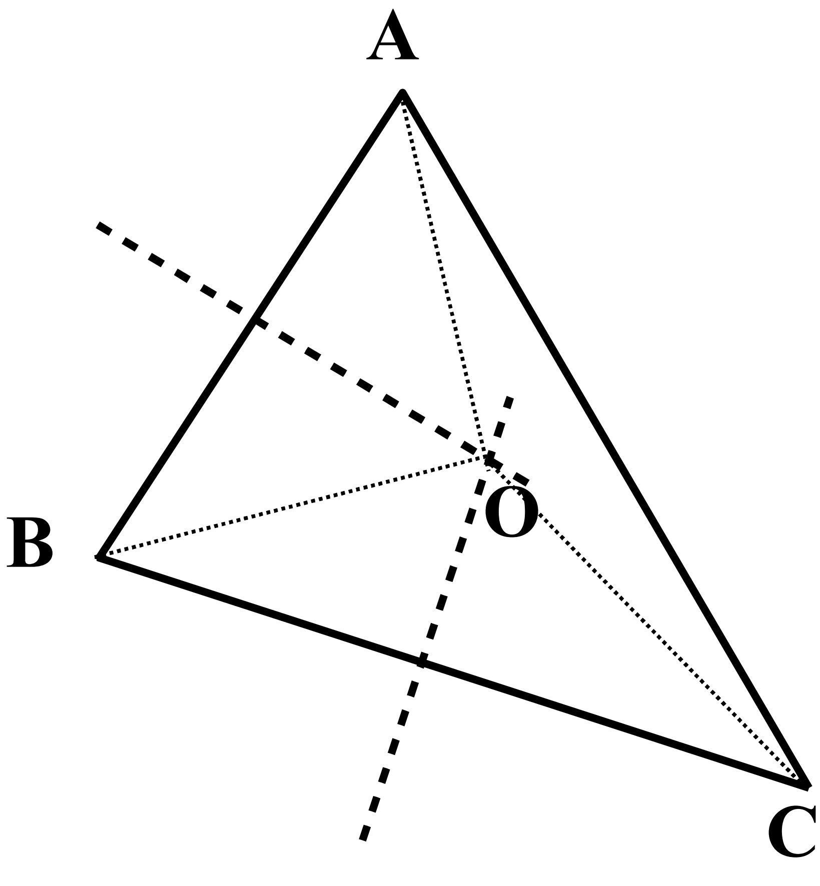


Δ I ≅ Δ II (Hyp-Leg)

AQ = QB

Circumcenter

Thm 4.1 : The perpendicular bisectors of the sides of a triangle are concurrent at a point called the ***circumcenter*** (**O**).



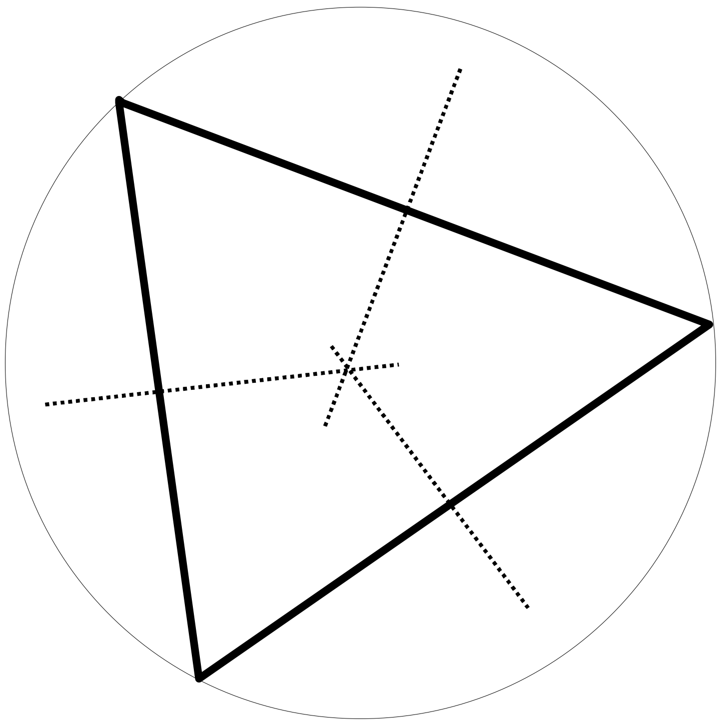
Draw two perpendicular bisectors of the sides. Label the point where they meet O (why must they meet?)

Now, OA = OB, and OB = OC (why?) so OA = OC and O is on the perpendicular bisector of side AC. ■

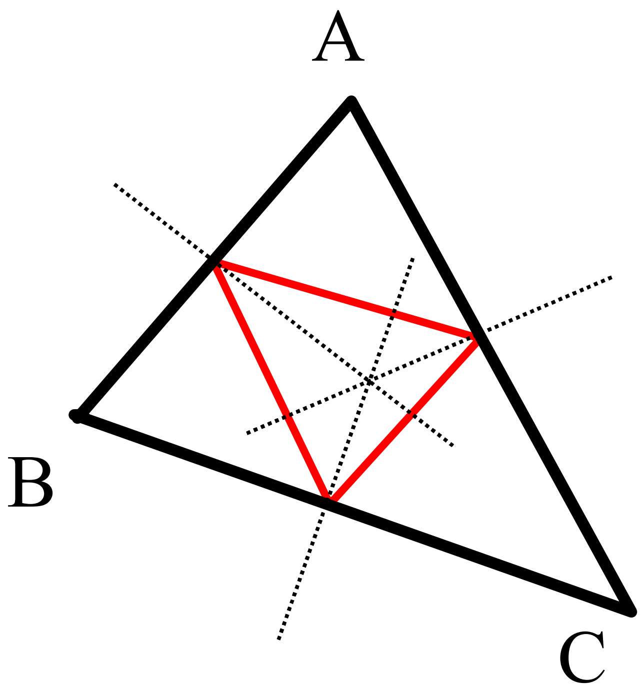
The circle with center O, radius OA passes through all the vertices and is called the *circumscribed circle* of the triangle.

Circumcenter (O)

Examples:



Orthocenter



The triangle formed by joining the midpoints of the sides of ΔABC is called the ***medial triangle*** of ΔABC.

The sides of the medial triangle are parallel to the original sides of the triangle.

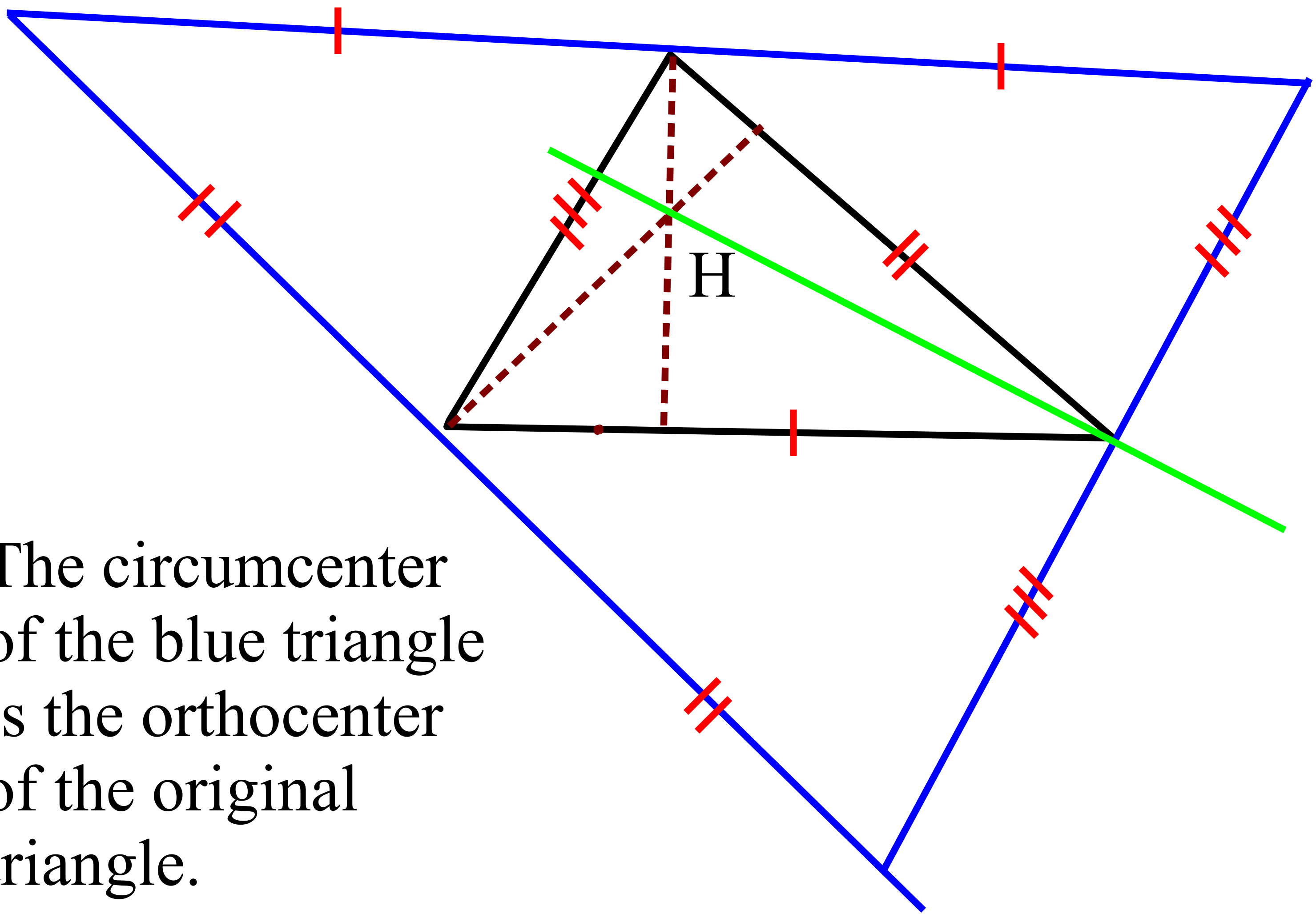
A line drawn from a vertex to the opposite side of a triangle and perpendicular to it is an ***altitude***.

Note that in the *medial triangle* the perp. bisectors are altitudes.

Thm 4.2: The altitudes of a triangle are concurrent at a point called the ***orthocenter*** (**H**).

Orthocenter (H)

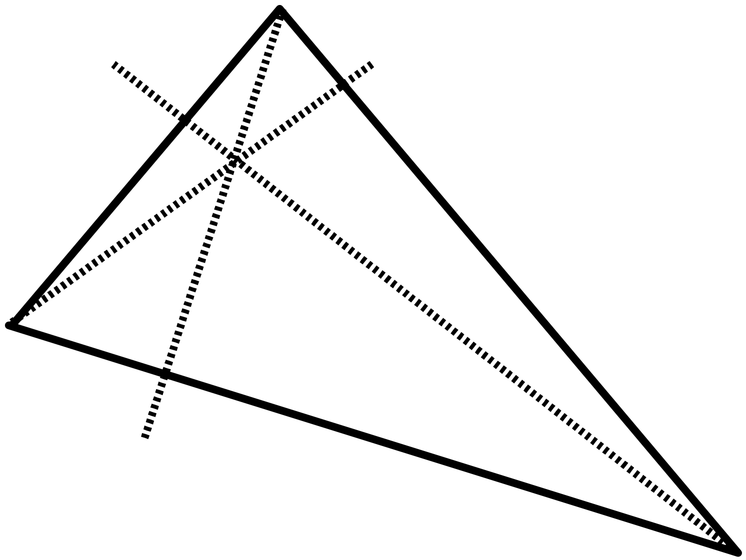
Thm 4.2: The altitudes of a triangle are concurrent at a point called the ***orthocenter*** (**H**).



The circumcenter of the blue triangle is the orthocenter of the original triangle.

