Advanced Ordinary Differential Equations

Third Edition

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DEDICATION

To the memory of my father Yorgos
To my mother Andromachi

PREFACE

This book has been designed for a two-semester course in Advanced Ordinary Differential Equations. It is based on the author's lectures on the subject at the University of South Florida. Although written primarily for graduate, or advanced undergraduate, students of mathematics, the book is certainly quite useful to engineers, physicists, and other scientists who are interested in various stability, asymptotic behaviour, and boundary value problems in the theory of differential systems.

The only prerequisites for the book are a first course in Ordinary Differential Equations and a course in Advanced Calculus.

The exercises at the end of each chapter are of varying degree of difficulty, and several among them are basic theoretical results. The bibliography contains references to most of the books and related papers which have been used in the text.

The author maintains that this functional-analytic treatment is a solid introduction to various aspects of Nonlinear Analysis.

Barring some instances in Chapter Nine, no knowledge of Measure Theory is required. The Banach spaces of continuous functions have sufficient structure for the development of the basic principles of Functional Analysis needed for the present theory.

Finally, the author is indebted to Hindawi Publishing Corporation for the publication of the book.

A. G. Kartsatos
Tampa, Florida

CONTENTS

CHAPTER 1

BANACH SPACES

In this chapter, we develop the main machinery that is needed throughout the book. We first introduce the concept of a real Banach space. Banach spaces are of particular importance in the field of differential equations. Many problems in differential equations can actually be reduced to finding a solution of an equation of the form $Tx = y$. Here, T is a certain operator mapping a subset of a Banach space X into another Banach space Y, and y is a known element of Y. We next establish some fundamental properties of the Euclidean space of dimension n, as well as real $n \times n$ matrices. Then we introduce the concept and some properties of a bounded linear operator mapping a normed space into another normed space. We conclude this chapter by providing some examples of important Banach spaces of continuous functions as well as bounded linear operators in such spaces.

1. PRELIMINARIES

In what follows, the symbol $x \in M$, or $M \ni x$, means that x is an element of the set M. By the symbol $A \subset B$, or $B \supset A$, we mean that the set A is a subset of the set B. The symbol $f : A \to B$ means that the function f is defined on the set A and takes values in the set B. By ∂M, $\bar{M}$, and $\operatorname{int} M$ we denote the boundary, the closure, and the interior of the set M, respectively. We use the symbol $\{x_n\} \subset M$ to denote the fact that the sequence $\{x_n\}$ has all of its terms in the set M. The symbol $\emptyset$ represents the empty set.

We denote by $\mathbb{R}$ the real line, and by $\mathbb{R}_+$, $\mathbb{R}_-$ the intervals $[0, \infty)$, $(-\infty, 0]$, respectively. The interval $[a, b] \subset \mathbb{R}$ will always be finite, that is, $-\infty < a < b < +\infty$. We use the symbol $B_r(x_0)$ to denote the open ball of $\mathbb{R}^n$ with center at x_0 and radius $r > 0$. The domain and the range of a mapping f are symbolized by $D(f)$ and $\mathbb{R}(f)$, respectively. The function $\operatorname{sgn} x$ is defined by

$$\operatorname{sgn} x = \begin{cases} \dfrac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.1}$$

By a *subspace* of the vector space X we mean a subset M of X which is itself a vector space with the same operations. The abbreviation "w.r.t." means "with respect to."

2. THE CONCEPT OF A REAL BANACH SPACE; THE SPACE $\mathbb{R}^n$

Definition 1.1. Let X be a vector space over $\mathbb{R}$. Let $\|\cdot\| : X \to \mathbb{R}_+$ have the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for every $\alpha \in \mathbb{R}$, $x \in X$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$ (*triangle inequality*).

Then the function $\|\cdot\|$ is said to be a *norm* on X and X is called a *real normed space*.

The mapping $d(x, y) \equiv \|x - y\|$ is a distance function on X. Thus, (X, d) is a metric space. In what follows, the topology of a real normed space is assumed to be the one induced by the distance function d. This is the *norm topology*. We also use the term *normed space* instead of *real normed spaces*. Without further mention, the symbol $\|\cdot\|$ always denotes the norm of the underlying normed space.

We have Definitions 1.2, 1.3, and 1.4 concerning convergence in a normed space.

Definition 1.2. Let X be a normed space. The sequence $\{x_n\} \subset X$ *converges* to $x \in X$ if the numerical sequence $\|x_n - x\|$ converges to zero as $n \to \infty$.

Definition 1.3. A sequence $\{x_n\} \subset X$, X a normed space, is said to be a *Cauchy sequence* if

$$\lim_{m,n\to\infty} ||x_m - x_n|| = 0. \tag{1.2}$$

Definition 1.4. A normed space X is said to be *complete* if every Cauchy sequence in X converges to some element of X. A complete normed space is called a *Banach space*.

The Euclidean space of dimension n is denoted by $\mathbb{R}^n$. We let $\mathbb{R} = \mathbb{R}^1$. Unless otherwise specified, the vectors in $\mathbb{R}^n$ are assumed to be column vectors, that is, vectors of the type

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \tag{1.3}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Sometimes we also use the notation $(x_1, x_2, \ldots, x_n)$ for such a vector. The basis of $\mathbb{R}^n$ is always assumed to be the ordered n-tuple $\{e_1, e_2, \ldots, e_n\}$, where e_i has its ith coordinate equal to 1 and the rest 0.

Three different norms on $\mathbb{R}^n$ are given in Example 1.5.

Example 1.5. The Euclidean space $\mathbb{R}^n$ is a Banach space if it is associated with any one of the following norms:

$$\begin{aligned} \|x\|_1 &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}, \\ \|x\|_2 &= \max\{|x_1|, |x_2|, \ldots, |x_n|\}, \\ \|x\|_3 &= |x_1| + |x_2| + \cdots + |x_n|. \end{aligned} \tag{1.4}$$

Unless otherwise specified, $\mathbb{R}^n$ will always be assumed to be associated with the first norm above, which is called the *Euclidean norm*.

Definition 1.6. Let X be a normed space. Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on X are said to be *equivalent* if there exist positive constants m, M such that

$$m\|x\|_a \leq \|x\|_b \leq M\|x\|_a \tag{1.5}$$

for every $x \in X$.

The following theorem shows that any two norms on $\mathbb{R}^n$ are equivalent.

Theorem 1.7. *If $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on $\mathbb{R}^n$, then they are equivalent.*

Proof. We recall that $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Then

$$x = \sum_{i=1}^{n} x_i e_i. \tag{1.6}$$

Taking a-norms of both sides of (1.6), we get

$$\|x\|_a \leq \sum_{i=1}^{n} |x_i| \, ||e_i||_a. \tag{1.7}$$

This inequality implies

$$\|x\|_a \leq M\|x\|_1, \tag{1.8}$$

where $\|\cdot\|_1$ is the Euclidean norm and

$$M = \left[\sum_{i=1}^{n} ||e_i||_a^2\right]^{1/2}. \tag{1.9}$$

Here, we have used the Cauchy-Schwarz inequality (see also Theorem 1.9). It follows that for every $x, y \in \mathbb{R}^n$, we have

$$\big| \|x\|_a - \|y\|_a \big| \leq \|x - y\|_a \leq M\|x - y\|_1. \tag{1.10}$$

The first inequality in (1.10) is given as an exercise (see Exercise 1.1). From (1.10) we conclude that the function $f(x) \equiv \|x\|_a$ is continuous on $\mathbb{R}^n$ w.r.t. the Euclidean norm. Since the sphere

$$S = \{u \in \mathbb{R}^n : \|u\|_1 = 1\} \tag{1.11}$$

is compact, the function f attains its minimum $m > 0$ on S. Consequently, for every $u \in S$ we have $\|u\|_a \geq m$. Now, let $x \in \mathbb{R}^n$ be given with $x \neq 0$. Then $x/\|x\|_1 \in S$ and

$$\left\| \frac{x}{\|x\|_1} \right\|_a \geq m, \tag{1.12}$$

which gives

$$\|x\|_a \geq m\|x\|_1. \tag{1.13}$$

Since (1.13) holds also for $x = 0$, we have that (1.13) is true for all $x \in \mathbb{R}^n$. Inequalities (1.8) and (1.13) show that every norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm. This proves our assertion. □

The following definition concerns itself with the *inner product* in $\mathbb{R}^n$. Theorem 1.9 contains the fundamental properties of the inner product. Its proof is left as an exercise (see Exercise 1.13).

Definition 1.8. The space $\mathbb{R}^n$ is associated with the *inner product* $\langle \cdot, \cdot \rangle$ defined as follows:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i. \tag{1.14}$$

Here, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Theorem 1.9. *For every x, $y \in \mathbb{R}^n$, α, $\beta \in \mathbb{R}$, and for the Euclidean norm, we have*

(1) $\langle x, y \rangle = \langle y, x \rangle$;
(2) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$;
(3) $\|x\|^2 = \langle x, x \rangle \geq 0$;
(4) $\langle x, x \rangle = 0$ *if and only if* $x = 0$;
(5) $|\langle x, y \rangle| \leq \|x\| \|y\|$ *(Cauchy-Schwarz inequality)*;
(6) $\langle Ax, y \rangle = \langle x, A^T y \rangle$, *where A^T denotes the transpose of the matrix $A \in M_n$.*

From linear algebra we recall the following definitions, theorems, and auxiliary facts. We denote by $\mathbb{C}$ the complex plane, $\mathbb{C}^n$ the space of all complex n-vectors, and M_n the real vector space of all real $n \times n$ matrices. For $A \in M_n$ we have $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$, or simply $A = [a_{ij}]$.

Definition 1.10. Two vectors $x, y \in \mathbb{R}^n$ are called *orthogonal* if $\langle x, y\rangle = 0$. A finite set $U = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^n$ is called *orthonormal* if the vectors in U are mutually orthogonal and $\|x_i\| = 1$, $i = 1, 2, \dots, n$.

Definition 1.11. The number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix $A \in M_n$ if

$$|A - \lambda I| = 0, \tag{1.15}$$

where $|\cdot|$ denotes determinant and I the identity matrix in M_n. If λ is an eigenvalue of A, then the equation $(A - \lambda I)x = 0$ has at least one nonzero solution in $\mathbb{C}^n$. Such a solution is called an *eigenvector* of A.

Theorem 1.12. *A symmetric matrix $A \in M_n$ ($A^T = A$) has only real eigenvalues. Moreover, A has a set of n linearly independent eigenvectors in $\mathbb{R}^n$ which is orthonormal.*

Definition 1.13. A symmetric matrix $A \in M_n$ is said to be *positive definite* if

$$\langle Ax, x\rangle > 0 \quad \forall x \in \mathbb{R}^n \text{ with } x \neq 0. \tag{1.16}$$

Assume now that A is a symmetric matrix in M_n. Then the continuous function

$$\phi(u) \equiv \langle Au, u\rangle \tag{1.17}$$

attains its maximum λ_M and its minimum λ_m on the unit sphere

$$S = \{u \in \mathbb{R}^n : \|u\| = 1\}. \tag{1.18}$$

Let $\lambda_M = \langle Au_0, u_0\rangle$ and $\lambda_m = \langle Av_0, v_0\rangle$ for some $u_0, v_0 \in S$. Consider the function $g(x) \equiv x_1^2 + x_2^2 + \cdots + x_n^2 - 1$. It is easy to see that $\nabla\phi(x) = 2Ax$ and $\nabla g(x) = 2x$ for all $x \in \mathbb{R}^n$. Since S is the set of all points $x \in \mathbb{R}^n$ such that $g(x) = 0$ and $\nabla g(x) \neq 0$, it follows from a well-known theorem of advanced calculus (see, for example, Edwards [15, page 108]) that there exist real numbers λ, μ such that

$$\nabla\phi(u_0) = \lambda\nabla g(u_0), \qquad \nabla\phi(v_0) = \mu\nabla g(v_0), \tag{1.19}$$

or $Au_0 = \lambda u_0$ and $Av_0 = \mu v_0$. Since $\langle Au_0, u_0\rangle = \lambda$, $\langle Av_0, v_0\rangle = \mu$, we have $\lambda = \lambda_M$ and $\mu = \lambda_m$. We have proved the following theorem.

Theorem 1.14. *If λ_m, λ_M are the smallest and largest eigenvalues of a symmetric matrix $A \in M_n$, respectively, then*

$$\lambda_m = \min_{\|u\|=1} \langle Au, u\rangle, \qquad \lambda_M = \max_{\|u\|=1} \langle Au, u\rangle. \tag{1.20}$$

If A is positive definite, then all the eigenvalues of A are positive.

Definition 1.15. A matrix $P \in M_n$ is called a *projection matrix* if $P^2 = P$.

It is easy to see that if P is a projection matrix, then $I - P$ is also a projection matrix.

3. BOUNDED LINEAR OPERATORS

In what follows, an *operator* is simply a function mapping a subset of a normed space into another normed space. In this section we obtain some elementary information concerning bounded linear operators. We also provide three norms for the space M_n which correspond to the norms given for $\mathbb{R}^n$ in Example 1.5. In particular, we recall some facts concerning linear operators mapping $\mathbb{R}^n$ into itself.

We often omit the parentheses in $T(x)$ for operators that are considered in the sequel.

Definition 1.16. Let X, Y be two normed spaces, and let V be a subset of X. An operator $T : V \to Y$ is *continuous at* $x_0 \in V$ if for every sequence $\{x_n\} \subset V$ with $x_n \to x_0$ we have $Tx_n \to Tx_0$. The operator T is *continuous on* V if it is continuous at each $x_0 \in V$.

Definition 1.17. Let X, Y be two normed spaces and V a subspace of X. Then $T : V \to Y$ is called *linear* if for every $\alpha, \beta \in \mathbb{R}$, $x, y \in V$, we have

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty. \tag{1.21}$$

Definition 1.18. Let X, Y be two normed spaces. A linear operator $T : X \to Y$ is called *bounded* if there exists a constant $K \geq 0$ such that $\|Tx\| \leq K\|x\|$ for every $x \in X$. If T is bounded, then the number

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \tag{1.22}$$

is called the *norm of* T.

We usually use the symbol $\|\cdot\|$ to denote the norm of all Banach spaces and bounded linear operators under consideration.

Theorem 1.19 characterizes the continuous linear operators in normed spaces.

Theorem 1.19. *A linear operator $T : X \to Y$, with X, Y normed spaces, is continuous on X if and only if it is bounded.*

Proof. *Sufficiency.* From the inequality

$$\|Tx\| \leq K\|x\|, \quad x \in X, \tag{1.23}$$

it follows immediately that

$$||Tx - Tx_0|| \leq K||x - x_0|| \tag{1.24}$$

for any $x_0, x \in X$. Thus, if $x_n \to x_0$, then $Tx_n \to Tx_0$.

Necessity. Suppose that T is continuous on X. We show that

$$K_0 = \sup_{\|x\|=1} \|Tx\| < +\infty. \tag{1.25}$$

In fact, let $K_0 = +\infty$. Then there exists a sequence $\{x_n\} \subset X$ such that $\|x_n\| = 1$ and $\|Tx_n\| \to \infty$. Let $\lambda_n = \|Tx_n\|$. We may assume that $\lambda_n > 0$ for all n. Let $\tilde{x}_n = x_n/\lambda_n$. Then $\|\tilde{x}_n\| = (1/\lambda_n)\|x_n\| \to 0$ and $\|T\tilde{x}_n\| = 1$, that is, a contradiction to the continuity of T. Therefore, $K_0 < +\infty$. Let $x \neq 0$ be a vector in X. Then $\tilde{x} = x/\|x\|$ satisfies $\|\tilde{x}\| = 1$. Thus, $\|T\tilde{x}\| = \|Tx\|/\|x\|$ and $\|T\tilde{x}\| \le K_0$. Consequently,

$$\|Tx\| \le K_0\|x\|. \tag{1.26}$$

Since (1.26) holds also for $x = 0$, we have shown (1.23) with $K = K_0$. □

THEOREM 1.20. *Let X, Y be two normed spaces. Let $T : X \to Y$ be a bounded linear operator. Then*

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\|. \tag{1.27}$$

PROOF. Obviously,

$$\sup_{\|x\|=1} \|Tx\| \le \sup_{\|x\|\le 1} \|Tx\|. \tag{1.28}$$

Let $x \in X$ be such that $\|x\| \le 1$ and $x \neq 0$. Then

$$\begin{gathered} \|Tx\| = \|x\|\left\|T\left(\frac{x}{\|x\|}\right)\right\| \le \sup_{\|x\|=1} \|Tx\|, \\ \sup_{\|x\|\le 1} \|Tx\| \le \sup_{\|x\|=1} \|Tx\|. \end{gathered} \tag{1.29}$$

□

Let X, Y be two normed spaces. The space of all bounded linear operators $T : X \to Y$ is a vector space under the obvious definitions of addition and multiplication by scalars (reals). This space becomes a normed space if it is associated with the norm of Definition 1.18. For $X = Y = \mathbb{R}^n$, we have the following example.

EXAMPLE 1.21. Let A be a matrix in M_n. Consider the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Tx = Ax, \quad x \in \mathbb{R}^n. \tag{1.30}$$

Then T is a linear operator. Now, let $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$. Then $\mathbb{R}^n$ associated with one of the three norms of Example 1.5 induces a norm on T according to Table 1:

TABLE 1

$\|x\|$	$\|T\|$
$\|x\|_1$	$\sqrt{\lambda}$ (λ = the largest eigenvalue of $A^T A$)
$\|x\|_2$	$\max_i \sum_j \|a_{ij}\|$
$\|x\|_3$	$\max_j \sum_i \|a_{ij}\|$

We prove the first assertion in Table 1. The other two are left as an exercise (see Exercise 1.11).

THEOREM 1.22. *Let $\mathbb{R}^n$ be associated with the Euclidean norm. Let T be the linear operator of Example 1.21. Then*

$$\|T\| = \sqrt{\lambda}, \tag{1.31}$$

where λ is the largest eigenvalue of $A^T A$.

PROOF. We assume first that A is symmetric and that λ_1 is an eigenvalue of A such that

$$|\lambda_1| = \max_i \{|\lambda_i|\}. \tag{1.32}$$

Here, λ_i, $i = 1, 2, \ldots, n$, are the eigenvalues of A with corresponding eigenvectors x_i, $i = 1, 2, \ldots, n$, which form an orthonormal set. It should be noted that these eigenvalues are not necessarily distinct. We show that these eigenvectors are linearly independent, although this fact has been stated in Theorem 1.12. Let

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0 \tag{1.33}$$

with c_i, $i = 1, 2, \ldots, n$, real constants. Then

$$c_1 \langle x_1, x_1 \rangle + c_2 \langle x_2, x_1 \rangle + \cdots + c_n \langle x_n, x_1 \rangle = 0, \tag{1.34}$$

showing that $c_1 = 0$. Similarly, $c_i = 0$, $i = 2, 3, \ldots, n$. Thus, the ordered set $\{x_1, x_2, \ldots, x_n\}$ is a basis for $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be given and let $\{c_1, c_2, \ldots, c_n\}$ be a set of real constants with

$$x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n. \tag{1.35}$$

Then we have

$$\begin{aligned} Tx &= T(c_1x_1 + c_2x_2 + \cdots + c_nx_n) \\ &= c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n, \\ \|Tx\|^2 &= \langle Tx, Tx\rangle = |c_1\lambda_1|^2 + |c_2\lambda_2|^2 + \cdots + |c_n\lambda_n|^2 \\ &\leq \lambda_1^2\left(|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2\right) = \lambda_1^2\|x\|^2. \end{aligned} \tag{1.36}$$

It follows that $\|Tx\| \leq |\lambda_1|\|x\|$ for every $x \in \mathbb{R}^n$. Since $\|Tx_1\| = |\lambda_1|$, we obtain $\|T\| = |\lambda_1|$. We also have the following characterization for $\|T\|$:

$$\|T\| = \max_{\|x\|=1} |\langle Ax, x\rangle| = \max_{\|x\|=1} |\langle Tx, x\rangle|. \tag{1.37}$$

Indeed, let $x \in \mathbb{R}^n$ be given with $\|x\| = 1$. Then

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\| = \|T\| = |\lambda_1|. \tag{1.38}$$

Moreover, we have $|\langle Tx, x\rangle| = |\lambda_1|$ for $x = x_1$. Consequently,

$$\max_{\|x\|=1} |\langle Tx, x\rangle| = |\lambda_1|, \tag{1.39}$$

proving (1.37).

Now, let $A \in M_n$ be arbitrary. We have

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 = \sup_{\|x\|=1} \langle Tx, Tx\rangle \\ &= \max_{\|x\|=1} \langle Ax, Ax\rangle = \max_{\|x\|=1} \langle A^TAx, x\rangle = \lambda, \end{aligned} \tag{1.40}$$

where $\lambda = |\lambda|$ is the largest eigenvalue of A^TA. This eigenvalue is nonnegative because $\langle A^TAx, x\rangle \geq 0$ (see Theorem 1.14). $\square$

In the following discussion we identify A and T in Example 1.21 and we assume (unless otherwise specified) that $A \in M_n$ has norm $\|A\| = \|T\| = \sqrt{\lambda}$ as in Theorem 1.22. It is easy to see that M_n is a Banach space with any one of the norms given in Table 1.

Let $P \in M_n$ be a projection matrix. We also use the symbol P to denote the linear operator defined by P as in Example 1.21. Then

$$\mathbb{R}^n = P\mathbb{R}^n \oplus (I - P)\mathbb{R}^n, \tag{1.41}$$

that is, $\mathbb{R}^n$ is the direct sum of the subspaces $P\mathbb{R}^n$, $(I - P)\mathbb{R}^n$. The equation $\mathbb{R}^n = M \oplus N$, with M, N subspaces of $\mathbb{R}^n$, means that every $x \in \mathbb{R}^n$ can be written in a unique way as $y + z$, where $y \in M$, $z \in N$. We first show that $P\mathbb{R}^n \cap (I - P)\mathbb{R}^n = \{0\}$. Assume that $x \in P\mathbb{R}^n \cap (I - P)\mathbb{R}^n$. Then there exist y, z in $\mathbb{R}^n$ such that $x = Py = (I - P)z$. This implies that

$$Px = P^2y = Py = P(I - P)z = (P - P^2)z = (P - P)z = 0. \tag{1.42}$$

Thus, $x = Py = 0$.

Assume now that $x = y + z = y_1 + z_1$ with $y, y_1 \in P\mathbb{R}^n$ and $z, z_1 \in (I - P)\mathbb{R}^n$. Then

$$P\mathbb{R}^n \ni y - y_1 = z_1 - z \in (I - P)\mathbb{R}^n \tag{1.43}$$

implies that $y - y_1 = z_1 - z = 0$. This says that $y = y_1$ and $z = z_1$.

We summarize the above in the following theorem.

THEOREM 1.23. *Let $P \in M_n$ be a projection matrix. Then $\mathbb{R}^n = P\mathbb{R}^n \oplus (I - P)\mathbb{R}^n$.*

Now, we give a meaning to the symbol e^A, where A is a matrix in M_n. We consider the series

$$I + \sum_{m=1}^{\infty} \frac{A^m}{m!}. \tag{1.44}$$

Since M_n is complete, the convergence of the series (1.44) will be shown if we prove that the sequence of partial sums $\{S_m\}_{m=1}^{\infty}$ with

$$S_m = I + \sum_{k=1}^{m} \frac{A^k}{k!} \tag{1.45}$$

is a Cauchy sequence. To this end, we observe that

$$||S_{\bar{m}} - S_m|| \leq \sum_{i=m+1}^{\bar{m}} \frac{\|A\|^i}{i!} \tag{1.46}$$

for every $\bar{m} > m$ and that

$$\sum_{i=0}^{\infty} \frac{\|A\|^i}{i!} = e^{\|A\|}. \tag{1.47}$$

It follows that $\{S_m\}$ is a Cauchy sequence. We denote its limit by e^A. It can be shown (see Exercise 1.27) that if $A, B \in M_n$ commute ($AB = BA$), then $e^{A+B} = e^A e^B$. From this equality we obtain

$$e^A e^{-A} = e^{-A} e^A = e^0 = I. \tag{1.48}$$

Thus, if A^{-1} denotes the inverse of the matrix A, we obtain

$$[e^A]^{-1} = e^{-A} \quad \forall A \in M_n. \tag{1.49}$$

We now recall some elementary facts from advanced calculus. Let J be a real interval. Let $f : J \to \mathbb{R}^n$ be such that $f(t) = (f_1(t), f_2(t), \dots, f_n(t))$, $t \in J$. Then f is differentiable at $t_0 \in J$ if and only if each function f_i is differentiable at t_0. We have

$$f'(t_0) = (f_1'(t_0), f_2'(t_0), \dots, f_n'(t_0)). \tag{1.50}$$

Similarly, f is (Riemann) integrable on $[a,b] \subset J$ if and only if each function f_i is integrable on $[a,b]$, and we have

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \dots, \int_a^b f_n(t)dt\right). \tag{1.51}$$

An analogous situation exists for functions $A : J \to M_n$ with $A(t) = [a_{ij}(t)]$, $i, j = 1, 2, \dots, n$. Exercises 1.6, 1.7, and 1.24 contain several differentiation and integration properties of vector-valued function $f(t)$. All these properties are assumed to hold without further mention.

4. EXAMPLES OF BANACH SPACES AND LINEAR OPERATORS

If J is any interval of $\mathbb{R}$, finite or infinite, and $f : J \to \mathbb{R}^n$ is a bounded function, we set

$$\|f\|_\infty = \sup_{t \in J} \|f(t)\|. \tag{1.52}$$

Whenever the symbol $\|\cdot\|_\infty$ is used without reference to some interval J, it will be assumed that J is the domain of the function under consideration. We let $\mathbb{C}_n(J)$ denote the vector space of all bounded continuous functions $f : J \to \mathbb{R}^n$. If the interval J is denoted by two points a, b with $|a|, |b| \le \infty$ and $a < b$, then we often drop the parentheses in $\mathbb{C}_n(J)$ and other spaces. Thus we have the spaces $\mathbb{C}_n[a,b)$, $\mathbb{C}_n[a,b]$, and so forth.

Example 1.24. The space $\mathbb{C}_n(J)$, with J any interval of $\mathbb{R}$, is a Banach space with norm

$$\|f\|_\infty = \sup_{t \in J} \|f(t)\|. \tag{1.53}$$

Example 1.25. Let T be a positive number and let $P_n(T)$ be the space of all continuous T-periodic functions with values in $\mathbb{R}^n$, that is,

$$P_n(T) = \{x \in \mathbb{C}_n(\mathbb{R});\ x(t+T) = x(t),\ t \in \mathbb{R}\}. \tag{1.54}$$

Then $P_n(T)$ is a Banach space with norm

$$\|f\|_\infty = \max_{t\in[0,T]} ||f(t)||. \tag{1.55}$$

The space $P_n(T)$ can be identified with the space

$$\{x \in \mathbb{C}_n[0,T] : x(0) = x(T)\}. \tag{1.56}$$

In fact, every function $f \in P_n(T)$ uniquely determines a function in (1.56)—its restriction $\bar{f}$ on the interval $[0,T]$. Conversely, every function $\bar{u}$ in the space (1.56) can be extended uniquely to a function $u \in P_n(T)$ in an obvious way. The correspondence $f \to \bar{f}$ is an isomorphism onto and such that

$$\|f\|_\infty = ||\bar{f}||_\infty. \tag{1.57}$$

The space $P_n(T)$ is thus a closed subspace of the Banach space $\mathbb{C}_n[0,T]$.

Example 1.26. The space $\mathbb{C}_n^l$ of all functions $f \in \mathbb{C}_n(\mathbb{R}_+)$ with finite limit as $t \to \infty$ is a closed subspace of $\mathbb{C}_n(\mathbb{R}_+)$. Thus, it is a Banach space with norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}_+} ||f(t)||. \tag{1.58}$$

The space $\mathbb{C}_n^l$ contains the space $\mathbb{C}_n^0$ of all functions $f \in \mathbb{C}_n^l$ such that

$$\lim_{t\to\infty} f(t) = 0. \tag{1.59}$$

The space $\mathbb{C}_n^0$ is a closed subspace of both $\mathbb{C}_n^l$ and $\mathbb{C}_n(\mathbb{R}_+)$. Thus it is a Banach space with the sup-norm. The interval $\mathbb{R}_+$ in this example can be replaced by any other interval $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$.

Definition 1.27. A function $f \in \mathbb{C}_n(\mathbb{R})$ is called *almost periodic* if for every $\epsilon > 0$ there exists a number $L(\epsilon) > 0$ such that every interval of length $L(\epsilon)$ contains at least one number τ with

$$||f(t+\tau) - f(t)|| < \epsilon \tag{1.60}$$

for every $t \in \mathbb{R}$.

The following theorem gives a characterization of almost periodicity.

Theorem 1.28. *A function $f \in \mathbb{C}_n(\mathbb{R})$ is almost periodic if and only if every sequence $\{f(t+\tau_m)\}_{m=1}^\infty$ of translates of f contains a uniformly convergent subsequence. Here, $\{\tau_m\}$ is any sequence of numbers in $\mathbb{R}$.*

EXAMPLE 1.29. The space AP_n of all almost periodic functions $f \in \mathbb{C}_n(\mathbb{R})$ is a closed subspace of $\mathbb{C}_n(\mathbb{R})$, and thus a Banach space with norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} ||f(t)||. \tag{1.61}$$

Several properties of almost periodic functions can be found in exercises at the end of this chapter.

EXAMPLE 1.30. Let $J \subset \mathbb{R}$ be an interval. We define the space $\mathbb{C}_n^k(J)$, $k = 1, 2, \ldots,$ as follows:

$$\mathbb{C}_n^k(J) = \{f \in \mathbb{C}_n(J) : f^{(k)} \in \mathbb{C}_n(J)\}. \tag{1.62}$$

This space is a Banach space with norm

$$\|f\|_k = \sum_{i=0}^{k} \sup_{t\in J} ||f^{(i)}(t)|| = \sum_{i=0}^{k} ||f^{(i)}||_\infty. \tag{1.63}$$

It should be noted here that the derivatives at finite left (right) endpoints of intervals of $\mathbb{R}$ are right (left) derivatives. We also set $f^{(0)} = f$.

We now give some examples of bounded linear operators. An operator T from a Banach space into $\mathbb{R}$ is also called a *functional*. A bounded linear functional is given in Example 1.31.

EXAMPLE 1.31. Consider the operator $T : \mathbb{C}_1[a, b] \to \mathbb{R}$ with

$$Tx = \sum_{i=1}^{n} c_i x(t_i), \tag{1.64}$$

where $c_1, c_2, \ldots, c_n$ are fixed real constants and $t_1, t_2, \ldots, t_n$ are fixed points in $[a, b]$ with $t_1 < t_2 < \cdots < t_n$. Then T is a bounded linear functional with norm

$$\|T\| = \sum_{i=1}^{n} |c_i|. \tag{1.65}$$

In fact,

$$|Tx| \le M\|x\|_\infty \quad \forall x \in \mathbb{C}_1[a, b], \tag{1.66}$$

where M is the number in the right-hand side of (1.65). Now, it is easy to find some $\bar{x} \in \mathbb{C}_1[a, b]$ such that $\|\bar{x}\|_\infty = 1$ and $|T\bar{x}| = M$. This shows that (1.65) holds.

Example 1.32. Let T denote the operator which maps every function $x \in \mathbb{C}_n[a,b]$ into the function

$$y(t) = \int_a^b K(t,s)x(s)ds. \tag{1.67}$$

Here, $K : [a,b] \times [a,b] \to M_n$ is a continuous function, that is, all the entries of K are continuous on $[a,b] \times [a,b]$. Then T is linear and

$$\|Tx\|_\infty \leq \max_{t\in[a,b]} \int_a^b ||K(t,s)||ds\|x\|_\infty. \tag{1.68}$$

Thus,

$$\|T\| \leq \max_{t\in[a,b]} \int_a^b ||K(t,s)||ds. \tag{1.69}$$

Example 1.33. Consider the operator $T : \mathbb{C}_2^2[a,b] \to \mathbb{C}_2[a,b]$ with

$$(Tx)(t) = x''(t) + x(t), \quad t \in [a,b]. \tag{1.70}$$

Then $\|Tx\|_\infty \leq \|x\|_2$, where $\|\cdot\|_2$ is the norm in $\mathbb{C}_2^2[a,b]$. Let $\bar{x} \in \mathbb{C}_2^2[a,b]$ be given by

$$\bar{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in [a,b]. \tag{1.71}$$

Then $\|\bar{x}\|_2 = 1$ and $\|T\bar{x}\|_\infty = 1$. Consequently, $\|T\| = 1$.

