Advanced Microeconomic Theory

Chapter 6: Partial and General Equilibrium

Outline

- Partial Equilibrium Analysis
- General Equilibrium Analysis
- Comparative Statics
- Welfare Analysis

Partial Equilibrium Analysis

- In a competitive equilibrium (CE), all agents must select an optimal allocation given their resources:
  - Firms choose profit-maximizing production plans given their technology;
  - Consumers choose utility-maximizing bundles given their budget constraint.
- A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans to coincide with the firms' production decision.

Partial Equilibrium Analysis

- ***Firm***:
  - Given the price vector p^*, firm j's equilibrium output level q_j^* must solve
$$\max_{q_j \geq 0} \quad p^* q_j - c_j(q_j)$$
which yields the necessary and sufficient condition
$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$
  - That is, every firm j produces until the point in which its marginal cost, $c_j'(q_j^*)$, coincides with the current market price.

Partial Equilibrium Analysis

- ***Consumers***:
  - Consider a quasilinear utility function

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

    where m_i denotes the numeraire, and $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ for all $x_i > 0$.
  - Normalizing, $\phi_i(0) = 0$. Recall that with quasilinear utility functions, the wealth effects for all non-numeraire commodities are zero.

Partial Equilibrium Analysis

– Consumer i's UMP is

$$\max_{m_i \in \mathbb{R}_+,\, x_i \in \mathbb{R}_+} \quad m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad \underbrace{m_i + p^* x_i}_{\text{Total expend.}} \leq \underbrace{w_{m_i} + \sum_{j=1}^{J} \theta_{ij} (\underbrace{p^* q_j^* - c_j(q_j^*)}_{\text{Profits}})}_{\text{Total resources (endowment+profits)}}$$

– The budget constraint must hold with equality (by Walras' law). Hence,

$$m_i = -p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right) \right]$$

Partial Equilibrium Analysis

– Substituting the budget constraint into the objective function,

$$\max_{x_i \in \mathbb{R}_+} \quad \phi_i(x_i) - p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right) \right]$$

– FOCs wrt x_i yields

$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$

– That is, consumer increases the amount he buys of good x until the point in which the marginal utility he obtains exactly coincides with the market price he has to pay for it.

Partial Equilibrium Analysis

– Hence, an allocation $(x_1^*, x_2^*, \dots, x_I^*, q_1^*, q_2^*, \dots, q_J^*)$ and a price vector $p^* \in \mathbb{R}^L$ constitute a CE if:

$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$

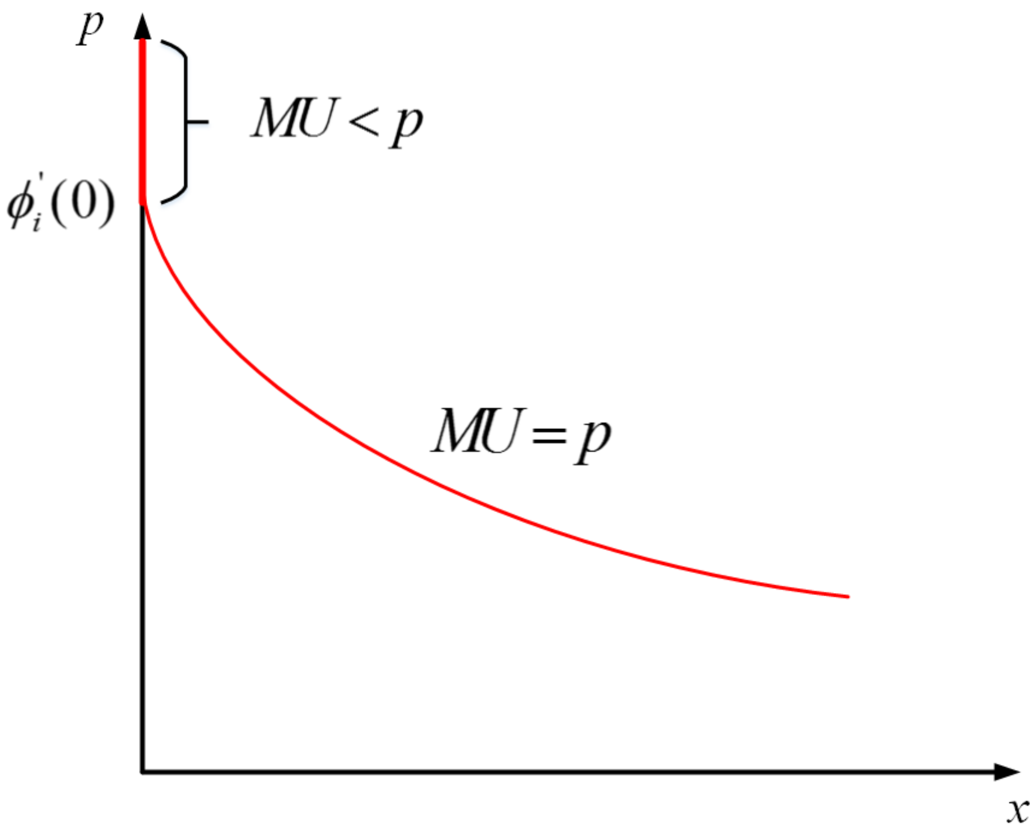
$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

– Note that the these conditions do not depend upon the consumer's initial endowment.

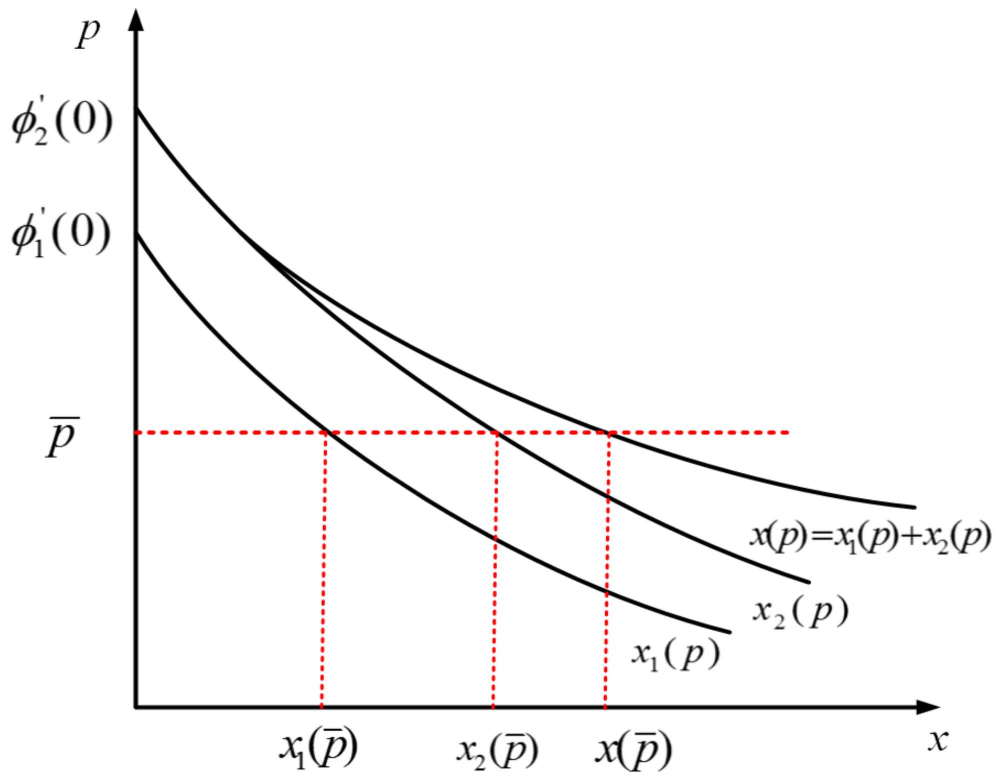
Partial Equilibrium Analysis

- The individual demand curve, where $\phi_i'(x_i^*) \leq p^*$



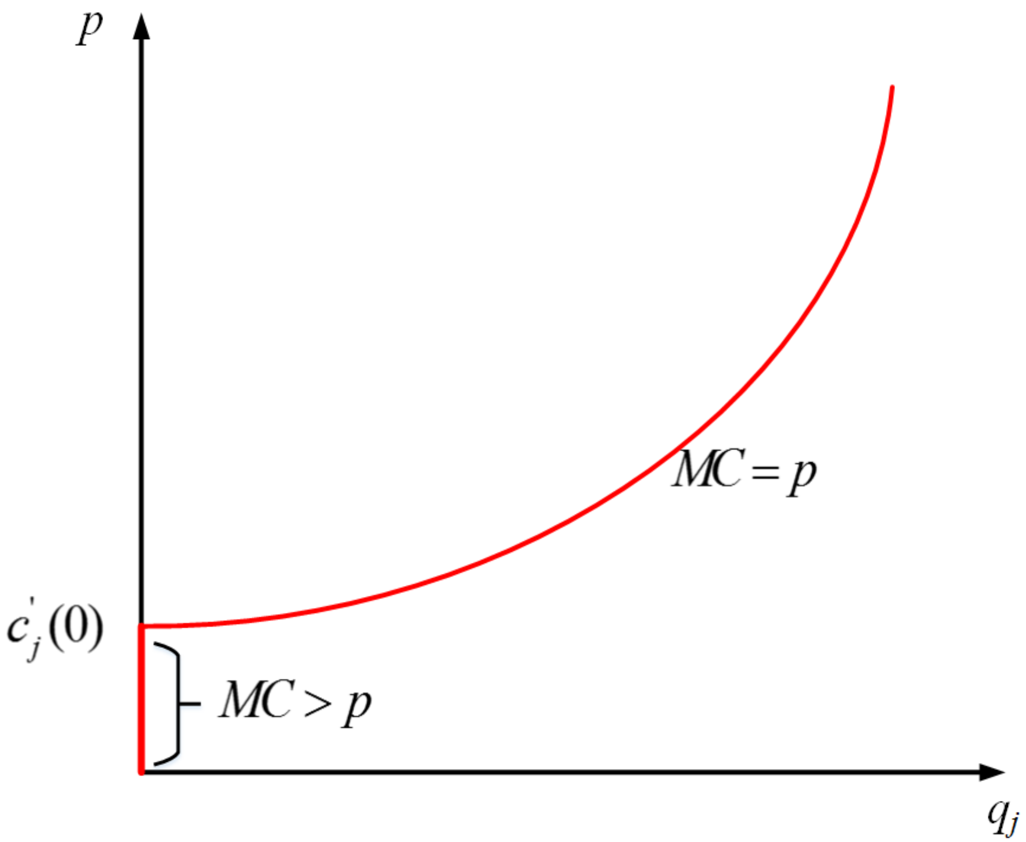
Partial Equilibrium Analysis

- Horizontally summing individual demand curves yields the aggregate demand curve.



Partial Equilibrium Analysis

- The individual supply curve, where $p^* \leq c_j'(q_j^*)$



Partial Equilibrium Analysis

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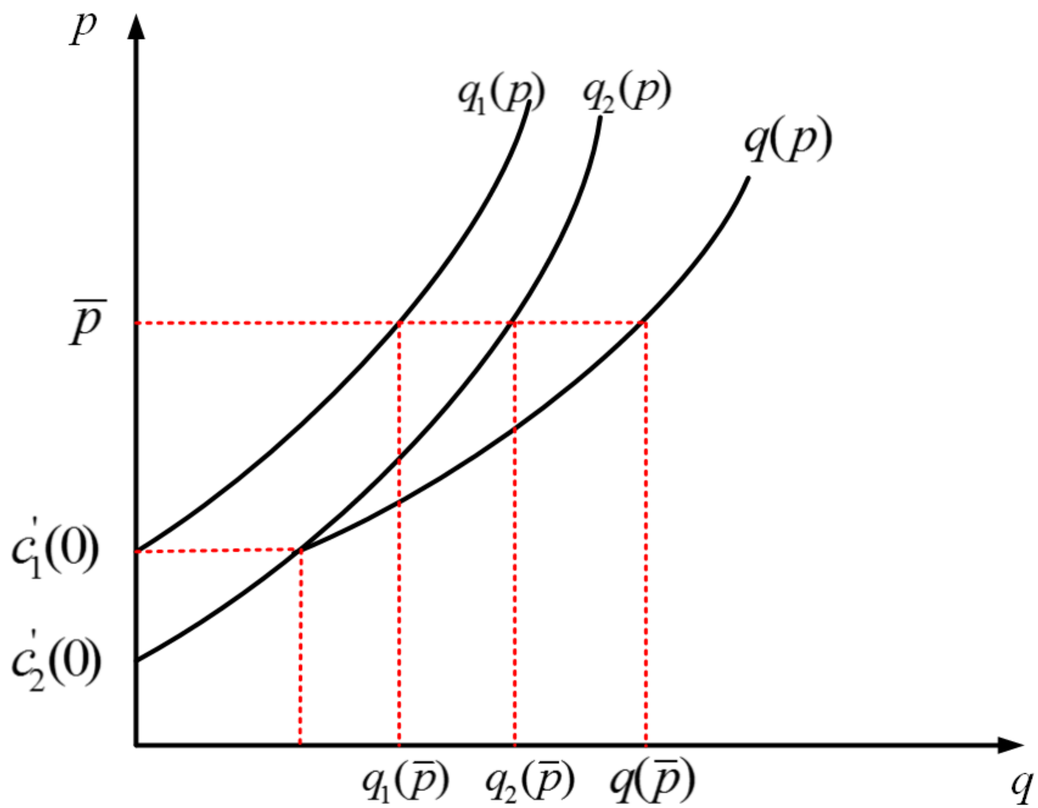
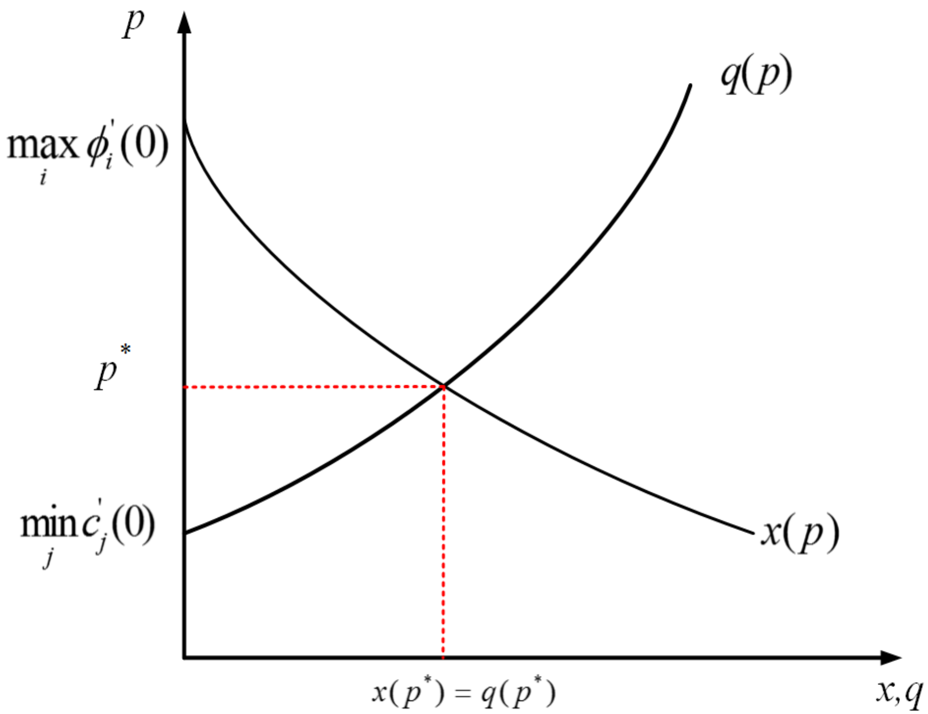


Partial Equilibrium Analysis

- Superimposing aggregate demand and aggregate supply curves, we obtain the CE allocation of good x.

- To guarantee that a CE exists, the equilibrium price p^* must satisfy

$$\max_i \phi_i'(0) \geq p^* \geq \min_j c_j'(0)$$



Partial Equilibrium Analysis

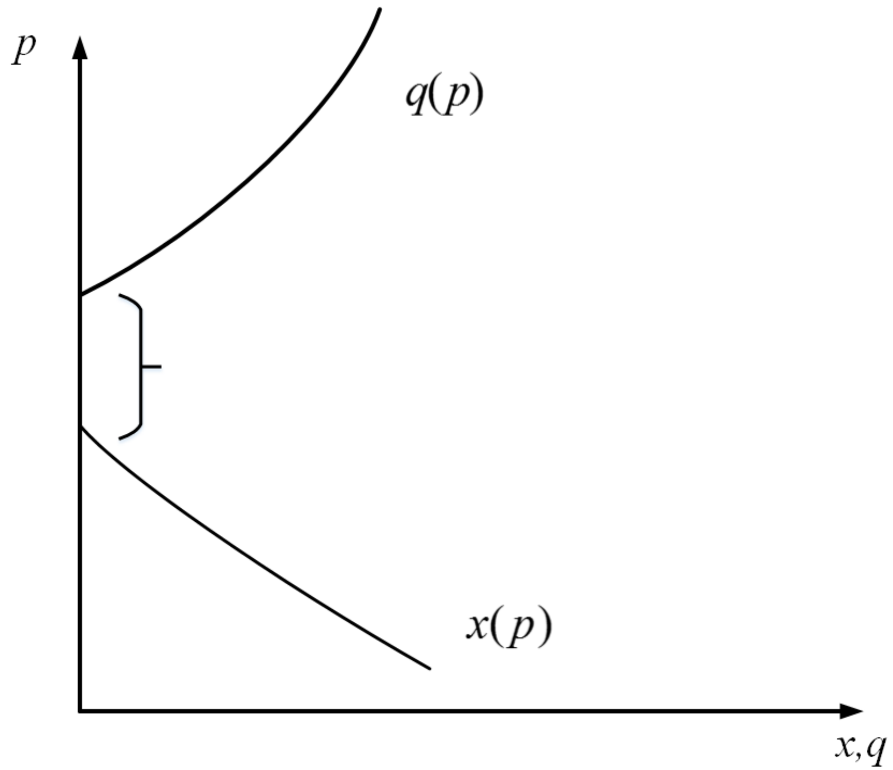
- Also, since $\phi_i'(x_i)$ is downward sloping in x_i, and $c_j'(q_i)$ is upward sloping in q_i, then aggregate demand and supply cross at a unique point.
  – Hence, the CE allocation is unique.

Partial Equilibrium Analysis

- If we have

  $$\max_i \phi_i'(0) < \min_j c_j'(0),$$

  then there is *no* positive production or consumption of good x.



Partial Equilibrium Analysis

- ***Example 6.1***:
  - Assume a perfectly competitive industry consisting of two types of firms: 100 firms of type *A* and 30 firms of type *B.*
  - Short-run supply curve of type *A* firm is
$$s_A(p) = 2p$$
  - Short-run supply curve of type *B* firm is
$$s_B(p) = 10p$$
  - The Walrasian market demand curve is
$$x(p) = 5000 - 500p$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Summing the individual supply curves of the 100 type-*A* firms and the 30 type-*B* firms,
    $$S(p) = 100 \cdot 2p + 30 \cdot 10p = 500p$$
  - The short-run equilibrium occurs at the price at which quantity demanded equals quantity supplied,
    $$5000 - 500p = 500p, \text{ or } p = 5$$
  - Each type-*A* firm supplies: $s_A(p) = 2 \cdot 5 = 10$
  - Each type-*B* firm supplies: $s_B(p) = 10 \cdot 5 = 50$

Comparative Statics

Comparative Statics

- Let us assume that the consumer's preferences are affected by a vector of parameters $\alpha \in \mathbb{R}^M$, where $M \leq L$.
  - Then, consumer i's utility from good x is $\phi_i(x_i, \alpha)$.
- Similarly, firms' technology is affected by a vector of parameters $\beta \in \mathbb{R}^S$, where $S \leq L$.
  - Then, firm j's cost function is $c_j(q_j, \beta)$.

- Notation:
  - $\hat{p}_i(p, t)$ is the effective price paid by the consumer
  - $\hat{p}_j(p, t)$ is the effective price received by the firm
  - Per unit tax: $\hat{p}_i(p, t) = p + t$.
    - Example: $t = \$2$, regardless of the price p
  - Ad valorem tax (sales tax): $\hat{p}_i(p, t) = p + pt = p(1 + t)$
    - Example: $t = 0.1$ (10%).

Comparative Statics

- If consumption and production are strictly positive in the CE, then

$$\phi_i'(x_i^*, \alpha) = \hat{p}_i(p^*, t) \text{ for every consumer } i$$

$$c_j'(q_j^*, \beta) = \hat{p}_j(p^*, t) \text{ for every firm } j$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Then we have $I + J + 1$ equations, which depend on parameter values α, β and t.
- In order to understand how x_i^* or q_j^* depends on parameters α and β, we can use the ***Implicit Function Theorem***.
  - The above functions have to be differentiable.

