Advanced Macroeconomics
Problem Set #1: Solutions

1. **Solow model in continuous time.** Consider the Solow model in continuous time with production function $y = f(k)$ satisfying the usual properties, constant savings rate s, depreciation rate δ, productivity growth g and employment growth n.

   (a) Use the implicit function theorem to show how an increase in s affects the steady state values k^*, y^*, c^*. Does this change in s increase or decrease long run output and consumption per worker? Explain.

   Now consider the special case of a Cobb-Douglas production function $f(k) = k^\alpha$.

   (b) Derive expressions for the *elasticities* of capital and output with respect to the savings rate

   $$\frac{d \log k^*}{d \log s}, \qquad \frac{d \log y^*}{d \log s}$$

   How do these depend on the curvature of the production function α? Explain.

   (c) Derive an exact solution for the time path $k(t)$ of capital per effective worker.

   Now consider the specific numerical values $\alpha = 0.3$, $s = 0.2$, $\delta = 0.05$, $g = 0.02$, $n = 0.03$.

   (d) Calculate and plot the time paths of $k(t), y(t), c(t)$ starting from the initial condition $k(0) = k^*/2$. How long is the *half-life* of convergence?

   (e) Now suppose that we are in steady state $k(0) = k^*$ when the savings rate suddenly increases to $s = 0.22$. Calculate and plot the time paths of $k(t), y(t), c(t)$ in response to this change. Explain both the short-run and long-run dynamics of $k(t), y(t), c(t)$. What if instead the savings rate had increased to $s = 0.30$? Do you think these are large or small effects on output? Explain.

   SOLUTIONS:

   (a) Steady state capital k^* solves

   $$sf(k^*) = (\delta + g + n)k^*$$

   This implicitly determines k^* as a function of the savings rate s, say $k^* = k(s)$. Write this

   $$sf(k(s)) = (\delta + g + n)k(s)$$

   Differentiating both sides with respect to s gives

   $$f(k(s)) + sf'(k(s))k'(s) = (\delta + g + n)k'(s)$$

Solving for $k'(s)$ then gives

$$k'(s) = -\frac{f(k^*)}{sf'(k^*) - (\delta + g + n)} > 0$$

which is positive since at steady state the slope of the savings curve $sf(k)$ is flatter than the depreciation line $(\delta + g + n)k$, that is $sf'(k^*) < (\delta + g + n)$. Now observe that steady state output is given by $y^* = f(k^*) = f(k(s)) \equiv y(s)$ and steady state consumption is given by $c^* = (1 - s)y^* = (1 - s)y(s) \equiv c(s)$ with

$$y'(s) = f'(k)k'(s) > 0$$

and

$$c'(s) = (1 - s)y'(s) - y(s)$$

Since a higher savings rate increases steady state capital, it also increases steady state output. The effect on consumption is ambiguous and depends whether the savings rate s is greater or lower than the 'golden rule' level (as discussed in class).

(b) With $y = f(k) = k^\alpha$ we have the solutions

$$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

and

$$y^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{\alpha}{1-\alpha}}$$

Hence

$$\frac{d \log k^*}{d \log s} = \frac{1}{1 - \alpha} > 1$$

and

$$\frac{d \log y^*}{d \log s} = \frac{\alpha}{1 - \alpha}$$

Notice that an increase in the saving rate has a 'multiplier-like' $\frac{1}{1-\alpha}$ effect on steady state capital. A higher savings rate leads to more capital which leads to more output which leads to more saving which leads to more capital, etc, cumulating in a total increase of $1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}$.

(c) With $y = f(k) = k^\alpha$ the capital stock $k(t)$ solves the nonlinear differential equation

$$\dot{k}(t) = sk(t)^\alpha - (\delta + g + n)k(t)$$

But this implies a *linear* differential equation in the capital/output ratio $x(t) \equiv k(t)/y(t) = k(t)^{1-\alpha}$. To see this, observe

$$\begin{aligned}
\dot{x}(t) &= (1 - \alpha)k(t)^{-\alpha}\dot{k}(t) \\
&= (1 - \alpha)k(t)^{-\alpha}\Big[sk(t)^\alpha - (\delta + g + n)k(t)\Big] \\
&= (1 - \alpha)\Big[s - (\delta + g + n)k(t)^{1-\alpha}\Big] \\
&= (1 - \alpha)\Big[s - (\delta + g + n)x(t)\Big]
\end{aligned}$$

This is a stable linear differential equation in $x(t)$ with steady state

$$x^* = \frac{k^*}{y^*} = \frac{s}{\delta + g + n}$$

and unique solution

$$x(t) = e^{\lambda t}x(0) + (1 - e^{\lambda t})x^*, \qquad \lambda \equiv -(1-\alpha)(\delta + g + n) < 0$$

So the solution for $k(t)$ is

$$k(t) = \left(e^{\lambda t}k(0)^{1-\alpha} + (1 - e^{\lambda t}){k^*}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

(d) Figure 1 below shows the transitional dynamics of $k(t), y(t), c(t)$ for the given parameter values. Notice that $k^* = 2.6918$, $y^* = 1.3459$ (so the capital/output ratio is $x^* = s/(\delta + g + n) = 2$) and $c^* = 1.0767$ so that $c^*/y^* = 1 - s = 0.80$.