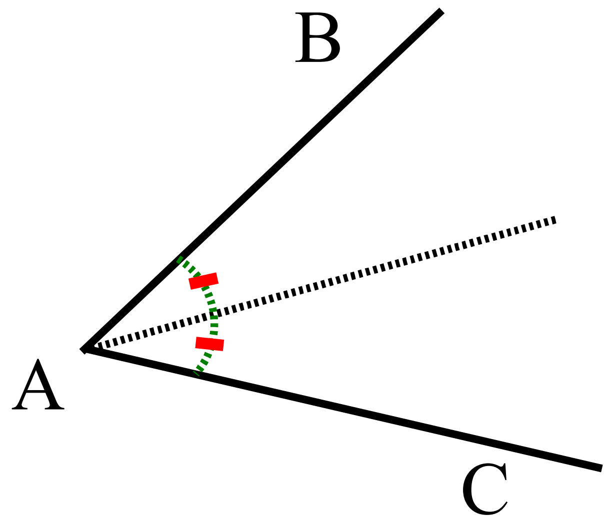
Orthocenter (H)

Examples:

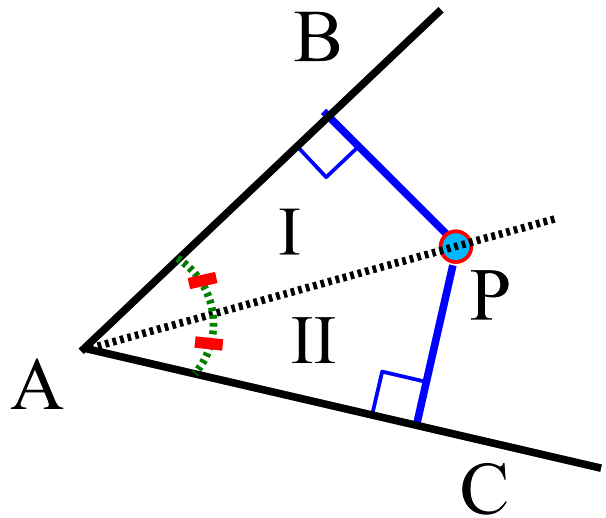


Incenter

Equally far from the sides?

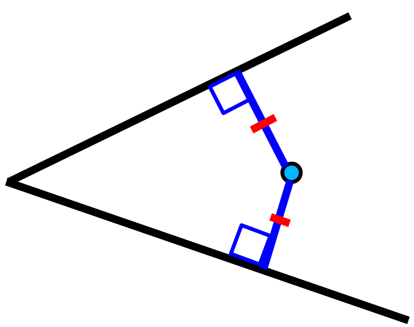


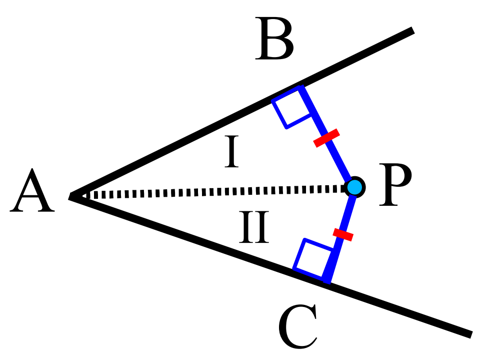
Points which are equally far from the sides of an angle are on the angle bisector.



$\Delta I \cong \Delta II$ (AAS)

PB = PC



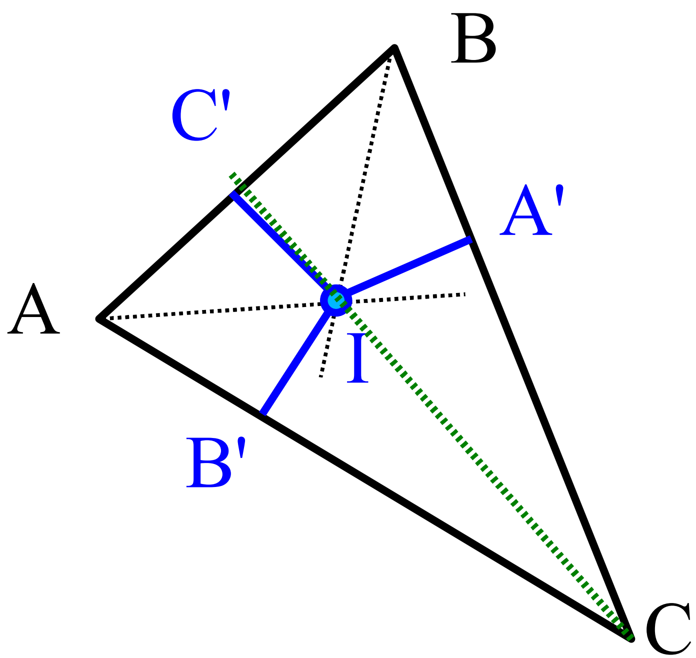


$\Delta I \cong \Delta II$ (leg-hypotenuse)

$\angle BAP \cong \angle CAP$

Incenter (I)

Thm 4.3 : The internal bisectors of the angles of a triangle meet at a point called the ***incenter*** (**I**).



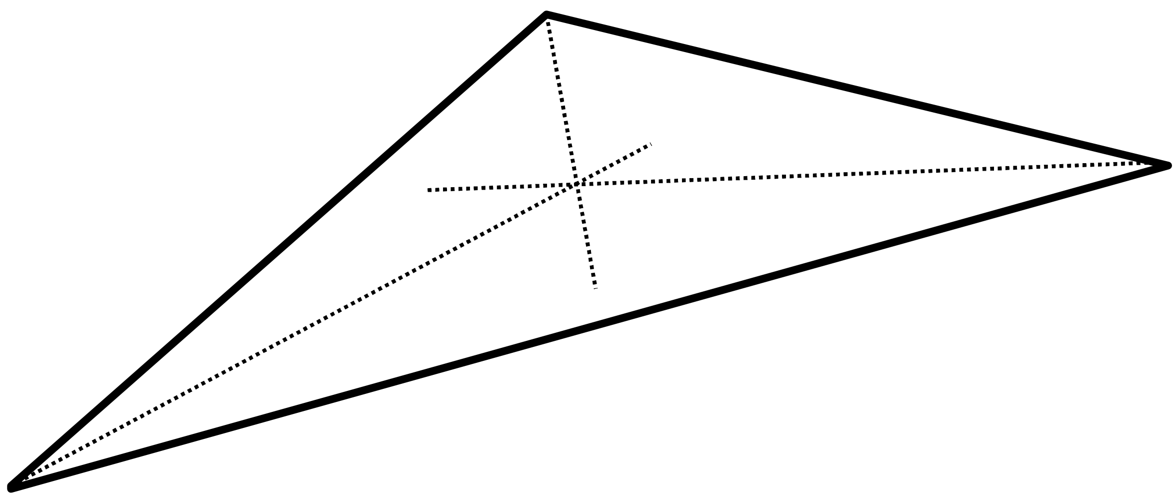
Draw two internal angle bisectors, let I be the point of their intersection (why does I exist?)

Drop perpendiculars from I to the three sides of the triangle.
IA' = IC' and IB' = IC' so IA' = IB' and I is on the angle bisector at C.

The incenter is the center of the ***inscribed circle***, the circle tangent to each of the sides of the triangle.

Incenter (I)

Examples:



Centroid (G)

How about the center of gravity?

A ***median*** of a triangle is a line segment joining a vertex to the midpoint of the opposite side of the triangle.

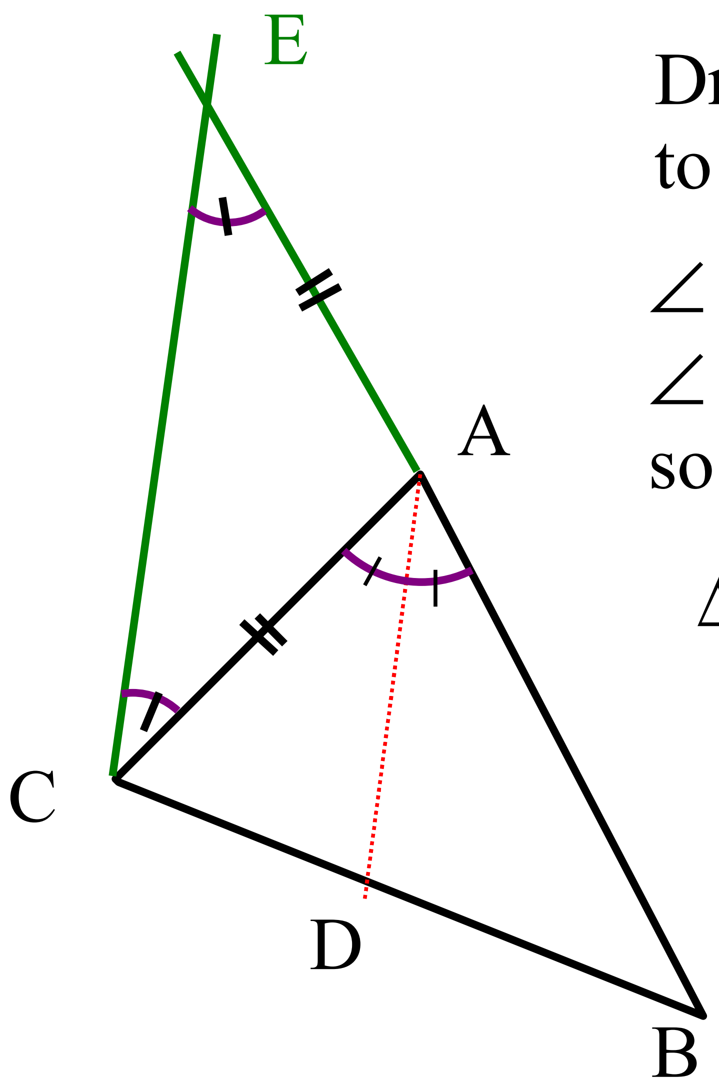
Examples:

Thm 4.4 : The medians of a triangle meet at a point called the ***centroid*** (**G**).

We can give an ugly proof now
or
a pretty proof later. ✓

Internal Bisectors

Thm 4.5: The internal bisector of an angle of a triangle divides the opposite side into two segments proportional to the sides of the triangle adjacent to the angle.



Draw parallel to AD through C, extend AB to E.

$\angle$ BAD = $\angle$ AEC (corr. angles of $\|$ lines)
$\angle$ ECA = $\angle$ CAD (alt. int. angles of $\|$'s)
so AE = AC (isoceles triangle)

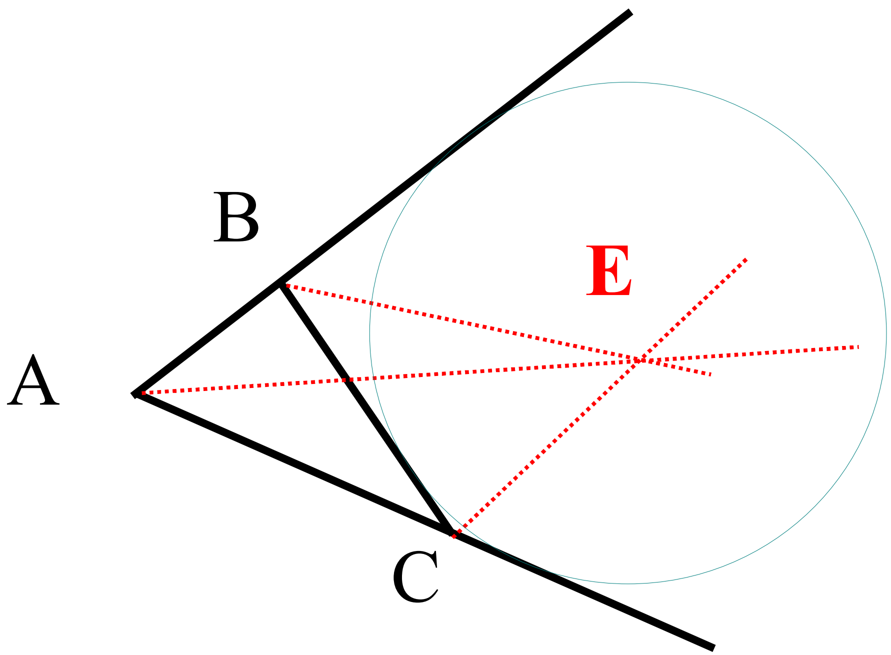
$\Delta ABD \sim \Delta EBC$ (AA)

$$\frac{BA + AE}{BA} = \frac{BD + DC}{BD}$$

$$\Rightarrow \quad \frac{AC}{BA} = \frac{DC}{BD}$$

Excenter

Thm 4.6: The external bisectors of two angles of a triangle meet the internal bisector of the third angle at a point called the ***excenter***.



