Advanced Microeconomics I

Old transparencies (Sommersemester 2016)

Organizational preliminaries

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- Office hours:
  - Monday, 2 – 4 pm;
    please contact: vwl4@uni-bayreuth.de (Heidi Rossner)
- Downloads and information:
  https://elearning.uni-bayreuth.de/
- Two “identical” classes per week; 1–2 weeks delay to lectures
- One-open-book exam will be posed in English;
  can be answered in English or German
  (same for optional midterm exam)

Textbooks

- The reference (consider *buying* it):
  - Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995). *Microeconomic Theory*. New York, NY: Oxford University Press. (≡ *MWG*)
- Other recommended textbooks:
  - Jehle, Geoffrey A., and Philip J. Reny (2011). *Advanced Microeconomic Theory*, 3rd edition. Amsterdam: Addison-Wesley.
  - Rubinstein, Ariel (2012). *Lecture Notes in Microeconomic Theory: The Economic Agent*, 2nd edition. Princeton, NJ: Princeton University Press.
    [it's free: http://arielrubinstein.tau.ac.il/]
  - Varian, Hal R. (1992). *Microeconomic Analysis*, 3rd edition. New York, NY: W. W. Norton & Company.

Goals and structure

- Goals of this course:
  - Introduce key concepts of advanced microeconomic analysis
  - Aid the self study of MWG
  - Prepare for possible PhD studies:
    we pick a level below typical PhD programs, but familiarize ourselves with the standard textbook
    → you may skip the small print and most proofs for now
- Structure follows MWG

Tentative schedule for lectures

#	**Topic**	**Chs.**
1	Introduction	
2	Preference and choice	1.A-D
3	Consumer choice	2.A-F
4	Classical demand theory	3.A-E, G
5	Aggregate demand	3.I; 4.A-D
6	Choice under risk	6.A-D, F
7	Static games of complete information	7.A-E; 8.A-D, F
8	Dynamic games of complete information	9.A-B; 12.App. A
9	Games of incomplete information	8.E, 9.C
10	Competitive Markets	10.A-G
11	Market power	12.A-F
12	Question session for exam	

… blood, toil, tears, and sweat

- This course is different …
  - Lectures will *not* provide a self-contained treatment of *all* material
  - Strenuous self-study cannot be avoided (workload still *much* lower than in a UK/US research MSc/PhD program; NB: 8 ECTS points imply *8 h of homework / week*, plus 4 h attendance!*)*
- Mixture of slides and chalk & talk
- Optional midterm exam:
  - Two problems on topics of sessions #1 – #6, each graded in a binary fashion ("+" or "o")
  - Each "+" earns 5 bonus points for the 60-point final exam
- Most of what you learn in this course will be learned by *doing problems*, i.e., preparing for weekly classes and exams

1. Introduction

- Microeconomics studies the behavior of individuals or groups, how they interact and bring about collective outcomes
- We will look at models of
  – Preferences, consumer choice, demand, choice under risk
  – Strategic decision making (= game theory)
  – Perfectly and imperfectly competitive markets
  – Market failure, asymmetric information, and mechanism design
  – General equilibrium
  – Social choice and welfare

Models

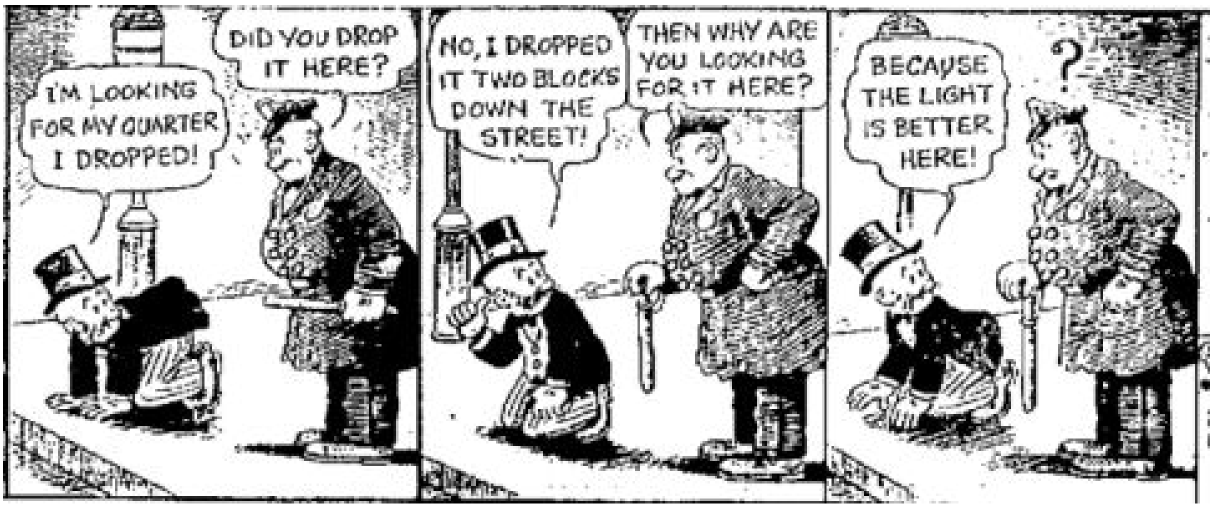
- "Models" are simplified descriptions of a part of reality (≡ the domain in which we observe and take measurements)
- Their purposes include
  – description per se
  – explanation and prediction
  – justification
  – decision support
- They can be represented in different ways, e.g.,
  – verbally
  – graphically or mechanically
  – mathematically
  – in a programming language
- All representations boil down to a system of assumptions, axioms, premises, or initial conditions $\{A_1, \ldots, A_n\}$

Models

- The system of assumptions, axioms, etc. should
  - be logically consistent, irreducible, and comprehensible (A. Einstein: "… as simple as possible, but not simpler!")
  - relevant for the model's purpose, relate to reality, and have at least some empirical support
- The advantage of stating $\{A_1, \ldots, A_n\}$ *mathematically* instead of in everyday language or software is that the model is particularly
  - concise and transparent
  - easy to check for consistency
  - amenable to formal manipulations and logical deduction
- Mathematical models are constructed with manipulability in mind; this implies a delicate trade-off with realism

(Danger: "Searching where the light is rather than where the keys were lost…")

"Searching where the light is …"



http://quoteinvestigator.com/2013/04/11/better-light/

Models and economic theory

- Some philosophers of science (Hempel, Oppenheim) argued that the distinctive feature of a *theory* (vs. a model) is: at least some A_i is a *universal law*, i.e., a time and space-invariant, necessary connection between certain phenomena
- Such requirements would preclude any economic theory …
- Social scientists have to contend themselves with restricted regularities or mere tendencies (vs. laws of mechanics)
  - e.g., that individuals can usually decide between two available options and mostly do so in a consistent fashion
- Economics is more difficult than physics also because it involves interpretation of phenomena created by objects of study (individuals, firms, …) who also base their actions on interpretations of reality, possibly influenced by economic theory

Do economics, not mathematics!

- Most microeconomic analysis uses mathematical language and techniques
- We need to do the maths because even trained economic intuition is sometimes wrong:
  - One obtains a 'counter-intuitive' result doing the maths, and only facing it realizes that some (ex post: intuitive) causal relations were overlooked
- It is essential to focus on the economics in what you read and do, even though the maths tend to be more time-consuming
- A good intuition about agents' economic incentives is more useful than superb knowledge of Kuhn-Tucker conditions or semidefiniteness of matrices, even in optimization problems

Example (1)

- Consider the following simple microeconomic problem:
  - Julian wants to buy spoons and forks
  - Each pair of one spoon and one fork gives Julian 1 unit of utility
  - A spoon not matched with a fork gives him only a units of utility, where $0 \leq a < ½$; a fork not matched with a spoon also gives a
  - Let p_1 be the price of spoons, p_2 the price of forks, and w the wealth that Julian plans to spend on spoons and forks
  - Assume he wants to get the most utility for each euro he spends
- Find Julian's demand functions for spoons and forks!

2. Preference and choice

- The basic constituent of most economic models is the neoclassical "economic man" or *homo economicus*
- He or she is a highly stylized model of real decision makers
- "*economicus*" refers to "the economic way" of decision making, not to the context of decisions
- Broadly speaking, *homo economicus* is assumed to
  - deliberately choose the most suitable means to his or her ends
  - evaluate options according to their anticipated consequences (decisions are made in the "shadow of the future")
  - weigh the costs and the benefits of a particular choice

  … or rather behave "as if" he or she would be doing so

Preliminary remarks

- While the rationality embodied by *homo economicus* is the key assumption of most of modern economics, it should not be taken too literally
- Hardly any economist thinks that real people *are* as deliberate, future-oriented, and clever as will commonly be assumed
- Most would hold that people are behaving *as if* they were "economically rational" sufficiently often to derive useful conclusions from correspondingly pragmatic models
- See
  - Ariely, Dan (2008). *Predictably Irrational*. London: Harper Collins.
  - Kahneman, Daniel (2011). *Thinking, Fast and Slow*. New York: Farrar, Straus and Giroux.

  for illuminating accounts of the "biases" of real decision makers

Choosing between several alternatives

- Consider an agent who needs to choose between several actions
- Suppose each action is associated with a particular outcome, and these outcomes are all that the agent cares about
- Denote the set of all possible, mutually exclusive outcomes (or options or alternatives) by X
  - Options can be very concrete, like
    X = {go to law school in Berlin, study economics in Bayreuth, ...},
    or, for us, abstract like $X = \{x,y,z\}$.
- Economics presumes that whenever she chooses from the subset $X' \subseteq X$, the agent picks an option $x \in X'$ which serves his or her goals best (whatever they may be…)

$\Rightarrow$ If we observe that the agent chooses x from X', *we conclude* that x was amongst the best options in X' for this agent

Preferences vs. choice rules

- There are two main approaches to modeling choice behavior:
  - Binary preference relations
  - Choice rules
- Preference relations are less general, but more handy
  (with many further restrictions imposed to make them even more handy)
- Observing the choice of x when X' was available reveals that x is *weakly preferred* to any other element $y \in X'$ *when a choice must be made from X'*
- The preference approach entails the simplifying assumption:
  x is weakly preferred to y *independently* of the presence or absence of any other alternatives $z \in X$, i.e., also when a choice must be made from $X'' \neq X'$

Preference relations

- Given such context-independence, an agent's full choice behavior is well-defined by his choices from *binary* subsets $X' = \{x, y\}$
- When x is weakly preferred to y, we write: $x \succsim y$
- $\succsim$ puts (some) *pairs* of elements $x, y \in X$ into a specific relationship;
  it is known mathematically as a *binary relation*
- A binary relation is formally just a subset of $X \times X$;
  some authors write $(x,y) \in \succsim$ instead of $x \succsim y$
  (NB: a function f: $X \to Y$ can similarly be viewed as just a subset of $X \times Y$)

Other relations derived from $\succsim$

- If sometimes *x* and sometimes *y* is chosen out of $X' = \{x, y\}$, then the agent is said to be *indifferent* between *x* and *y*, i.e.,
$$x \succsim y \;\wedge\; y \succsim x \;\Leftrightarrow:\; x \sim y$$
- If the agent (weakly) prefers *x* over *y* and is not indifferent, he is said to *strictly prefer* *x* over *y*, i.e.,
$$x \succsim y \;\wedge\; \neg(y \succsim x) \;\Leftrightarrow:\; x \succ y$$
- $x \succ y$ is equivalent to saying:
"The agent never chooses *y* when *x* is available"

Rational preferences

- Economics does not care about *why* somebody prefers *x* to *y*; neither does it proclaim which option the agent *should* prefer
- The common requirement for calling an individual *rational* is that his choices reflect preferences that are "complete" and "transitive"
- *Complete* means that for any two options $x, y \in X$, the agent either weakly prefers *x* or weakly prefers *y* or both, i.e.,
$$\forall x, y \in X:\; x \succsim y \;\vee\; y \succsim x$$
- Completeness formalizes that the agent can reach a decision facing any binary choice problem
- *Transitive* means that a preference for *x* over *y* together with a preference for *y* over *z* also entails a preference for *x* over *z*, i.e.,
$$x \succsim y \;\wedge\; y \succsim z \;\Rightarrow\; x \succsim z$$
- Transitivity rules out cycles, which would, e.g., preclude a decision facing $X' = \{x, y, z\}$

Violations of transitivity

- An argument against persistent intransitivity of real people is that one might (or the market would) ruin them with a *money pump*:
  - Suppose your colleague has intransitive preferences:

    apple $\succ$ banana $\succ$ citrus fruit $\succ$ apple
  - Give him an apple for free
  - Then offer to sell him a citrus fruit for the apple and, e.g., 1 cent; he will accept because he strictly prefers the citrus fruit
  - Next sell him a banana for the citrus fruit and 1 cent
  - Now sell him an apple for the banana and 1 cent, and repeat the cycle …
- However, this ignores transaction costs, and the possibility that an intransitivity may be corrected (only) if someone exploits it big time
- Intransitivity is normal when alternatives are very finely graded:
  - $\forall k \in \mathbf{N}_0$: coffee with k grains of sugar ~ coffee with k+1 grains of sugar $\Rightarrow$ coffee without sugar ~ coffee with 100g of sugar ?

Utility representation

- If set of alternatives X is *finite* (or *countably infinite*) and the agent has a complete and transitive preference relation $\succsim$ over it, then the agent's preferences over X can be *represented* by a *utility function* $u : X \to \mathbf{R}$, i.e., we can find real numbers $u(x)$ such that

$$x \succsim y \quad \Leftrightarrow \quad u(x) \geq u(y)$$

- Note that if $u(\cdot)$ represents the agent's preferences, then so does any $v(\cdot)$ which is a strictly increasing transformation of $u(\cdot)$
- The latter implies that the difference or ratio between utility levels of x and y do *not* mean anything:
  $u(\cdot)$ only allows conclusions about the order of x and y, and is therefore called an *ordinal* utility function

Utility representation

- If the set of alternatives X is *uncountably infinite*, then completeness and transitivity of a preference relation are not sufficient in order to guarantee existence of a utility representation
- In particular, *lexicographic preferences* $\succsim_L$ over bundles of two goods $(x_1, x_2) \in \mathbf{R}_+^2$, defined by

$$(x_1, x_2) \succ_L (y_1, y_2) \;:\Leftrightarrow\; x_1 > y_1 \vee \{ x_1 = y_1 \wedge x_2 > y_2 \},$$

  and

$$(x_1, x_2) \sim_L (y_1, y_2) \;:\Leftrightarrow\; x_1 = y_1 \wedge x_2 = y_2$$

  do not possess a utility representation

Utility representation

- A preference relation $\succsim$ is called *continuous* if

  whenever $x_k \succsim y$ (resp. $y \succsim x_k$) holds for all elements x_k of a sequence $\{x_k\}_{k=1, 2, \ldots}$ with limit point x^*
  then $x^* \succsim y$ (resp. $y \succsim x^*$) must also be true
- Continuity rules out that negligible changes completely reverse the ordering of two options:
  - Lexicographic preferences rank $(2+1/k, 1)$ strictly higher than $(2, 2)$ for every $k = 1, 2, \ldots$
  - The limit point (2,1), however, is ranked strictly lower than (2, 2)
- An important result in decision theory states:

  If $\succsim$ is a complete, transitive, and continuous preference relation on the arbitrary set of outcomes X, then
  - $\succsim$ can be represented by an ordinal utility function $u : X \rightarrow \mathbf{R}$
  - $u(\cdot)$ can be chosen to be continuous (but not necessarily also differentiable, or even C^1)

Remarks

- Economic rationality itself does not require existence of a utility representation of an agent's preferences
- Only for convenience is economic rationality sometimes equated with utility-maximizing behavior, but inaccurately so
- In any case, assuming utility maximization does *not* require agents to "know their utility function" and "try to maximize"; as it happens, if their preferences satisfy completeness and transitivity (plus continuity), they act exactly *as if* they did …
- Use of a particular utility function (e.g., $u(x_1,x_2)=x_1+x_2$) amounts to an additional assumption *on top* of that of a *homo economicus*

Remarks

- Preferences are psychological or biological entities that economists take as given and fixed
- We tend to ignore preferences'
  - origin or causes
  - intensity
  - possible dynamics
- There are, however, also economic studies that investigate preference saliences, likely mechanisms of preference change, and their effects on decisions in markets or outside
- A key problem of changing / reference-dependent preferences is the welfare interpretation of outcomes

Choice structures and choice rules

- Recall that the move from observing choice x from X' towards a binary preference relation entailed a presumption of context-independence regarding greater desirability of x than $y \in X'$
- If one does not want to impose this restriction, one can work with so-called *choice structures*
- A choice structure $(\mathscr{B}, C(\cdot))$ has two ingredients:
  - $\mathscr{B} \subseteq 2^X$ is a family of nonempty subsets of X;
    elements $B \in \mathscr{B}$ are called *budget sets*,
    $\mathscr{B}$ is meant to describe all choice experiments that could be posed to the decision maker
  - The so-called *choice rule* or *choice correspondence* $C(\cdot)$ maps any budget set $B \in \mathscr{B}$ to a (nonempty) subset $C(B) \subseteq B$;
    it lists all alternatives that the decision maker might choose from B (i.e., he finds equally acceptable from B)

Example

- Suppose that X ={BT, KU, N} and $\mathscr{B}$ = {{KU, N}, {BT, KU, N}}
- A possible choice structure is $(\mathscr{B}, C_1(\cdot))$, where
  - C_1({KU, N}) = {KU}
  - C_1({BT, KU, N}) = {KU}

$\rightarrow$ Kulmbach is the preferred location no matter what other alternatives are in the budget set

- Another possible choice structure is $(\mathscr{B}, C_2(\cdot))$, where
  - C_2({KU, N}) = {N}
  - C_2({BT, KU, N}) = {KU}

$\rightarrow$ He prefers the location in the budget set which is second-closest to Bayreuth

Weak axiom

- A common restriction on choice structures ($\mathcal{B}$, $C(\cdot)$), which rules out behavior of the latter kind, is the *weak axiom of revealed preference* (*WARP* or *WA*):
  - If x is chosen for a $B \in \mathcal{B}$ that also contains y,
    and y is chosen for another $B' \in \mathcal{B}$ that also contains both,
    then x must be equally acceptable in B', i.e.,

    $x,y \in B,\ x \in C(B)$ and $x,y \in B',\ y \in C(B') \ \Rightarrow\ x \in C(B')$

- If we interpret the existence of a budget set $B \ni x,y$ with $x \in C(B)$ as: "x is *revealed weakly preferred* to y (for some budget set)", WARP can simply be phrased as follows:
  - If x is revealed weakly preferred to y, then y cannot be revealed strictly preferred to x

Relationship of preferences and choice rules (1)

- Two natural questions arise about WARP:
  1. If a decision maker has a rational preference ordering $\succsim$, do her decisions – when facing choices from budget sets in $\mathcal{B}$ – necessarly generate a choice structure that satisfies WARP?
  2. If an individual's choice behavior for a family of budget sets $\mathcal{B}$ is captured by a choice structure ($\mathcal{B}$, $C(\cdot)$) that satisfies WARP, does any rational preference relation $\succsim$ exist which is consistent with these choices (i.e., which *rationalizes* $C(\cdot)$ relative to $\mathcal{B}$)?