The half-life of convergence is the time t^* it takes to close half of the initial deviation from steady state. Since the differential equation for the capital/output ratio is linear, we have a simple formula for the half-life in the capital/output ratio. Using the solution

$$x(t) = e^{\lambda t}x(0) + (1 - e^{\lambda t})x^* = x^* + e^{\lambda t}(x(0) - x^*)$$

we look for the value of t such that $x(t) - x^* = (x(0) - x^*)/2$. This is given by

$$t^* = -\frac{\log 2}{\lambda} = \frac{2}{(1-\alpha)(\delta + g + n)} > 0$$

With the given parameters, this works out to be

$$t^* = \frac{0.69}{(1 - 0.30)(0.05 + 0.02 + 0.03)} = 9.90$$

(measured in years, if the growth rates are annual growth rates).

Note: does this give the half-life for $k(t)$ too? Or $y(t)$? Not in general, because of the transformation $k(t) = x(t)^{\frac{1}{1-\alpha}}$, the gap $k(t) - k^*$ is not half of $k(0) - k^*$ when $x(t) - x^*$ is half of $x(0) - x^*$. But near the steady state k^* (i.e., for small deviations), the speed of adjustment in $k(t)$ is given by

$$t^* = -\frac{\log 2}{sf'(k^*) - (\delta + g + n)} = \frac{2}{(1-\alpha)(\delta + g + n)}$$

just as for the capital/output ratio.

(e) With $s = 0.22$ the long-run values increase to $k^* = 3.0844$, $y^* = 1.4020$ and $c^* = 1.0936$. This is a 10% increase in the savings rate (from 0.2 to 0.22) leading to an approximately $\frac{1}{1-\alpha} \times 10 = 14\%$ increase in capital (from 2.6918 to 3.0844) and an approximately $\frac{\alpha}{1-\alpha} \times 10 = 4\%$ increase in output (from 1.3459 to 1.4020), as expected from the elasticities in part (b). Similarly with $s = 0.3$ the long-run values increase to $k^* = 4.8040$, $y^* = 1.6013$ and $c^* = 1.0936$. This is a 50% increase in the savings rate (from 0.2 to 0.3) leading to an approximately $\frac{1}{1-\alpha} \times 50 = 71\%$ increase in capital (from 2.6918 to 4.8040) and an approximately $\frac{\alpha}{1-\alpha} \times 50 = 21\%$ increase in output (from 1.3459 to 1.6013). I would say these are relatively small effects on output, even a 50% increase in national savings is only increasing long-run output by about 21%.

Figure 2 below compares the transitional dynamics of $k(t), y(t), c(t)$ for $s = 0.20$, $s = 0.22$ and $s = 0.30$.

2. **Natural resource depletion in the Solow model.** Consider a Solow model where output is given by the CRS production function

$$Y(t) = K(t)^{\alpha} R(t)^{\phi} (A(t)L(t))^{1-\alpha-\phi}, \qquad 0 < \alpha, \phi < 1$$

where $R(t)$ denotes a stock of resources that depletes at rate $\theta > 0$

$$\dot{R}(t) = -\theta R(t)$$

The rest of the model is as standard with constant savings rate s, depreciation rate δ, productivity growth g and employment growth n.

(a) Let $g_Y(t)$ and $g_K(t)$ denote the growth rates of output and the capital stock. Derive a formula for $g_Y(t)$ in terms of $g_K(t)$.

(b) Let g_Y^* and g_K^* denote the growth rates of output and the capital stock along a balanced growth path. Show that along any balanced growth path $g_K^* = g_Y^*$. Solve for this growth rate.

(c) Does the economy necessarily converge to a balanced growth path? Explain.

(d) Now suppose instead that resources $R(t)$ grew in line with population, $\dot{R}(t) = nR(t)$. Compare the long-run growth rate of the economy with resource depletion from part (b) to the long growth rate of this alternative economy without resource depletion. What would make this gap between the growth rates large? Explain.