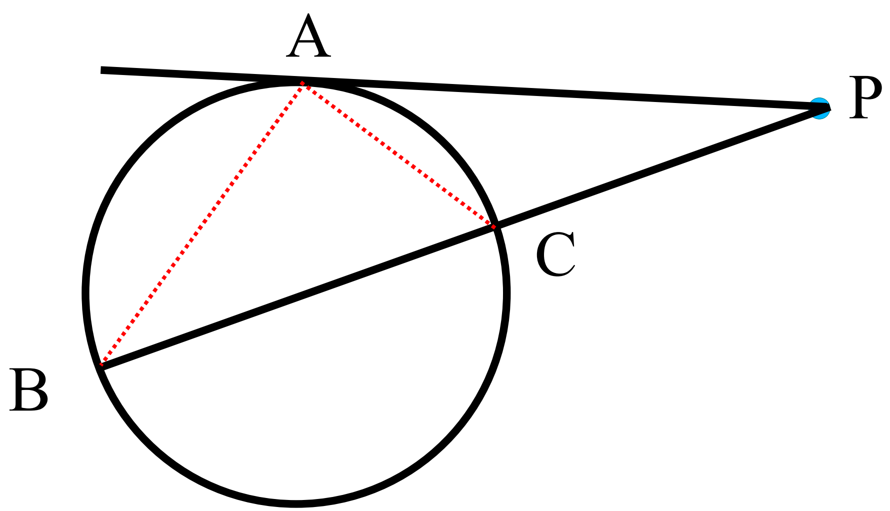
There are 3 excenters of a triangle.

An excenter is the center of an excircle, which is a circle exterior to the triangle that is tangent to the three sides of the triangle.

At a vertex, the internal angle bisector is perpendicular to the external angle bisector.

A Circle Theorem

Thm 4.9: The product of the lengths of the segments from an exterior point to the points of intersection of a secant with a circle is equal to the square of the length of a tangent to the circle from that point.



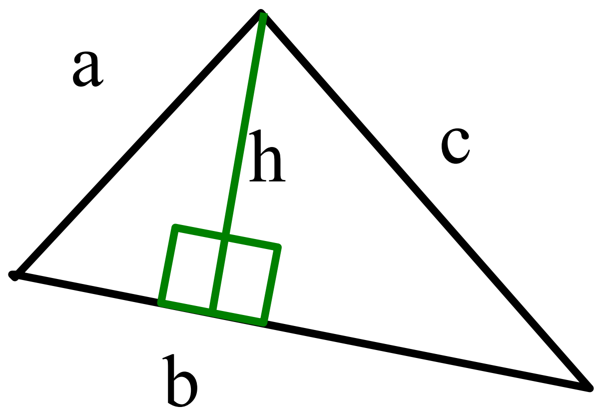
$\angle PAC \cong \angle ABC$ since both are measured by ½ arc AC.

$\Delta PAC \sim \Delta ABP$ (AA)

$$\frac{PC}{PA} = \frac{PA}{PB}$$

$$(PA)^2 = (PC)(PB).$$

Area Formulas



$$A = \tfrac{1}{2}bh$$

Heron's formula for the area of a triangle.

$$A=\sqrt{s(s-a)(s-b)(s-c)},\ \textit{where}\ s=\frac{a+b+c}{2}\quad \textit{the semiperimeter}$$

Brahmagupta's formula for the area of a cyclic quadrilateral.

$$A=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Directed Segments

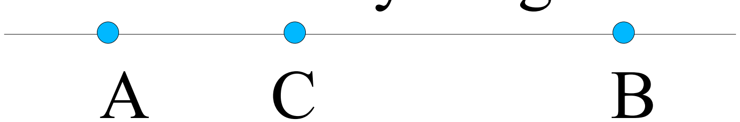
The next few theorems involve the lengths of line segment and we want to permit directed lengths (positive and negative).

By convention we assign to each line an independent direction. Each length measured in the same direction as the assigned one is positive and those in the opposite direction are negative.

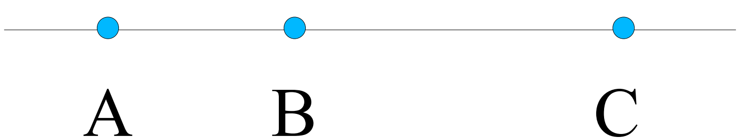
AB positive, BA negative

AB negative, BA positive

With this convention, internal ratios are always positive and external ratios are always negative.



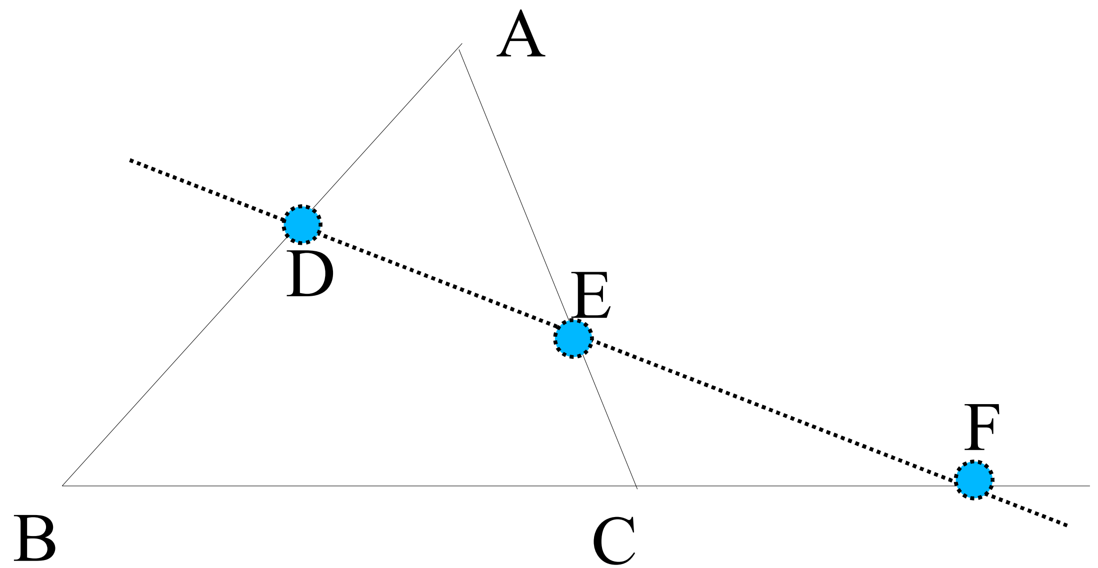
$$\frac{AC}{CB} > 0$$



$$\frac{AC}{CB} < 0$$

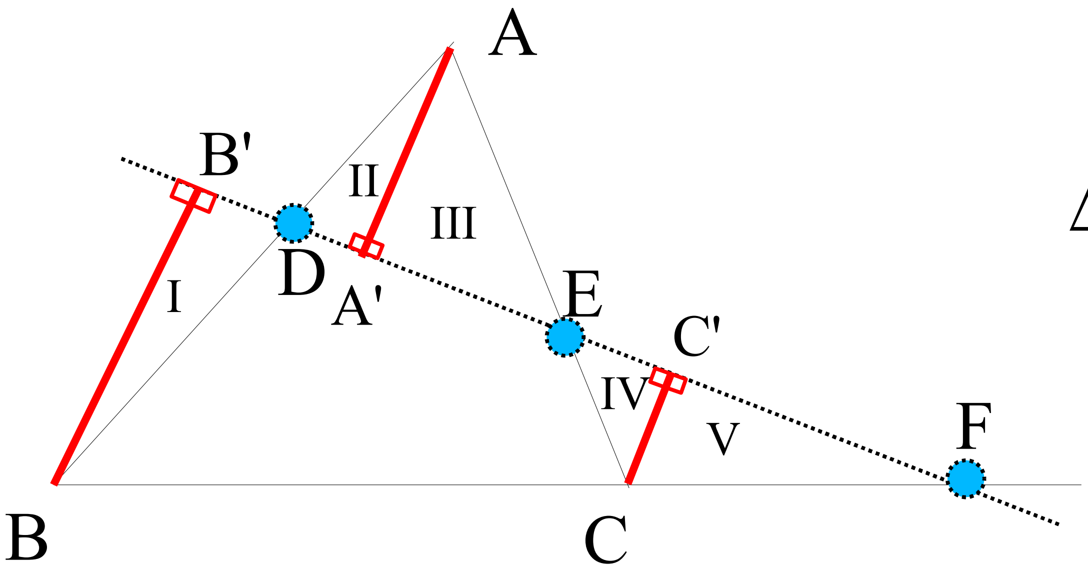
Menelaus' Theorem

Thm 4.10 : Menelaus's Theorem. If three points, one on each side of a triangle are collinear, then the product of the ratios of the division of the sides by the points is -1. (Alexandria, ~ 100 AD)



$$\frac{AD}{DB}\frac{BF}{FC}\frac{CE}{EA} = -1$$

Menelaus' Theorem



ΔI ~ ΔII, ΔIII~ ΔIV, and
ΔV ~ ΔFB'B

$$\frac{AD}{DB} = \frac{AA'}{BB'}$$

$$\frac{CE}{EA} = \frac{CC'}{AA'}$$

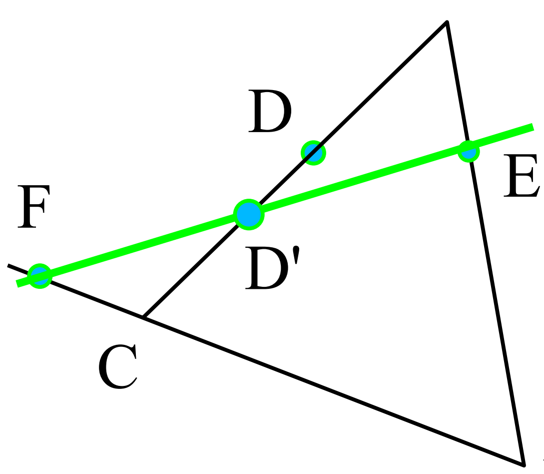
$$\frac{BF}{CF} = \frac{BB'}{CC'} \quad so \quad \frac{BF}{FC} = -\frac{BB'}{CC'}$$

$$Thus \quad \frac{AD}{DB}\frac{BF}{FC}\frac{CE}{EA} = \frac{AA'}{BB'}\frac{CC'}{AA'}\frac{-BB'}{CC'} = -1.$$

Converse of Menelaus

Thm 4.11 : Converse of Menelaus's Theorem.
If the product of the ratios of division of the sides by three points, one on each side of a triangle, extended if necessary, is -1, then the three points are collinear.

Pf:



We assume that $\frac{AD}{DC}\frac{CF}{FB}\frac{BE}{EA} = -1$

Join E and F by a line which intersects AC at D'. We will show that D' = D, proving the theorem. By Menelaus we have:

$$\frac{AD'}{D'C}\frac{CF}{FB}\frac{BE}{EA} = -1, \quad so$$

$$\frac{AD'}{D'C}\frac{CF}{FB}\frac{BE}{EA} = \frac{AD}{DC}\frac{CF}{FB}\frac{BE}{EA}$$