Relationship of preferences and choice rules (2)

- Both questions can basically be answered affirmatively:
  1. A choice structure which is generated by a rational preference ordering $\succsim$ automatically satisfies WARP
  2. That a choice structure $(\mathscr{B}, C(\cdot))$ satisfies WARP is sufficient for the existence of an (even unique) preference relation $\succsim$ which rationalizes it *if* $\mathscr{B}$ includes all subsets $X' \subseteq X$ with $|X'| \leq 3$
     (only then does WARP guarantee transitivity)
- So, if choices are defined on *all* subsets of X and satisfy WARP, then both preference and choice rule-based approaches to modeling behavior are equivalent
- NB: Consumer decisions described by a demand function $\boldsymbol{x}(\boldsymbol{p},w)$ are defined only for special subsets of X;
  then, stronger properties than WARP are needed to guarantee the rationalizability of choices (in an economic sense)

3. Choice-based demand theory

- Now study *homo economicus* as a consumer in a (competitive) market economy;
  adopt a choice-based perspective ($\leftrightarrow$ preference-based in 4.)
- Choice of quantities of goods or services provided by the market, called *commodities*, subject to physical and economic constraints
- Any particular quantity combination $(x_1, x_2, \ldots, x_L)$ of L different commodities corresponds to a point $\boldsymbol{x}$ in *commodity space* $\mathbf{R}^L$
- Definition of the relevant commodities comes with great flexibility: same good delivered at different points in time, different locations, or in distinct 'states of the world' are just different commodities
- Physical restrictions on bundles that the individual can consume are reflected by restricting $\mathbf{R}^L$ to a *consumption set* $X \subseteq \mathbf{R}^L$

Divisibility and price taking

- For simplicity, we consider $\mathbf{R}_+^L$ as agents' consumption set; this is a *convex* set, i.e., we assume *perfect divisibility*
- We also assume the existence of a *complete market*, i.e., every commodity $i = 1, \ldots, L$ is traded
  (property rights are well-defined for every relevant good)
- The considered consumer is presumed to be a *price taker*, i.e., cannot affect prices by his decisions
- Suppliers use linear price schemes, i.e., sell at constant unit price
  (vs. *non-linear pricing*: two-part tariffs, quantity discounts, …),
  e.g., because there is perfect competition
- For convenience, let the price of any good i be positive, i.e., $p_i > 0$ for $i = 1, \ldots, L$

Walrasian budget sets

- The economic constraint faced by the agent is that he must afford any commodity bundle $\boldsymbol{x} \in \mathbf{R}_+^L$ which he intends to pick, i.e., for a given price vector $\boldsymbol{p} \in \mathbf{R}_+^L$ total expenditure
  $$\boldsymbol{p} \cdot \boldsymbol{x} := p_1 x_1 + \ldots + p_L x_L$$
  cannot exceed wealth $w > 0$
- The set of affordable, physically feasible bundles for given $\boldsymbol{p}$ and w is the consumer's *Walrasian* or *competitive budget set*
  $$B_{\boldsymbol{p},w} := \{\boldsymbol{x} \in \mathbf{R}_+^L : \boldsymbol{p} \cdot \boldsymbol{x} \leq w\}$$
- The consumer's choice problem is thus:
  "Choose a consumption bundle $\boldsymbol{x}$ from $B_{\boldsymbol{p},w}$"

Budget hyperplane

- The set $\{\boldsymbol{x} \in \mathbf{R}^L : \boldsymbol{p} \cdot \boldsymbol{x} = w\}$ is known as the *budget line;* or for $L>2$ as the *budget hyperplane*; it is the upper boundary of $B_{\boldsymbol{p},w}$
- It's respective intercepts are w/p_i, i.e., the maximal affordable quantity if only good i is purchased
- The fact that $\boldsymbol{p} \cdot \boldsymbol{x} = w$ and $\boldsymbol{p} \cdot \boldsymbol{x'} = w$ for any two points $\boldsymbol{x}$ and $\boldsymbol{x'}$ on the budget hyperplane implies that $\boldsymbol{p}$ is orthogonal to it

  [Recall that the dot product of any vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbf{R}^L$ satisfies:

  $$\boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{x}| \cdot |\boldsymbol{y}| \cdot \cos(\theta)$$

  where θ denotes the angle between the vectors;
  in particular, $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ iff $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal]

Walrasian demand

- Set $\mathcal{B}^w = \{ B_{\boldsymbol{p},w} : \boldsymbol{p} \in \mathbf{R}_{++}{}^L \wedge w>0 \}$ is just a family of budget sets
- At least in principle, we can observe a consumer's choices $C(B) \subseteq B$ for any budget set $B = B_{\boldsymbol{p},w} \in \mathcal{B}$
- These choices are called the *(Walrasian) demand* of the consumer and we refer to

  $$x(\boldsymbol{p}, w) := C(B_{\boldsymbol{p},w})$$

  as the consumer's *Walrasian demand correspondence*
- We often focus on case in which $C(B_{\boldsymbol{p},w})$ is singleton-valued, i.e., the consumer picks a unique element in any Walrasian budget set
- $x(\boldsymbol{p}, w)$ is then called the Walrasian demand *function*
  (we then drop the brackets around $\{\boldsymbol{x}^*\}$)

Homogeneity of Walrasian demand

- A function f: $X \to Y$ (analogously, a correspondence F: $X \rightrightarrows Y$) between vector spaces X and Y is called *homogeneous of degree r* $\Leftrightarrow$ $\forall\lambda>0$: $\forall x\in X$: $f(\lambda x) = \lambda^r f(x)$
- Demand is *homogeneous of degree zero* iff $x(\lambda \boldsymbol{p}, \lambda w) \equiv x(\boldsymbol{p}, w)$, i.e., when prices and wealth all change by the same factor then demand does not change ($\to$ only relative prices matter)
- We will assume that the individual cares only about the commodities, and doesn't suffer from money illusion

$\Rightarrow$ Choice depends only on which bundles are affordable and so the fact that $B_{\boldsymbol{p},w} \equiv B_{\lambda\boldsymbol{p},\lambda w}$ implies $x(\lambda \boldsymbol{p}, \lambda w) \equiv x(\boldsymbol{p}, w)$

Notes on homogeneity of degree zero

- Given that we can scale prices and wealth up or down by $\lambda>0$ without affecting demand, it is often convenient to normalize such that $w = 1$, or such that $p_i = 1$ for some good i and then all prices are expressed in units of this *numeraire good*

Technical note:

- If $x(\boldsymbol{p}, w)$ is differentiable, then Euler's theorem for homogeneous functions implies that

$$D_{\boldsymbol{p}}x(\boldsymbol{p}, w)\, \boldsymbol{p} + D_w x(\boldsymbol{p}, w)\, w = \mathbf{0},$$

  where $D_{\boldsymbol{p}}$ and D_w refer to the Jacobian (sub-)matrices of partial derivatives w.r.t. $p_1, \ldots, p_L$, and w:
  price and wealth derivatives of demand for any good i, when weighted by these prices and wealth, sum to zero
- This can also be stated in terms of elasticities (divide each row i by $x_i(\boldsymbol{p}, w)$): the effects on demand for good i of (i) an equal percentage change in all prices, and (ii) in wealth, cancel

Walras' law

- We say that a Walrasian demand function (or correspondence) $x(\boldsymbol{p}, w)$ satisfies *Walras' law* (or is *budget balancing*) iff it is an element of the budget hyperplane for all $\boldsymbol{p}$ and w, i.e.,

$$\boldsymbol{x} = x(\boldsymbol{p}, w) \quad \Rightarrow \quad \boldsymbol{p}\cdot\boldsymbol{x} = w$$

(or $\boldsymbol{x} \in x(\boldsymbol{p}, w)$)

- Walras' law says that the consumer fully expends his wealth
- When understood in a broad way (e.g., as applying to the entire lifetime of an agent, with bequests viewed as commodities, too), this does not amount to a very restrictive assumption

Technical note on Walras' law

- In case of a differentiable demand function the requirement

$$\boldsymbol{p}\cdot x(\boldsymbol{p}, w) = w$$

imposed by Walras' law yields

$$\sum_{k=1}^{L} p_k \frac{\partial x_k(\mathbf{p}, w)}{\partial p_i} + x_i(\mathbf{p}, w) = 0,$$

after differentiation w.r.t. the price of good i

- This is a differential statement of the fact that total expenditure cannot change after a change in some price (… it equals w)
- Similarly,

$$\sum_{k=1}^{L} p_k \frac{\partial x_k(\boldsymbol{p}, w)}{\partial w} = 1$$

captures that increasing w raises expenditure exactly by the increment

Comparative statics w.r.t. wealth

- How do observed choices vary with changes in wealth and prices?
- Examination of outcome changes due to a change in underlying economic parameters is known as *comparative statics* analysis
- The *wealth effect* for good i at $(\boldsymbol{p}, w)$ is simply $\partial x_i(\boldsymbol{p}, w)/\partial w$
- Commodity i is *normal* at $(\boldsymbol{p}, w)$ if the wealth effect for it is positive, i.e., demand increases in wealth;
  i is *inferior* at $(\boldsymbol{p}, w)$ if the wealth effect is negative
- If all commodities are normal at all $(\boldsymbol{p}, w)$, demand is called normal
- If we fix prices $\boldsymbol{p}'$ then $x(\boldsymbol{p}', w)$ is called the consumer's *Engel function* and $x_i(\boldsymbol{p}', w)$ his *Engel curve* for good i;
  the image of $x(\boldsymbol{p}', w)$ is known as the *wealth expansion path*

Comparative statics w.r.t. prices

- Derivative $\partial x_i(\boldsymbol{p}, w)/\partial p_k$ is the *price effect* of p_k on demand for good i at $(\boldsymbol{p}, w)$;
  the Jacobian matrix $D_{\boldsymbol{p}}x(\boldsymbol{p}, w)$ collects these in a compact form
- Good i is said to be a *Giffen good* at $(\boldsymbol{p},w)$ if $\partial x_i(\boldsymbol{p}, w)/\partial p_i > 0$, i.e., a drop in i's price reduces the demand for it
- Under WARP and Walras' law, a commodity can only be Giffen if it is also (very) inferior,
  e.g., very low-quality good purchased by a poor consumer
- We commonly plot $x_i(\boldsymbol{p}, w)$ as a function of p_i for fixed $\boldsymbol{p}_{-i}$ and w;
  the image of $x(\boldsymbol{p}, w)$ in, e.g., x_1-x_2-space when only p_i is varied is known as an *offer curve*

Minimal condition for rationalizing demand

- $\mathscr{B}^{w} = \{ B_{\boldsymbol{p},w}: \boldsymbol{p} \in \mathbf{R}_{++}^{L} \wedge w>0 \}$ and $x(\boldsymbol{p}, w)$ define a choice structure
- If $x(\boldsymbol{p}, w)$ is single-valued, i.e., a function, then WARP becomes:

$$\boldsymbol{p} \cdot x(\boldsymbol{p}', w') \leq w \wedge x(\boldsymbol{p}', w') \neq x(\boldsymbol{p}, w) \Rightarrow \boldsymbol{p}' \cdot x(\boldsymbol{p}, w) > w'$$

- That is:
  If $x(\boldsymbol{p}', w')$ is affordable in price-wealth situation $(\boldsymbol{p}, w)$ but ignored, then choice of $x(\boldsymbol{p}', w')$ at $(\boldsymbol{p}', w')$ requires that $x(\boldsymbol{p}, w)$ would blow the budget in situation $(\boldsymbol{p}', w')$
  (If $x(\boldsymbol{p}, w)$ is revealed preferred to $x(\boldsymbol{p}', w')$ then $x(\boldsymbol{p}', w')$ must not be revealed preferred to $x(\boldsymbol{p}, w)$! – Choice of $x(\boldsymbol{p}', w')$ at $(\boldsymbol{p}', w')$ would do so if $x(\boldsymbol{p}, w)$ were also affordable at $(\boldsymbol{p}', w')$.)
- NB:
  WARP is not sufficient to conclude that demand can be rationalized by a preference relation over commodity bundles (why?)

Slutsky wealth compensation

- A price change has two effects:
  1. It alters the relative price of different commodities
  2. It changes the consumer's real wealth (affordability)
- Weak axiom restricts demand changes in response to price changes when taking affordability into account
- One can isolate the effect of relative price changes by adjusting the budget in a way that keeps the baseline bundle just affordable, i.e., consider $w' = \boldsymbol{p}' \cdot x(\boldsymbol{p}, w)$
- This adjustment is known as a *Slutsky wealth compensation*, resulting in *Slutsky compensated price changes*

WARP ≈ compensated law of demand

- Provided that the Walrasian demand function $x(\mathbf{p}, w)$ is homogeneous of degree zero and satisfies Walras' law, WARP is equivalent to the *compensated law of demand (CLD)*:

  $x(\mathbf{p}, w)$ satisfies WARP

  $\Leftrightarrow$ For any compensated price change from $(\mathbf{p}, w)$ to

  $$(\mathbf{p}', w') = (\mathbf{p}', \mathbf{p}' \cdot x(\mathbf{p}, w)),$$

  we have

  $$(\mathbf{p}' - \mathbf{p}) \cdot [x(\mathbf{p}', w') - x(\mathbf{p}, w)] \leq 0$$

  with strict inequality whenever $x(\mathbf{p}', w') \neq x(\mathbf{p}, w)$

- This 'law' implies that price p_i and *compensated* demand x_i always move in opposite directions;
  $\Delta\mathbf{p} = (\mathbf{p}' - \mathbf{p}) = (0, \ldots, 0, \Delta p_i, 0, \ldots, 0)$ implies $\Delta p_i \Delta x_i \leq 0$
- Question:
  Should the same be true for uncompensated demand?