SOLUTIONS:

(a) Taking logs of the production function

$$\log Y(t) = \alpha \log K(t) + \phi \log R(t) + (1-\alpha-\phi)(\log A(t) + \log L(t))$$

Then differentiating with respect to t

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha\frac{\dot{K}(t)}{K(t)} + \phi\frac{\dot{R}(t)}{R(t)} + +(1-\alpha-\phi)\left(\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right)$$

Plugging in the given growth rates we then have

$$g_Y(t) = \alpha g_K(t) - \phi\theta + (1-\alpha-\phi)(g+n)$$

(b) Since the savings rate is constant we can write

$$\dot{K}(t) = sY(t) - \delta K(t)$$

or

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = s\frac{Y(t)}{K(t)} - \delta$$

Hence along any balanced growth path where capital grows at a constant rate g_K^* we must have

$$g_K^* = s\frac{Y(t)}{K(t)} - \delta$$

That is, along a balanced growth path the capital/output ratio $x(t) \equiv K(t)/Y(t)$ ratio must be constant, in other words output must be growing at the same rate as the capital stock, $g_Y^* = g_K^*$. Let this common growth rate be g^*. From part (a) this g^* satisfies

$$g^* = \alpha g^* - \phi\theta + (1-\alpha-\phi)(g+n)$$

which solves for

$$g^* = \frac{1}{1-\alpha}\left(-\phi\theta + (1-\alpha-\phi)(g+n)\right)$$

Observe that for any t the capital/output ratio $x(t)$ is strictly decreasing in the growth rate of the capital stock

$$x(t) = \frac{K(t)}{Y(t)} = \frac{s}{g_K(t)+\delta}$$

And along a balanced growth path

$$x^* = \frac{s}{g^*+\delta}$$

Importantly, we will have $x(t) > x^*$ if and only if $g_K(t) < g^*$.

NOTE: In what follows we will presume that

$$-\phi\theta + (1-\alpha-\phi)(g+n) + (1-\alpha)\delta > 0$$

so that $g^* + \delta > 0$ even if $g^* < 0$.

(c) From parts (a) and (b) we have

$$g_Y(t) = \alpha g_K(t) - \phi\theta + (1-\alpha-\phi)(g+n)$$

and

$$g^* = \alpha g^* - \phi\theta + (1-\alpha-\phi)(g+n)$$

Taking the difference between these expressions

$$g_Y(t) - g^* = \alpha(g_K(t) - g^*)$$

Using this we can write the growth rate in the capital/output ratio as

$$g_K(t) - g_Y(t) = (g_K(t) - g^*) - (g_Y(t) - g^*) = (1-\alpha)(g_K(t) - g^*)$$

We will now use this calculation to argue that the balanced growth path is stable. To see this, first suppose that $g_K(t) < g^*$. Then $g_K(t) < g_Y(t)$ and since the capital/ output ratio $x(t) = K(t)/Y(t)$ is strictly decreasing in $g_K(t)$ we also know that $x(t) > x^*$ so the capital/output ratio is falling towards the balanced growth path x^* from above. Alternatively, suppose that $g_K(t) > g^*$. Then $g_K(t) > g_Y(t)$ and since the capital/output ratio is strictly decreasing in $g_K(t)$ we also know that $x(t) < x^*$ so the capital/output ratio is rising towards the balanced growth path x^* from below. In this sense, the balanced growth path is stable (i.e., the growth rate is 'mean reverting' towards g^*).

(d) Now let $\theta = -n$. This makes the growth rate

$$\hat{g}^* = \frac{1}{1-\alpha}\left(-\phi\theta + (1-\alpha-\phi)(g+n)\right)\Big|_{\theta=-n}$$

$$= \frac{1}{1-\alpha}\left(\phi n + (1-\alpha-\phi)(g+n)\right)$$

Compare this to the growth rate g^* from part (b)

$$g^* = \frac{1}{1-\alpha}\left(-\phi\theta + (1-\alpha-\phi)(g+n)\right)$$

so that

$$g^* - \hat{g}^* = -\frac{\phi}{1-\alpha}(\theta+n) < 0$$

That is, the growth rate in the economy with resource depletion is less than the growth rate without resource depletion and the size of the gap between the growth rates is

$$\frac{\phi}{1-\alpha}(\theta+n)$$

This gap is large when for example ϕ is large (resources are important in the production function) or when θ is large (resources deplete at a faster rate).

3. **Transitional dynamics in the Ramsey-Cass-Koopmans model.** Suppose the planner seeks to maximize the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t\, u\Big(\frac{C_t}{L}\Big)L, \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$C_t + K_{t+1} = F(K_t, L) + (1-\delta)K_t, \qquad 0 < \delta < 1$$

given initial $K_0 > 0$. The production function has the Cobb-Douglas form

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}, \qquad 0 < \alpha < 1$$

Suppose that productivity $A > 0$ and the labor force $L > 0$ are constant. Let $c_t = C_t/L$, $k_t = K_t/L$, $y_t = Y_t/L$ etc denote consumption, capital, output etc in *per worker* units. Suppose that the period utility function is strictly increasing and strictly concave.

(a) Derive optimality conditions that characterize the solution to the planner's problem. Give intuition for those optimality conditions. Explain how these optimality conditions pin down the dynamics of c_t and k_t.

(b) Solve for the steady state values c^*, k^*, y^* in terms of the parameters. How do these steady state values depend on the level of A?