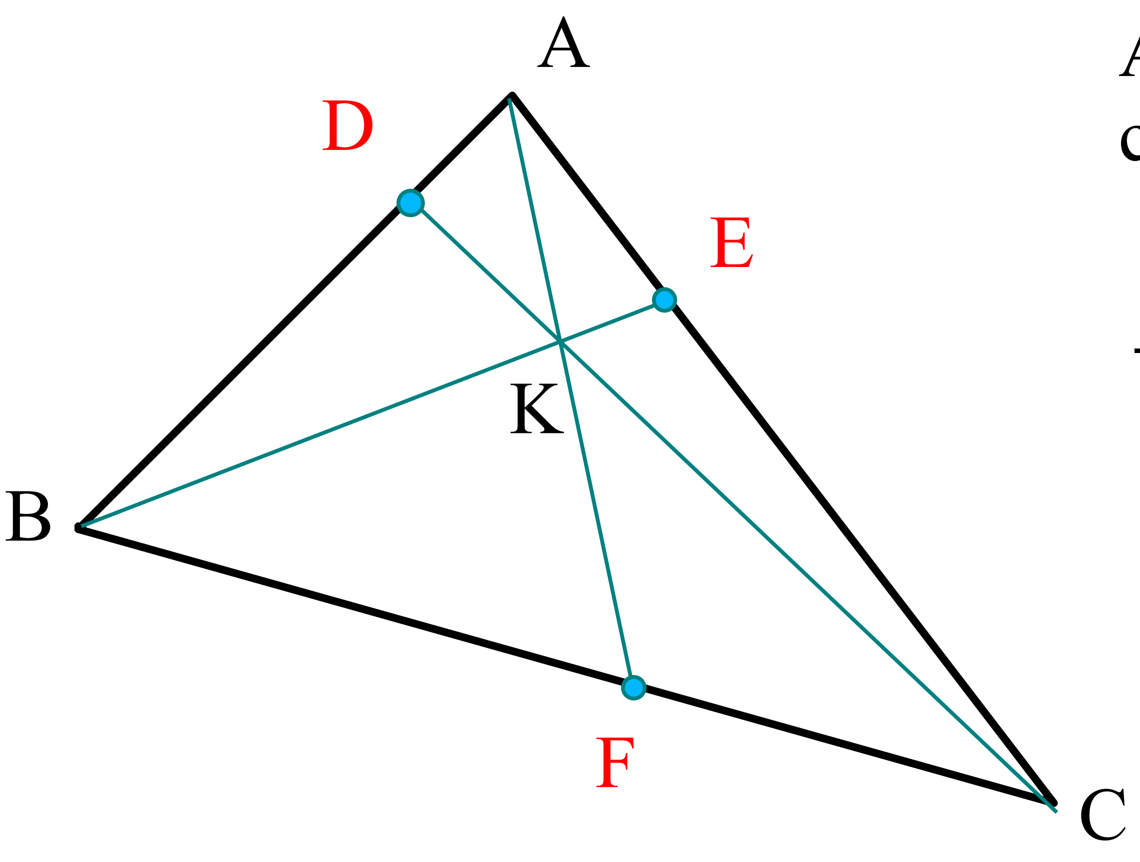
$$\frac{AD'}{D'C} = \frac{AD}{DC} \Rightarrow \frac{AD'}{D'C} + 1 = \frac{AD}{DC} + 1$$

$$\frac{AD'+D'C}{D'C} = \frac{AD+DC}{DC} \Rightarrow \frac{AC}{D'C} = \frac{AC}{DC}$$

$$Thus,\ D'C = DC \ \ so\ \ D'=D.$$

Ceva's Theorem

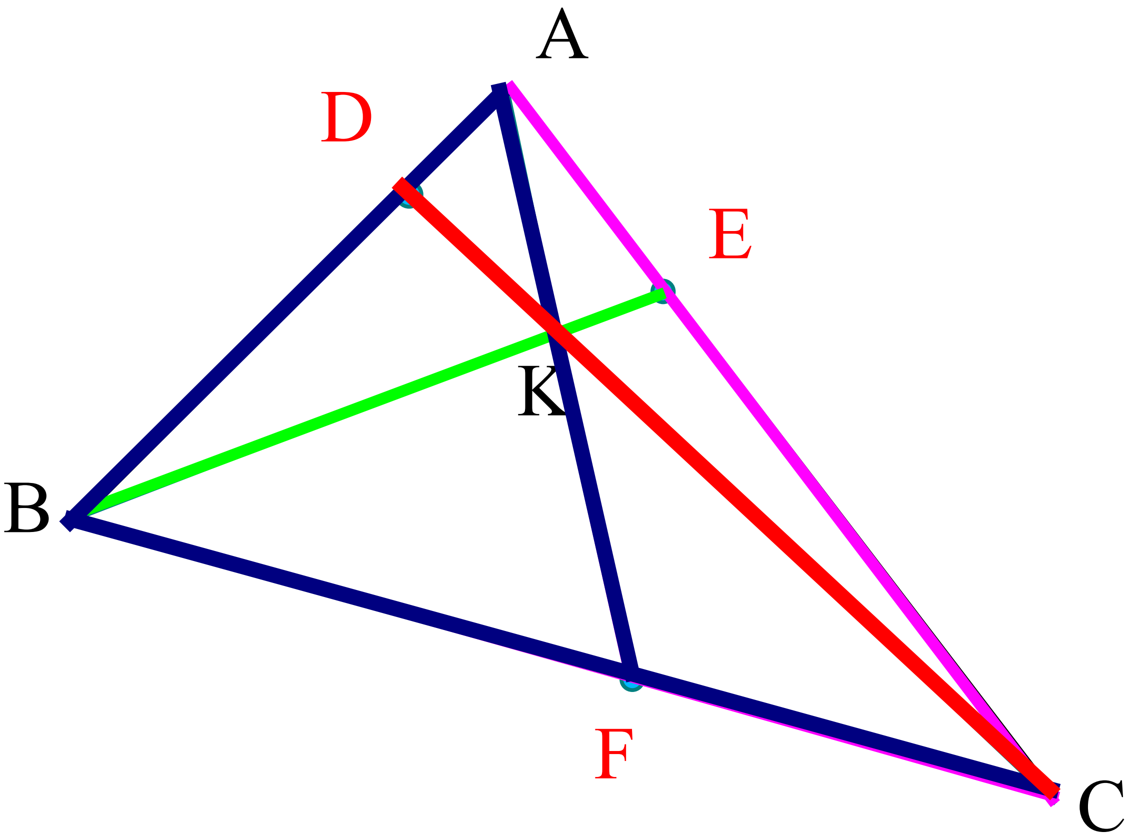
Thm 4.13 : Ceva's Theorem. Three lines joining vertices to points on the opposite sides of a triangle are concurrent if and only if the product of the ratios of the division of the sides is 1. (Italy, 1678)



AF, BE and CD are concurrent at K iff

$$\frac{AD}{DB}\frac{BF}{FC}\frac{CE}{EA} = 1.$$

Ceva's Theorem



Assume the lines meet at K.

Apply Menelaus to ΔAFC:

$$\frac{AK}{KF}\frac{FB}{BC}\frac{CE}{EA} = -1$$

Apply Menelaus to ΔAFB:

$$\frac{AD}{DB}\frac{BC}{CF}\frac{FK}{KA} = -1$$

Now multiply, cancel and be careful with sign changes to get:

$$\frac{AD}{DB}\frac{BF}{FC}\frac{CE}{EA} = 1.$$

Ceva's Theorem

Note that the text does not provide a proof of the converse of Ceva's theorem (although it is given as an iff statement).

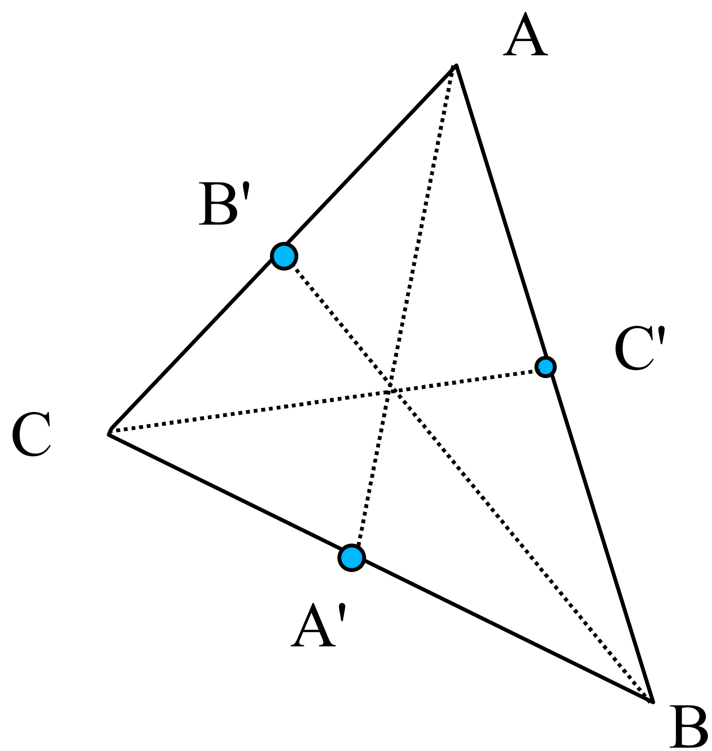
This converse is proved in a manner very similar to that used for the proof of the converse of Menelaus' theorem. I think this is a very good exercise to do, so consider it a homework assignment.

This converse is often used to give very elegant proofs that certain lines in a triangle are concurrent. We will now give two examples of this.

Centroid Again

Thm 4.4 : The medians of a triangle meet at a point called the ***centroid*** (**G**).

Pf:



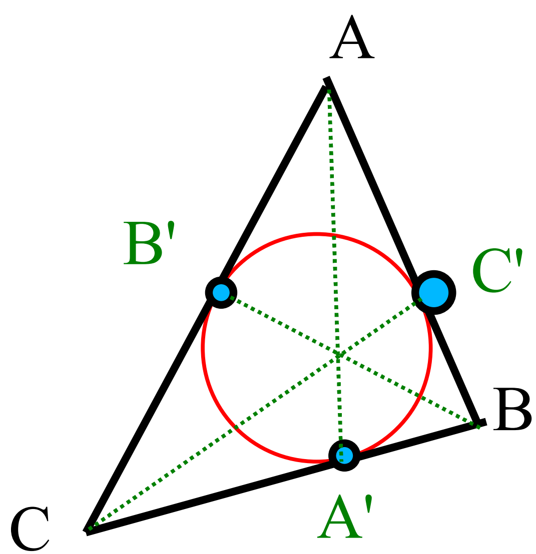
Since A', B' and C' are midpoints, we have AB' = B'C, CA' = A'B and BC' = C'A. Thus,

$$\frac{AB'}{B'C}\frac{CA'}{A'B}\frac{BC'}{C'A} = (1)(1)(1) = 1$$

So, by the converse of Ceva's theorem, AA', BB' and CC' (the medians) are concurrent. ❑

Gergonne Point

Thm 4.14: The lines from the points of tangency of the incircle to the vertices of a triangle are concurrent (***Gergonne point***).



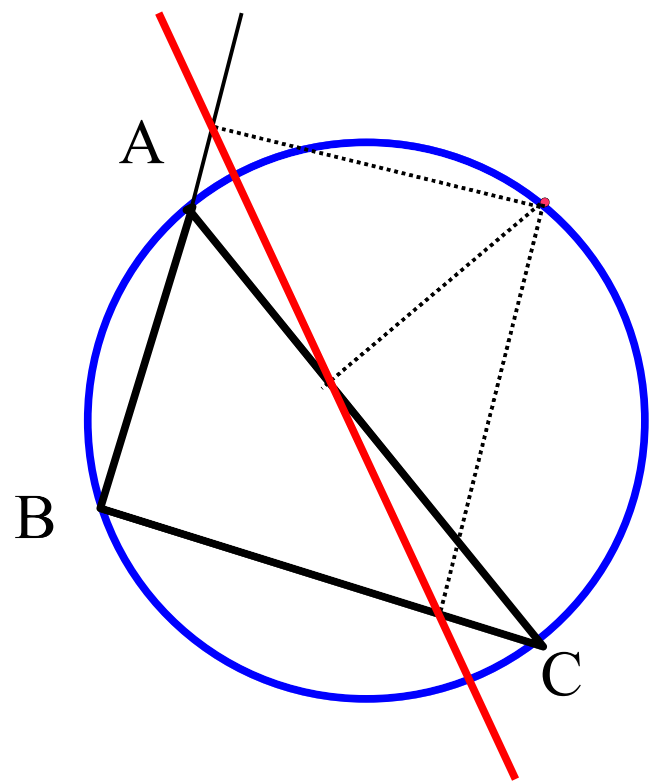
Pf: Since the sides of the triangle are tangents to the incircle, AB' = AC', BC' = BA', and CA' = CB'. So,

$$\frac{AB'}{B'C}\frac{CA'}{A'B}\frac{BC'}{C'A} = 1$$

So, by the converse of Ceva's theorem, the lines are concurrent. ❑

Simson Line

Thm 4.15: The three perpendiculars from a point on the circumcircle to the sides of a triangle meet those sides in collinear points. The line is called the ***Simson line***. (No proof)



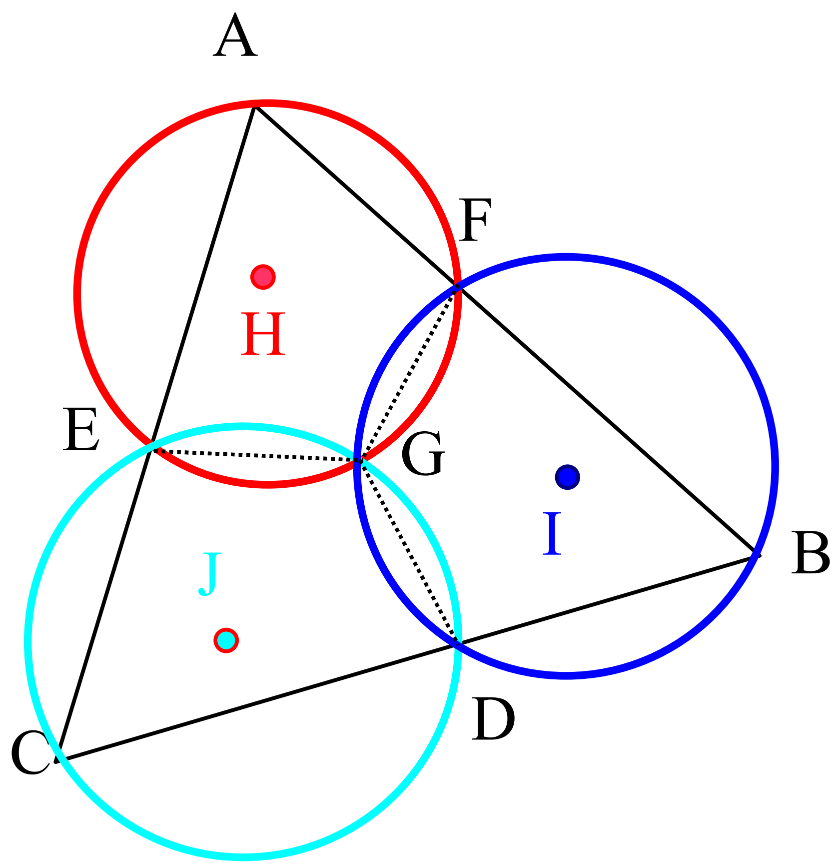
Miquel Point

Thm 4.19 : If three points are chosen, one on each side of a triangle then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the ***Miquel point***.