Substitution and income effects

- Let us fix a reference bundle $\mathbf{z} = x(\mathbf{p}^0, w^0)$ and look at the *Slutsky compensated demand function* $x^s(\mathbf{p}, \mathbf{z}) \equiv x(\mathbf{p}, \mathbf{p} \cdot \mathbf{z})$
- As prices vary, $x^s(\mathbf{p}, \mathbf{z})$ changes;
  this change reflects a pure *substitution effect*:
  the consumer responds to new relative prices, while his real wealth has stayed constant (in the sense of $\mathbf{z}$ still being affordable)
- A change Δx_i in uncompensated demand can be decomposed into such a (virtual) substitution effect $\Delta x_i^{sub.}$ and the *income effect* $\Delta x_i^{inc.}$ from the (virtual) change in income from $\mathbf{p} \cdot \mathbf{z}$ to w^0
- Taking the derivative of $x_i^s(\mathbf{p}, \mathbf{z}) \equiv x_i(\mathbf{p}, \mathbf{p} \cdot \mathbf{z})$ w.r.t p_k at $\mathbf{p}^0$, one obtains the *Slutsky equation*

$$\partial x_i^s(\mathbf{p}^0, \mathbf{z}) / \partial p_k = \partial x_i(\mathbf{p}^0, w^0)/\partial p_k + \partial x_i(\mathbf{p}^0, w^0)/\partial w \cdot x_k(\mathbf{p}^0, w^0)$$

Slutsky matrix

- These pure substitution effects (of a change in p_k on demand for commodity i) can be collected in an $L\times L$-matrix, known as the *substitution* or *Slutsky matrix* $S(\boldsymbol{p}, w)$ [$= D_{\boldsymbol{p}} x^s(\boldsymbol{p}, \boldsymbol{z})$ with $\boldsymbol{z} = x(\boldsymbol{p}, w)$]
- Multiplying $\partial x_i^s(\boldsymbol{p}, \boldsymbol{z})/\partial p_k$ with the change Δp_k for k=1, …, L and adding these changes up, we obtain the total change Δx_i caused by a compensated price change $\Delta \boldsymbol{p}$ (infinitesimal units)
- Doing this for all i = 1, …, L, we get the change in compensated demand $\Delta \boldsymbol{x} = S(\boldsymbol{p}, w)\Delta \boldsymbol{p}$ caused by price change $\Delta \boldsymbol{p}$
- The compensated law of demand, namely $\Delta \boldsymbol{p} \cdot \Delta \boldsymbol{x} \leq 0$, thus requires that

$$\Delta \boldsymbol{p} \cdot S(\boldsymbol{p}, w)\Delta \boldsymbol{p} \leq 0$$

holds for any $\Delta \boldsymbol{p} \in \mathbf{R}^L$

Negative semidefiniteness of Slutsky matrix

- So the assumptions of Walras' law, homogeneity of degree zero, and WARP (↔CLD) imply that above quadratic form is never positive, i.e., $S(\boldsymbol{p}, w)$ is *negative semidefinite*
(mathematicians sometimes restrict the term to symmetric matrices; but symmetry of $S(\boldsymbol{p}, w)$ is *not* implied by Walras' law, WARP and homogeneity for L>2)
- Negative semidefiniteness requires that, in particular, $s_{i,i} = \partial x_i^s/\partial p_i$ is negative for every i
(echoing that the compensated law of demand requires $\Delta p_i \Delta x_i \leq 0$)
- Given that the virtual substitution effects $\partial x_i^s(\boldsymbol{p}, \boldsymbol{z})/\partial p_k$ can be inferred from real and, at least in principle, observable price and wealth effects at $(\boldsymbol{p}, w)$, the joint hypothesis of a consumer's behavior satisfying Walras' law, homogeneity of degree zero, and WARP can be tested empirically

Remarks

- Negative semidefiniteness of $S(\boldsymbol{p},w)$ is a necessary implication of WARP (given Walras' law and homogeneity), but not yet sufficient to guarantee that a differentiable demand function satisfies WARP
  (sufficiency requires that $\Delta\boldsymbol{p}\cdot S(\boldsymbol{p}, w)\Delta\boldsymbol{p} \leq 0$ holds strictly if $\Delta\boldsymbol{p}$ is not proportional to $\boldsymbol{p}$)
  (BTW: homogeneity and Walras' law imply that each row in $S(\boldsymbol{p}, w)$ is orthogonal to $\boldsymbol{p}$, i.e., $S(\boldsymbol{p}, w)\boldsymbol{p} = \mathbf{0} = \Delta\boldsymbol{x}$:
  a change $\Delta\boldsymbol{p} = \lambda\boldsymbol{p}$ preserves relative prices; it doesn't change *compensated* demand since it corresponds to a mere rescaling of prices and wealth)
- A theory of consumer demand based on the assumption of homogeneity of degree zero, Walras' law, and WARP is a bit less restrictive than one based on rational preference maximization; as we'll see in next chapter, the latter forces the Slutsky matrix to be *symmetric* at all $(\boldsymbol{p},w)$

4. Preference-based demand theory

- The classical approach to consumer theory tries to explain demand by rational preferences $\succsim$ over commodity bundles
  (vs. description of choices from Walrasian budget sets by $(\mathscr{B}^w, x(\cdot))$)
- We'll assume that $\succsim$ can be represented by a utility function u, and that u is sufficiently "smooth"/differentiable
- A rational consumer's demand can be seen as the result of
  - maximizing utility under the constraint that a given budget is not blown

  or of
  - minimizing expenditure under the constraint of a target utility level
- The latter perspective will be useful for comparing individual welfare across different price vectors (e.g., different policies)

4.1 Preference relations and utility

- Many qualitative properties of $\succsim$ imply analogue properties of u:
  - $\succsim$ is *monotone* $:\Leftrightarrow \{\boldsymbol{y} \geq \boldsymbol{x} \wedge \boldsymbol{y} \neq \boldsymbol{x} \Rightarrow \boldsymbol{y} \succ \boldsymbol{x}\}$
    $\Leftrightarrow$ u is strictly increasing
  - $\succsim$ is *locally nonsatiated*
    $:\Leftrightarrow \ \forall \boldsymbol{x} \in X: \forall \varepsilon > 0: \exists \boldsymbol{y} \in U_\varepsilon(\boldsymbol{x}): \boldsymbol{y} \succ \boldsymbol{x}$;
    this is implied by monotonicity
  - $\succsim$ is *convex* $:\Leftrightarrow$ upper contour sets $\{\boldsymbol{y} \in X: \boldsymbol{y} \succsim \boldsymbol{x}\}$ are convex
    $\Leftrightarrow \{\boldsymbol{y} \succsim \boldsymbol{x} \wedge \boldsymbol{z} \succsim \boldsymbol{x} \Rightarrow \forall \alpha \in (0,1): \alpha \boldsymbol{y} + (1-\alpha)\boldsymbol{z} \succsim \boldsymbol{x}\}$
    $\Leftrightarrow$ u is quasiconcave*
  - $\succsim$ is *strictly convex*
    $:\Leftrightarrow \boldsymbol{y} \succsim \boldsymbol{x} \wedge \boldsymbol{z} \succsim \boldsymbol{x} \Rightarrow \forall \alpha \in (0,1): \alpha \boldsymbol{y} + (1-\alpha)\boldsymbol{z} \succ \boldsymbol{x}\}$
    $\Leftrightarrow$ u is strictly quasiconcave

*$:\Leftrightarrow$ upper level sets $\{\boldsymbol{x} \in X: u(\boldsymbol{x}) \geq a\}$ are convex for all $a \in \mathbf{R}$
$\Leftrightarrow \forall \boldsymbol{x} \neq \boldsymbol{y}: \forall \lambda \in (0,1): u(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \min\{u(\boldsymbol{x}), u(\boldsymbol{y})\}$

4.1 Preference relations and utility

- $\succsim$ is *homothetic* $:\Leftrightarrow \{\boldsymbol{x} \sim \boldsymbol{y} \Rightarrow \forall \alpha \geq 0: \alpha \boldsymbol{x} \sim \alpha \boldsymbol{y}\}$
  $\Leftrightarrow \exists u$: u is homogeneous of degree 1*
- $\succsim$ is *quasilinear* w.r.t. good i
  $:\Leftrightarrow$ {good i is desirable** $\wedge$
  $\boldsymbol{x} \sim \boldsymbol{y} \Rightarrow \forall \alpha \in \mathbf{R}: (\boldsymbol{x} + \alpha \boldsymbol{e}_i) \sim (\boldsymbol{y} + \alpha \boldsymbol{e}_i)$ }
  $\Leftrightarrow \exists u: u(\boldsymbol{x}) = x_i + \phi(\boldsymbol{x}_{-i})$

*: $\Leftrightarrow \forall \boldsymbol{x}: \forall \lambda > 0: u(\lambda \cdot \boldsymbol{x}) = \lambda \cdot u(\boldsymbol{x})$
**: $\Leftrightarrow \forall \boldsymbol{x}: \forall \alpha > 0: (\boldsymbol{x} + \alpha \boldsymbol{e}_i) \succ \boldsymbol{x}$

4.2 Utility maximization problem

- If $p \gg 0$ and u is continuous, then the consumer's *utility maximization problem*

$$\max_{\boldsymbol{x} \geq 0} u(\boldsymbol{x}) \qquad \text{s.t.} \quad \boldsymbol{p}\cdot\boldsymbol{x} \leq w \qquad\qquad \text{(UMP)}$$

  has (by Weierstrass's theorem) a solution $\boldsymbol{x}(\boldsymbol{p},w)$, namely, the consumer's (*Walrasian* or *Marshallian) demand*
- Assume u represents locally nonsatiated preferences $\succsim$ then $\boldsymbol{x}(\boldsymbol{p},w)$
  - is convex-valued *if* u is quasiconcave ($\succsim$ convex)
  - is singleton-valued, i.e., a function, and continuous at all $(\boldsymbol{p},w) \gg 0$ *if* u is strictly quasiconcave ($\succsim$ strictly convex)
  - satisfies Walras' law and is homogeneous of degree 0
- NB: Lagrange multiplier in (UMP) is the marginal utility of wealth
- The utility value of (UMP), $v(\boldsymbol{p},w) := u(\boldsymbol{x}(\boldsymbol{p},w))$, is the consumer's *indirect utility function*

4.3 Expenditure minimization problem

- The *expenditure minimization problem*

$$\min_{\boldsymbol{x} \geq 0} \boldsymbol{p}\cdot\boldsymbol{x} \qquad \text{s.t.} \quad u(\boldsymbol{x}) \geq u' \qquad\qquad \text{(EMP)}$$

  is related to (UMP), often called its "dual problem"
- Its cost value $e(\boldsymbol{p}, u')$ is the consumer's *expenditure function*
- Analogously to a firm's cost function, if u is continuous and $\succsim$ locally nonsatiated then $e(\boldsymbol{p}, u')$ is strictly increasing in u', homogeneous of degree 1 in $\boldsymbol{p}$, nondecreasing in p_i, and weakly concave in $\boldsymbol{p}$
  (intuition for the latter: 1. linearly raise expenditure by sticking to the old consumption quantities at new prices; 2. lower costs by re-optimizing)
- Note that $e(\boldsymbol{p}, v(\boldsymbol{p}, w)) = w$ and $v(\boldsymbol{p}, e(\boldsymbol{p}, u')) = u'$

Hicksian demand

- (EMP)'s solution bundle(s) constitute the *Hicksian demand* (or *Hicks compensated demand*) $h(\mathbf{p}, u')$
- For strictly convex $\succsim$, $h(\mathbf{p}, u')$ is a function;
  it is homogeneous of degree zero in p, and satisfies the *compensated law of demand*

$$(\mathbf{p}' - \mathbf{p})[h(\mathbf{p}', u) - h(\mathbf{p}, u)] \leq \mathbf{0}$$

- Goods l and k are called *substitutes* if $\partial h_l(\mathbf{p}, u)/ \partial p_k > 0$
- Goods l and k are called *complements* if $\partial h_l(\mathbf{p}, u)/ \partial p_k < 0$

Hicksian demand and expenditure function

- Even though Hicks compensation (keeping utility constant) and Slutsky compensation (keeping the old bundle affordable) produce different demand changes for a discrete price change, they coincide for *marginal* price changes
- In particular, the Slutsky matrix $S(\mathbf{p},w)$ equals the Jacobian of both $x(\mathbf{p}, \mathbf{p}{\cdot}x(\mathbf{p},w))$ and $h(\mathbf{p},v(\mathbf{p},w))$ w.r.t. $\mathbf{p}$
- Note that $e(\mathbf{p}, u') = \mathbf{p}{\cdot}h(\mathbf{p}, u')$ implies

$$\partial e(\cdot)/\partial p_i = h_i(\mathbf{p}, u')$$

  (where $[+ \Sigma\, p_j \cdot \partial h_j/\partial p_i] = 0$ because $(h_1^*, \ldots, h_L^*)$ is chosen optimally, i.e., $p_j = \lambda^{-1} \cdot \partial u/\partial x_j|_{xj=hj(\cdot)}$, and so [...] equals λ^{-1} · total utility change from quantity adjustment which, for constant u', must be zero)
- So the marginal expenditure change that is required to keep utility constant after a change of p_i is just equal to current quantity consumed of good i
  (this mimicks *Shepard's lemma* in the theory of production)

Symmetry of (UMP)/(EMP)-implied Slutsky matrix

- Assuming $e(\boldsymbol{p}, u')$ is twice differentiable, we have
$$\partial^2 e(\cdot)/\, \partial p_i \partial p_j = \partial h_i(\cdot)/\, \partial p_j = \partial h_j(\cdot)/\, \partial p_i$$
or in matrix notation
$$D^2_{\boldsymbol{p}}\, e(\boldsymbol{p}, u') = D_{\boldsymbol{p}}\, h(\boldsymbol{p}, u') = S(\boldsymbol{p}, e(\boldsymbol{p}, u'))$$
- So the Hesse matrix $D^2_{\boldsymbol{p}}\, e(\boldsymbol{p}, u') = S(\boldsymbol{p}, e(\boldsymbol{p}, u'))$ is symmetric, i.e., the Slutsky matrix is *symmetric*
- Since $e(\boldsymbol{p}, u')$ is concave in $\boldsymbol{p}$, Slutsky matrix must moreover be *negative semidefinite*
- As preference-based (or utility-maximizing) demand implies symmetry of the Slutsky matrix, it is more restrictive than choice-based demand satisfying Walras‘ law, WARP and homogeneity of degree zero

Remarks

- Revealed choice-based demand can be rationalized if it satisfies Walras’ law, WARP, zero-homogeneity *and* has a symmetric substitution matrix
(latter is equivalent to satisfying Houthakker’s SARP instead of WARP)
- That the derivative of (EMP)‘s value function is simply (EMP)‘s solution vector cannot have a direct equivalent in the (UMP): indirect utility $v(\boldsymbol{p},w)$ is ordinal while $x(\boldsymbol{p},w)$ is cardinal
- But there exists a close analogue, in which marginal (indirect) utility is “normalized”, known as *Roy‘s identity*
$$x_i(\boldsymbol{p},w) = -\frac{\partial v(\boldsymbol{p},w) / \partial p_i}{\partial v(\boldsymbol{p},w) / \partial w}$$
- This makes indirect utility functions convenient to work with: demand can be computed w/o solving an optimization problem

4.4 Individual welfare evaluation

- We can evaluate whether a consumer is better off under price vector $\boldsymbol{p}'$ or $\boldsymbol{p}''$ by checking if $v(\boldsymbol{p}',w) - v(\boldsymbol{p}'',w)$ is positive or negative
- Recall that we obtain an equivalent (indirect) utility function $\tilde{u}$ ($\tilde{v}$) if we apply a strictly increasing transformation to u (v); e.g., $e(\boldsymbol{p}', v(\boldsymbol{p},w))$ is also an indirect utility function
- It is *money metric*: evaluates $\boldsymbol{p}$-vectors by the euro amount that the consumer would need to get ($\boldsymbol{p},w$)-situation utility under fixed reference prices $\boldsymbol{p}'$:
  - If under $\boldsymbol{p}'$, say, 100€ would be needed to obtain utility $v(\boldsymbol{p}^0,w)$ while 120€ would be needed to obtain $v(\boldsymbol{p}^1,w)$, then welfare can, loosely speaking, be said to be 20€ higher for $\boldsymbol{p}^1$ than for $\boldsymbol{p}^0$

Compensating variation

- Suppose we want to use $e(\boldsymbol{p}', v(\boldsymbol{p},w))$ in order to quantify the change in a consumer's welfare caused by going from $\boldsymbol{p}^0$ to $\boldsymbol{p}^1$; what should be the reference price?
- One natural choice is $\boldsymbol{p}' = \boldsymbol{p}^1$, i.e., we use new prices as our reference
- The change $CV(\boldsymbol{p}^0, \boldsymbol{p}^1,w) := e(\boldsymbol{p}^1, v(\boldsymbol{p}^1,w)) - e(\boldsymbol{p}^1, v(\boldsymbol{p}^0,w))$

  $w - e(\boldsymbol{p}^1, v(\boldsymbol{p}^0,w))$

  is known as the *compensating variation*
- It measures the welfare effect of $\boldsymbol{p}^0 \rightarrow \boldsymbol{p}^1$ on the consumer by answering the question:
  How much money could be extracted from the consumer (would need to be paid to him) under the more (less) favorable $\boldsymbol{p}^1$ in order for him to be indifferent to the change, i.e., to be fully compensated under the new situation?

Equivalent variation

- Another natural choice is $\boldsymbol{p}' = \boldsymbol{p}^0$, i.e., we use old prices as our reference
- The change $EV(\boldsymbol{p}^0, \boldsymbol{p}^1, w) := e(\boldsymbol{p}^0, v(\boldsymbol{p}^1, w)) - e(\boldsymbol{p}^0, v(\boldsymbol{p}^0, w))$

  $e(\boldsymbol{p}^0, v(\boldsymbol{p}^1, w)) - w$

  is known as the *equivalent variation*
- It measures the welfare effect of $\boldsymbol{p}^0 \to \boldsymbol{p}^1$ on the consumer by answering the question:

  How much money would need to be paid to the consumer (could be extracted from him) under $\boldsymbol{p}^0$ in order for him to be indifferent to the change to a more (less) favorable $\boldsymbol{p}^1$, i.e., to render the old situation equivalent to the prospective new one?