Example 1.34. Let the function $A : \mathbb{R}_+ \to M_n$ be continuous and such that

$$\int_0^\infty ||A(t)||dt < +\infty. \tag{1.72}$$

Consider the operator $T : \mathbb{C}_n(\mathbb{R}_+) \to \mathbb{C}_n^l$ defined as follows:

$$(Tx)(t) = L + \int_t^\infty A(s)x(s)ds. \tag{1.73}$$

Here, L is a fixed vector in $\mathbb{R}^n$. It is easy to see that $(Tx)(t) \to L$ as $t \to \infty$. We also have

$$\begin{aligned} ||(Tx)(t)|| &\leq \|L\| + \left|\left| \int_t^\infty A(s)x(s)ds \right|\right| \\ &\leq \|L\| + \int_t^\infty ||A(s)||\,||x(s)||ds \\ &\leq \|L\| + \int_0^\infty ||A(s)||ds\|x\|_\infty. \end{aligned} \tag{1.74}$$

If $L = 0$, we obtain

$$\|Tx\|_\infty \leq M\|x\|_\infty \tag{1.75}$$

with

$$M = \int_0^\infty ||A(t)||dt. \tag{1.76}$$

This shows that, for $L = 0$, T is a bounded linear operator with $\|T\| \leq M$.

EXERCISES

1.1. Let X be a normed space. Let $x, y \in X$. Show that

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|. \tag{1.77}$$

1.2. For $f \in \mathbb{C}_1[a, b]$, let

$$\|f\| = \left(\int_a^b |f(t)|^2 dt\right)^{1/2}. \tag{1.78}$$

Show that $\mathbb{C}_1[a, b]$ is not complete w.r.t. this norm.

1.3. Show that the spaces $\mathbb{C}_n^k[a, b]$, $\mathbb{C}_n(\mathbb{R}_+)$, $\mathbb{C}_n^l$, $\mathbb{C}_n^0$ are Banach spaces with norms as in the respective examples of Section 4.

1.4. Let X, Y be normed spaces. Let $T : X \to Y$ be a linear operator. Show that T is continuous at $x_0 \in X$ if and only if it is continuous at 0.

1.5. Let X, Y, Z be normed spaces. Let $T : X \to Y$, $U : Y \to Z$ be bounded linear operators. Show that

$$\|UT\| \leq \|U\|\|T\|. \tag{1.79}$$

Here, UT denotes composition, that is, $(UT)(x) = U(Tx)$, $x \in X$. Conclude that for matrices $A, B \in M_n$ we have

$$\|AB\| \leq \|A\|\|B\|. \tag{1.80}$$

1.6. Let $f \in \mathbb{C}_n[a, b]$. Prove the following inequality, and then state and prove an analogous inequality for M_n-valued functions

$$\left|\left|\int_a^b f(t)dt\right|\right| \leq \int_a^b ||f(t)||dt. \tag{1.81}$$

1.7. Assume that $x, y \in \mathbb{C}_n[a,b]$ and $A, B : [a,b] \to M_n$ are differentiable at the point $t \in [a,b]$. Prove the following differentiation rules:

(i) $[A(t) + B(t)]' = A'(t) + B'(t)$.
(ii) $[cA(t)]' = cA'(t)$, $c \in \mathbb{R}$.
(iii) $[A(t)B(t)]' = A'(t)B(t) + A(t)B'(t)$.
(iv) $[A(t)x(t)]' = A'(t)x(t) + A(t)x'(t)$.
(v) $\langle x(t), y(t)\rangle' = \langle x'(t), y(t)\rangle + \langle x(t), y'(t)\rangle$.
(vi) $[A^{-1}(t)]' = -A^{-1}(t)A'(t)A^{-1}(t)$ if $A(t)$ is nonsingular on $[a,b]$.

Show that integration properties hold analogous to those of (i) and (ii).

1.8. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded linear functional, there exist constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{1.82}$$

for every $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

1.9. Show that if x_0 is a given vector in $\mathbb{R}^n$, then there exists a bounded linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

1.10. Let $x \in \mathbb{C}_n^1[a,b]$ and $t_0, t_1 \in [a,b]$ be given. Show that there exists a number $\bar{t}$, properly between t_0 and t_1, such that

$$\|x(t_1) - x(t_0)\| \le \|x'(\bar{t})\| \, |t_1 - t_0|. \tag{1.83}$$

Hint. Apply the mean value theorem to the function $t \to f(x(t))$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a linear functional with

$$f(x(t_1) - x(t_0)) = \|x(t_1) - x(t_0)\|, \quad \|f\| = 1. \tag{1.84}$$

See Exercise 1.9.

1.11. Prove the last two cases of Example 1.21.

1.12. Let $T : X \to Y$ (X, Y normed spaces) be a bounded linear operator mapping X onto Y. Show that the inverse operator $T^{-1} : Y \to X$ exists and is bounded if and only if there exists a positive constant m with the property

$$\|Tx\| \ge m\|x\|, \quad x \in X. \tag{1.85}$$

1.13. Prove Theorem 1.9. Then show that the matrix $A \in M_n$ is symmetric if and only if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \tag{1.86}$$

for every $x, y \in \mathbb{R}^n$.

1.14. Let $P \in M_n$ be a projection matrix. Show that

$$\langle Px, (I - P)y \rangle = 0 \tag{1.87}$$

for every x, $y \in \mathbb{R}^n$ if and only if P is symmetric. Hint. Use Exercise 1.13.

1.15. Complete the proof of Example 1.31. Find a function $\bar{x}$ there with the desired properties.

1.16. Prove Theorem 1.28.

1.17. Show that every $f \in AP_n$ is uniformly continuous.

1.18. Show that every T-periodic function is almost periodic.

1.19. Exercise 1.18 implies that for every positive number q the function $f(t) \equiv \sin t + \sin(qt)$ is almost periodic. Show that $f(t)$ is not T-periodic, for any number $T > 0$, if q is a positive irrational number.

1.20. Show that given two functions $f, g \in AP_n$ we have the following property: for every $\epsilon > 0$ there exists $L(\epsilon) > 0$ such that: every interval of length $L(\epsilon)$ contains at least one number τ with

$$||f(t+\tau) - f(t)|| < \epsilon, \qquad ||g(t+\tau) - g(t)|| < \epsilon \tag{1.88}$$

for every $t \in \mathbb{R}$. Then show that AP_n is a Banach space. Hint. Consider the function

$$t \longrightarrow \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \in \mathbb{R}^{2n}. \tag{1.89}$$

Show that this function is almost periodic by using the result of Theorem 1.28.

1.21. Let M be a subspace of $\mathbb{R}^n$. Show that there exists a subspace N of $\mathbb{R}^n$ such that $\langle x, y \rangle = 0$ for $x \in M$, $y \in N$, and $\mathbb{R}^n = M \oplus N$.

1.22. Consider the function $P : \mathbb{R}^2 \to \mathbb{R}^2$ which maps every point $(x_1, x_2) \in \mathbb{R}^2$ into the point Q of the line $x_2 = 3x_1$ such that the distance between (x_1, x_2) and Q is a minimum. Show that P is a projection operator, that is, the operator P is linear and represented by a projection matrix in M_2.

1.23. Let $A : [a, b] \to M_n$ be a continuous function such that $A(t)$ is symmetric for every $t \in [a, b]$. Show that the largest and the smallest eigenvalues of $A(t)$ are continuous on $[a, b]$.

1.24. Given $A : [a, \infty) \to M_n$ continuous, we have

$$\int_a^t A(s)ds = \left[\int_a^t a_{ij}(s)ds \right]_{i,j=1}^n, \quad t \in [a, \infty), \tag{1.90}$$

where $A(t) = [a_{ij}(t)]$, $i, j = 1, 2, \ldots, n$, $t \in [a, \infty)$. Let

$$\int_a^\infty ||A(s)||ds < +\infty. \tag{1.91}$$

Show that

$$\lim_{t\to\infty} \int_a^t A(s)ds = \int_a^\infty A(s)ds \tag{1.92}$$

exists as a finite matrix and

$$\int_a^\infty A(s)ds = \left[\int_a^\infty a_{ij}(s)ds\right]_{i,j=1}^n. \tag{1.93}$$

1.25. Show that if $\mathbb{C}_2^1[0,1]$ is associated with the norm of $\mathbb{C}_2[0,1]$, then it is not complete.

1.26. Let $\mathbb{C}_2^1[0,1]$ be associated with the sup-norm on $[0,1]$. Let $T : \mathbb{C}_2^1[0, 1] \to \mathbb{C}_2[0,1]$ be given by $(Tx)(t) = 2x'(t) - x(t)$, $t \in [0,1]$. Show that T is not a bounded linear operator.

1.27. Show that if $A, B \in M_n$ commute, then $e^A e^B = e^{A+B}$.

1.28. Let $\text{Lip}[a,b]$ denote the space of all Lipschitz-continuous functions in $\mathbb{C}_1[a,b]$, that is, given $f \in \text{Lip}[a,b]$, there exists a constant $K = K_f > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad x, y \in [a,b]. \tag{1.94}$$

Show that $\text{Lip}[a,b]$ is a Banach space with norm

$$\|f\|_{\text{Lip}} = \|f\|_\infty + \sup_{\substack{x,y\in[a,b]\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|}. \tag{1.95}$$

Generalize this to functions $f : \mathbb{R}^n \supset \bar{\Omega} \to \mathbb{R}^n$, where Ω is open and bounded.

1.29. For $p \in (1, \infty)$, $p \neq 2$, and $x \in \mathbb{R}^n$, let

$$\|x\|_p = \left\{\sum_{i=1}^n |x_i|^p\right\}^{1/p}. \tag{1.96}$$

Show that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

1.30. Let $\|x\|_p$ be as in Exercise 1.29, but for $p \in (0,1)$. Show that $\|\cdot\|_p$ is not a norm on $\mathbb{R}^n$.

1.31. Let $T : X \to Y$ be a bounded linear operator, where X, Y are normed spaces. Is it always true that

$$\|T\| = \sup_{\|x\|<1} \|Tx\|? \tag{1.97}$$

1.32. Let $U \subset \mathbb{R}^n$ be bounded ($\|x\| \le K$ for all $x \in U$, where $K > 0$ is a constant), open, convex and symmetric ($x \in U$ implies $-x \in U$) with $0 \in U$. Let

$$M_x = \left\{ t > 0 : \frac{x}{t} \in U \right\}, \quad x \in \mathbb{R}^n, \tag{1.98}$$

and $\|x\|_U = \inf M_x$. Show that $\|\cdot\|_U$ is a norm on $\mathbb{R}^n$ and U is the open unit ball $\{x \in \mathbb{R}^n : \|x\|_U < 1\}$ for this norm. Hint. To show the triangle inequality, pick two vectors $x, y \in \mathbb{R}^n$ and two numbers $s > 0$, $t > 0$. Then

$$\frac{1}{s+t}(x+y) = \frac{s}{s+t}\left(\frac{1}{s}x\right) + \frac{t}{s+t}\left(\frac{1}{t}y\right). \tag{1.99}$$

Show that $M_x = \{t \in \mathbb{R}_+ : t > \|x\|_U\}$ and that $s \in M_x$, $t \in M_y$ imply $s + t \in M_{x+y}$. Thus, $\inf M_{x+y} \le \inf M_x + \inf M_y$.

CHAPTER 2

FIXED POINT THEOREMS; THE INVERSE FUNCTION THEOREM

In Chapter 1, we stated that a large number of problems in the field of differential equations can be reduced to the problem of finding a solution x to an equation of the form $Tx = y$. The operator T maps a subset of a Banach space X into some other Banach space Y and y is a known element of Y. If $y = 0$ and $Tx \equiv Ux - x$, for some other operator U, then the equation $Tx = y$ is equivalent to the equation $Ux = x$. Naturally, in order to solve $Ux = x$, we must assume that the domain $D(U)$ and the range $R(U)$ have points in common. Points x for which $Ux = x$ are called *fixed points* of the operator U.

In this chapter, we state the fixed point theorems which are most widely used in the field of differential equations. These are the Banach contraction principle, the Schauder-Tychonov theorem, and the Leray-Schauder theorem. We give a proof of the Banach contraction principle in Section 1. The Schauder-Tychonov theorem and the Leray-Schauder theorem are proven in Chapter 9.

In Section 2, we state and prove the inverse function theorem in Banach spaces. This theorem generalizes the well-known theorem of advanced calculus and has been an important tool in the applications of nonlinear functional analysis to the field of differential equations.

1. THE BANACH CONTRACTION PRINCIPLE

Theorem 2.1 (Banach's contraction principle). *Let X be a Banach space and M a nonempty, closed subset of X. Let $T : M \to M$ be such that, there exists a constant $k \in [0, 1)$ with the property*

$$\|Tx - Ty\| \le k\|x - y\| \quad \forall x, y \in M. \tag{2.1}$$

Then T has a unique fixed point in M.

Proof. Let $x_0 \in M$ be given with $Tx_0 \neq x_0$. Define the rest of the sequence $\{x_m\}_{m=0}^{\infty}$ as follows:

$$x_j = Tx_{j-1}, \quad j = 1, 2, \ldots. \tag{2.2}$$

Then we have

$$\begin{aligned} ||x_{j+1} - x_j|| &\leq k||x_j - x_{j-1}|| \leq k^2||x_{j-1} - x_{j-2}|| \\ &\leq \cdots \leq k^j||x_1 - x_0|| \end{aligned} \tag{2.3}$$

for every $j \geq 1$. Thus, if $m > n \geq 1$, we obtain

$$\begin{aligned} ||x_m - x_n|| &\leq ||x_m - x_{m-1}|| + ||x_{m-1} - x_{m-2}|| + \cdots + ||x_{n+1} - x_n|| \\ &\leq k^{m-1}||x_1 - x_0|| + k^{m-2}||x_1 - x_0|| + \cdots + k^n||x_1 - x_0|| \\ &\leq k^n(1 + k + k^2 + \cdots + k^{m-n-1})||x_1 - x_0|| \\ &\leq \left(\frac{k^n}{1-k}\right)||x_1 - x_0||. \end{aligned} \tag{2.4}$$

Since $k^n \to 0$ as $n \to \infty$, it follows that $\{x_m\}$ is a Cauchy sequence. Since X is complete, there exists $\bar{x} \in X$ such that $x_m \to \bar{x}$. Obviously, $\bar{x} \in M$ because M is closed. Taking limits as $j \to \infty$ in (2.2), we obtain $\bar{x} = T\bar{x}$.

To show uniqueness, let y be another fixed point of T in M. Then

$$\|\bar{x} - y\| = \|T\bar{x} - Ty\| \leq k\|\bar{x} - y\|, \tag{2.5}$$

which implies that $\bar{x} = y$. This completes the proof of the theorem. □

An operator $T : M \to X$, $M \subset X$, satisfying (2.1) on M is called a *contraction operator* (or *mapping*) *on* M.

Example 2.2. Let X be a Banach space and let $T : X \to X$ be a bounded linear operator such that $\|T\| < 1$. Then T is a contraction operator on X. This implies immediately that the equation $x = y + Tx$, for a fixed element $y \in X$, has a unique solution $x_0 \in X$. In fact, the mapping $x \to y + Tx$ is also a contraction operator on X and has a unique fixed point in X by the above theorem. Other contraction operators can also be found in Examples 1.32 and 1.34.

Example 2.3. Let $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that

$$||F(t, x_1) - F(t, x_2)|| \leq \lambda(t)||x_1 - x_2|| \tag{2.6}$$

for every $t \in \mathbb{R}_+$, $x_1, x_2 \in \mathbb{R}^n$, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies

$$L = \int_0^\infty \lambda(t)dt < +\infty. \tag{2.7}$$

Assume further that

$$\int_0^\infty ||F(t,0)||dt < +\infty. \tag{2.8}$$

Then the operator T with

$$(Tx)(t) = \int_t^\infty F(s, x(s))ds, \quad t \in \mathbb{R}_+, \tag{2.9}$$

maps the space $\mathbb{C}_n(\mathbb{R}_+)$ into itself and is a contraction operator on $\mathbb{C}_n(\mathbb{R}_+)$ if $L < 1$. In fact, let $x, y \in \mathbb{C}_n(\mathbb{R}_+)$ be given. Then we have

$$\begin{aligned} \|Tx\|_\infty &\le \int_0^\infty ||F(t, x(t))||dt \\ &\le \int_0^\infty ||F(t, x(t)) - F(t,0)||dt + \int_0^\infty ||F(t,0)||dt \\ &\le \int_0^\infty \lambda(t)||x(t)||dt + \int_0^\infty ||F(t,0)||dt \\ &\le \int_0^\infty \lambda(t)dt\|x\|_\infty + \int_0^\infty ||F(t,0)||dt, \end{aligned} \tag{2.10}$$

which shows that $T\mathbb{C}_n(\mathbb{R}_+) \subset \mathbb{C}_n(\mathbb{R}_+)$. We also have

$$\begin{aligned} \|Tx - Ty\|_\infty &\le \int_0^\infty ||F(t, x(t)) - F(t, y(t))||dt \\ &\le \int_0^\infty \lambda(t)||x(t) - y(t)||dt \\ &\le L\|x - y\|_\infty. \end{aligned} \tag{2.11}$$

It follows that, for $L < 1$, the equation $Tx = x$ has a unique solution $\bar{x}$ in $\mathbb{C}_n(\mathbb{R}_+)$. Thus, there is a unique $\bar{x} \in \mathbb{C}_n(\mathbb{R}_+)$ such that

$$\bar{x}(t) = \int_t^\infty F(s, \bar{x}(s))ds, \quad t \in \mathbb{R}_+. \tag{2.12}$$

It is obvious that we actually have $\bar{x} \in \mathbb{C}_n^0$.

It is easily seen that under the above assumptions on F and L, the equation

$$x(t) = f(t) + \int_t^\infty F(s, x(s))ds \tag{2.13}$$

also has a unique solution in $\mathbb{C}_n(\mathbb{R}_+)$ if f is a given function in $\mathbb{C}_n(\mathbb{R}_+)$. This solution belongs to $\mathbb{C}_n^l$ ($\mathbb{C}_n^0$) if $f \in \mathbb{C}_n^l$ ($f \in \mathbb{C}_n^0$).

2. THE SCHAUDER-TYCHONOV THEOREM

Before we state the Schauder-Tychonov theorem, we characterize the compact subsets of $\mathbb{C}_n[a,b]$. This characterization, which is contained in Theorem 2.5, allows us to detect the relative compactness of the range of an operator defined on a subset of $\mathbb{C}_n(J)$ and having values in $\mathbb{C}_n(J)$, where J in an interval.

Definition 2.4. Let X be a Banach space. A subset M of X is called *bounded* if there exists $K > 0$ such that $\|x\| \le K$ for every $x \in M$. A subset M of X is said to be *compact* if every sequence $\{x_n\} \subset M$ contains a subsequence which converges to a vector in M. The set $M \subset X$ is said to be *relatively compact* if every sequence $\{x_n\} \subset M$ contains a subsequence which converges to a vector in X.

It is obvious from this definition that M is relatively compact if and only if its closure $\bar{M}$ is compact. The following theorem characterizes the compact subsets of $\mathbb{C}_n[a,b]$.

Theorem 2.5 (Arzelà-Ascoli). *Let M be a subset of $\mathbb{C}_n[a,b]$. Then M is relatively compact if and only if the following statements hold:*

(i) *there exists a constant K such that*

$$\|f\|_\infty \le K, \quad f \in M, \tag{2.14}$$

that is, M is bounded;

(ii) *the set M is* equicontinuous, *that is, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ (depending only on ϵ) such that $\|f(t_1) - f(t_2)\| < \epsilon$ for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\epsilon)$ and all $f \in M$.*

The proof is based on Lemma 2.7. We need another definition.

Definition 2.6. Let M be a subset of the Banach space X and let $\epsilon > 0$ be given. Then the set $M_1 \subset X$ is said to be an *ϵ-net* of M if for every point $x \in M$ there exists $y \in M_1$ such that $\|x - y\| < \epsilon$.

Lemma 2.7. *Let M be a subset of a Banach space X. Then M is relatively compact if and only if for every $\epsilon > 0$ there exists a finite ϵ-net of M in X.*

Proof. *Necessity.* Assume that M is relatively compact and that the condition in the statement of the lemma is not satisfied. Then there exists some $\epsilon_0 > 0$ for which there is no finite ϵ_0-net of M. Choose $x_1 \in M$. Then the set $\{x_1\}$ is not an ϵ_0-net of M. Consequently, $\|x_2 - x_1\| \ge \epsilon_0$ for some $x_2 \in M$. Consider the set $\{x_1, x_2\}$. Again, this set is not an ϵ_0-net of M. Thus, there exists $x_3 \in M$ such that $\|x_3 - x_i\| \ge \epsilon_0$ for $i = 1, 2$. Continuing in the same way, we construct by induction a sequence $\{x_n\}_{n=1}^\infty$ such that $\|x_m - x_n\| \ge \epsilon_0$ for $m \ne n$. Thus, $\{x_n\}$ cannot contain any Cauchy sequence. It follows that no convergent sequence can be extracted from $\{x_n\}$. This is a contradiction to the relative compactness of M. We conclude that for every $\epsilon > 0$ there is a finite ϵ-net of M.

Sufficiency. We assume that M is an infinite set. If M is finite, our conclusion is trivial. Suppose that for every $\epsilon > 0$ there exists a finite ϵ-net of M, and consider a strictly decreasing sequence $\{\epsilon_n\}$, $n = 1, 2, \ldots$, of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then, for each $n = 1, 2, \ldots$, there exists a finite ϵ_n-net of M. We construct open balls with centers at the points of the ϵ_1-net and radii equal to ϵ_1, and we observe that every point of M belongs to one of these balls.

Now, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in M. Obviously, there exists a subsequence of $\{x_n\}$, say $\{x_n'\}$, which belongs to one of these ϵ_1-balls. Let $B(y_1)$ be this ball with center at y_1. Now, consider the ϵ_2-net of M. The sequence $\{x_n'\}$ has a subsequence $\{x_n''\}$, $n = 1, 2, \ldots$, which is contained in some ϵ_2-ball. We call this ball $B(y_2)$ (with center at y_2). Continuing in the same way, we obtain by induction a sequence of balls $\{B(y_n)\}_{n=1}^{\infty}$ with centers at the points y_n, radii equal to ϵ_n, and with the following property: the intersection of any finite number of such balls contains a subsequence of $\{x_n\}$. Thus, we may choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows:

$$x_{n_1} \in B(y_1), x_{n_2} \in B(y_2) \cap B(y_1), \ldots, x_{n_j} \in \bigcap_{i=1}^{j} B(y_i) \tag{2.15}$$

with $n_j > n_{j-1} > \cdots > n_1$. Since $x_{n_j}, x_{n_k} \in B(y_k)$ for $j \geq k$, we must have

$$||x_{n_j} - x_{n_k}|| \leq ||x_{n_j} - y_k|| + ||y_k - x_{n_k}|| < 2\epsilon_k. \tag{2.16}$$

This implies easily that $\{x_{n_j}\}$ is a Cauchy sequence and, since X is complete, it converges to a point in X. This completes the proof. □

Proof of Theorem 2.5. *Necessity.* It suffices to give the proof for $n = 1$. We assume that M is relatively compact. Lemma 2.7 implies the existence of a finite ϵ-net of M for any $\epsilon > 0$. Let $x_1(t), x_2(t), \ldots, x_n(t)$, $t \in [a, b]$, be the functions of such an ϵ-net. For every $x \in M$ there exists $x_k(t)$ for which $\|x - x_k\|_\infty < \epsilon$. This implies that

$$\begin{aligned} |x(t)| &\leq |x_k(t)| + |x(t) - x_k(t)| \\ &\leq ||x_k||_\infty + ||x - x_k||_\infty \\ &< ||x_k||_\infty + \epsilon. \end{aligned} \tag{2.17}$$

Choose $K = \max_k\{\|x_k\|_\infty\} + \epsilon$. Thus, M is bounded.

Since each function $x_k(t)$ is uniformly continuous on $[a, b]$, there exists $\delta_k(\epsilon) > 0$, $k = 1, 2, \ldots, n$, such that

$$|x_k(t_1) - x_k(t_2)| < \epsilon \quad \text{for } |t_1 - t_2| < \delta_k(\epsilon). \tag{2.18}$$

Let $\delta = \min_k\{\delta_k\}$. Suppose that $x \in M$ and let x_j be a function of the ϵ-net for which $\|x - x_j\|_\infty < \epsilon$. Then

$$\begin{aligned} |x(t_1) - x(t_2)| &\le |x(t_1) - x_j(t_1)| + |x_j(t_1) - x_j(t_2)| + |x_j(t_2) - x(t_2)| \\ &\le \|x - x_j\|_\infty + |x_j(t_1) - x_j(t_2)| + \|x - x_j\|_\infty \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon \end{aligned} \tag{2.19}$$

for all $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta(\epsilon)$. It follows that M is equicontinuous.

Sufficiency. Fix $\epsilon > 0$ and pick $\delta = \delta(\epsilon) > 0$ from the condition of equicontinuity. We are going to show the existence of a finite ϵ-net of M. Divide $[a,b]$ into subintervals $[t_{k-1}, t_k]$, $k = 1,2,\dots,n$, with $t_0 = a$, $t_n = b$ and $t_k - t_{k-1} < \delta$. Define a family P of polygons on $[a,b]$ as follows: the function $f : [a,b] \to [-K,K]$ belongs to P if and only if f is a line segment on $[t_{k-1}, t_k]$, for $k = 1,2,\dots,n$, and f is continuous. This implies that if $f \in P$, its vertices (endpoints of its line segments) can appear only at the points $(t_k, f(t_k))$, $k = 0,1,\dots,n$. It is easy to see that P is a compact set in $\mathbb{C}_1[a,b]$. We show that P is a compact ϵ-net of M. To this end, let $t \in [a,b]$. Then $t \in [t_{j-1}, t_j]$ for some $j = 1,2,\dots,n$. Let M_j and m_j be the maximum and the minimum values of $x \in M$ on $[t_{j-1}, t_j]$, respectively. Assume that $\bar{x}_0 : [a,b] \to [-K,K]$ is a polygon in P such that $\bar{x}_0(t_k) = x(t_k)$, $k = 0,1,2,\dots,n$. Then we have

$$m_j \le x(t) \le M_j, \qquad m_j \le \bar{x}_0(t) \le M_j \tag{2.20}$$

for all $t \in [t_{j-1}, t_j]$. This yields

$$|x(t) - \bar{x}_0(t)| \le M_j - m_j < \epsilon, \quad t \in [t_{j-1}, t_j], \tag{2.21}$$

which shows that P is an ϵ-net of the set M. It is easy to check now that since P itself has a finite ϵ-net, say N, the same set N will be a finite 2ϵ-net of M. This ends the proof. □

The following two examples contain relatively compact subsets of functions in $\mathbb{C}_n[a,b]$.

Example 2.8. Let $M \subset \mathbb{C}_n^1[a,b]$. Assume that K and L are two positive constants such that, every $x \in M$ satisfies the following properties:

(i) $\|x(t)\| \le K$, $t \in [a,b]$;
(ii) $\|x'(t)\| \le L$, $t \in [a,b]$.

Then M is a relatively compact subset of $\mathbb{C}_n[a,b]$. In fact, the equicontinuity of M follows from the mean value theorem for scalar-valued functions or from Exercise 1.10.

EXAMPLE 2.9. Consider the operator T of Example 1.32. Let $M \subset \mathbb{C}_n[a,b]$ be such that, there exists $L > 0$ with the property

$$\|x\|_\infty \le L \quad \forall x \in M. \tag{2.22}$$

Then the set $S = \{Tu : u \in M\}$ is a relatively compact subset of $\mathbb{C}_n[a,b]$. In fact, if

$$N = \sup_{t\in[a,b]} \int_a^b ||K(t,s)||ds, \tag{2.23}$$

then $\|f\|_\infty \le LN$ for any $f \in S$. Moreover, for $f = Tx$ we have

$$\begin{aligned} ||f(t_1) - f(t_2)|| &= \left\| \int_a^b [K(t_1,s) - K(t_2,s)]x(s)ds \right\| \\ &\le L \int_a^b ||K(t_1,s) - K(t_2,s)||ds. \end{aligned} \tag{2.24}$$

This implies easily the equicontinuity of S.

We now give an example of a bounded sequence in $\mathbb{C}_1(\mathbb{R}_+)$ which is not equicontinuous.

EXAMPLE 2.10. The sequence $\{f_n(t)\}_{n=1}^\infty$ with

$$f_n(t) = \sin nt, \quad t \ge 0, \ n = 1, 2, \ldots, \tag{2.25}$$

does not have any pointwise convergent subsequence on $\mathbb{R}_+$. Thus, it cannot be equicontinuous on $\mathbb{R}_+$.

The next two definitions are needed in the statement of the Schauder-Tychonov theorem.

DEFINITION 2.11. Let X be a Banach space and M a subset of X. Then M is called *convex* if $\lambda x + (1-\lambda)y \in M$ for all $x, y \in M$ and all $\lambda \in [0,1]$.

DEFINITION 2.12. Let X, Y be Banach spaces and M a subset of X. An operator $T : M \to Y$ is called *compact* if it is continuous and maps bounded subsets of M onto relatively compact subsets of Y.

THEOREM 2.13 (Schauder-Tychonov). *Let X be a Banach space. Let $M \subset X$ be closed, convex, and bounded. Assume that $T : M \to M$ is compact. The T has a fixed point in M.*

For a proof of this theorem the reader is referred to Theorem 9.32 in Chapter 9. It should be noted here that the fixed point of T in the above theorem is not necessarily unique. In the proof of the contraction mapping principle we saw that the unique fixed point of a contraction operator T can be approximated by the terms of a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_j = Tx_{j-1}$, $j = 1, 2, \ldots$. Unfortunately, no general approximation methods are known for fixed points of operators T as in Theorem 2.13.

We give below an application of the Schauder-Tychonov theorem.

EXAMPLE 2.14. Consider the operator $T : \mathbb{C}_n[a,b] \to \mathbb{C}_n[a,b]$ defined by

$$(Tx)(t) = f(t) + \int_a^b K(t,s)x(s)ds, \tag{2.26}$$

where $f \in \mathbb{C}_n[a,b]$ is fixed and $K : [a,b] \times [a,b] \to M_n$ is continuous. It is easy to show, as in Examples 1.32 and 2.9, that T is continuous on $\mathbb{C}_n[a,b]$ and that every bounded set $M \subset \mathbb{C}_n[a,b]$ is mapped by T onto the set TM which is relatively compact. Thus, T is compact. Now, let

$$M = \{u \in \mathbb{C}_n[a,b] : \|u\|_\infty \le L\}, \tag{2.27}$$

where L is a positive constant. Moreover, assume that $K + LN \le L$, where

$$K = \|f\|_\infty, \quad N = \sup_{t \in [a,b]} \int_a^b ||K(t,s)||ds. \tag{2.28}$$

Then M is a closed, convex, and bounded subset of $\mathbb{C}_n[a,b]$ with $TM \subset M$. By the Schauder-Tychonov theorem, there exists at least one $x_0 \in \mathbb{C}_n[a,b]$ such that $x_0 = Tx_0$. For this x_0 we have

$$x_0(t) = f(t) + \int_a^b K(t,s)x_0(s)ds, \quad t \in [a,b]. \tag{2.29}$$

COROLLARY 2.15 (Brouwer). *Let $x_0 \in \mathbb{R}^n$ and $r > 0$ be fixed. Let $f : \overline{B_r(x_0)} \to \overline{B_r(x_0)}$ be continuous. Then f has a fixed point in $\overline{B_r(x_0)}$.*

PROOF. This is a trivial consequence of Theorem 2.13, because every continuous function $f : \overline{B_r(x_0)} \to \overline{B_r(x_0)}$ is compact. $\square$

3. THE LERAY-SCHAUDER THEOREM

THEOREM 2.16 (Leray-Schauder). *Let X be a Banach space and consider the operator $S : [0,1] \times X \to X$ and the equation*

$$x - S(t,x) = 0 \tag{2.30}$$

under the following hypotheses:

(i) *$S(t,\cdot)$ is compact for all $t \in [0,1]$. Moreover, for every bounded set $M \subset X$ and every $\epsilon > 0$ there exists $\delta(\epsilon, M) > 0$ such that, $\|S(t_1,x) - S(t_2,x)\| < \epsilon$ for every $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta(\epsilon, M)$ and every $x \in M$;*

(ii) $S(t_0, x) = 0$ *for some* $t_0 \in [0,1]$ *and every* $x \in X$;

(iii) *there exists a constant* $K > 0$ *such that* $\|x_t\| \le K$ *for every solution* x_t *of* (2.30).

Then equation (2.30) *has a solution for every* $t \in [0,1]$.

The operator S in the above theorem is what we call a *homotopy of compact operators* in Chapter 9. The main difficulty in applying the Leray-Schauder theorem lies in the verification of the uniform boundedness of the solutions (condition (iii)). The reader should bear in mind that obtaining a priori bounds of the solutions of ordinary and partial differential equations can be a painstaking process.

As an easy application of Theorem 2.16, we provide an example in $\mathbb{R}^n$.

Example 2.17. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that

$$\langle F(x), x\rangle \le \|x\|^2 \quad \text{whenever } \|x\| > r, \tag{2.31}$$

where r is a positive constant. Then F has at least one fixed point in the ball $\overline{B_r(0)}$.