Example:

Miquel Point

Pf:



In circle J,
∠EGD + ∠C = π.

In circle I,
∠FGD + ∠B = π.

Since ∠EGF + ∠FGD + ∠EGD + ∠A + ∠B + ∠C = 2π + π, by subtracting we get ∠EGF + ∠A = π and so A,E,G and F are concyclic (on the same circle). ❑

Feuerbach's Circle

Thm 4.16 : The midpoints of the sides of a triangle, the feet of the altitudes and the midpoints of the segments joining the orthocenter and the vertices all lie on a circle called the ***nine-point circle***.

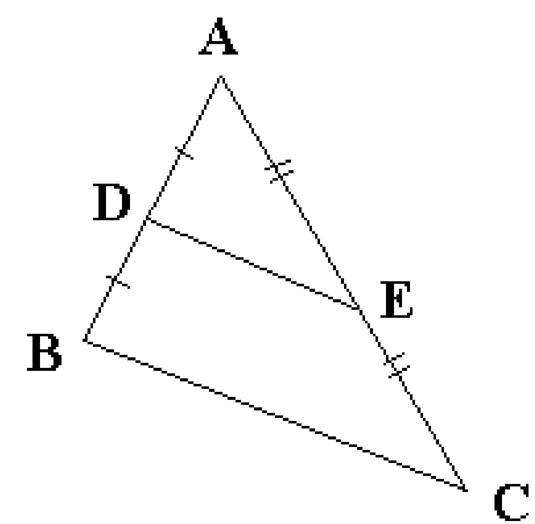
Example:

The 9-Point Circle Worksheet

We will start by recalling some high school geometry facts.

1. *The line joining the midpoints of two sides of a triangle is parallel to the third side and measures 1/2 the length of the third side of the triangle.*

a) Why is the ratio of side AD to side AB 1:2?
b) In the diagram, ΔDAE is similar to ΔBAC because ....

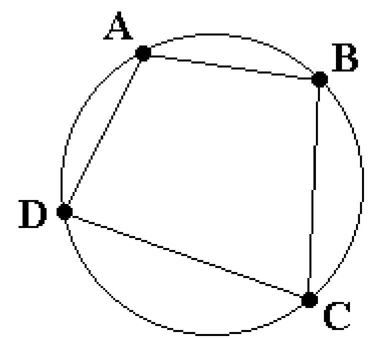


Since similar triangles have congruent angles, we have that $\angle ADE \cong \angle ABC$.

c) Line DE is parallel to BC because ....
d) Since the ratio of corresponding sides of similar triangles is constant, what is the ratio of side DE to side BC?

The 9-Point Circle Worksheet

2. *Four points, forming the vertices of a quadrilateral, lie on a circle if and only if the sum of the opposite angles in the quadrilateral is 180^o.*



e) An angle inscribed in a circle has measure equal to 1/2 the measure of the arc it subtends. In the diagram above, what arc does the $\angle$DAB subtend? What arc does the opposite $\angle$BCD subtend?

f) Since the total number of degrees of the arc of the full circle is 360^o, what is the sum of the measures of $\angle$DAB and $\angle$BCD?

9-Point Circle Worksheet

Three points, not on a line, determine a unique circle. Suppose we have four points, no three on a line. We can pick any three of these four and construct the circle that contains them. The fourth point will either lie inside, on or outside of this circle. Let's say that the three points determining the circle are A, B and C. Call the fourth point D. We have already seen that if D lies on the circle, then $m(\angle ADC) + m(\angle ABC) = 180^o$.

g) What can you say about the size of $\angle$ADC if D lies inside the circle? What is then true about $m(\angle ADC) + m(\angle ABC)$?

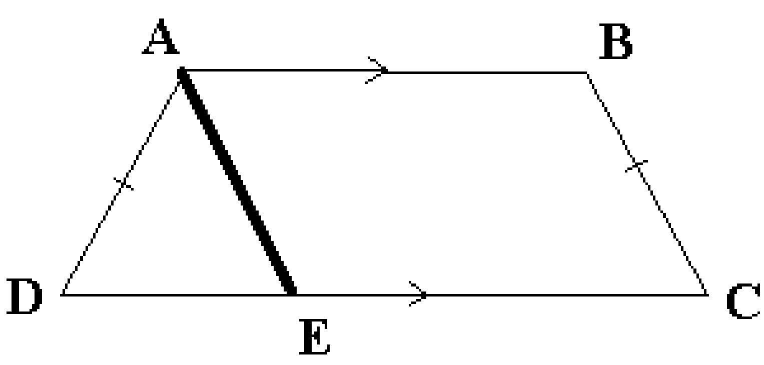
h) What can you say about the size of $\angle$ADC if D lies outside the circle? What is then true about $m(\angle ADC) + m(\angle ABC)$?

9-Point Circle Worksheet

A ***trapezoid*** is a quadrilateral with two parallel sides. An ***isosceles trapezoid*** is one whose non-parallel sides are congruent.

3. *The vertices of an isosceles trapezoid all lie on a circle.*

Consider the isosceles trapezoid ABCD below, and draw the line through A which is parallel to BC. This line meets CD in a point that we label E.



i) What kind of quadrilateral is ABCE?
j) What does this say about the sides AE and BC?
k) ΔDAE is what kind of triangle?

9-Points Circle Worksheet

Because AE is parallel to BC, $\angle BCE \cong \angle AED$ (corresponding angles of parallel lines). And so, by k) this means that $\angle BCD \cong \angle ADE$.

l) Show that $\angle DAB \cong \angle ABC$.

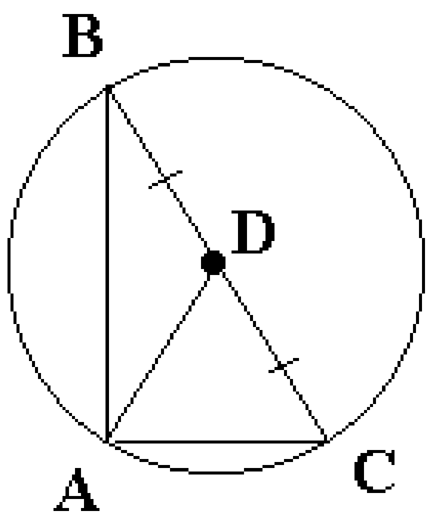
This means that the sums of the opposite angles of the isosceles trapezoid are equal.

m) Show that the sums of the opposite angles of an isosceles trapezoid are 180^o.

9-Points Circle Worksheet

4. *In a right triangle, the line joining the right angle to the midpoint of the hypotenuse has length equal to 1/2 the hypotenuse.*

Draw the circumscribed circle O about the right triangle ABC with right angle A.

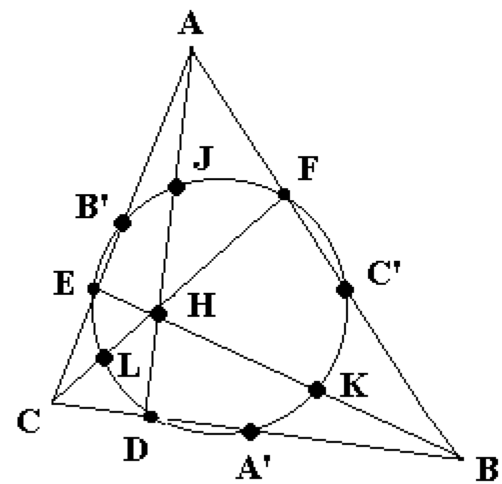


n) What is the measure of the arc subtended by angle A?
o) What kind of line is BC with respect to this circle?
p) Where is the center of the circle?
q) Prove the theorem.

9-Points Circle Worksheet

We are now ready to discuss the Feuerbach circle.

For an arbitrary triangle, the 3 midpoints of the sides, the 3 feet of the altitudes and the 3 points which are the midpoints of the segments joining the orthocenter to the vertices of the triangle all lie on a circle, called *the nine-points circle*.



There is a circle passing through the 3 midpoints of the sides of the triangle, A', B' and C'. We shall show that the other 6 points are on this circle also. Let D be the foot of the altitude from A. Consider the quadrilateral A'DB'C'. We claim that this is an isoceles trapezoid.

r) Why is A'D parallel to B'C' ?

s) Why is DB' equal in length to 1/2 AC?

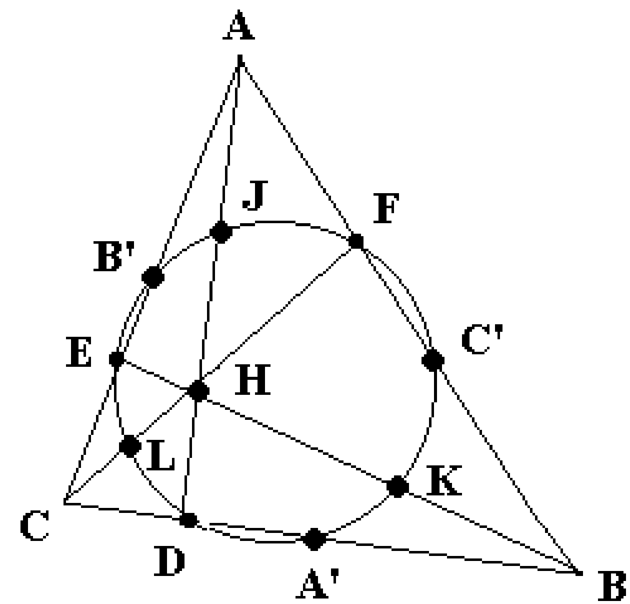
t) Why is A'C' equal in length to 1/2 AC?

9-Points Circle Worksheet

Since A'DB'C' is an isosceles trapezoid, D must be on the same circle as A', B' and C'. The other altitude feet (E and F) are dealt with similarly.

u) Determine the isosceles trapezoid that contains E and the one that contains F.

Now consider J, the midpoint of the segment joining the orthocenter H to the vertex A. Draw the circle that has A'J as its diameter.



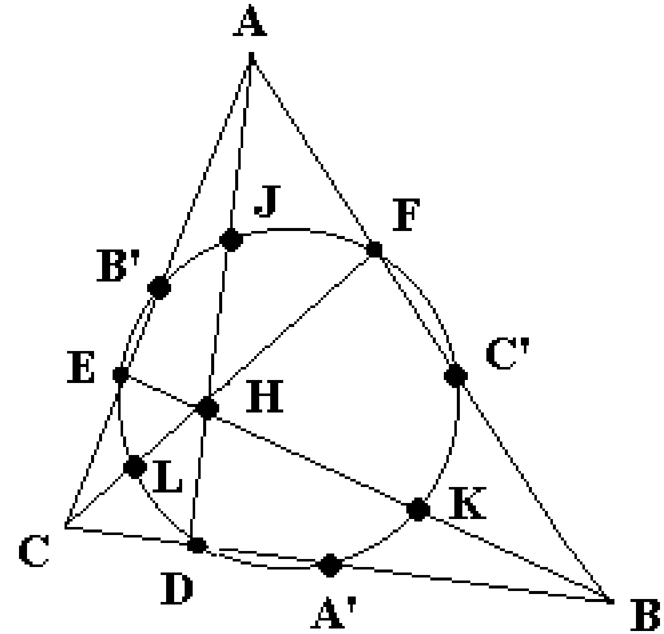
v) Why is A'B' parallel to the side AB?
w) Why is JB' parallel to the altitude CF?
x) Show that A'B' is perpendicular to JB'.