Consumer surplus

- If $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ differ only in the price of a normal good *i* then

  $$CV(\boldsymbol{p}^0, \boldsymbol{p}^1, w) < \Delta CS < EV(\boldsymbol{p}^0, \boldsymbol{p}^1, w)$$

  where ΔCS is the change in (Marshallian) *consumer surplus*
- *CS* adds up *marginal willingness to pay* for the next marginal unit of good *i* and subtracts actual payment for $x_i(\boldsymbol{p}, w)$:

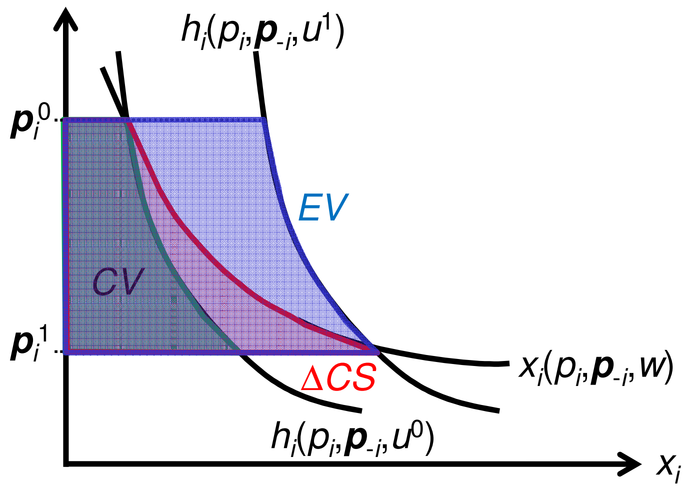
  denoting by $p_i(x_i, w)$ the price at which the consumer would buy x_i units of good *i,* she would strictly prefer to buy the last marginal unit of total x_i if $p_i < p_i(x_i, w)$ but is indifferent if $p_i = p_i(x_i, w)$, i.e.,

  $$MWTP_i(x_i) = p_i(x_i, w).$$
- Remark:

  ΔCS evaluation is not „fully consistent“ as $MWTP_i(x_i)$ evaluates an additional unit at a time for a smoothly varying price p_i, not additional units from a $\boldsymbol{p}^0$ or $\boldsymbol{p}^1$ perspective

  (instead of compensating a gradual drop from p_i^0 to p_i^1 by taking released funds away, ΔCS lets the consumer “become richer” along the way)

CV, EV, and consumer surplus



- If there is *no* wealth effect for good *i* (e.g., $\succsim$ is quasilinear w.r.t some good $j \neq i$, so that any extra utility from $w\uparrow$ comes via $x_j\uparrow$), then $h_i(p,u^1) = h_i(p,u^0)$ and all three measures coincide

5. Aggregate demand

- Aggregate demand in an economy is readily obtained by adding individual demand $x^i(\boldsymbol{p},w^i)$ across all individuals, i.e.,

$$x(\boldsymbol{p},w^1, \ldots, w^I) = \Sigma_i\, x^i(\boldsymbol{p},w^i)$$

- Tracking the wealth vector $(w^1, \ldots, w^I)$ in, e.g., comparative static analysis is cumbersome;
  one is tempted to work with aggregate wealth $w = \Sigma_i\, w^i$ and to pretend that $x(\boldsymbol{p},w)$ is the demand of a single agent
- This raises questions:
  - When is it possible to work with w instead of the full wealth distribution $(w^1, \ldots, w^I)$?
  - Assuming that individual demands are preference-based and $(\boldsymbol{p},w)$ determines aggregate demand, are the choices $x(\boldsymbol{p},w)$ compatible with existence of a single *rational* representative consumer?
  - Can the representative consumer's (money-metric) indirect utility function be used for welfare statements?

5.1 When doesn‘t the wealth distribution matter?

- Total demand $x(\boldsymbol{p},w^1, \ldots, w^I) = \Sigma_i\, x^i(\boldsymbol{p},w^i)$ can be expressed as a function $x(\boldsymbol{p},w)$ of total wealth $w = \Sigma_i\, w^i$ only in special cases
- Distribution independence requires that individual wealth effects exactly cancel out as we shift Δw between consumers i and j, i.e.,

$$\partial x^i_k/\partial w = \partial x^j_k/\partial w \text{ for all } k \text{ and arbitrary } i, j\ w^i, \text{ and } w^j$$

- This necessitates that consumers (for the relevant wealth range) have parallel straight lines as their wealth expansion paths
- That turns out to be equivalent to each $\succsim_i$ admitting a utility representation s.t. indirect utility functions are of the *Gorman form*

$$v_i(\boldsymbol{p},w^i) = a_i(\boldsymbol{p}) + b(\boldsymbol{p})\cdot w_i$$

with *identical* wealth multiplier $b(\boldsymbol{p})$ for all i

When doesn‘t the (initial) wealth distribution matter?

- This is the case (mainly) if
  - all $\succsim_i$ equal the *same* homothetic $\succsim$ (e.g., Cobb-Douglas, perfect substitutes, or complements)

  or
  - all $\succsim_i$ are quasilinear w.r.t. the same good k and we only consider sufficiently big wealth levels
- We can also, trivially, drop $(w^1, \ldots, w^I)$ and simply write $x(\boldsymbol{p},w)$ *if* each w^i can be expressed as a function $w^i(\boldsymbol{p},w)$ of $\boldsymbol{p}$ and w (e.g., because of wealth redistribution according to a particular given rule, or as an empirical “regularity“)

5.2 Aggregate demand $\overset{?}{=}$ demand of a single $\succsim$

- That each $x^i(\boldsymbol{p},w^i)$ satisfies WARP, or even results from a rational $\succsim_i$, does *not* guarantee that $\Sigma_i\, x^i(\boldsymbol{p},w^i)$ satisfies WARP, or comes from a "representative" rational $\succsim$ even when $\Sigma_i\, x^i(p,w^i) = x(\boldsymbol{p},w)$
- WARP ($\triangleq$ compensated law of demand) doesn't "aggregate": a price-wealth change that is compensated for the aggregate may fail to be compensated for some individuals …
- The stronger *uncompensated law of demand* (ULD)

$$(\boldsymbol{p}' - \boldsymbol{p})\cdot[x^i(\boldsymbol{p}',w^i) - x^i(\boldsymbol{p},w^i)] \leq \boldsymbol{0}$$

  does aggregate when $w^i \equiv \alpha_i \cdot w$
- So, if all $x^i(\cdot)$ satisfy ULD (and hence also CLD), $x(\cdot)$-induced choice structure will satisfy WARP (example: all $\succsim_i$ are homothetic)

Positive representative consumer

- We say that a *positive representative consumer* exists for a given economy if one can find a fictional individual whose optimal behavior would result in aggregate demand $x(\boldsymbol{p}, w^1,\ldots, w^I)$ if he could spend the society's budget $w = \Sigma w^i$
- Existence requires that
  - wealth distribution $(w^1,\ldots, w^I)$ doesn't matter, so that $x(\boldsymbol{p},w^1,\ldots, w^I) = x(\boldsymbol{p},w)$, and
  - $x(\boldsymbol{p},w)$ satisfies WARP
    (in fact, even Houthakker's SARP)
- Note that it is also possible that aggregate demand satisfies more stringent 'consistency requirements' than individual demands: individual violations of, say, ULD may "average out"

5.3 Aggregate welfare evaluation

- A social planner, who evaluates different ($\boldsymbol{p}$,w)-situations for society as a whole, presumably looks at a *social welfare function* W: $\mathbf{R}^I \rightarrow \mathbf{R}$, which is defined on (indirect) utility vectors (u_1, ..., u_I) and required to be non-decreasing in every u_i
- Prominent examples:
  - *utilitarian welfare* $W^U(u_1, ..., u_I) = \Sigma_i u_i$
  - *"Rawlsian" welfare* $W^R(u_1, ..., u_I) = \min\{u_1, ..., u_I\}$
- Note that such a social aggregation rule implicitly requires interpersonal comparability of utility

Normative representative consumer

- To what extent can social welfare evaluation be simplified to individual welfare evaluation for the representative consumer?
- Answer depends on the considered social welfare function
- The positive representative consumer with preferences $\succsim$ is called a *normative representative consumer* relative to social welfare function $W(\cdot)$ if the value function $W^*(\boldsymbol{p},w)$ of the planner‘s welfare maximization problem

$$\max_{w^1,\ldots,w^I} W(v^1(\boldsymbol{p},w^1),\ldots, v^I(\boldsymbol{p},w^I))$$
$$\text{s.t. } \Sigma w^i \leq w,$$

  is an indirect utility function for $\succsim$,
  i.e., if the representative consumer‘s demand corresponds to the aggregate demand which would result from utility-maximizing individual demands after an optimal wealth redistribution

Welfare vs. normative representative consumer

- If a normative representative consumer exists, we can, in principle, say that $\boldsymbol{p}^0 \to \boldsymbol{p}^1$ is socially beneficial or detrimental by looking at $CV(\boldsymbol{p}^0, \boldsymbol{p}^1, w)$, $EV(\boldsymbol{p}^0, \boldsymbol{p}^1, w)$, or ΔCS for that consumer
- But:
  w's optimal distribution $(w^{1*},\ldots, w^{I*})$, which maximizes $W(\cdot)$, generally depends on $\boldsymbol{p}$;
  hence, saying "$\boldsymbol{p}^0 \to \boldsymbol{p}^1$ is socially beneficial because $\Delta CS > 0$" is only warranted in the sense that there *exists* a redistribution scheme s.t. welfare is higher under $\boldsymbol{p}^1$
  ("potential welfare" $W^*(\boldsymbol{p},w)$ is higher while actual welfare $W(v^1(\boldsymbol{p},w^1),\ldots, v^I(\boldsymbol{p},w^I))$ may be *lower* under $\boldsymbol{p}^1$ than before if wealth is not redistributed)

Existence of a normative representative consumer

- Conditions for existence of a positive representative consumer were already very demanding
- And if a positive representative consumer happens to exist, there is no guarantee that he is also a normative one for the considered welfare function $W(\cdot)$;
  it is even possible that his preferences have no normative content for *any* social welfare function
- However, if all consumers have indirect utility of the Gorman form *with identical* $b(\boldsymbol{p})$, then the positive representative consumer also is a normative one
  (the Gorman form imposes sufficient structure for $v(\boldsymbol{p},w) = \Sigma_i\, a_i(\boldsymbol{p}) + b(\boldsymbol{p})\cdot w$ to be a strictly increasing transformation of the planner's value function for any social welfare function $W(\cdot)$)

6. Choice under risk and uncertainty

- Section 2 considered preferences and choice w/o specific assumptions re. the considered alternatives $X=\{x_1, x_2\ldots\}$; they might involve risk, uncertainty, different points in time, space, etc.
- We now specifically consider *risky* alternatives, i.e., options associated with known objective probability distributions over deterministic outcomes (vs. *uncertain* / *ambiguous* alternatives)
- One may distinguish between *simple lotteries* $L = (\pi_1, \ldots, \pi_N)$ over deterministic outcomes $y_1, \ldots, y_N$, and *compound lotteries* ('lotteries over lotteries')
- From a consequentialist perspective, a compound lottery can be equated with the simple lottery which it induces; hence, we focus on choice between simple lotteries

6.1 Expected utility representations

- We know that if agent has complete, transitive and continuous preferences $\succsim$ over the space of all (simple) lotteries L, then preferences can be represented by a utility function $U(L)$
- Here, *continuity* may, e.g., be simplified to:

  $\forall L,L',L''$: $\{\alpha\in[0,1]: \alpha L\oplus(1-\alpha)L' \succsim L''\}$ and
  $\{\alpha\in[0,1]: L'' \succsim \alpha L\oplus(1-\alpha)L'\}$ are closed sets,
- The function $U(\cdot)$ which maps each distribution L to a number may be highly complicated and unwieldy
  (e.g., involve a "Choquet integral" w.r.t. a "capacity" derived from L)
- However, if $\succsim$ additionally satisfies the *von Neumann-Morgenstern independence axiom*

  $\forall L,L',L''$: $\forall\alpha\in(0,1)$: $L \succsim L' \Leftrightarrow \alpha L\oplus(1-\alpha)L'' \succsim \alpha L'\oplus(1-\alpha)L''$,

  then $U(\cdot)$ can be chosen to have a simple functional form

Von Neumann-Morgenstern expected utility

- In particular, $U(\cdot)$ can be chosen to have the *v.N.-M.-expected utility* form, that is:
  there exists a *(Bernoulli) utility function* $u(y)$ defined only for deterministic outcomes such that:
  $$U(L) = \Sigma\, \pi_i \cdot u(y_i) = \mathbf{E}_L[u(y)] \qquad [= \int u(y)\, dL(y)]$$
- $\succsim$'s Bernoulli utility function $u(\cdot)$ is unique up to an order-preserving affine transformation, i.e.,
  $u(\cdot)$ can be chosen as Bernoulli utility function for $\succsim$
  $\Leftrightarrow$ $\alpha u(\cdot)+\beta$ for $\alpha>0$ can also be chosen
- $u(\cdot)$ is a *cardinal* utility function over deterministic outcomes
- $u(x) - u(y) > u(z) - u(w)$ now has the interpretation that x is a bigger improvement on y than z is on w:
  - one could mix x with a greater probability for a bad outcome q and the agent still prefers this to y …
  - … than one could mix z with q and retain preference over w

Remarks on independence axiom

- Requiring "independence" when "adding" lottery L'' to L and L' makes normative sense because there is no (obvious) complementarity or substitutability for mutually exclusive events
- An agent whose $\succsim$ violates independence may be "Dutch-booked", i.e., some money can be extracted from her at no risk:
  - Suppose $L_1 \succ L_2$, but $\alpha L \oplus (1-\alpha) L_1 \prec \alpha L \oplus (1-\alpha) L_2$
  - Let her own $\alpha L \oplus (1-\alpha) L_1$, while you own $\alpha L \oplus (1-\alpha) L_2$
  - Trade lotteries with her, collect a fee, and wait
  - If L isn't realized, then trade L_1 for L_2 and collect another fee

  $\Rightarrow$ Your position is exactly as without the trades (L with prob. α, L_2 with prob. $1-\alpha$), but you additionally collect a fee one or two times (one might even play this game ex ante and repeat it)
- Independence is equivalent to $\succsim$'s indifference curves being parallel straight lines in the probability simplex

Allais paradox

- Though normatively appealing, real people frequently violate the independence axiom
- This is illustrated, e.g., by the *Allais paradox*:
  For $(y_1, y_2, y_3) = (50€, 40€, 0€)$ many people reveal
  (1) $L_1 = (0, 1, 0) \succ L_2 = (0.1, 0.89, 0.01)$
  (2) $L_3 = (0, 0.11, 0.89) \prec L_4 = (0.1, 0, 0.9)$
- If this satisfied the v.N.-M. axioms, we could choose $u(0) = 0$, and then infer
  - from (1): $[1 - 0.89]\cdot u(40€) > 0.1\cdot u(50€)$
  - from (2): $0.11\cdot u(40€) < 0.1\cdot u(50€)$

  (L_1 and L_2 lie on a parallel line to L_3 and L_4 in probability simplex; so 1st choice fixes 2nd one under v.N-M. axioms: *all* indifference lines either have greater, smaller, or same slope as these two lines)

6.2 Money lotteries and risk attitudes

- Consider lotteries over interval $[a,\infty)$ of final wealth levels as described by random variable X with cumulative distribution functions $F(x) = Pr(X \leq x)$, and v.N.-M. utility function $U(\cdot)$ with increasing Bernoulli utility $u(\cdot)$ such that $\mathbf{E}_F[u(X)]$ is finite
- The agent is said to be
  - *risk neutral* $\Leftrightarrow$ she is indifferent between lottery F and receiving $\mathbf{E}_F[X]$ for sure, i.e., $\forall F$: $\mathbf{E}_F[u(X)] = u(\mathbf{E}_F[X])$
  - *(strictly) risk averse* $\Leftrightarrow$ she (strictly) prefers $\mathbf{E}_F[X]$ for sure to F
  - *(strictly) risk loving* $\Leftrightarrow$ she (strictly) prefers F to $\mathbf{E}_F[X]$ for sure
- By *Jensen's inequality*, u is concave iff

$$\int u(x) f(x) dx \leq u\left(\int x f(x) dx\right)$$

- So (strict) risk aversion is equivalent to (strict) concavity of u
- It is also equivalent to the *certainty equivalent*, i.e., sure payment $c(F,u)$ that renders agent indifferent to F, being (strictly) smaller than $\mathbf{E}_F[X]$

Quantifying and comparing risk aversion

- Risk attitudes of two individuals, or the same individual at different levels of wealth x can be compared by the *Arrow-Pratt coefficient of absolute risk aversion*

$$r_A(x; u) = - u''(x)/u'(x)$$

- $u_2(\cdot)$ is *more risk averse than* $u_1(\cdot)$
  $\Leftrightarrow$ $r_A(x; u_2) \geq r_A(x; u_1)$ for all x
  $\Leftrightarrow$ $c(F; u_2) \leq c(F; u_1)$ for any lottery F
  $\Leftrightarrow$ u_2 is "more concave" than u_1, i.e., there exists an increasing concave transformation $k(\cdot)$ s.t. $u_2(x) = k(u_1(x))$

Common assumptions about risk aversion

- It is often plausible to assume that $u(\cdot)$ has *decreasing absolute risk aversion* in wealth (DARA), i.e., that $r_A(x;u)$ decreases in x
- Moreover, one often assumes that $u(\cdot)$ has *nonincreasing relative risk aversion*, i.e., the *coefficient of relative risk aversion*

$$r_R(x;u) = - x \cdot u''(x)/u'(x)$$

  is constant or decreasing (CRRA or DRRA)
- This captures the regularity that, as an individual becomes richer, a greater absolute amount is invested in risky assets (DARA), and this amount corresponds to a weakly greater share of total wealth (CRRA or DRRA)
- Remarks:
  – $r_A(x;u) \equiv \gamma \neq 0$ (CARA) $\Leftrightarrow$ $u(x)= a_1 - a_2 \cdot e^{-\gamma x}$ (with $a_2 > 0$)
  – $r_R(x;u) \equiv \delta$ (CRRA) $\Leftrightarrow$ $\delta=1$: $u(x)= a_1 + a_2 \cdot \ln(x)$
  $\delta \neq 1$: $u(x)= a_1 + a_2 \cdot x^{1-\delta}$

(Partial) orderings of random variables

- Any two agents, who like higher x better, agree that lottery F_1 is better than lottery F_2 if $F_1(x) \leq F_2(x)$ for all x, i.e., F_1 places less probability on small realizations of X than F_2