Comparative Statics

- ***Implicit Function Theorem***:
  - Let $u(x, y)$ be a utility function, where x and y are amounts of two goods.
  - If $\frac{\partial u(\bar{x},\bar{y})}{\partial x} \neq 0$ when evaluated at $(\bar{x}, \bar{y})$, then

$$\frac{\partial u(\bar{x},\bar{y})}{\partial x} dx + \frac{\partial u(\bar{x},\bar{y})}{\partial y} dy = 0$$

which yields

$$\frac{dy(\bar{x})}{dx} = -\frac{\frac{\partial u(\bar{x},\bar{y})}{\partial x}}{\frac{\partial u(\bar{x},\bar{y})}{\partial y}}$$

Comparative Statics

– Similarly, if $\frac{\partial u(\bar{x},\bar{y})}{\partial y} \neq 0$ when evaluated at $(\bar{x}, \bar{y})$, then

$$\frac{dx(\bar{y})}{dy} = -\frac{\frac{\partial u(\bar{x},\bar{y})}{\partial y}}{\frac{\partial u(\bar{x},\bar{y})}{\partial x}}$$

for all $(\bar{x}, \bar{y})$.

Comparative Statics

– Similarly, if $u(x, \alpha)$ describes the consumption of a single good x,where α determines the consumer's preference for x, and $\frac{\partial u(x,\alpha)}{\partial \alpha} \neq 0$, then

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\frac{\partial u(x,\alpha)}{\partial \alpha}}{\frac{\partial u(x,\alpha)}{\partial x}}$$

– The left-hand side is unknown

– The right-hand side is, however, easier to find.

Comparative Statics

- ***Sales tax*** (Example 6.2):
  - The expression of the aggregate demand is now $x(p+t)$, because the effective price that the consumer pays is actually $p+t$.
  - In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy
$$x(p^*(t)+t) = q(p^*(t))$$
  - if the sales tax is marginally increased, and functions are differentiable at $p = p^*(t)$,
$$x'(p^*(t)+t) \cdot (p^{*\prime}(t)+1) = q'(p^*(t)) \cdot p^{*\prime}(t)$$

Comparative Statics

– Rearranging, we obtain

$$p^{*\prime}(t) \cdot \left[x'(p^*(t) + t) - q'\big(p^*(t)\big)\right]$$
$$= -x'(p^*(t) + t)$$

– Hence,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\big(p^*(t)\big)}$$

– Since $x(p)$ is decreasing in prices, $x'(p^*(t) + t) < 0$, and $q(p)$ is increasing in prices, $q'\big(p^*(t)\big) > 0$,

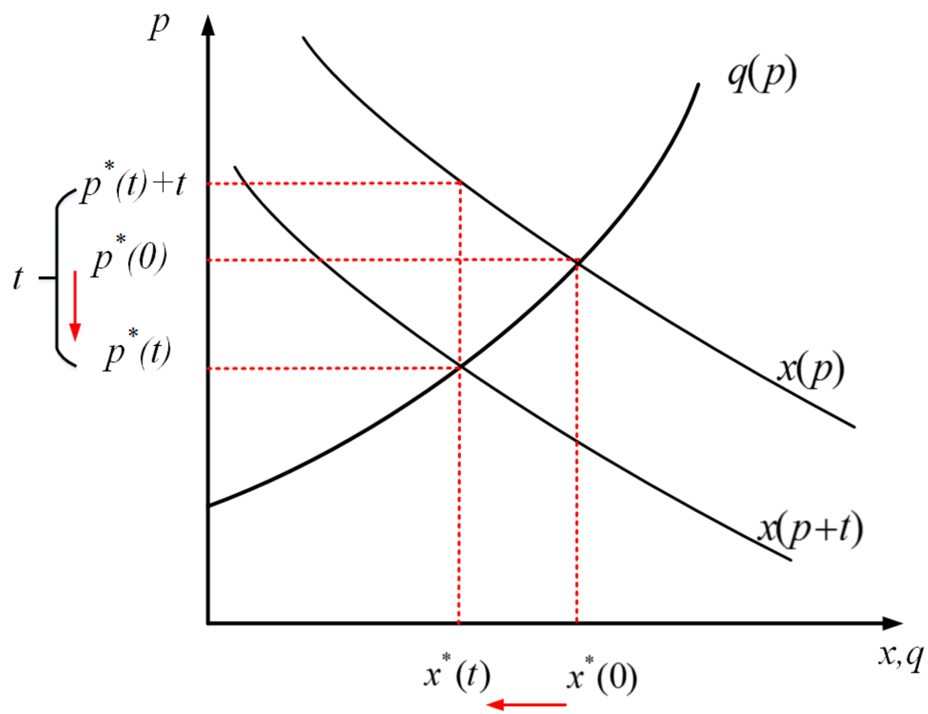
$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{\underbrace{x'(p^*(t)+t)}_{-}-\underbrace{q'\big(p^*(t)\big)}_{+}} = -\frac{-}{-} = -$$

Comparative Statics

- Hence, $p^{*\prime}(t) < 0$.
- Moreover, $p^{*\prime}(t) \in (-1,0]$.
- Therefore, $p^*(t)$ decreases in t.
  - That is, the price received by producers falls in the tax, but less than proportionally.
- Additionally, since $p^*(t) + t$ is the price paid by consumers, then $p^{*\prime}(t) + 1$ is the marginal increase in the price paid by consumers when the tax marginally increases.
  - Since $p^{*\prime}(t) \geq 1$, then $p^{*\prime}(t) + 1 \geq 0$, and consumers' cost of the product also raises less than proportionally.

Comparative Statics

- *No tax:*
  - CE occurs at $p^*(0)$ and $x^*(0)$
- *Tax:*
  - x^* decreases from $x^*(0)$ to $x^*(t)$
  - Consumers now pay $p^*(t) + t$
  - Producers now receive $p^*(t)$ for the $x^*(t)$ units they sell.



Comparative Statics

- ***Sales Tax*** (Extreme Cases):
  a) *The supply is very responsive to price changes, i.e., $q'\left(p^*(t)\right)$ is large.*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\left(p^*(t)\right)} \to 0$$

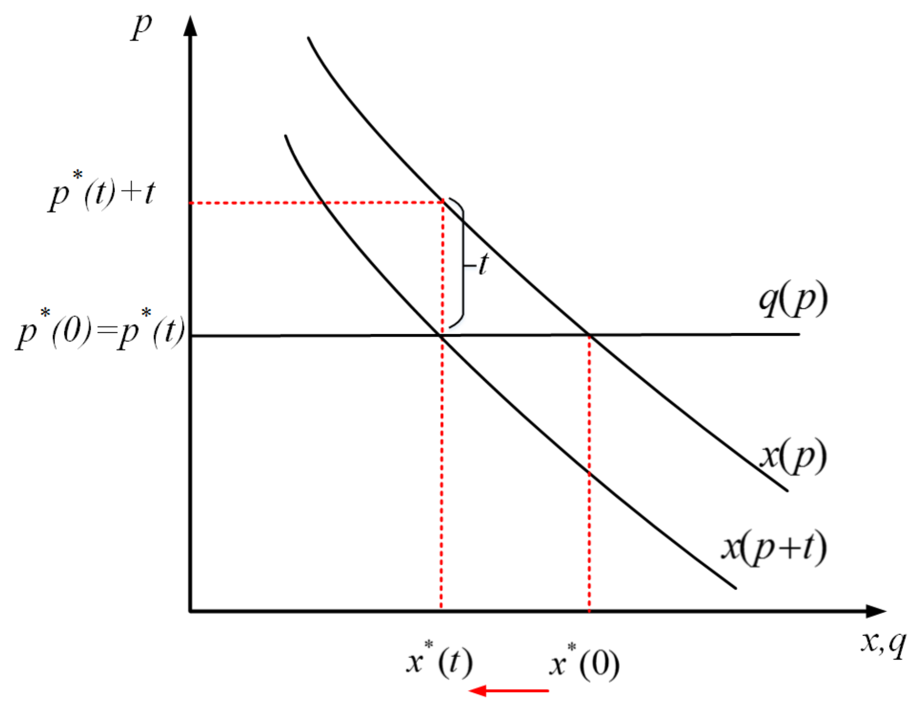
  - Therefore, $p^{*\prime}(t) \to 0$, and the price received by producers does not fall.
  - However, consumers still have to pay $p^*(t) + t$.
    - A marginal increase in taxes therefore provides an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = 0 + 1 = 1$$

    - The tax is solely borne by consumers.

Comparative Statics

- A very elastic supply curve
  - The price received by producers almost does not fall.
  - But, the price paid by consumers increases by exactly the amount of the tax.



Comparative Statics

b) *The supply is not responsive to price changes, i.e., $q'\left(p^*(t)\right)$ is close to zero.*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\left(p^*(t)\right)} = -1$$

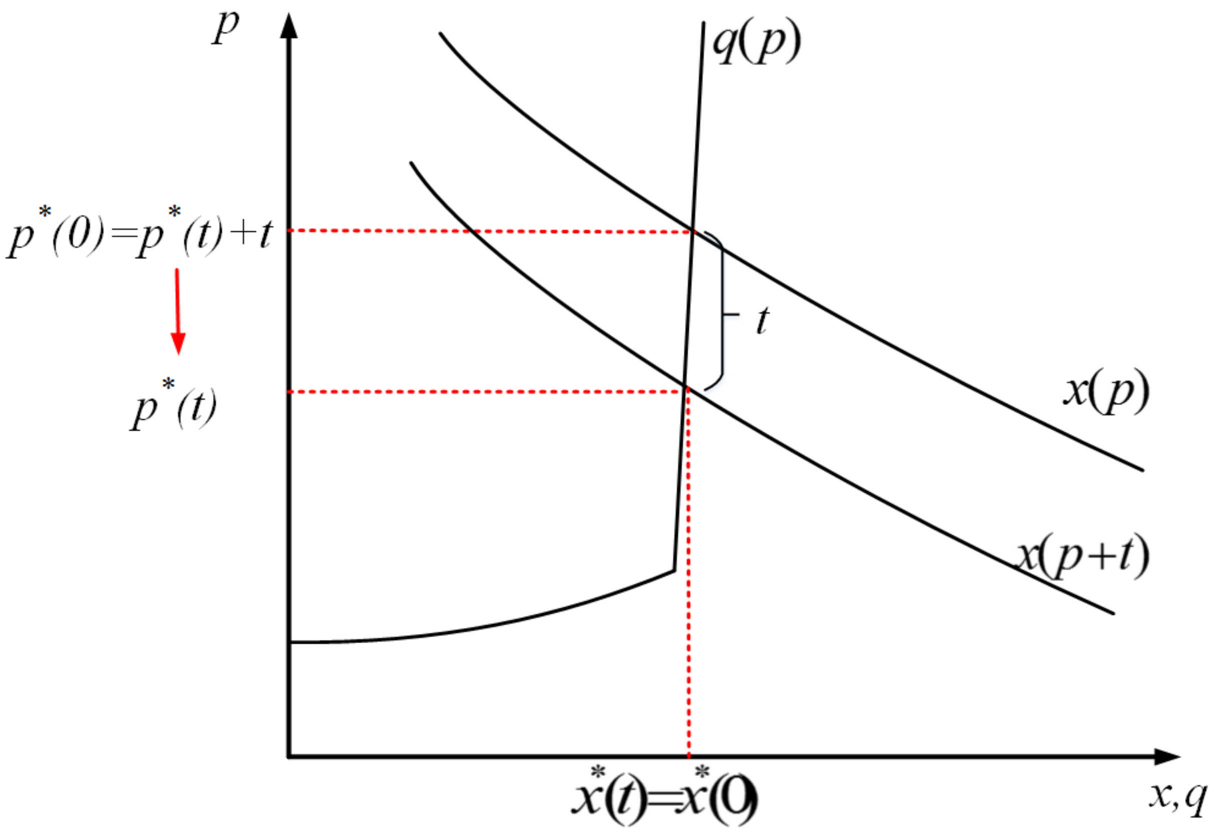
- Therefore, $p^{*\prime}(t) = -1$, and the price received by producers falls by $1 for every extra dollar in taxes.
  - Producers bear most of the tax burden
- In contrast, consumers pay $p^*(t) + t$
  - A marginal increase in taxes produces an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = -1 + 1 = 0$$

  - Consumers do not bear tax burden at all.

Comparative Statics

- Inelastic supply curve



Comparative Statics

- ***Example 6.3***:
  - Consider a competitive market in which the government will be imposing an ad valorem tax of t.
  - Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve is $q(p) = ap^{\gamma}$, where $a > 0$ and $\gamma > 0$.
  - Let us evaluate how the equilibrium price is affected by a marginal increase in the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - The change in the price received by producers at $t = 0$ is

$$p^{*\prime}(0) = -\frac{x'(p^*)}{x'(p^*) - q'(p^*)}$$
$$= -\frac{A\varepsilon p^{*\varepsilon-1}}{A\varepsilon p^{*\varepsilon-1} - a\gamma p^{*\gamma-1}} = -\frac{A\varepsilon p^{*\varepsilon}}{A\varepsilon p^{*\varepsilon} - a\gamma p^{*\gamma}}$$
$$= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}$$

  - The change in the price paid by consumers at $t = 0$ is

$$p^{*\prime}(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}$$

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\gamma = 0$ (i.e., supply is perfectly inelastic), the price paid by consumers in unchanged, and the price received by producers decreases be the amount of the tax.
    - That is, producers bear the full effect of the tax.
  - When $\varepsilon = 0$ (i.e., demand is perfectly inelastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.
    - That is, consumers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\varepsilon \to -\infty$ (i.e., demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax.
  - When $\gamma \to +\infty$(i.e., supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.

Welfare Analysis

Welfare Analysis

- Let us now measure the changes in the aggregate social welfare due to a change in the competitive equilibrium allocation.
- Consider the aggregate surplus

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

- Take a differential change in the quantity of good k that individuals consume and that firms produce such that $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$.
- The change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} \phi_i'(x_i) dx_i - \sum_{j=1}^{J} c_j'(q_j) dq_j$$

Welfare Analysis

- Since
  - $\phi_i'(x_i) = P(x)$ for all consumers; and
    - That is, every individual consumes until MB=p.
  - $c_j'(q_j) = C'(q)$ for all firms
    - That is, every firm's MC coincides with aggregate MC)

then the change in surplus can be rewritten as

$$dS = \sum_{i=1}^{I} P(x) dx_i - \sum_{j=1}^{J} C'(q) dq_j$$

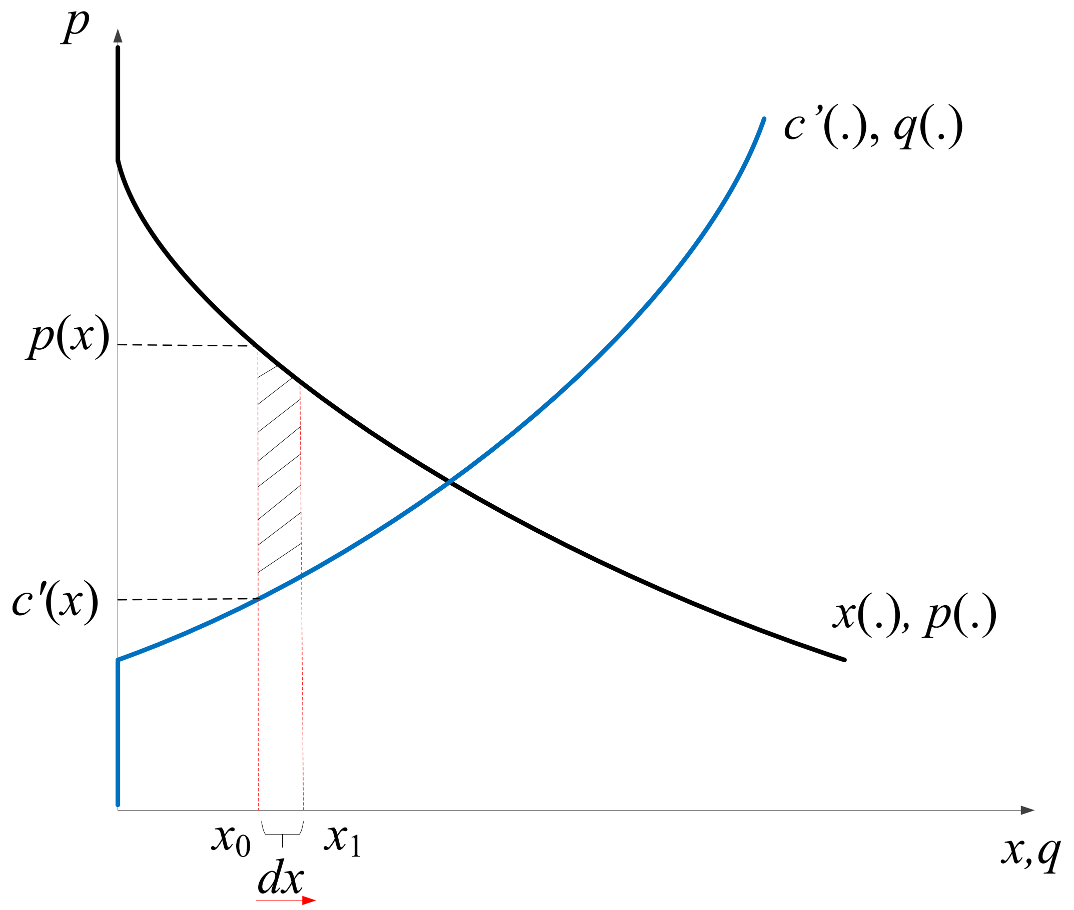
$$= P(x) \sum_{i=1}^{I} dx_i - C'(q) \sum_{j=1}^{J} dq_j$$

Welfare Analysis

- But since $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j = dx$, and $x = q$ by market feasibility, then
$$dS = [P(x) - C'(q)]dx$$
- *Intuition*:
  - The change in surplus of a marginal increase in consumption (and production) reflects the difference between the consumers' additional utility and firms' additional cost of production.

Welfare Analysis

- Differential change in surplus



Welfare Analysis

- We can also integrate the above expression, eliminating the differentials, in order to obtain the total surplus for an aggregate consumption level of x:

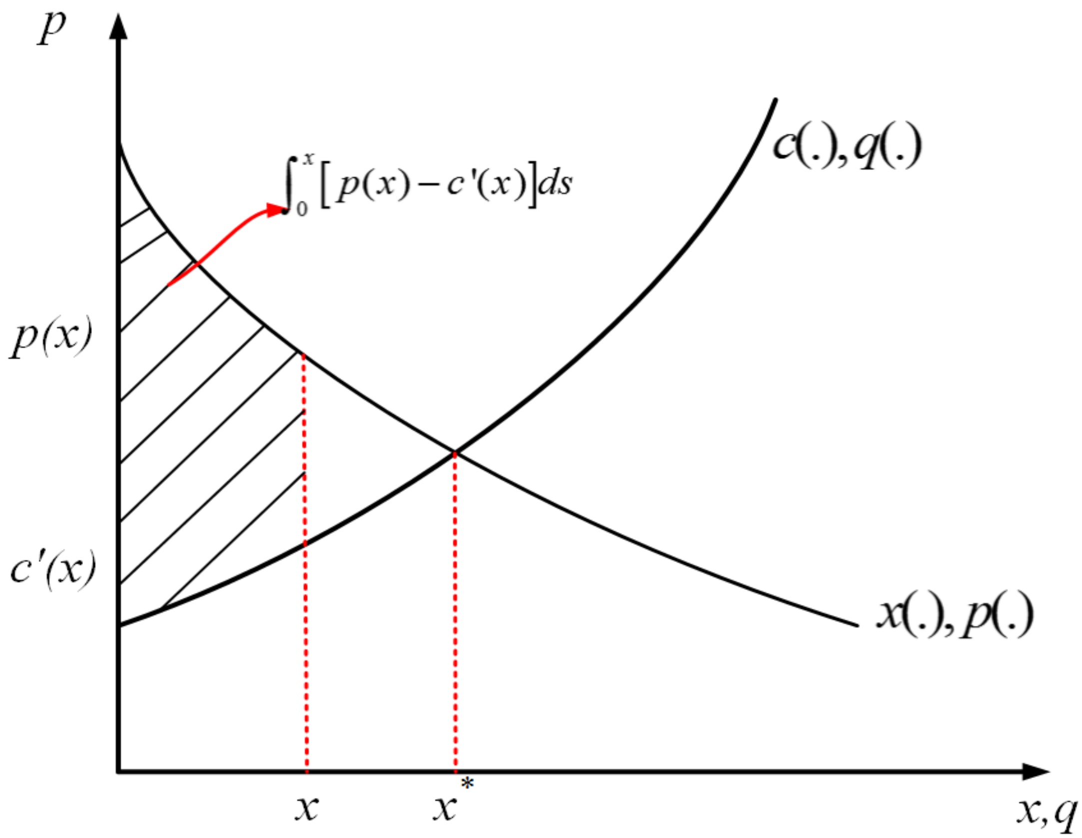
$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds$$

where $S_0 = S(0)$ is the constant of integration, and represents the aggregate surplus when aggregate consumption is zero, $x = 0$.

  – $S_0 = 0$ if the intercept of the marginal cost function satisfies $c_j'(0) = 0$ for all J firms.

Welfare Analysis

- Surplus at aggregate consumption x



Welfare Analysis

- For which consumption level is aggregate surplus $S(x)$ maximized?
  - Differentiating $S(x)$ with respect to x,

$$S'(x) = P(x^*) - C'(x^*) \leq 0$$

$$\text{or,} \quad P(x^*) \leq C'(x^*)$$

  - The second order (sufficient) condition is

$$S''(x) = \underbrace{P'(x^*)}_{-} - \underbrace{C''(x^*)}_{+} < 0$$

    - Hence, $S(x^*)$ is concave.
    - Then, when $x^* > 0$, aggregate surplus $S(x)$ is maximized at $P(x^*) = C'(x^*)$.