Proof. We consider the equation

$$(1+\epsilon)x - tF(x) = 0 \tag{2.32}$$

with $t \in [0,1]$, $\epsilon > 0$. Since every continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$ is compact, the assumptions of Theorem 2.16 will be satisfied for (2.32), with $S(t,x) \equiv [t/(1+\epsilon)]F(x)$, if we show that all possible solutions of $x - S(t,x)$ (or (2.32)) are in the ball $\overline{B_r(0)}$. Indeed, let $\bar{x}_t$ be a solution of (2.32) such that $\|\bar{x}_t\| > r$. Then

$$\langle tF(\bar{x}_t) - (1+\epsilon)\bar{x}_t, \bar{x}_t\rangle = 0 \tag{2.33}$$

or

$$\langle tF(\bar{x}_t), \bar{x}_t\rangle = (1+\epsilon)\langle \bar{x}_t, \bar{x}_t\rangle = (1+\epsilon)\|\bar{x}_t\|^2. \tag{2.34}$$

This implies

$$\langle F(\bar{x}_t), \bar{x}_t\rangle \ge (1+\epsilon)\|\bar{x}_t\|^2, \tag{2.35}$$

which is a contradiction to (2.31).

Theorem 2.16 implies that for $t = 1$ and every $\epsilon > 0$, equation (2.32) has a solution x_ϵ, such that $\|x_\epsilon\| \le r$. Let $\epsilon_m = 1/m$, $m = 1, 2, \ldots$, and $x_m = x_{\epsilon_m}$. Since the sequence $\{x_m\}$ is bounded ($\|x_m\| \le r$), it contains a convergent subsequence, say, $\{x_{m,k}\}$. Let $x_{m,k} \to x_0 \in \overline{B_r(0)}$ as $k \to \infty$. We have

$$\left(1 + \frac{1}{m_k}\right)x_{m,k} - F(x_{m,k}) = 0, \quad k = 1, 2, \ldots, \tag{2.36}$$

which, by the continuity of F, gives $F(x_0) = x_0$. □

4. THE INVERSE FUNCTION THEOREM

The inverse function theorem is an important tool in the theory of differential equations. It ensures the existence of solutions x of the equation $Tx = y$ under certain differentiability properties of the operator T. Although T is not explicitly assumed to be a contraction operator, the Banach contraction principle (Theorem 2.1) plays an important role in the proof of this result (Theorem 2.27). This section is also a good introduction to the properties of the Fréchet derivative of a nonlinear mapping.

We start with the definition of the Fréchet derivative.

Definition 2.18. Let X, Y be Banach spaces and S an open subset of X. Let $f : S \to Y$, $u \in S$ be such that

$$f(u+h) - f(u) = f'(u)h + w(u,h) \tag{2.37}$$

for every $h \in X$ with $u + h \in S$, where $f'(u) : X \to Y$ is a linear operator and

$$\lim_{\|h\|\to 0} \frac{\|w(u,h)\|}{\|h\|} = 0. \tag{2.38}$$

Then $f'(u)h$ is the *Fréchet differential of f at u with increment h*, the operator $f'(u)$ is the *Fréchet derivative of f at u* (see Theorem 2.19 for the uniqueness of $f'(u)$), and f is called *Fréchet differentiable at u*. If f is Fréchet differentiable at every u in an open set S, we say that *f is Fréchet differentiable on S*.

We should note here that the magnitude of the open set S plays no role in the above definition. All that is needed here is that equation (2.37) be satisfied for all h is a small neighborhood of zero.

The uniqueness of the Fréchet derivative is covered by the following theorem.

Theorem 2.19 (uniqueness of the Fréchet derivative). *Let $f : S \to Y$ be given, where S is an open subset of the Banach space X and Y is another Banach space. Suppose further that f is Fréchet differentiable at $u \in S$. Then the Fréchet derivative of f at u is unique.*

Proof. Suppose that $D_1(u)$, $D_2(u)$ are Fréchet derivatives of f at u with *remainders* $w_1(u,h)$, $w_2(u,h)$, respectively. Then

$$D_1(u)h + w_1(u,h) = D_2(u)h + w_2(u,h) \tag{2.39}$$

for every $h \in X$ with $u + h \in S_1$, where S_1 is an open subset of S containing u. It follows that, for $h \neq 0$,

$$\begin{aligned} \frac{\|D_1(u)h - D_2(u)h\|}{\|h\|} &= \frac{\|w_1(u,h) - w_2(u,h)\|}{\|h\|} \\ &\leq \frac{\|w_1(u,h)\|}{\|h\|} + \frac{\|w_2(u,h)\|}{\|h\|}. \end{aligned} \tag{2.40}$$

The last two terms of (2.40) tend to zero as $\|h\| \to 0$. Let

$$Tx = [D_1(u) - D_2(u)]x, \quad x \in X. \tag{2.41}$$

Then T is a linear operator on X such that

$$\lim_{\|x\|\to 0} \frac{\|Tx\|}{\|x\|} = 0. \tag{2.42}$$

Thus, given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|Tx\|/\|x\| < \epsilon$ for every $x \in X$ with $\|x\| \in (0, \delta(\epsilon))$. Given $y \in X$ with $y \neq 0$, let $x = \delta(\epsilon)y/(2\|y\|)$. Then $\|x\| < \delta(\epsilon)$ and $\|Tx\|/\|x\| < \epsilon$, or $\|Ty\| < \epsilon\|y\|$. Since ϵ is arbitrary, we obtain $Ty = 0$ for every $y \in X$. We conclude that $D_1(u) = D_2(u)$. □

The boundedness of the Fréchet derivative $f'(u)$ is equivalent to the continuity of f at u. This is the content of the next theorem.

Theorem 2.20. *Let $f : S \to Y$ be given, where S is an open subset of a Banach space X and Y is another Banach space. Let f be Fréchet differentiable at $u \in S$. Then f is continuous at u if and only if $f'(u)$ is a bounded linear operator.*

Proof. Let f be continuous at $u \in S$. Then for each $\epsilon > 0$ there exists $\delta(\epsilon) \in (0, 1)$ such that

$$\begin{aligned} ||f(u+h) - f(u)|| &< \frac{\epsilon}{2}, \\ ||f(u+h) - f(u) - f'(u)h|| &< \left(\frac{\epsilon}{2}\right)\|h\| < \frac{\epsilon}{2} \end{aligned} \tag{2.43}$$

for all $h \in X$ with $u + h \in S$ and $\|h\| \in (0, \delta(\epsilon))$. Therefore,

$$||f'(u)h|| < \left(\frac{\epsilon}{2}\right) + ||f(u+h) - f(u)|| < \epsilon \tag{2.44}$$

for $\|h\| \in (0, \delta(\epsilon))$, $u + h \in S$, implies that the linear operator $f'(u)$ is continuous at the point 0. Exercise 1.4 says the $f'(u)$ is continuous on X. Thus, $f'(u)$ is bounded by Theorem 1.19. The converse is left as an exercise (see Exercise 2.15). □

Theorems 2.22, 2.23, and 2.24 establish further important properties of Fréchet derivatives. They are presented here in order to provide a better understanding of Fréchet differentiation. We denote by X^* the space of all continuous linear functionals on the Banach space X, that is, the space of all bounded linear operators $x^* : X \to \mathbb{R}$. The term *continuously Fréchet differentiable*, referring to a function f and a point x_0 in its domain S, means that the function f has a Fréchet derivative $f'(x_0)$ which is continuous at x_0 with the range of $f'(x_0)$ associated with the *bounded operator topology*, that is, given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$||f'(x) - f'(x_0)|| \leq \epsilon \tag{2.45}$$

for every $x \in S$ with $\|x - x_0\| \leq \delta(\epsilon)$. If f is continuously Fréchet differentiable at x_0 as above, then

$$||(f'(x) - f'(x_0))h|| \leq \epsilon\|h\| \tag{2.46}$$

for every $x \in S$ with $\|x - x_0\| \leq \delta(\epsilon)$ and every $h \in X$. Naturally, the assumption of the continuity of the Fréchet derivative of f at x_0 implies that the Fréchet derivative $f'(x_0)$ is a bounded linear operator. This fact implies, by Theorem 2.20, that f is continuous at x_0. The term *continuously Fréchet differentiable* on a set S is defined accordingly. This term is used in Theorems 2.23, 2.24, and 2.27.

Lemma 2.21. *Let X be a Banach space. Then for each $x \in X$ with $x \neq 0$ there exists $x^* \in X^*$ such that $x^*(x) = \|x\|$ and $\|x^*\| = 1$.*

For a proof of this lemma the reader is referred to [42, page 161] (see also Exercise 1.9 for $X = \mathbb{R}^n$). For x, y in the Banach space X, with $x \neq y$, we denote by $[x, y]$ the *line segment* $\{tx + (1 - t)y; t \in [0, 1]\}$.

Theorem 2.22 (mean value theorem for real-valued functions). *Let X be a Banach space and let S be an open subset of X. Assume further that $f : S \to \mathbb{R}$ is Fréchet differentiable on $[x, x + h] \subset S$, where x is a point in S and $h \neq 0$ a point in X. Then there exists a number $\theta \in (0, 1)$ such that*

$$f(x + h) - f(x) = f'(x + \theta h)h. \tag{2.47}$$

Proof. Consider the function $g(t) = f(x + th)$, $t \in [0, 1]$. It is easy to see that g is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $g'(t) = f'(x + th)h$. Applying the mean value theorem for real-valued functions of a real variable, we obtain

$$g(1) - g(0) = g'(\theta), \tag{2.48}$$

for some $\theta \in (0, 1)$. We have

$$f(x + h) - f(x) = f'(x + \theta h)h. \tag{2.49}$$

□

Theorem 2.23 (mean value theorem in Banach spaces). *Let X, Y be Banach spaces and S an open subset of X. Let $f : S \to Y$ be continuously Fréchet differentiable on $[x, x + h] \subset S$. Then*

$$||f(x + h) - f(x)|| \leq \sup_{\theta \in (0,1)} \{||f'(x + \theta h)||\}\|h\|. \tag{2.50}$$

PROOF. Let $y^* \in Y^*$ be such that

$$y^*(f(x+h) - f(x)) = ||f(x+h) - f(x)||, \quad ||y^*|| = 1. \tag{2.51}$$

Such a functional exists by Lemma 2.21. Applying Theorem 2.22 to the function $g(x) = y^*(f(x))$, we obtain

$$\begin{aligned} g(x+h) - g(x) &= y^*(f(x+h)) - y^*(f(x)) \\ &= y^*(f(x+h) - f(x)) = g'(x+\theta h)h, \end{aligned} \tag{2.52}$$

where $\theta = \theta(y^*) \in (0,1)$. However, $g'(x+\theta h) = y^*(f'(x+\theta h))$ (cf. Exercise 2.6). Thus,

$$\begin{aligned} ||f(x+h) - f(x)|| &= y^*(f'(x+\theta h))h \\ &\le ||y^*|| \, ||f'(x+\theta h)|| \, \|h\| \\ &= ||f'(x+\theta h)|| \, \|h\|. \end{aligned} \tag{2.53}$$

□

THEOREM 2.24. *Let X, Y be Banach spaces. Let $S \subset X$ be open and $x_0 \in S$. Assume that $f : S \to Y$ is continuously Fréchet differentiable on S. Then for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$||w(x_0, h)|| \le \sup_{u \in [x_0, x_0+h]} \{||f'(u) - f'(x_0)||\} \, \|h\| \tag{2.54}$$

for all $h \in X$ with $\|h\| \le \delta(\epsilon)$, where w is the remainder in (2.37).

PROOF. Consider the mapping $g : x \to f(x) - f'(x_0)x$. We have $g'(x) = f'(x) - f'(x_0)$. Given $\epsilon > 0$, let $\delta(\epsilon) > 0$ be such that $x_0 + h \in S$ for all $h \in X$ with $\|h\| \in (0, \delta(\epsilon)]$. Applying the mean value theorem (Theorem 2.23) to the function g, for such h, we obtain

$$\begin{aligned} ||w(x_0, h)|| &= ||g(x_0 + h) - g(x_0)|| \\ &\le \sup_{u \in [x_0, x_0+h]} \{||g'(u)||\} \, \|h\| \\ &= \sup_{u \in [x_0, x_0+h]} \{||f'(u) - f'(x_0)||\} \, \|h\| \end{aligned} \tag{2.55}$$

because

$$\begin{aligned} [x_0, x_0 + h] &= \{x \in S : x = \theta x_0 + (1-\theta)(x_0 + h) \text{ for some } \theta \in [0,1]\} \\ &= \{x \in S : x = x_0 + \theta h \text{ for some } \theta \in [0,1]\}. \end{aligned} \tag{2.56}$$

□

The following result is a well-known theorem of linear functional analysis. It is called the *bounded inverse theorem* (see Schechter [51, Theorem 4.11]).

Lemma 2.25 (bounded inverse theorem). *Let X, Y be Banach spaces and let $T : X \to Y$ be linear, bounded, one-to-one, and onto. Then the inverse T^{-1} of T is a bounded linear operator on Y.*

The concept of local invertibility, which is the content of the inverse function theorem, is the purpose of the next definition. In the rest of this chapter, $B^\alpha(u_0)$ denotes the open ball of a Banach space with center at u_0 and radius $\alpha > 0$.

Definition 2.26. Let S be an open subset of the Banach space X and let f map S into the Banach space Y. Fix a point $u_0 \in S$ and let $v_0 = f(u_0)$. Then f is said to be *locally invertible at u_0* if there exist two numbers $\alpha > 0$, $\beta > 0$ with the following property: for every $v \in \overline{B^\beta(v_0)} \subset Y$ there exists a unique $u \in \overline{B^\alpha(u_0)} \subset S$ such that $f(u) = v$.

We are now ready for the main result of this section.

Theorem 2.27 (inverse function theorem). *Let X, Y be Banach spaces and S an open subset of X. Let $f : S \to Y$ be continuously Fréchet differentiable on S. Moreover, assume that the Fréchet derivative $f'(u_0)$ is one-to-one and onto at some point $u_0 \in S$. Then the function f is locally invertible at u_0.*

Proof. We may assume that $u_0 = 0$ and $f(u_0) = 0$. If this is not true, we can consider instead the mapping $\widetilde{f}(u) \equiv f(u + u_0) - f(u_0)$ on the set $\widetilde{S} = S - u_0 = \{u - u_0; u \in S\}$. We have that $\widetilde{S}$ is open, $0 \in \widetilde{S}$, $\widetilde{f}(0) = 0$, and $\widetilde{f}'(0) = f'(u_0)$.

Let $D = f'(0)$. Then the operator D^{-1} exists on all of Y and is bounded (Lemma 2.25). Thus, the equation $f(u) = v$ is equivalent to the equation $D^{-1}f(u) = D^{-1}v$. Fix $v \in Y$ and define the operator $U : S \to X$ as follows:

$$Uu = u + D^{-1}[v - f(u)], \quad u \in S. \tag{2.57}$$

Obviously, the fixed points of U are solutions of the equation $f(u) = v$. By a suitable choice of v, we show that there exists a closed ball inside S with center at 0 on which U is a contraction operator. To this end, we Fréchet-differentiate U to obtain

$$U'(u) = I - D^{-1}f'(u) = D^{-1}[f'(0) - f'(u)], \quad u \in S \tag{2.58}$$

(see Exercise 2.6). Since $f'(u)$ is continuous in u, given $\epsilon \in (0, 1)$ there exists $\alpha = \alpha(\epsilon) > 0$ such that $\|U'(u)\| \le \epsilon$ for all $u \in S$ with $\|u\| \le \alpha$. Fix such an ϵ and choose α so that $\overline{B^\alpha(0)} \subset S$. By Theorem 2.23, we also have

$$\begin{aligned} ||Uu_1 - Uu_2|| &= ||U(u_2 + (u_1 - u_2)) - Uu_2|| \\ &\le \sup_{\theta \in (0,1)} ||U'(u_2 + \theta(u_1 - u_2)|| \, ||u_1 - u_2|| \\ &\le \epsilon||u_1 - u_2|| \end{aligned} \tag{2.59}$$

for all $u_1, u_2 \in \overline{B^\alpha(0)}$, because $u_2 + \theta(u_1 - u_2) = \theta u_1 + (1-\theta)u_2 \in \overline{B^\alpha(0)}$ for all $\theta \in (0,1)$. Let

$$\beta = \frac{(1-\epsilon)\alpha}{||D^{-1}||} \tag{2.60}$$

and $\|v\| \le \beta$. Then

$$\begin{aligned} \|Uu\| &\le ||Uu - U(0)|| + ||U(0)|| \\ &\le \epsilon\|u\| + ||D^{-1}||\,\|v\| \\ &\le \epsilon\alpha + \left[\frac{(1-\epsilon)\alpha}{||D^{-1}||}\right]||D^{-1}|| \\ &= \alpha \end{aligned} \tag{2.61}$$

whenever $\|u\| \le \alpha$. It follows that, for $v \in \overline{B^\beta(0)}$, the operator U maps the closed ball $\overline{B^\alpha(0)}$ into itself. By the Banach contraction principle, U has a unique fixed point in $\overline{B^\alpha(0)}$. We have shown that f is locally invertible, that is, for every $v \in \overline{B^\beta(0)}$ there exists a unique $u \in \overline{B^\alpha(0)}$ such that $f(u) = v$. □

In the following example we find the Fréchet derivative of a large class of functions which are important for the applications of the inverse function theorem to differential equations.

Example 2.28. Let $J = [a,b]$ and let $B^r(0)$ be the open ball of $\mathbb{C}_n(J)$ with center at 0 and radius $r > 0$. We consider a continuous function $F : J \times \overline{B_r(0)} \to \mathbb{R}^n$ and the operator $U : B^r(0) \to \mathbb{C}_n(J)$ defined as follows:

$$(Ux)(t) = F(t, x(t)), \quad t \in J,\ x \in B^r(0). \tag{2.62}$$

We note first that U is continuous on $B^r(0)$. In fact, since F is uniformly continuous on the compact set $J \times \overline{B_r(0)}$, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$||F(t,u) - F(t,v)|| < \epsilon \tag{2.63}$$

for every $u, v \in \overline{B_r(0)}$ with $\|u - v\| < \delta(\epsilon)$ and every $t \in J$. This implies that

$$\|Ux - Uy\|_\infty < \epsilon \tag{2.64}$$

whenever $x, y \in B^r(0)$ with $\|x - y\|_\infty < \delta(\epsilon)$. In order to compute the Fréchet derivative of U, we assume that the Jacobian matrix

$$F_x(t,u) = \left[\frac{\partial F_i}{\partial x_j}(t,u)\right], \quad i, j, = 1, 2, \dots, n, \tag{2.65}$$

exists and is continuous on $J \times B_r(0)$. Then, given a function $x_0 \in B^{r_1}(0)$ (where $r_1 \in (0,r)$), we have that $x_0 + h \in B^{r_1}(0)$ for all sufficiently small $h \in C_n(J)$. For such functions h we have

$$\begin{aligned}
&\|U(x_0+h) - Ux_0 - F_x(\cdot, x_0(\cdot))h(\cdot)\|_\infty \\
&\quad = \sup_{t\in J} \|F(t,x_0(t)+h(t)) - F(t,x_0(t)) - F_x(t,x_0(t))h(t)\| \\
&\quad \leq \sup_{t\in J} \left\{ \left\| \left[\frac{\partial F_i}{\partial x_j}(t, x_0(t)+\theta_i(t)h(t)) \right]_{i,j=1}^n - F_x(t,x_0(t)) \right\| \right\} \|h\|_\infty,
\end{aligned} \tag{2.66}$$

where the functions $\theta_i(t)$, $i = 1,2,\ldots,n$, are lying in the interval $(0,1)$. Here, we have used the mean value theorem for real functions on $B_r(0)$ as follows:

$$F_i(t,x_0(t)+h(t)) - F_i(t,x_0(t)) = \langle \nabla F_i(t,z_i(t)), h(t)\rangle, \quad i = 1,2,\ldots,n, \tag{2.67}$$

where $z_i(t) = x_0(t) + \theta_i(t)h(t)$. From the uniform continuity of the functions $\nabla F_i(t,u)$, $i = 1,2,\ldots,n$, on $J \times \overline{B_{r_1}(0)}$ and (2.66), it follows that the Fréchet derivative $U'(x_0)$ exists and is a bounded linear operator given by the formula

$$[U'(x_0)h](t) = F_x(t,x_0(t))h(t) \tag{2.68}$$

for every $h \in C_n(J)$ and every $t \in J$.

EXERCISES

2.1. Let $f \in \mathbb{C}_1^1(\mathbb{R})$ be such that $f(0) = 0$ and $f'(0) = 0$. Show that there exists a $q > 0$ such that $f : [-q,q] \to [-q,q]$ and f is a contraction mapping on $[-q,q]$. Can you generalize this to the case of functions $f : \mathbb{R}^n \to \mathbb{R}^n$ which are continuously differentiable on $\mathbb{R}^n$?

2.2. Consider the operator T with

$$(Tx)(t) = f(t) + y(t)\int_t^\infty e^{3s}x(s)ds, \quad t \in \mathbb{R}_+, \tag{2.69}$$

where $y \in \mathbb{C}_1(\mathbb{R}_+)$ satisfies

$$\sup_{t\in R_+} \{e^{4t}|y(t)|\} < 1. \tag{2.70}$$

and $f \in \mathbb{C}_1(\mathbb{R}_+)$. Obviously, T is not well defined for all $x \in \mathbb{C}_1(\mathbb{R}_+)$. We consider the "weighted" norm

$$\|x\|_e = \sup_{t\in\mathbb{R}_+} \{e^{4t}|x(t)|\} \tag{2.71}$$

and the space

$$\mathbb{C}_e = \{u \in \mathbb{C}_1(\mathbb{R}_+) : \|u\|_e < +\infty\} \tag{2.72}$$

associated with this norm. Show that $\mathbb{C}_e$ is a Banach space. Using the contraction mapping principle, show that for every $f \in \mathbb{C}_e$ the operator T has a unique fixed point in $\mathbb{C}_e$.

2.3. Let X, Y, Z be Banach spaces. Let $T : X \to Y$, $U : Y \to Z$ be compact. Show that the operator $UT : X \to Z$ is compact. Here, UT is the composition of U and T.

2.4. Let X be a Banach space and $S_0 : X \to X$ a compact operator. Let

$$\|S_0 x\| \leq \lambda \|x\| + m, \tag{2.73}$$

where $\lambda < 1$, m are positive constants. Show that S_0 has at least one fixed point in X. Hint. Apply the Leray-Schauder Theorem to the operator $S(x,\mu) = \mu S_0 x$, $\mu \in [0,1]$. Actually, here we may also apply the Schauder-Tychonov theorem. (How?)

2.5. Let $M \subset \mathbb{C}_n^l$ have the following properties:

(i) for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|f(t) - f(t')\| < \epsilon \tag{2.74}$$

for every $t, t' \in \mathbb{R}_+$ with $|t - t'| < \delta(\epsilon)$ and every $f \in M$;

(ii) there exists a constant $L > 0$ such that $\|f\|_\infty \leq L$ for every $f \in M$;

(iii) let l_f denote the limit of $f(t)$ as $t \to +\infty$. Then for every $\epsilon > 0$ there exists $Q(\epsilon) > 0$ such that

$$\|f(t) - l_f\| < \epsilon \quad \forall t > Q(\epsilon),\ f \in M. \tag{2.75}$$

Show that M is relatively compact.

Hint. Consider the mapping $T : \mathbb{C}_n^l \to \mathbb{C}_n[0,1]$ such that $g = Tf$ with

$$g(t) = \begin{cases} f\left(\dfrac{t}{1-t}\right), & t \in [0,1), \\ l_f, & t = 1. \end{cases} \tag{2.76}$$

2.6. Let X, Y be Banach spaces and S an open subset of X. Let $T : X \to Y$ be a bounded linear operator and let $f : S \to X$ have Fréchet derivative $f'(u_0)$ at $u_0 \in S$. Show that $Tf : S \to Y$ is Fréchet differentiable at u_0 with Fréchet derivative

$$(Tf)'(u_0) = T(f'(u_0)). \tag{2.77}$$

2.7. Consider the operator $T : \mathbb{C}_1[0,1] \to \mathbb{C}_1[0,1]$ defined by

$$(Tx)(t) = \int_0^t \left[1 - x^2(s)\right]^{1/3} ds. \tag{2.78}$$

Show that T has a fixed point.

2.8. Let the operator T be defined on $\mathbb{C}_1(\mathbb{R}_+)$ as follows:

$$(Tx)(t) = f(t) + \left(\frac{1}{2}\right) \int_0^t e^{-s} \sin\left(x(s)\right) ds. \tag{2.79}$$

Here, $f \in \mathbb{C}_1(\mathbb{R}_+)$ is a given function. Show that T has a unique fixed point in $\mathbb{C}_1(\mathbb{R}_+)$.

2.9. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that

$$\|F(x)\| \le \|x\| \quad \text{whenever } \|x\| > r, \tag{2.80}$$

where r is a positive constant. Show that F has a fixed point in the ball $\overline{B_r(0)}$.

2.10. Let $T : \mathbb{C}_3[a,b] \to \mathbb{C}_1[a,b]$ be defined as follows:

$$(Tx)(t) = x_1^2(t) + \sin\left(x_2(t)\right) - x_1(t) \exp\left\{x_3(t)\right\}, \quad t \in [a,b], \tag{2.81}$$

where $x(t) \equiv (x_1(t), x_2(t), x_3(t))$. Find a formula for the Fréchet derivative $T'(x_0)$ at any $x_0 \in \mathbb{C}_3[0,1]$.

2.11. Let X, Y be Banach spaces and let $f : X \to Y$ be compact and Fréchet differentiable at $x_0 \in X$. Show that $f'(x_0) : X \to Y$ is a compact linear operator. Hint. Assume that the conclusion is false. Then, for some $\epsilon > 0$ and some $\{x_n\} \subset X$ with $\|x_n\| = 1$,

$$\|f'(x_0)x_n - f'(x_0)x_m\| > 3\epsilon, \quad n \ne m. \tag{2.82}$$

Show that, for some $\delta > 0$ such that $\|w(x_0,h)\| \le \epsilon\|h\|$ for all $h \in \overline{B_\delta(0)}$, we have

$$\|f(x_0 + \delta x_m) - f(x_0 + \delta x_n)\| > \delta\epsilon, \quad m \ne n, \tag{2.83}$$

which contradicts the compactness of f.

2.12. Let S be an open subset of a Banach space X with norm $\|\cdot\|$. Let $f : S \to X$ have Fréchet derivative $f'(u_0)$ at some point $u_0 \in S$. Show that if $\|\cdot\|_a$ is another norm of X, equivalent to $\|\cdot\|$, then the Fréchet derivative of f w.r.t. $\|\cdot\|_a$ at u_0 is also $f'(u_0)$.

2.13. Let the operator T be defined as follows:

$$(Tx)(t) = f(t) + \int_a^t K(t,s)F(s,x(s))\,ds, \tag{2.84}$$

where $f \in \mathbb{C}_1[a,b]$ and $K : [a,b] \times [a,b] \to \mathbb{R}$, $F : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Provide further conditions so that the Fréchet derivative of T exists on $\mathbb{C}_1[a,b]$, and obtain a formula for it.

2.14. In the setting of Theorem 2.27, show that there exist open neighborhoods $N(u_0)$ and $N(v_0)$ such that f maps $N(u_0)$ homeomorphically onto $N(v_0)$.

2.15. In the setting of Theorem 2.20, prove that if $f'(u)$ is a bounded linear operator, then f is continuous at u.

2.16. Show that there is a unique function $x \in \mathbb{C}_1(\mathbb{R}_+)$ such that

$$x(t) + \sin\left(0.4x(t)\right) + \tan^{-1}\left(0.5x(t)\right) = e^{-t}, \quad t \in \mathbb{R}_+. \tag{2.85}$$

2.17 (directional derivative). Let X, Y be Banach spaces and let S be an open subset of X. Fix $x_0 \in S$, $h \in X$. We say that $f : S \to Y$ has a directional derivative at x_0 in the direction h, if the limit

$$\lim_{t\to 0} \frac{f(x_0+th) - f(x_0)}{t} \tag{2.86}$$

exists. This limit is called the *directional derivative* of f at x_0 and is denoted by $D_h f(x_0)$. In general, the operator $D_h f(x)$ is neither linear nor continuous w.r.t. the variable h. Show that if the Fréchet derivative $f'(x_0)$ exists, then $D_h f(x_0)$ exists in every direction h and $D_h f(x_0) = f'(x_0)h$.

2.18 (Gâteaux derivative). In the setting of Exercise 2.17, if there exists a bounded linear operator $f'(x_0) : X \to Y$ such that $D_h f(x_0) = f'(x_0)h$, we say that f is *Gâteaux differentiable* at x_0 and we call the operator $f'(x_0)$ the *Gâteaux derivative* of f at x_0. Show that if the Gâteaux derivative $f'(x)$ of f exists on a neighborhood $N(x_0)$ of the point $x_0 \in S$ and is continuous at x_0 ($\|f'(x) - f'(x_0)\| \to 0$ as $x \to x_0$), then the Fréchet derivative also exists at x_0 and equals $f'(x_0)$. Hint. Let the Gâteaux derivative $f'(x)$ exist on $N(x_0)$ and be continuous at x_0. Let $[x_0, x_0+h] \subset S$. Let $y^* \in Y^*$ and $g(t) = y^*(f(x_0+th))$, $t \in [0,1]$. Then $g'(t) = y^*(f'(x_0+th)h)$ and $g'(\tilde{t}) = g(1) - g(0)$. Thus,

$$y^*\left(f(x_0+h) - f(x_0)\right) = y^*\left(f'(x_0+\tilde{t}h)h\right). \tag{2.87}$$

Add $-y^*(f'(x_0)h)$ above to get

$$y^*\left(f(x_0+h) - f(x_0) - f'(x_0)h\right) = y^*\left(\left(f'(x_0+\tilde{t}h) - f'(x_0)\right)h\right). \tag{2.88}$$

Picking a more appropriate functional y^*, as in Lemma 2.21, show that

$$||f(x_0+h) - f(x_0) - f'(x_0)h|| \leq ||f'(x_0+\tilde{t}h) - f'(x_0)|| \|h\|. \tag{2.89}$$

Use the continuity of f' at x_0 to show that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\frac{||f(x_0+h) - f(x_0) - f'(x_0)h||}{\|h\|} < \epsilon \tag{2.90}$$

for $\|h\| \in (0, \delta(\epsilon))$.

2.19 (zero Fréchet derivative). Assume that X, Y are Banach spaces and that $S \subset X$ is open and connected. Let $f : S \to Y$ be Fréchet differentiable on S. Show that if $f'(x) = 0$ on S, then f is constant on S. Hint. Fix $x_0 \in S$ and consider the set

$$M = \{x \in S; f(x) = f(x_0)\}. \tag{2.91}$$

Show that M is closed in S, and then show that M is open in S. To do the latter, pick a point $\tilde{x}_0 \in M$ and a ball $B_r(\tilde{x}_0) \subset S$. Using the Mean Value Theorem on an appropriate line segment $[\tilde{x}_0, \tilde{x}_0 + h]$, conclude that $B_r(\tilde{x}_0) \subset M$, for all sufficiently small $r > 0$. Since S is connected, the only subsets of S which are both open and closed in S are the empty set and S itself. Thus, $M = S$.

2.20. Which theorems on the Fréchet derivative (in Section 4 above) are actually true in any normed spaces X, Y?

CHAPTER 3

EXISTENCE AND UNIQUENESS; CONTINUATION; BASIC THEORY OF LINEAR SYSTEMS

In this chapter, we study systems of the form

$$x' = F(t, x), \tag{3.1}$$

where $F : J \times M \to \mathbb{R}^n$ is continuous. Here, J is an interval of $\mathbb{R}$ and M is a subset of $\mathbb{R}^n$.

In Section 1, we state and prove the fundamental theorem of Peano. This theorem ensures the existence of local solutions of (3.1) under the mere assumption of continuity of F. The uniqueness of the local solution then follows from an assumed Lipschitz condition on F. This is the Picard-Lindelöf theorem (Theorem 3.2), which we prove by using the method of successive approximations as in the scalar case.

In Section 2, we concern ourselves with the problem of continuation of solutions of (3.1). Roughly speaking, we show that the boundedness of the solution $x(t)$, $t \in [a, b)$, or the boundedness of the function $F(t, u)$ on an appropriate subset of $\mathbb{R}^n$, implies the continuation of $x(t)$ to the point b, that is, the existence of an extension of $x(t)$ to $t = b$ which is still a solution of (3.1). A similar situation exists for left end-points of existence of $x(t)$.

Section 3 is devoted to the establishment of some elementary properties of linear systems. These properties are used in later chapters in order to obtain further information about such systems or perturbed linear systems.