9-Points Circle Worksheet

Angle JB'A' is thus a right angle and so, B' must be on the circle with diameter A'J.

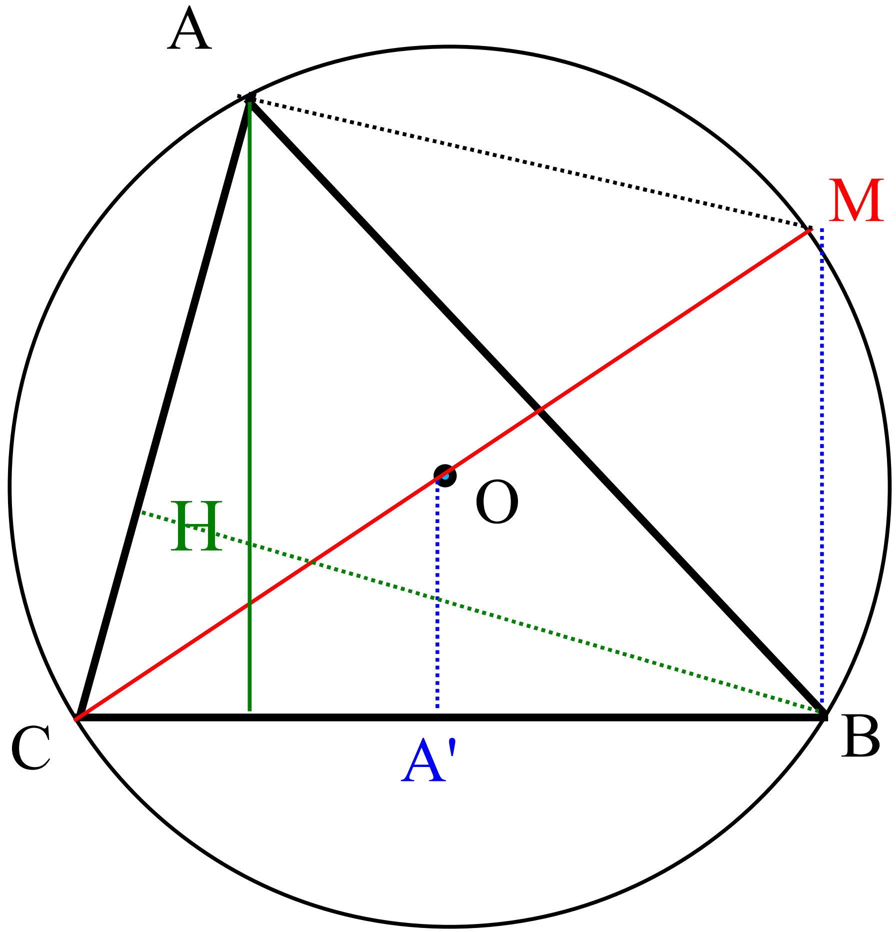
y) In an analogous manner, prove that C' is on the circle with diameter A'J.

We therefore have J, A', B' and C' all on the same circle, which is the circle we started with.
By repeating this argument starting with the circles having diameters KB' and LC', we can show that K and L are also on this circle.



9-point Circle

Thm 4.17: The centroid of a triangle trisects the segment joining the circumcenter and the orthocenter. [***Euler line***]



Draw circumcircle and extend radius CO to diameter COM.

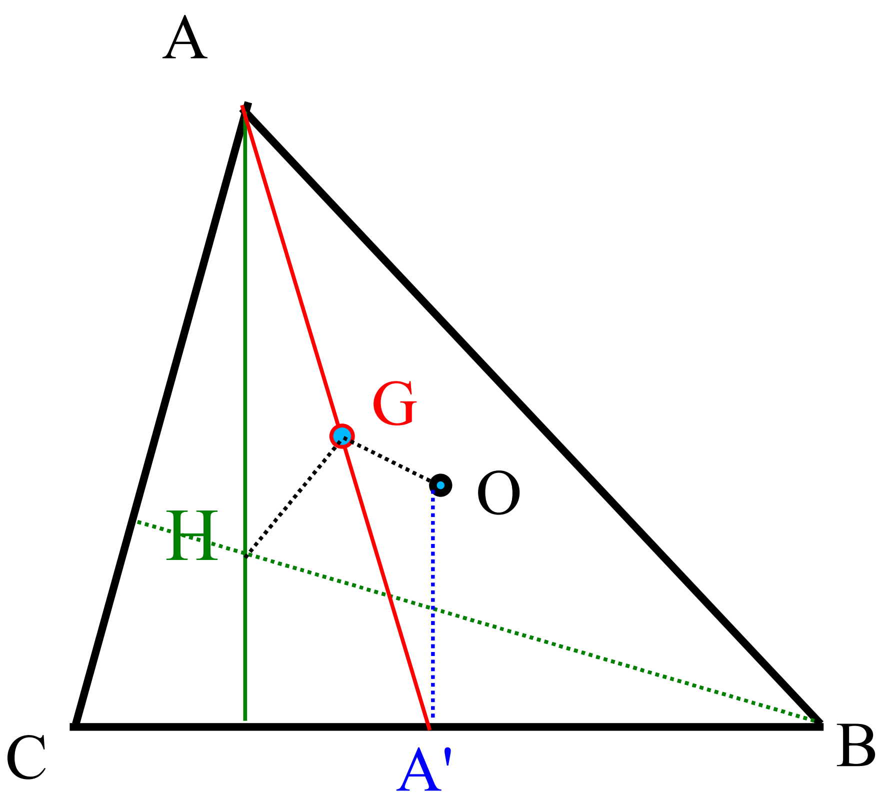
$\Delta CBM \sim \Delta CA'O$
so $OA' = \frac{1}{2}MB$

Drop altitude from A, find orthocenter H.

AMBH is a parallelogram, so $AH = MB = 2OA'$.

9-point Circle

Thm 4.17: The centroid of a triangle trisects the segment joining the circumcenter and the orthocenter. [***Euler line***]



Draw centroid G on median AA', 2/3 of the way from A.

ΔHAG ~ ΔOA'G because

AH = 2OA'

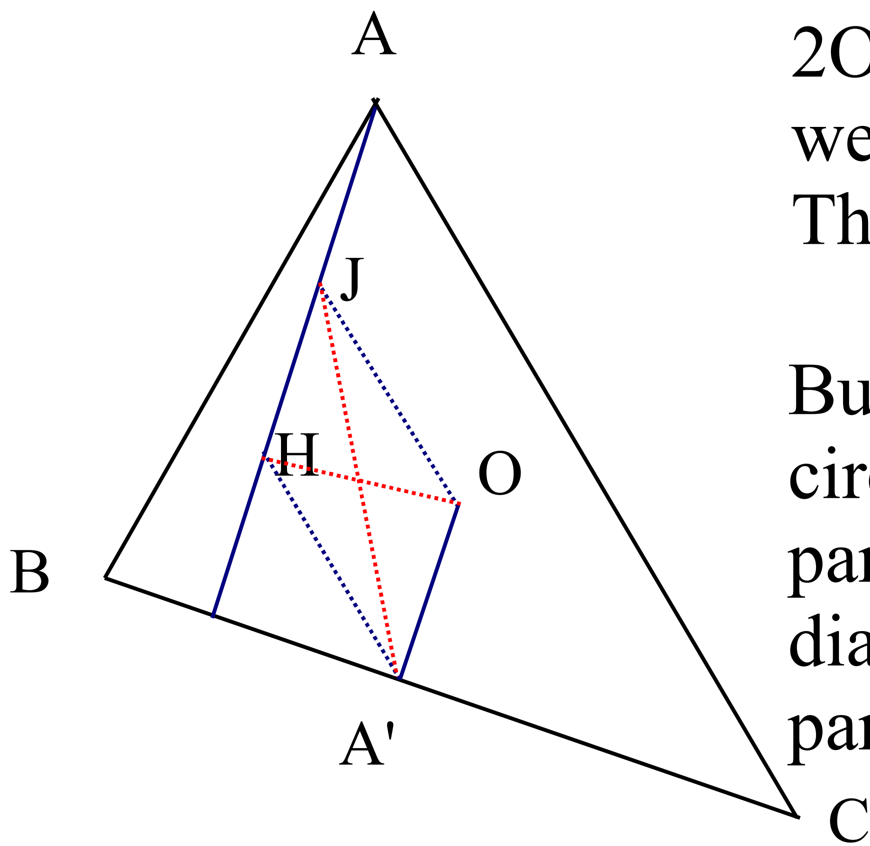
AG = 2GA' and

∠HAG ≅ ∠OA'G

thus ∠HGA ≅ ∠OGA', and so H,G, and O are collinear with HG = 2 GO.

9-point Circle

Thm 4.18: The center of the nine-point circle bisects the segment of the Euler line joining the orthocenter and the circumcenter of a triangle.

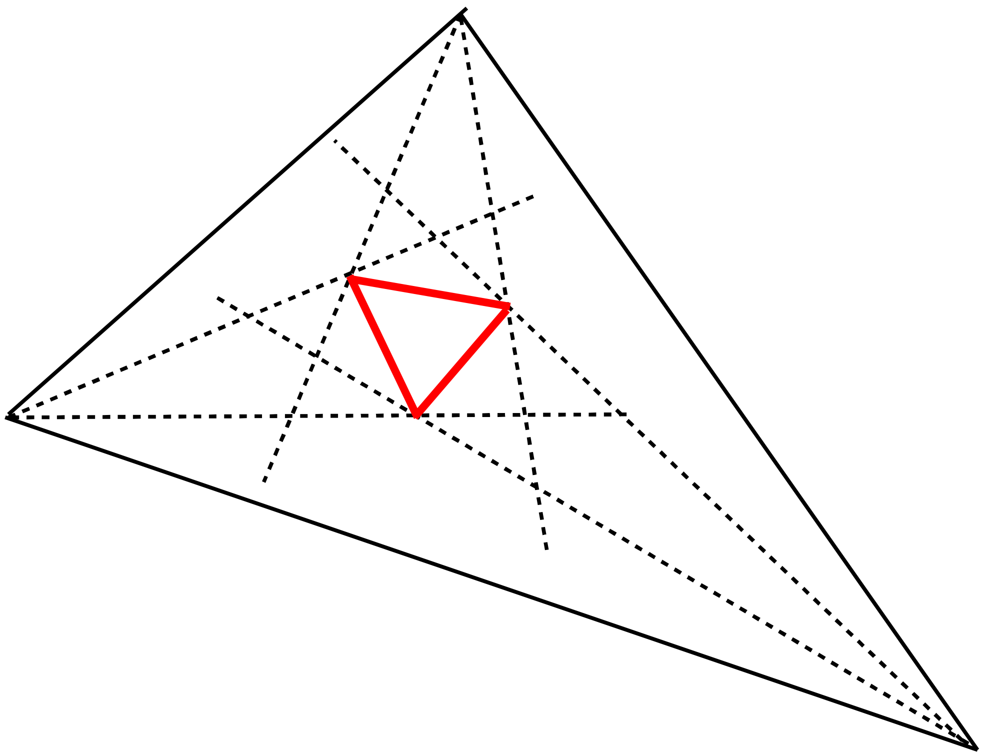


Recall from the last proof that AH = 2OA'. Since J is the midpoint of AH, we have JH = OA'.
Thus, JOA'H is a parallelogram.

But JA' is a diameter of the 9-point circle, and also a diagonal of this parallelogram (HO being the other diagonal). Since the diagonals of a parallelogram bisect each other we have the conclusion.

Morley's Theorem

Thm 4.26: (Morley's Theorem) The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.