  $\Leftrightarrow$ F_1 *first-order stochastically dominates* F_2
- Any two risk averters agree that lottery F_1 is better than lottery F_2 if F_1 and F_2 have the same mean ($\triangleq$ expected value) and F_2 can be generated from F_1 by shifting probability towards the extremes,

  $\Leftrightarrow$ F_2 is a *mean-preserving spread* of F_1

  $\Leftrightarrow$ F_1 *second-order stochastically dominates* F_2

6.3 Subjective probability theory

- If agents choose between *uncertain* prospects for which *no* objective probabilities are given, their behavior may still be represented in "as-if"-fashion as expected utility maximization for *subjective probabilities*
- The key requirements for this to be possible is *Savage's sure thing principle*: the ranking of two *prospects* P_1 and P_2 ($\triangleq$ mappings from states of the world to, e.g, wealth) depends only on provisions for states in which P_1 and P_2 actually differ
- In particular,

P_1:

s_1	s_2	s_3
x	y	z

$\succsim$ P_2:

s_1	s_2	s_3
x'	y'	z

if and only if

P_3:

s_1	s_2	s_3
x	y	z'

$\succsim$ P_4:

s_1	s_2	s_3
x'	y'	z'

Ellsberg paradox

- Intuitively reasonable choices under uncertainty can violate subjective expected utility maximization
  (e.g., because the latter doesn‘t allow for *ambiguity aversion*)
- Example:
  Suppose a ball is drawn from an urn with 30 red balls, and 60 white or blue balls in unknown proportion
  – Many people prefer P_1 in
    P_1: 100€ for red, 0€ otherwise
    vs. P_2: 100€ for blue, 0€ otherwise
  - And they prefer P_4 in
    P_3: 100€ for red or white, 0€ otherwise,
    vs. P_4: 100€ for blue or white, 0€ otherwise
- The first choice indicates $\pi_{blue} < 1/3 = \pi_{red}$;
  the second one indicates $2/3 > 1 - \pi_{blue} \quad \Leftrightarrow \quad \pi_{blue} > 1/3$

7. Static games of complete information

- GT ≡ multiperson decision theory
- Each agent’s utility possibly depends on actions of other agents; optimal decisions thus depend on individual beliefs about other agents’ choices (which depend on their beliefs)
- GT works with models of real-life situations, called “games”;
  to these, it applies “solution concepts”
- GT helps to understand how decision makers interact if they are *rational* and reason *strategically*,
  i.e., if they pursue a well-defined objective and make optimal use of their knowledge about other decision makers

Some distinctions

- There are two main branches of GT
  - *non-cooperative* GT:
    Players may communicate before acting but cannot sign binding contracts or irrevocably commit to some action;
    order of moves and players' information is explicitly specified
  - *cooperative* GT:
    Players can make binding agreements;
    "details" of the game are unspecified
- Players' information in a game can be
  - *complete*:
    all know the game's structure and everybody's preferences (though maybe not all of others' actions prior to a move)
  - *incomplete*:
    at least one player lacks information, e.g., about others' preferences

Some distinctions

- A non-cooperative game can be
  - *in normal form* or *static* or *simultaneous-move*:
    players choose a strategy (= a complete plan of action covering all contingencies) once and "simultaneously"
  - *in extensive form* or *dynamic*:
    players act sequentially based on *perfect* or *imperfect* information about what has happened so far
- An extensive form game can be translated into normal form, and vice versa;
  dynamic information is often useful, but sometimes also distracting

Basic notation

- Notation:
  - $N = \{1,2,\ldots, n\}$: set of agents or *players*
  - S_i: set of *(pure) strategies* available to player i
  - $s_i \in S_i$: a strategy of player i
  - $\boldsymbol{S} \equiv S_1 \times \ldots \times S_n$: strategy space of the game
  - $\boldsymbol{s}=(s_1,\ldots,s_n) \in \boldsymbol{S}$: a *strategy profile*
  - $\boldsymbol{s}_{-i}=(s_1,\ldots,s_{i-1}, s_{i+1},\ldots, s_n)$: profile of all except player i's strategy choices
  - $\boldsymbol{S}_{-i} \equiv S_1 \times \ldots S_{i-1} \times S_{i+1} \times \ldots \times S_n$
  - u_i: $\boldsymbol{S} \rightarrow \mathbf{R}$: player i's v.N.-M. utility or *payoff function*
  - $\boldsymbol{u}$: $\boldsymbol{S} \rightarrow \mathbf{R}^n$ with $\boldsymbol{u}(\boldsymbol{s}) \equiv (u_1(\boldsymbol{s}), \ldots, u_n(\boldsymbol{s}))$
  - $\Delta(S_i)$: set of all probability distributions over S_i (= i's *mixed strategies*)
  - $\sigma_i \in \Delta(S_i)$: a mixed strategy of i
  - σ, σ_{-i}: analogous

Normal form

- The *normal* or *strategic form* of a game is a triplet $\langle N, \boldsymbol{S}, \boldsymbol{u} \rangle$ specifying the players, their strategies and payoff functions
- The *mixed extension* of $\langle N, \boldsymbol{S}, \boldsymbol{u} \rangle$, denoted by $\langle N, \Sigma, \boldsymbol{u} \rangle$ with $\Sigma=\Delta(S_1)\times \ldots \times \Delta(S_n)$, explicitly allows the use of mixed strategies, i.e., players can *independently* randomize over their pure strategies
- Remarks:
  - Pure strategies are just particular (degenerate) mixed strategies
  - Often the analysis concerns $\langle N, \Sigma, \boldsymbol{u} \rangle$, but only $\langle N, \boldsymbol{S}, \boldsymbol{u} \rangle$ is mentioned
  - Utility on $\boldsymbol{S}$ naturally extends to Σ by the assumption of v.N.-M. utilities

Complete information and common knowledge

- Unless otherwise stated, we will consider games of *complete information*, i.e., we assume that $\langle N, \boldsymbol{S}, \boldsymbol{u} \rangle$ and the rationality underlying u are *common knowledge*
- Some fact x is called *common knowledge* if
  - everybody knows x,
  - everybody knows that everybody knows x,
  - everybody knows that everybody knows that everybody knows x,
  - etc. ad infinitum
- We presume that with any facts x, y, and z players know all the logical implications of x, y, and z, too

Dominant strategies and rationalizability

- Question: Which predictions can be made just based on (common knowledge of) rationality?
- Strategy $\sigma_i^* \in \Sigma_i$ is
  - *strictly dominating* strategy $s_i' \in S_i$ (or s_i' is *strictly dominated* by σ_i^*)
    $\Leftrightarrow \quad \forall\ \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}: u_i(\sigma_i^*, \boldsymbol{s}_{-i}) > u_i(s_i', \boldsymbol{s}_{-i})$,

    i.e., σ_i^* is always strictly better than s_i' no matter what (player i believes that) the other players do
  - *weakly dominating* s_i' (or s_i' is *weakly dominated* by σ_i^*)
    $\Leftrightarrow \quad \forall\ \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}: u_i(\sigma_i^*, \boldsymbol{s}_{-i}) \geq u_i(s_i', \boldsymbol{s}_{-i})$
    $\wedge \quad \exists\ \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}: u_i(\sigma_i^*, \boldsymbol{s}_{-i}) > u_i(s_i', \boldsymbol{s}_{-i})$

    i.e., σ_i^* is never worse than s_i' and sometimes strictly better
- s_i^* is *strictly dominant* if it strictly dominates all other $s_i' \in S_i$
- If a strictly dominant strategy exists, rationality dictates its use
- For n=2, a profile $\boldsymbol{\sigma}$ is consistent with common knowledge of rationality, i.e., is *rationalizable* iff all involved s_i survive *iterated elimination of strictly dominated strategies*

Nash equilibrium

- When many strategy profiles are rationalizable, more specific predictions can be obtained if players are assumed to have beliefs consistent with each other, i.e., i's beliefs about s_{-i} are correct for every $i \in N$
- NB: this is *not* implied by common knowledge of rationality and the game, but requires extra motivation!

- Strategy profile $\boldsymbol{s}^* = (s_1^*, \ldots, s_n^*) \in S$ is a *Nash equilibrium* (NE)

  $\Leftrightarrow \quad \forall i \in N: \forall s_i \in S_i: u_i(s_i^*, \boldsymbol{s}_{-i}^*) \geq u_i(s_i, \boldsymbol{s}_{-i}^*)$,

  i.e., everybody plays a *best response*[1] to (his correct beliefs about the) strategy choices $\boldsymbol{s}_{-i}^*$ of everybody else.

  [[1] There may be others!]

Remarks

- Mixed strategy NE: same for profiles $\sigma^* \in \Delta(S_1) \times \ldots \times \Delta(S_n)$
- A strategy profile $\boldsymbol{s}^*$ is a *strict Nash equilibrium* iff it is a NE and above inequality is strict, i.e., everyone has a *unique* best response to $\boldsymbol{s}_{-i}^*$ [NB: a game may have several strict NE]

- Why game theorists care about NE so much:
  - Though NE is not implied by rationality, it is “focal” amongst all rationalizable profiles: only NE involve *consistent beliefs*
  - If there is any “unique predicted outcome” or a stable social convention for playing a particular game (w/o external coordination), then it must be a NE
  - A NE may be viewed as a “steady state” of play where an unspecified dynamic process has brought about correct expectations;
    many learning dynamics or evolutionary processes converge to a NE
  - If players can talk prior to the game and agree on some profile $\boldsymbol{s}$ *without* exogenous commitment or coordination, only NE are *self-enforcing*

Mixed-strategy NE

Proposition

Consider the mixed extension of finite game $\langle N,\boldsymbol{S},\boldsymbol{u}\rangle$.

σ^* is a NE of $\langle N,\Sigma,\boldsymbol{u}\rangle$

$\Leftrightarrow$ For all $i \in N$, every pure strategy s_i played with positive probability under σ_i^* ($\equiv s_i$ is in the *support* of σ_i^*) is a best response to $\boldsymbol{\sigma}_{-i}^*$

Proof:

It is always true that $u_i(\boldsymbol{\sigma}) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \boldsymbol{\sigma}_{-i})$

"$\Rightarrow$" Assume some s_i in supp(σ_i^*) is *no* best response to $\boldsymbol{\sigma}_{-i}^*$.
Then $u_i(\sigma^*)$ can be *increased* by shifting probability from s_i to some s_i' that is a best response. ↯ to "σ_i^* is a best response"

"$\Leftarrow$" Assume $\boldsymbol{\sigma}^*$ is *no* NE, i.e., for some i, σ_i^* is no best response to $\boldsymbol{\sigma}_{-i}^*$
Some s_i' in supp(σ_i') with σ_i' being a best response to $\boldsymbol{\sigma}_{-i}^*$ gives higher payoff against $\boldsymbol{\sigma}_{-i}^*$ than some s_i in supp(σ_i^*). ↯

Mixed-strategy NE

- That truly mixed NE involve indifference reduces their appeal
- Defense of mixed NE:
  - In some games, players try to be unpredictable and mixed NE has some empirical support
    (penalty kicks, tennis serves, R-S-P game, …)
  - In zero-sum games, σ_i^* maximizes i's guaranteed expected payoff, i.e., is a "safe" strategy with minimal knowledge requirements
  - A mixed NE may describe a large population where individuals are randomly matched and play pure strategies in the "right" population proportions
  - A mixed NE can be viewed as approximating a pure (Bayesian) NE of a game in which part of players' payoffs is private knowledge (*purification* of mixed NE proposed by Harsanyi)

Existence of NE

- Games with infinite pure strategy spaces may fail to have any NE
- Nash (1950) proved that every *finite* game has “an equilibrium point” (=mixed NE)
- The proof involves showing that in such games
  - all players have at least one best response to any σ_{-i}; if i has multiple BRs, they form a convex set
  - $BR_i(\cdot)$ has a closed graph (i.e., is *upper semicontinuous*)
- It follows that $BR(\cdot) \equiv BR_1(\cdot) \times \ldots \times BR_n(\cdot)$ is a u.s.c., nonempty and convex-valued correspondence from the non-empty, convex, and compact set $\Sigma \equiv \Delta(S_1) \times \ldots \times \Delta(S_n)$ to itself

$\Rightarrow$ *Kakutani's fixed point theorem* guarantees existence of a fixed point $\sigma \in BR(\sigma)$, which is a NE

- Nash’s result can be extended to games with general convex strategy spaces, to symmetric NE, or pure-strategy NE

Equilibrium selection and refinement

- The key “problem” is usually not existence but multiplicity of NE
- Consider

a)

1 \ 2	F	H
F	7,7	0,0
H	0,0	9,9

b)

1 \ 2	F	H
F	7,7	8,0
H	0,8	9,9

$\rightarrow$ What would *you* play?

c)

1 \ 2	f	h
F	3,1	0,0
H	2,2	2,2

d) $S_1 = S_2 = [0,100]$, $u_i(s_i, s_j) = s_i$ if $s_i + s_j = 100$, and 0 otherwise

Equilibrium refinement

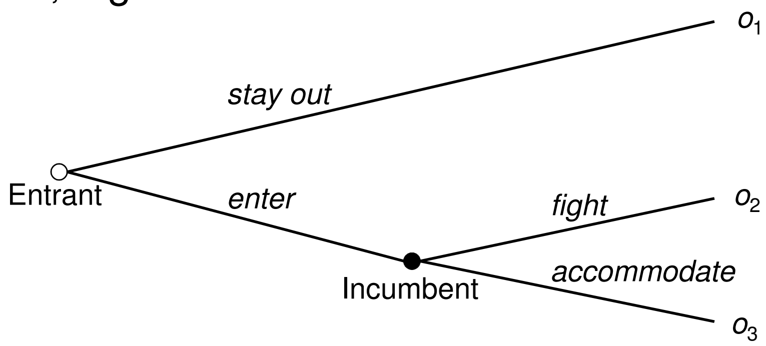
- A large literature has tried to build plausibility or robustness considerations into the equilibrium concept itself
- Prominent *refinements of NE* include:
  - *(trembling-hand) perfect equilibrium*
    - A NE σ is trembling-hand perfect iff each σ_i is still optimal against *some* completely mixed strategy profile "nearby", i.e., each player i sticks to σ_i even if he expects others to "tremble" and play *any* of their pure strategies with at least (some particular) small positive probability
    - This rules out the use of weakly dominated strategies; *strict NE* and *NE involving only completely mixed strategies* are automatically perfect
  - *strictly perfect equilibrium*
    - As above, but robustness against *all*, not just some "trembles" is required
  - *essential equilibrium*
    - Requires robustness against payoff perturbations
- NB: there are also plausible generalizations of NE, esp. the notion of a *correlated equilibrium*

8. Dynamic games of complete information

- A *dynamic* or *sequential-move* or *extensive form* game adds to the information provided in static games an explicit description of
  - the timing of players' actions
  - the information about play so far on which actions can be conditioned
- We keep the assumption of complete information, i.e., the game (incl. all preferences) is common knowledge

Game tree

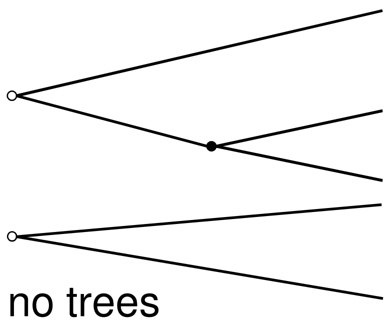
- Central to the modeling of dynamic games is the concept of a *game tree*, e.g.