Welfare Analysis

- Therefore, the CE allocation maximizes aggregate surplus.
- This is the ***First Welfare Theorem***:
  - Every CE is Pareto optimal (PO).

Welfare Analysis

- ***Example 6.4***:
  - Consider an aggregate demand $x(p) = a - bq$ and aggregate supply $y(p) = J \cdot \frac{p}{2}$, where J is the number of firms in the industry.
  - The CE price solves

$$a - bp = J \cdot \frac{p}{2} \quad \text{or} \quad p = \frac{2a}{2b+J}$$

  - Intuitively, as demand increases (number of firms) increases (decreases) the equilibrium price increases (decreases, respectively).

Welfare Analysis

- ***Example 6.4*** (continued):
  - Therefore, equilibrium output is

$$x^* = a - b\frac{2a}{2b + J} = \frac{aJ}{2b + J}$$

  - Surplus is

$$S(x^*) = \int_0^{x^*} p(x) - C'(x)dx$$

where $p(x) = \frac{a-x}{b}$ and $C'(x) = \frac{2x}{J}$.

  - Thus,

$$S(x^*) = \int_0^{x^*} \left(\frac{a - x}{b} - \frac{2x}{J}\right) dx = \frac{a^2 J}{4b^2 + 2bJ}$$

which is increasing in the number of firms J.

General Equilibrium

General Equilibrium

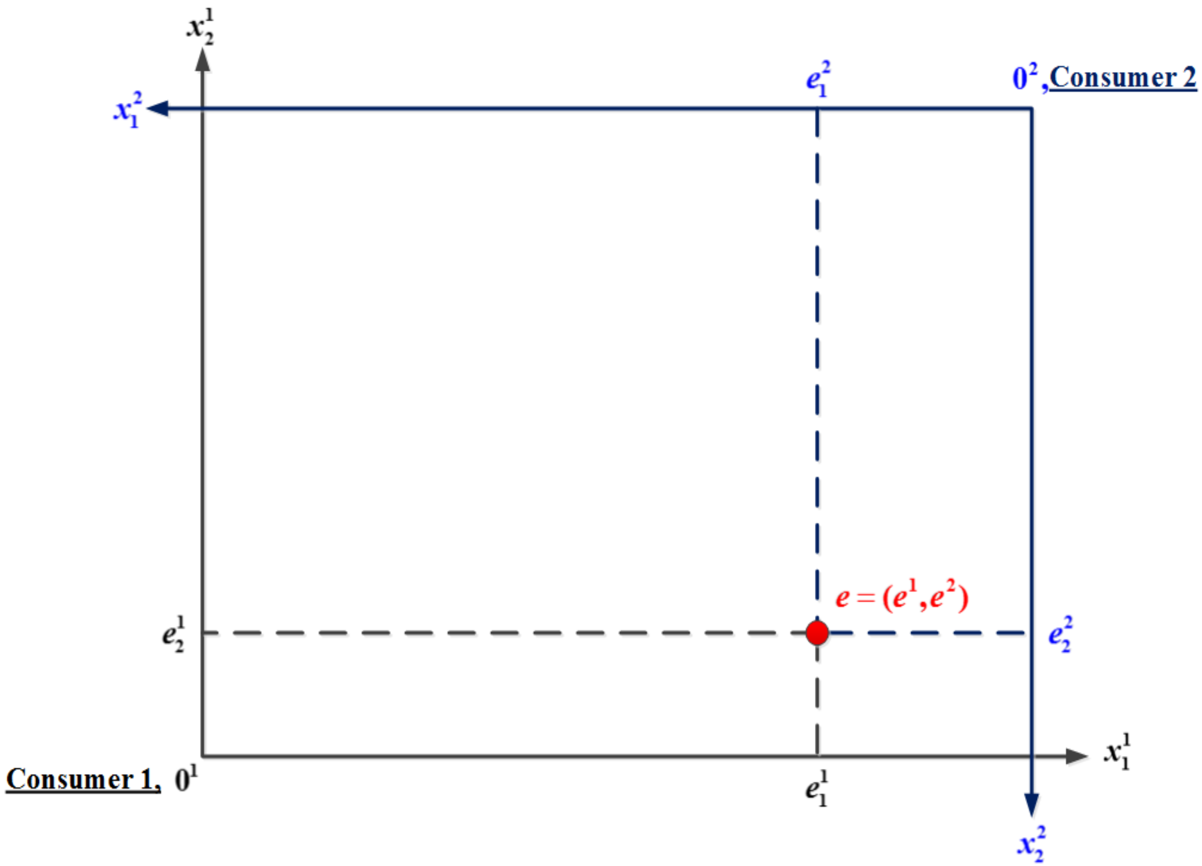
- So far, we explored equilibrium conditions in a single market with a single type of consumer.
- Now we examine settings with markets for different goods and multiple consumers.

General Equilibrium: No Production

- Consider an economy with two goods and two consumers, $i = \{1,2\}$.
- Each consumer is initially endowed with $\mathbf{e}^i \equiv (e_1^i, e_2^i)$ units of good 1 and 2.
- Any other allocations are denoted by $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

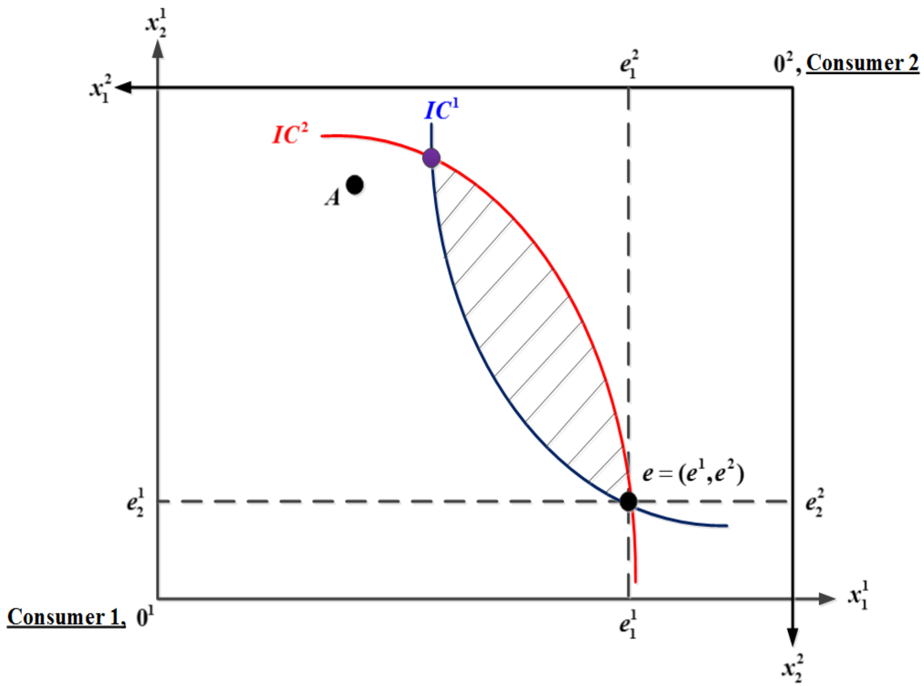
- ***Edgeworth box***:



General Equilibrium: No Production

- IC^i is the indifference curve of consumer i, which passes through his endowment point $\mathbf{e}^i$.
- The shaded area represents the set of bundles (x_1^i, x_2^i) for consumer i satisfying

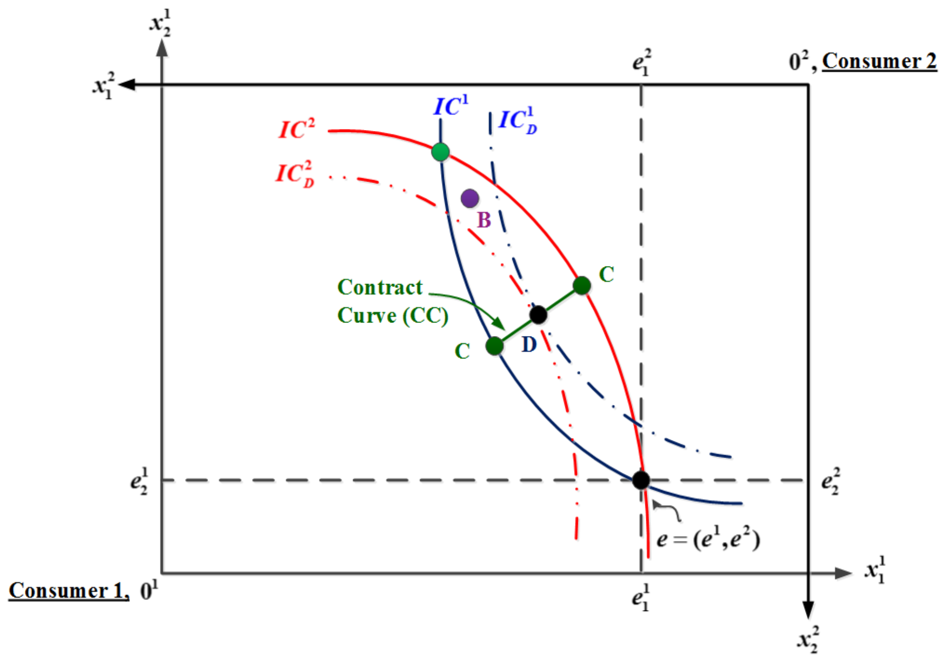
$$u^1(x_1^1, x_2^1) \geq u^1(e_1^1, e_2^1)$$
$$u^2(x_1^2, x_2^2) \geq u^2(e_1^2, e_2^2)$$



- Bundle A cannot be a barter equilibrium:
  - Consumer 1 does not exchange $\mathbf{e}$ for A.

General Equilibrium: No Production

- Not all points in the lens-shaped area is a barter equilibrium!
- Bundle B lies inside the lens-shaped area
  - Thus, it yields a higher utility level than the initial endowment $\mathbf{e}$ for both consumers.
- Bundle D, however, makes both consumers better off than bundle B.
  - It lies on "***contract curve***," in which the indifference curves are tangent to one another.
  - It is an equilibrium, since Pareto improvements are no longer possible



General Equilibrium: No Production

- ***Feasible allocation***:
  - An allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^I)$ is *feasible* if it satisfies

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$$

  - That is, the aggregate amount of goods in allocation $\mathbf{x}$ does not exceed the aggregate initial endowment $\mathbf{e} \equiv \sum_{i=1}^{I} \mathbf{e}^i$.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - A feasible allocation $\mathbf{x}$ is *Pareto efficient* if there is no other feasible allocation $\mathbf{y}$ which is weakly preferred by all consumers, i.e.,$\mathbf{y}^i \gtrsim \mathbf{x}^i$ for all $i \in I$, and at least strictly preferred by one consumer, $\mathbf{y}^i \succ \mathbf{x}^i$.
  - That is, allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ making all individuals at least as well off as under $\mathbf{x}$ and making one individual strictly better off.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - The set of Pareto efficient allocations $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ solves

$$\max_{\mathbf{x}^1,\ldots,\mathbf{x}^I \geq 0} \; u^1(\mathbf{x}^1)$$

$$\text{s.t.} \quad u^j(\mathbf{x}^j) \geq u^{-j} \text{ for } j \neq 1, \text{ and}$$

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i \quad \text{(feasibility)}$$

  where $\mathbf{x}^i = (x_1^i, x_2^i)$.

  - That is, allocations $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ are Pareto efficient if they maximizes individual 1's utility without reducing the utility of all other individuals below a given level u^{-j}, and satisfying feasibility.

General Equilibrium: No Production

– The Lagrangian is

$$L(\mathbf{x}^1, \ldots, \mathbf{x}^I; \lambda^2, \ldots, \lambda^I, \mu) =$$
$$u^1(\mathbf{x}^1) + \lambda^2[u^2(\mathbf{x}^2) - u^{-2}] + \cdots$$
$$+\lambda^I[u^I(\mathbf{x}^I) - u^{-I}] + \mu\left[\sum_{i=1}^{I} \mathbf{e}^i - \sum_{i=1}^{I} \mathbf{x}^i\right]$$

– FOC wrt $\mathbf{x}^1 = (x_1^1, x_2^1)$ yields

$$\frac{\partial L}{\partial x_k^1} = \frac{\partial u^1(\mathbf{x}^1)}{\partial x_k^1} - \mu \leq 0$$

for every good k of consumer 1.

– For any individual $j \neq 1$, the FOCs become

$$\frac{\partial L}{\partial x_k^j} = \frac{\partial u^j(\mathbf{x}^j)}{\partial x_k^1} - \mu \leq 0$$

General Equilibrium: No Production

– FOCs wrt λ^j and μ yield $u^j(\mathbf{x}^j) \geq u^{-j}$ and $\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$, respectively.

– In the case of interior solutions, a compact condition for Pareto efficiency is

$$\frac{\frac{\partial u^1(\mathbf{x}^1)}{\partial x_k^1}}{\frac{\partial u^1(\mathbf{x}^1)}{\partial x_2^1}} = \frac{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_k^1}}{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_2^j}} \text{ or } MRS_{1,2}^1 = MRS_{1,2}^j$$

for every consumer $j \neq 1$.

– Graphically, consumers' indifference curves become tangent to one another at the Pareto efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (Pareto efficiency):
  - Consider a barter economy with two goods, 1 and 2, and two consumers, A and B, each with the initial endowments of $\mathbf{e}^A = (100,350)$ and $\mathbf{e}^B = (100,50)$, respectively.
  - Both consumers' utility function is a Cobb-Douglas type given by $u^i(x_1^i, x_2^i) = x_1^i x_2^i$ for all individual $i = \{A, B\}$.
  - Let us find the set of Pareto efficient allocations.

General Equilibrium: No Production

- ***Example*** (continued):
  - Pareto efficient allocations are reached at points where the $MRS^A = MRS^B$. Hence,

$$MRS^A = MRS^B \Longrightarrow \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} \text{ or } x_2^A x_1^B = x_2^B x_1^A$$

  - Using the feasibility constraints for good 1 and 2, i.e.,

$$e_1^A + e_1^B = x_1^A + x_1^B$$
$$e_2^A + e_2^B = x_2^A + x_2^B$$

we obtain

$$x_1^B = e_1^A + e_1^B - x_1^A$$
$$x_2^B = e_2^A + e_2^B - x_2^A$$

General Equilibrium: No Production

- ***Example*** (continued):
  - Combining the tangency condition and feasibility constraints yields

$$x_2^A \underbrace{(e_1^A + e_1^B - x_1^A)}_{x_1^B} = \underbrace{(e_2^A + e_2^B - x_2^A)}_{x_2^B} x_1^A$$

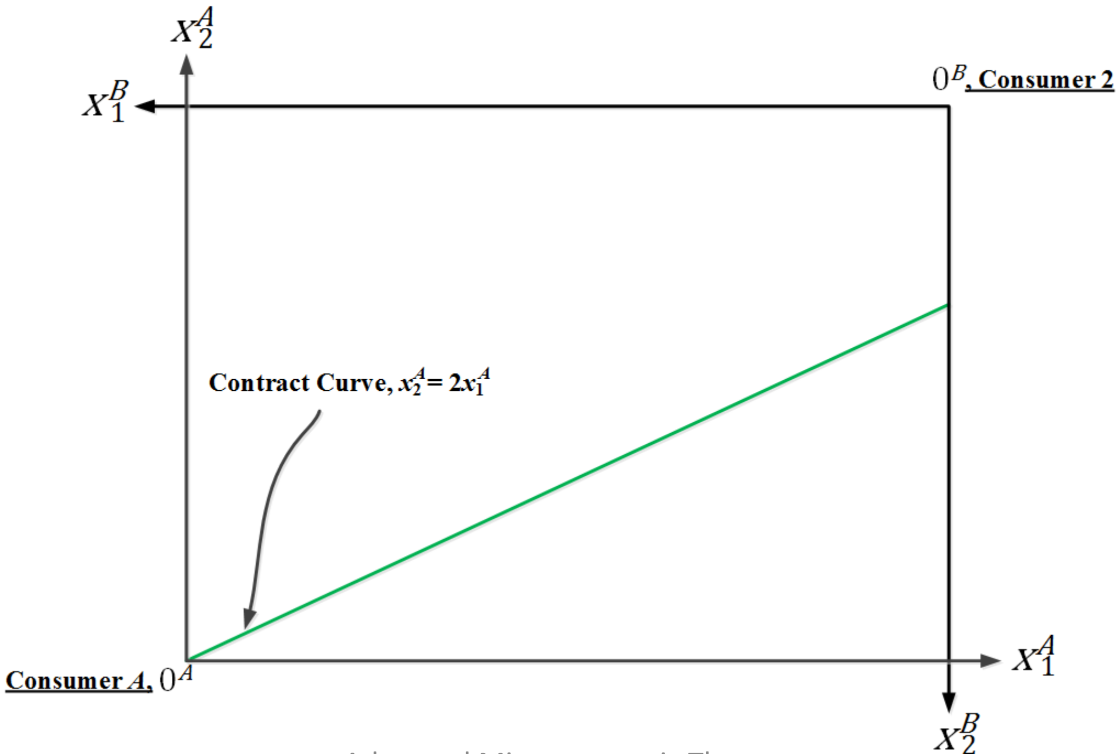
which can be re-written as

$$x_2^A = \frac{e_2^A + e_2^B}{e_1^A + e_1^B} x_1^A = \frac{350+50}{100+100} x_1^A = 2x_1^A$$

for all $x_1^A \in [0,200]$.

General Equilibrium: No Production

- ***Example*** (continued):
  - The line representing the set of Pareto efficient allocations



General Equilibrium: No Production

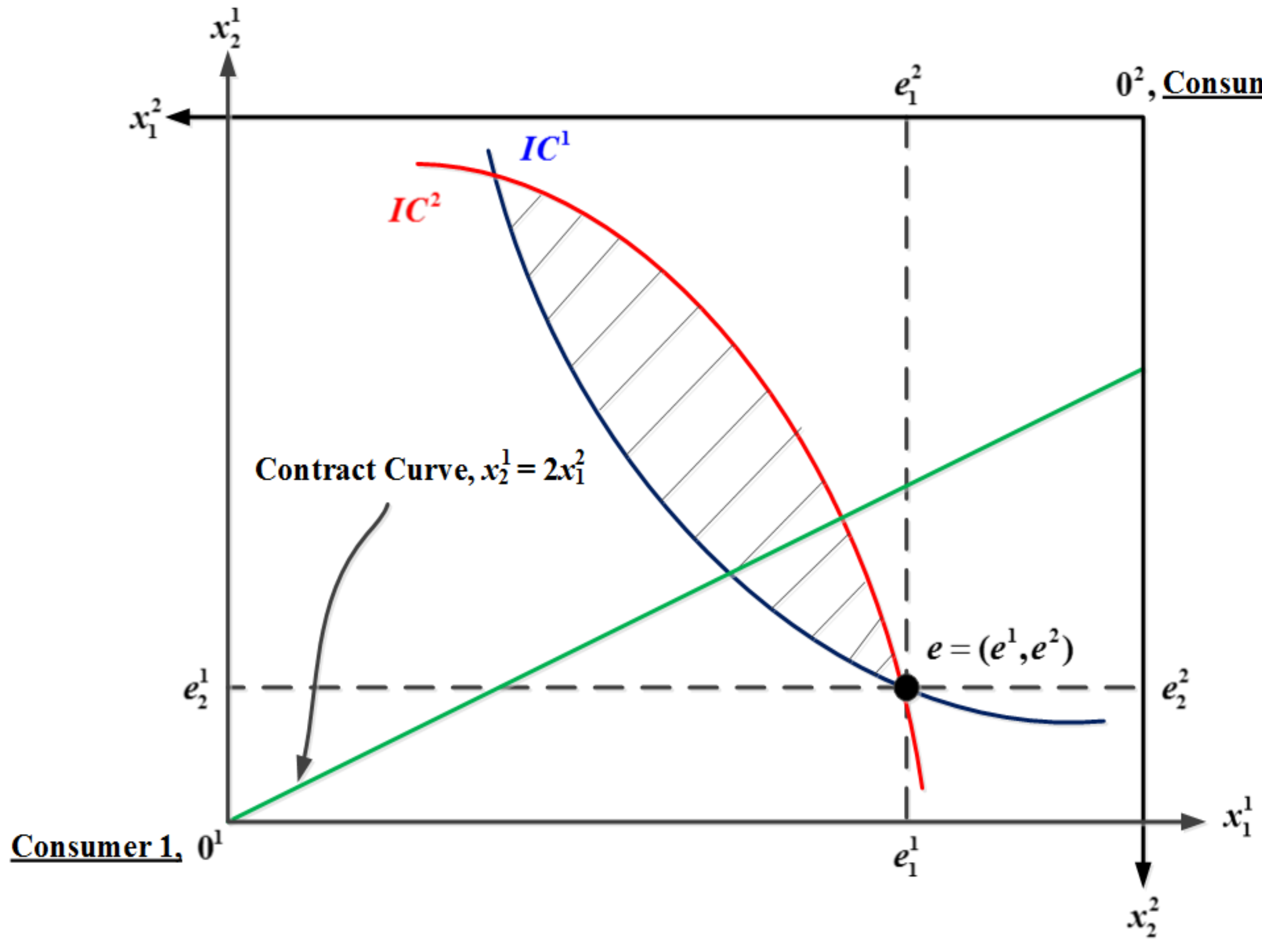
- ***Blocking coalitions***: Let $S \subset I$ denote a coalition of consumers. We say that S *blocks* the feasible allocation $\mathbf{x}$ if there is an allocation $\mathbf{y}$ such that:

  1) *Allocation is feasible for S*. The aggregate amount of goods that individuals in S enjoy in allocation $\mathbf{y}$ coincides with their aggregate initial endowment, i.e., $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$; and
  2) *Preferable*. Allocation $\mathbf{y}$ makes all individuals in the coalition weakly better off than under $\mathbf{x}$, i.e., $\mathbf{y}^i \succsim \mathbf{x}^i$ where $i \in S$, but makes at least one individual strictly better off, i.e., $\mathbf{y}^i \succ \mathbf{x}^i$.