1. EXISTENCE AND UNIQUENESS

Theorem 3.1 (Peano). *Let (t_0, x_0) be a given point in $\mathbb{R} \times \mathbb{R}^n$. Let $J = [t_0 - a, t_0 + a]$, $D = \{x \in \mathbb{R}^n : \|x - x_0\| \le b\}$, where a, b are positive numbers. For the system* (3.1) *assume the following: $F : J \times D \to \mathbb{R}^n$ is continuous with $\|F(t, u)\| \le L$, $(t, u) \in J \times D$, where L is a nonnegative constant. Then there exists a solution $x(t)$*

of (3.1) *with the following property:* $x(t)$ *is defined and satisfies* (3.1) *on* $S = \{t \in J : |t - t_0| \leq \alpha\}$ *with* $\alpha = \min\{a, b/L\}$. *Moreover,* $x(t_0) = x_0$ *and* $\|x(t) - x_0\| \leq b$ *for all* $t \in [t_0 - \alpha, t_0 + \alpha]$.

Proof. If $L = 0$, then $F(t,u) \equiv 0$ and $x(t) \equiv x_0$ is the only solution to the problem. Hence, we assume that $L \neq 0$. We apply the Schauder-Tychonov theorem (Theorem 2.13). To this end, we consider first the operator

$$(Tu)(t) = x_0 + \int_{t_0}^{t} F(s, u(s))ds, \quad t \in J_1, \tag{3.2}$$

where $J_1 = [t_0, t_0 + \alpha]$. Let

$$D_0 = \{u \in C_n(J_1) : ||u - x_0||_\infty \leq b\}. \tag{3.3}$$

We note that D_0 is a closed and convex set. To show that T maps D_0 into itself, let $u \in D_0$. Then

$$||(Tu)(t) - x_0|| \leq \int_{t_0}^{t_0+\alpha} ||F(s, u(s))||ds \leq \alpha L \leq b. \tag{3.4}$$

Thus, $TD_0 \subset D_0$. To show that TD_0 is equicontinuous, let $u \in D_0$. Then

$$||(Tu)(t_1) - (Tu)(t_2)|| \leq \left| \int_{t_1}^{t_2} ||F(s, u(s))||ds \right| \leq L|t_1 - t_2|, \quad t_1, t_2 \in J_1. \tag{3.5}$$

According to Theorem 2.5, TD_0 is relatively compact.

Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $u, v \in D_0$, $\|u - v\|_\infty < \delta(\epsilon)$ imply that $\|F(\cdot, u(\cdot)) - F(\cdot, v(\cdot))\|_\infty < \epsilon$. This follows from the uniform continuity of F on the set $D_1 = [t_0, t_0+\alpha] \times \{x \in \mathbb{R}^n : \|x - x_0\| \leq b\}$. For the proof of the continuity of T on D_0, let $u_n, u \in D_0$ be such that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Then given $\epsilon > 0$, there exists $N(\epsilon) > 0$ with $\|u_n - u\|_\infty < \delta(\epsilon)$ for $n > N(\epsilon)$. Thus, we easily obtain $\|Tu_n - Tu\|_\infty < \alpha\epsilon$ for $n > N(\epsilon)$. The Schauder-Tychonov theorem applies now and ensures the existence of a fixed point of T, that is, a function $u \in D_0$ such that $Tu = u$ or

$$u(t) = x_0 + \int_{t_0}^{t} F(s, u(s))ds, \quad t \in [t_0, t_0 + \alpha]. \tag{3.6}$$

The same method can be applied to show the existence of a solution $\bar{u}(t)$ of (3.6) on the interval $[t_0 - \alpha, t_0]$. Both functions $u(t)$ and $\bar{u}(t)$ satisfy (3.1) on their respective domains. Now, consider the function

$$x(t) = \begin{cases} \bar{u}(t), & t \in [t_0 - \alpha, t_0], \\ u(t), & t \in [t_0, t_0 + \alpha]. \end{cases} \tag{3.7}$$

This function satisfies the equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\,ds, \quad t \in J, \tag{3.8}$$

and it is the desired solution to (3.1). $\square$

The uniqueness of the above solution can be achieved by assuming a Lipschitz condition on F w.r.t. its second variable. This is the content of Theorem 3.2.

THEOREM 3.2 (Picard-Lindelöf). *Consider system* (3.1) *under the assumptions of Theorem 3.1. Let $F : J \times D \to \mathbb{R}^n$ satisfy the* Lipschitz condition

$$||F(t, x_1) - F(t, x_2)|| \le k||x_1 - x_2|| \tag{3.9}$$

for every $(t, x_1), (t, x_2) \in J \times D$, where k is a positive constant. Then there exists a unique solution $x(t)$ satisfying the conclusion of Theorem 3.1.

PROOF. We give a proof of the existence of $x(t)$ which is independent of the result in Theorem 3.1. The method employed here uses *successive approximations* as in the scalar case. In fact, consider the sequence of functions

$$\begin{aligned} y_0(t) &= x_0, \\ y_{n+1}(t) &= x_0 + \int_{t_0}^{t} F(s, y_n(s))\,ds, \quad n = 0, 1, \ldots, \end{aligned} \tag{3.10}$$

for $t \in [t_0, t_0 + \alpha]$. Then it is easy to show that for $n \ge 1$ we have

$$\begin{aligned} ||y_n(t) - x_0|| &\le b, \\ ||y_{n+1}(t) - y_n(t)|| &\le \frac{Lk^n (t - t_0)^{n+1}}{(n+1)!}, \end{aligned} \tag{3.11}$$

for every $t \in [t_0, t_0 + \alpha]$. It follows that the series

$$x_0 + \sum_{n=0}^{\infty} [y_{n+1}(t) - y_n(t)], \tag{3.12}$$

whose partial sum S_n equals y_n, $n = 1, 2, \ldots$, converges uniformly on $[t_0, t_0 + \alpha]$ to a function $y(t), t \in [t_0, t_0 + \alpha]$. In fact, this is a consequence of the Weierstrass M-Test and the fact that the terms of the above series are bounded above by the corresponding terms of the series

$$||x_0|| + \frac{L}{k} \sum_{n=0}^{\infty} \frac{(k\alpha)^{n+1}}{(n+1)!}, \tag{3.13}$$

which converges to the number $\|x_0\| + (L/k)(e^{k\alpha} - 1)$. This shows the uniform convergence of $\{y_n(t)\}$ to $y(t)$ as $n \to \infty$. This function $y(t)$ satisfies

$$y(t) = x_0 + \int_{t_0}^{t} F(s, y(s))ds, \quad t \in [t_0, t_0 + \alpha]. \tag{3.14}$$

Thus, $y(t_0) = x_0$, $\|y(t) - x_0\| \le b$, $t \in [t_0, t_0 + \alpha]$, and $y(t)$ satisfies (3.1) on the above interval. This process can now be repeated on the interval $[t_0 - \alpha, t_0]$ to obtain a solution $\bar{y}(t)$ with the required properties on this interval. The function $x(t), t \in [t_0 - \alpha, t_0 + \alpha]$, which is identical to $\bar{y}(t)$ on $[t_0 - \alpha, t_0]$, and $y(t)$ on $[t_0, t_0 + \alpha]$, is a solution on $[t_0 - \alpha, t_0 + \alpha]$. Now, let $x_1(t)$ be another solution having the same properties as $x(t)$. Then we can use the equation

$$x_1(t) = x_0 + \int_{t_0}^{t} F(s, x_1(s))ds \tag{3.15}$$

to obtain by induction that

$$||y_n(t) - x_1(t)|| \le \frac{L}{k}\frac{(k\alpha)^{n+1}}{(n+1)!}, \quad t \in [t_0, t_0 + \alpha]. \tag{3.16}$$

Taking limits as $n \to \infty$, we obtain $x(t) = x_1(t)$, $t \in [t_0, t_0+\alpha]$. One argues similarly on the interval $[t_0 - \alpha, t_0]$. This completes the proof of the theorem. □

Remark 3.3. It is obvious that the interval J in Theorems 3.1 and 3.2 may be replaced by one of the intervals $[t_0 - \alpha, t_0]$ or $[t_0, t_0 + \alpha]$, in which case the solution found to exist will be defined on $[t_0 - \alpha, t_0]$ or $[t_0, t_0 + \alpha]$, respectively. We will refer to Peano's theorem in the sequel even in cases where our assumptions involve only one of these two intervals.

By a *solution* of equation (3.1), we mean a continuously differentiable function $x(t)$ which is defined on an interval $J_1 \subset J$ and satisfies (3.1) on J_1.

The uniqueness part of Theorem 3.2 can be shown with the help of an inequality—Gronwall's inequality. In Theorem 3.2 we gave a proof using successive approximations in order to exhibit the method in $\mathbb{R}^n$. This method actually goes over to Banach spaces, where a continuous function $F(t, x)$ does not necessarily give rise to a compact integral operator as in the case of Theorem 3.1.

Gronwall's inequality, which will be needed several times in the sequel, is contained in the following lemma. Its proof is given as an exercise (see Exercise 3.1).

Lemma 3.4 (Gronwall's inequality). *Let $u, g : [a, b] \to \mathbb{R}_+$ be continuous and such that*

$$u(t) \le K + \int_{a}^{t} g(s)u(s)ds, \quad t \in [a, b], \tag{3.17}$$

where K is a nonnegative constant. Then

$$u(t) \leq K \exp\left\{\int_a^t g(s)ds\right\}, \quad t \in [a,b]. \tag{3.18}$$

2. CONTINUATION

In this section, we study the problem of continuation (or extendability) of the solutions whose existence is ensured by Theorems 3.1 and 3.2. In what follows, a *domain* is an open, connected set. We have the following definition.

Definition 3.5. A solution $x(t)$, $t \in [a,b)$, $a < b < +\infty$, of system (3.1) is said to be *continuable to* $t = b$, if there exists another solution $\bar{x}(t)$, $t \in [a,c]$, $c \geq b$, of system (3.1) such that $\bar{x}(t) = x(t)$, $t \in [a,b)$. A solution $x(t)$, $t \in [a,b)$, $a < b < +\infty$, of system (3.1) is said to be *continuable to* $t = c$ $(b < c < +\infty)$, if it is continuable to $t = b$, and whenever we assume that $x(t)$ is a solution on $[a,d)$, for any $d \in (b,c]$, we can show that $x(t)$ is continuable to the point $t = d$. Such a solution is *continuable to* $+\infty$ if it is continuable to $t = c$ for any $c > b$. Similarly one defines continuation to the left.

The words "extendable" and "continuable" are interchangeable in the sequel.

Theorem 3.6. *Suppose that D is a domain of* $\mathbb{R} \times \mathbb{R}^n$ *and that* $F : D \to \mathbb{R}^n$ *is continuous. Let* (t_0, x_0) *be a point in D and assume that system* (3.1) *has a solution* $x(t)$ *defined on a finite interval* (a,b) *with* $t_0 \in (a,b)$ *and* $(t, x(t)) \in D$, $t \in (a,b)$. *Then if F is bounded on D, the limits*

$$x(a^+) = \lim_{t \to a^+} x(t), \qquad x(b^-) = \lim_{t \to b^-} x(t) \tag{3.19}$$

exist as finite vectors. If the point $(a, x(a^+))$ $((b, x(b^-)))$ *is in D, then* $x(t)$ *is continuable to* $t = a$ $(t = b)$.

Proof. In order to show that the first limit in (3.19) exists, we first note that

$$x(t) = x_0 + \int_{t_0}^t F(s, x(s))ds, \quad t \in (a,b). \tag{3.20}$$

Let $\|F(t,x)\| \leq L$ for $(t,x) \in D$, where L is a positive constant. Then if $t_1, t_2 \in (a,b)$, we see that

$$\|x(t_1) - x(t_2)\| \leq \left| \int_{t_1}^{t_2} \|F(s, x(s))\| ds \right| \leq L|t_1 - t_2|. \tag{3.21}$$

Thus, $x(t_1) - x(t_2)$ converges to zero as t_1 and t_2 converge to the point $t = a$ from the right. Applying the Cauchy criterion for functions, we obtain our assertion. One argues similarly for the second limit of (3.19).

We assume now that the point $(b, x(b^-))$ belongs to D and consider the function

$$\bar{x}(t) = \begin{cases} x(t), & t \in (a,b), \\ x(b^-), & t = b. \end{cases} \tag{3.22}$$

This function is a solution of (3.1) on $(a,b]$. In fact, (3.20) implies

$$\bar{x}(t) = x_0 + \int_{t_0}^{t} F(s, \bar{x}(s))\,ds, \quad t \in (a,b], \tag{3.23}$$

which in turn implies the existence of the left-hand derivative $\bar{x}'_-(b)$ of $\bar{x}(t)$ at $t = b$.

Thus, we have

$$\bar{x}'_-(b) = F(b, \bar{x}(b)), \tag{3.24}$$

which completes the proof for $t = b$. A similar argument holds for $t = a$. □

It should be noted that if the point $(a, x(a^+))$ is not in D, but $F(a, x(a^+))$ can be defined so that F is continuous at $(a, x(a^+))$, then $x(t)$ is continuable to $t = a$ with value $x(a^+)$ there. A similar situation exists at $(b, x(b^-))$.

In the rest of this chapter we are mainly concerned with the continuation of a solution to the right of its interval of existence. The reader should bear in mind that corresponding results cover the continuation to the left of that interval. The following continuation theorem is needed for the proof of Theorem 3.8. More general theorems can be found in Chapter 5.

Theorem 3.7. *Let $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that $\|F(t, x)\| \le L$ for all $(t,x) \in [a,b] \times \mathbb{R}^n$, where L is a positive constant. Then every solution $x(t)$ of* (3.1), *defined on an interval to the left of b, is continuable to $t = b$.*

Proof. Let $x(t)$ be a solution of (3.1) passing through the point $(t_0, x_0) \in [a,b) \times \mathbb{R}^n$. Assume that $x(t)$ is defined on the interval $[t_0, c)$, where c is some point with $c \le b$. Then, as in the proof of Theorem 3.6, $x(c^-)$ exists and $x(t)$ is continuable to $t = c$. If $c = b$, the proof is complete. If $c < b$, then Peano's theorem (Theorem 3.1), applied on $[c, b] \times D$, with D a sufficiently large closed ball with center at $x(c^-)$, ensures the existence of a solution $\bar{x}(t)$, $t \in [c, b]$, such that $\bar{x}(c) = x(c^-)$. Thus, the function

$$x_0(t) = \begin{cases} x(t), & t \in [t_0, c], \\ \bar{x}(t), & t \in [c, b], \end{cases} \tag{3.25}$$

is the desired continuation of $x(t)$. □

Theorem 3.8 says that the uniform boundedness of all solutions through a certain point implies their extendability.

Theorem 3.8. *Let $F : [a, b] \times M \to \mathbb{R}^n$ be continuous, where M is the closed ball $\overline{B_r(0)}$ (or $\mathbb{R}^n$). Assume that $(t_0, x_0) \in [a, b] \times M$ is given and that every solution $x(t)$ of* (3.1) *passing through (t_0, x_0) satisfies $\|x(t)\| < \lambda$ for as long as it exists to the right of t_0. Here, $\lambda \in (0, r]$ (or $\lambda \in (0, \infty)$). Then every solution $x(t)$ of* (3.1) *with $x(t_0) = x_0$ is continuable to $t = b$.*

Proof. We give the proof for $M = \overline{B_r(0)}$. An even easier proof applies to the case $M = \mathbb{R}^n$. Let $x(t)$ be a solution of (3.1) with $x(t_0) = x_0$ and assume that $x(t)$ is defined on $[t_0, c)$ with $c < b$. Since F is continuous on $[a, b] \times \overline{B_\lambda(0)}$, there exists $L > 0$ such that $\|F(t, x)\| \le L$ for all $(t, x) \in [a, b] \times \overline{B_\lambda(0)}$. Now, consider the function

$$F_1(t, x) = \begin{cases} F(t, x), & (t, x) \in [a, b] \times \overline{B_\lambda(0)}, \\ \dfrac{\lambda}{\|x\|} F\left(t, \dfrac{\lambda x}{\|x\|}\right), & t \in [a, b],\ \|x\| \ge \lambda. \end{cases} \tag{3.26}$$

It is easy to see that F_1 is continuous and such that $\|F_1(t, x)\| \le L$ on $[a, b] \times \mathbb{R}^n$. Consequently, Theorem 3.7 implies that every solution of the system

$$x' = F_1(t, x) \tag{3.27}$$

is continuable to $t = b$. Naturally, $x(t)$ is a solution of (3.27) defined on $[t_0, c)$ because $F_1(t, x) = F(t, x)$ for $\|x\| \le \lambda$. Therefore, there exists a solution $x_1(t)$, $t \in [t_0, b]$, of (3.27) such that $x_1(t) = x(t)$, $t \in [t_0, c)$. We claim that $\|x_1(t)\| < \lambda$, $t \in [t_0, b]$. We already know that $\|x_1(t)\| = \|x(t)\| < \lambda$ for all $t \in [t_0, c)$. Assume that there is $t_1 \in [c, b]$ such that $\|x_1(t_1)\| = \lambda$. Then, for some $t_2 \in [c, t_1]$, $\|x_1(t_2)\| = \lambda$ and $\|x_1(t)\| < \lambda$ for all $t \in [t_0, t_2)$. Obviously, $x_1(t)$ satisfies system (3.1) on $[t_0, t_2]$. Since $\|x_1(t_2)\| = \lambda$, we have a contradiction to our assumption that $\|x(t)\| < \lambda$ for all solutions with $x(t_0) = x_0$. Thus, $\|x_1(t)\| < \lambda$ for all $t \in [t_0, b]$, which implies that $x(t)$ is continuable to the point $t = b$. □

The following theorem is an important tool in dealing with various problems on infinite intervals. It guarantees the existence of a bounded solution on an infinite interval under the assumption that for every finite interval of $\mathbb{R}$ there exists a solution of (3.1) defined on that interval and bounded there by a fixed positive constant.

Theorem 3.9. *Let the function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ ($F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$) be continuous. For each $m = 1, 2, \ldots$, assume the existence of a solution $x_m(t)$, $t \in [-m, m]$ ($t \in [0, m]$), of* (3.1) *such that $\|x_m\|_\infty \le K$, where K is a fixed positive constant. Then system* (3.1) *has at least one solution $x(t)$, $t \in \mathbb{R}$ ($t \in \mathbb{R}_+$), such that $\|x\|_\infty \le K$.*

Proof. We give the proof for $t \in \mathbb{R}$. The proof for $t \in \mathbb{R}_+$ follows similarly. Let

$$q_m = \sup_{\substack{|t| \le m \\ \|u\| \le K}} \|F(t, u)\|, \quad m = 1, 2, \ldots. \tag{3.28}$$

Then, for every $m = 1, 2, \ldots$, we have

$$x_m(t) = x_m(0) + \int_0^t F(s, x_m(s))\, ds, \quad t \in [-m, m]. \tag{3.29}$$

This yields, for $m \ge k$,

$$\|x_m(t) - x_m(t')\| \le q_k |t - t'|, \quad t, t' \in [-k, k]. \tag{3.30}$$

It follows that the sequence $\{x_m(t)\}_{m=k}^{\infty}$ is uniformly bounded and equicontinuous on the interval $[-k, k]$ for all $k = 1, 2, \ldots$. Theorem 2.5 implies the existence of a subsequence $\{x_{1m}(t)\}_{m=1}^{\infty}$ of $\{x_m(t)\}$ which converges uniformly to a function $\bar{x}_1 \in C_n[-1, 1]$ as $m \to \infty$. This sequence $\{x_{1m}(t)\}$ has a subsequence $\{x_{2m}(t)\}_{m=1}^{\infty}$ which converges uniformly to a function $\bar{x}_2 \in C_n[-2, 2]$. By induction, we can construct a subsequence $\{x_{j+1,m}(t)\}_{m=1}^{\infty}$ of the sequence $\{x_{j,m}(t)\}$ which converges uniformly to the function $\bar{x}_{j+1} \in C_n[-(j+1), j+1]$. All functions $\bar{x}_m(t)$ satisfy $\|\bar{x}_m\|_\infty \le K$, $m = 1, 2, \ldots$. Given an integer $r > 0$, let J denote the interval $[-r, r]$. The diagonal sequence $\{x_{mm}(t)\}$ is defined on J for all $m \ge r$ and converges uniformly to the desired solution on J. In fact, we have

$$x_{mm}(t) = x_{mm}(0) + \int_0^t F(s, x_{mm}(s))\, ds, \quad t \in J,\ m \ge r. \tag{3.31}$$

Taking the limit of each side of (3.31) as $m \to \infty$, we get

$$x_r(t) = x_r(0) + \int_0^t F(s, x_r(s))ds \tag{3.32}$$

or

$$x_r'(t) = F(t, x_r(t)), \quad t \in [-r, r], \tag{3.33}$$

where $x_r(t)$ is the uniform limit of $\{x_{mm}(t)\}_{m=r}^{\infty}$ on J. Since r is arbitrary, we have constructed a function $x(t)$ which is the amalgamation of uniform limits $x_{r_n}(t)$ of the sequence $\{x_{mm}(t)\}$ (for $m \geq r_n$) on the intervals $[-r_n, r_n]$, respectively, where r_n is any sequence of positive integers with $r_n \to \infty$ as $n \to \infty$. This completes the proof. $\square$

3. LINEAR SYSTEMS

We now consider systems of the form

$$x' = A(t)x, \tag{S}$$

$$x' = A(t)x + f(t). \tag{S_f}$$

Here, $A(t)$ is an $n \times n$ matrix of continuous functions on a real interval J and $f(t)$ is an n-vector of continuous functions on J.

There is a very simple existence and continuation theory about these *linear systems*. Theorem 3.10 below shows this fact and suggests a more general result (Theorem 3.11) for system (3.1) with Lipschitz functions F. We give an independent proof of Theorem 3.10 in order to exhibit the method.

Theorem 3.10. *Let $A : J \to M_n$, $f : J \to \mathbb{R}^n$ be continuous. Let (t_0, x_0) be a point in $J \times \mathbb{R}^n$. Then there exists a unique solution $x(t)$ of (S_f) which is defined on J and satisfies the condition $x(t_0) = x_0$.*

Proof. Let $[a, b]$ be a closed interval contained in J and containing t_0. Let $t_0 \neq a, b$. Then we can show the existence of a unique solution $x(t)$ of system (S_f) defined on $[a, b]$ and such that $x(t_0) = x_0$. In fact, consider first the interval $[a, t_0]$ and the operator T defined on $C_n[a, t_0]$ as follows:

$$(Tx)(t) = x_0 + \int_{t_0}^t [A(s)x(s) + f(s)]ds, \quad t \in [a, t_0]. \tag{3.34}$$

In order to apply the contraction principle, we modify the norm of $C_n[a, t_0]$. We use instead the so-called *Bielecki* norm (see also Exercise 2.2) which is defined, for $u \in C_n[a, t_0]$, to be

$$\|u\|_e = \sup_{a \leq t \leq t_0} e^{-k(t_0 - t)} ||u(t)||. \tag{3.35}$$

Here, k is a fixed positive constant with $k > r$, where

$$r = \max_{a \le t \le t_0} \|A(t)\|. \tag{3.36}$$

The space $C_n[a, t_0]$ is a Banach space with the norm $\|\cdot\|_e$. We have

$$\begin{aligned} \|(Tu_1)(t) - (Tu_2)(t)\| &\le \int_t^{t_0} \|A(s)(u_1(s) - u_2(s))\| ds \\ &\le \int_t^{t_0} \|A(s)\| \|u_1(s) - u_2(s)\| ds \\ &\le r \int_t^{t_0} \|u_1(s) - u_2(s)\| ds \\ &\le r\|u_1 - u_2\|_e \int_t^{t_0} e^{k(t_0 - s)} ds. \end{aligned} \tag{3.37}$$

If we multiply the above inequality by $e^{-k(t_0-t)}$, we obtain

$$\begin{aligned} e^{-k(t_0-t)}\|(Tu_1)(t) - (Tu_2)(t)\| &\le r\|u_1 - u_2\|_e \int_t^{t_0} e^{k(t-s)} ds \\ &\le \left(\frac{r}{k}\right)\|u_1 - u_2\|_e. \end{aligned} \tag{3.38}$$

Thus,

$$\|Tu_1 - Tu_2\|_e \le \left(\frac{r}{k}\right)\|u_1 - u_2\|_e, \quad u_1, u_2 \in C_n[a, t_0]. \tag{3.39}$$

Since $r/k < 1$, T is a contraction on $C_n[a, t_0]$, and an application of the contraction mapping principle (Theorem 2.1) yields a unique fixed point u of T, which is the unique solution of (S_f) on $[a, t_0]$. One works similarly on the interval $[t_0, b]$ to obtain the unique solution $\bar{u}(t)$ of (S_f) on this interval with the property $\bar{u}(t_0) = x_0$. Joining these two solutions together, we obtain the unique solution $x(t)$ of (S_f) on the interval $[a, b]$. Naturally, if $t_0 = a$ or $t_0 = b$, then one of these two solutions equals $x(t)$ on $[a, b]$. Now, we prove the theorem in the case $J = (-\infty, b)$. All the other cases, finite or infinite, can be handled similarly. Suppose that $x(t)$ is a solution of (S_f), with $x(t_0) = x_0$, which cannot be continued to $-\infty$. Then $x(t)$ is defined on a largest interval (c, d). Using the above methods of existence on closed intervals, we can show that (S_f) has a unique solution $\bar{x}(t)$ on the interval $[c, t_0]$ such that $\bar{x}(t_0) = x_0$. Obviously, $\bar{x}(t) = x(t)$, $t \in (c, t_0]$, which proves that $x(t)$ is continuable to the point $t = c$, that is, a contradiction. A similar argument applies to the case $d < b$. Thus, $x(t)$ is the unique solution of (S_f) on $J = (-\infty, b)$. This finishes the proof. □

The method of the proof of the above theorem can now be applied to obtain a more general result.

Theorem 3.11. *Let $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume further that for every interval $[a,b] \subset J$ there exists a constant $k > 0$ depending on $[a,b]$ and such that*

$$||F(t,u_1) - F(t,u_2)|| \le k||u_1 - u_2||, \quad t \in [a,b],\ u_1,u_2 \in \mathbb{R}^n. \tag{3.40}$$

Then if $(t_0,x_0) \in J \times \mathbb{R}^n$, system (3.1) possesses a unique solution $x(t)$ defined on J and such that $x(t_0) = x_0$.

We now consider the matrix system

$$X' = A(t)X, \quad t \in J, \tag{S_A}$$

where J is an interval of $\mathbb{R}$ and $A : J \to M_n$ is continuous. Here, we seek a solution $X : J \to M_n$. The existence and uniqueness theory of systems of the form (S_A) is identical to the corresponding theory of systems of the form (S). This is stated in the following theorem.

Theorem 3.12. *Consider (S_A) with $A : J \to M_n$ continuous. Fix $t_0 \in J$ and $B \in M_n$. Then there exists a unique solution $X(t)$ of (S_A) which is defined on J and satisfies $X(t_0) = B$.*

The following theorem says that the space of solutions of (S) is a vector space of dimension n.

Theorem 3.13. *Let $A : J \to M_n$. Then the set of all solutions of (S) is an n-dimensional vector space, that is, there exists a set P of n linearly independent solutions of (S) such that every other solution of (S) is a linear combination of the solutions in P.*

Proof. Obviously, the set of all solutions of (S) is closed under addition and scalar multiplication (by real scalars). Thus, it is a real vector space. We first show that there exist at least n linearly independent solutions of (S). To this end, let $\bar{x}_i,\ i = 1,2,\ldots,n$, denote the n-vectors (in $\mathbb{R}^n$)

$$\bar{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \bar{x}_3 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.41}$$

respectively. Let t_0 be a point in J and consider the solutions $x_i(t)$, $i = 1,2,\ldots,n$, of (S) which satisfy $x_i(t_0) = \bar{x}_i$. These solutions exist on J and are unique by Theorem 3.10. Now, assume that the function set $\{x_i : i = 1,2,\ldots,n\}$ is linearly dependent on J. Then there are constants c_i, $i = 1,2,\ldots,n$, not all zero, such that

$$\sum_{i=1}^{n} c_i x_i(t) = 0, \quad t \in J. \tag{3.42}$$

In particular,

$$\sum_{i=1}^{n} c_i\bar{x}_i = \sum_{i=1}^{n} c_i x_i(t_0) = 0. \tag{3.43}$$

This, however, is a contradiction because the vectors $\bar{x}_i$, $i = 1, 2, \ldots, n$, are linearly independent. Now, let $\bar{x}(t)$ be any solution of (S) and consider the algebraic system (in c_i)

$$\sum_{i=1}^{n} c_i\bar{x}_i = \sum_{i=1}^{n} c_i x_i(t_0) = \bar{x}(t_0). \tag{3.44}$$

This system has a unique solution, say $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n$, because the determinant of its coefficients equals 1. Define the function $y(t)$ as follows:

$$y(t) = \sum_{i=1}^{n} \bar{c}_i x_i(t), \quad t \in J. \tag{3.45}$$

Then $y(t)$ satisfies (S) on J and $y(t_0) = \bar{x}(t_0)$. Since the solutions of (S) are unique w.r.t. initial conditions, $y(t) = \bar{x}(t)$, $t \in J$, which shows that $\bar{x}(t)$ is a linear combination of the functions $x_i(t)$. This ends the proof. □

Definition 3.14. Consider the system (S) with $A : J \to M_n$ continuous. Let $x_i(t)$, $i = 1, 2, \ldots, n$, be any linearly independent solutions of (S). Then the matrix $X(t)$, $t \in J$, whose columns are the n solutions $x_i(t)$ is called a *fundamental matrix of* (S).

The connection between the system (S) and the system (S_A) is established in the following theorem.

Theorem 3.15. *Let $A : J \to M_n$ be continuous. Then every fundamental matrix $X(t)$ of* (S) *satisfies the matrix system* (S_A). *Moreover, if $X(t_0) = B$ for some $t_0 \in J$ and $B \in M_n$, then $X(t)$ is the unique solution of* (S_A) *taking the value B at $t = t_0$.*

Proof. It is easy to see that

$$X'(t) = A(t)X(t), \quad t \in J. \tag{3.46}$$

The rest of the proof follows from Theorem 3.12. □

Under the assumptions of Theorem 3.15, let $X(t)$, $t \in J$, be a fundamental matrix of (S). Then it is a simple fact that $x(t) \equiv X(t)x_0$ is a solution of (S) for any $x_0 \in \mathbb{R}^n$. Moreover, $x(t_0) = X(t_0)x_0$. Thus, if we show that $X^{-1}(t_0)$ exists for any fundamental matrix $X(t)$ of (S), then the solution $x(t)$ of (S) with $x(t_0) = u_0$, for some $(t_0, u_0) \in J \times \mathbb{R}^n$, will be given by $x(t) \equiv X(t)X^{-1}(t_0)u_0$. To this end, assume that $X(t_0)$ is singular at some $t_0 \in J$. Then the equation $X(t_0)y = 0$ has a nonzero solution $y_0 \in \mathbb{R}^n$. But this implies that the function $y(t) = X(t)y_0$, $t \in J$, is a solution of (S) with $y(t_0) = X(t_0)y_0 = 0$. Since the solutions of (S) are unique w.r.t. initial conditions, we must have $y(t) \equiv X(t)y_0 \equiv 0$. This implies that the columns of $X(t)$ are linearly dependent on J, that is, a contradiction. Therefore, $X^{-1}(t_0)$ exists.

The following theorem summarizes the above and gives an expression for the general solution of (S_f) in terms of a fundamental matrix X and the function f.

Theorem 3.16. *Let $A : J \to M_n$, $f : J \to \mathbb{R}^n$ be continuous. Let $(t_0, x_0) \in J \times \mathbb{R}^n$ be given. Then the unique solution $u(t), t \in J$, of the linear system* (S), *such that $u(t_0) = x_0$, is given by $u(t) \equiv X(t)X^{-1}(t_0)x_0$, where $X(t)$ is any fundamental matrix of* (S). *Furthermore, the unique solution $x(t)$ of* (S_f), *such that $x(t_0) = x_0$, is given by the formula*

$$x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t} X^{-1}(s)f(s)ds, \quad t \in J. \tag{3.47}$$

Proof. Since $x(t_0) = x_0$, it suffices to show that the function

$$v(t) \equiv X(t)\int_{t_0}^{t} X^{-1}(s)f(s)ds \tag{3.48}$$

is a particular solution of (S_f). In fact,

$$\begin{aligned} v'(t) &= X'(t)\int_{t_0}^{t} X^{-1}(s)f(s)ds + X(t)X^{-1}(t)f(t) \\ &= A(t)X(t)\int_{t_0}^{t} X^{-1}(s)f(s)ds + f(t) \\ &= A(t)v(t) + f(t). \end{aligned} \tag{3.49}$$

□

Equation (3.47) is called the *variation of constants formula* for system (S_f). For a constant matrix $A \in M_n$, we have the following theorem.