(c) Suppose the economy is initially in the steady state you found in (b). Then suddenly there is a permanent increase in productivity from A to $A' > A$. Use a phase diagram to explain both the short-run and long-run dynamics of c_t and k_t in response to this increase in productivity. Does c_t increase or decrease? Explain.

Now consider the specific utility function $u(c) = \log(c)$.

(b) Log-linearize the planner's optimality conditions around the steady-state. Guess that in log-deviations capital satisfies

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$$

and that consumption satisfies

$$\hat{c}_t = \psi_{ck}\hat{k}_t$$

Use the method of undetermined coefficients to determine ψ_{kk} and ψ_{ck} in terms of model parameters. How if at all do these depend on the level of A?

Now consider the specific numerical values $\alpha = 0.3$, $\beta = 1/1.05$, $\delta = 0.05$ and $A = 1$.

(c) Calculate the values of ψ_{kk} and ψ_{ck}. Suppose the economy is at steady state when suddenly at $t = 0$ there is a 5% permanent increase in the level of productivity from $A = 1$ to $A' = 1.05$. Calculate the transitional dynamics of the economy as it adjusts to its new long run values. In particular, calculate and plot the time-paths of capital, output, and consumption until they have converged to their new steady state levels.

(e) How if at all would your answers to parts (b) through (d) change if σ was lower, say $\sigma = 0.5$? Or higher, say $\sigma = 2$? Give intuition for your answers.

Solutions:

(a) The planner's problem is to maximize intertemporal utility (per worker)

$$\sum_{t=0}^{\infty} \beta^t \, u(c_t)$$

subject to the sequence of resource constraints

$$c_t + k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t$$

Setting up the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [Ak_t^{\alpha} + (1-\delta)k_t - c_t - k_{t+1}]$$

The key first order conditions for this problem are, for consumption,

$$c_t : \qquad \beta^t u'(c_t) = \lambda_t$$

and for capital,

$$k_{t+1} : \qquad \lambda_t = \lambda_{t+1}\left[\alpha A k_{t+1}^{\alpha-1} + 1 - \delta\right]$$

and for the multipliers,

$$\lambda_t : \qquad c_t + k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t$$

We also have the transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

Eliminating the multipliers λ_t gives the consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})[\alpha A k_{t+1}^{\alpha-1} + 1 - \delta]$$

The consumption Euler equation and the resource constraint are two nonlinear difference equations in c_t, k_t. To pin down the dynamics of c_t, k_t we also need two boundary conditions. One of these is the given initial condition $k_0 > 0$. The other is the transversality condition given above.

(b) In a steady state with $c_t = c_{t+1} = c^*$ the consumption Euler equation implies

$$1 = \beta[\alpha A k^{*\,\alpha-1} + 1 - \delta]$$

which can be solved to get steady state capital per worker

$$k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}}, \qquad \rho \equiv \frac{1}{\beta} - 1 > 0$$

Steady state output per worker is then found from the production function

$$y^* = Ak^{*\,\alpha} = A\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{\alpha}{1-\alpha}} = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}}$$

so that the capital/output ratio is

$$\frac{k^*}{y^*} = \frac{\alpha}{\rho + \delta}$$

and hence the consumption/output ratio is

$$\frac{c^*}{y^*} = 1 - \delta\frac{k^*}{y^*} = 1 - \delta\frac{\alpha}{\rho + \delta} = \frac{\rho + (1-\alpha)\delta}{\rho + \delta}$$

Steady state output per worker is therefore

$$c^* = \left(\frac{\rho + (1-\alpha)\delta}{\rho + \delta}\right)\left(\frac{\alpha}{\rho + \delta}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}}$$

Notice that the long run levels c^*, k^*, y^* are all proportional to the level of productivity via $A^{\frac{1}{1-\alpha}}$ but the long run ratios c^*/y^*, k^*/y^* are independent of productivity.

(c) Using the expressions in part (b) above, it is clear that an increase in the level of productivity from A to $A' > A$ increases steady state consumption from c^* to $c^{*\prime}$, say, increases steady state capital from k^* to $k^{*\prime}$, say, and increases steady state output from y^* to $y^{*\prime}$. To see this in a phase diagram, first note that an increase in A shifts the $\Delta c = 0$ locus to the right and shifts up the $\Delta k = 0$ locus (i.e., the curve $C(k) = Ak^\alpha - \delta k$ shifts up). Thus in the long run consumption, output and capital per worker all increase.

Relative to these new steady state levels the economy 'begins' with initial capital per worker $k_0 = k^* < k^{*\prime}$. On 'impact' the level of consumption immediately jumps *up* to $c(0) > c^*$ on the new stable arm going through the new steady state. As discussed below, this new stable arm is approximately *parallel* to the old stable arm (going through the old steady state). The level of output also jumps up on impact because of the change in productivity. Capital does not jump on impact because it is predetermined. On impact, consumption jumps by less than the jump in output with the difference being saved. This increase in savings/investment is what allows the economy to build up a new higher level of capital in the long run. As the economy transitions to its new long run, consumption and output continue to rise with the new higher levels of capital.