General Equilibrium: No Production

- ***Equilibrium in a barter economy***: A feasible allocation $\mathbf{x}$ is an equilibrium in the exchange economy with initial endowment $\mathbf{e}$ if $\mathbf{x}$ is *not blocked* by any coalition of consumers.
- ***Core***: The core of an exchange economy with endowment $\mathbf{e}$, denoted $C(\mathbf{e})$, is the set of all *unblocked* feasible allocations $\mathbf{x}$.
  - Such allocations:
    a) mutually beneficial for all individuals (i.e., they lie in the lens-shaped area)
    b) do not allow for further Pareto improvements (i.e., they lie in the contract curve)

General Equilibrium: No Production



General Equilibrium: Competitive Markets

- Barter economy did not require prices for an equilibrium to arise.
- Now we explore the equilibrium in economies where we allow prices to emerge.
- Order of analysis:
  - consumers' preferences
  - the excess demand function
  - the equilibrium allocations in competitive markets (i.e., *Walrasian equilibrium allocations*)

General Equilibrium: Competitive Markets

- ***Consumers***:
  - Consider consumers' utility function to be continuous, strictly increasing, and strictly quasiconcave in $\mathbb{R}^n_+$.
  - Hence the UMP of every consumer i, when facing a budget constraint

    $$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i \text{ for all price vector } \mathbf{p} \gg \mathbf{0}$$

    yields a unique solution, which is the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$.
  - $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in the price vector $\mathbf{p}$.

General Equilibrium: Competitive Markets

– Intuitively, individual i's income comes from selling his endowment $\mathbf{e}^i$ at market prices $\mathbf{p}$, producing $\mathbf{p} \cdot \mathbf{e}^i = p_1 e_1^i + \cdots + p_k e_k^i$ dollars to be used in the purchase of allocation $\mathbf{x}^i$.

General Equilibrium: Competitive Markets

- ***Excess demand***:
  - Summing the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ for good k of every individual in the economy, we obtain the *aggregate demand* for good k.
  - The difference between the aggregate demand and the aggregate endowment of good k yields the *excess demand of good* k:

$$z_k(\mathbf{p}) = \sum_{i=1}^{I} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i=1}^{I} e_k^i$$

where $z_k(\mathbf{p}) \in \mathbb{R}$.

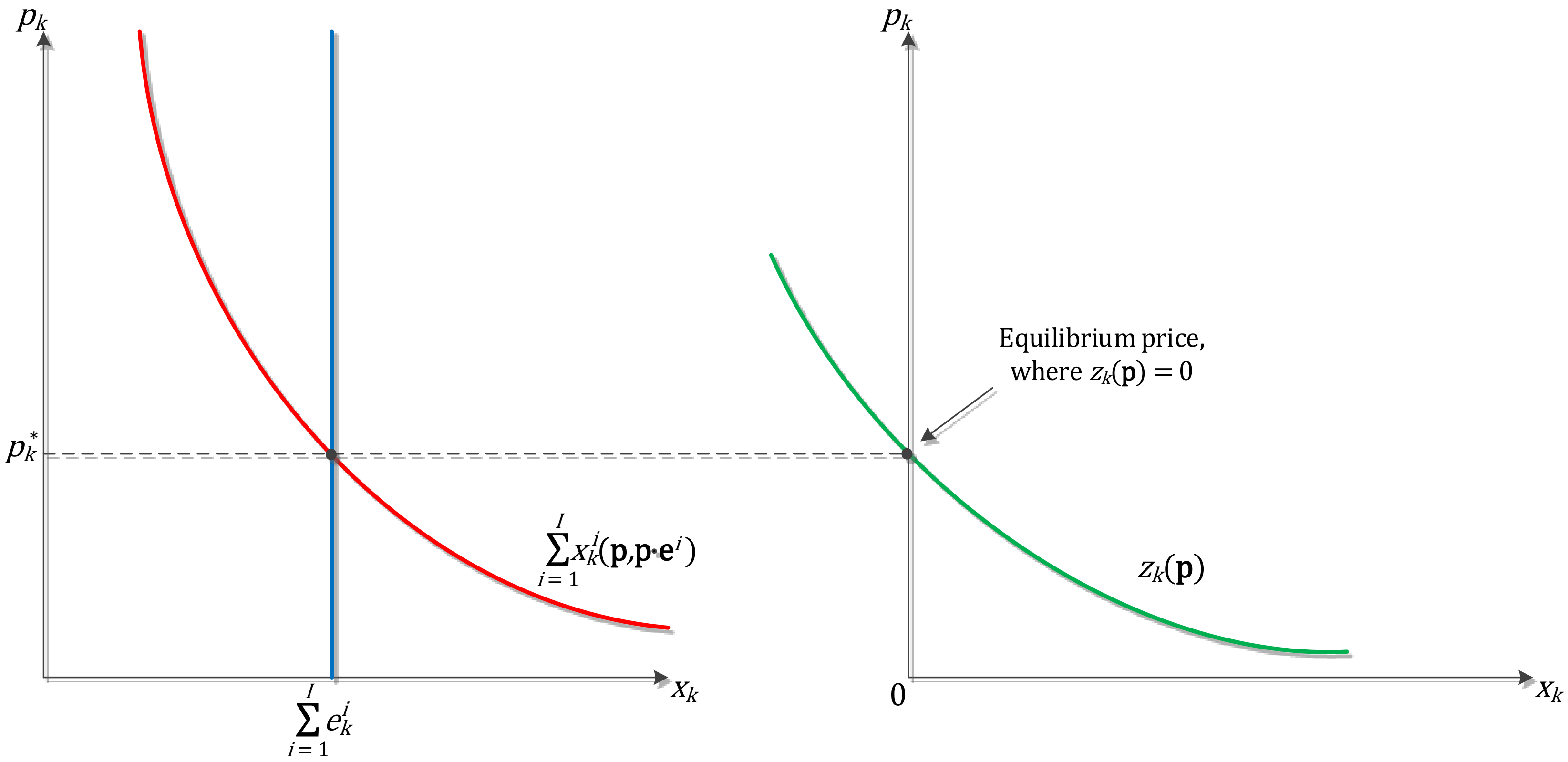
General Equilibrium: Competitive Markets

– When $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds its aggregate endowment.
  - Excess demand of good k

– When $z_k(\mathbf{p}) < 0$, the aggregate demand for good k falls short of its aggregate endowment.
  - Excess supply of good k

General Equilibrium: Competitive Markets

- Difference in demand and supply, and excess demand



General Equilibrium: Competitive Markets

- The excess demand function $\mathbf{z}(\mathbf{p}) \equiv \left(z_k(\mathbf{p}), z_k(\mathbf{p}), \dots, z_k(\mathbf{p})\right)$ satisfies following properties:

  1) ***Walras' law***: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.
     - Since every consumer $i \in I$ exhausts all his income,

$$\sum_{k=1}^{n} p_k \cdot x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{k=1}^{n} p_k e_k^i \;\Leftrightarrow$$

$$\sum_{k=1}^{n} p_k \left[x_k^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right) - e_k^i\right] = 0$$

General Equilibrium: Competitive Markets

– Summing over all individuals,

$$\sum_{i=1}^{I} \sum_{k=1}^{n} p_k \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i \right] = 0$$

– We can re-write the above expression as

$$\sum_{k=1}^{n} \sum_{i=1}^{I} p_k \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i \right] = 0$$

which is equivalent to

$$\sum_{k=1}^{n} p_k \underbrace{\left(\sum_{i=1}^{I} \left[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) \right] - \sum_{i=1}^{I} e_k^i \right)}_{z_k(\mathbf{p})} = 0$$

– Hence,

$$\sum_{k=1}^{n} p_k \cdot z_k(\mathbf{p}) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$$

General Equilibrium: Competitive Markets

– In a two-good economy, Walras' law implies

$$p_1 \cdot z_1(\mathbf{p}) = -p_2 \cdot z_2(\mathbf{p})$$

– *Intuition*: if there is excess demand in market 1, $z_1(\mathbf{p}) > 0$, there must be excess supply in market 2, $z_2(\mathbf{p}) < 0$.

– Hence, if market 1 is in equilibrium, $z_1(\mathbf{p}) = 0$, then so is market 2, $z_2(\mathbf{p}) = 0$.

– More generally, if the markets of $n - 1$ goods are in equilibrium, then so is the nth market.

General Equilibrium: Competitive Markets

2) ***Continuity***: $\mathbf{z}(\mathbf{p})$ is continuous at $\mathbf{p}$.
   - This follows from individual Walrasian demands being continuous in prices.

3) ***Homegeneity***: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
   - This follows from Walrasian demands being homogeneous of degree zero in prices.

- We now use excess demand to define a Walrasian equilibrium allocation.

General Equilibrium: Competitive Markets

- ***Walrasian equilibrium***:
  - A price vector $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if aggregate excess demand is zero at that price vector, $\mathbf{z}(\mathbf{p}^*) = 0$.
    - In words, price vector $\mathbf{p}^*$ clears all markets.
  - Alternatively, $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if:
    1) Each consumer solves his UMP, and
    2) Aggregate demand equals aggregate supply

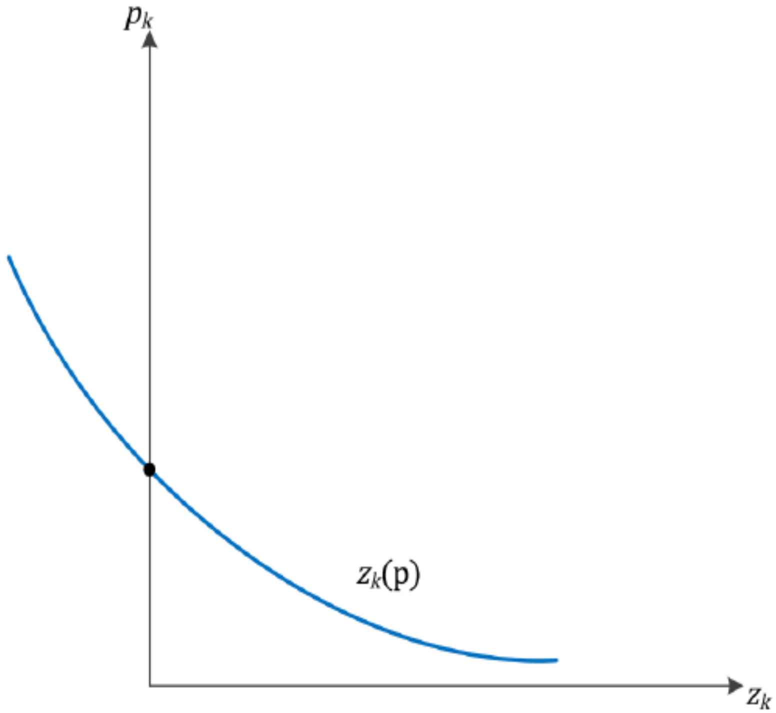
$$\sum_{i=1}^{I} x^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{i=1}^{I} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

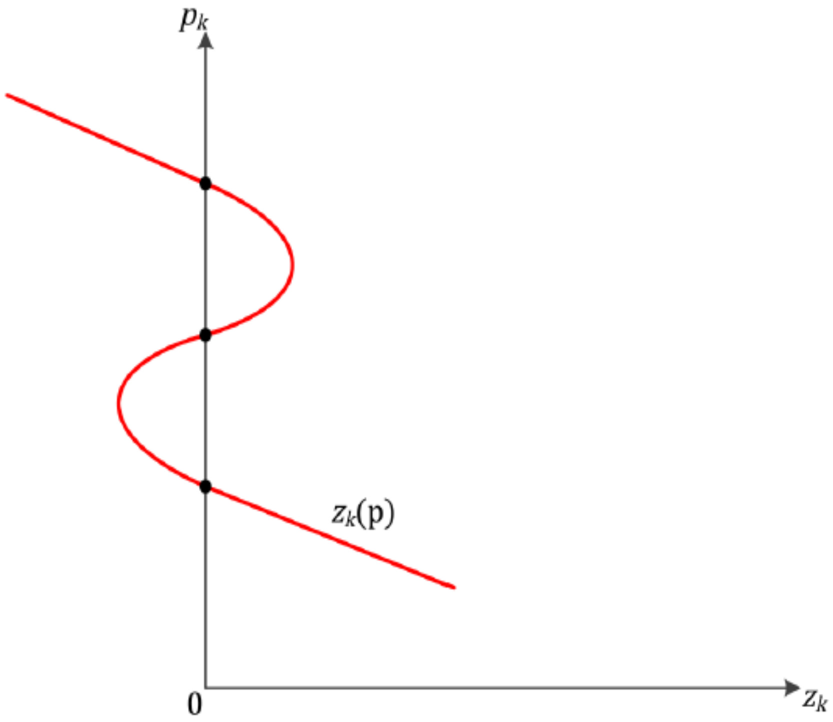
- ***Existence of a Walrasian equilibrium***:
  - A Walrasian equilibrium price vector $\mathbf{p}^* \in \mathbb{R}^n_{++}$, i.e., $\mathbf{z}(\mathbf{p}^*) = 0$, exists if the excess demand function $\mathbf{z}(\mathbf{p})$ satisfies *continuity* and *Walras' law* (Varian, 1992).

General Equilibrium: Competitive Markets

- ***Uniqueness of equilibrium prices***:



satisfied



violated

General Equilibrium: Competitive Markets

- ***Example 6.6*** (Walrasian equilibrium allocation):
  - In example 6.1, we determined that

$$MRS^A = MRS^B = \frac{p_1}{p_2}$$

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}$$

  - Let us determine the Walrasian demands of each good for each consumer.
  - Rearranging the second equation above, we get

$$p_1 x_1^A = p_2 x_2^A$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting this into consumer A's budget constraint yields
$$p_1 x_1^A + p_1 x_1^A = p_1 \cdot 100 + p_2 \cdot 350$$
$$\text{or } \; x_1^A = 50 + 175\frac{p_2}{p_1}$$
    which is consumer A's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^A = p_2 x_2^A$ yields
$$p_1\left(50 + 175\frac{p_2}{p_1}\right) = p_2 x_2^A$$
$$\text{or } \; x_2^A = 175 + 50\frac{p_1}{p_2}$$
    which is consumer A's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - We can obtain consumer B's demand in an analogous way. In particular, substituting $p_1 x_1^B = p_2 x_2^B$ into consumer B's budget constraint yields
$$p_1 x_1^B + p_1 x_1^B = p_1 \cdot 100 + p_2 \cdot 50$$
$$\text{or } \; x_1^B = 50 + 25\frac{p_2}{p_1}$$
which is consumer B's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^B = p_2 x_2^B$ yields
$$p_1\left(50 + 25\frac{p_2}{p_1}\right) = p_2 x_2^B$$
$$\text{or } \; x_2^B = 25 + 50\frac{p_1}{p_2}$$
which is consumer B's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - For good 1, the feasibility constraint is

$$x_1^A + x_1^B = 100 + 100$$

$$\left(50 + 175\frac{p_2}{p_1}\right) + \left(50 + 25\frac{p_2}{p_1}\right) = 200$$

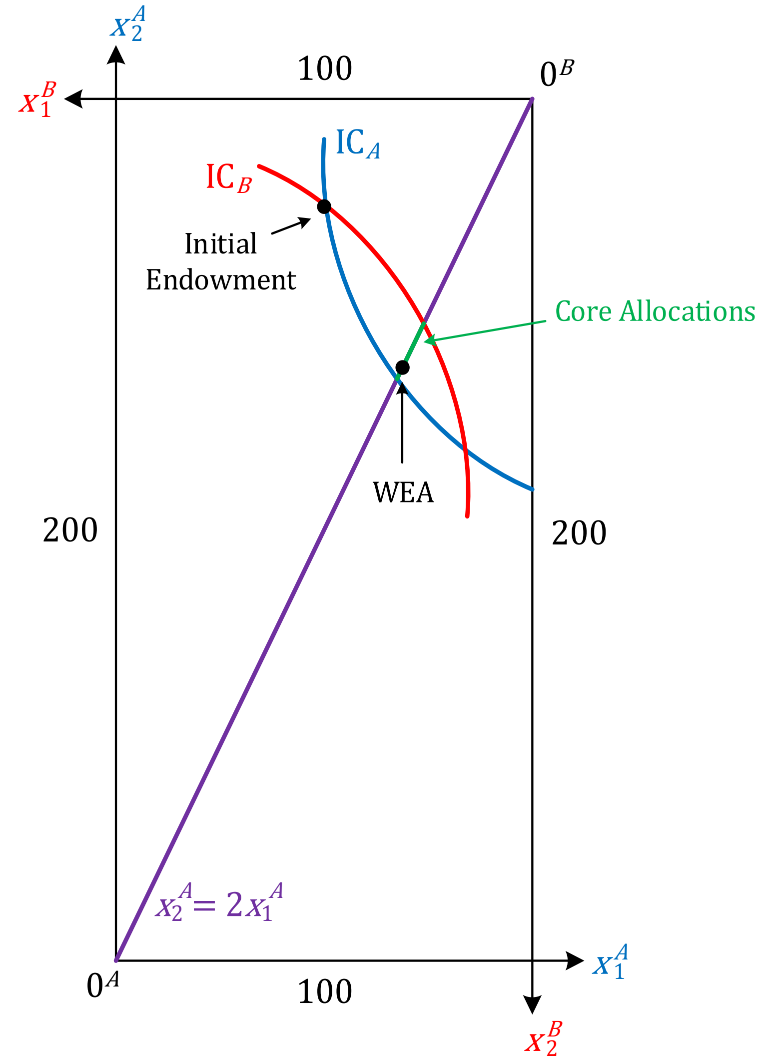
$$\frac{p_2}{p_1} = \frac{1}{2}$$

  - Plugging the relative prices into the Walrasian demands yields Walrasian equilibrium:

$$\left(x_1^{A,*}, x_2^{A,*}, x_1^{B,*}, x_2^{B,*}; \frac{p_1}{p_2}\right) = (137.5, 275, 62.5, 125; 2)$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Initial allocation,
  - Core allocation, and
  - Walrasian equilibrium allocations (WEA).



General Equilibrium: Competitive Markets

- ***Equilibrium allocations must be in the Core***:
  – If each consumer's utility function is strictly increasing, then every WEA is in the Core, i.e., $W(\mathbf{e}) \subset C(\mathbf{e})$.

- *Proof* (by contradiction):
  – Take a WEA $\mathbf{x}(\mathbf{p}^*)$ with equilibrium price $\mathbf{p}^*$, but $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.
  – Since $\mathbf{x}(\mathbf{p}^*)$ is a WEA, it must be feasible.
  – If $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition of individuals S and another allocation $\mathbf{y}$ such that

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \text{ for all } i \in S$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - The above expression:
    - holds with strict inequality for at least one individual in the coalition
    - is feasible for the coalition, i.e., $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$.
  - Multiplying both sides of the feasibility condition by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

  - However, the most preferable vector $\mathbf{y}^i$ must be more costly than $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$:

$$\mathbf{p}^* \mathbf{y}^i \geq \mathbf{p}^* \mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \cdot \mathbf{e}^i$$

with strict inequality for at least one individual.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Hence, summing over all consumers in the coalition S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

    which contradicts $\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$.