Theorem 3.17. *Let $A \in M_n$ be given. Then the fundamental matrix $X(t)$, $t \in \mathbb{R}$, of the system*

$$x' = Ax, \tag{3.50}$$

with the property $X(0) = I$, is given by the formula $X(t) \equiv e^{tA}$. Moreover, the variation of constants formula (3.47) *becomes now*

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}f(s)ds. \tag{3.51}$$

PROOF. Let $X(t) = e^{tA}$. Then, for $h \neq 0$, we have

$$X(t+h) - X(t) = e^{(t+h)A} - e^{tA} = (e^{hA} - I)e^{tA}. \tag{3.52}$$

However,

$$\begin{aligned} e^{hA} - I &= hA + \frac{(hA)^2}{2!} + \cdots \\ &= hA + hL(h, A), \end{aligned} \tag{3.53}$$

where $L(h, A) \to 0$ as $h \to 0$. Consequently,

$$\lim_{h \to 0} \frac{X(t+h) - X(t)}{h} = X'(t) = Ae^{tA} = AX(t). \tag{3.54}$$

Since (3.50) has unique solutions w.r.t. initial conditions, the first part of the proof is complete. The second part follows trivially from (3.47). □

EXERCISES

3.1. Prove Lemma 3.4.

3.2. Let $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy the following Lipschitz condition:

$$||F(t, u_1) - F(t, u_2)|| \leq L||u_1 - u_2||, \quad t \in [a, b],\ u_1, u_2 \in \mathbb{R}^n, \tag{3.55}$$

where L is a positive constant. Use Gronwall's inequality to show that, for every $x_0 \in \mathbb{R}^n$, the problem

$$x' = F(t, x), \qquad x(a) = x_0 \tag{3.56}$$

can have at most one solution on $[a, b]$.

3.3 (continuity w.r.t. initial conditions). Let F be as in Exercise 3.2. Moreover, let the sequence of terms $\bar{x}_m \in \mathbb{R}^n$ converge to $\bar{x} \in \mathbb{R}^n$ as $m \to \infty$. Show that the solution $x_m(t)$, $t \in [a, b]$, of the problem

$$x' = F(t, x), \qquad x(a) = \bar{x}_m \tag{3.57}$$

converges as $m \to \infty$ to the unique solution of the problem

$$x' = F(t, x), \qquad x(a) = \bar{x} \tag{3.58}$$

uniformly on $[a, b]$.

3.4. Let $A : J \to M_n$ (J is a subinterval of $\mathbb{R}$) be continuous. Let $t_0 \in J$ be given and $X(t)$ be the fundamental matrix of system (S) with $X(t_0) = I$. Furthermore, let $X_1(t)$ denote the fundamental matrix of the system $x' = -A^T(t)x$ (called the *adjoint system*) with $X_1(t_0) = I$. Show that $X_1^T(t) = X^{-1}(t)$ for every $t \in J$. Moreover, if $-A^T(t) \equiv A(t)$, show that $\|y(t)\|$ is constant for any solution $y(t)$ of (S).

3.5. Let $A(t)$ be as in the first part of Exercise 3.4 and let $X(t)$ be a fundamental matrix of (S). Let $Y(t)$ be another fundamental matrix of (S). Show that there exists a constant nonsingular matrix B such that $X(t)B = Y(t)$, $t \in J$.

3.6 (Liouville-Jacobi). Let $A : [a,b] \to M_n$ be continuous. Let $X(t)$ be a fundamental matrix of (S) and let $t_0 \in [a,b]$. Then

$$|X(t)| = |X(t_0)| \exp\left\{\int_{t_0}^{t} \operatorname{tr} A(s)ds\right\}, \quad t \in [a,b], \tag{3.59}$$

where $|A|$ denotes the determinant and $\operatorname{tr} A$ denotes the trace of the matrix A. Hint. Show that $u(t) \equiv |X(t)|$ satisfies the differential equation $u' = \operatorname{tr} A(t)u$.

3.7. Let $A : \mathbb{R} \to M_n$ be continuous and such that $A(t+T) = A(t)$, $t \in \mathbb{R}$, where T is some positive constant. Show that in order to obtain a solution $x(t)$ of (S) with $x(t+T) = x(t)$, $t \in \mathbb{R}$, it suffices to show the existence of $x(t)$, $t \in [0,T]$, with $x(0) = x(T)$. Furthermore, show that $x(t) \equiv 0$ is the only such solution if and only if $I - X(T)$ is nonsingular. Here, $X(t)$ is the fundamental matrix of (S) with $X(0) = I$.

3.8. Let $A : \mathbb{R}_+ \to M_n$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ be continuous and such that

$$\int_0^\infty ||A(t)||dt < +\infty, \qquad \int_0^\infty ||f(t)||dt < +\infty. \tag{3.60}$$

Using Gronwall's inequality, show that every solution $x(t)$ of (S_f) is bounded, that is, there exists $K > 0$ (depending on the solution x) such that $\|x(t)\| \le K$, $t \in \mathbb{R}_+$. Then show that every solution of (S_f) belongs to C_n^l. Moreover, given $\xi \in \mathbb{R}^n$, there exists a solution $x_\xi(t)$ of (S_f) such that $\lim_{t\to\infty} x_\xi(t) = \xi$.

3.9. Let $A : \mathbb{R}_+ \to M_n$ be continuous and such that $\|A(t)\| \le K$, $t \ge 0$, where K is a positive constant. Prove that every solution $x(t) \not\equiv 0$ of (S) satisfies

$$\limsup_{t\to\infty} \frac{\ln(||x(t)||)}{t} < +\infty. \tag{3.61}$$

3.10. Consider the *sublinear* scalar problem

$$x' = |x|^\beta \operatorname{sgn} x, \qquad x(0) = 0, \tag{3.62}$$

where $\beta \in (0,1)$ is a constant. Show that every function of the type

$$x(t) = \begin{cases} 0, & t \le c, \\ [(1-\beta)(t-c)]^{(1/(1-\beta))}, & t \ge c, \end{cases} \tag{3.63}$$

is a solution to this problem. Here, c is any positive constant. Conclude that $f(x) \equiv |x|^\beta \operatorname{sgn} x$ cannot satisfy a Lipschitz condition on any interval containing zero.

3.11. Consider the scalar problem

$$x' = f(x), \qquad x(0) = 0, \qquad t \ge 0, \tag{3.64}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(-x) = -f(x)$, $x \in \mathbb{R}$, and $xf(x) > 0$ for $x \neq 0$. Show that this problem has infinitely many solutions if

$$\int_{0^+}^{\epsilon} \frac{du}{f(u)} < +\infty, \qquad \int_{\epsilon}^{\infty} \frac{du}{f(u)} = +\infty, \tag{3.65}$$

for some $\epsilon > 0$. If the first integral equals $+\infty$, then the only solution to the problem is the zero solution. Hint. For every $t_0 > 0$, let $u(t) = 0$, $t \in [0, t_0]$, and, for $t > t_0$, let $u = u(t)$ be the unique positive solution of

$$\int_{0^+}^{u} \frac{dv}{f(v)} = t - t_0. \tag{3.66}$$

3.12. For the system (3.1), assume that $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and such that

$$||F(t,x)|| \le p(t)\|x\| + q(t), \tag{3.67}$$

where $p, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous. Apply Gronwall's inequality and Theorem 3.8 to conclude that all local solutions of (3.1) are continuable to $+\infty$.

3.13. Using Exercise 3.12, discuss the continuation to the right of the local solutions of the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} t \sin x_1 \\ (\cos t) \ln(|x_2| + 1) \\ e^{-t} x_3 \end{bmatrix} + \begin{bmatrix} t^2 \\ \sin t \\ e^{2t} \end{bmatrix}. \tag{3.68}$$

3.14. The scalar equation

$$y'' + y = f(t) \tag{3.69}$$

can be written in system form as follows:

$$x_1' = x_2, \qquad x_2' = -x_1 + f(t). \tag{3.70}$$

Here, $x_1 = y$. Using the variation of constants formula, express the general solution of this system in terms of its initial condition $x(0) = x_0$, the fundamental matrix $X(t)$ ($X(0) = I$) of the linear system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.71}$$

and the function f.

3.15. Examine the local existence, uniqueness and extendability to the right of solutions of the *superlinear* scalar equation

$$x' = |x|^{\beta} \operatorname{sgn} x, \tag{3.72}$$

where $\beta > 1$ is a constant. Compare this situation to that of Exercise 3.10 for the initial condition $x(0) = 0$.

3.16. Solve the problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.73}$$

using the variation of constants formula.

3.17. Let the assumptions of Exercise 3.2 be satisfied with $a = 0$. Using Gronwall's inequality, show that the problem

$$x' = F(t, x), \qquad x(b) = x_1 \tag{3.74}$$

has a unique solution defined on the interval $[0, b]$. Provide and prove a statement concerning *continuity w.r.t. final conditions* $x_n(b) = \bar{x}_n$ analogous to that of Exercise 3.3.

3.18. Using Theorem 3.8, show that every solution to the scalar problem

$$x' = (\sin x)\left(1 - x^2\right)^{1/2}, \qquad x(0) = \frac{1}{2} \tag{3.75}$$

is continuable to $t = 1/3$.

3.19 (generalized Gronwall's inequality). Let $J \subset \mathbb{R}$ be an interval, $t_0 \in J$, and $a, b, u : J \to \mathbb{R}_+$ continuous. Assume that

$$u(t) \le a(t) + \left| \int_{t_0}^{t} b(s)u(s)ds \right|, \quad t \in J. \tag{3.76}$$

Show that

$$u(t) \le a(t) + \left| \int_{t_0}^{t} a(s)b(s) \exp\left\{ \left| \int_{s}^{t} b(\sigma)d\sigma \right| \right\} ds \right|, \quad t \in J. \tag{3.77}$$

3.20. Let F be as in Exercise 3.2. Let $x_i \in C_n^1[a,b]$, $i = 1,2$, be such that $\|x_1(a) - x_2(a)\| \le \delta$, for a positive constant δ, and

$$\left\|x_i'(t) - F(t, x_i(t))\right\| \le \epsilon_i, \quad i = 1,2,\ t \in [a,b], \tag{3.78}$$

where ϵ_i is a positive constant for $i = 1,2$. Show that

$$\left\|x_1(t) - x_2(t)\right\| \le \delta e^{L(t-a)} + \frac{(\epsilon_1 + \epsilon_2)\left[e^{L(t-a)} - 1\right]}{L} \tag{3.79}$$

for all $t \in [a,b]$. Hint. Use Exercise 3.19.

3.21 (Green's formula). Let $A : [a,b] \to M_n$, $f : [a,b] \to \mathbb{R}^n$, $g : [a,b] \to \mathbb{R}^n$ be continuous. Assume that $x(t)$, $t \in [a,b]$, is a solution of (S_f) and $y(t)$, $t \in [a,b]$, is a solution of

$$y' = -A^T(t)y - g(t). \tag{3.80}$$

Then

$$\int_a^t \left[\langle f(s), y(s)\rangle - \langle x(s), g(s)\rangle\right]ds = \langle x(t), y(t)\rangle - \langle x(a), y(a)\rangle. \tag{3.81}$$

3.22 (invariant sets). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy a Lipschitz condition on every ball $B_r(0)$. We denote by $x(t, x_0)$ the solution of the *autonomous* $(f(t,x) \equiv f(x))$ system

$$x' = f(x) \tag{3.82}$$

with $x(0) = x_0$. Fix such a solution $x(t) \equiv x(t, x_0)$, defined on $\mathbb{R}_+$, and assume that $\Omega(x)$ is the set of all $\tilde{x} \in \mathbb{R}^n$ with the property: there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to \infty$ and $x(t_n) \to \tilde{x}$. Show that $\Omega(x)$ is a closed set which is invariant, that is, if $\tilde{x} \in \Omega(x)$, then $x(t, \tilde{x}) \in \Omega(x)$ for all $t \in \mathbb{R}_+$. Hint. Show first that $x(t, x(s, x_0)) = x(t+s, x_0)$, $s, t \ge 0$. Fix $\tilde{x} \in \Omega(x)$ with $x(t_n, x_0) \to \tilde{x}$. Consider the sequence $x(t + t_n, x_0)$.

3.23 (Green's function for an initial value problem). Let P_1, P_2 be two projection matrices in M_n with $P_1 + P_2 = I$. Let $A : [a,b] \to M_n$, $f : [a,b] \to \mathbb{R}^n$ be continuous. Show that the function

$$x(t) = \int_a^b G(t,s)f(s)ds, \quad t \in [a,b], \tag{3.83}$$

is a solution of the system (S_f). Here, for $t, s \in [a,b]$ with $t \neq s$,

$$G(t,s) \equiv \begin{cases} X(t)P_1X^{-1}(s), & t > s, \\ -X(t)P_2X^{-1}(s), & t < s. \end{cases} \tag{3.84}$$

Hint. We actually have

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= A(t)G(t,s), \\ \frac{\partial G(t,s)}{\partial s} &= -G(t,s)A(s) \end{aligned} \tag{3.85}$$

for $t \neq s$, and

$$\begin{aligned} G(t^+,t) - G(t^-,t) &= I, \\ G(s,s^+) - G(s,s^-) &= -I, \end{aligned} \tag{3.86}$$

for t, s in the interior of $[a,b]$. Use

$$x(t) \equiv \int_a^t G(t,s)f(s)ds + \int_t^b G(t,s)f(s)ds. \tag{3.87}$$

3.24 (evolution identity). Let $W(t,t_0)x_0$ denote the right-hand side of the variation of constants formula (3.47). Show that

$$W(t,s)W(s,r) = W(t,r) \tag{3.88}$$

for all $(t,s,r) \in \mathbb{R}^3_+$ with $r \leq s \leq t$.

CHAPTER 4

STABILITY OF LINEAR SYSTEMS; PERTURBED LINEAR SYSTEMS

In this chapter, we study the stability of systems of the form

$$x'(t) = F(t, x), \tag{E}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

A solution $x_0(t)$, $t \in \mathbb{R}_+$, of the system (E) is stable, if the solutions of (E) which start close to $x_0(t)$ at the origin remain close to $x_0(t)$ for $t \in \mathbb{R}_+$. This actually means that small disturbances in the system that cause small perturbations to the initial conditions of solutions close to $x_0(0)$ do not really cause a considerable change to these solutions over the interval $\mathbb{R}_+$.

Although there are numerous types of stability, we present here only the five types that are most important in the applications of linear and perturbed linear systems. Other stability results can be found in Chapters 5, 7, and 8.

1. DEFINITIONS OF STABILITY

In the following definitions $x_0(t)$ denotes a fixed solution of (E) defined on $\mathbb{R}_+$.

Definition 4.1. The solution $x_0(t)$ is called *stable* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that every solution $x(t)$ of (E) with $\|x(0) - x_0(0)\| < \delta(\epsilon)$ exists and satisfies $\|x(t) - x_0(t)\| < \epsilon$ on $\mathbb{R}_+$. The solution $x_0(t)$ is called *asymptotically stable* if it is stable and there exists a constant $\eta > 0$ such that $x(t) - x_0(t) \to 0$ as $t \to \infty$ whenever $\|x(0) - x_0(0)\| \leq \eta$. The solution $x_0(t)$ is called *unstable* if it is not stable.

Definition 4.2. The solution $x_0(t)$ is called *uniformly stable* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that every solution $x(t)$ of (E) with $\|x(t_0) - x_0(t_0)\| < \delta(\epsilon)$, for some $t_0 \geq 0$, exists and satisfies $\|x(t) - x_0(t)\| < \epsilon$ on $[t_0, \infty)$. It is called *uniformly asymptotically stable* if it is uniformly stable and there exists $\eta > 0$ with the property: for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $\|x(t_0) - x_0(t_0)\| < \eta$, for some $t_0 \geq 0$, implies $\|x(t) - x_0(t)\| < \epsilon$ for every $t \geq t_0 + T(\epsilon)$.

It is obvious that uniform stability implies stability and that uniform asymptotic stability implies asymptotic stability.

Definition 4.3. The solution $x_0(t)$ is called *strongly stable* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that every solution $x(t)$ of (E) with $\|x(t_0) - x_0(t_0)\| < \delta(\epsilon)$, for some $t_0 \geq 0$, exists and satisfies $\|x(t) - x_0(t)\| < \epsilon$ on $\mathbb{R}_+$.

Naturally, strong stability implies uniform stability. We should mention here that the interval $[0, \infty)$ in the definitions of stability can be replaced by any (but fixed) interval $[t_1, \infty)$ of the real line. We use the interval $[0, \infty)$ only for convenience. We should also have in mind that $x_0(t)$ can be considered to be the zero solution. In fact, if (E) does not have the zero solution, then the transformation $u(t) = x(t) - y(t)$, where $y(t)$ is a fixed solution of (E) and $x(t)$ is any other solution, takes (E) into the system

$$u' = F(t, u + y(t)) - F(t, y(t)) \equiv G(t, u). \tag{4.1}$$

This system has the function $u(t) \equiv 0$ as a solution. The stability properties of this solution correspond to the stability properties of the solution $y(t)$.

2. LINEAR SYSTEMS

In this section, we study the stability properties of the linear systems

$$x' = A(t)x, \tag{S}$$

$$x' = A(t)x + f(t), \tag{S$_f$}$$

where $A : \mathbb{R}_+ \to M_n$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ are continuous. It is clear that the solution $x_0(t)$ of (S_f) satisfies one of the definitions of stability of the previous section if and only if the zero solution of (S) has the same property. This follows from the fact that the concept of stability involves differences of solutions combined with the superposition principle. Consequently, we may talk about the stability of (S_f) instead of the stability of one of its particular solutions. This will be done in the sequel even if $f \equiv 0$.

Theorem 4.4. *Let $X(t)$ be a fundamental matrix of* (S). *Then* (S) *is stable if and only if there exists a constant $K > 0$ with*

$$||X(t)|| \leq K, \quad t \in \mathbb{R}_+. \tag{4.2}$$

The system (S) *is asymptotically stable if and only if*

$$||X(t)|| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{4.3}$$

The system (S) *is uniformly stable if and only if there exists a constant* $K > 0$ *such that*

$$||X(t)X^{-1}(s)|| \leq K, \quad 0 \leq s \leq t < +\infty. \tag{4.4}$$

The system (S) *is uniformly asymptotically stable if and only if there exist constants* $\alpha > 0$, $K > 0$ *with*

$$||X(t)X^{-1}(s)|| \leq Ke^{-\alpha(t-s)}, \quad 0 \leq s \leq t < +\infty. \tag{4.5}$$

The system (S) *is strongly stable if and only if there exists a constant* $K > 0$ *such that*

$$||X(t)|| \leq K, \quad ||X^{-1}(t)|| \leq K, \quad t \in \mathbb{R}_+. \tag{4.6}$$

Proof. We may assume that $X(0) = I$ because the conditions in the hypotheses hold for any fundamental matrix of (S) if they hold for a particular one. Assume first that (4.2) holds and let $x(t)$, $t \in \mathbb{R}_+$, be a solution of (S) with $x(0) = x_0$. Then, since $x(t) \equiv X(t)x_0$, given $\epsilon > 0$ we choose $\delta(\epsilon) = K^{-1}\epsilon$ to obtain

$$||x(t)|| = ||X(t)x_0|| < \epsilon \tag{4.7}$$

whenever $\|x_0\| < \delta(\epsilon)$. Thus, system (S) is stable.

Conversely, suppose that (S) is stable and fix $\epsilon > 0$, $\delta(\epsilon) > 0$ with the property

$$||X(t)x_0|| < \epsilon \tag{4.8}$$

for every $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta(\epsilon)$. For a fixed $t \in \mathbb{R}_+$ we get

$$\left[\frac{1}{\delta(\epsilon)}\right]||X(t)x_0|| = \left|\left|X(t)\left(\frac{x_0}{\delta(\epsilon)}\right)\right|\right| < \frac{\epsilon}{\delta(\epsilon)}. \tag{4.9}$$

Since $x_0/\delta(\epsilon)$ ranges over the interior of the unit ball, we obtain

$$||X(t)|| = \sup_{\|u\|<1} ||X(t)u|| \leq \frac{\epsilon}{\delta(\epsilon)}. \tag{4.10}$$

This completes the proof of the first case because (4.10) holds for arbitrary $t \in \mathbb{R}_+$.

Now, assume that (4.3) holds. Then (4.2) holds for some $K > 0$ and

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} X(t)x_0 = 0 \tag{4.11}$$

for any solution $x(t)$ of (S) with $x(0) = x_0$. Thus, (S) is asymptotically stable. Conversely, assume that (S) is asymptotically stable. Then there exists $\eta > 0$ such that $X(t)x_0 \to 0$ as $t \to \infty$ for every $x_0 \in \mathbb{R}^n$ with $\|x_0\| \le \eta$. Choose $x_0 = [\eta, 0, \dots, 0]^T$. Since $X(t)x_0 = \eta y(t)$, where $y(t)$ is the first column of $X(t)$, we obtain that every entry of the first column of $X(t)$ tends to zero as $t \to \infty$. Similarly one concludes that every entry of $X(t)$ tends to zero as $t \to \infty$. This completes the proof of this case.

In order to prove the third conclusion of the theorem, let (4.4) hold and let $t_0 \in \mathbb{R}_+$ be given. Then $x(t) = X(t)X^{-1}(t_0)x_0$ is the solution of (S) with $x(t_0) = x_0$. Thus,

$$||x(t)|| \le ||X(t)X^{-1}(t_0)||\,||x_0|| \le K||x_0|| \tag{4.12}$$

for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| < K^{-1}\epsilon$, proves the uniform stability of (S) with $\delta(\epsilon) = K^{-1}\epsilon$. Now, assume that (S) is uniformly stable. Fix $\epsilon > 0$, $\delta(\epsilon) > 0$ such that $\|X(t)X^{-1}(t_0)x_0\| < \epsilon$ for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta(\epsilon)$, any $t_0 \in \mathbb{R}_+$ and any $t \ge t_0$. From this point on, the proof follows as the sufficiency part of the first case and is therefore omitted.

In the fourth case, let (4.5) hold. Then (S) is uniformly stable by virtue of (4.4). Now, let $\epsilon \in (0, K)$, $t_0 \in \mathbb{R}_+$ be given, and let $x(t)$ be a solution of (S) with $\|x(t_0)\| = \|x_0\| < 1$. Then

$$||x(t)|| = ||X(t)X^{-1}(t_0)x_0|| < Ke^{-\alpha(t-t_0)} \le \epsilon \tag{4.13}$$

for every $t \ge t_0 + T(\epsilon)$, where $T(\epsilon) = -\alpha^{-1}\ln(\epsilon/K)$. Consequently, system (S) is uniformly asymptotically stable with the constant η of Definition 4.2 equal to 1.

Conversely, let (S) be uniformly asymptotically stable. Fix $\eta, \epsilon \in (0, \eta)$, $T = T(\epsilon)$ as in Definition 4.2. Then $\|x_0\| < \eta$ implies

$$||X(t)X^{-1}(t_0)x_0|| < \epsilon, \quad t \ge t_0 + T. \tag{4.14}$$

Thus, working as in the first case, we find

$$||X(t)X^{-1}(t_0)|| \le \mu < 1, \quad t \ge t_0 + T, \tag{4.15}$$

where $\mu = \epsilon/\eta$. Now, (4.5) implies the existence of a constant $K > 0$ such that

$$||X(t)X^{-1}(t_0)|| \le K, \quad t \ge t_0. \tag{4.16}$$

Given $t \geq t_0$, there exists an integer $m \geq 0$ such that $t_0 + mT \leq t < t_0 + (m+1)T$. This implies

$$\begin{aligned}||X(t)X^{-1}(t_0)|| &= ||X(t)X^{-1}(t_0+mT)X(t_0+mT)X^{-1}(t_0+(m-1)T)\\&\quad\cdot X(t_0+(m-1)T)\cdots X^{-1}(t_0+T)X(t_0+T)X^{-1}(t_0)||\\&\leq ||X(t)X^{-1}(t_0+mT)||\\&\quad\cdot ||X(t_0+mT)X^{-1}(t_0+(m-1)T)||\\&\quad\cdot\cdots\cdot ||X(t_0+2T)X^{-1}(t_0+T)||\,||X(t_0+T)X^{-1}(t_0)||\\&\leq K\mu^m.\end{aligned} \tag{4.17}$$

If we take $\alpha = -T^{-1}\ln\mu$, then

$$\begin{aligned}||X(t)X^{-1}(t_0)|| &\leq \mu^{-1}K\mu^{m+1} = \mu^{-1}Ke^{-(m+1)\alpha T}\\&< \mu^{-1}Ke^{-\alpha(t-t_0)}\end{aligned} \tag{4.18}$$

for every $t \geq t_0$. This finishes the proof of the case of uniform asymptotic stability.

Assume now that (4.6) holds. Given $\epsilon > 0$, choose $\delta(\epsilon) = K^{-2}\epsilon$. Then we have

$$||X(t)X^{-1}(t_0)x_0|| \leq ||X(t)||\,||X^{-1}(t_0)||\,||x_0|| \leq K^2||x_0|| < \epsilon \tag{4.19}$$

whenever $\|x_0\| < K^{-2}\epsilon$ and $t, t_0 \in \mathbb{R}_+$. Thus, system (S) is strongly stable. To show the converse, let (S) be strongly stable and fix $\epsilon > 0$, $\delta(\epsilon) > 0$ so that

$$||X(t)X^{-1}(t_0)x_0|| < \epsilon, \quad t, t_0 \in \mathbb{R}_+, \tag{4.20}$$

whenever $\|x_0\| < \delta(\epsilon)$. This implies that for arbitrary $t, t_0 \in \mathbb{R}_+$ we have

$$||X(t)x_0|| < \epsilon, \qquad ||X^{-1}(t_0)x_0|| < \epsilon \tag{4.21}$$

provided that $\|x_0\| < \delta(\epsilon)$. In fact, this follows from (4.20) if we take $t_0 = 0$, $t = 0$, respectively. Thus, as above,

$$||X(t)|| \leq \frac{\epsilon}{\delta(\epsilon)}, \qquad ||X^{-1}(t)|| \leq \frac{\epsilon}{\delta(\epsilon)}. \tag{4.22}$$

This says that (4.6) holds with $K = \epsilon/\delta(\epsilon)$. □

Before we consider system (S) with a constant matrix A, we should note that in the case of an *autonomous* system (i.e., $F(t,x) \equiv F(x)$) stability is equivalent to uniform stability and asymptotic stability is equivalent to uniform asymptotic stability. This is a consequence of the fact that in this case $y(t) \equiv x(t+\alpha)$ is a solution of (E) whenever $x(t)$ is a solution of the same equation. This is true for any number $\alpha \in \mathbb{R}$. Now, consider the system

$$x' = Ax \tag{4.23}$$

with $A \in M_n$. If λ is an eigenvalue of A, then the dimension of the eigenspace of λ (i.e., the subspace of $\mathbb{C}^n$ generated by the eigenvectors of A corresponding to λ) is called the *index* of λ. The following theorem characterizes the fundamental matrices of (4.23) (cf. Cole [8, pages 89, 90, 100]).

THEOREM 4.5. *Let $X(t) \equiv e^{tA}$ be the fundamental matrix of* (4.23) *with $X(0) = I$. Then every entry of $X(t)$ takes the form $e^{\alpha t}(p(t)\cos\beta t - q(t)\sin\beta t)$ or the form $e^{\alpha t}(p(t)\sin\beta t + q(t)\cos\beta t)$, where $\lambda = \alpha + \beta i$ is some eigenvalue of A and p, q are real polynomials in t. The degree d of the polynomial $p(t) + iq(t)$ lies in $[0, m-r]$, where m is the multiplicity of λ and r its index. Furthermore, if $m \neq r$, there is at least one entry of $X(t)$ such that $d \neq 0$.*

THEOREM 4.6. *System* (4.23) *is stable if and only if every eigenvalue of A that has multiplicity m equal to its index r has nonpositive real part, and every other eigenvalue has negative real part. The system* (4.23) *is asymptotically stable if and only if every eigenvalue of A has negative real part. It is strongly stable if and only if every eigenvalue of A is purely imaginary and has multiplicity equal to its index.*

PROOF. Let $X(t) \equiv e^{tA}$, $t \in \mathbb{R}_+$. Then (4.23) is stable if and only if $\|X(t)\| \leq K$ for all $t \in \mathbb{R}_+$, where K is a positive constant (see Theorem 4.4). Let $\lambda = \alpha + \beta i$ be an eigenvalue of A. Then every entry of $X(t)$ corresponding to λ will be bounded if and only if $\alpha \leq 0$ for $m = r$ and $\alpha < 0$ for $m > r$. This completes the proof of our first assertion. System (4.23) is asymptotically stable if and only if $e^{tA} \to 0$ as $t \to \infty$. This is of course possible if and only if every eigenvalue of A has negative real part. The system is strongly stable if and only if there exists a constant $K > 0$ such that $\|e^{tA}\| \leq K$ and $\|e^{-tA}\| \leq K$ for every $t \in \mathbb{R}_+$. Since e^{-tA} solves the system $X' = -AX$ and λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of $-A$, these inequalities can hold if and only if every eigenvalue of A has real part zero and $m = r$. □

3. THE MEASURE OF A MATRIX; FURTHER STABILITY CRITERIA

DEFINITION 4.7. Let $A \in M_n$. Then $\mu(A)$ denotes the *measure* of A which is defined by

$$\mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}. \tag{4.24}$$

THEOREM 4.8. *The measure $\mu(A)$ exists as a finite number for every $A \in M_n$.*

Proof. Let $\epsilon \in (0,1)$ be given and consider the function

$$g(h) = \frac{\|I + hA\| - 1}{h}, \quad h > 0. \tag{4.25}$$

We have

$$\|I + \epsilon hA\| = \big\|\epsilon(I + hA) + (1-\epsilon)I\big\| \le \epsilon\|I + hA\| + (1-\epsilon) \tag{4.26}$$

or

$$g(\epsilon h) = \frac{\|I + \epsilon hA\| - 1}{\epsilon h} \le \frac{\|I + hA\| - 1}{h} = g(h). \tag{4.27}$$

Thus, $g(h)$ is an increasing function of h. On the other hand,

$$\left|\frac{\|I + hA\| - 1}{h}\right| = \left|\frac{\|I + hA\| - \|I\|}{h}\right| \le \frac{\|I + hA - I\|}{h} = \|A\|. \tag{4.28}$$

This implies the existence of the limit of $g(h)$ as $h \to 0^+$. It follows that $\mu(A)$ exists and is finite. □

Theorem 4.9. *Let $A \in M_n$ be given. The $\mu(A)$ has the following properties:*

(i) $\mu(\alpha A) = \alpha\mu(A)$ *for all* $\alpha \in \mathbb{R}_+$;
(ii) $|\mu(A)| \le \|A\|$;
(iii) $\mu(A + B) \le \mu(A) + \mu(B)$;
(iv) $|\mu(A) - \mu(B)| \le \|A - B\|$.

Proof. Case (i) is trivial and (ii) follows from the fact that $|g(h)| \le \|A\|$ for all $h > 0$, where g is the function in the proof of Theorem 4.8. Inequality (iii) follows from

$$\begin{aligned}\frac{\big\|I + h(A+B)\big\| - 1}{h} &\le \frac{\big\|(1/2)I + hA\big\| - (1/2)}{h} + \frac{\big\|(1/2)I + hB\big\| - (1/2)}{h} \\ &= \frac{\|I + 2hA\| - 1}{2h} + \frac{\|I + 2hB\| - 1}{2h}.\end{aligned} \tag{4.29}$$

Inequality (iv) follows easily from (ii) and (iii). □

The following theorem establishes the relationship between the solutions of (S) and the measure of the matrix $A(t)$.

THEOREM 4.10. *Let $A : \mathbb{R}_+ \to M_n$ be continuous. Then for every $t_0, t \in \mathbb{R}_+$ with $t \geq t_0$ we have*

$$||x(t_0)|| \exp\left[-\int_{t_0}^{t} \mu(-A(s))\,ds\right] \leq ||x(t)|| \leq ||x(t_0)|| \exp\left[\int_{t_0}^{t} \mu(A(s))\,ds\right], \tag{4.30}$$

where $x(t)$ is any solution of (S).

Before we provide a proof of Theorem 4.10, we give the following auxiliary lemma.