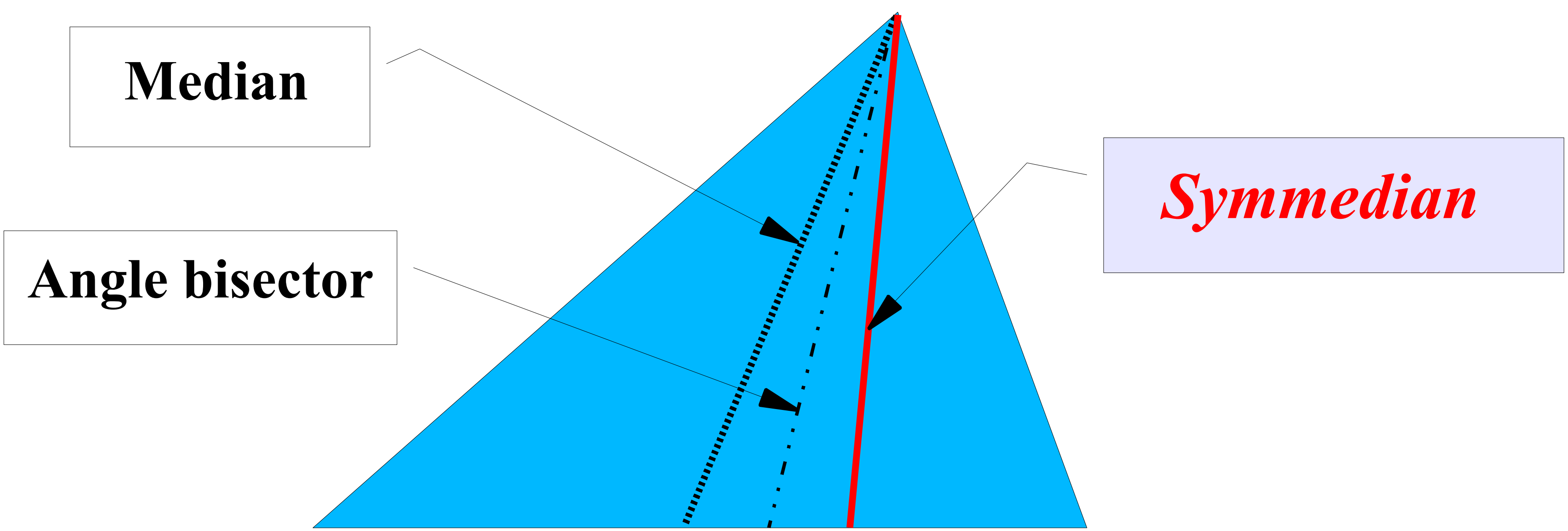


Symmedian

In a triangle, lines through a vertex that are symmetrically placed around the angle bisector of that vertex are called ***isogonal lines***. One of these lines is called the ***isogonal conjugate*** of the other.

Note that the angle bisector is also the angle bisector of the angle formed by a pair of isogonal lines.

A ***symmedian*** is the isogonal conjugate of a median.



LeMoine Point

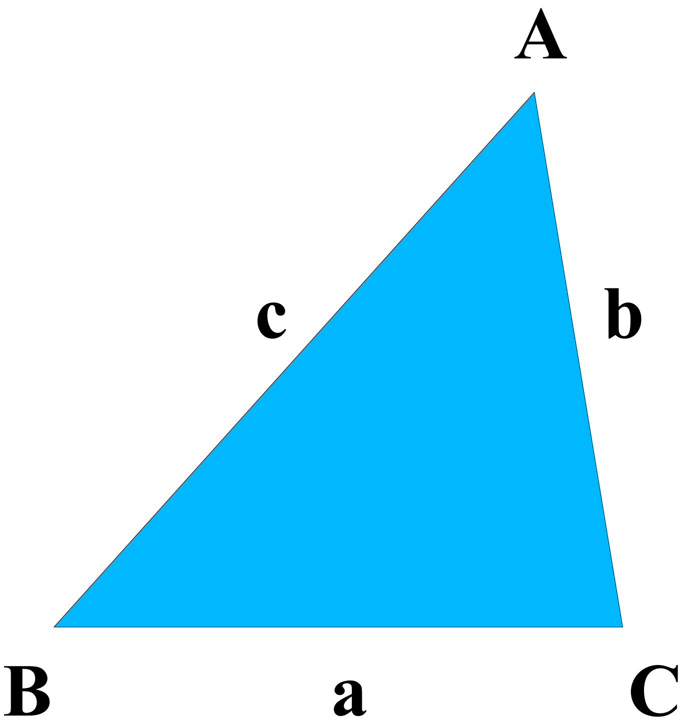
The symmedians of a triangle are concurrent at the ***LeMoine point*** (also called the symmedian point).

This is a consequence of a more general result, namely:

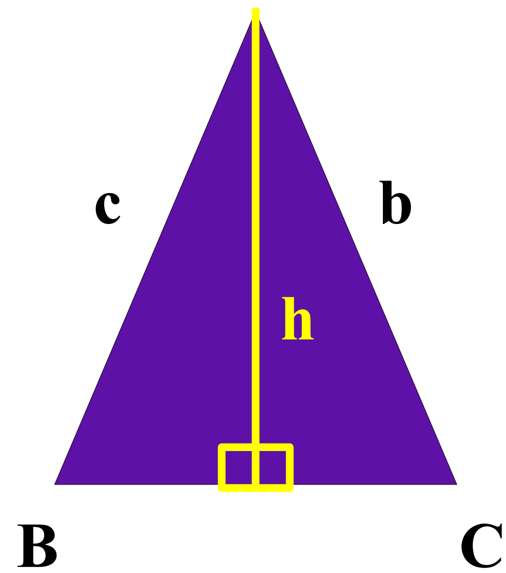
Theorem: The isogonal conjugates of a set of concurrent segments from the vertices to the opposite sides of a triangle are also concurrent.

To prove this theorem we should recall the Law of Sines for a triangle.

Law of Sines



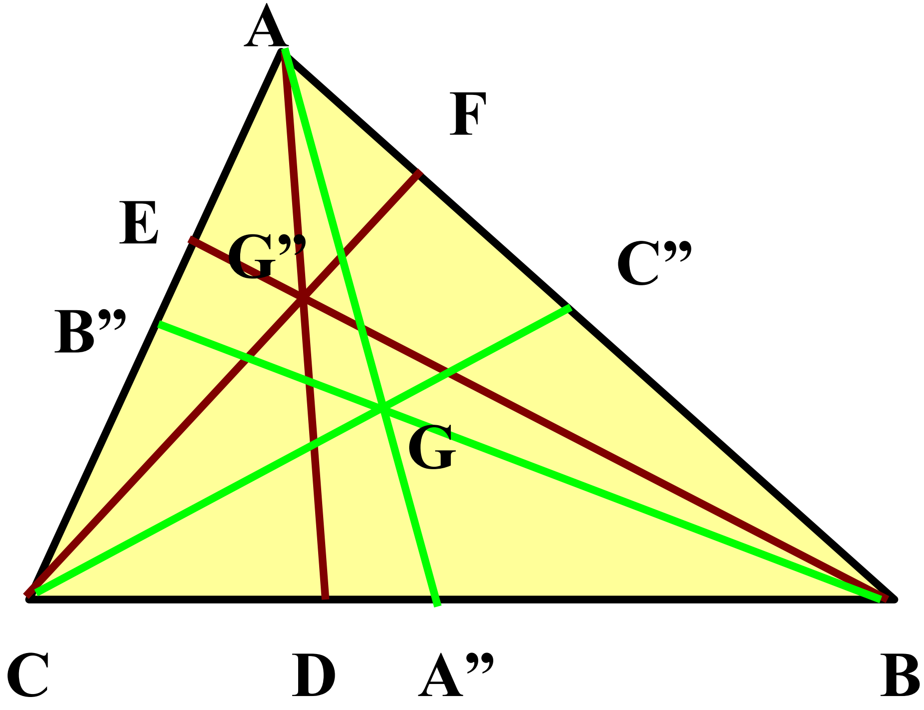
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



c Sin (B) = h = b Sin (C)

Theorem

Theorem: The isogonal conjugates of a set of concurrent segments from the vertices to the opposite sides of a triangle are also concurrent.



Pf: Assume G is point of concurrency of AA", BB" and CC". Let AD, BE and CF be the isogonal conjugates of these lines.

The Law of Sines says that in triangle AA"C,

$$\frac{CA''}{\sin(CAA'')} = \frac{CA}{\sin(CA''A)}$$

or

$$CA'' = \frac{CA\sin(CAA'')}{\sin(CA''A)}$$

Theorem

Theorem: The isogonal conjugates of a set of concurrent segments from the vertices to the opposite sides of a triangle are also concurrent.

Pf (cont): Similarly, in triangle AA”B, the law of sines leads to

$$A''B = \frac{AB \sin(BAA'')}{\sin(BA''A)}$$

Using the fact that supplementary angles CA”A and BA”A have the same sine, we can write.

$$\frac{CA''}{A''B} = \frac{CA \sin(CAA'')}{AB \sin(BAA'')}.$$

Repeating this for the other ratios of divisions leads to the trigonometric form of Ceva's theorem (and its converse):

Theorem

Theorem: The isogonal conjugates of a set of concurrent segments from the vertices to the opposite sides of a triangle are also concurrent.

Pf (cont):

$$\frac{\sin(CAA'')}{\sin(BAA'')} \cdot \frac{\sin(BCC'')}{\sin(ACC'')} \cdot \frac{\sin(ABB'')}{\sin(CBB'')} = 1 .$$

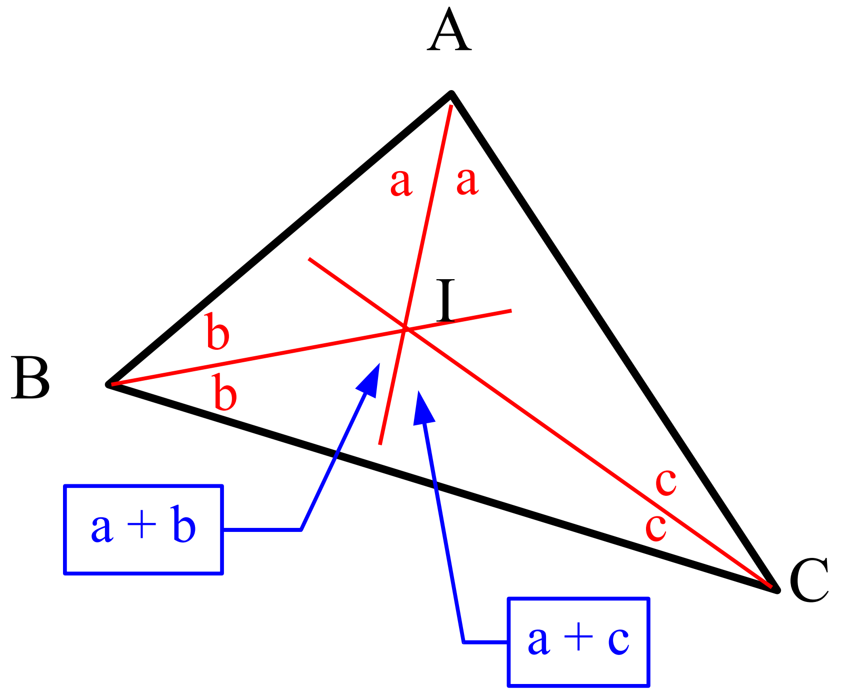
Due to the isogonal conjugates we have the following equalities amongst the angles, BAA" = CAD, BCC" = ACF and CBB" = ABE. Also, CAA" = BAD, ACC" = BCF, and ABB" = CBE. So,

$$\frac{\sin(BAD)}{\sin(CAD)} \cdot \frac{\sin(ACF)}{\sin(BCF)} \cdot \frac{\sin(CBE)}{\sin(ABE)} = 1 ,$$

showing that AD, BE and CF are concurrent.

Morley's Theorem

The incenter I of triangle ABC lies on the angle bisector of angle A at a point where $\angle BIC = 90^o + \frac{1}{2}\angle A$.