  - Therefor, all WEAs must be part of the Core, i.e.,

$$\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$$

General Equilibrium: Competitive Markets

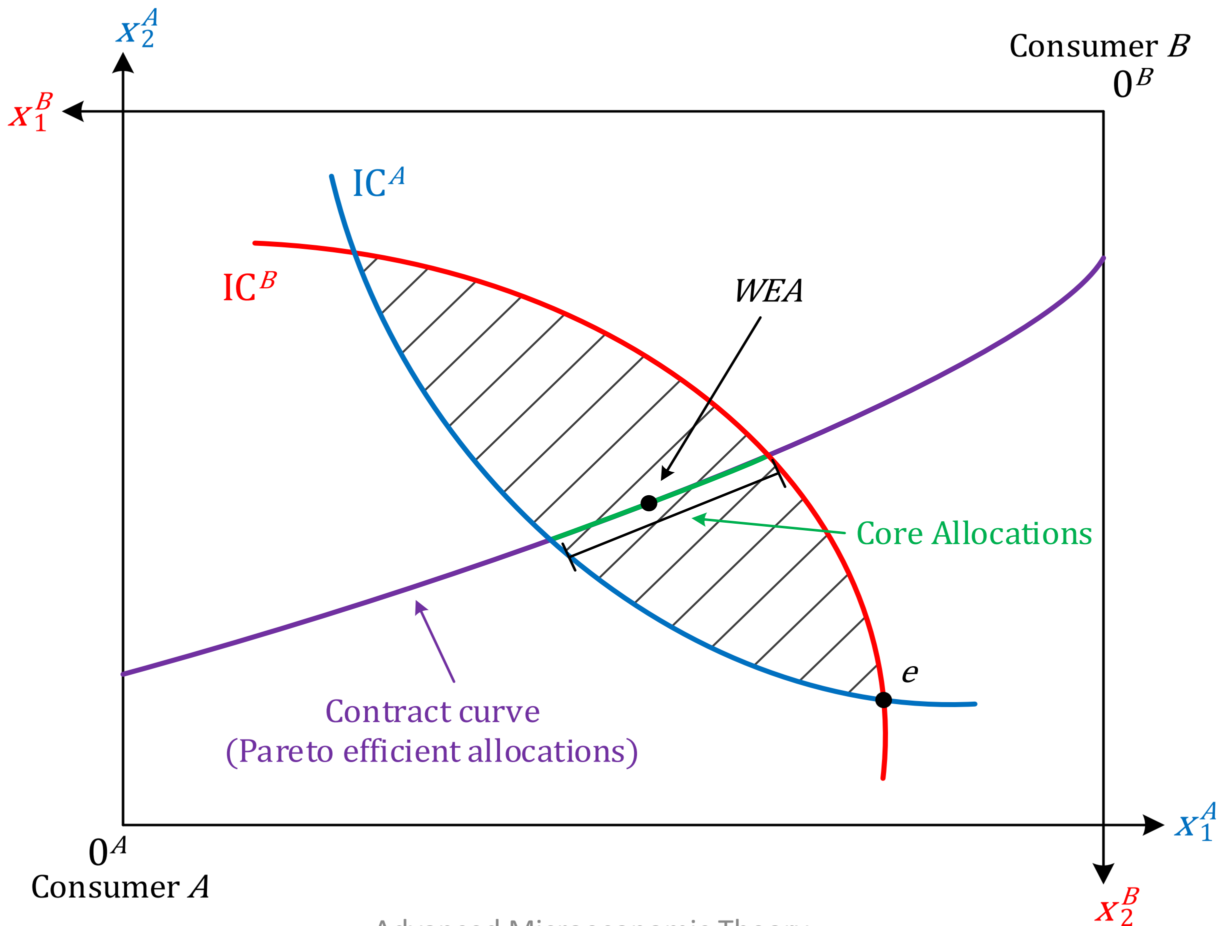
- *Remarks*:
  1) The Core $C(\mathbf{e})$ contains the WEA (or WEAs)
     - That is, the Core is nonempty.
  2) Since all core allocations are Pareto efficient (i.e., we cannot increase the welfare of one consumer without decreasing that of other consumers), then all WEAs (which are part of the Core) are also Pareto efficient.

General Equilibrium: Competitive Markets

- ***First Welfare Theorem***: Every WEA is Pareto efficient.
  - The WEA lies on the core (the segment of the contract curve within the lens-shaped area),
  - The core is a subset of all Pareto efficient allocations.

General Equilibrium: Competitive Markets

- First Welfare Theorem



General Equilibrium: Competitive Markets

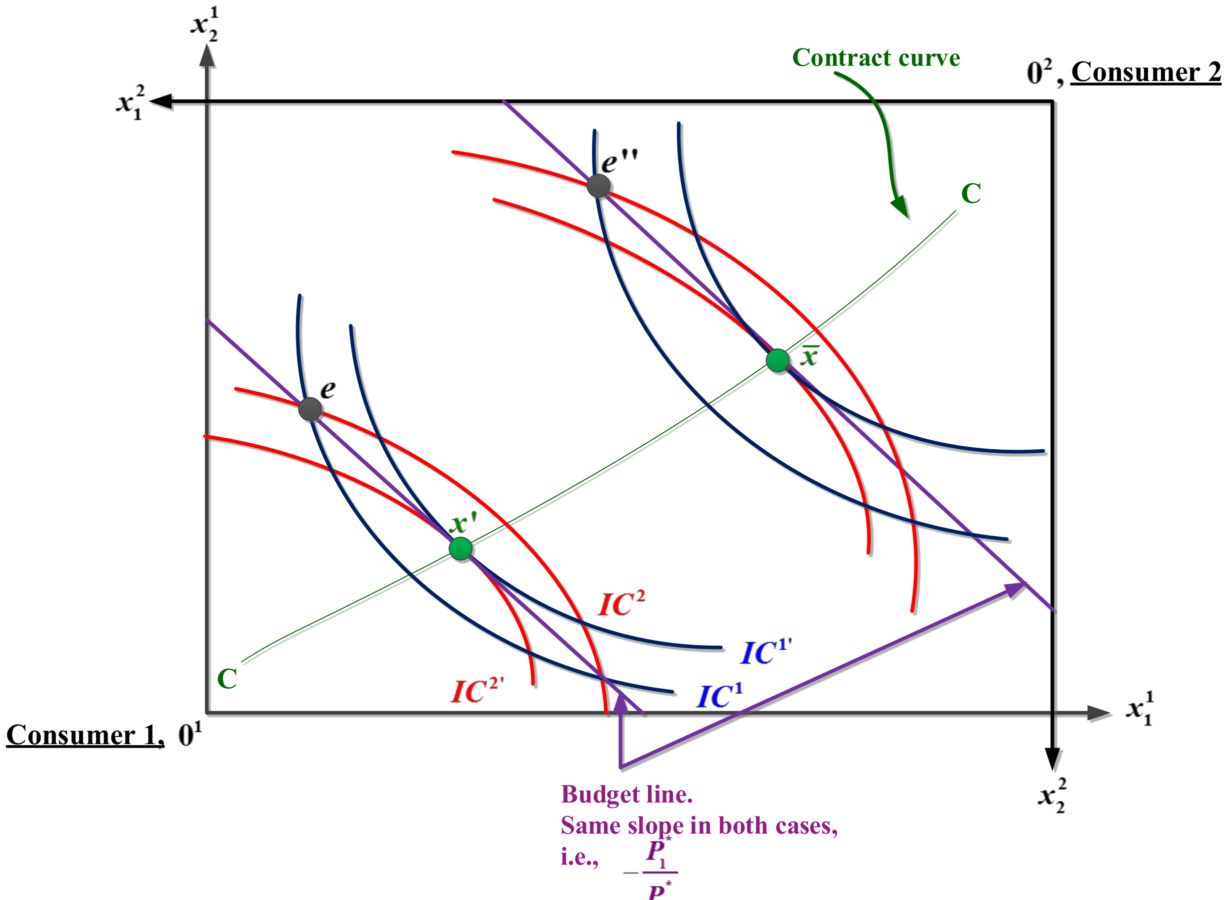
- ***Second Welfare Theorem***:
  - Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation (i.e., it lies on the contract curve), and that endowments are redistributed so that the new endowment vector $\mathbf{e}^{*i}$ lies on the budget line, thus satisfying

$$\mathbf{p}^* \cdot \mathbf{e}^{*i} = \mathbf{p}^* \cdot \bar{\mathbf{x}}^i \text{ for every consumer } i$$

  - Then, the Pareto-efficient allocation $\bar{\mathbf{x}}$ is a WEA given the new endowment vector $\mathbf{e}^*$.

General Equilibrium: Competitive Markets

- Second welfare theorem



General Equilibrium: Competitive Markets

- ***Example 6.7*** (WEA and Second welfare theorem):
  - Consider an economy with utility functions $u^A = x_1^A x_2^A$ for consumer A and $u^B = \{x_1^B, x_2^B\}$ for consumer B.
  - The initial endowments are $\mathbf{e}^A = (3,1)$ and $\mathbf{e}^B = (1,3)$.
  - Good 2 is the numeraire, i.e., $p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto Efficient Allocations:

  – Consumer B's preferences are perfect complements. Hence, he consumes at the kink of his indifference curves, i.e.,

$$x_1^B = x_2^B$$

  – Given feasibility constraints

$$x_1^A + x_1^B = 4$$
$$x_2^A + x_2^B = 4$$

  substitute x_2^B for x_1^B in the first constraint to get

$$x_2^B = 4 - x_1^A$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) *Pareto Efficient Allocations*:

     – Substituting the above expression in the second constraint yields

$$x_2^A + \underbrace{(4 - x_1^A)}_{x_2^B} = 4 \iff x_2^A = x_1^A$$

     – This defines the contract curve, i.e., the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Consumer A's maximization problem is

$$\max_{x_1^A, x_2^A} \quad x_1^A x_2^A$$
$$\text{s.t.} \quad p_1 x_1^A + x_2^A \leq p_1 \cdot 3 + 1$$

  – FOCs:

$$x_2^A - \lambda p_1 = 0$$
$$x_1^A - \lambda = 0$$
$$p_1 x_1^A + x_2^A = 3p_1 + 1$$

where λ is the lagrange multiplier.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Combining the first two equations,

$$\lambda = \frac{x_2^A}{p_1} = x_1^A \quad \text{or} \quad p_1 = \frac{x_2^A}{x_1^A}$$

  – From Pareto efficiency, we know that $x_2^A = x_1^A$. Hence,

$$p_1 = \frac{x_2^A}{x_1^A} = 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Substituting both the price and Pareto efficient allocation requirement into the budget constraint,

$$1 \cdot x_1^A + x_1^A = 1 \cdot 3 + 1$$

$$\text{or } \; x_1^{A*} = x_2^{A*} = 2$$

  - Using the feasibility constraint,

$$\underbrace{2}_{x_1^A} + x_1^B = 4 \;\text{ or }\; x_1^{B*} = x_2^{B*} = 2$$

  - Thus, the WEA is

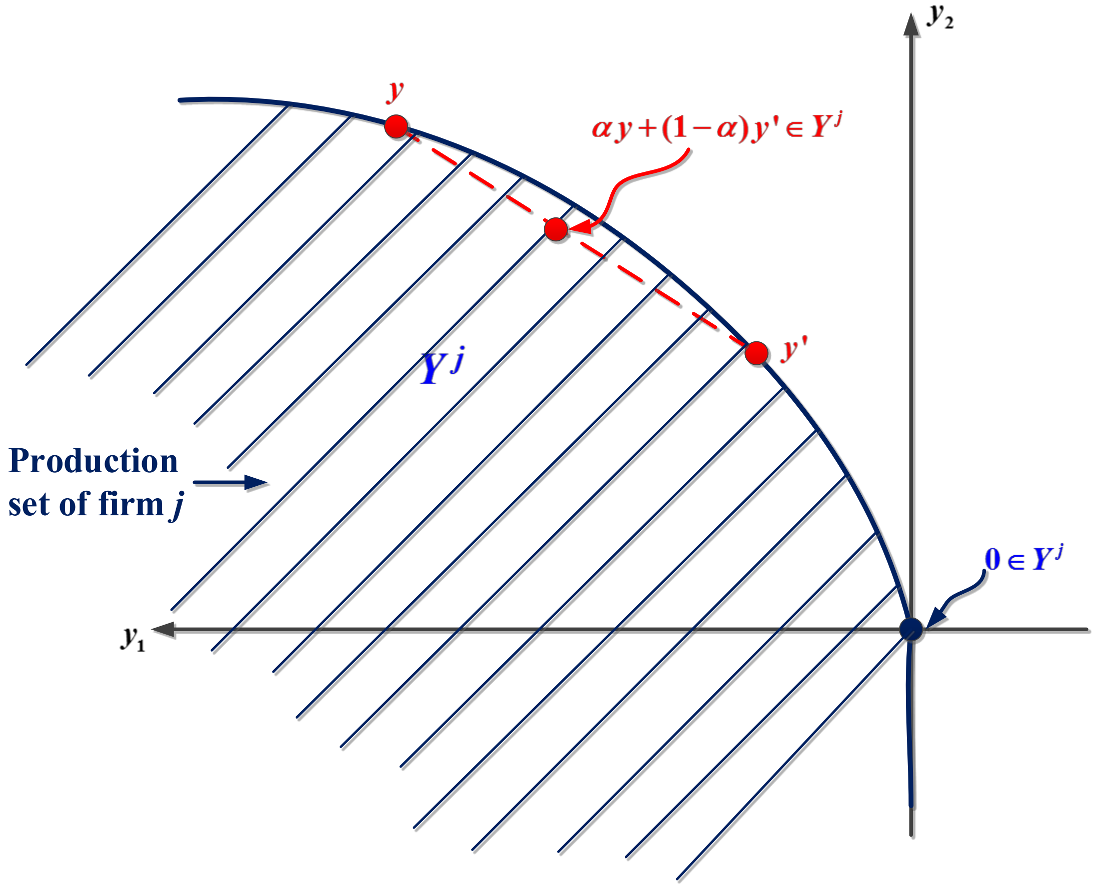
$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (2{,}2; 2{,}2; 1)$$

General Equilibrium: Production

- Let us now extend our previous results to setting where firms are also active.
- Assume J firms in the economy, each with production set Y^j, which satisfies:
  - Inaction is possible, i.e., $\mathbf{0} \in Y^j$.
  - Y^j is closed and bounded, so points on the production frontier are part of the production set and thus feasible.
  - Y^j is strictly convex, so linear combinations of two production plans also belong to the production set.

General Equilibrium: Production

- Production set Y^j for a representative firm



General Equilibrium: Production

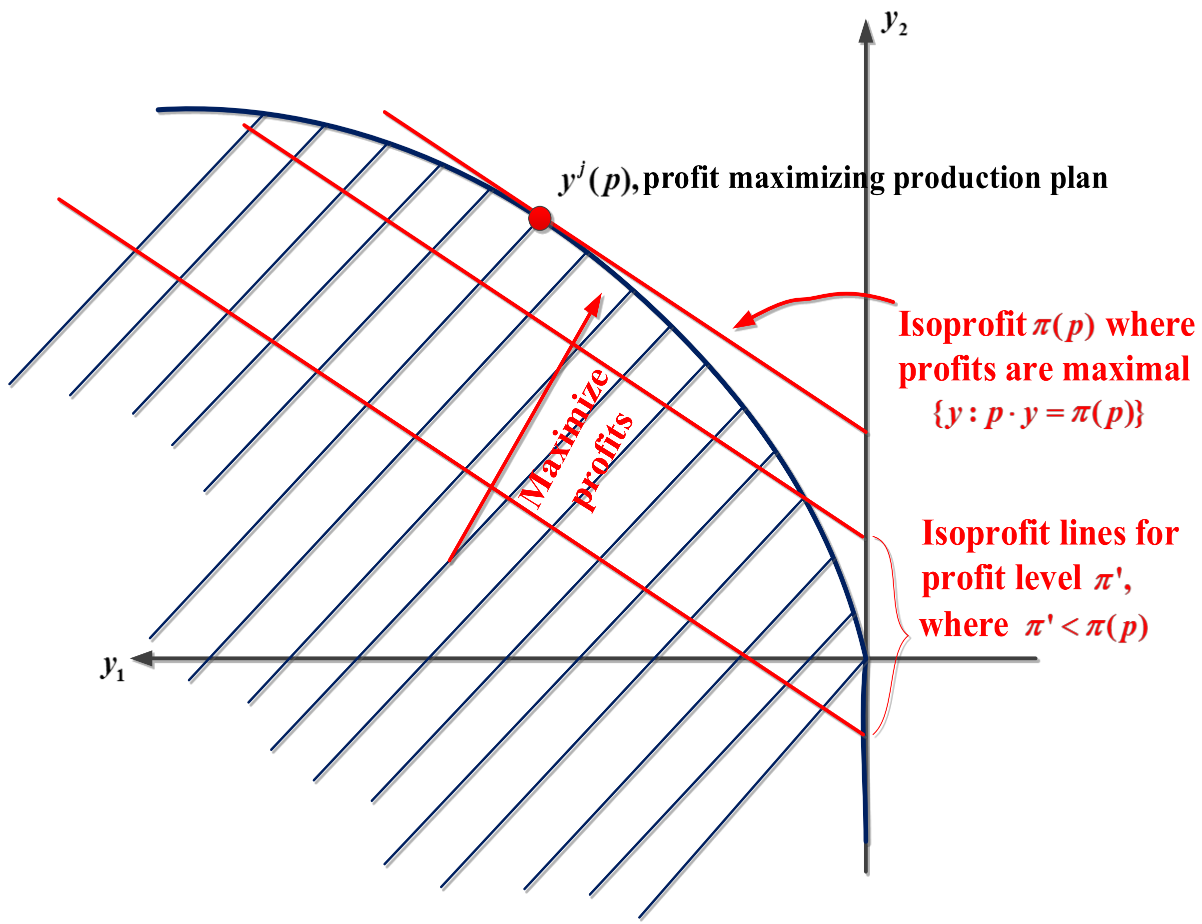
- Every firm j facing a fixed price vector $\mathbf{p} \gg 0$ independently and simultaneously solves

$$\max_{y_j \in Y^j} \quad \mathbf{p} \cdot y_j$$

- A profit-maximizing production plan $y_j(\mathbf{p})$ exists for every firm j, and it is unique.
- By the theorem of the maximum, both the argmax, $y_j(\mathbf{p})$, and the value function, $\pi_j(\mathbf{p}) \equiv \mathbf{p} \cdot y_j(\mathbf{p})$, are continuous in p.

General Equilibrium: Production

- $y^j(p)$ exists and is unique



General Equilibrium: Production

- ***Aggregate production set***:
  - The aggregate production set is the sum of all firms' production plans (either profit maximizing or not):

$$Y = \left\{\mathbf{y} | \mathbf{y} = \sum_{j=1}^{J} y_j \text{ where } y_j \in Y^j\right\}$$

  - A joint-profit maximizing production plan $\mathbf{y}(\mathbf{p})$ is the sum of each firm's profit-maximizing plan, i.e.,

$$\mathbf{y}(\mathbf{p}) = y_1(\mathbf{p}) + y_2(\mathbf{p}) + \cdots + y_J(\mathbf{p})$$

General Equilibrium: Production

- In an economy with J firms, each of them earning $\pi_j(\mathbf{p})$ profits in equilibrium, how are profits distributed?
  - Assume that each individual i owns a share θ^{ij} of firm j's profits, where $0 \leq \theta^{ij} \leq 1$ and $\sum_{i=1}^{I} \theta^{ij} = 1$.
  - This allows for multiple sharing profiles:
    - $\theta^{ij} = 1$: individual i owns all shares of firm j
    - $\theta^{ij} = 1/I$: every individual's share of firm j coincides

General Equilibrium: Production

– Consumer's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi_j(\mathbf{p})$$

where $\sum_{j=1}^{J} \theta^{ij} \pi_j(\mathbf{p})$ is new relative to the standard budget constraint.

– Let us express the budget constraints as

$$\mathbf{p} \cdot \mathbf{x}^i \leq \underbrace{\mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi_j(\mathbf{p})}_{m^i(\mathbf{p})}$$

$$\Longrightarrow \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$$

where $m^i(\mathbf{p}) > 0$ (given assumptions on Y^j).

General Equilibrium: Production

- ***Equilibrium with production***:
  - We start defining excess demand functions and use such a definition to identify the set of equilibrium allocations.
  - *Excess demand*: The excess demand function for good k is

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{i=1}^{I} e_k^i - \sum_{j=1}^{J} y_k^j(\mathbf{p})$$

where $\sum_{j=1}^{J} y_k^j(\mathbf{p})$ is a new term relative to the analysis of general equilibrium without production.

General Equilibrium: Production

– Hence, the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p}))$$

– *WEA with production*: If the price vector is strictly positive in all of its components, $\mathbf{p}^* \gg 0$, a pair of consumption and production bundles $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a WEA if:

1) Each consumer i solves his UMP, which becomes the ith entry of $\mathbf{x}(\mathbf{p}^*)$, i.e., $\mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*))$;
2) Each firm j solves its PMP, which becomes the jth entry of $\mathbf{y}(\mathbf{p}^*)$, i.e., $\mathbf{y}^j(\mathbf{p}^*)$;

General Equilibrium: Production

3) Demand equals supply

$$\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*)) = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p}^*)$$

which is the market clearing condition.

– *Existence*: Assume that
  - consumers' utility functions are continuous, strictly increasing and strictly quasiconcave;
  - every firm j's production set Y^j is closed and bounded, strictly convex, and satisfies inaction being possible;
  - every consumer is initially endowed with positive units of at least one good, so the sum $\sum_{i=1}^{I} \mathbf{e}^i \gg 0$.

Then, there is a price vector $\mathbf{p}^* \gg 0$ such that a WEA exisits, i.e., $z(\mathbf{p}^*) = 0$.