LEMMA 4.11. *Let $r : [t_0, b) \to \mathbb{R}_+$ and $\phi : [t_0, b) \to \mathbb{R}$ $(0 \leq t_0 < b \leq +\infty)$ be continuous and such that*

$$r'_+(t) \leq \phi(t) r(t), \quad t \in [t_0, b), \tag{4.31}$$

where r'_+ denotes the right derivative of the function $r(t)$. Then $r(t) \leq u(t)$, $t \in [t_0, b)$, where $u(t)$ is the solution of

$$u' = \phi(t) u, \qquad u(t_0) = r(t_0). \tag{4.32}$$

PROOF. Let $t_1 \in (t_0, b)$ be arbitrary. We will show that $r(t) \leq u(t)$ on the interval $[t_0, t_1]$. Consider first the solution $u_n(t)$, $t \in [t_0, t_1]$, $n = 1, 2, \ldots$, of the problem

$$u' = \phi(t) u + \frac{1}{n}, \qquad u(t_0) = r(t_0). \tag{4.33}$$

Fix n and assume the existence of a point $t_2 \in (t_0, t_1)$ such that $r(t_2) > u_n(t_2)$. Then there exists $t_3 \in [t_0, t_2)$ such that $r(t_3) = u_n(t_3)$ and $r(t) > u_n(t)$ on $(t_3, t_2]$. From (4.33) we obtain

$$\begin{aligned} u'_n(t_3) &= \phi(t_3) u_n(t_3) + \frac{1}{n} \\ &= \phi(t_3) r(t_3) + \frac{1}{n} \\ &\geq r'_+(t_3) + \frac{1}{n} \\ &> r'_+(t_3). \end{aligned} \tag{4.34}$$

Consequently, $u_n(t) > r(t)$ in a small right neighborhood of the point t_3. This is a contradiction to the fact that $r(t) > u_n(t)$, $t \in (t_3, t_2]$. Thus, $r(t) \leq u_n(t)$ for any $t \in [t_0, t_1]$ and any $n = 1, 2, \ldots$. Now, we use Gronwall's inequality (Lemma 3.4) to show that (4.33) actually implies

$$|u_n(t) - u_m(t)| \leq 2t_1 \left|\frac{1}{m} - \frac{1}{n}\right| \exp\left[\int_{t_0}^{t_1} |\phi(s)|\,ds\right], \quad t \in [t_0, t_1], \tag{4.35}$$

for all $m, n \geq 1$. Thus, $\{u_n(t)\}$, $n = 1, 2, \ldots$, is a Cauchy sequence. It follows that $u_n(t) \to u(t)$ as $n \to \infty$ uniformly on $[t_0, t_1]$, where $u(t)$ is the solution of problem (4.32) on the interval $[t_0, b)$. Since t_1 is arbitrary, we have $r(t) \leq u(t)$, $t \in [t_0, b)$. □

It should be noted that a corresponding inequality holds if $r'_-(t) \geq \phi(t)r(t)$, where $r'_-(t)$ is the left derivative of $r(t)$ on $(t_0, b]$.

Proof of Theorem 4.10. Let $r(t) = \|x(t)\|$. We are planning to show that

$$r'_+(t) \leq \mu(A(t))r(t). \tag{4.36}$$

To this end, we first notice that for any two vectors $x_1, x_2 \in \mathbb{R}^n$, the limit

$$\lim_{h \to 0^+} \frac{||x_1 + hx_2|| - ||x_1||}{h} \tag{4.37}$$

exists as a finite number. To see this, it suffices to show that the function

$$g_1(h) = \frac{||x_1 + hx_2|| - ||x_1||}{h} \tag{4.38}$$

is increasing and bounded by $\|x_2\|$ on $(0, \infty)$. We omit the proof of these properties because we have already done this for the function $g(h)$ in the proof of Theorem 4.8. It follows that the limit

$$\lim_{h \to 0^+} \frac{||x(t) + hx'(t)|| - ||x(t)||}{h} \tag{4.39}$$

exists and is a finite number. We will show that this number equals $r'_+(t)$. In fact, let $h > 0$ be given. Then we have

$$\begin{aligned} &\left| \frac{||x(t+h)|| - ||x(t)||}{h} - \frac{||x(t) + hx'(t)|| - ||x(t)||}{h} \right| \\ &\quad = \left| \frac{||x(t+h)|| - ||x(t) + hx'(t)||}{h} \right| \\ &\quad \leq \frac{||x(t+h) - x(t) - hx'(t)||}{h} \longrightarrow 0 \end{aligned} \tag{4.40}$$

as $h \to 0^+$, which proves that $r'_+(t)$ equals the limit in (4.39). Consequently,

$$\begin{aligned} r'_+(t) &= \lim_{h \to 0^+} \frac{||x(t) + hA(t)x(t)|| - ||x(t)||}{h} \\ &\leq \lim_{h \to 0^+} \frac{||I + hA(t)|| - 1}{h} r(t) \\ &= \mu(A(t))r(t). \end{aligned} \tag{4.41}$$

Applying Lemma 4.11, we obtain

$$||x(t)|| = r(t) \le ||x(t_0)|| \exp\left[\int_{t_0}^{t} \mu(A(s))\,ds\right] \tag{4.42}$$

for every $t \ge t_0$. In order to find a corresponding lower bound of $\|x(t)\|$, let $u = -t$, $u_0 = -t_0$. Then $y(u) = x(-u)$, $u \in (-\infty, u_0]$, satisfies the system

$$y' = -A(-u)y. \tag{4.43}$$

Thus, as in (4.42), we get

$$\begin{aligned} ||y(u_0)|| &\le ||y(u)|| \exp\left[\int_{u}^{u_0} \mu(-A(-s))\,ds\right] \\ &= ||y(u)|| \exp\left[-\int_{-u}^{-u_0} \mu(-A(v))\,dv\right], \quad u_0 \ge u, \end{aligned} \tag{4.44}$$

or

$$||x(t_0)|| \le ||x(t)|| \exp\left[\int_{t_0}^{t} \mu(-A(s))\,ds\right], \quad t \ge t_0. \tag{4.45}$$

This completes the proof. □

We are now ready for the main theorem of this section.

Theorem 4.12. *Consider system* (S) *with* $A : \mathbb{R}_+ \to M_n$ *continuous. If*

$$\liminf_{t\to\infty} \int_0^t \mu(-A(s))\,ds = -\infty, \tag{4.46}$$

then (S) *is unstable. If*

$$\limsup_{t\to\infty} \int_0^t \mu(A(s))\,ds < +\infty, \tag{4.47}$$

then (S) *is stable. If*

$$\lim_{t\to\infty} \int_0^t \mu(A(s))\,ds = -\infty, \tag{4.48}$$

then (S) *is asymptotically stable. If*

$$\mu(A(t)) \le 0, \quad t \ge 0, \tag{4.49}$$

then (S) *is uniformly stable. If, for some* $r > 0$,

$$\mu(A(t)) \le -r, \quad t \ge 0, \tag{4.50}$$

then (S) *is uniformly asymptotically stable.*

Table 1 provides formulas for $\mu(A)$ corresponding to the three different norms of Example 1.5.

TABLE 1

$\|x\|$	$\mu(A)$
$\|x\|_1$	largest eigenvalue of $(1/2)(A+A^T)$
$\|x\|_2$	$\max_i\{a_{ii}+\sum_{j,j\neq i}\|a_{ij}\|\}$
$\|x\|_3$	$\max_j\{a_{jj}+\sum_{i,i\neq j}\|a_{ij}\|\}$

4. PERTURBED LINEAR SYSTEMS

In this section, we study the stability of systems of the form

$$x' = A(t)x + F(t,x), \tag{S_F}$$

where $A : \mathbb{R}_+ \to M_n$ and $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions with $F(t,0) \equiv 0$, $t \in \mathbb{R}_+$. We start with a theorem concerning the asymptotic stability of (S_F). The proof of this theorem is based on Lemma 4.13 below.

LEMMA 4.13. *Let $X(t)$ be a fundamental matrix of the system* (S). *Assume further that there exists a constant $K > 0$ such that*

$$\int_0^t ||X(t)X^{-1}(s)||ds \le K, \quad t \ge 0. \tag{4.51}$$

Then there exists a constant $M > 0$ such that

$$||X(t)|| \le Me^{-K^{-1}t}, \quad t \ge 0. \tag{4.52}$$

PROOF. Let $u(t) \equiv \|X(t)\|^{-1}$. Then we have

$$\left(\int_0^t u(s)ds\right)X(t) = \int_0^t u(s)X(t)X^{-1}(s)X(s)ds, \tag{4.53}$$

from which we obtain

$$\int_0^t u(s)ds||X(t)|| \le \int_0^t ||X(t)X^{-1}(s)||||X(s)||u(s)ds \le K, \quad t \ge 0, \tag{4.54}$$

or

$$u(t) \ge K^{-1}\int_0^t u(s)ds. \tag{4.55}$$

Now, let $\lambda(t)$ denote the integral on the right-hand side of (4.55). Then we have

$$\lambda'(t) \geq K^{-1}\lambda(t), \quad t \geq 0. \tag{4.56}$$

Dividing (4.56) by $\lambda(t)$ and integrating from $t_0 > 0$ to $t \geq t_0$, we obtain

$$\lambda(t) \geq \lambda(t_0)e^{K^{-1}(t-t_0)}, \quad t \geq t_0. \tag{4.57}$$

Consequently,

$$||X(t)|| = [u(t)]^{-1} \leq K[\lambda(t)]^{-1} \leq \left[\frac{K}{\lambda(t_0)}\right]e^{-K^{-1}(t-t_0)} \tag{4.58}$$

for every $t \geq t_0$. We choose M so large that

$$\begin{gathered} M \geq \left[\frac{K}{\lambda(t_0)}\right]e^{K^{-1}t_0}, \\ ||X(t)|| \leq Me^{-K^{-1}t_0}, \quad 0 \leq t \leq t_0. \end{gathered} \tag{4.59}$$

This completes the proof. □

Theorem 4.14. *Let $X(t)$ be a fundamental matrix of system* (S) *such that*

$$\int_0^t ||X(t)X^{-1}(s)||ds \leq K, \quad t \geq 0. \tag{4.60}$$

Moreover, let

$$||F(t,x)|| \leq \mu\|x\|, \quad t \geq 0, \tag{4.61}$$

with $\mu \in [0, K^{-1})$. Then the zero solution of (S_F) *is asymptotically stable.*

Proof. Let $X(t)$ be the fundamental matrix of (S) with $X(0) = I$. Then since $X(t)X^{-1}(s) = Y(t)Y^{-1}(s)$ for any other fundamental matrix $Y(t)$ of (S), it follows that our assumed inequality on $X(t)$ holds for this particular fundamental matrix and so does Lemma 4.13. Thus, $X(t) \to 0$ as $t \to \infty$. If $x(t)$ is a local solution of (S_F) defined to the right of $t = 0$, then $x(t)$ satisfies the system

$$u' = A(t)u + F(t, x(t)). \tag{4.62}$$

Using the variation of constants formula (3.47) for this system, we obtain

$$x(t) = X(t)x(0) + \int_0^t X(t)X^{-1}(s)F(s, x(s))ds. \tag{4.63}$$

Letting $L > 0$ be such that $\|X(t)\| \leq L$ for $t \geq 0$, we obtain

$$||x(t)|| \leq L||x(0)|| + \mu K \max_{0 \leq s \leq t} ||x(s)||, \tag{4.64}$$

which implies

$$\max_{0 \leq s \leq t} ||x(s)|| \leq (1 - \mu K)^{-1} L||x(0)||. \tag{4.65}$$

It follows that

$$||x(t)|| \leq (1 - \mu K)^{-1} L||x(0)|| \tag{4.66}$$

as long as $x(t)$ is defined. This implies that $x(t)$ is continuable to $+\infty$ (see Theorem 3.8), and that the zero solution is stable. Now, we show that $x(t) \to 0$ as $t \to \infty$. To this end, let

$$c = \limsup_{t \to \infty} ||x(t)|| \tag{4.67}$$

and pick $d \in (\mu K, 1)$. We are going to show that $c = 0$. Assume that $c > 0$. Then, since $d^{-1}c > c$, there exists $t_0 \geq 0$ such that

$$||x(t)|| \leq d^{-1}c \tag{4.68}$$

for every $t \geq t_0$. Thus, (4.63) implies

$$\begin{aligned}
||x(t)|| &= \left|\left| X(t)x(0) + X(t)\int_0^t X^{-1}(s)F(s, x(s))\,ds \right|\right| \\
&= \left|\left| X(t)x(0) + X(t)\int_0^{t_0} X^{-1}(s)F(s, x(s))\,ds \right.\right. \\
&\qquad \left.\left. + X(t)\int_{t_0}^t X^{-1}(s)F(s, x(s))\,ds \right|\right| \\
&\leq ||X(t)||\,||x(0)|| + ||X(t)|| \int_0^{t_0} ||X^{-1}(s)F(s, x(s))||\,ds \\
&\qquad + \int_{t_0}^t ||X(t)X^{-1}(s)||\,||F(s, x(s))||\,ds \\
&\leq ||X(t)||\,||x(0)|| + ||X(t)|| \int_0^{t_0} ||X^{-1}(s)F(s, x(s))||\,ds + \mu K d^{-1}c.
\end{aligned} \tag{4.69}$$

Taking the lim sup above as $t \to \infty$, we obtain $c \leq \mu K d^{-1}c$, that is, a contradiction. Thus, $c = 0$ and the proof is finished. □

Theorem 4.15. *Let $X(t)$ be a fundamental matrix of system* (S) *such that*

$$||X(t)X^{-1}(s)|| \le K, \quad t \ge s \ge 0, \tag{4.70}$$

where K is a positive constant. Moreover, let

$$||F(t,x)|| \le \lambda(t)\|x\|, \tag{4.71}$$

where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that

$$\int_0^\infty \lambda(t)dt < +\infty. \tag{4.72}$$

Let

$$M = K \exp\left\{K\int_0^\infty \lambda(t)dt\right\}. \tag{4.73}$$

Then every local solution $x(t)$ of (S_F)*, defined to the right of the point $t_0 \ge 0$, is continuable to $+\infty$ and satisfies*

$$||x(t)|| \le M||x(t_0)|| \tag{4.74}$$

for every $t \ge t_0$.

Proof. From the variation of constants formula (3.47) we have

$$||x(t)|| \le K||x(t_0)|| + K\int_{t_0}^t \lambda(s)||x(s)||ds \tag{4.75}$$

for all $t \ge t_0$. Applying Gronwall's inequality (Lemma 3.4), we obtain

$$||x(t)|| \le K||x(t_0)|| \exp\left\{K\int_{t_0}^t \lambda(s)ds\right\} \le M||x(t_0)|| \tag{4.76}$$

for $t \ge t_0$. Consequently, by Theorem 3.8, $x(t)$ is continuable to $+\infty$ and (4.76) holds for every $t \ge t_0$. □

Corollary 4.16. *If system* (S) *is uniformly stable and F is as in Theorem 4.15, then the zero solution of* (S_F) *is uniformly stable. In particular, the uniform stability of* (S) *implies the uniform stability of the system*

$$x' = [A(t) + B(t)]x, \tag{4.77}$$

where $B : \mathbb{R}_+ \to M_n$ is continuous and such that

$$\int_0^\infty ||B(t)||dt < +\infty. \tag{4.78}$$

The corollary to Theorem 4.17 below shows that uniform asymptotic stability of linear systems (S) is maintained under the effect of small perturbations $F(t,x)$.

THEOREM 4.17. *Let $X(t)$ be a fundamental matrix of* (S) *such that*

$$||X(t)X^{-1}(s)|| \le Ke^{-\mu(t-s)}, \quad t \ge s \ge 0, \tag{4.79}$$

where K and μ are positive constants. Let

$$||F(t,x)|| \le \lambda\|x\| \tag{4.80}$$

with $\lambda \in (0, K^{-1}\mu)$. Then if $c = \mu - \lambda K$, every solution $x(t)$ of (S_F)*, defined on a right neighborhood of $t_0 \ge 0$, exists for all $t \ge t_0$ and satisfies*

$$||x(t)|| \le Ke^{-c(t-s)}||x(s)|| \tag{4.81}$$

for every t, s with $t \ge s \ge t_0$.

PROOF. From the variation of constants formula,

$$x(t) = X(t)X^{-1}(t_0)x(t_0) + \int_{t_0}^{t} X(t)X^{-1}(s)F(s,x(s))\,ds, \tag{4.82}$$

in a right neighborhood of the point $t_0 \ge 0$, we obtain

$$||x(t)|| \le Ke^{-\mu(t-t_0)}||x(t_0)|| + \lambda K \int_{t_0}^{t} e^{-\mu(t-s)}||x(s)||\,ds, \quad t \ge t_0. \tag{4.83}$$

Letting $z(t) \equiv e^{\mu(t-t_0)}\|x(t)\|$, we have

$$z(t) \le Kz(t_0) + \lambda K \int_{t_0}^{t} z(s)\,ds, \quad t \ge t_0. \tag{4.84}$$

An application of Gronwall's inequality (Lemma 3.4) yields

$$\begin{aligned} z(t) &\le Kz(t_0)e^{\lambda K(t-t_0)}, \quad t \ge t_0, \\ ||x(t)|| &\le K||x(t_0)||e^{-c(t-t_0)}. \end{aligned} \tag{4.85}$$

Obviously, $x(t)$ is continuable to $+\infty$ (see Theorem 3.8). The conclusion of the theorem for points t, s with $t \ge s \ge t_0$, follows exactly as above. □

COROLLARY 4.18. *If* (S) *is uniformly asymptotically stable and if* (4.80) *holds for a sufficiently small $\lambda > 0$, then the zero solution of* (S_F) *is also uniformly asymptotically stable. In particular, the uniform asymptotic stability of system* (S) *implies the same property for system* (4.77)*, where $B : \mathbb{R}_+ \to M_n$ is continuous and such that $B(t) \to 0$ as $t \to \infty$.*

In the second part of Corollary 4.18 we use an interval $[t_1, \infty)$, for a sufficiently large $t_1 \ge 0$, instead of the interval $[0, \infty)$ of the definitions of stability.

EXERCISES

4.1. Consider the scalar equation

$$x' = b(t)x, \tag{4.86}$$

where $b : \mathbb{R}_+ \to \mathbb{R}$ is continuous. Show that (4.86) is asymptotically stable if and only if

$$\int_0^\infty b(t)dt = -\infty. \tag{4.87}$$

4.2. Prove Theorem 4.12.

4.3. Show that if

$$x' = A(t)x, \tag{S}$$

with $A : \mathbb{R}_+ \to M_n$ continuous, is stable, and if

$$\int_0^t \operatorname{tr} A(s)ds \geq m, \quad t \geq 0, \tag{4.88}$$

with m constant, then (S) is strongly stable. Hint. Use the Liouville-Jacobi Formula of Exercise 3.6 and the fact that $X^{-1}(t) \equiv \widetilde{X}(t)/\det X(t)$, where $\widetilde{X}(t)$ is the matrix whose (i, j)th entry is the cofactor of the (j, i)th entry of $X(t)$.

4.4. Consider System (S_F) with $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ continuous. Furthermore, assume the existence of a positive constant c such that $\|F(t,x)\| \leq \lambda(t)\|x\|$ for every $x \in \mathbb{R}^n$ with $\|x\| \leq c$, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that

$$\int_0^\infty \lambda(t)dt = d < +\infty. \tag{4.89}$$

Then the uniform stability of the homogeneous system (S), assuming (4.4), implies the stability of the zero solution of system (S_F). Hint. Show that if $\|x_0\| < \mu$, where

$$\mu = \min\{c, cK^{-1}e^{-dK}\}, \tag{4.90}$$

then $\|x(t)\| < c$ as long as $x(t)$ exists. Here, $x(t)$ is a solution of (S_F) with $x(0) = x_0$. To this end, assume that there is a point $\bar{t} \in (0, \infty)$ such that $\|x(\bar{t})\| = c$ and $\|x(t)\| < c$ on $[0, \bar{t})$. Then use the growth condition on F along with Gronwall's inequality to obtain the contradiction: $\|x(t)\| \leq K\|x_0\|e^{dK} < c$ on $[0, \bar{t}]$. This last inequality on $[0, \infty)$ implies the stability of the zero solution of (S_F).

4.5. Consider the scalar equation

$$x'' + x = (1 + t^3)^{-1} |x|^{\alpha} \operatorname{sgn} x, \quad t \geq 0, \tag{4.91}$$

where $\alpha > 1$ is a constant. Show that the system corresponding to this equation has its zero solution stable. Hint. Apply the result of Exercise 4.4.

4.6. Consider the "autonomous" system $x' = F(x)$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and such that $F(0) = 0$. Show that the stability of the zero solution of this system is equivalent to its uniform stability.

4.7. How small should $|m|$ be (m constant) so that the zero solution of the equation

$$x'' + 2x' + x = \left[\frac{m}{(1 + t^2)} \right] |x|, \tag{4.92}$$

written as a system, is asymptotically stable?

4.8. Let (S) be uniformly asymptotically stable. Let

$$\int_t^{t+1} ||f(s)|| ds \leq C, \quad t \geq 0, \tag{4.93}$$

where $f : \mathbb{R}_+ \to \mathbb{R}^n$ is continuous and C is a positive constant. Show that every solution of (S_f) is bounded. Hint. Show first that (4.93) implies

$$e^{-\alpha t} \int_0^t e^{\alpha s} ||f(s)|| ds \leq C (1 - e^{-\alpha})^{-1}, \quad t \geq 0, \tag{4.94}$$

for any $\alpha > 0$. In fact,

$$e^{-\alpha t} \int_0^t e^{\alpha s} ||f(s)|| ds \leq e^{-\alpha t} \sum_{j=0}^{[t]} e^{\alpha(t-j)} C, \tag{4.95}$$

where $[t]$ is the greatest integer function at t.

4.9. Consider the system (S) with $A : \mathbb{R}_+ \to M_n$ continuous. Let $\lim_{t \to \infty} A(t) = A_0$. Then, assuming "smallness" conditions on

$$||A(t) - A_0|| \qquad \text{or} \qquad \int_0^{\infty} ||A(t) - A_0|| dt, \tag{4.96}$$

obtain stability properties for (S) based on those of $x' = A_0 x$.

4.10. Let $A_0 \in M_n$ and let $A : \mathbb{R}_+ \to M_n$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ be continuous. Show that if all solutions of $x' = A_0 x$ are bounded, then all solutions of (S_f) are bounded, provided that

$$\int_0^\infty \|A(t) - A_0\| dt < +\infty, \qquad \int_0^\infty \|f(t)\| dt < +\infty. \tag{4.97}$$

4.11. Consider (S_f) with $A : \mathbb{R}_+ \to M_n$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ continuous. Assume that system (S) is uniformly asymptotically stable. Show that if $\|f(t)\| \to 0$ as $t \to \infty$, then every solution of (S_f) tends to zero as $t \to \infty$.

4.12. Let $A \in M_n$ and $F : R_+ \times R^n \to R^n$ be continuous in (S_F). Assume that the system (S) is asymptotically stable and that

$$\lim_{\|u\| \to 0} \frac{\|F(t,u)\|}{\|u\|} = 0, \quad \text{uniformly w.r.t. } t \in \mathbb{R}_+. \tag{4.98}$$

Show that the zero solution of (S_F) is asymptotically stable.

4.13. Assume that $X(t)$ satisfies (4.6) and that F is as in Theorem 4.15. Let t_0 be an arbitrary point in $\mathbb{R}_+$. Show that every solution $x(t)$ of (S_F), defined on a right neighborhood of t_0, is continuable to $+\infty$ and satisfies

$$\|x(t)\| \le M \|x(t_0)\|, \quad t \in [t_0, \infty), \tag{4.99}$$

where M is a positive constant independent of the solution $x(t)$. Then show that the zero solution of (S_F) is strongly stable. Hint. Use the variation of constants formula to get (4.99) for $M = LK^2$, where L is the constant determined below. Then differentiate the function $q(t) = \|X^{-1}(t)x(t)\|$ from the left, by using

$$|q(t) - q(t-h)| \le K^2 \int_{t-h}^t \lambda(s) q(s) ds, \quad 0 < t-h < t, \tag{4.100}$$

to obtain locally

$$-K^2 \lambda(t) q(t) \le q'_-(t) \le K^2 \lambda(t) q(t). \tag{4.101}$$

Use a suitable version of Lemma 4.11 to obtain integral inequalities as in Theorem 4.10:

$$q(t_1) \exp\left\{ -K^2 \int_{t_1}^{t_2} \lambda(t) dt \right\} \le q(t_2) \le q(t_1) \exp\left\{ K^2 \int_{t_1}^{t_2} \lambda(t) dt \right\} \tag{4.102}$$

for $t_2 \ge t_1 \ge t_0$. Show that

$$\|X^{-1}(t)x(t)\| \le L \|X^{-1}(s)x(s)\|, \quad t, s \ge t_0, \tag{4.103}$$

where

$$L = \exp\left\{K^2 \int_{t_0}^{\infty} \lambda(t)dt\right\}. \tag{4.104}$$

4.14. Using Exercise 4.13, show that the strong stability of the system (S) implies the strong stability of (4.77), where $B : \mathbb{R}_+ \to M_n$ is continuous and such that

$$\int_0^{\infty} ||B(t)||dt < +\infty. \tag{4.105}$$

4.15. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable with $F(0) = 0$. Assume that the system

$$x' = F'(0)x \tag{4.106}$$

is asymptotically stable, where $F'(0)$ is the Jacobian matrix of F at 0. Show that the zero solution of the system

$$x' = F(x) \tag{4.107}$$

is asymptotically stable. How would you extend this result to time-dependent systems of the type $x' = G(t,x)$ and Jacobian matrices $G_x(t,0)$? Hint. Prove that

$$F(x) = F'(0)x + W(x), \tag{4.108}$$

where $W(x)/\|x\| \to 0$ as $x \to 0$. Then use Exercise 4.12.

4.16. Show that $\mu(A(t))$ is a continuous function of t for any continuous $A : J \to M_n$, where J is an interval of $\mathbb{R}$.

CHAPTER 5

LYAPUNOV FUNCTIONS IN THE THEORY OF DIFFERENTIAL SYSTEMS; THE COMPARISON PRINCIPLE

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $\lim_{\|x\| \to \infty} V(x) = +\infty$, and let $x(t)$ be a solution of the system

$$x' = F(t, x) \tag{E}$$

defined on $\mathbb{R}_+$. In order to show that $x(t)$ is bounded on $\mathbb{R}_+$, it suffices to show that the function $V(x(t))$ is bounded above on $\mathbb{R}_+$. In fact, let $V(x(t)) \leq K$, $t \in \mathbb{R}_+$, where K is a positive constant, and assume that $\|x(t_m)\| \to +\infty$ as $m \to \infty$, for some sequence $\{t_m\}_{m=1}^{\infty}$. Then $V(x(t_m)) \to +\infty$, which is a contradiction. It follows that $x(t)$ is bounded.

The main observation here is that local or global information about the solutions of (E) may be obtained from certain scalar functions which, in some sense, are associated with system (E). The study of this situation is the subject of this chapter.

In Section 1, we introduce the concept of such *Lyapunov functions V* and show their main connection with system (E). Section 2 establishes the existence of *maximal* and *minimal* solutions of first-order scalar problems

$$u' = \gamma(t, u), \qquad u(t_0) = u_0, \tag{5.1}$$

and their relationships to system (E) by means of

$$V_E'(t, u) \leq \gamma(t, V(t, u)), \qquad V(t_0, u_0) \leq u_0, \tag{5.2}$$

where V is a Lyapunov function associated with (E) and V_E' is a certain derivative of V *along the system* (E). This inequality, which provides information about

$V(t,x(t))$, is the main ingredient of the well-known *comparison principle*. Naturally, the scalar function $\gamma(t,u)$ above is intimately related to system (E). Such a relationship could be given, for example, by an inequality of the type

$$||F(t,u)|| \leq \gamma(t,\|u\|) \tag{5.3}$$

on a subset of $\mathbb{R}_+ \times \mathbb{R}^n$.

An application of these considerations to an existence theorem on $\mathbb{R}_+$ is given in Section 3, and Section 4 concerns itself with stability properties of the zero solution of (E).

1. LYAPUNOV FUNCTIONS

Definition 5.1. Let $S = \mathbb{R}_+ \times \mathbb{R}^n$ and let $V : S \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition w.r.t. its second variable in every compact subset of S. This means that if $K \subset S$ is compact, then there exists a constant $L_K > 0$ such that

$$||V(t,u_1) - V(t,u_2)|| \leq L_K||u_1 - u_2|| \tag{5.4}$$

for every $(t,u_1),(t,u_2) \in K$. Then V is called a *Lyapunov function*.

In what follows, the function $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ in equation (E) will be assumed continuous. Let $V(t,u)$ be a Lyapunov function. We define V_E' as follows:

$$V_E'(t,u) = \limsup_{h\to 0^+} \frac{V(t+h,u+hF(t,u)) - V(t,u)}{h}. \tag{5.5}$$

If $x(t)$ is a solution of (E) such that $(t,x(t)) \in S$, $t \in [a,b)$ $(0 \leq a < b \leq +\infty)$, then we define

$$V^+(t,x(t)) = \limsup_{h\to 0^+} \frac{V(t+h,x(t+h)) - V(t,x(t))}{h} \tag{5.6}$$

for every $t \in [a,b)$. In our argument below, the point $t \in \mathbb{R}_+$ is fixed. Moreover, h is a sufficiently small positive number, and L is a (local) Lipschitz constant for the Lyapunov function V. We first note that

$$\frac{x(t+h) - x(t)}{h} - F(t,x(t)) = \epsilon(h), \tag{5.7}$$

where $\epsilon(h) \to 0$ as $h \to 0^+$. Since

$$|V(t+h,x(t)+hF(t,x(t))+h\epsilon(h)) - V(t+h,x(t)+hF(t,x(t)))| \leq Lh||\epsilon(h)||, \tag{5.8}$$

it follows that

$$
\begin{aligned}
&V(t+h, x(t+h)) - V(t, x(t)) \\
&\qquad = V(t+h, x(t) + hF(t, x(t)) + h\epsilon(h)) - V(t, x(t)) \\
&\qquad \le V(t+h, x(t) + hF(t, x(t))) + Lh\|\epsilon(h)\| - V(t, x(t)).
\end{aligned}
\tag{5.9}
$$

Dividing above by $h > 0$ and then taking the lim sup as $h \to 0^+$, we obtain

$$
V^+(t, x(t)) \le V'_E(t, x(t)). \tag{5.10}
$$

Similarly, we have

$$
\begin{aligned}
V(t+h, x(t+h)) - V(t, x(t)) \ge V(t+h, x(t) + hF(t, x(t))) \\
- Lh\|\epsilon(h)\| - V(t, x(t)),
\end{aligned}
\tag{5.11}
$$

which yields

$$
V'_E(t, x(t)) \le V^+(t, x(t)). \tag{5.12}
$$

Thus,

$$
V'_E(t, x(t)) = V^+(t, x(t)) \tag{5.13}
$$

at all points $t \in [a, b)$. Evidently, we also have

$$
\begin{aligned}
&\liminf_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \\
&\qquad = \liminf_{h \to 0^+} \frac{V(t+h, x(t) + hF(t, x(t))) - V(t, x(t))}{h}.
\end{aligned}
\tag{5.14}
$$

We have shown the following lemma.

LEMMA 5.2. *Let $S = \mathbb{R}_+ \times \mathbb{R}^n$ and let $(t, x(t)) \in S$ for every $t \in [a, b)$ ($0 \le a < b \le +\infty$), where $x(t)$ is a solution of system* (E). *Assume further that V is a Lyapunov function. Then*

$$
V'_E(t, x(t)) = V^+(t, x(t)) \tag{5.15}
$$

for every $t \in [a, b)$.

Naturally, if V is differentiable, then

$$
\begin{aligned}
V^+(t, x(t)) = \frac{d}{dt} V(t, x(t)) = V_t(t, x(t)) \\
+ \langle \nabla V(t, x(t)), F(t, x(t)) \rangle,
\end{aligned}
\tag{5.16}
$$

where ∇V denotes the gradient of $V(t, u)$ w.r.t. u.

2. MAXIMAL AND MINIMAL SOLUTIONS; THE COMPARISON PRINCIPLE

The four Dini derivatives of a function $u(t)$ are defined as follows:

$$\begin{aligned} D^{+}u(t) &= \limsup_{h\to 0^{+}} \frac{u(t+h)-u(t)}{h}, \\ D_{+}u(t) &= \liminf_{h\to 0^{+}} \frac{u(t+h)-u(t)}{h}, \\ D^{-}u(t) &= \limsup_{h\to 0^{-}} \frac{u(t+h)-u(t)}{h}, \\ D_{-}u(t) &= \liminf_{h\to 0^{-}} \frac{u(t+h)-u(t)}{h}. \end{aligned} \tag{5.17}$$

These functions may assume the values $+\infty$ and $-\infty$.

Definition 5.3. Consider the equation

$$u' = \gamma(t,u), \tag{5.18}$$

where γ is a continuous real-valued function defined on a suitable subset of $\mathbb{R}_{+}\times\mathbb{R}$. Let $u(t)$ be a solution of (5.18) defined on an interval $I = [t_0, t_0+a)$ $(0 < a \le +\infty)$ or $I = [t_0, t_0+a]$ with $t_0 \ge 0$. Then $u(t)$ is said to be a *maximal solution* of (5.18) on I, if for any other solution $v(t)$ of (5.18) with $v(t_0) = u(t_0)$ and domain $I_1 \subset I$ we have

$$u(t) \ge v(t), \quad t \in I_1. \tag{5.19}$$

Similarly one defines a *minimal solution*.

Obviously, maximal and minimal solutions are unique w.r.t. their initial conditions. We are planning to show that there are always local maximal and minimal solutions of (5.18) with any appropriate initial conditions. Before we state the relevant result, we need the following auxiliary theorem.