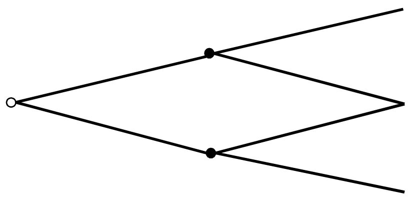


- A tree is a particular type of *directed graph*, with *nodes* (or vertices) and *edges*, each connecting two nodes

Game tree

- Formally, a *tree* is defined by
  - a set of nodes N
  - a transitive and asymmetric (i.e., $a \prec b \Rightarrow \neg(b \prec a)$) precedence relation $\prec$ satisfying the *arborescence properties*:
    - there is a unique *initial node* $n^0 \in$ N without predecessor
    - if n and n' precede n'', then *either* $n \prec n'$ *or* $n' \prec n$ (in particular, every node except n^0 has a unique direct predecessor)
- For example,

1. 

or 2. 

are no trees

Game tree

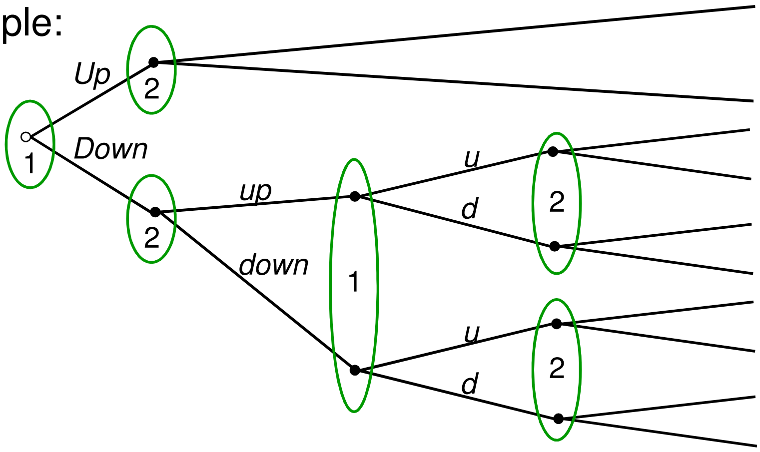
- Nodes without successors are called *terminal nodes*;
  all non-terminal nodes are called *decision nodes*
- Given N and $\prec$ with decision nodes D, a function
  $$\iota: D \to N \cup \{\text{Nature}\}$$
  for every decision node specifies the player who has to move
- The additional player "Nature" is a trick to model chance moves (if needed)
- For $n \in$ D, $A(n)$ denotes the set of *actions* available to player $\iota(n)$
- Each $a \in A(n)$ leads to a different direct successor n' of n as defined by a function
  $$\alpha(n): A(n) \to \text{Succ}(n)$$
  [i.e., each non-initial node n' is reached from a unique n by a unique action $a \in A(n)$]

Information sets

- The player $\iota(n)$ who has to move at n may not know that the game is currently exactly at n, e.g., because moves of other players are imperfectly observed
- This is reflected by a partition ***P*** of D into *information sets* $\{n^0\}$, P^2, …, $P^k \in \boldsymbol{P}$ that capture what players know when moving

Information sets

- Example:



- Here:
  - 1^{st}-moving player 1 (always) knows the entire “history”
  - Player 1 does not know whether 2 played *up* or *down*
  - Player 2 knows 1’s choice when making his first choice, but not his second

Information partition

- The *information partition* ***P*** of D into information sets must satisfy the following conditions:
  - the same player $\iota(n)$ (or $\iota(P^j)$) and action set $A(n)$ (or $A(P^j)$) are assigned to all $n \in P^j$
  - if $n \in P^j$, then no successor of n is also contained in P^j

- Player $\iota(P^j)$ called to select an action $a \in A(P^j)$ at node P^j knows that moves leading to $P^i \neq P^j$ were not played, but need not be able to identify the particular move which has led to P^j

Extensive game

- Formally, the collection $\langle N, \mathrm{N}, \prec, \iota, \{A(n)\}_{n\in \mathrm{N}}, \{\alpha(n)\}_{n\in \mathrm{N}}, \boldsymbol{P} \rangle$ defines an *extensive game form*.
  An extensive game form together with
  - v.N.-M. utilities u_i over all (lotteries over) terminal nodes for all $i\in N$
  - a probability distribution $\rho(n)$ on $A(n)$ for each n at which Nature "moves"

  defines an *extensive (form) game.*

- Remarks:
  - Above 9-tuple (or 10-tuple in MWG) is rarely written down; usually Γ is "defined" by a diagram or verbal description
  - We assume that players have *perfect recall*, i.e., do not forget what they learned at some stage (this restricts possible partitions $\boldsymbol{P}$)
  - If all information sets are singletons then we speak of a *game of perfect information*, otherwise of a *game of imperfect information*

Strategies in extensive games

- In extensive games, *actions* (at some information set) need to be clearly distinguished from *strategies*;
  strategies are complete plans that prescribe an action *for every contingency* calling a player to move

- Denoting the set of information sets P such that $\iota(P)=i$ by $\boldsymbol{P}_i$, a *(pure) strategy* of player i in an extensive game is a function

$$s_i\colon \boldsymbol{P}_i \to \cup_{P\in \boldsymbol{P}_i} A(P)$$

  which maps each of i's information sets $P\in \boldsymbol{P}_i$ to a feasible action $s_i(P)\in A(P)$
- A player may randomize either over his pure strategies ($\to$ *mixed strategy*) or independently over feasible actions at each information set ($\to$ *behavior strategy*)

Backward induction

- Extensive games of perfect information can be solved by *backward induction* if there is a "last period", i.e., if every possible history is finite:
  - One determines optimal choices for the respective last-moving players in all next-to-terminal nodes
  - One replaces these decision nodes by the selected terminal nodes (or marks the corresponding edges appropriately), and then repeats the exercise until the initial node is reached
- Every finite game of perfect information has a solution to backward induction;
  for "generic" games – i.e., if no two payoffs are the same – the solution is unique

Subgame perfect equilibrium

- The idea of players behaving rationally (and others anticipating this) throughout the entire game (= *sequential rationality*) *can* also be applied to games of imperfect information or without "last period" …

- A *subgame* Γ_n of an extensive game Γ is an extensive game starting in a singleton information set $\{n\}$ (of Γ), containing exactly all successors of n as its other nodes, not cutting through any of Γ's information sets and inheriting payoffs, information sets, etc. from Γ
- A strategy profile $\boldsymbol{s}^*$ of Γ is a *subgame perfect equilibrium* (SPE) iff $\boldsymbol{s}^*$ induces a NE in every subgame of Γ
- In games with finitely many stages, SPE can be found by (a generalization of) backward induction

One-deviation principle

- Consider a game of perfect information or one where at each stage players move simultaneously and afterwards observe all actions:
  - Obviously, $\boldsymbol{s}^*$ is a SPE *only if* no player i has a strategy s_i' differing from s_i^* in *just one* information set $P \in \boldsymbol{P}_i$ and doing strictly better than s_i^* conditional on P being reached
  - The reverse is also true and known as the

- <u>One-deviation principle</u>:
  $\boldsymbol{s}^*$ is a SPE *if* no player i has a strategy s_i' differing from s_i^* in *just one* information set $P \in \boldsymbol{P}_i$ and doing strictly better than s_i^* conditional on P being reached

Finitely repeated games

- Suppose that extensive game Γ^T consists of $T<\infty$ iterations of *exactly* the same normal form game $\Gamma=\langle N,\boldsymbol{S},\boldsymbol{u}\rangle$ and players try to maximize their undiscounted sum of payoffs
- Knowing the NE of Γ, what can we say about SPE of Γ^T?

- If stage game Γ has a *unique* NE $\boldsymbol{s}^*$ then T-fold play of $\boldsymbol{s}^*$ independently of the current history is Γ^T's *unique* SPE
- If $\boldsymbol{s}^*$ is *any* NE of stage game Γ, then T-fold play of $\boldsymbol{s}^*$ independently of the current history is *a* SPE of Γ^T
- In case of multiple stage game NE, there may also exist other SPE which are history-dependent and involve play of a stage game NE only in an "end-game" phase

Infinitely repeated games

- Let Γ^∞ denote the *infinite* repetition of normal form game $\Gamma=\langle N,\boldsymbol{S},\boldsymbol{u}\rangle$ in which players maximize their discounted sum of payoffs
  (with common discount factor $\delta\in(0,1)$)
- A payoff vector $\boldsymbol{x}$ is called *strictly individually rational* iff for every player i, x_i strictly exceeds i's *minmax payoff* M_i in Γ,
  i.e, the lowest payoff that players $-i$ can impose as punishment on a player i who correctly anticipates σ_{-i} and best-responds to it

- *Nash Folk Theorem / Perfect Folk Theorem:*
  Let $\boldsymbol{x}$ be feasible and strictly individually rational. Then, for δ sufficiently close to 1, there exists a *NE* / *SPE* of Γ^∞ with average payoff $\cong \boldsymbol{x}$.
  (For games with $n>2$ players, an additional technical condition related to reward opportunities has to be satisfied for the Perfect Folk Theorem)

9. Games of incomplete information

- So far, we assumed that players have complete information about the game;
  in particular, every player knows
  - every other player's preferences (incl. the rationality associated to that)
  - every other player's strategy space
  - every other player's information partition
- What use are NE or SPE, which rest on correct beliefs about others' behavior in the game, when there is *incomplete information* on one of the above aspects, i.e., about which game is played?

Harsanyi's transformation

- John C. Harsanyi (1967/68) proposed a powerful framework for analysis of games of incomplete information
  1. Introduce different *types* of each player:
  – A particular type θ_i of player *i* is identified with a particular preference, strategy space and information partition
  – Each player *i* knows own type θ_i but possibly not that of other players
  2. Introduce an additional player, called *Nature*:
  – *Nature* moves first and assigns each player *i* his type $\theta_i \in \Theta_i$
  – Nature's move is a random draw from an exogenous and commonly known joint probability distribution ρ on $\Theta \equiv \Theta_1 \times \ldots \times \Theta_n$
  – Each player *i* rationally updates the common prior ρ after learning θ_i
- Thus, a game of *incomplete information* is transformed into an (extensive) game with *complete (but imperfect) information*

Example

- Suppose a potential entrant and the market's incumbent simultaneously decide about whether to enter and whether to boost capacity, respectively
- Cost of a capacity increase is either high or low, and *private information* of the incumbent
- Profits are

Incumbent \ Entrant	*Enter*	*Stay out*
Invest	0, -1	2, 0
Don't invest	2, 1	3, 0

in case of *high* costs

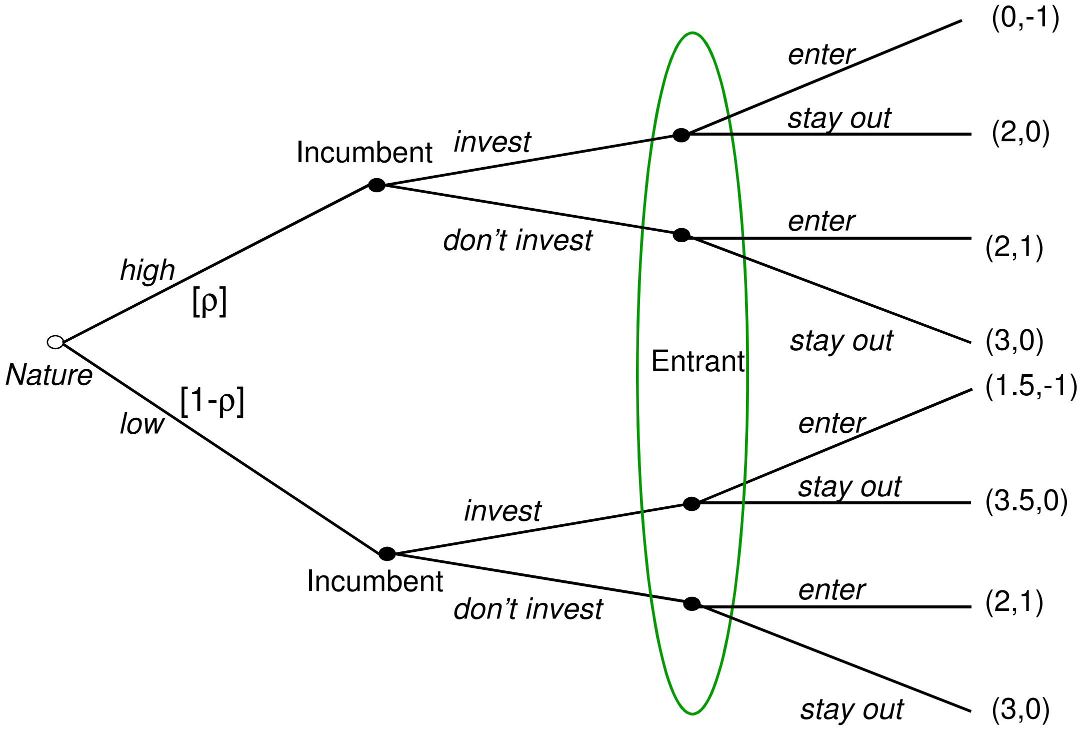
and

Incumbent \ Entrant	*Enter*	*Stay out*
Invest	1.5, -1	3.5, 0
Don't invest	2, 1	3, 0

in case of *low* costs

Example

- *Nature* 'selects' high costs with probability ρ, i.e., we obtain:



Bayesian games

- A *Bayesian (normal form) game* is a collection $\langle N, \Theta, \rho, \boldsymbol{A}, \boldsymbol{u} \rangle$ where
  - type space $\boldsymbol{\Theta} \equiv \Theta_1 \times \ldots \times \Theta_n$ specifies all possible types of players $i \in N$
  - actual types are drawn from joint probability distribution ρ on $\boldsymbol{\Theta}$
  - players' (pure) strategy spaces S_i are *implicitly* defined as the set of all functions s_i: $\Theta_i \rightarrow A_i$ which map every possible type of player i, θ_i, to an action $s_i(\theta_i) \in A_i$
    (elements of A_i are strategies in the original game of incomplete information)
  - u_i is defined on $\boldsymbol{A} \times \Theta_i$

- We assume that $\langle N, \Theta, \rho, \boldsymbol{A}, \boldsymbol{u} \rangle$ is common knowledge
  ⇒ Rational players update the the prior ρ using *Bayes' rule*:

$$Pr(A/B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Best responses

- Comparing two actions a_i, $a_i' \in A_i$, player i with type θ_i will (in equilibrium: correctly) anticipate some *strategy* profile s_{-i} but – in the spirit of players having incomplete information – must treat other players' *types* and hence actions as random variables
- So player i's type θ_i compares
$$\mathbf{E}u_i(a_i, s_{-i}, \theta_i) \equiv \sum_{\theta_{-i}\in\Theta_{-i}} \rho(\theta_{-i}/\theta_i)\cdot u_i\left(a_i, s_{-i}(\theta_{-i}), \theta_i\right)$$
to $\mathbf{E}u_i(a_i', s_{-i}, \theta_i)$
- If players use mixed strategies, then $u_i(a_i', s_{-i}(\theta_{-i}), \theta_i)$ is simply replaced by expected payoff $u_i(a_i', \sigma_{-i}(\theta_{-i}), \theta_i)$
- Strategy s_i^* of player i (in a Bayesian game) is a *best response* to s_{-i} iff it specifies an optimal action $s_i^*(\theta_i) \in A_i$ for *each* type θ_i that player i might happen to be,
i.e., $\forall \theta_i \in \Theta_i$: $\forall a_i' \in A_i$: $\mathbf{E}u_i(s_i^*(\theta_i), s_{-i}, \theta_i) \geq \mathbf{E}u_i(a_i', s_{-i}, \theta_i)$

Bayesian Nash equilibrium

- A *Bayesian Nash equilibrium (BNE)* of the game $\langle N, \Theta, \rho, \boldsymbol{A}, \boldsymbol{u}\rangle$ is a strategy profile $\boldsymbol{s}^* = (s_1^*, \ldots, s_n^*)$ such that for each player $i \in N$ the strategy s_i^* is a best response to s_{-i}^*, i.e.,
$$\forall \theta_i \in \Theta_i: a_i = s_i^*(\theta_i) \in A_i \text{ maximizes } \mathbf{E}u_i(a_i, s_{-i}^*, \theta_i)$$
(with expectation $\mathbf{E}$ based on $\rho(\theta_{-i} \mid \theta_i)$)
- A mixed-strategy BNE σ^* is defined analogously
- As in games of complete information, mixed strategy σ_i^* is a best response to σ_{-i} iff *each* action a_i played with a probability $\sigma_i(\theta_i)(a_i) > 0$ maximizes $\mathbf{E}u_i(a_i, \sigma_{-i}, \theta_i)$
- Proofs of existence of BNE are analogous to those for NE
(i's best response correspondence is $BR_i \equiv BR_i(\theta_1) \times BR_i(\theta_2) \times \ldots \times BR_i(\theta_{ki})$)

Example

- Again consider

1_h / 1_l \\ 2	*enter*	*stay out*
invest	0/1.5,-1	2/3.5,0
don't invest	2/2,1	3/3,0

with probability $\rho \in [0,1]$ for firm 1 having high costs

- Given ρ=0.5,
  - σ^*=(($1_h \mapsto$ *don't invest*, $1_l \mapsto$ *don't invest*); *enter*) and
  - every σ^{**}= (($1_h \mapsto$ *don't invest*,$1_l \mapsto$ *invest*); (q,1-q)) with $q \in [0,1/2]$

  are BNE

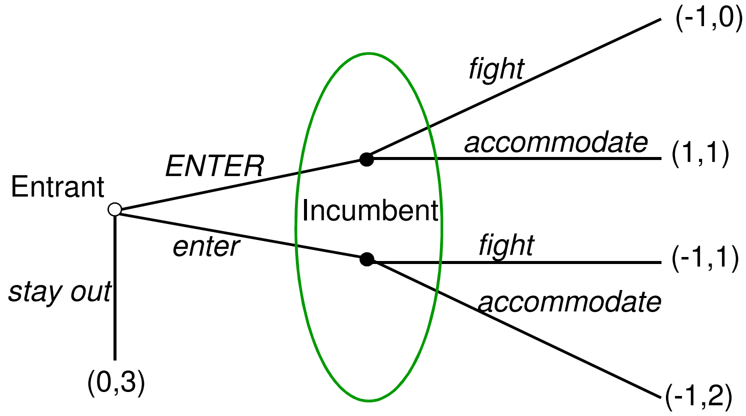
  (q refers to probability of *enter*)

Dynamic games of incomplete information

- Two complications arise when we apply the Harsanyi transformation to an extensive game of incomplete information:
  1. If θ_i is private information, -i's information sets are never singletons
     $\Rightarrow$ there are no proper subgames started by -i's moves
     $\Rightarrow$ *subgame perfection* does not restrict -i's moves off the NE path
     $\Rightarrow$ sequentially irrational behavior can survive (e.g., empty threats)
  2. While -i's beliefs about θ_i should be updated after any of i's moves a_i^t, *Bayes' rule* only defines the conditional probability $\rho(\theta_i | a_i^t, \theta_{-i})$ after moves a_i^t which have positive probability under strategy profile σ^*

Example

- Consider the following game of complete but imperfect information (not even involving a move by *Nature*):



- (*ENTER*, *accommodate*) and (*stay out*, *fight*) are NE
- For the incumbent, *fight* is strictly dominated; still, (*stay out*, *fight*) is SPE because the game is its only subgame

$\Rightarrow$ We need a better formalization of (sequential) rationality than SPE