(d) Proceeding as in Lecture 5 slides 8–11 we have the log-linearized resource constraint

$$c^*\hat{c}_t + k^*\hat{k}_{t+1} = \frac{1}{\beta}k^*\hat{k}_t$$

and the log-linearized consumption Euler equation

$$\hat{c}_{t+1} = \hat{c}_t + \beta\frac{f''(k^*)k^*}{\sigma}\hat{k}_{t+1}$$

Plugging in our guesses and rearranging terms in this version of the resource constraint

$$\left[c^*\psi_{ck} + k^*\psi_{kk} - \frac{1}{\beta}k^*\right]\hat{k}_t = 0$$

This has to hold for any $\hat{k}_t$ hence we must have

$$c^*\psi_{ck} + k^*\psi_{kk} - \frac{1}{\beta}k^* = 0$$

or

$$\boxed{\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*}}$$

Likewise plugging in our guesses and rearranging terms in the consumption Euler equation

$$\left[\psi_{ck}\psi_{kk} - \psi_{ck} - \beta\frac{f''(k^*)k^*}{\sigma}\psi_{kk}\right]\hat{k}_t = 0$$

This has to hold for any $\hat{k}_t$ hence we must also have

$$\boxed{\psi_{ck}\psi_{kk} - \psi_{ck} - \beta\frac{f''(k^*)k^*}{\sigma}\psi_{kk} = 0}$$

Combining the two expressions in boxes gives a familiar looking quadratic in ψ_{kk}

$$\psi_{kk}^2 - \left(1 + \frac{1}{\beta} - \beta\frac{f''(k^*)c^*}{\sigma}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

The roots of this quadratic *are* the eigenvalues of this dynamic system. There is one stable and one unstable root. Let ψ_{kk} denote the stable root. We can then recover ψ_{ck} from the first boxed equation. To see how ψ_{kk} and ψ_{ck} depend on productivity A, let's use the given functional forms. We have $u(c) = \log c$ so $\sigma = 1$. And we have $f(k) = Ak^\alpha$ so $f''(k) = \alpha(\alpha-1)Ak^{\alpha-2}$ so that $f''(k)k = \alpha(\alpha-1)Ak^{\alpha-1} = (\alpha-1)f'(k)$. So we can write the quadratic

$$\psi_{kk}^2 - \left(1 + \frac{1}{\beta} - \beta(\alpha-1)f'(k^*)\frac{c^*}{k^*}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

Moreover from part (b) we know that in steady state

$$f'(k^*) = \rho + \delta$$

and

$$\frac{c^*}{k^*} = \frac{c^*/y^*}{k^*/y^*} = \frac{\rho + (1-\alpha)\delta}{\alpha}$$

Hence the quadratic does not depend on the level of productivity A and so A will not affect the eigenvalues of this system. Let ψ_{kk} denote the stable eigenvalue of this system. The stable arm of the system is then given by

$$\hat{c}_t = \psi_{ck}\hat{k}_t$$

with slope

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*}$$

Since the stable eigenvalue ψ_{kk} is independent of productivity A and the consumption/capital ratio c^*/k^* is independent of A so too is the *slope* of the stable arm ψ_{ck} independent of A.

But note the *level* of the stable arm depend on the steady state value. That is, writing things in log-levels as opposed to log-deviations

$$\log c_t = \log c^* + \psi_{ck}(\log k_t - \log k^*)$$

While the slope ψ_{ck} is independent of A, the level of the stable arm does depend on A via the steady state terms. After all, as we saw in part (b) above, an increase in A increases both c^* and k^*. In this sense, the new stable arm going through the new steady state with productivity level $A' > A$ approximately parallel to the old stable arm. Because of this, when the shock from A to A' hits the economy, consumption jumps *up* on impact.

(e) With these parameter values we get $k^* = 4.8040$, $y^* = 1.6013$ and $c^* = 1.3611$ and the roots of the quadratic are 0.8928 and 1.1761 so we set $\psi_{kk} = 0.8928$. The slope of the stable arm is then

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*} = (1.05 - .8928)\,\frac{4.8040}{1.3611} = 0.5550$$

With the shift to $A' = 1.05$ the new steady state values are $k^{*\prime} = 5.1508$, $y^{*\prime} = 1.7169$ and $c^{*\prime} = 1.4594$. Relative to these *new* steady state values our initial condition is the old steady state $k_0 = k^* = 4.8040$ so our initial log deviation is $\log(4.8040/5.1508) = -0.0697$, i.e., we begin about 7% below the *new* steady state. Figure 3 below shows the transitional dynamics to this new steady state. But note that this means consumption immediately jumps above the *old* steady state level, an initial log deviation of $\hat{c}_0 = -0.0387$, about 4% below the new steady state consumption, is equivalent to a consumption level of $c_0 = \exp(\hat{c}_0)c^{*\prime} = 1.4040$ which is a jump up from the old steady state of $c^* = 1.3611$, indeed it is a jump up of about 3% in consumption in response to the 5% increase in productivity.