Theorem 5.4. *Let $M \subset \mathbb{R}^2$ be open and let $\gamma : \bar{M} \to \mathbb{R}$ be a continuous function. Let $v, w : [t_0, t_0+a) \to \mathbb{R}$ $(0 < a \le +\infty)$ be continuous and such that $(t, v(t)), (t, w(t)) \in \bar{M}$ for all $t \in [t_0, t_0+a)$. Furthermore, assume that $v(t_0) < w(t_0)$ and*

$$\begin{aligned} D_{-}v(t) &< \gamma(t, v(t)), \\ D_{-}w(t) &\ge \gamma(t, w(t)) \end{aligned} \tag{5.20}$$

for every $t \in (t_0, t_0+a)$. Then

$$v(t) < w(t), \quad t \in [t_0, t_0+a). \tag{5.21}$$

Proof. Assume that the conclusion is not true. Then

$$N = \{t \in [t_0, t_0 + a) : v(t) \geq w(t)\} \neq \emptyset. \tag{5.22}$$

Let $t_1 = \inf\{t : t \in N\}$. By the continuity of v, w, and the inequality $v(t_0) < w(t_0)$, we have $t_1 > t_0$, $v(t_1) = w(t_1)$, and

$$v(t) < w(t), \quad t \in [t_0, t_1). \tag{5.23}$$

Thus, for $h < 0$ and $-h$ sufficiently small, we have

$$\frac{v(t_1 + h) - v(t_1)}{h} > \frac{w(t_1 + h) - w(t_1)}{h}, \tag{5.24}$$

which yields $D_- v(t_1) \geq D_- w(t_1)$. This and (5.20) imply $\gamma(t_1, v(t_1)) > \gamma(t_1, w(t_1))$, that is, a contradiction to $v(t_1) = w(t_1)$. It follows that N is the empty set. □

Theorem 5.5 (existence of maximal and minimal solutions). *Let $D = [t_0, t_0 + a] \times \{u \in \mathbb{R} : |u - u_0| \leq b\}$, where $t_0 \geq 0$, $u_0 \in \mathbb{R}$ and $b > 0$ are fixed. Let $\gamma : D \to \mathbb{R}$ be continuous and such that $|\gamma(t, u)| \leq K$ (K constant) for every $(t, u) \in D$. Then the scalar problem*

$$u' = \gamma(t, u), \qquad u(t_0) = u_0, \tag{*}$$

has a maximal and a minimal solution on $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, b/(b + 2K)\}$.

Above, and in what follows, a maximal (minimal) solution of (*) is just a maximal (minimal) solution $u(t)$ of (5.18) with initial condition $u(t_0) = u_0$.

Proof of Theorem 5.5. We will only be concerned with the existence of a maximal solution. Dual arguments cover the existence of a minimal solution. Let $\epsilon \in (0, b/2]$ and consider the scalar problem

$$u' = \gamma(t, u) + \epsilon, \qquad u(t_0) = u_0 + \epsilon. \tag{5.25$_\epsilon$}$$

We notice that the function $\gamma(t, u) + \epsilon$ is continuous on the set $D_\epsilon = [t_0, t_0 + a] \times \{u \in \mathbb{R} : |u - (u_0 + \epsilon)| \leq b/2\}$. We also notice that $D_\epsilon \subset D$ and $|\gamma(t, u) + \epsilon| \leq K + (b/2)$ on D_ϵ. Consequently, the Peano theorem (Theorem 3.1) ensures the existence of a solution $u_\epsilon(t)$ of the problem $(5.25)_\epsilon$ which exists on $[t_0, t_0 + \alpha]$. The boundedness of $\gamma(t, u)$ on D implies the equicontinuity of the family of functions $u_\epsilon(t), \epsilon \in (0, b/2]$, $t \in [t_0, t_0 + \alpha]$. Since these functions are also uniformly bounded, we may choose a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ and $s(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[t_0, t_0 + \alpha]$ (see Theorem 2.5). Letting $n \to \infty$ in

$$u_{\epsilon_n}(t) = u_0 + \epsilon_n + \int_{t_0}^{t} [\gamma(s, u_{\epsilon_n}(s)) + \epsilon_n] ds \tag{5.26}$$

and taking into consideration the fact that $\gamma(t, u_{\epsilon_n}(t)) \to \gamma(t, s(t))$ uniformly on $[t_0, t_0 + \alpha]$, we obtain that $s(t)$ is a solution of $(^*)$ on the interval $[t_0, t_0 + \alpha]$. To show that $s(t)$ is actually the maximal solution of $(^*)$ on $[t_0, t_0 + \alpha]$, let $u(t)$ be any other solution of $(^*)$ on $[t_0, t_0 + \alpha']$, $\alpha' \le \alpha$. Then we have

$$\begin{aligned} u(t_0) &= u_0 < u_0 + \epsilon = u_\epsilon(t_0), \\ u'(t) &< \gamma(t, u(t)) + \epsilon, \\ u'_\epsilon(t) &= \gamma(t, u_\epsilon(t)) + \epsilon \end{aligned} \tag{5.27}$$

for every $t \in [t_0, t_0 + \alpha']$, $\epsilon \in (0, b/2]$. Theorem 5.4 applies now to obtain $u(t) < u_\epsilon(t)$, $t \in [t_0, t_0 + \alpha']$. This of course implies that $u(t) \le s(t)$ on $[t_0, t_0 + \alpha']$ and completes the proof. □

It is well known that a continuous function $f : [a, b] \to \mathbb{R}$ is decreasing on $[a, b]$ if and only if $D_- f(x) \le 0$ ($D_+ f(x) \le 0$) for all $x \in (a, b)$. Similarly, f is increasing on $[a, b]$ if and only if $D^+ f(x) \ge 0$ ($D^- f(x) \ge 0$) for all $x \in (a, b)$. The following three lemmas will be used in the proof of the *Comparison theorem* (Theorem 5.10). The letter D denotes any one of the Dini derivatives.

Lemma 5.6. *Let $u : [t_0, t_0 + a] \to \mathbb{R}$, $t_0 \ge 0$, be continuous and such that $Du(t) \le 0$ for $t \in [t_0, t_0 + a] \setminus S$, where S is a countable set. Then $u(t)$ is decreasing on $[t_0, t_0 + a]$.*

The proof of this lemma follows easily from the proof of Theorem 34.1 in McShane [41, page 200].

Lemma 5.7. *Let $v, w : [t_0, t_0 + a] \to \mathbb{R}$, $t_0 \ge 0$, be continuous and such that $Dv(t) \le w(t)$ for every $t \in [t_0, t_0 + a] \setminus S$, where S is a countable set. Then $D_- v(t) \le w(t)$ for every $t \in (t_0, t_0 + a)$.*

Proof. Let

$$y(t) = v(t) - \int_{t_0}^{t} w(s)\,ds, \quad t \in [t_0, t_0 + a]. \tag{5.28}$$

Then $Dy(t) = Dv(t) - w(t) \le 0$, $t \in [t_0, t_0 + a] \setminus S$. Consequently, Lemma 5.6 implies that $y(t)$ is decreasing on $[t_0, t_0 + a]$, which in turn implies

$$D_- y(t) = D_- v(t) - w(t) \le 0, \quad t \in (t_0, t_0 + a) \tag{5.29}$$

(see McShane [41, page 191]). This completes the proof of the lemma. □

Remark 5.8. It is important to note that, in view of the above lemma, the derivative D_- in Theorem 5.4 can actually be replaced by any other Dini derivative.

Lemma 5.9. *Let $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous and let $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ be fixed. Let $s(t)$ be the maximal solution of* (*) *on the interval $[t_0, t_0 + a)$, for some $a \in (0, +\infty)$, and fix $t_1 \in (t_0, t_0 + a)$. Then there exists $\epsilon_0 > 0$ such that, if $\epsilon \in (0, \epsilon_0)$, then the maximal solution $s_\epsilon(t)$ of the problem*

$$u' = \gamma(t, u) + \epsilon, \qquad u(t_0) = u_0 + \epsilon, \tag{5.25$_\epsilon$}$$

exists on $[t_0, t_1]$ and satisfies

$$\lim_{\epsilon \to 0} s_\epsilon(t) = s(t) \tag{5.30}$$

uniformly on $[t_0, t_1]$.

Proof. Let $M \subset \mathbb{R}_+ \times \mathbb{R}$ be compact and such that, there exists $b > 0$ such that

$$D_\epsilon^t = [t, t + b] \times \{u \in \mathbb{R} : \left| u - (s(t) + \epsilon) \right| \leq b\} \subset M \tag{5.31}$$

for every $t \in [t_0, t_1]$, $\epsilon \in (0, b/2]$. Suppose that $|\gamma(t, u)| \leq K$ on M, where K is a positive constant. Then we have

$$\left| \gamma(t, u) + \epsilon \right| \leq K + \frac{b}{2} \tag{5.32}$$

on D_ϵ^t for every $t \in [t_0, t_1]$, $\epsilon \in (0, b/2]$. Theorem 5.5 applied to the rectangle $D_\epsilon^{t_0}$ ensures the existence of a maximal solution $s_\epsilon(t)$ of $(5.25)_\epsilon$ on the interval $[t_0, t_0 + \alpha]$ with $\alpha = \min\{b, (b/b + 2K)\}$. The number α does not depend on ϵ. Now, we choose a positive integer N such that

$$\alpha' = \frac{t_1 - t_0}{N} \leq \alpha, \tag{5.33}$$

and we proceed as in Theorem 5.5 to conclude that

$$\lim_{\epsilon \to 0} s_\epsilon(t) = s(t) \tag{5.34}$$

uniformly on $[t_0, t_0 + \alpha']$. Here, we have used the fact that a maximal solution is unique and that for every sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ and $\lim_{n \to \infty} \epsilon_n = 0$ there exists a subsequence $\{\epsilon_n'\}$ such that $s_{\epsilon_{n'}}(t) \to s(t)$ as $n \to \infty$ uniformly on $[t_0, t_0 + \alpha']$. It follows that $s_\epsilon(t_0 + \alpha') \to s(t_0 + \alpha')$ as $\epsilon \to 0$. Thus, there exists a positive $\epsilon_1 \leq b/2$ such that

$$0 < \mu(\epsilon) = s_\epsilon(t_0 + \alpha') - s(t_0 + \alpha') \leq \frac{b}{2}, \quad \epsilon \in (0, \epsilon_1]. \tag{5.35}$$

Repeating the above argument on the rectangle $D_{\mu(\epsilon)}^{t_0+\alpha'} \subset M$, for $\epsilon < \epsilon_1$, we obtain that the problem

$$u' = \gamma(t,u) + \epsilon, \qquad u(t_0 + \alpha') = s(t_0 + \alpha') + \mu(\epsilon) \tag{5.36}$$

has its maximal solution $\bar{s}_\epsilon(t)$ existing on $[t_0 + \alpha', t_0 + 2\alpha']$. We can extend the function $s_\epsilon(t)$ to the interval $[t_0 + \alpha', t_0 + 2\alpha']$ by defining

$$s_\epsilon(t) = \bar{s}_\epsilon(t), \quad t \in [t_0 + \alpha', t_0 + 2\alpha'], \tag{5.37}$$

for $\epsilon < \epsilon_1$. Obviously, this extended function $s_\epsilon(t)$ is the maximal solution of $(5.25)_\epsilon$ on the interval $[t_0, t_0+2\alpha']$ and converges to $s(t)$ uniformly on this interval as $\epsilon \to 0$. Similarly, using finite induction, we show that there is $\epsilon_0 = \epsilon_{N-1}$ such that the maximal solution $s_\epsilon(t)$ of $(5.25)_\epsilon$ exists on $[t_0, t_0 + N\alpha'] = [t_0, t_1]$ for $\epsilon \in (0, \epsilon_0)$, and converges to $s(t)$ uniformly on $[t_0, t_1]$ as $\epsilon \to 0$. The proof is finished. □

Theorem 5.10 (the comparison theorem). *Let $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous and fix $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$, $a \in (0, +\infty]$. Let $s(t)$ be the maximal solution of $(*)$ on the interval $[t_0, t_0 + a)$. Let $u : [t_0, t_0 + a) \to \mathbb{R}$ be continuous and such that $u(t_0) \leq u_0$ and*

$$Du(t) \leq \gamma(t, u(t)), \quad t \in [t_0, t_0 + a) \setminus S, \tag{5.38}$$

where S is a countable set. Then

$$u(t) \leq s(t), \quad t \in [t_0, t_0 + a). \tag{5.39}$$

Proof. We first notice that, by Lemma 5.7, we have

$$D_- u(t) \leq \gamma(t, u(t)), \quad t \in (t_0, t_0 + a). \tag{5.40}$$

If t_1 is a point in $(t_0, t_0 + a)$, then the above lemma ensures the existence of the maximal solution $s_\epsilon(t)$ of $(5.25)_\epsilon$ on $[t_0, t_1]$ for all sufficiently small $\epsilon > 0$. We also have that $s_\epsilon(t) \to s(t)$ uniformly on $[t_0, t_1]$ as $\epsilon \to 0$. Combining $(5.25)_\epsilon$, (5.40), and Theorem 5.4, we find that $u(t) < s_\epsilon(t)$ on $[t_0, t_1]$. This implies that $u(t) \leq s(t)$, $t \in [t_0, t_1]$. Since $t_1 \in (t_0, t_0 + a)$ is arbitrary, the theorem is proved. □

We are now ready to state and prove the *comparison principle* mentioned in the introduction. This principle is the source of an abundance of local and asymptotic properties of solutions of (E). Some of its applications will be given in Sections 3 and 4.

Theorem 5.11 (the comparison principle). *Let V be a Lyapunov function defined on $\mathbb{R}_+ \times \mathbb{R}^n$ and assume that*

$$V_E'(t,u) \le \gamma(t, V(t,u)), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5.41}$$

where $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and such that problem $(^*)$ has the maximal solution $s(t)$ on the interval $[t_0, T)$ $(0 \le t_0 < T \le +\infty)$. Let $x(t)$, $t \in [t_0, T)$, be any solution of (E) *with $V(t_0, x(t_0)) \le u_0$. Then $V(t, x(t)) \le s(t)$ for every $t \in [t_0, T)$.*

Proof. Let $\lambda(t) = V(t, x(t))$, $t \in [t_0, T)$. Then $\lambda(t_0) \le u_0$ and, by Lemma 5.2,

$$D^+\lambda(t) \le \gamma(t, \lambda(t)), \quad t \in [t_0, T). \tag{5.42}$$

Theorem 5.10 implies $\lambda(t) \le s(t)$, $t \in [t_0, T)$. □

3. EXISTENCE ON $\mathbb{R}_+$

In this section, we employ the comparison principle in order to establish an existence theorem on $\mathbb{R}_+$ for system (E). As before, we assume that $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. The following lemma is an easy consequence of the proofs of Theorems 3.6 and 3.7.

Lemma 5.12. *Let $x(t)$, $t \in [t_0, t_1)$ $(0 \le t_0 < t_1 < +\infty)$, be a solution of system* (E)*. Then $x(t)$ is extendable to the point $t = t_1$ if and only if it is bounded on $[t_0, t_1)$.*

Definition 5.13. Let $x(t)$, $t \in [t_0, T)$, $(0 \le t_0 < T \le +\infty)$, be a solution of system (E). Then $x(t)$ is said to be *noncontinuable* or *nonextendable to the right* if T equals $+\infty$ or $x(t)$ cannot be continued to $t = T$.

Theorem 5.14 below says that every extendable to the right solution of (E) is the restriction of a noncontinuable to the right solution of the same system.

Theorem 5.14. *Let $x(t)$, $t \in [t_0, t_1)$ $(t_1 > t_0 \ge 0)$, be an extendable to the right solution of system* (E)*. Then there is a noncontinuable to the right solution of* (E) *which extends $x(t)$. This means that there is a solution $y(t)$, $t \in [t_0, t_2)$, such that $t_2 > t_1$, $y(t) = x(t)$, $t \in [t_0, t_1)$, and $y(t)$ is noncontinuable to the right. Here, t_2 may equal $+\infty$.*

Proof. It suffices to assume that $t_0 > 0$. Let $Q = (0, \infty) \times \mathbb{R}^n$ and, for $m = 1, 2, \ldots$, let $Q_m = \{(t,u) \in Q : t^2 + \|u\|^2 \le m, t \ge 1/m\}$. Then $Q_m \subset Q_{m+1}$ and $\bigcup Q_m = Q$. Furthermore, each set Q_m is a compact subset of Q. By Lemma 5.12, a solution $y(t)$, $t \in [t_0, T)$, is noncontinuable to the right if its graph $\{(t, y(t)) : t \in [t_0, T)\}$ intersects all the sets Q_m. We are going to construct such a solution $y(t)$

which extends $x(t)$. Since $x(t)$ is continuable to the right, we may consider it defined and continuous on the interval $[t_0, t_1]$. Now, since the graph $G = \{(t, x(t)) : t \in [t_0, t_1]\}$ is compact, there exists an integer m_1 such that $G \subset Q_{m_1}$. If the number $\alpha > 0$ is sufficiently small, then for every $(a, u) \in Q_{m_1}$ the set

$$M_{a,u} = \{(t, x) \in \mathbb{R}^{n+1} : |t - a| \le \alpha, \|x - u\| \le \alpha\} \tag{5.43}$$

is contained in the set Q_{m_1+1}. Let $\|F(t, x)\| \le K$ on the set Q_{m_1+1}, where K is a positive constant. By Peano's theorem (Theorem 3.1), for every point $(a, u) \in Q_{m_1}$ there exists a solution $x(t)$ of system (E) such that $x(a) = u$, defined on the interval $[a, a + \beta]$ with $\beta = \min\{\alpha, \alpha/K\}$. This number β does not depend on the particular point $(a, u) \in Q_{m_1}$. Consequently, since $(t_1, x(t_1)) \in Q_{m_1}$, there exists a solution $x_1(t)$ of (E) which continues $x(t)$ to the point $t = t_1 + \beta$. Repeating this process, we eventually have a solution $x_q(t)$ (q a positive integer) of system (E), which continues $x(t)$ to the point $t_1 + q\beta$ and has its graph in the set Q_{m_1+1}, but not entirely inside the set Q_{m_1}. In this set Q_{m_1+1} we repeat the continuation process as we did for the set Q_{m_1}. Thus, we eventually obtain a solution $y(t)$ which intersects all the sets Q_m, $m \ge m_1$, for some m_1, and is a noncontinuable extension of the solution $x(t)$. □

The theorem below shows that a noncontinuable to the right solution of (E) actually *blows up* at the right endpoint T of the interval of its existence, provided that this point T is finite.

Theorem 5.15. *Let $x(t)$, $t \in [t_0, T)$ $(0 \le t_0 < T < +\infty)$, be a noncontinuable to the right solution of* (E). *Then*

$$\lim_{t \to T^-} ||x(t)|| = +\infty. \tag{5.44}$$

Proof. Assume that our assertion is false. Then there exists an increasing sequence $\{t_m\}_{m=1}^{\infty}$ such that $t_m \in [t_0, T)$, $\lim_{m\to\infty} t_m = T$ and $\lim_{m\to\infty} \|x(t_m)\| = L < +\infty$. Since $\{x(t_m)\}$, $m = 1, 2, \ldots$, is bounded, there exists a subsequence $\{x(t'_m)\}$ such that $x(t'_m) \to y$, as $m \to \infty$, with $\|y\| = L$ and t'_m increasing. Let M be a compact subset of $\mathbb{R}_+ \times \mathbb{R}^n$ such that the point (T, y) is an interior point of M. We may assume that $(t'_m, x(t'_m)) \in \operatorname{int} M$ for all $m = 1, 2, \ldots$. We will show that, for infinitely many m, there exists $\bar{t}_m$ such that

$$t'_m < \bar{t}_m < t'_{m+1}, \quad (\bar{t}_m, x(\bar{t}_m)) \in \partial M. \tag{5.45}$$

In fact, if this were not true, there would be some $\epsilon \in (0, T)$ such that $(t, x(t)) \in \operatorname{int} M$ for all $t \in (T - \epsilon, T)$. But then Lemma 5.12 would imply the extendability of $x(t)$ to $t = T$, which is a contradiction. Let (5.45) hold for a subsequence $\{m'\}$ of the positive integers so that $\bar{t}_{m'}$ is the smallest number with the property

$$t'_{m'} < \bar{t}_{m'} < t'_{m'+1}, \quad (\bar{t}_{m'}, x(\bar{t}_{m'})) \in \partial M. \tag{5.46}$$

Then we have

$$\lim_{m'\to\infty} (\bar{t}_{m'}, x(\bar{t}_{m'})) = (T, y). \tag{5.47}$$

This is a consequence of the fact that $\bar{t}_{m'} \to T$ as $m' \to \infty$ and the inequality

$$||x(\bar{t}_{m'}) - x(t'_{m'})|| \leq \mu(\bar{t}_{m'} - t'_{m'}), \tag{5.48}$$

where μ is an upper bound for the function $F(t, x)$ on M. However, since ∂M is a closed set, we have $(T, y) \in \partial M$. This is a contradiction to our assumption. The proof is complete. □

It should be noted that a local maximal solution of (*) can always be extended to a noncontinuable to the right maximal solution. This is a consequence of Theorems 5.5 and 5.14.

We are now ready for the main result of this section which says that every solution of system (E) is extendable to $+\infty$ if this is true for an associated scalar differential equation.

Theorem 5.16 (the comparison principle and existence on $\mathbb{R}_+$). *Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function satisfying*

$$V'_E(t, u) \leq \gamma(t, V(t, u)), \quad (t, u) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{5.49}$$

and $V(t, u) \to +\infty$ as $\|u\| \to +\infty$ uniformly w.r.t. t lying in any compact set. Here, $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and such that for every $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the problem (*) *has a maximal solution defined on $[t_0, +\infty)$. Then every solution of* (E) *is extendable to $+\infty$.*

Proof. Let $[t_0, T)$ be the maximal interval of existence of a solution $x(t)$ of (E) and assume that $T < +\infty$. Let $y(t)$ be the maximal solution of (*) with $y(t_0) = V(t_0, x(t_0))$. Then Theorem 5.11 ensures that

$$V(t, x(t)) \leq y(t), \quad t \in [t_0, T). \tag{5.50}$$

On the other hand, since $x(t)$ is a noncontinuable to the right solution, we must have

$$\lim_{t\to T^-} ||x(t)|| = +\infty \tag{5.51}$$

by Theorem 5.15. This implies that $V(t, x(t))$ converges to $+\infty$ as $t \to T^-$, but (5.50) implies that

$$\limsup_{t\to T^-} V(t, x(t)) \leq y(T). \tag{5.52}$$

Thus, $T = +\infty$. □

It is easy to see that (5.49) is not needed for all $(t,u) \in \mathbb{R}_+ \times \mathbb{R}^n$. It can be assumed instead that it holds for all $u \in \mathbb{R}^n$ such that $\|u\| > \alpha$, where α is a positive constant. Having this in mind, we establish the following important corollary.

Corollary 5.17. *Assume that there exists $\alpha > 0$ such that*

$$||F(t,u)|| \le \gamma(t, \|u\|), \quad t \in \mathbb{R}_+,\ \|u\| > \alpha, \tag{5.53}$$

where $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that for every $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}_+$, problem $(*)$ has a maximal solution defined on $[t_0, \infty)$. Then every solution of (E) is extendable to $+\infty$.

Proof. Here, it suffices to take $V(t,u) \equiv \|u\|$. In fact, from the proof of Theorem 4.10 we obtain

$$\begin{aligned} V_E'(t,x(t)) &= \lim_{h\to 0^+} \frac{||x(t) + hF(t,x(t))|| - ||x(t)||}{h} \\ &\le ||F(t,x(t))|| \le \gamma(t, ||x(t)||) = \gamma(t, V(t,x(t))), \end{aligned} \tag{5.54}$$

provided that $\|x(t)\| > \alpha$. Now, let $x(t)$, $t \in [t_0, T)$, be a noncontinuable to the right solution of (E) such that $T < +\infty$. Then, for t sufficiently close to T from the left, we have $\|x(t)\| > \alpha$. The rest of the proof follows as in Theorem 5.16. □

4. COMPARISON PRINCIPLE AND STABILITY

In this section, we establish stability properties of the zero solution of a system of the form (E) by assuming the same stability properties of the zero solution of an associated scalar equation of the form $(*)$.

Definition 5.18. Let the function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be strictly increasing, continuous and such that $\lambda(0) = 0$. Then λ is called a *Q-function*. The set of all Q-functions will be denoted by QF.

Definition 5.19. A Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ with $V(t,0) \equiv 0$ is said to be *Q-positive* if there exists $\lambda \in QF$ such that

$$V(t,u) \ge \lambda(\|u\|), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{5.55}$$

Definition 5.20. A Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is said to be *Q-bounded* if there exists $\lambda \in QF$ such that

$$V(t,u) \le \lambda(\|u\|), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{5.56}$$

We are now ready for our first stability results via the comparison method.

Theorem 5.21. *Let $F(t,0) \equiv 0$ and $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function with the property*

$$V_E'(t,u) \le \gamma(t, V(t,u)), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5.57}$$

where $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and $\gamma(t,0) \equiv 0$. If the zero solution of

$$u' = \gamma(t,u) \tag{5.58}$$

is stable (asymptotically stable) and V is Q-positive, then the zero solution of (E) *is stable (asymptotically stable).*

Proof. Assume the stability of the zero solution of (5.58). The Q-positiveness of V implies the existence of $\lambda \in QF$ such that

$$V(t,u) \ge \lambda(\|u\|), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{5.59}$$

Given $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that $|y(t)| < \lambda(\epsilon)$ for all $t \in \mathbb{R}_+$, where $y(t)$ is any solution of (5.58) with $|y(0)| < \eta(\epsilon)$. This property follows from the stability assumption on (5.58). On the other hand, since V is continuous and $V(t,0) \equiv 0$, there exists $\delta(\epsilon) > 0$ such that

$$V(0,x_0) < \eta(\epsilon) \quad \text{whenever } ||x_0|| < \delta(\epsilon). \tag{5.60}$$

Now, fix x_0 with $\|x_0\| < \delta(\epsilon)$, let $x(t)$ be a solution of (E) with $x(0) = x_0$, and let $u(t)$ be the maximal solution of (5.58) with the property $u(0) = V(0,x_0)$. Then, by Theorem 5.11, we have

$$\lambda(||x(t)||) \le V(t,x(t)) \le u(t) < \lambda(\epsilon), \quad t \in \mathbb{R}_+, \tag{5.61}$$

which, along with the fact that λ is strictly increasing, implies that

$$||x(t)|| < \epsilon, \quad t \in \mathbb{R}_+. \tag{5.62}$$

This proves the stability of the zero solution of (E).

If we assume the asymptotic stability of the zero solution of (5.58), then the asymptotic stability of the zero solution of (E) follows from

$$\lambda(||x(t)||) \le u(t), \quad t \in \mathbb{R}_+, \tag{5.63}$$

which holds for $\|x(0)\| = \|x_0\|$ sufficiently small. Since $\lim_{t\to\infty} u(t) = 0$, we also have

$$\lim_{t\to\infty} ||x(t)|| \le \lim_{t\to\infty} \lambda^{-1}(u(t)) = 0. \tag{5.64}$$

This is the end of the proof. □

The uniform stability cases are covered by Theorem 5.22.

THEOREM 5.22. *Let $F(t,0) \equiv 0$ and $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function with the property*

$$V_E'(t,u) \le \gamma(t, V(t,u)), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5.65}$$

where $\gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and $\gamma(t,0) \equiv 0$. Assume that V is Q-positive and Q-bounded. Then if the zero solution of (5.58) *is uniformly (uniformly asymptotically) stable, the same fact is true for* (E).

PROOF. Let $\lambda, \mu \in QF$ be such that

$$\lambda(\|u\|) \le V(t,u) \le \mu(\|u\|), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5.66}$$

and let (5.58) have its zero solution uniformly stable. Then given $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists $\eta(\epsilon) > 0$, independent of t_0, such that $|y(t)| < \lambda(\epsilon)$ for all $t \ge t_0$, where $y(t)$ is any solution of (5.58) with $|y(t_0)| < \eta(\epsilon)$. Now, let $\delta(\epsilon) > 0$ satisfy $\delta(\epsilon) < \mu^{-1}(\eta(\epsilon))$. Then if $x_0 \in R^n$ is given with $\|x_0\| < \delta(\epsilon)$, we have

$$V(t,x_0) \le \mu(||x_0||) < \eta(\epsilon) \tag{5.67}$$

for every $t \ge 0$. From this point on, the proof of our first assertion follows as the proof of Theorem 5.21 by arguing at $t = t_0$ instead of $t = 0$.

To prove our second assertion, let (5.58) have its zero solution uniformly asymptotically stable, and let λ, μ be as in (5.66). Then there exists $\eta_0 > 0$ with the following property: given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that every solution $y(t)$ of (5.58) with $|y(t_0)| < \eta_0$, for some $t_0 \ge 0$, satisfies

$$|y(t)| < \lambda(\epsilon), \quad t \ge t_0 + T(\epsilon). \tag{5.68}$$

Let $\delta_0 > 0$ be such that $\mu(\delta_0) < \eta_0$. Then if x_0 is a vector in $\mathbb{R}^n$ with $\|x_0\| < \delta_0$, we have

$$V(t,x_0) \le \mu(||x_0||) < \mu(\delta_0) < \eta_0 \tag{5.69}$$

for every $t \in \mathbb{R}_+$. Again, the proof continues as the proof of Theorem 5.21 by arguing at $t = t_0 + T(\epsilon)$ instead of $t = 0$. □

EXERCISES

5.1. Consider (E) with $F(t,0) \equiv 0$. Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be an appropriate Lypunov function satisfying

$$a\|u\| \le V(t,u) \le b\|u\|, \qquad V_E'(t,u) \le -cV(t,u), \tag{5.70}$$

for every $(t,u) \in \mathbb{R}_+ \times \mathbb{R}^n$, where a, b, c are positive constants. Show that there exists a constant $K > 0$ such that

$$||x(t)|| \le K||x_0||e^{-c(t-t_0)}, \quad t \ge t_0 \ge 0, \tag{5.71}$$

where $x(t)$ is any solution of (E) with $x(t_0) = x_0$. This phenomenon is called *exponential asymptotic stability* of the zero solution of (E).

5.2. Consider (E) with $F(t,0) \equiv 0$. Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a Q-positive Lyapunov function satisfying

$$V_E'(t,u) \le -p(t)q(V(t,u)) \tag{5.72}$$

for all $(t,u) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies

$$\int_0^\infty p(t)dt = +\infty. \tag{5.73}$$

Provide conditions on $q : \mathbb{R} \to \mathbb{R}$ so that the zero solution of (E) is asymptotically stable.

5.3. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 - g(x_2), \end{aligned} \tag{A}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous with $g(0) = 0$. Provide conditions on g that ensure the stability of the zero solution of (A). Hint. Use the Lyapunov function $V(t,u) \equiv \|u\|^2$.

5.4. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -p(x_1,x_2), \end{aligned} \tag{B}$$

where $p : \mathbb{R}^2 \to \mathbb{R}$ is continuous and such that $\operatorname{sgn} p(u_1,0) = \operatorname{sgn} u_1$ for any $u_1 \in \mathbb{R}$,

$$\begin{gathered} u_2[p(u_1,0) - p(u_1,u_2)] \le 0, \quad (u_1,u_2) \in \mathbb{R}^2, \\ \lim_{|v|\to\infty} \int_0^v p(s,0)ds = +\infty. \end{gathered} \tag{5.74}$$

Show that all solutions of (B) are bounded. Hint. Consider the Lyapunov function

$$V(t,u) \equiv u_2^2 + 2\int_0^{u_1} p(s,0)ds, \tag{5.75}$$

where $u = (u_1,u_2) \in \mathbb{R}^2$.

5.5. Study the stability properties of the linear system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} b(t) & a(t) \\ -a(t) & b(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.76}$$

with $a, b : \mathbb{R}_+ \to \mathbb{R}$ continuous, by imposing general conditions on the functions a, b. Use the Lyapunov function $V(u) \equiv \|u\|^2$.

5.6. Show that the local Lipschitz continuity assumption in the definition of a Lyapunov function (Definition 5.1) cannot be omitted. Consider the Lyapunov function $V(x) \equiv |x|^{1/2} \operatorname{sgn} x$. Compute $V^+(0, x(0))$ and $V_E'(0, x(0))$ for the solution $x(t) \equiv t^2$ of the equation $x' = 2t$.

5.7. Let F in (E) be such that $\|F(t,u)\| \le p(t)q(\|u\|)$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ and $q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous with $q(0) = 0$ and $q(s) > 0$ for $s > 0$. Furthermore, let

$$\int_v^\infty \frac{ds}{q(s)} = +\infty, \tag{5.77}$$

for some $v > 0$. Using the problem

$$u' = p(t)q(u), \qquad u(t_0) = u_0 > 0, \tag{5.78}$$

show that for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a solution $x(t)$, $t \in \mathbb{R}_+$, of (E) with $x(t_0) = x_0$.