Strategies and beliefs

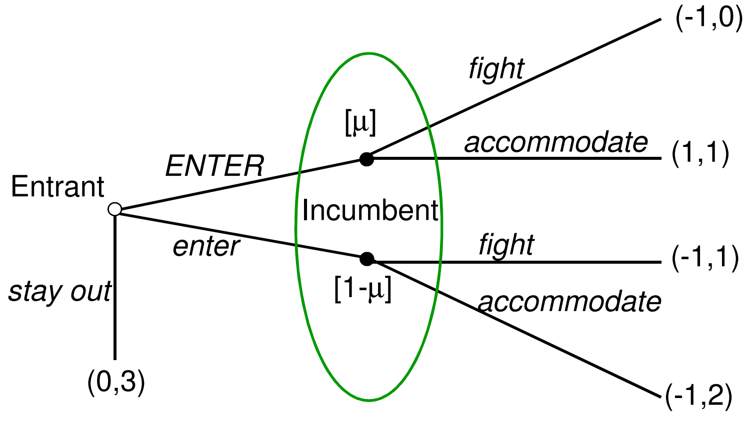
- More refined equilibrium concepts try to formalize optimal behavior in every "continuation game", i.e., in whatever follows a possible history, rather than only proper subgames
- For player *i* to be able to identify an *optimal* action in an arbitrary information set P^j at which he has the move he must
  - anticipate a particular (mixed) strategy $\boldsymbol{\sigma}_{-i}$ played by other players
  - have conditional beliefs $\mu_i(\cdot|P^j)$ about which decision node $n \in P^j$ he is in (= a probability distribution μ_i on P^j) given that P^j was reached
- The beliefs held by any player *i* and the equilibrium strategy profile σ^* depend on each other:
  - each player *i*'s strategy σ_i must maximize expected utility given μ_i
  - beliefs μ_i must be consistent with prior ρ and anticipated strategies $\boldsymbol{\sigma}_{-i}$

Perfect Bayesian equilibrium

- A *(weak) Perfect Bayesian (Nash) equilibrium (PBE)* of the game $\Gamma = \langle N, \Theta, \mathrm{N}, \prec, \iota, \{A(n)\}_{n\in \mathrm{N}}, \{\alpha(n)\}_{n\in \mathrm{N}}, \boldsymbol{P}, \{\rho(n)\}, \boldsymbol{u}\rangle$ is a combination $(\boldsymbol{\sigma}^*, \boldsymbol{\mu}^*)$ of a strategy profile $\boldsymbol{\sigma}^*=(\sigma_1^*,\ldots,\sigma_n^*)$ and a system of beliefs $\boldsymbol{\mu}^*=(\mu_1^*,\ldots, \mu_n^*)$ such that for each player $i \in N$
  1. strategy s_i^* is *sequentially rational*,
     i.e., it prescribes a best response to σ_{-i}^* in any information set $P^j \in \boldsymbol{P}_i$ given the system of beliefs μ_i^*, i.e.,
     $\forall \theta_i \in \Theta_i$: $\forall P^j \in \boldsymbol{P}_i$: $\sigma_i^*(\theta_i) \in \Delta(A_i)$ maximizes $\mathbf{E}u_i(a_i,\sigma_{-i}^*,\theta_i | P^j)$
     (with expectation $\mathbf{E}$ based on μ_i^*, and $\sigma_i^*(\theta_i)(a_i) > 0$)
  2. system of beliefs μ_i^* is *consistent with* $\boldsymbol{\sigma}^*$,
     i.e., it is derived from $\boldsymbol{\sigma}^*$ and Bayes' rule (where it can be applied; that is: for information sets which have positive probability under $\boldsymbol{\sigma}^*$)
- A combination of a strategy profile and a system of beliefs, $(\boldsymbol{\sigma},\boldsymbol{\mu})$, is also called an *assessment*;
  so a PBE is a sequentially rational and consistent assessment

Example

- Consider



- When the incumbent's information set is reached, sequential rationality requires *accommodate* for *any* belief $(\mu,1-\mu)$ about the true history
- Anticipating $\sigma_2^*=$*accommodate*, rationality requires $\sigma_1^*=$*ENTER*
- Anticipating σ_1^*, incumbent must believe that *ENTER* was played with probability 1

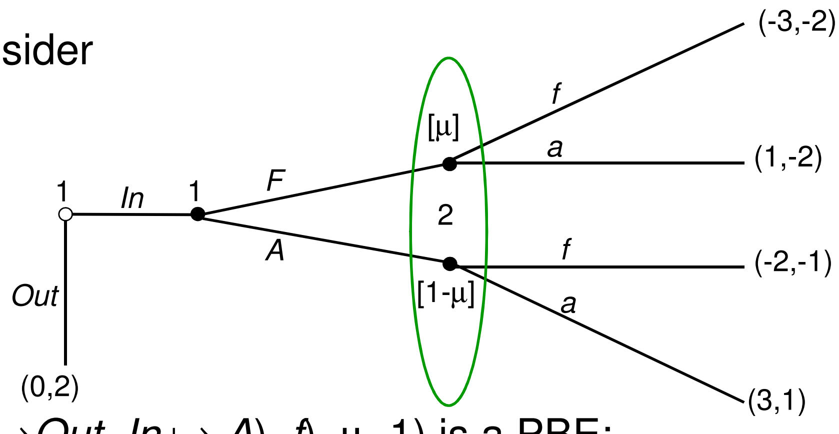
$\Rightarrow$ $(\boldsymbol{\sigma}^*, \mu^*)$ with $\boldsymbol{\sigma}^*=($*ENTER*, *accommodate*$)$ and $\mu^*=1$ is the unique PBE

Remarks

- If players use completely mixed strategies in a PBE, every information set is reached with positive probability and the system of beliefs is well-defined by Bayes' rule everywhere
- Otherwise, there is no restriction on conditional beliefs in information sets reached only after a deviation, i.e., the respective player *i* who has the move is free to interpret -*i*'s deviation as, for example, a fully informative indication of any particular type θ_{-i}, or as not revealing any information, or …

Problematic example

- Consider



- $(((\emptyset \mapsto \mathit{Out},\ \mathit{In} \mapsto A), f), \mu=1)$ is a PBE:
  - If 1 anticipates that 2 would *fight*, it is best to choose *Out* and to *Accommodate* after involuntary entry
  - Anticipating that 1 will stay out, Bayes' rule doesn't restrict 2's beliefs for the zero-probability event that 2 has to make a move; 2 *may* think that 1 made another "mistake", so that μ=1
  - Based on μ=1, *fight* is indeed optimal for 2
- This implausible beliefs-based PBE isn't even a SPE: (A,f) is no NE of the subgame started by *In*

Sequential equilibrium

- Kreps and Wilson (1982) proposed to avoid complete arbitrariness of beliefs in information sets reached with probability zero by requiring existence of some fully mixed strategy profiles – which reach every information set with positive probability – that "justify" the beliefs in (σ^*,μ^*)

- A *sequential equilibrium (SE)* of the (mixed extension of) game Γ is an assessment (σ^*,μ^*)
  1. which constitutes a perfect Bayesian equilibrium
  2. for which a sequence $\{\sigma^k\}_{k=1,2,...}$ of completely mixed strategy profiles with $\sigma^k \rightarrow \sigma^*$ exists such that the sequence of beliefs implied by σ^k and Bayes' rule, $\{\mu^k\}_{k=1,2,...}$, converges to μ^*

Remarks

- Any SE is a PBE, but the reverse is not true;
  in particular, SE requires two players to have consistent beliefs about a third player also after he deviated
- Every finite game has at least one SE
- In games in which only players' types are private information but all actions are observed, PBE and SE coincide
  - if each player has at most two possible types or
  - if the game has only two periods (e.g., simple signaling games)
- NB: The sequence $\{\sigma^k\}_{k=1,2,...}$ need not consist of equilibria; requiring that each (σ^k, μ^k) also forms a PBE leads to (*trembling-hand) perfect equilibria (PE)* in extensive games, which are a "refinement" of SE introduced by Selten (1975)
- PE and SE are not the end of the PBE refinement story …
  (e.g., the "Dominance Criterion" asks that, if possible, beliefs place zero probability on nodes reached by a strictly dominated action)

10. Competitive markets

- In a *perfectly competitive economy*, every relevant good is traded, voluntarily and without transaction costs, by agents without market power nor informational advantages
- A *general competitive equilibrium* is an allocation and a price vector s.t.
  1. all firms‘ production and factor demand plans maximize their respective profits,
  2. all consumers’ consumption and factor supply plans maximize their respective utility,
  3. and these plans match, i.e., all markets clear
- Properties of competitive equilibria have fundamental importance:
  – Do market allocations satisfy “minimal quality standards” from a collective point of view?
  – How do competitive market interaction and social objectives relate?

Two requirements for market outcomes

- A first minimal requirement is that the allocations brought about by the market are *Pareto efficient*
- NB: Pareto efficiency doesn‘t involve any equitability concerns
- So, a second ambition is that specific normatively desired allocations somehow can be brought about by the market, too …
- These issues are addressed for the economy as a whole by *general equilibrium theory*;
  we here restrict attention to a single market which constitutes a small part of the overall economy, i.e., *partial equilibrium*

10.1 Partial equilibrium competitive analysis

- Generally, a consumer's welfare depends on the optimal use of her endowments (time, talents, goods, ...) , and thus on *all* prices in the economy
- We study a good k on which consumers spend only a small part of their budgets
- Then it is reasonable to ignore the effect of, e.g., a tax on this good on the price of other goods and any wealth effects
- Without wealth effects, the distribution of wealth amongst consumers doesn't matter, i.e., we could assume a representative consumer;
  moreover, equivalent variation and compensating variation coincide with Marshallian consumer surplus

Partial equilibrium competitive analysis

- Fixed prices for all other goods and no wealth effects can most easily be captured by assuming quasilinear utility

$$u_i(x_i,m_i) = \phi_i(x_i) + m_i$$

  for sufficiently rich consumers $i = 1, ..., I$, where m_i captures i's expenditure on "other goods" (treated as a composite *numeraire good*)
- The price of the numeraire is usually normalized to equal 1; the considered good k has price p

Optimization by firms

- Assuming that consumers have no initial endowment of good k, all consumption has to be produced by profit-maximizing firms
- Capturing firm j's transformation of the numeraire into good k by cost function $c_j(q_j)$, with $c_j' > 0$ and $c_j'' \geq 0$, the necessary and sufficient condition for

$$\max_{q_j \geq 0} \; p^* \cdot q_j - c_j(q_j)$$

is

(I) $p^* \leq c_j'(q_j^*)$, with equality for positive output $q_j^* > 0$

Optimization by consumers

- Consumer i chooses consumption (x_i, m_i) to solve

$$\max_{x_i, m_i \geq 0} \; \phi_i(x_i) + m_i$$

$$\text{s.t.} \qquad m_i + p^* \cdot x_i \leq \omega_{mi} + \Sigma\theta_{ij} \cdot (p^* \cdot q_j - c_j(q_j))$$

(ω_{mi} is i's endowment of the numeraire good, θ_{ij} is i's share of firm j's profits)

- Monotonicity of preferences implies that the budget is exhausted, and

$$\max_{x_i \geq 0} \; \phi_i(x_i) + [\omega_{mi} + \Sigma\theta_{ij}(p^* \cdot q_j - c_j(q_j))] - p^* \cdot x_i$$

calls for

(II) $\phi_i'(x_i^*) \leq p^*$, with equality if $x_i^* > 0$

(unique when assuming that $\phi''_i(\cdot) < 0$)

Competitive equilibrium

- Conditions (I) for all firms $j = 1, ..., J$
  (II) for all consumers $i = 1, ..., I$, and
  (III) $\Sigma x_i^* = \Sigma q_j^*$
  define a *competitive equilibrium (CE)*
- For quasilinear preferences, consumers' shares of firm θ_{ij} and their initial numeraire endowments play no role for p^*, total consumption and production
- *Market demand* for and *supply* of the good are defined by (II) and (I) for arbitrary p
- The inverse of the supply function, $q^{-1}(\cdot)$, can be viewed as the *industry marginal cost function* $C'(\cdot)$
  (with the next unit produced by the most efficient firm)
- The inverse $P(x) = x^{-1}(x)$ of the demand function corresponds to the *marginal social benefit* of the next unit of the good *if* the quantity x is distributed efficiently amongst consumers

10.2 Fundamental Welfare Theorems

- For any fixed consumption and production plans, $\boldsymbol{x}$ and $\boldsymbol{q}$, and (sufficient) total endowments ω_m of the numeraire, any utility vector in set
  $$\{(u_1, ..., u_I) \mid \Sigma u_i \leq \Sigma\phi_i(x_i) + \omega_m - \Sigma c_j(q_j)\}$$
  could be realized by appropriate transfers of the numeraire in the considered quasilinear case
  (numeraire has same constant marginal utility for everyone)
- For given $\boldsymbol{x}$ and $\boldsymbol{q}$, the RHS above is a constant, so the boundary of this utility possibility set is a hyperplane with normal vector (1,1, ..., 1);
  variations of $\boldsymbol{x}$ and $\boldsymbol{q}$ imply parallel shifts of it

Pareto optimal plans

- Plans $\boldsymbol{x}^*$ and $\boldsymbol{q}^*$ are Pareto-optimal iff they maximize the RHS, i.e., they solve

$$\max_{\boldsymbol{x},\boldsymbol{q} \geq 0} \; \Sigma\phi_i(x_i) + \omega_m - \Sigma c_j(q_j)$$

$$\text{s.t. } \Sigma x_i - \Sigma q_j = 0.$$

- Given our convexity assumptions ($c_j'' \geq 0$, $\phi_i'' \leq 0$), the maximization of the Lagrangean

$$L(x_1, \ldots, x_I, q_1, \ldots, q_J, \lambda) = \Sigma\phi_i(x_i) - \Sigma c_j(q_j) - \lambda\cdot(\Sigma x_i - \Sigma q_j)$$

yields the necessary and sufficient conditions ($j=1, \ldots, J$; $i=1, \ldots, I$)

(i) $-c_j'(q_j^*) + \lambda \leq 0 \Leftrightarrow \lambda \leq c_j'(q_j^*)$, with equality for $q_j^* > 0$

(ii) $\phi_i'(x_i^*) - \lambda \leq 0 \Leftrightarrow \phi_i'(x_i^*) \leq \lambda$, with equality for $x_i^* > 0$,

(iii) $\Sigma x_i^* = \Sigma q_j^*$

- These correspond exactly to the conditions which characterize a competitive equilibrium, with λ replacing p^*

First Fundamental Welfare Theorem

- Hence, if price p^* and allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ constitute a CE, then this allocation is Pareto optimal
- This result is also known as the *First Fundamental Theorem of Welfare Economics*

- Good k's price p^* in a CE exactly reflects the good's marginal social value (in units of the numeraire), i.e., the "shadow price" of the resource constraint:
  in its resp. profit maximization, each firm equates its marginal production cost to the marginal social value of its output
- Similarly, consumers consume up to the point where their marginal utility equals the marginal cost of production (in units of the numeraire)
- The theorem vindicates Adam Smith's "invisible hand" for perfectly competitive markets, and holds more generally than considered here

Remarks

- Market power, information imperfections or market incompleteness can yield very different conclusions …
- Nothing is said yet about actual *existence* of a CE, or how it might be reached (if at all) by a dynamic adaptation or *tâtonnement* process with decentralized information …
- In the quasilinear case, CE price p^* and individually consumed and produced quantities of good k do not depend on the distribution of total endowment ω_m
  (NB: except for corner solutions, in which some agents are too poor to consume both good k *and* the numeraire)

Second Fundamental Welfare Theorem

- So, ignoring corner solutions, changing the initial distribution $(\omega_{m1}, ..., \omega_{mI})$ changes individual consumption of the numeraire but not $(x_1^*, ..., x_I^*, q_1^*, ..., q_J^*)$:
  one moves within the Pareto efficient hyperplane
- For *any* Pareto optimal levels of utility $(u_1^*, ..., u_I^*)$, there are transfers $(T_1, ..., T_I)$ of the numeraire good with $\Sigma T_i = 0$ such that a CE reached from the redistributed endowments $(\omega_{m1} + T_1, ..., \omega_{mI} + T_I)$ yields exactly the utilities $(u_1^*, ..., u_I^*)$
- This result is also known as the *Second Fundamental Theorem of Welfare Economics*
- Hence, pursuing a particular distributional goal does *not* conflict with having competitive markets: one can achieve the goal by appropriate endowment transfers and then "let the market work"
- This result generalizes, too, but not as much as the First Theorem
  (in particular, preferences and technology need to be convex)

10.3 Welfare Analysis in Partial Equilibrium

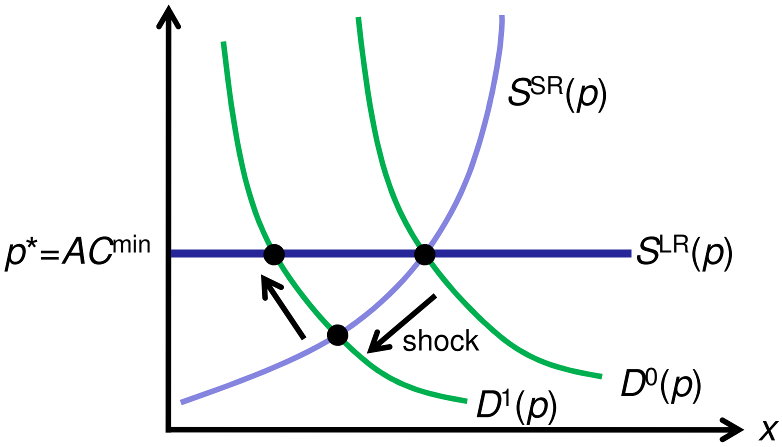
- What "yardstick" can we use for comparing different allocations (esp. Pareto-incomparable ones)?
- The value of $\Sigma\phi_i(x_i) - \Sigma c_j(q_j)$ in the maximization problem which characterizes Pareto efficient allocations is known as the (*Marshallian*) *aggregate surplus*
- It is an indicator of social welfare under *any* (increasing) social welfare function $W(u_1, ..., u_I)$ in the quasilinear case:
  - greater surplus implies a larger utility possibility set
  - the planner can select a utility vector with a greater (maximized) *W*-value through appropriate endowment transfers
- Aggregate surplus can be derived very simply from market demand and supply functions;
  it is thus a convenient tool and used in many applications