General Equilibrium: Production

- ***Example 6.8*** (Equilibrium with production):
  - Consider a two-consumer, two-good economy where consumer $i = \{A, B\}$ has utility function $u^i = x_1^i x_2^i$.
  - There are two firms in this economy, and each of them use capital (K) and labor (L) to produce one of the consumption goods each.
  - Firm 1 produces good 1 according to $y_1 = K_1^{0.75} L_1^{0.25}$.
  - Firm 2 produces good 2 according to $y_2 = K_2^{0.25} L_2^{0.75}$.
  - Consumer A is endowed with $(K^A, L^A) = (1,1)$, while consumer B is endowed with $(K^B, L^B) = (2,1)$.
  - Let us find a WEA in this economy with production.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - ***UMPs***: Consumer i's maximization problem is

$$\max_{x_1^i,\, x_2^i} \; x_1^i x_2^i$$
$$\text{s.t.} \;\; p_1 x_1^i + x_2^i = rK^i + wL^i$$

    where r and w are prices for capital and labor, respectively.
  - FOC:

$$\frac{p_1}{p_2} = MRS_{1,2}^i \implies \frac{p_1}{p_2} = \frac{x_2^i}{x_1^i} \implies p_1 x_1^i = p_2 x_2^i$$

    for $i = \{A, B\}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Taking the above equation for consumers A and B, and adding them together yields

$$p_1(x_1^A + x_1^B) = p_2(x_2^A + x_2^B)$$

  where $x_1^A + x_1^B$ is the left side of the feasibility condition $x_1^A + x_1^B = y_1 = K_1^{0.75} L_1^{0.25}$.
  - Substituting both feasibility conditions into the above expression, and re-arranging, yields

$$\frac{p_1}{p_2} = \frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - ***PMPs***: Firm 1's maximization problem is

$$\max_{K_1, L_1} \quad p_1 K_1^{0.75} L_1^{0.25} - rK_1 - wL_1$$

  - FOCs:

$$r = 0.75 p_1 K_1^{-0.25} L_1^{0.25}$$
$$w = 0.25 p_1 K_1^{0.75} L_1^{-0.75}$$

  - Combining these conditions gives the tangency condition for profit maximization

$$\frac{r}{w} = MRTS_{L,K}^{1} \implies \frac{r}{w} = 3\,\frac{L_1}{K_1}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Likewise, firm 2's PMP gives the following FOCs:

$$r = 0.25 p_2 K_2^{-0.75} L_2^{0.75}$$
$$w = 0.75 p_2 K_2^{0.25} L_2^{-0.25}$$

  - Combining these conditions gives the tangency condition for profit maximization

$$\frac{r}{w} = MRTS_{L,K}^{2} \implies \frac{r}{w} = \frac{1}{3}\frac{L_2}{K_2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Combining both $MRTS$ yields,

$$3\frac{L_1}{K_1} = \frac{1}{3}\frac{L_2}{K_2} \implies \frac{K_1}{L_1} = 9\frac{K_2}{L_2}$$

    - *Intuition*: firm 1 is more capital intensive than firm 2, i.e., its capital to labor ratio is higher.
  - Setting both firm's price of capital, r, equal to each other yields

$$0.75p_1 K_1^{-0.25} L_1^{0.25} = 0.25p_2 K_2^{-0.75} L_2^{0.75}$$

$$\implies \frac{p_1}{p_2} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firm's price of labor, w, equal to each other yields

$$0.25p_1K_1^{0.75}L_1^{-0.75} = 0.75p_2K_2^{0.25}L_2^{-0.25}$$

$$\Longrightarrow \frac{p_1}{p_2} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25}$$

  - Setting price ratio from consumers' UMP equal to the first price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75} \Longrightarrow K_1 = 3K_2$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - By the feasibility conditions, we know that $K_1 + K_2 = K^A + K^B = 3$ or $K_2 = 3 - K_1$.
  - Substituting the above expression into $K_1 = 3K_2$, we find the profit-maximizing demands for capital use by firms 1 and 2:

$$K_1 = 3(3 - K_1) \implies K_1^* = \frac{9}{4}$$

$$K_2^* = \frac{1}{3} K_1^* = \frac{3}{4}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting price ratio from consumers' UMP equal to the second price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25} \Longrightarrow L_1 = \frac{1}{3}L_2$$

  - By the feasibility conditions, we know that $L_1 + L_2 = L^A + L^B = 2$ or $L_2 = 2 - L_1$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the above expression into $L_1 = \frac{1}{3}L_2$, we find the profit-maximizing demands for labor use by firms 1 and 2:

$$L_1 = \frac{1}{3}(2 - L_1) \implies L_1^* = \frac{1}{2}$$

$$L_2^* = 3L_1^* = \frac{3}{2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the capital and labor demands for firm 1 and 2 into the price ratio from consumers' UMP yields

$$\frac{p_1}{p_2} = \frac{\left(\frac{3}{4}\right)^{0.25}\left(\frac{3}{2}\right)^{0.75}}{\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{0.25}} = 0.82$$

where normalizing the price of good 2, i.e., $p_2 = 1$, gives $p_1 = 0.82$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Furthermore, substituting our calculated values into the price of capital and labor yields

$$r^* = 0.75(0.82)\left(\frac{9}{4}\right)^{-0.25}\left(\frac{1}{2}\right)^{0.25} = 0.42$$

$$w^* = 0.25(0.82)\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{-0.75} = 0.63$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Using consumer A's tangency condition, we know
$$x_2^A = \frac{p_1}{p_2} x_1^A \implies x_2^A = 0.82 x_1^A$$
  - Substituting this value into consumer A's budget constraint yields
$$p_1 x_1^A + p_2 (0.82 x_1^A) = rK^A + wL^A$$
  - Plugging in our calculated values and solving for x_1^A yields
$$x_1^{A,*} = 0.64$$
$$x_2^{A,*} = 0.82 x_1^{A,*} = 0.53$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Performing the same process with the tangency condition of consumer B yields
$$x_1^{B,*} = 0.90$$
$$x_2^{B,*} = 0.74$$
  - Thus, our WEA is
$$\left(x_1^A, x_2^A; x_1^B, x_2^B; \frac{p_1}{p_2}; L_1, L_2; K_1, K_2\right) =$$
$$\left(0.64,0.53; 0.90,0.74; 0.82; \frac{1}{2}, \frac{3}{2}, \frac{9}{4}, \frac{3}{4}\right)$$

General Equilibrium: Production

- ***Equilibrium with production – Welfare***:
  - We extend the First and Second Welfare Theorems to economies with production, connecting WEA and Pareto efficient allocations.
  - *Pareto efficiency*: The feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that

$$u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$$

for every consumer $i \in I$, with $u^i(\bar{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

General Equilibrium: Production

– In an economy with two goods, two consumers, two firms and two inputs (labor and capital), the set of Pareto efficient allocations solves

$$\max_{x_1^1, x_2^1, x_1^2, x_2^2, L_1, K_1, L_2, K_2 \geq 0} \quad u^1(x_1^1, x_2^1)$$

$$\text{s.t.} \quad u^2(x_1^2, x_2^2) \geq \bar{u}^2$$

$$\left.\begin{aligned} x_1^1 + x_2^1 &\leq F_1(L_1, K_1) \\ x_1^2 + x_2^2 &\leq F_2(L_2, K_2) \end{aligned}\right\} \text{tech. feasibility}$$

$$\left.\begin{aligned} L_1 + L_2 &\leq \bar{L} \\ K_1 + K_2 &\leq \bar{K} \end{aligned}\right\} \text{input feasibility}$$

General Equilibrium: Production

– The Lagrangian is

$$\mathcal{L} = u^1(x_1^1, x_2^1) + \lambda[u^2(x_1^2, x_2^2) - \bar{u}^2] + \mu_1[F_1(L_1, K_1) - x_1^1 - x_2^1] + \mu_2[F_2(L_2, K_2) - x_1^2 - x_2^2] + \delta_L[\bar{L} - L_1 - L_2] + \delta_K[\bar{K} - K_1 - K_2]$$

– In the case of interior solutions, the set of FOCs yield a condition for efficiency in consumption similar to barter economics:

$$MRS_{1,2}^1 = MRS_{1,2}^2$$

General Equilibrium: Production

– FOCs wrt L_j and K_j, where $j = \{1,2\}$, yield a condition for efficiency that we encountered in production theory

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

– That is, the $MRTS_{L,K}$ between labor and capital must coincide across firms.

– Otherwise, welfare could be increased by assigning more labor to the firm with the highest $MRTS_{L,K}$.

General Equilibrium: Production

- Combining the above two conditions for efficiency in consumption and production, we obtain

$$\frac{\frac{\partial U^i}{\partial x_1^i}}{\frac{\partial U^i}{\partial x_2^i}} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

- That is, $MRS_{1,2}^{i}$ must coincide with the rate at which units of good 1 can be transformed into units of good 2, i.e., the marginal rate of transformation $MRT_{1,2}$.

General Equilibrium: Production

- If we move labor from firm 2 to firm 1, the production of good 2 increases by $\frac{\partial F_2}{\partial L}$ while that of good 1 decreases by $\frac{\partial F_1}{\partial L}$. Hence, in order to increase the total output of good 1 by one unit we need $\frac{\partial F_2}{\partial L} / \frac{\partial F_1}{\partial L}$ units of good 2.
- *Intuition*: for an allocation to be efficient we need that the rate at which consumers are willing to substitute goods 1 and 2 coincides with the rate at which good 1 can be transformed into good 2.

General Equilibrium: Production

- ***First Welfare Theorem with production***: if the utility function of every individual i, u^i, is strictly increasing, then every WEA is Pareto efficient.

- *Proof* (by contradiction):
  – Suppose that $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is *not* Pareto efficient.
  – Since $(\mathbf{x}, \mathbf{y})$ is a WEA, then it must be feasible

$$\sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j$$

General Equilibrium: Production

– Because $(\mathbf{x}, \mathbf{y})$ is *not* Pareto efficient, there exists some other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$$

for every consumer $i \in I$, with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

- That is, the alternative allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ makes at least one consumer strictly better off than WEA .

– But this implies that bundle $\hat{\mathbf{x}}^i$ is more costly than $\mathbf{x}^i$,

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i$$

for every individual i (with at least one strictly inequality).

General Equilibrium: Production

– Summing over all consumers yields

$$\mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i$$

which can be re-written as

$$\mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j \right) > \mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j \right)$$

or re-arranging

$$\mathbf{p}^* \cdot \sum_{j=1}^{J} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j=1}^{J} \mathbf{y}^j$$

– However, this result implies that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j.

General Equilibrium: Production

- This indicates that production plan $\mathbf{y}^j$ was not profit-maximizing and, as a consequence, it cannot be part of a WEA.

- We therefore reached a contradiction.

- This implies that the original statement was true: if an allocation $(\mathbf{x}, \mathbf{y})$ is a WEA, it must also be Pareto efficient.

General Equilibrium: Production

- ***Example 6.9*** (WEA and PE with production):
  - Consider the setting described in example 6.8.
  - The set of Pareto efficient allocations must satisfy

$$MRS^A_{1,2} = MRS^B_{1,2} \text{ and } MRTS^1_{L,K} = MRTS^2_{L,K}$$

  - We can show that

$$MRS^A_{1,2} = \frac{x^A_2}{x^A_1} = \frac{0.53}{0.64} = 0.82$$

$$MRS^B_{1,2} = \frac{x^B_2}{x^B_1} = \frac{0.74}{0.90} = 0.82$$

which implies that $MRS^A_{1,2} = MRS^B_{1,2}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - We can also show that

$$MRTS^1_{L,K} = 3\frac{L_1}{K_1} = 3\left(\frac{1}{2}/\frac{9}{4}\right) = \frac{2}{3}$$

$$MRTS^2_{L,K} = 3\frac{L_2}{K_2} = 3\left(\frac{3}{2}/\frac{3}{4}\right) = \frac{2}{3}$$

    which implies that $MRTS^1_{L,K} = MRTS^2_{L,K}$.

  - Since both of these conditions hold, our WEA from example 6.8 is Pareto efficient.

General Equilibrium: Production

- ***Second Welfare Theorem with production***:
  - Consider the assumptions on consumers and producers described above.
  - Then, for every Pareto efficient allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ we can find:
    a) a profile of income transfers $(T_1, T_2, \ldots, T_I)$ redistributing income among consumers, i.e., satisfying $\sum_{i=1}^{I} T_i = 0$;
    b) a price vector $\overline{\mathbf{p}}$,

    such that:

General Equilibrium: Production

1) Bundle $\hat{\mathbf{x}}^i$ solves the UMP

$$\max_{\mathbf{x}^i} \quad u^i(\mathbf{x}^i)$$

$$\text{s.t.} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\bar{\mathbf{p}}) + T_i \text{ for every } i \in I$$

where individual i's original income $m^i(\bar{\mathbf{p}})$ is increased (decreased) if the transfer T_i is positive (negative).

2) Production plan $\hat{\mathbf{y}}^j$ solves the PMP

$$\max_{y^j} \quad \bar{\mathbf{p}} \cdot \mathbf{y}^j$$

$$\text{s.t.} \quad \mathbf{y}^j \in Y^j \text{ for every } j \in J$$

General Equilibrium: Production

- ***Example 6.10*** (Second Welfare Theorem with production):
  - Consider an alternative allocation in the set of Pareto efficient allocations identified in example 6.9.
  - Such as, $\left(\hat{x}_1^A, \hat{x}_2^A; \hat{x}_1^B, \hat{x}_2^B\right) = (0.82,1; 0.79,0.65)$.
  - Consumer A's budget constraint becomes
$$p_1\hat{x}_1^A + p_2\hat{x}_2^A = rK^A + wL^A + T_1$$
  - Recall that
$$(p_1, p_2; K^A, L^A; r, w) = (0.82,1; 1,1; 0.42,0.63)$$
    remains unchanged.

General Equilibrium: Production

– Substituting these values into consumer A's budget constraint

$$0.82\hat{x}_1^A + \hat{x}_2^A = 1.05 + T_1$$

– Recall that

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^A}{\hat{x}_1^A} \implies \hat{x}_2^A = 0.82\hat{x}_1^A$$

– Substituting

$$2(0.82)\underbrace{(0.75)}_{\hat{x}_1^A} = 1.05 + T_1 \implies T_1 = 0.17$$

General Equilibrium: Production

- Likewise for consumer B, his budget constraint becomes

$$p_1 \hat{x}_1^B + p_2 \hat{x}_2^B = rK^B + wL^B + T_2$$

- Substituting the unchanged values $(p_1, p_2; K^A, L^A; r, w) = (0.82,1; 1,1; 0.42,0.63)$,

$$0.82\hat{x}_1^B + \hat{x}_2^B = 1.47 + T_2$$

- Recall that

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^B}{\hat{x}_1^B} \Longrightarrow \hat{x}_2^B = 0.82\hat{x}_1^B$$

General Equilibrium: Production

– Substituting

$$2(0.82)\underbrace{(0.79)}_{\hat{x}_1^B} = 1.47 + T_1 \implies T_1 = -0.17$$

– Clearly, $T_1 + T_2 = 0$

– Thus these transfers will allow for the new allocation to be a WEA.

Comparative Statics

Comparative Statics

- We analyze how equilibrium outcomes are affected by an increase in:
  - the price of one good
  - the endowment of one input
- Consider a setting with two goods, each being produced by two factors 1 and 2 under constant returns to scale (CRS).
- A necessary condition for input prices (w_1^*, w_2^*) to be in equilibrium is

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - That is, firms produce until their marginal costs equal the price of the good.

Comparative Statics

- Let $z_{1j}(w)$ denote firm j's demand for factor 1, and $z_{2j}(w)$ be its demand for factor 2.
  - This is equivalent to the factor demand correspondences $z(w, q)$ in production theory.
- The production of good 1 is relatively more intense in factor 1 than is the production of good 2 if

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

where $\frac{z_{1j}(w)}{z_{2j}(w)}$ represents firm j's demand for input 1 relative to that of input 2.

Comparative Statics: Price Change

1) ***Changes in the price of one good,*** p_j **(Stolper-Samuelson theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - If the price of good j, p_j, increases, then:
     a) the equilibrium price of the factor more intensively used in the production of good increases; while
     b) the equilibrium price of the other factor decreases.

Comparative Statics: Price Change

- *Proof*:
  - Let us first take the equilibrium conditions

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - Differentiating them yields

$$\frac{\partial c_1(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_1(w_1,w_2)}{\partial w_2} dw_2 = dp_1$$

$$\frac{\partial c_2(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_2(w_1,w_2)}{\partial w_2} dw_2 = dp_2$$

  - Applying Shephard's lemma, we obtain

$$z_{11}(w) dw_1 + z_{12}(w) dw_2 = dp_1$$

$$z_{21}(w) dw_1 + z_{22}(w) dw_2 = dp_2$$

Comparative Statics: Price Change

– If only price p_1 varies, then $dp_2 = 0$.

– Hence, we can rewrite the second expression as

$$z_{21}(w)dw_1 + z_{22}(w)dw_2 = 0$$

$$\Longrightarrow \quad dw_1 = -\frac{z_{22}}{z_{21}}dw_2$$

– Solving for $\frac{dw_1}{dp_1}$ in the first expression yields

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22}-z_{12}z_{21}}$$

– Solving, instead, for $\frac{dw_2}{dp_1}$ yields

$$\frac{dw_2}{dp_1} = -\frac{z_{21}}{z_{11}z_{22}-z_{12}z_{21}}$$

Comparative Statics: Price Change

- From the factor intensity condition, $\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$, we know that $z_{11}z_{22} - z_{12}z_{21} > 0$.
- Hence, the denominator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is positive.
- The numerator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is also positive (they are just factor demands).
- Thus, $\frac{dw_1}{dp_1} > 0$ and $\frac{dw_2}{dp_1} < 0$.

Comparative Statics: Price Change

- ***Example 6.11***:
  - Let us solve for the input demands in Example 6.8:

$$r_1 = p_1 0.75 K_1^{-0.25} L_1^{0.25} \Longrightarrow z_{11} = K_1 = \left(\frac{3p_1}{4r}\right)^4 L_1$$

$$w_1 = p_1 0.25 K_1^{0.75} L_1^{-0.75} \Longrightarrow z_{21} = L_1 = \left(\frac{p_1}{4w}\right)^{\frac{4}{3}} K_1$$

$$r_2 = p_2 0.25 K_2^{-0.75} L_2^{0.75} \Longrightarrow z_{12} = K_2 = \left(\frac{p_2}{4r}\right)^{\frac{4}{3}} L_2$$

$$w_2 = p_2 0.75 K_2^{0.25} L_2^{-0.25} \Longrightarrow z_{22} = L_2 = \left(\frac{3p_2}{4w}\right)^4 K_2$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  – Since firm 1 is more capital intensive than firm 2, then $z_{11}z_{22} - z_{12}z_{21} > 0$ must hold, i.e.,

$$\left(\frac{3p_1}{4r}\right)^4 L_1 \left(\frac{3p_2}{4w}\right)^4 K_2 - \left(\frac{p_2}{4r}\right)^{\frac{4}{3}} L_2 \left(\frac{p_1}{4w}\right)^{\frac{4}{3}} K_1 > 0$$

  – From example 8.4, $\frac{K_1}{L_1} = 9\frac{K_2}{L_2} \Longrightarrow K_1 L_2 = 9K_2 L_1$.
  – Substituting this value into the above expression,

$$36.33\left(\frac{p_1 p_2}{rw}\right)^{\frac{8}{3}} - 1 > 0 \;\Longrightarrow\; \frac{p_1 p_2}{rw} > 0.26$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - In our solution, $\frac{p_1 p_2}{rw} = 3.08$, hence this condition is satisfied.
  - Next, observe that both z_{11} and z_{22} are trivially positive.
  - Applying the ***Stolper-Samuelson theorem*** yields

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}} > 0$$

$$\frac{dw_2}{dp_1} = -\frac{z_{21}}{z_{11}z_{22} - z_{12}z_{21}} < 0$$

Comparative Statics: Endowment Change

2) ***Changes in endowments*** **(Rybczynski theorem)**:
   – Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   – If the endowment of a factor increases, then
      a) production of the good that uses this factor more intensively increases; whereas
      b) the production of the other good decreases.

Comparative Statics: Endowment Change

- *Proof*:
  - Consider an economy with two factors, labor and capital, and two goods, 1 and 2.
  - Let $z_{Lj}(w)$ denote firm j's factor demand for labor (when producing one unit of output)
  - Similarly, let $z_{Kj}(w)$ denote firm j's factor demand for capital.
  - Then, factor feasibility requires

$$L = z_{L1}(w) \cdot y_1 + z_{L2}(w) \cdot y_2$$
$$K = z_{K1}(w) \cdot y_1 + z_{K2}(w) \cdot y_2$$

Comparative Statics: Endowment Change

– Differentiating the first condition

$$dL = z_{L1} \cdot \frac{\partial y_1}{\partial L} + z_{L2} \cdot \frac{\partial y_2}{\partial L}$$

– Dividing both sides by L yields

$$\frac{dL}{L} = \frac{z_{L1}}{L} \cdot \frac{\partial y_1}{\partial L} + \frac{z_{L2}}{L} \cdot \frac{\partial y_2}{\partial L}$$

– Multiplying the first term by $\frac{y_1}{y_1}$ and the second term by $\frac{y_2}{y_2}$, we obtain

$$\frac{dL}{L} = \frac{z_{L1} \cdot y_1}{L} \cdot \frac{\frac{\partial y_1}{\partial L}}{y_1} + \frac{z_{L2} \cdot y_2}{L} \cdot \frac{\frac{\partial y_2}{\partial L}}{y_2}$$

Comparative Statics: Endowment Change

– We can express:

a) $\frac{z_{Li}(w)\cdot y_i}{L} \equiv \gamma_{Li}$, i.e., the share of labor used by firm i;

b) $\frac{\frac{\partial y_i}{\partial L}}{y_i} \equiv \%\Delta y_i$, i.e., the percentage increase in the production of firm i brought by the increase in the endowment of labor;

c) $\frac{dL}{L} \equiv \%\Delta L$, i.e., the percentage increase in the endowment of labor in the economy.