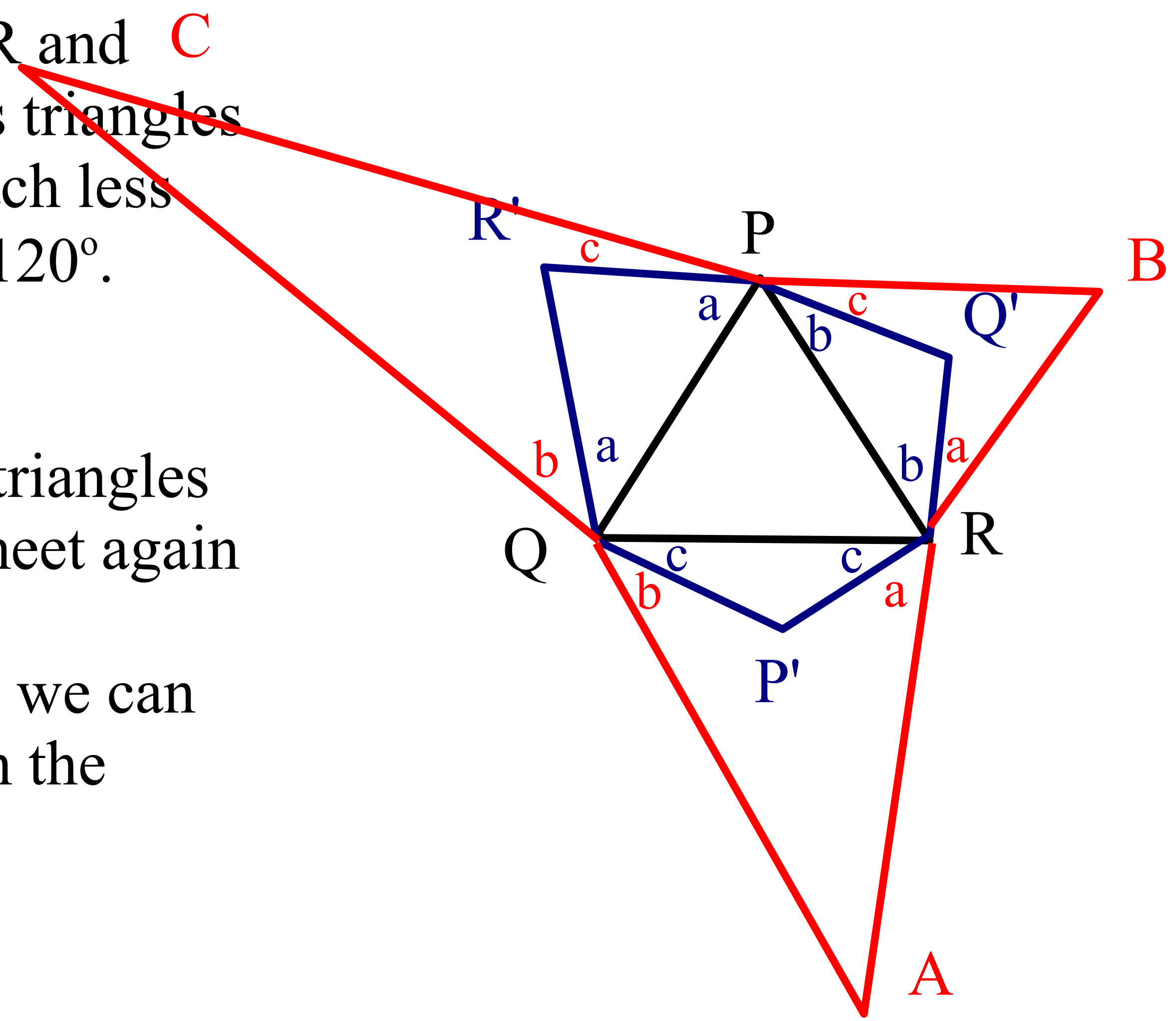


$$a + b + a + c = a + b + c + a = \tfrac{1}{2}(180^o) + a = 90^o + a$$

Morley's Theorem

A direct proof of Morley's Theorem is difficult, so we will give an indirect proof which essentially works backwards.

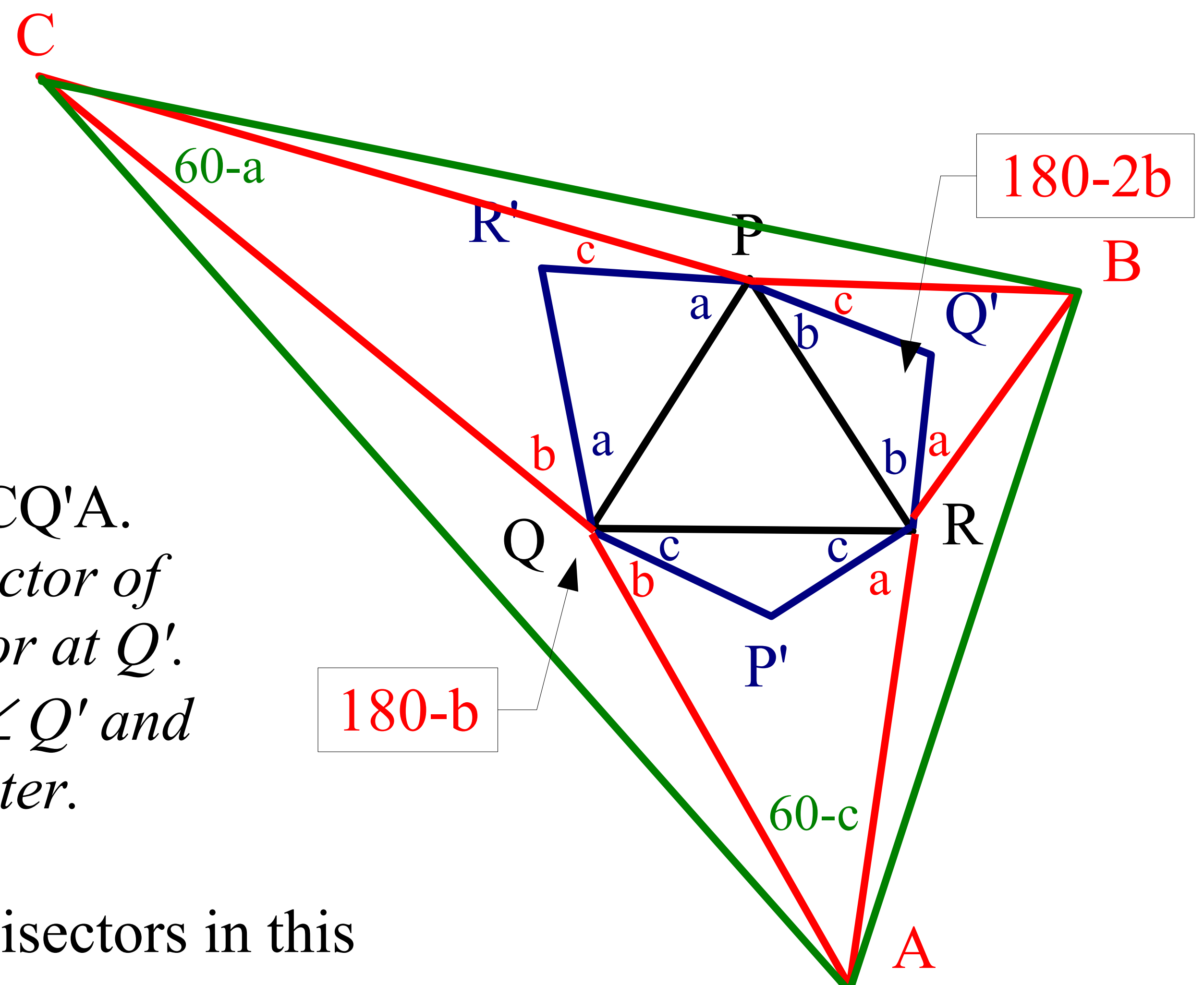
Start with an equilateral ΔPQR and construct on its sides isosceles triangles with base angles a, b and c, each less than 60^{o} and with $a + b + c = 120^{o}$.



Extend the sides of isosceles triangles below their bases until they meet again at points A, B and C.

Since $a + b + c + 60 = 180$, we can calculate some other angles in the figure.

Morley's Theorem



Draw the sides of ΔABC and note the marked angles.

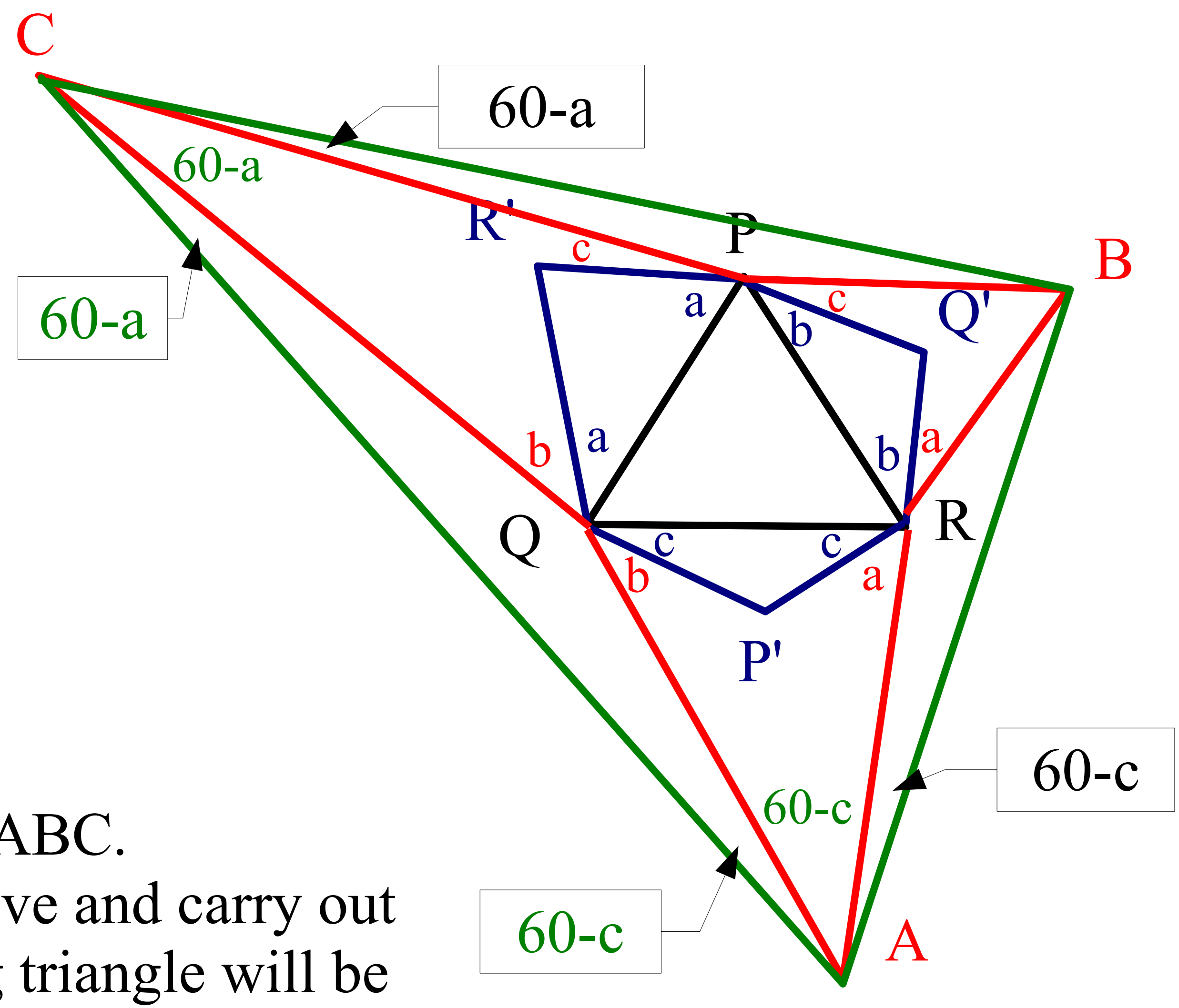
Note the angles at Q and Q'.

Claim: Q is the incenter of ΔCQ'A.
QQ' is the perpendicular bisector of PR and so is the angle bisector at Q'.
The angle at Q is = 90º + ½∠Q' and this means that Q is the incenter.

Thus, QC and QA are angle bisectors in this triangle, so ∠QCA = 60-a and ∠QAC = 60-c.

Morley's Theorem

Similarly, P is the incenter of ΔBP'C and R is the incenter of ΔAR'B. Thus, ...



The angles at A, B and C are trisected. We then have:

a = 60º – 1/3∠C

b = 60º – 1/3∠B and

c = 60º – 1/3∠A

Now, start with an arbitrary ΔABC. Determine a, b and c from above and carry out the construction. The resulting triangle will be similar to the original and the statement of the theorem will be true for it. □