(f) If $\sigma = 0.5$, consumption is *more substitutable* over time — i.e., the intertemporal elasticity of substitution is relatively high, $1/\sigma = 2$. In this case, the consumption smoothing motive is *weak* and the planner instead transitions the economy to its new steady state (which is the same as in part (e), since these values don't depend on σ) more quickly. To be specific, we now get $\psi_{kk} = 0.84$ and the convergence to the new steady state is faster, as shown in Figure 4.

On the other hand, if $\sigma = 2$, consumption is *more complementary* over time — i.e., the intertemporal elasticity of substitution is relatively low, $1/\sigma = 0.5$. In this case, the consumption smoothing motive is *strong* and the planner smooths consumption over a longer period than in parts (c) and (d). To be specific, we now get $\psi_{kk} = 0.93$ and the convergence to the new steady state is slower, as shown in Figure 5.

To summarize, while σ is irrelevant for the steady state values, the magnitude of σ plays an important role in determining the transitional dynamics of the economy around steady state. Even though the $\sigma = 0.5$ and $\sigma = 2$ cases have the same steady state, the former will generally be closer to steady state since its transitions are relatively quick while the latter can exhibit long, persistent deviations from steady state since its transitions are relatively slow.

Figure 1: Savings rate $s = 0.20$

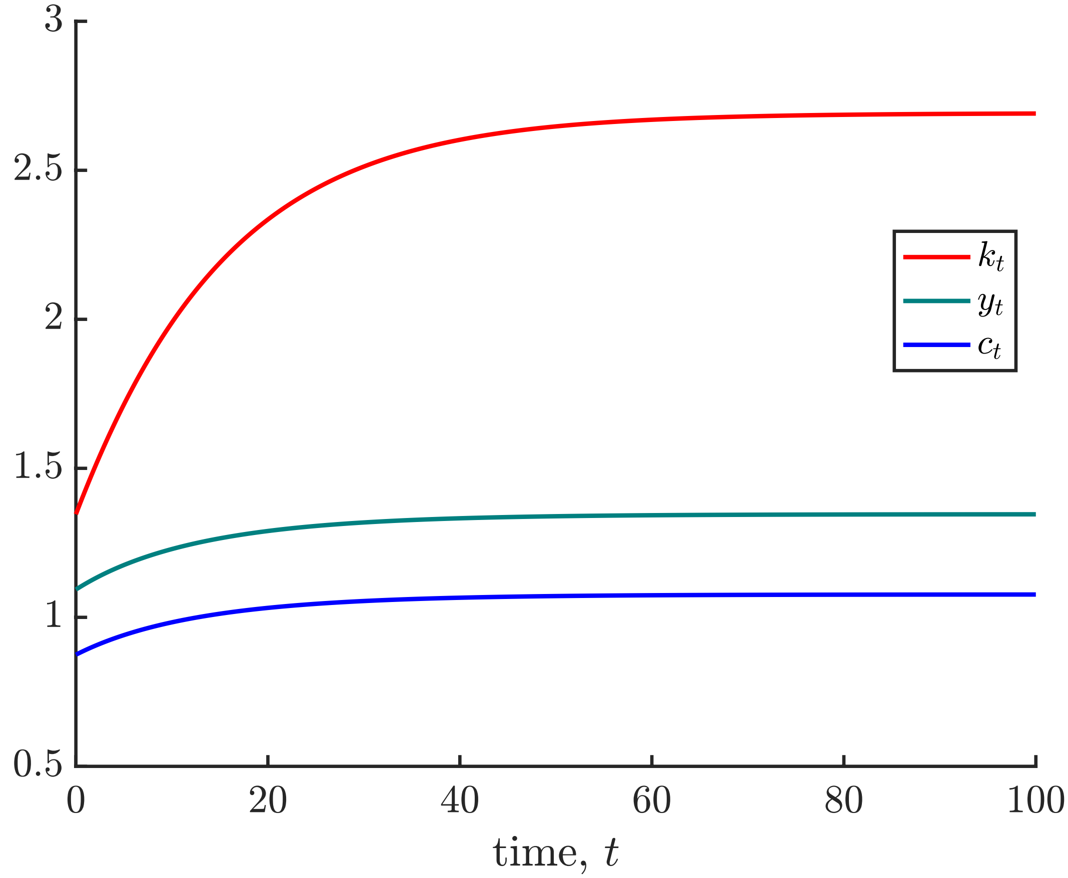

Figure 2: Savings rates compared

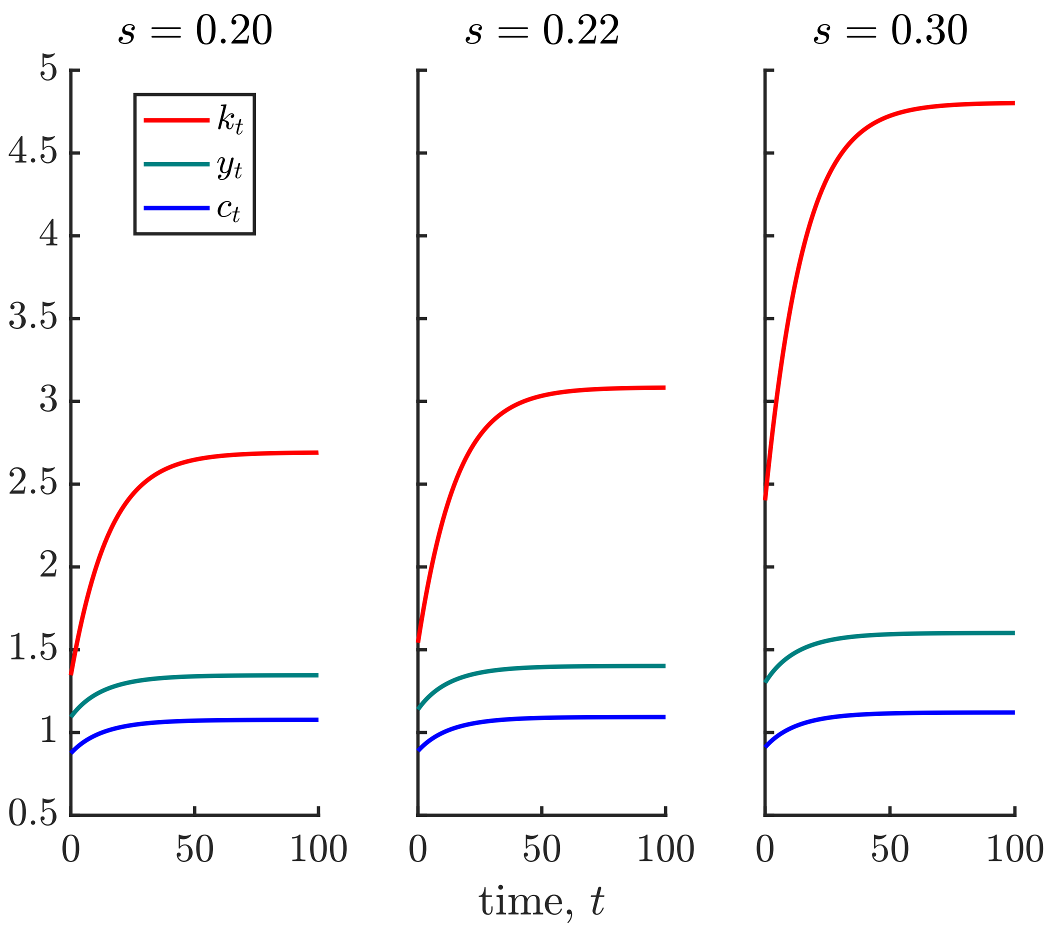

Figure 3: $\sigma = 1$ giving $\psi_{kk} = 0.89, \psi_{ck} = 0.56$

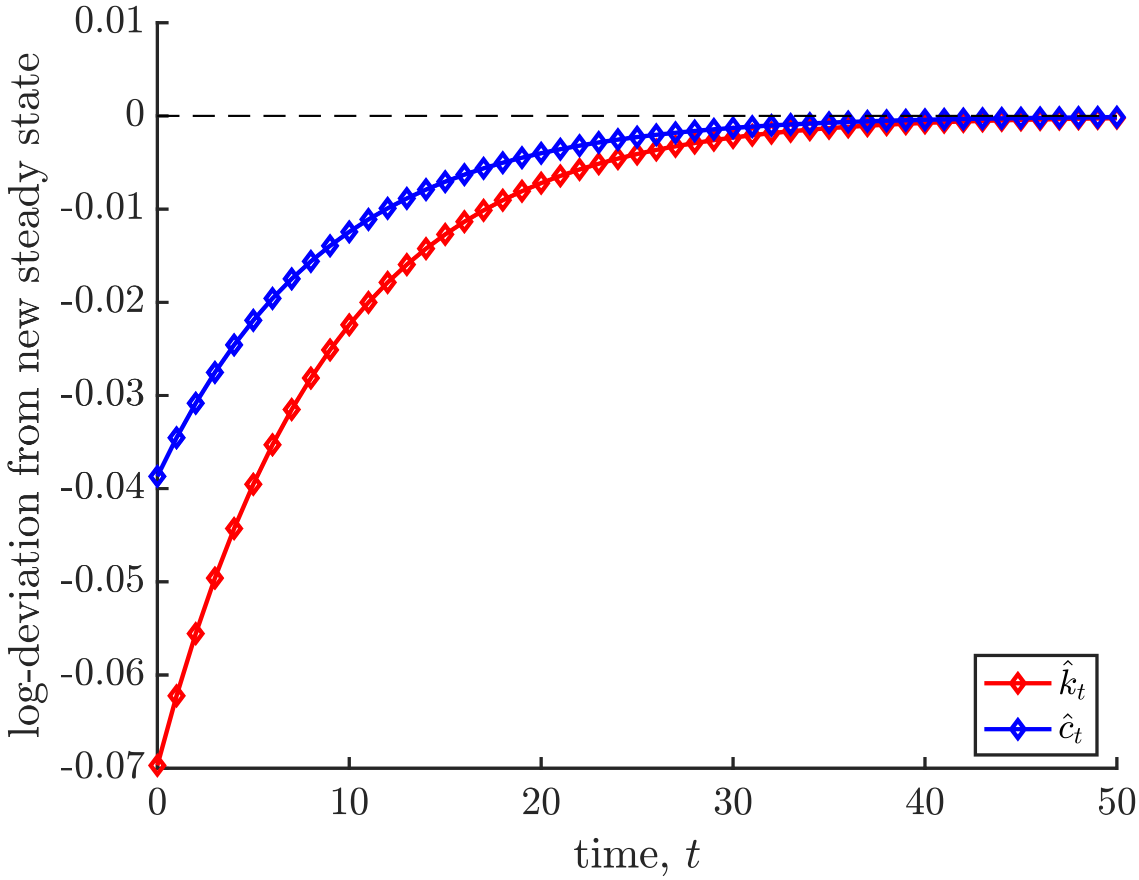

Figure 4: $\sigma = 0.5$ giving $\psi_{kk} = 0.84, \psi_{ck} = 0.72$

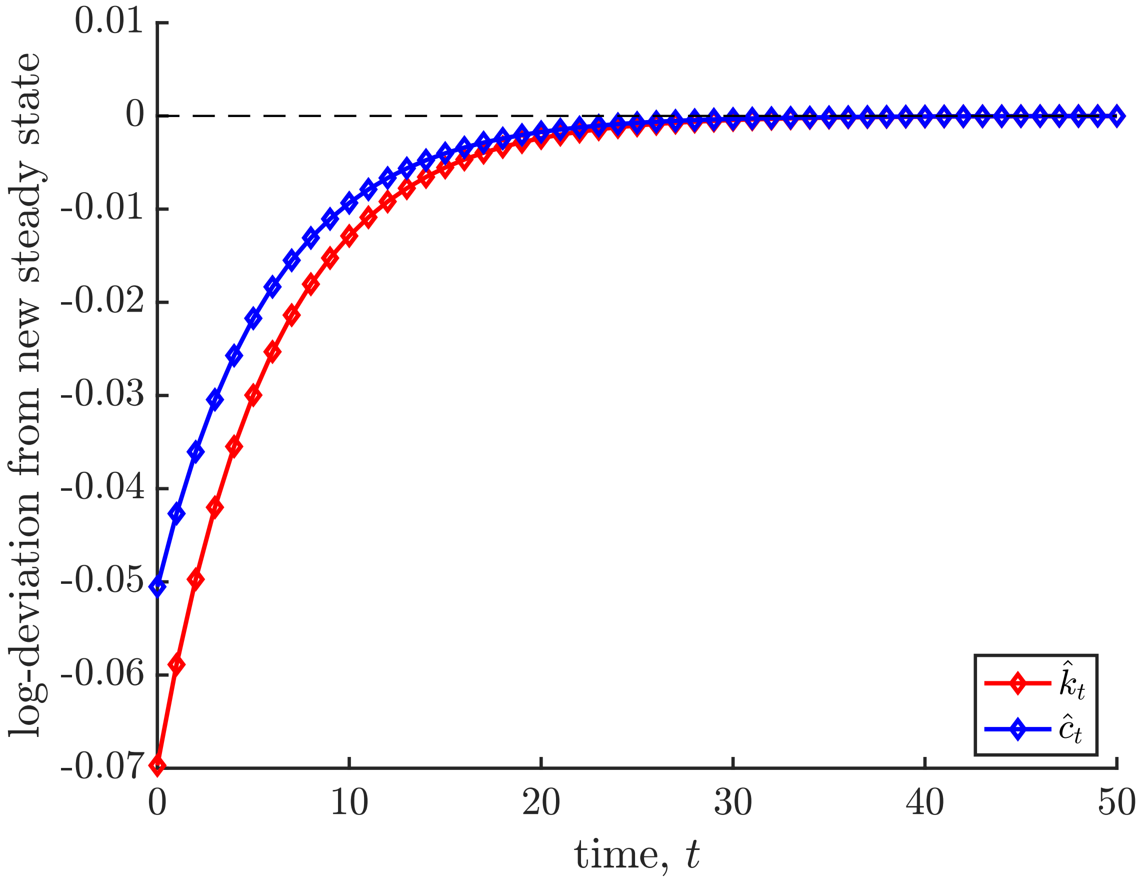

Figure 5: $\sigma = 2$ giving $\psi_{kk} = 0.93, \psi_{ck} = 0.43$