Comparative Statics: Endowment Change

– Hence, the above expression becomes

$$\%\Delta L = \gamma_{L1} \cdot (\%\Delta y_1) + \gamma_{L2} \cdot (\%\Delta y_2)$$

– A similar expression can be obtained for the endowment of capital:

$$\%\Delta K = \gamma_{K1} \cdot (\%\Delta y_1) + \gamma_{K2} \cdot (\%\Delta y_2)$$

– Note that $\gamma_{L1}, \gamma_{L2} \in (0,1)$

- Hence, $\%\Delta L$ is a linear combination of $\%\Delta y_1$ and $\%\Delta y_2$.

– Similar argument applied to $\%\Delta K$, where $\gamma_{K1}, \gamma_{K2} \in (0,1)$.

Comparative Statics: Endowment Change

– Capital is assumed to be more intensively used in firm 1, i.e.,

$$\frac{K_1}{L_1} > \frac{K_2}{L_2} \text{ or}$$

$$\gamma_{K1} > \gamma_{L1} \text{ for firm 1 and } \gamma_{K2} < \gamma_{L2} \text{ for firm 2}$$

– Hence, if capital becomes relatively more abundant than labor, i.e., $\%\Delta K > \%\Delta L$, it must be that $\%\Delta y_1 > \%\Delta y_2$.

Comparative Statics: Endowment Change

– That is

$$\begin{array}{ccccccccc} \%\Delta L & = & \gamma_{L1} & \cdot & (\%\Delta y_1) & + & \gamma_{L2} & \cdot & (\%\Delta y_2) \\ \wedge & & \wedge & & \| & & \vee & & \| \\ \%\Delta K & = & \gamma_{K1} & \cdot & (\%\Delta y_1) & + & \gamma_{K2} & \cdot & (\%\Delta y_2) \end{array}$$

– *Intuition*: the change in the input endowment produces a more-than-proportional increase in the good whose production was intensive in the use of that input.

Comparative Statics: Endowment Change

- ***Example 6.12*** (Rybczynski Theorem):
  - Consider the production decisions of the two firms in Example 6.8, where we found that $K_1 = 3K_2$ and $K_1 + K_2 = \bar{K} = 3$.
  - Assume that total endowment of capital increases to $\bar{K} = 5$, i.e., $K_2 = 5 - K_1$.
  - The profit maximizing demands for capital are

$$K_1 = 3(5 - K_1) \implies K_1^* = \frac{15}{4}$$

$$K_2 = \frac{1}{3}K_1^* = \frac{5}{4}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Similarly, for labor we found that $L_1 = \frac{1}{3}L_2$ and $L_1 + L_2 = \bar{L} = 2$.
  - We do not alter the aggregate endowment of labor, $\bar{L} = 2$.
  - Hence, capital use by firm 1 increases from $K_1^* = \frac{9}{4}$ to $\frac{15}{4}$.
  - Firm 1 uses capital more intensively than firm 2 does, i.e., $\frac{K_1}{L_1} > \frac{K_2}{L_2}$, since $\frac{\frac{9}{4}}{\frac{1}{2}} > \frac{\frac{3}{4}}{\frac{3}{2}}$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - The factor demands for each good are

$$z_{K1} = \left(\frac{3r}{w}\right)^{-0.75} \text{ and } z_{L1} = \left(\frac{3r}{w}\right)^{0.25}$$

$$z_{K2} = \left(\frac{r}{3w}\right)^{-0.75} \text{ and } z_{L2} = \left(\frac{r}{3w}\right)^{0.25}$$

  - Using the values from example 6.8, we can assign following values:

$$(\gamma_{K1}, \gamma_{L1}, \gamma_{K2}, \gamma_{L2}) = (0.75, 0.25, 0.25, 0.75)$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Our two equations then become
$$0 = 0.25 \cdot (\%\Delta y_1) + 0.75 \cdot (\%\Delta y_2)$$
$$0.66 = 0.75 \cdot (\%\Delta y_1) + 0.25 \cdot (\%\Delta y_2)$$
  - Solving the above equations simultaneously yields
$$\%\Delta y_1 = 1 = 100\%$$
$$\%\Delta y_2 = -0.3333 = -33.33\%$$
  - *Intuition*: an increase in the endowment of capital by $\frac{5-3}{3} = 0.66 = 66\%$ entails an increase in good 1's output by 100% while that of good 2 decreases by 33.33%.

Introducing Taxes

Introducing Taxes: Tax on Goods

- Assume that a sales tax t_k is imposed on good k.
- Then the price paid by consumers increases by $p_k^C = (1 + t_k)p_k^P$, where p_k^P is the price received by producers.

- If the tax on good 1 and 2 coincides, i.e., $t_1 = t_2$, the price ratio consumers and producers face is unaffected:

$$\frac{p_1^C}{p_2^C} = \frac{(1+t_1)p_1^P}{(1+t_2)p_2^P} = \frac{p_1^P}{p_2^P}$$

  – Hence, the after-tax allocation is still Pareto efficient.

Introducing Taxes: Tax on Goods

- However, if only good 1 is affected by the tax, i.e., $t_1 > 0$ while $t_2 = 0$ (i.e., $t_1 \neq t_2$), then the allocation will not be Pareto efficient.
  - In this setting, the $MRTS_{L,K}$ is still the same as before the introduction of the tax:

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L}{w_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

  - Therefore, the allocation of inputs still achieves productive efficiency.

Introducing Taxes: Tax on Goods

– Similarly, the $MRT_{1,2}$ still coincides with the price ratio of goods 1 and 2:

$$\frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{p_1^P}{p_2} = \frac{\frac{\partial F_2}{\partial K}}{\frac{\partial F_1}{\partial K}}$$

where the price received by the producer, p_1^P, is the same before and after introducing the tax.

Introducing Taxes: Tax on Goods

- However, while the $MRS_{1,2}$ is equal to the price ratio that consumers face, i.e., $\frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2}$, it now becomes larger than the price ratio that producers face, $\frac{p_1^P}{p_2}$:

$$MRS_{1,2} = \frac{p_1^C}{p_2} = \frac{(1 + t_1)p_1^P}{p_2} > \frac{p_1^P}{p_2}$$

- *Intuition*:
  - The rate at which consumers are willing to substitute good 1 for 2 is larger than the rate at which firms can transform good 1 for 2.
  - Thus, the production of good 1 should decrease and that of good 2 increase.

Introducing Taxes: Tax on Inputs

- Similar arguments extend to the introduction of taxes on inputs
- Price paid by producers is $w_m^P = (1 + t_m)w_m^C$ for input $m = \{L, K\}$.
- If both inputs are subject to the same tax, i.e., $t_L = t_K = t$, the input price ratio consumers and producers face coincides:

$$\frac{w_L^P}{w_K^P} = \frac{(1+t)w_L^C}{(1+t)w_K^C} = \frac{w_L^C}{w_K^C}$$

  – Hence, the efficiency conditions is unaffected

Introducing Taxes: Tax on Inputs

- However, when taxes differ, $t_L \neq t_K$, productive efficiency no longer holds under such condition:
  - While input consumers satisfy

$$\frac{w_L^C}{w_K^C} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}}$$

    and input producers satisfy

$$\frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

Introducing Taxes: Tax on Inputs

the input price ratios they face do not coincide

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L^C}{w_K^C} \neq \frac{(1+t_L)w_L^C}{(1+t_K)w_K^C} = \frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

– For instance:
  - If $t_L > t_K$, the $MRTS_{L,K}$ is larger for firm 1 than 2,
  - Thus the allocation of inputs is inefficient, i.e., the marginal productivity of additional units of labor (relative to capital) is larger in firm 1 than in 2.

Appendix A:
Large Economies and the Core

Large Economies and the Core

- We know that equilibrium allocations (WEAs) are part of the Core.
- We now show that, as the economy becomes larger, the Core shrinks until exactly coinciding with the set of WEAs.

Large Economies and the Core

- Let us first consider an economy with I consumers, each with utility function u^i and endowment vector $\mathbf{e}^i$.
- Consider this economy's replica by doubling the number of consumers to $2I$, each of them still with utility function u^i and endowment vector $\mathbf{e}^i$.
  - There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.
- Define an r-fold replica economy $\mathcal{E}_r$, having consumers of each type, for a total of rI consumers.
  - For any consumer type $i \in I$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i \gg 0$.

Large Economies and the Core

- When comparing two replica economies, the largest will be that having more of every type of consumer.
- Allocation $\mathbf{x}^{iq}$ indicates the vector of goods for the qth consumer of type i.
- The feasibility condition is

$$\sum_{i=1}^{I} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i=1}^{I} \mathbf{e}^{i}$$

Large Economies and the Core

- ***Equal treatment at the Core***: If $\mathbf{x}$ is an allocation in the Core of the r-fold replica economy $\mathcal{E}_r$, then every consumer of type i must have the same bundle, i.e.,

$$\mathbf{x}^{iq} = \mathbf{x}^{iq'}$$

for every two "twins" q and q' of type i, $q \neq q' \in \{1,2, \dots, r\}$, and for every type $i \in I$.

  – That is, in the r-fold replica economy, not only similar type of consumers start with the same endowment vector $\mathbf{e}^i$, but they also end up with the same allocation at the Core.

Large Economies and the Core

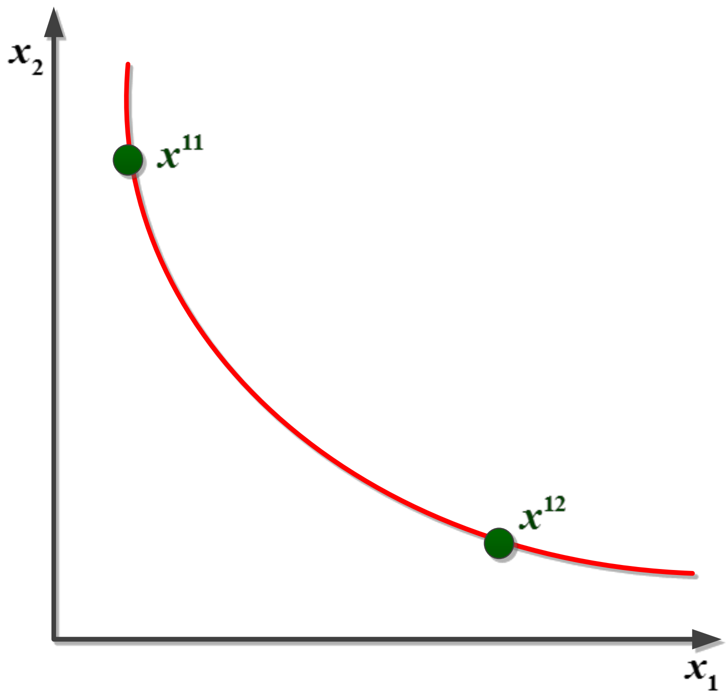
- *Proof* (by contradiction):
  - Consider a two-fold replica economy $\mathcal{E}_2$
    - The results can be generalized to r-fold replicas).
  - Suppose that allocation $\mathbf{x} \equiv \{\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22}\}$ is at the core of $\mathcal{E}_2$.
  - Since $\mathbf{x}$ is in the core, then it must be feasible, i.e.,
$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$$
  - Assume that allocation $\mathbf{x}$ does not assign the same consumption vectors to the two twins of type-1, i.e., $\mathbf{x}^{11} \neq \mathbf{x}^{12}$.

Large Economies and the Core

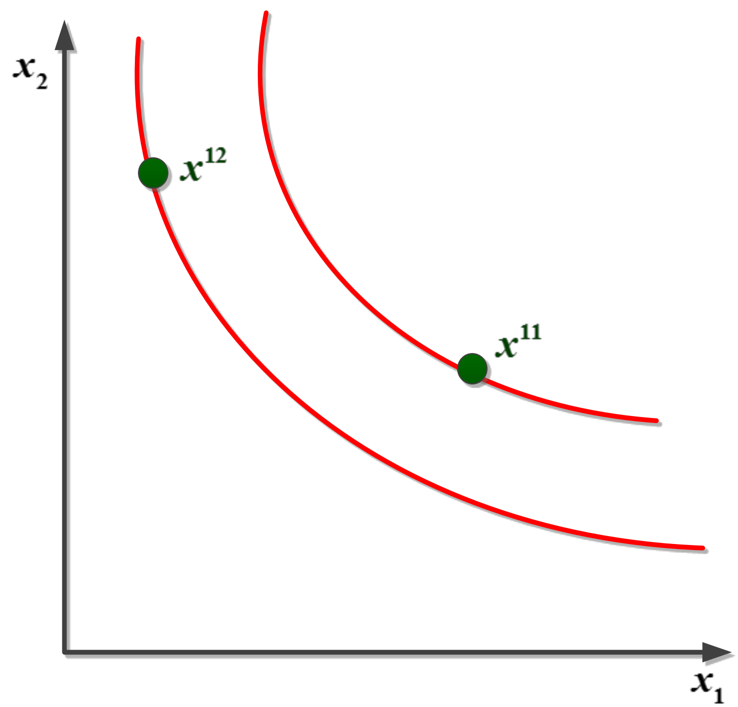
– Assume that type-1 consumer weakly prefers $\mathbf{x}^{11}$ to $\mathbf{x}^{12}$, i.e., $\mathbf{x}^{11} \succsim^{1} \mathbf{x}^{12}$.
  - This is true for both type-1 twins, since they have the same preferences.
  - A similar result emerges if we instead assume $\mathbf{x}^{12} \succsim^{1} \mathbf{x}^{11}$.

Large Economies and the Core

- Unequal treatment at the core for type-1 consumers



In this case $x^{11} \sim^1 x^{12}$ since both bundles lie on the same indifference curve



In this case $x^{11} \succ^1 x^{12}$

Large Economies and the Core

– Consider that for type-2 consumers we have $\mathbf{x}^{21} \succsim^{2} \mathbf{x}^{22}$.

– Hence, consumer 12 is the worst off type-1 consumer and consumer 22 is the worst off type 2 consumer.

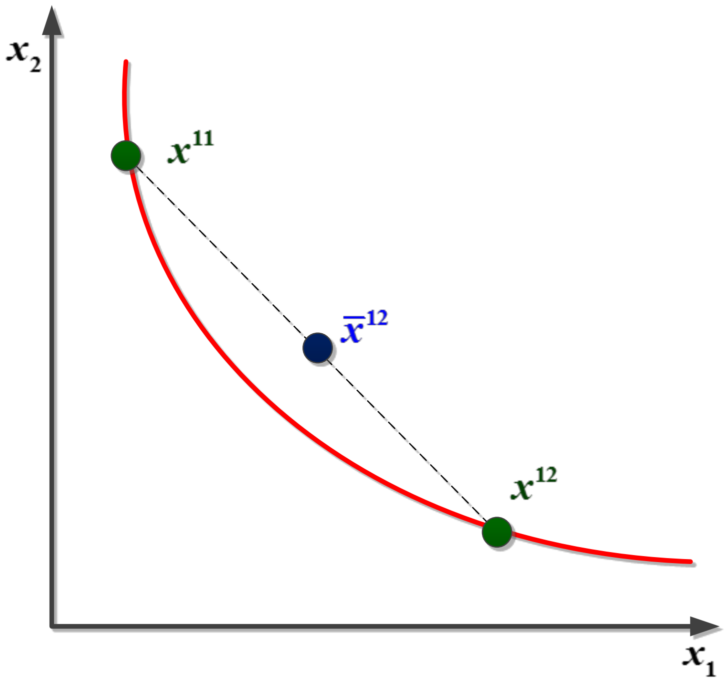
– Let us take these two "poorly treated" consumers of each type, and check if they can form a blocking coalition to oppose allocation $\mathbf{x}$.

– The average bundles for type-1 and type-2 consumers are

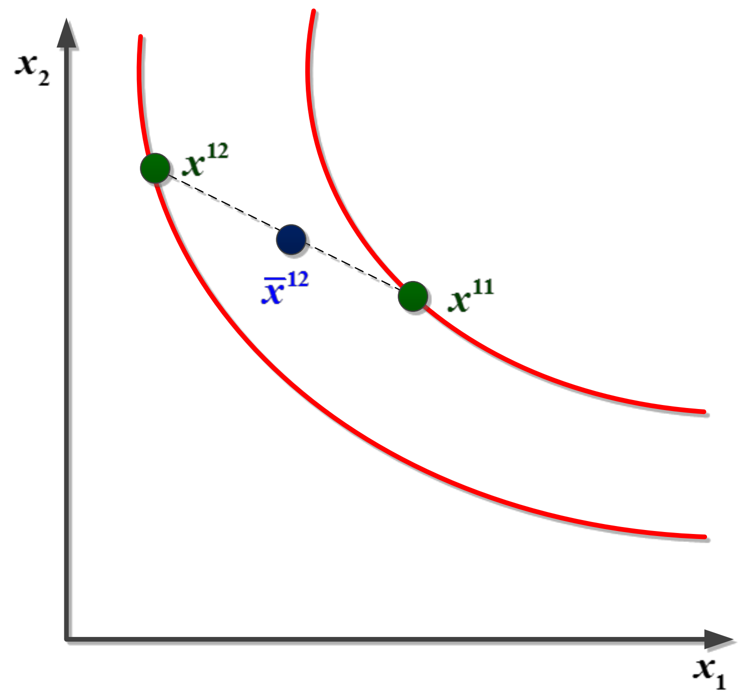
$$\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2} \text{ and } \bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2}$$

Large Economies and the Core

- Average bundles leading to a blocking coalition



In this case $x^{11} \sim^1 x^{12}$ but we can find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^1 x^{12}$



In this case $x^{11} \succ^1 x^{12}$, but we can still find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^1 x^{12}$

Large Economies and the Core

- *Desirability*. Since preferences are strictly convex, the worst off type-1 consumer prefers $\bar{\mathbf{x}}^{12} \succsim^1 \mathbf{x}^{12}$,
  - That is, a linear combination between $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ is preferred to the original bundle $\mathbf{x}^{12}$.
- A similar argument applies to the worst off type-2 consumer, i.e., $\bar{\mathbf{x}}^{22} \succsim^2 \mathbf{x}^{22}$.
- Hence, $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ makes both consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.

Large Economies and the Core

- *Feasibility*. Can consumers 12 and 22 achieve $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$?
- Sum the amount of goods consumers 12 and 22 need to achieve $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ to obtain

$$\bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2}$$
$$= \frac{1}{2}(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22})$$
$$= \frac{1}{2}(2\mathbf{e}^{1} + 2\mathbf{e}^{2}) = \mathbf{e}^{1} + \mathbf{e}^{2}$$

- Hence, the pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ is feasible.

Large Economies and the Core

– In summary, pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$:
  - makes consumers 12 and 22 better off than the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$
  - is feasible

– Thus, these consumers can block $(\mathbf{x}^{12}, \mathbf{x}^{22})$.
  - The original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$ cannot be at the Core.

– Therefore, if an allocation is at the Core of the replica economy, it must give consumers of the same type the exact same bundle.

Large Economies and the Core

- If $\mathbf{x}$ is in the core of a r-fold replica economy $\mathcal{E}_r$, i.e., $\mathbf{x} \in C_r$, then (by the equal treatment property) allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = (\underbrace{\mathbf{x}^1, \dots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \dots, \mathbf{x}^2}_{r \text{ times}}, \dots, \underbrace{\mathbf{x}^I, \dots, \mathbf{x}^I}_{r \text{ times}})$$

  – All consumers of the same type must receive the same bundle.

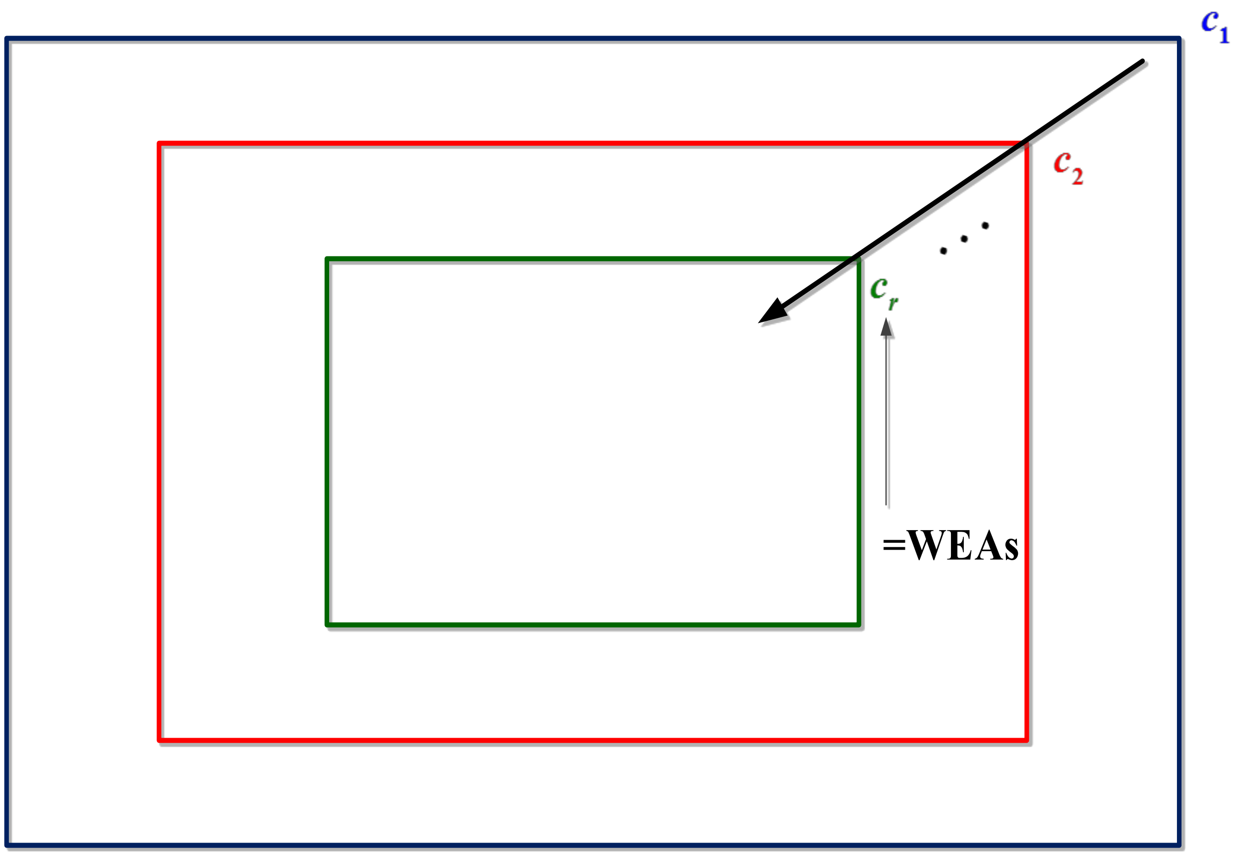
  – Core allocations in $\mathcal{E}_r$ are r-fold copies of allocations in $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^I)$.