Aggregate surplus and CE

- Starting from (possibly non-CE) total consumption and production $x = \Sigma x_i = \Sigma q_j = q$, increases by $(\Delta x_1, ..., \Delta x_I)$ and $(\Delta q_1, ..., \Delta q_J)$ such that $\Sigma\Delta x_i = \Sigma\Delta q_j \equiv \Delta x > 0$ change surplus by
$$\Delta S \approx \Sigma\phi_i'(x_i)\cdot\Delta x_i - \Sigma c_j'(q_j)\cdot\Delta q_j$$
- For given x, the planner maximizes surplus by allocating consumption and production s.t. $\phi_i'(x_i) = P(x)$ and $c_j'(q_j) = C'(x)$ for all i, j
- Then $\Delta S \approx [P(x) - C'(x)]\cdot\Delta x$ or
  $dS/dx = P(x) - C'(x)$ for marginal changes
- So aggregate surplus under an optimal distribution of output x is
$$S(x) = S(0) + \int_0^x [P(s) - C'(s)]ds$$
- $S(0)$ reflects possible fixed costs of production;
  $S(x)$-$S(0)$ is the area between market demand and supply curves
- $S(x)$ increases up to x^* s.t. $P(x^*) = C'(x^*)$, i.e., the CE level
  $\Rightarrow$ surplus is maximal in the undistorted laissez-faire CE
  (but: given one distortion, adding another *may raise* surplus …)

10.4 Free-entry long run equilibria

- For strictly convex costs, there is “no” long-run free entry equilibrium because any firm would produce zero (at minimal MC)
- Otherwise, the demand curve intersects with an approximately horizontal LR industry supply curve
  (resulting either from CRS, with then an indeterminate industry structure, or from r firms each producing at an efficient scale)
- This LR equilibrium can differ from the SR equilibrium, in which the number of firms is fixed and a SR supply curve slopes upwards



11. Market power

- Price-taking behavior is implausible if there are only a few producers (or consumers)
- Several “workhorse” models of *industrial organization* capture the performance differences that market power can cause
- A first benchmark is an uncontested *monopolist* who can
  - produce quantity x of a good at cost $C(x)$, and
  - sell it at a constant unit price p to consumers, whose demand is described by demand curve $D(p)$
- The monopolist maximizes $\Pi(p) = p \cdot D(p) - C(D(p))$
- The necessary condition for an interior profit maximum is

$$[p - C'(D(p))] \cdot D'(p) = -D(p)$$

$$\Leftrightarrow \quad [p - C'(D(p))]/p = -D(p) / [D'(p) \cdot p] = 1/|\varepsilon|$$

- In the monopolist's profit maximum, the mark-up over price ratio $[p^m - C']/p^m$ (a.k.a. *Lerner index*) equals the inverse of the (absolute) price elasticity $|\varepsilon| = - D'(p^m) \cdot p^m / D(p^m)$

Deadweight loss of monopoly

- Except for perfectly elastic demand ("$|\varepsilon|=\infty$"), $p^m > C'(D(p^m))$; quantity $x^m=D(p^m)$ is weakly smaller than x^* in the CE
- Any quantity $x < x^*$ results in an inefficient allocation and entails a *deadweight welfare loss*: surplus which could be generated by further trade is left unrealized
  - Having sold $D(p^m)$ units at price p^m, the monopolist would gain from selling *additional* units at any price p with $C'(D(p))<p<p^m$
  - Consumers with willingness to pay v satisfying $p<v<p^m$ would gain from buying these additional units
- If the monopolist could *perfectly discriminate* between consumers, i.e., confront each consumer i with an individual price-payment bundle (x_i, T_i), then he could capture *all* surplus
- It would then become optimal to maximize surplus; then there would be *no* deadweight loss

Further remarks on monopolies

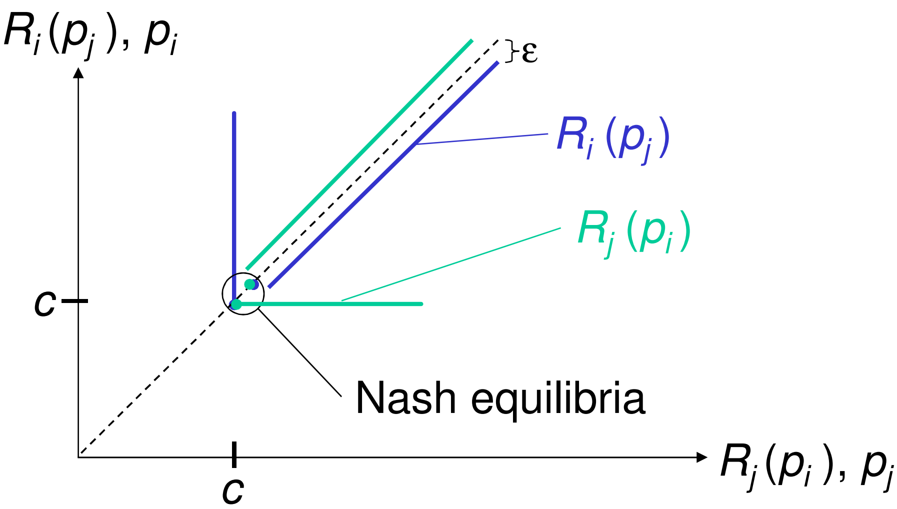
- In addition to the indicated *allocative inefficiency* $x^m < x^*$, a monopolistic market structure has further welfare costs
  - With neither product market nor capital market benchmarks, agency problems inside a monopolistic firm are harder to tackle, and tend to result in inefficient organization of production (*X-inefficiency*)
  - Firms engage in unproductive fights to secure monopoly rents (*rent-seeking*)
  - A monopolist has smaller incentives to invest in R&D than firms without market power (*dynamic inefficiency*)
- Situation can be better if the monopoly market is *contestable*
- The profit maximization problem of a *multi-product monopolist* differs from the standard case in that
  - (dis-)economies of scope in production and
  - positive or negative cross-price effects (complements/substitutes)

  need to be taken into account

Bertrand competition

- The *Bertrand duopoly model* considers
  - two firms who simultaneously announce their respective price p_j for a homogenous good, which – in the baseline case – can be produced at identical constant marginal cost c without capacity constraints, and
  - consumers who buy only at the cheaper firm if $p_1 \neq p_2$, and otherwise split their demand $D(p)$ equally between firms 1 and 2
- If costs and prices have to be multiples of a smallest currency unit ε, firm i's best response correspondence R_i to p_j is
  - $R_i(p_j) = \{p_j - \varepsilon\}$ for $p_j > c + \varepsilon$
  - $R_i(p_j) = \{c + \varepsilon\}$ for $p_j = c + \varepsilon$
  - $R_i(p_j) = \{p_i : \ p_i \geq c\}$ for $p_j = c$

NE in the discrete Bertrand game



- The discrete Bertrand game has two NE:
  - $(p_1^*, p_2^*) = (c, c)$
  - $(p_1^{**}, p_2^{**}) = (c + \varepsilon, c + \varepsilon)$

Bertrand paradox

- If p_1 and p_2 are chosen from interval $[0, \infty)$, then $(p_1^b, p_2^b)=(c, c)$ becomes the unique NE
- The *"Bertrand paradox"*:
  Price competition between two symmetric firms with CRS results in the same market outcome as perfect competition, namely $p^*=c$
- If firm j has a *non-drastic* cost advantage over its competitor(s), it supplies the entire market at price $p_j^b = \min_{k \neq j} c_k$ (or ε below); for a *drastic* advantage, it chooses $p_j^b = p^m$
- Even symmetric firms can avoid the paradox
  - if technology commits them not to undercut their rival for some $p > C'$ (e.g., for strictly convex $C(\cdot)$ or capacity constraints)
  - if they differentiate their products, i.e., make them imperfect substitutes
  - if they collude

Edgeworth competition

- As a limit case of strictly convex $C(\cdot)$, consider price competition with exogenous capacities q_1 and $q_2 < D(c)$, i.e., a single firm cannot serve the whole market at $p^* = c$
- If firm i's capacity q_i is already exhausted for $p_i = p_j$, it will not undercut firm j
- If capacities q_1 and q_2 are small enough ($\leq x_i^c$ in Cournot NE), equilibrium prices $p_1^e = p_2^e = p^e$ are defined by $D(p^e) = q_1 + q_2$:
  - Unilateral undercutting of p^e is unprofitable because the firm's capacity is already exhausted
  - A unilateral increase of p^e (i.e., selling below capacity) is unprofitable given that one's quantity/capacity is already small

Cournot competition

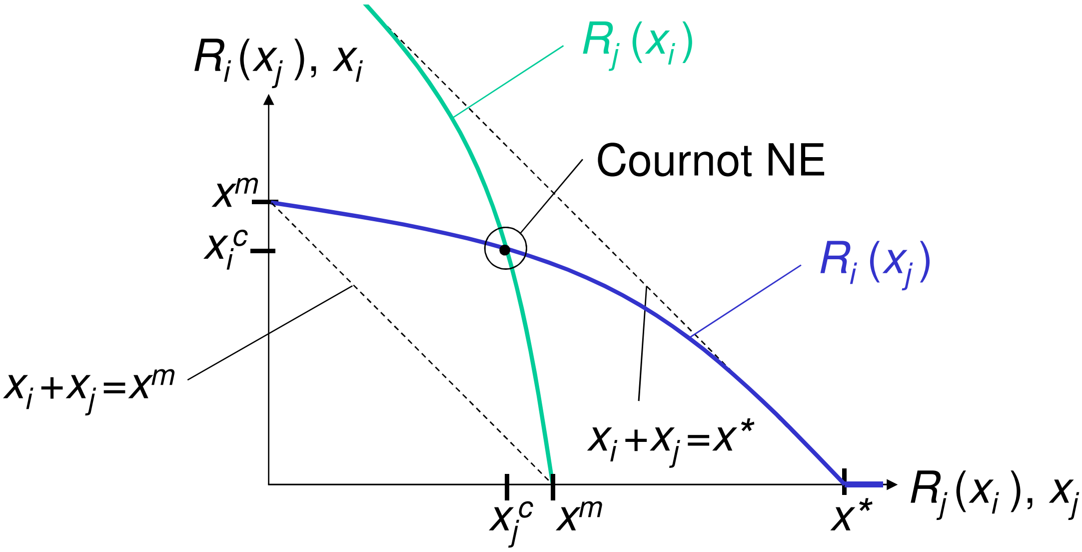
- The *Cournot duopoly model* considers
  - two firms who simultaneously produce a respective output x_j of a homogenous good at cost $C_j(x_j)$, and
  - market clearing at price $p=P(x_1+x_2)$, i.e., such that $D(p)=x_1+x_2$
- While it is usually unspecified how market clearing is brought about, Kreps and Scheinkman (1983) have shown that the Cournot game can be interpreted as the reduced form of a two-stage extensive game in which
  - first, firms invest in capacities x_i, incurring costs $C_i(x_i)$ for this (i=1,2)
  - second, they engage in Edgeworth competition with fixed capacities $q_i = x_i$ and zero costs of production
- We assume that no firm as a *drastic cost advantage*: costs are sufficiently similar that both firms want to produce in equilibrium

Reaction function of firm *i*

- Firm i maximizes
$$\Pi_i(x_i, x_j)=P(x_i+x_j)\cdot x_i - C_i(x_i)$$
- The (singleton-valued) best response $x_i^* = R_i(x_j)$ to the anticipated competitor output x_j is defined by (i=1,2):
$$P(x_i+x_j)+P'(x_i+x_j)\cdot x_i = C'_i(x_i)$$
- For x_j=0, i should behave like a monopolist, i.e., $R_i(0)=x_i^m$
- If the competitor already produces the CE quantity $x_j=x^*$, it is optimal not to produce, i.e., $R_i(x^*)=0$
- Under the usual assumptions – with $P''(x)\leq 0$ and $C_i''(x_i)\geq 0$ – the reaction function $R_i(x_j)$ is strictly decreasing on $(0, x^*)$
- This means firms' quantity decisions are *strategic substitutes*: firm i reacts to a larger output x_j with a reduction of own output x_i

Nash equilibrium in the Cournot game

Symmetric case:



n-firm Cournot game

- With $x_\Sigma = \Sigma x_i$, the NE $\boldsymbol{x}^c = (x_1{}^c, \ldots, x_n{}^c)$ of Cournot competition between n firms is characterized by:
$$P(x_\Sigma) + P'(x_\Sigma) \cdot x_i = C'_i(x_i) \text{ for } i \in \{1, \ldots, n\}$$
or, expressed in market shares $s_i = x_i / x_\Sigma$ and with $p^c = P(x_\Sigma{}^c)$,
$$[p^c - C'_i(x_i^c)]/p^c = s_i / |\varepsilon(p^c)|$$
- In the Cournot NE, the Lerner index (a measure of market power and *profitability*) of firm i is proportional to its *market share* s_i; market shares are linked to *technology* differences
- For symmetric firms, $s_i = 1/n$ with mark-up to price ratio $1/[n \cdot |\varepsilon(p^c)|]$; so the market price p^c in NE $\boldsymbol{x}^c$ converges to $p^* = c$ as $n \to \infty$

Prototypes of product differentiation

- Vertically differentiated products:
  - All consumers have the same preferences over products as such, i.e., they would all buy the same one(s) *for identical prices* $p_1 = \ldots = p_n$
  - Different preference intensities (marginal rate of substitution between wealth and the differentiating characteristic) explain different purchase behavior for non-identical prices
- Horizontally differentiated products:
  - Different consumers prefer different products for identical prices; they have different notions of the "ideal" location of the good in an abstract or physical product space
- Representative consumer with love of variety:
  - A representative consumer obtains utility from the numeraire and an "index" of his consumption of goods 1, …, n, to which all goods contribute symmetrically, and where usually the first unit of any good has infinite marginal utility

Hotelling competition

- The *Hotelling duopoly model* considers
  - a continuum of consumers who want to buy at most one unit of a differentiated good, regarding which they have uniformly distributed ideal points in the one-dimensional product space $X=[0,1]$, and
  - two firms who are – in the baseline case – located at the extremes of X, and simultaneously announce the respective price p_j for their good, which has constant marginal cost c
- We look at the case in which consumers suffer from a quadratic disutility of distance, tx^2 or $t(1-x)^2$ for $t>0$, and have a sufficiently high valuation for the good
- They will always buy from the firm for which the total disutility ($\triangleq$ price plus transportation cost) is minimal

Hotelling model with fixed locations

- The consumer at location x buys from 1 if $p_1 + tx^2 \leq p_2 + t(1-x)^2$, otherwise from 2

$\Rightarrow$ Firm 1 faces demand $D_1(p_1, p_2) = (p_2 - p_1 + t)/2t$,
firm 2 faces $D_2(p_1, p_2) = 1 - D_1(p_1, p_2)$

- Maximization of $\Pi_i(p_1, p_2) = (p_i - c) \cdot D_i(p_1, p_2)$ yields the reaction functions $R_i(p_j) = 1/2 \cdot (p_j + c + t)$
(NB: firms' prices are *strategic complements*)

$\Rightarrow$ Nash equilibrium: $p_1^* = p_2^* = c + t$

- Firms' profits $\Pi_i(p_1^*, p_2^*) = t/2$ are positive and increasing in the differentiation parameter $t > 0$

Collusive behavior

- *Collusion* refers to anti-competitive coordination of firms' prices, quantities, etc. in markets where cartel agreements cannot be enforced in court
- Firms' always have an interest in full coordination:
they could *duplicate* the non-cooperative outcome, but in most cases strictly increase profits by *not* doing so
- Such coordination is, however, not self-enforcing if firms interact only once, or over a definite time-horizon
- If firms interact repeatedly over an in(de)finite time horizon, collusion can be supported by strategies that involve credible punishment of free-riding deviators
(provided that a deviator's forgone long-term collusion rents are important enough relative to the short-term deviation profit)

Collusion in the symmetric CRS Bertrand oligopoly

- In the symmetric Bertrand *n*-firm oligopoly with CRS, a firm's per-period profit is

$$\Pi^* \approx 0 \text{ if all firms compete,}$$
$$\Pi^k = \Pi^m / n \text{ if all firms collude, and}$$
$$\Pi^d \approx \Pi^m \text{ if the firm deviates}$$

- Collusion can be realized by an SPE in *Nash reversion strategies* iff firms discount future profits by a factor δ that is no smaller than the *critical discount factor*

$$\delta_{crit.}^{b} = (\Pi^d - \Pi^k) / (\Pi^d - \Pi^*) = (n-1)/n$$

- The critical discount factor increases in *n*, and converges to 1
- That repeated interaction allows for equilibria which have very different outcomes than single-shot interaction is formalized by game theory's (Nash or Perfect) *Folk Theorems*