Large Economies and the Core

- ***The core shrinks as the economy enlarges***. The sequence of core sets $C_1, C_2, \dots, C_r$ is decreasing.
- That is,
  - the core of the original (un-replicated) economy, C_1, is a superset of that in the 2-fold replica economy, C_2;
  - the core in the 2-fold replica economy, C_2, is a superset of the 3-fold replica economy, C_3;
  - etc.

Large Economies and the Core

- The Core shrinks as r increases



Large Economies and the Core

- *Proof*:
  - Since we seek to show that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{r-1} \supseteq C_r$, it suffices to show that, for any $r > 1$, $C_{r-1} \supseteq C_r$.
  - Suppose that allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^I) \in C_r$.
  - There is no blocking coalition to $\mathbf{x}$ in the r-fold replica economy $\mathcal{E}_r$.
  - We then need to show that $\mathbf{x}$ cannot be blocked by any coalition in the $(r-1)$-fold replica economy either.
    - If we could find a blocking coalition to $\mathbf{x}$ in $\mathcal{E}_{r-1}$, then we could also find a blocking coalition in $\mathcal{E}_r$.
    - All members in $\mathcal{E}_{r-1}$ are also present in the larger economy $\mathcal{E}_r$ and their endowments have not changed.

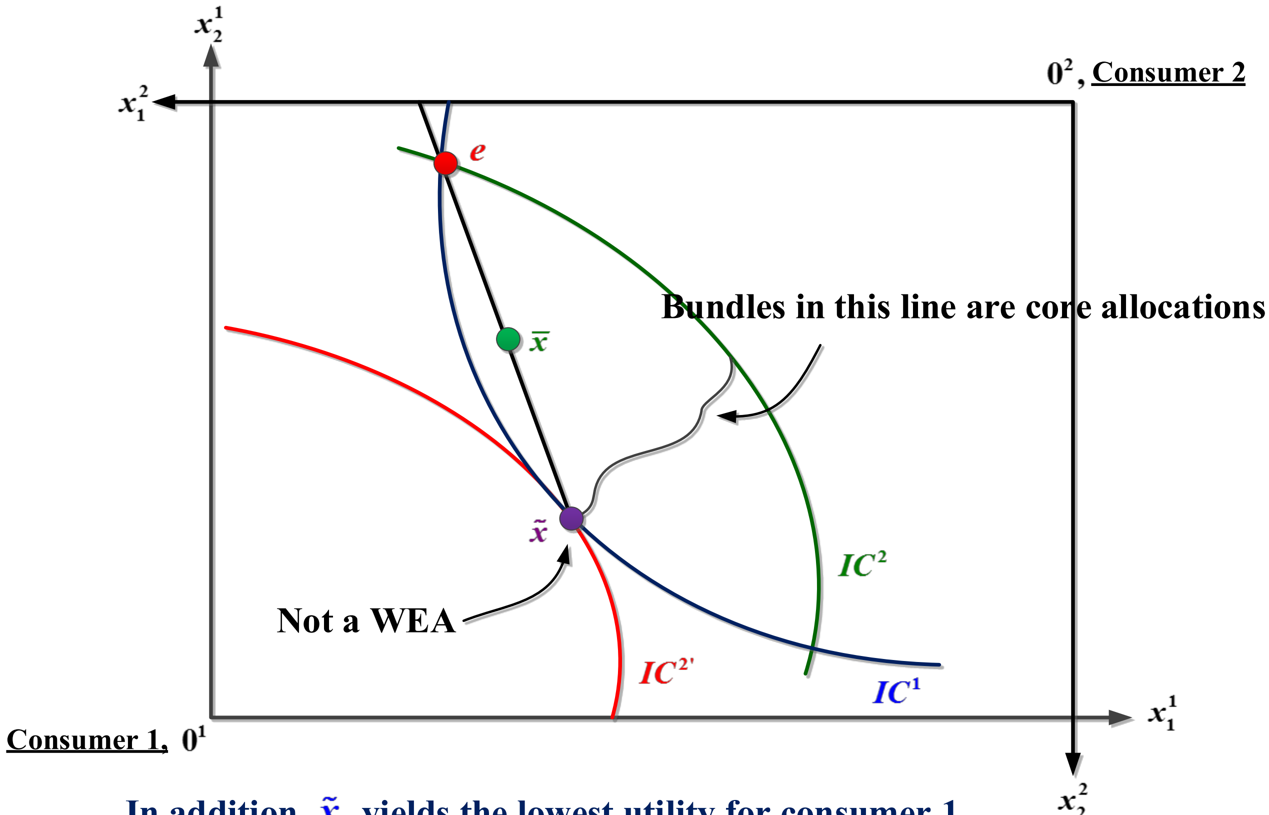
Large Economies and the Core

- Now we need to show that, as r increases, the core shrinks.

- We will do this be demonstrating that allocations at the frontier of C_1 do not belong to the core of the 2-fold replica economy, C_2.

Large Economies and the Core

- Un-replicated economy $\mathcal{E}_1$



In addition, $\tilde{x}$ yields the lowest utility for consumer 1, among all core allocations.

Large Economies and the Core

- The line between $\tilde{\mathbf{x}}$ and $\mathbf{e}$ includes core allocations.
  - All points in the line are part of the core.
  - However, not all points in this line are WEAs.
  - For instance: $\tilde{\mathbf{x}}$ is not a WEA since the price line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumer's indifference curve at $\tilde{\mathbf{x}}$.
- If the Core shrinks as the economy enlarges, we should be able to show that allocation $\tilde{\mathbf{x}} \notin C_2$.
- Let us build a blocking coalition against $\tilde{\mathbf{x}}$.

Large Economies and the Core

- *Desirability*. Consider the midpoint allocation $\bar{\mathbf{x}}$ and the coalition $S = \{11,12,21\}$. Such a midpoint in the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is strictly preferred by both types of consumer 1.

- If the midpoint allocation $\bar{\mathbf{x}}$ is offered to both types of consumer 1 (11 and 12), and to one of the consumer 2 types, they will all accept it:

$$\bar{\mathbf{x}}^{11} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) \succ^1 \tilde{\mathbf{x}}^{11}$$

$$\bar{\mathbf{x}}^{12} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}) \succ^1 \tilde{\mathbf{x}}^{12}$$

$$\tilde{\mathbf{x}}^{21} \sim^2 \tilde{\mathbf{x}}^{21}$$

Large Economies and the Core

– *Feasibility*. Since $\bar{\mathbf{x}}^{11} = \bar{\mathbf{x}}^{12}$, then the sum of the suggested allocations yields

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = 2\frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) + \tilde{\mathbf{x}}^{12}$$
$$= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{12}$$

– Recall that $\tilde{\mathbf{x}}$ is part of the un-replicated economy $\mathcal{E}_1$.

- Hence, it must be feasible, i.e., $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$.
- Therefore, $\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{12} = \mathbf{e}^1 + \mathbf{e}^2$.

Large Economies and the Core

– We can thus re-write the above equality as

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \underbrace{\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{12}}_{\mathbf{e}^1+\mathbf{e}^2}$$

$$= \mathbf{e}^1 + \mathbf{e}^1 + \mathbf{e}^2 = 2\mathbf{e}^1 + \mathbf{e}^2$$

which confirms the feasibility.

– Hence, the frontier allocation $\tilde{\mathbf{x}}$ in the core of the un-replicated economy does not belong to the core of the two-fold economy, $\tilde{\mathbf{x}} \notin C_2$.

- There is a blocking coalition $S = \{11,12,21\}$ and an alternative allocation $\bar{\mathbf{x}} = \{\bar{\mathbf{x}}^{11}, \bar{\mathbf{x}}^{12}, \tilde{\mathbf{x}}^{21}\}$ that they would prefer to $\tilde{\mathbf{x}}$ and that is feasible for the coalition members.

Large Economies and the Core

- ***WEA in replicated economies***:
  - Consider a WEA in the un-replicated economy $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^I)$.
  - An allocation $\mathbf{x}$ is a WEA for the r-fold replica economy $\mathcal{E}_r$ iff it is of the form

$$\mathbf{x} = (\underbrace{\mathbf{x}^1, \dots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \dots, \mathbf{x}^2}_{r \text{ times}}, \dots, \underbrace{\mathbf{x}^I, \dots, \mathbf{x}^I}_{r \text{ times}})$$

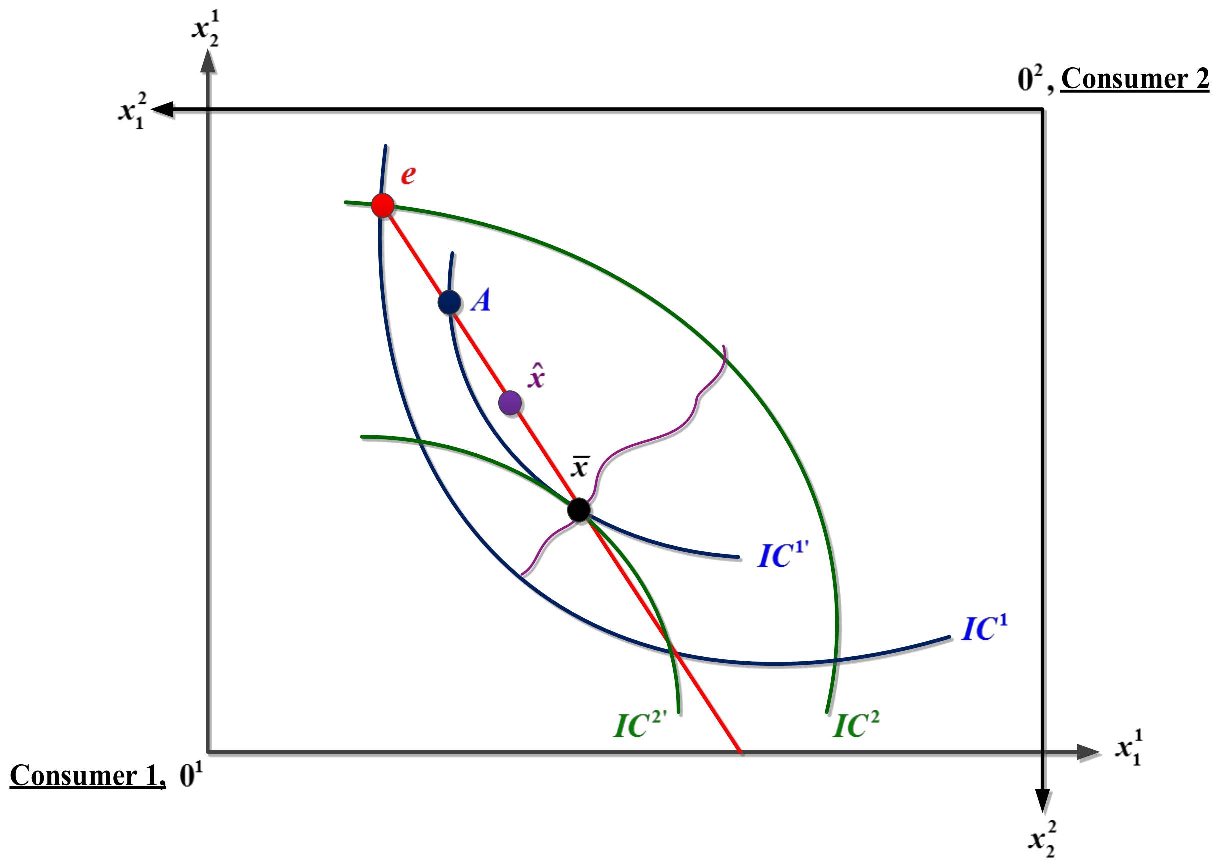
  - If $\mathbf{x}$ is a WEA for $\mathcal{E}_r$, then it also belongs to the core of that economy (by the "equal treatment at the core" property).

Large Economies and the Core

- ***A limit theorem on the Core***: If an allocation $\mathbf{x}$ belongs to the core of all r-fold replica economies then such allocation must be a WEA of the un-replicated economy $\mathcal{E}_1$.
- *Proof* (by contradiction):
  – Consider that an allocation $\tilde{\mathbf{x}}$ belongs to the core of the r-fold replica economy C_r but is *not* a WEA.
  – A core allocation for the un-replicated economy $\mathcal{E}_1$, $\tilde{\mathbf{x}} \in C_1$ satisfyies $\tilde{\mathbf{x}} \in C_r$ since $C_1 \supset C_r$.
  – Allocation $\tilde{\mathbf{x}}$ must then be within the lens-shaped area and on the contract curve.

Large Economies and the Core

- A core allocation $\tilde{\mathbf{x}}$ that is not WEA



Large Economies and the Core

- Consider now the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$.
- Since $\tilde{\mathbf{x}}$ is not a WEA, the budget line cannot be tangent to both consumers' indifference curves:

$$\frac{p_1}{p_2} > MRS \text{ or } \frac{p_1}{p_2} < MRS$$

- Can allocation $\tilde{\mathbf{x}}$ be at the Core C_r and yet not be a WEA?
- Let us show that if $\tilde{\mathbf{x}}$ is not a WEA it *cannot* be part of the Core C_r either.
  - To demonstrate that $\tilde{\mathbf{x}} \notin C_r$, let us find a blocking coalition

Large Economies and the Core

- By the convexity of preferences, we can find a set of bundles (such those between A and $\tilde{\mathbf{x}}$) that consumer 1 prefers to $\tilde{\mathbf{x}}$:

$$\hat{\mathbf{x}} \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1$$

  for some $r > 1$, where $\frac{1}{r} + \frac{r-1}{r} = 1$.
- Consider a coalition S with all r type-1 consumers and $r - 1$ type-2 consumers.
- Let us now show that allocation $\hat{\mathbf{x}}$ satisfies the properties of acceptance and feasibility for the blocking coalition S.

Large Economies and the Core

- *Acceptance*. If we give every type-1 consumer the bundle $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1$. Similarly, if we give every type-2 consumer in the coalition the bundle $\hat{\mathbf{x}}^2 = \tilde{\mathbf{x}}^2$, then $\hat{\mathbf{x}}^2 \sim^2 \tilde{\mathbf{x}}^2$.
- *Feasibility*. Summing over the consumers in coalition S, their aggregate allocation is

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = r\left[\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right] + (r-1)\tilde{\mathbf{x}}^2$$
$$= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)$$

- Since $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ is in the core of the un-replicated economy $\mathcal{E}_1$, then it must be feasible $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$.

Large Economies and the Core

– Combining the above two results, we find that

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = \mathbf{e}^1 + (r-1)\underbrace{(\mathbf{e}^1 + \mathbf{e}^2)}_{\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2}$$
$$= r\mathbf{e}^1 + r(\mathbf{e}^1 + \mathbf{e}^2) - (\mathbf{e}^1 + \mathbf{e}^2)$$
$$= r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$

which confirms feasibility.

– Hence, r type-1 consumers and $r-1$ type-2 consumers can get together in coalition and block allocation $\tilde{\mathbf{x}}$.

Large Economies and the Core

– Thus if $\tilde{\mathbf{x}}$ is not a WEA, then, $\tilde{\mathbf{x}}$ cannot be in the Core of the r-fold replica economy $\mathcal{E}_r$.

– As a consequence, if $\tilde{\mathbf{x}} \in C_r$ for all $r > 0$, then $\tilde{\mathbf{x}}$ must be a WEA.

Appendix B:
Marshall–Hicks Four Laws of Derived Demand

Marshall–Hicks Four Laws

- Consider a production function $q = f(K, L)$, with positive marginal products, $f_L, f_K > 0$.
- Assume that the supply of each input $(w(L), r(K))$ is positively sloped, $w'(L) > 0$ and $r'(K) > 0$.
- Demand for output is given by $q = g(p)$, which satisfies $g'(p) < 0$.
- The total cost is $w(L)L + r(K)K$.
- Assume that the capital market is perfectly competitive, but the labor and output markets are not necessarily competitive.

Marshall–Hicks Four Laws

- Define:
  - $\varepsilon_{q,p} = (\partial q/\partial p)(p/q)$ as the price elasticity of output
  - $s_{K,r} = (\partial K/\partial r)(r/K)$ as the elasticity of capital supply to a change in its price
  - $s_{L,r} = (\partial L/\partial r)(r/L)$ as the elasticity of labor supply to a change in the price of capital
  - $s_{L,w} = (\partial L/\partial w)(w/L)$ as the elasticity of labor supply to a change in its price
  - σ as the elasticity of substitution between inputs
- We use superscript i to refer to the elasticity that an individual firm faces ($\varepsilon_{q,p}^{i}$).
- The industry elasticities do not include superscripts ($\varepsilon_{q,p}$).

Marshall–Hicks Four Laws

- Let $\theta_L \equiv wL/pq$ and $\theta_K \equiv rK/pq$ be the cost of labor and capital, respectively, relative to total sales.
- This implies that $\theta_L = 1 - \theta_K$.
- For compactness, let us define

$$A \equiv 1 - (1/\varepsilon_{q,p}^{i})$$
$$B \equiv 1 + (1/s_{L,w}^{i})$$

Marshall–Hicks Four Laws

- Marshall, Hicks, and Allen analyze how the input demand of a perfectly competitive input, such as capital, is affected by a marginal change in the price of capital:

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} A + (\sigma \varepsilon_{q,p}/s_{L,w})A^2 + \theta_L AB\sigma}{(\theta_K + \theta_L B)^2 + \theta_K(\sigma/s_{L,w})A + \theta_L(\sigma/s_{L,w})AB}$$

Marshall–Hicks Four Laws

- Marshall–Hicks's four laws of input demand ("**derived demand**") state that an input demand becomes more elastic, whereby $s_{K,r}$ decreases, in
  1. the elasticity of substitution between inputs σ
  2. the price-elasticity of output demand $\varepsilon_{q,p}$
  3. the cost of the input relative to total sales θ_K
  4. the elasticity of the other input's supply to a change in its price $s_{L,w}$
- We analyze these four comparative statics under two market structures:
  1. **The Marshall's presentation**: $\varepsilon^i_{q,p} = s^i_{L,w} = \infty, \sigma = 0$
  2. **The Hick's presentation**: $\varepsilon^i_{q,p} = s^i_{L,w} = \infty$ (no assumptions on σ)

Marshall's Presentation

- Assumptions:
  - Output and inputs markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i
  - Inputs cannot be substituted in the production process, $\sigma = 0$
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w}}{s_{L,w} + \theta_L \varepsilon_{q,p}}$$

Marshall's Presentation

- The derivatives testing the laws are:

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K (s_{L,w})^2}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{s_{L,w} \cdot \varepsilon_{q,p}(s_{L,w} + \varepsilon_{q,p})}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L (\varepsilon_{q,p})^2}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

- If labor is a "normal" input, $s_{L,w} > 0$, the three derivatives are all negative (the three laws hold).
- If labor is inferior, $s_{L,w} < 0$, $s_{K,r}$ is still decreasing in $\varepsilon_{q,p}^i$ and in $s_{L,w}^i$, but not necessarily in θ_K.

Hick's Presentation

- Assumptions:
  - Output and inputs markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i
  - No condition imposed on the substitution of inputs (σ)
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w} - \sigma \varepsilon_{q,p} - \theta_L \sigma s_{L,w}}{s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}$$

Hick's Presentation

- The derivatives testing the laws are

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K (s_{L,w} + \sigma)^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{(\varepsilon_{q,p} + s_{L,w}) + \ (s_{L,w} + \sigma)(\varepsilon_{q,p} - \sigma)}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L (\varepsilon_{q,p} - \sigma)^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \sigma} = -\frac{\theta_L (\varepsilon_{q,p} + s_{L,w})^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

- Hence, $s_{K,r}$ decreases in $\varepsilon_{q,p}$, $s_{L,w}$, and σ (the three laws hold).
- $s_{K,r}$ also decreases in θ_K if the input is "normal", $s_{L,w} > 0$, and inputs are not extremely easy to substitute, $\varepsilon_{q,p} > \sigma$.