5.8 (integral inequalities). Let $K : J \times J \times \mathbb{R} \to \mathbb{R}$, $J = [t_0, \infty)$, be continuous and increasing in its third variable. Assume further that for three continuous functions $u, v, f : J \to \mathbb{R}$ we have

$$\begin{aligned} u(t) &< f(t) + \int_{t_0}^t K(t,s,u(s))\,ds, \\ v(t) &\ge f(t) + \int_{t_0}^t K(t,s,v(s))\,ds \end{aligned} \tag{5.79}$$

for every $t \ge t_0$ and $u(t_0) < v(t_0)$. Show that $u(t) < v(t)$, $t \ge t_0$. Hint. Assume that there is $t_1 > t_0$ with $u(t_1) = v(t_1)$ and $u(t) < v(t)$, $t \in [t_0, t_1)$. Prove that this assumption leads to the contradiction that $u(t_1) < v(t_1)$.

5.9 (local solutions to scalar integral equations). Consider the equation

$$x(t) = f(t) + \int_{t_0}^t K(t,s,x(s))\,ds, \tag{I}$$

where $f : [t_0, t_0 + a] \to \mathbb{R}$ is continuous. Furthermore, assume that $K : D \to \mathbb{R}$ be continuous, where

$$D = [t_0, t_0 + a] \times [t_0, t_0 + a] \times D_1, \tag{5.80}$$

where $D_1 = \{u \in \mathbb{R} : |u - f(t)| \le b, \text{ for some } t \in [t_0, t_0 + a]\}$, with b a positive constant. Show that (I) has a solution $x(t)$, $t \in [t_0, t_0+\alpha]$, where $\alpha = \min\{a, b/K_0\}$, $K_0 = \sup\{|K(t,s,u)|; (t,s,u) \in D\}$. Hint. Consider the sequence $\{x_m(t)\}_{m=1}^{\infty}$ with $x_1(t) = f(t)$, $t \in [t_0, t_0 + \alpha]$, and

$$\begin{aligned} x_m(t) &= f(t), \quad t \in \left[t_0, t_0 + \frac{\alpha}{m}\right], \\ x_m(t) &= f(t) + \int_{t_0}^{t-\alpha/m} K(t, s, x_m(s))\,ds, \quad t \in \left[t_0 + \frac{\alpha}{m}, t_0 + \alpha\right], \end{aligned} \tag{5.81}$$

for every $m = 2, 3, \ldots$. The first equation defines $\{x_m(t)\}$ on $[t_0, t_0 + (\alpha/m)]$. From the second equation we get

$$x_m(t) = f(t) + \int_{t_0}^{t-\alpha/m} K(t, s, f(s))\,ds, \quad t \in \left[t_0 + \frac{\alpha}{m}, t_0 + \frac{2\alpha}{m}\right]. \tag{5.82}$$

In the next step, $x_m(t)$ is defined on $[t_0 + (2\alpha/m), t_0 + (3\alpha/m)]$ and so on. Thus, the second equation of the definition of $x_m(t)$ defines $x_m(t)$ "piecewise" on the intervals $[t_0+(k\alpha/m), t_0+((k+1)\alpha/m)]$, $k = 1, 2, \ldots, m-1$. The resulting function is continuous on $[t_0, t_0 + \alpha]$. Show that $|x_m(t) - f(t)| \le b$ and use Theorem 2.5 to show the existence of a subsequence of $\{x_m(t)\}$ converging to the desired solution.

5.10 (Maximal solutions of integral equations). Let the assumptions of Exercise 5.9 be satisfied with the last factor of D replaced by $\mathbb{R}$. Assume further that K is increasing in its last variable. Then (I) has a maximal solution on some interval $[t_0, t_0 + \alpha]$, $\alpha > 0$. Hint. Model your proof after that of Theorem 5.5. Consider the integral equation

$$x(t) = f(t) + \int_{t_0}^{t} K(t, s, x(s))\,ds. \tag{5.83}$$

Apply the result of Exercise 5.9 to obtain an appropriate solution $x_\epsilon(t)$ on $[t_0, t_0+\alpha]$ as in Theorem 5.5.

5.11. Let $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}_+$ be continuous and such that $F(t, 0) \equiv 0$, $V(0) = 0$ and $V(u) > 0$ for $u \ne 0$. Assume that V is continuously differentiable on the ball $B_\alpha(0)$, for some $\alpha > 0$, and satisfies

$$\langle \nabla V(u), F(t, u)\rangle \le 0, \quad t \in \mathbb{R}_+,\ u \in B_\alpha(0). \tag{D}$$

Prove that the zero solution of

$$x' = F(t, x) \tag{E}$$

is stable by following these steps: letting $\epsilon \in (0, \alpha)$, show that V attains its minimum $m > 0$ on $\partial B_\epsilon(0)$. Then show that there is $\delta \in (0, \epsilon)$ such that $V(x) \leq m/2$ for $x \in \overline{B_\delta(0)}$. Thus, if $\|x(0)\| < \delta$, then the inequality

$$V(x(t)) \leq V(x(0)) \leq \frac{m}{2} \tag{5.84}$$

precludes $x(t)$ from reaching the surface $\partial B_\epsilon(0)$. This means that $\|x(t)\| < \epsilon$ for as long as $x(t)$ exists. (Naturally, this problem can be solved by considering the scalar equation $u' = 0$.)

5.12. Let the assumptions of Exercise 5.11 be satisfied with $F(u)$ instead of $F(t, u)$. Assume further that

$$T(u) \equiv \langle \nabla V(u), F(u) \rangle < 0 \tag{5.85}$$

for every $u \in B_\beta(0)$ such that $u \neq 0$. Here, $\beta \in (0, \alpha]$. Show that the zero solution of (E) is asymptotically stable. Hint. Let $\epsilon \in (0, \beta)$, $\|x(0)\| < \delta$, where δ is as in Exercise 5.11. Let $V(x(t)) \to L$ as $t \to \infty$. If $L = 0$, we are done. If $L > 0$, there exists $K > 0$ such that $\|x(t)\| \geq K$ and $\|x(t)\| < \epsilon$ for all large t. Use the fact that the function $T(u)$ attains its (negative) maximum on the set $S = \{u \in \mathbb{R}^n : K \leq \|u\| \leq \epsilon\}$, and an integration of $T(x(t))$ to get a contradiction to the positiveness of $V(x(t))$.

5.13. Consider Van der Pol's equation

$$x'' + k(1 - x^2)x' + x = 0, \tag{G}$$

where k is a constant.

(1) Determine k so that the zero solution of (G) (written as a system) is stable.
(2) Extend your result to Liénard's equation

$$x'' + f(x)x' + g(x) = 0 \tag{L}$$

with $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ continuous and $g(0) = 0$.
(3) Impose further conditions on f, g that guarantee the asymptotic stability of the zero solution of (L). Hint. You may consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -f(x_1)x_2 - g(x_1), \end{aligned} \tag{5.86}$$

or the system

$$x_1' = x_2 - \int_0^{x_1} f(u)du,$$
$$x_2' = -g(x_1), \tag{5.87}$$

in connection with the Lyapunov function

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} g(u)du \tag{5.88}$$

and Exercises 5.11 and 5.12.

5.14 (instability). Let $V : \mathbb{R}^n \to \mathbb{R}_+$, $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous with $F(0) = 0$, $V(0) = 0$ and $V(u) > 0$ for $u \neq 0$. Furthermore, assume that V is continuously differentiable on $B_\alpha(0)$, for some $\alpha \in (0, \infty)$, and

$$\langle \nabla V(u), F(u) \rangle > 0 \tag{5.89}$$

for all $u \in B_\alpha(0)$ with $u \neq 0$. Show that the zero solution of (E) is unstable. Hint. Let $\beta \in (0, \alpha)$. Choose $c \in \overline{B_\beta(0)}$ ($c \neq 0$), $\mu \in (0, \|c\|)$ such that $V(u) < V(c)$ for $\|u\| \leq \mu$. Let m be the minimum of $\langle \nabla V(u), F(u) \rangle$ on $\{u \in \mathbb{R}^n : \mu \leq \|u\| \leq \beta\}$. Show that if the solution $x(t)$ satisfies $x(0) = c$, then $dV(x(t))/dt \geq 0$ for as long as $\|x(t)\| \leq \beta$. This says that $V(x(t)) \geq V(c)$ and $\|x(t)\| \geq \mu$. It follows that

$$V(x(t)) \geq V(c) + tm \tag{5.90}$$

for as long as $\|x(t)\| \leq \beta$. Since the function $V(x(t))$ is bounded for as long as $\|x(t)\| \leq \beta$, the above inequality implies that $x(t)$ must go through the sphere $\partial B_\beta(0)$ at some time t_0. Thus, from each ball $B_\beta(0)$ starts a solution $x(t)$ of $x' = F(x)$ that penetrates the sphere $\partial B_\beta(0)$. This implies the instability of the zero solution of $x' = F(x)$.

5.15. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}_+$ be continuous and such that $F(t, u)$ is T-periodic in t ($T > 0$ is fixed and $F(t + T, u) = F(t, u)$, $(t, u) \in \mathbb{R} \times \mathbb{R}^n$). Moreover, $F(t, 0) \equiv 0$, $V(0) = 0$ and $V(u) > 0$ for $u \neq 0$. Assume that V is continuously differentiable on $\mathbb{R}^n$ and satisfies

$$\langle \nabla V(u), F(t, u) \rangle \leq 0 \tag{5.91}$$

for all $t \in [0, T]$, $u \in \mathbb{R}^n$. Assume further that given any constant $c > 0$, there is no solution $x(t)$ of (E) such that $V(x(t)) = c$, $t \in [0, T]$. Show that (E) has no T-periodic solutions $x(t) \not\equiv 0$.

5.16 (uniqueness via Lyapunov functions). Let $V : \mathbb{R}^n \to \mathbb{R}_+$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that $V(0) = 0$, $V(u) > 0$ for $u \neq 0$, and $F(t,0) \equiv 0$. Furthermore, let V be continuously differentiable on $\mathbb{R}^n$ and such that

$$\langle \nabla V(u-v), F(t,u) - F(t,v) \rangle \leq 0 \tag{5.92}$$

for any $t \in \mathbb{R}_+$, $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n$. Then if $x(t)$, $y(t)$, $t \in [t_0,\infty)$ ($t_0 \in \mathbb{R}_+$), are two solutions of (E) such that $x(t_0) = y(t_0)$, we have $x(t) = y(t)$, $t \in [t_0,\infty)$. Show this, and then consider an improvement of this result by replacing zero in the above inequality by $\gamma(t, V(u-v))$, for some suitable scalar function γ.

5.17. Extend the results of Exercises 5.12 and 5.14 to nonautonomous systems (E) (with F depending also on the variable t). Furthermore, examine the case of Lyapunov functions V depending also on the variable t.

5.18. Prove Lemma 5.12.

5.19. Show that the zero solution of the following system is stable:

$$x_1' = x_2 - x_1^3, \qquad x_2' = -x_1 - x_2^3. \tag{5.93}$$

Hint. Use the Lyapunov function $V(x) \equiv x_1^2 + x_2^2$ and Exercise 5.11.

5.20. Consider the system

$$x' = -\nabla V(x), \tag{5.94}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$ for $x \neq 0$. Show that $V(x)$ can be used as a Lyapunov function to show the stability of this system. Use Exercise 5.11.

5.21. Let $A : \mathbb{R}_+ \to M_n$ be continuous and consider the system

$$x' = A(t)x. \tag{S}$$

Assume that $A(t) = [a_{ij}(t)]_{i,j=1}^n$, where $a_{ij}(t) = -a_{ji}(t)$ for $i \neq j$ and $a_{ii}(t) \leq 0$. Show the stability of this system by using the Lyapunov function $V(x) \equiv \|x\|^2$ and Exercise 5.11. Compare with Exercise 5.5.

5.22. For the scalar equation

$$x'' + f(x,x')x' + g(x) = h(t), \tag{5.95}$$

assume the following:

(i) $f : \mathbb{R}^2 \to \mathbb{R}_+$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R}_+ \to \mathbb{R}$ are continuous;

(ii) $ug(u) > 0$ for $u \neq 0$ and $G(x) \to \infty$ as $|x| \to \infty$, where

$$G(x) \equiv \int_0^x g(u)du; \tag{5.96}$$

(iii)

$$\int_0^\infty |h(t)|dt < +\infty. \tag{5.97}$$

Show that all solutions of this equation and their derivatives are bounded on $\mathbb{R}_+$.

Hint. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -f(x_1, x_2)x_2 - g(x_1) + h(t), \end{aligned} \tag{5.98}$$

and the "Lyapunov" function

$$V(t,x) \equiv \sqrt{x_2^2 + 2G(x_1)} - \int_0^t |h(s)|ds. \tag{5.99}$$

Show that $(d/dt)V(t,x(t)) \leq 0$ whenever $t \geq 0$ and $x_1^2(t) + x_2^2(t) \neq 0$.

5.23. For the scalar equation

$$x'' + f(t,x,x')x' + g(x) = h(t), \tag{5.100}$$

assume that $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R}_+ \to \mathbb{R}$ are continuous and such that

$$\begin{aligned} &\lim_{|x|\to\infty} \left\{ G(x) \equiv \int_0^x g(u)du \right\} = +\infty, \\ &\lim_{t\to\infty} \left\{ E(t) \equiv \int_0^t |h(t)|dt \right\} < +\infty. \end{aligned} \tag{5.101}$$

Show that all solutions of this equation and their derivatives are bounded on $\mathbb{R}_+$. Hint. Consider the resulting system in $\mathbb{R}^2$ and the Lyapunov function

$$V(t,x) \equiv e^{-2E(t)} \left\{ G(x_1) + \frac{x_2^2}{2} + 1 \right\}. \tag{5.102}$$

CHAPTER 6

BOUNDARY VALUE PROBLEMS ON FINITE AND INFINITE INTERVALS

Let J be a subinterval of $\mathbb{R}$ and $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ continuous. Let B_J be a class of continuous $\mathbb{R}^n$-valued functions defined on J. Then the system

$$x' = F(t, x) \tag{E}$$

along with the condition $x \in B_J$ is a *boundary value problem* on the interval J. Naturally, the condition $x \in B_J$ is too general and contains the initial value problem on $J = [t_0, T]$, that is, B_J can be the class $\{x \in C_n[t_0, T] : x(t_0) = x_0\}$. A boundary value problem (b.v.p.) usually concerns itself with *boundary conditions* of the form $x \in B_J$ which involve values of the function x at more than one point of the interval J. One of the most important b.v.p.'s in the theory of ordinary differential equations is the problem concerning the existence of a T-periodic solution. This problem consists of (E) and the condition

$$x(t + T) = x(t) \quad \text{for every } t \in \mathbb{R}, \tag{6.1}$$

where T is a fixed positive number. Here, we usually assume that F is T-periodic in its first variable t which ranges over $\mathbb{R}$. In this case, this problem is actually reduced to the simple problem

$$x(0) - x(T) = 0 \tag{B_1}$$

(see Exercise 3.7). Another, more general, b.v.p. is the problem ((E), (B_2)), where

$$Mx(0) - Nx(T) = 0. \tag{B_2}$$

Here, M and N are known $n \times n$ matrices. Naturally, the points $0, T$ can be replaced above, and in what follows, by any other points $a, b \in \mathbb{R}$ with $b > a$. In the case

$M = N = I$, the condition (B_2) coincides with (B_1). An even more general problem is the problem ((E), (B_3)), where

$$Ux = r. \tag{B_3}$$

Here, $r \in \mathbb{R}^n$ is fixed and U is a linear operator with domain in $C_n[0, T]$ and values in $\mathbb{R}^n$. Such an operator U could be given, for example, by

$$Ux = \int_0^T V(s)x(s)ds, \tag{6.2}$$

where $V : [0, T] \to M_n$ is continuous. The operator U in (B_3) could be defined on a class of functions over an infinite interval. For example,

$$Ux = \int_0^\infty V(s)x(s)ds \tag{6.3}$$

or

$$Ux = Mx(0) - Nx(\infty), \tag{6.4}$$

where $x(\infty)$ denotes the limit of $x(t)$ at $t \to \infty$. In these two last conditions we may consider $B_J = \{u \in C_n^l : Uu = r\}$. Naturally, we must also assume a condition on V like

$$\int_0^\infty ||V(t)||dt < +\infty. \tag{6.5}$$

In this chapter, we study b.v.p.'s on finite as well as infinite intervals. We begin with the case of linear systems and continue with perturbed systems of the type discussed in Chapter 4. All boundary value problems on finite (closed) intervals are studied on $[0, T]$. Extensions to arbitrary finite (closed) intervals are obvious.

1. LINEAR SYSTEMS ON FINITE INTERVALS

We are interested in b.v.p.'s for linear systems of the forms

$$x' = A(t)x, \tag{S}$$

$$x' = A(t)x + f(t), \tag{S_f}$$

where $A : J \to M_n$, $f : J \to \mathbb{R}^n$ are assumed to be continuous on the interval $J = [0, T]$. Let $X(t)$ be the fundamental matrix of (S) with the property $X(0) = I$. Then the general solution of (S) is $X(t)x_0$, where x_0 is an arbitrary vector in $\mathbb{R}^n$. Let $U : C_n[0, T] \to \mathbb{R}^n$ be a bounded linear operator. Then $U(X(\cdot)x_0) = \widetilde{X}x_0$ for every $x_0 \in \mathbb{R}^n$, where $\widetilde{X}$ is the matrix whose columns are the values of U on the corresponding columns of $X(t)$. It is easy to prove that this equation holds. It is obvious that the homogeneous problem (S), with the homogeneous boundary conditions

$$Ux = 0, \tag{B$_4$}$$

is satisfied only by the zero solution if and only if $x_0 = 0$ is the only solution to the equation $\widetilde{X}x_0 = 0$, that is, if and only if the matrix $\widetilde{X}$ is nonsingular. Now, we look at the problem $((\mathrm{S}_f), (\mathrm{B}_3))$. The general solution of (S_f) is given by

$$x(t) = X(t)x_0 + p(t, f), \quad t \in [0, T], \tag{6.6}$$

where

$$p(t, f) = X(t)\int_0^t X^{-1}(s)f(s)ds. \tag{6.7}$$

The solution $x(t)$, with $x(0) = x_0$, satisfies (B_3) if and only if

$$Ux = r = \widetilde{X}x_0 + Up(\cdot, f). \tag{6.8}$$

This equation in x_0 has a unique solution in $\mathbb{R}^n$, for some $r \in \mathbb{R}^n$, $f \in C_n[0, T]$, if and only if $\widetilde{X}$ is nonsingular. This solution is given by

$$x_0 = \widetilde{X}^{-1}[r - Up(\cdot, f)]. \tag{6.9}$$

Consequently, we have shown the following theorem.

THEOREM 6.1. *Consider the problem* $((\mathrm{S}_f), (\mathrm{B}_3))$, *where* $U : C_n[0, T] \to \mathbb{R}^n$ *is a bounded linear operator. The following statements are equivalent:*

(i) *the problem* $((\mathrm{S}), (\mathrm{B}_4))$ *has only the zero solution;*
(ii) *the problem* $(((\mathrm{S}_f), (\mathrm{B}_3))$ *has a unique solution for every* $(r, f) \in \mathbb{R}^n \times C_n[0, T]$;
(iii) *the problem* $((\mathrm{S}_f), (\mathrm{B}_3))$ *has a unique solution for some* $(r, f) \in \mathbb{R}^n \times C_n[0, T]$;
(iv) $\widetilde{X}$ *is nonsingular.*

This theorem holds true if $C_n[0, T]$ is replaced by $C_n(\mathbb{R}_+)$ or C_n^l, provided of course that the matrix $\widetilde{X}$ and the vector $Up(\cdot, f)$ are well defined.

2. PERIODIC SOLUTIONS OF LINEAR SYSTEMS

We now show that the dimension of the vector space of T-periodic solutions (i.e., solutions $x(t)$ such that $x(0) = x(T)$) of (S) is the same as the dimension of the corresponding space of the *adjoint system*

$$y' = -yA(t) \qquad (\text{or } u' = -A^T(t)u), \tag{S$_a$}$$

where $y = [y_1, y_2, \dots, y_n]$ and $u = y^T$. Whenever we are dealing with T-periodicity, the functions $F(t,x)$ in (E) and $A(t)$, $f(t)$ in (S_f) will be assumed to be T-periodic in t. The reader should not confuse the period T with the letter T indicating transpose.

THEOREM 6.2. *Let m be the number of linearly independent T-periodic solutions of system* (S). *Then m is also the number of linearly independent T-periodic solutions of* (S_a).

PROOF. System (S) has a T-periodic solution with initial value x_0 if we have $X(T)x_0 = x_0$. From Exercise 3.4 we know that $X^{-1}(t)$ is the fundamental matrix of system (S_a) with $X^{-1}(0) = I$. Consequently, the general solution of (S_a) is $y(t) = y_0X^{-1}(t)$. System (S_a) will have a T-periodic solution with initial value y_0 if we have $y_0 = y_0X^{-1}(T)$ or $y_0X(T) = y_0$. Transposing this equation we obtain $X^T(T)y_0^T = y_0^T$. However, the matrices $X(T) - I$, $X^T(T) - I$ have the same rank. It follows that the equations

$$[X(T) - I]x_0 = 0, \qquad [X^T(T) - I]y_0^T = 0 \tag{6.10}$$

have the same number of linearly independent solutions. This implies that the systems (S) and (S_a) have the same number of linearly independent T-periodic solutions. □

Theorem 6.3 below is called the *Fredholm alternative* and provides a necessary and sufficient condition for the existence of T-periodic solutions of system (S_f). If $C \in M_n$ and D is an n-vector (row n-vector), the symbol $[C \mid D]$ denotes the augmented (row-augmented) matrix of C and D.

THEOREM 6.3 (Fredholm alternative). *A necessary and sufficient condition for the existence of a T-periodic solution of system (S_f) is that f be orthogonal to all T-periodic solutions of (S_a), that is,*

$$\int_0^T y_j(t)f(t)dt = 0, \quad j = 1, 2, \dots, m, \tag{6.11}$$

where $\{y_j\}_{j=1}^m$ is a basis for the vector space of T-periodic solutions of system (S_a).

PROOF. The general solution $x(t)$ of (S_f) is given by

$$x(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(s)f(s)ds \tag{6.12}$$

for every $t \in [0, T]$, $x_0 \in \mathbb{R}^n$. Thus,

$$x(T) = X(T)x_0 + X(T)\int_0^T X^{-1}(s)f(s)ds. \tag{6.13}$$

Hence, $x(t)$ will be T-periodic if and only if

$$[I - X(T)]x_0 = X(T) \int_0^T X^{-1}(s) f(s) ds \tag{6.14}$$

has solutions x_0. Assume that (S_f) does have a T-periodic solution $x(t)$ with initial condition $x(0) = x_0$. Let $y(t)$ be a T-periodic solution of (S_a) with $y(0) = y_0$. Then we have (see proof of Theorem 6.2) $y_0 = y_0 X(T)$ or

$$y_0[I - X(T)] = 0. \tag{6.15}$$

It follows that

$$y_0[I - X(T)]x_0 = 0. \tag{6.16}$$

Equations (6.14) and (6.16) yield

$$y_0 X(T) \int_0^T X^{-1}(s) f(s) ds = \int_0^T y_0 X(T) X^{-1}(s) f(s) ds = 0. \tag{6.17}$$

However, $y_0 X(T) = y_0$ and $y(t) \equiv y_0 X^{-1}(t)$ (see proof of Theorem 6.2). Hence,

$$\int_0^T y(s) f(s) ds = 0, \tag{6.18}$$

which shows that (6.11) is necessary.

Before we show that (6.11) is also sufficient, we first recall that a system

$$xC = D \quad (Cx = D) \tag{6.19}$$

with $C \in M_n$, x, D row n-vectors (column n-vectors), has at least one solution if and only if the rank of C equals the rank of the augmented (row-augmented) matrix $[C \mid D]$. Now, assume that (6.18) is true for every T-periodic solution $y(t)$ of (S_a). Then we have

$$y_0 X(T) \int_0^T X^{-1}(s) f(s) ds = 0 \tag{6.20}$$

for every solution y_0 of the system

$$y_0 = y_0 X(T). \tag{6.21}$$

This implies that the dimension of the solution space of (6.15) is the same as the dimension of the solution space of the system

$$y_0[I - X(T)] = 0, \qquad y_0 X(T) \int_0^T X^{-1}(s) f(s) ds = 0. \tag{6.22}$$

By the Kronecker-Capelli theorem, the rank of the matrix $I - X(T)$ is the same as the rank of the augmented matrix

$$\left[I - X(T) \mid X(T)\int_0^T X^{-1}(s)f(s)ds\right]. \tag{6.23}$$

Hence, system (6.14) has at least one solution x_0. □

The following important result says that if system (S_f) has at least one bounded solution on $\mathbb{R}_+$, then (S_f) has at least one T-periodic solution.

THEOREM 6.4. *If system (S_f) does not have any T-periodic solutions, then it has no bounded solutions.*

PROOF. Let $X(t)$ denote the fundamental matrix of system (S) with $X(0) = I$. Suppose that (S_f) does not have any T-periodic solutions. Then (S_a) must have at least one nontrivial T-periodic solution. If this were not true, then Theorem 6.1 would imply the existence of a unique T-periodic solution of (S_f) because the matrix $\widetilde{X} = X(0) - X(T)$ would be nonsingular. Thus, there must exist a nontrivial T-periodic solution $y(t)$ of (S_a) which can be chosen, by Theorem 6.3, to satisfy

$$\int_0^T y(t)f(t)dt \neq 0. \tag{6.24}$$

Let $y_0 = y(0)$. Then we have

$$y_0[I - X(T)] = 0, \qquad y_0\int_0^T X^{-1}(t)f(t)dt \neq 0 \tag{6.25}$$

because $y(t) \equiv y_0X^{-1}(t)$ and $y(t)$ is T-periodic. Now, let $x(t)$ be any solution of (S_f) with $x(0) = x_0$. Then the variation of constants formula (3.47) implies

$$x(T) = X(T)\left[x_0 + \int_0^T X^{-1}(s)f(s)ds\right], \tag{6.26}$$

which yields

$$\begin{aligned} y_0x(T) &= y_0X(T)x_0 + y_0X(T)\int_0^T X^{-1}(s)f(s)ds \\ &= y_0x_0 + y_0\int_0^T X^{-1}(s)f(s)ds. \end{aligned} \tag{6.27}$$

We also have

$$x(t+T) = X(t)x(T) + \int_0^t X(t)X^{-1}(s)f(s)ds \tag{6.28}$$

because both sides of (6.28) are solutions of (S_f) with the value $x(T)$ at $t = 0$. Thus,

$$x(2T) = X(T)\left[x(T) + \int_0^T X^{-1}(s)f(s)ds\right], \tag{6.29}$$

which, along with (6.26) and (6.27), implies

$$\begin{aligned} y_0x(2T) &= y_0x(T) + y_0\int_0^T X^{-1}(s)f(s)ds \\ &= y_0x_0 + 2y_0\int_0^T X^{-1}(s)f(s)ds. \end{aligned} \tag{6.30}$$

Similarly, by induction, we obtain

$$y_0x(nT) = y_0x_0 + ny_0\int_0^T X^{-1}(s)f(s)ds, \quad n = 1, 2, \dots. \tag{6.31}$$

It follows that $x(t)$ cannot be a bounded solution of (S_f) on $\mathbb{R}_+$. In fact, if $x(t)$ was bounded on $\mathbb{R}_+$, then the sequence $\{y_0x(nT)\}$ would be bounded. This in turn would imply that the sequence

$$ny_0\int_0^T X^{-1}(s)f(s)ds, \quad n = 1, 2, \dots, \tag{6.32}$$

is bounded, which is in contradiction with the second relation in (6.25). The proof is complete. □

3. DEPENDENCE OF $x(t)$ ON A, U

In this section, we study the dependence of the solution $x(t)$ of $((S_f), (B_3))$ on the matrix A and the operator U. We actually show that $x(t)$ is a continuous function of A, U in a certain sense. In what follows, U is assumed to be a bounded linear operator on $C_n[0, T]$ with values in $\mathbb{R}^n$. We recall that $C_n[0, T]$ is associated with the sup-norm. The norm of $\mathbb{R}^n$ in this section will be

$$\|x\| = \sum_{i=1}^{n} |x_i|. \tag{6.33}$$

The corresponding matrix norm is

$$\|A\| = \max_k \sum_i |a_{ik}| \tag{6.34}$$

(see Table 1). We use the symbol $\|A\|$ ($\|x\|$) to denote the sup-norm of a time dependent matrix (vector) $A(t)$ ($x(t)$). The proof of the following lemma is left as an exercise.

Lemma 6.5. *Let $A \in M_n$ be nonsingular with $\|A^{-1}\| = M$. Then any matrix $B \in M_n$ with $\|B - A\| < 1/M$ is also nonsingular.*

The fundamental matrix $X(t)$ of (S) with $X(0) = I$ will be denoted by $X_A(t)$. We also denote the matrix $\widetilde{X}$, introduced in Section 1, by $\widetilde{X}_A$. By (S_B) we denote the system (S_f) with $A(t)$ replaced by $B(t)$. The following lemma shows a continuity property of the matrix X_A as a function of the matrix A.

Lemma 6.6. *Let $A : [0,T] \to M_n$ be continuous. Then for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every continuous $B : [0,T] \to M_n$ with*

$$\int_0^T ||A(t) - B(t)||dt < \delta(\epsilon), \tag{6.35}$$

we have

$$||X_A - X_B|| < \epsilon. \tag{6.36}$$

Proof. From

$$X_B'(t) = A(t)X_B(t) + [B(t) - A(t)]X_B(t) \tag{6.37}$$

it is easy to see that

$$X_B(t) = X_A(t)\left[I + \int_0^t X_A^{-1}(s)[B(s) - A(s)]X_B(s)ds\right]. \tag{6.38}$$

Letting

$$K = \max_{t\in[0,T]} ||X_A(t)||, \qquad L = \max_{t\in[0,T]} ||X_A^{-1}(t)||, \tag{6.39}$$

we obtain

$$\begin{aligned} ||X_A(t) - X_B(t)|| &\le \left|\left|X_A(t)\int_0^t X_A^{-1}(s)[A(s) - B(s)][X_A(s) - X_B(s)]ds\right|\right| \\ &\quad + \left|\left|X_A(t)\int_0^t X_A^{-1}(s)[A(s) - B(s)]X_A(s)ds\right|\right| \\ &\le KL\int_0^t ||A(s) - B(s)||||X_A(s) - X_B(s)||ds \\ &\quad + K^2L\int_0^t ||A(s) - B(s)||ds \end{aligned} \tag{6.40}$$

for every $t \in [0,T]$. To get this inequality, we have added to and subtracted an obvious term from the right-hand side of (6.38). Applying Gronwall's inequality to (6.40), we obtain

$$||X_A(t) - X_B(t)|| \leq C \exp\left\{KL \int_0^t ||A(s) - B(s)||ds\right\}, \tag{6.41}$$

where

$$C = K^2 L \int_0^T ||A(t) - B(t)||dt. \tag{6.42}$$

Thus, it suffices to pick as $\delta(\epsilon)$ any positive number λ such that

$$\lambda K^2 L e^{\lambda KL} < \epsilon. \tag{6.43}$$

□

Lemma 6.7. *Let A be as in Lemma 6.6 with $\widetilde{X}_A$ nonsingular. Then there exists a number $\delta > 0$ such that for every continuous $B : [0, T] \to M_n$ with*

$$\int_0^T ||A(t) - B(t)||dt < \delta, \tag{6.44}$$

the matrix $\widetilde{X}_B$ is also nonsingular.

Proof. Let $V = \widetilde{X}_A - \widetilde{X}_B = [v_{ij}]$ and $X_A(t) \equiv [x_{ij}(t)]$, $X_B(t) \equiv [y_{ij}(t)]$. Then we have

$$\begin{aligned}
\sum_{i=1}^n |v_{ij}| &= \left\|U\left(\left[x_{1j}(\cdot) - y_{1j}(\cdot), \ldots, x_{nj}(\cdot) - y_{nj}(\cdot)\right]^T\right)\right\| \\
&\leq \|U\| \left\|\left[x_{1j}(\cdot) - y_{1j}(\cdot), \ldots, x_{nj}(\cdot) - y_{nj}(\cdot)\right]^T\right\| \\
&= \|U\| \max_{t\in[0,T]} \sum_{i=1}^n |x_{ij}(t) - y_{ij}(t)|.
\end{aligned} \tag{6.45}$$

Consequently,

$$\begin{aligned}
\|V\| &= \max_j \sum_{i=1}^n |v_{ij}| \\
&\leq \|U\| \max_j \max_{t\in[0,T]} \sum_{i=1}^n |x_{ij}(t) - y_{ij}(t)| \\
&= \|U\| \max_{t\in[0,T]} \max_j \sum_{i=1}^n |x_{ij}(t) - y_{ij}(t)| \\
&= \|U\| \max_{t\in[0,T]} ||X_A(t) - X_B(t)|| \\
&= \|U\| ||X_A - X_B||.
\end{aligned} \tag{6.46}$$

If we let $\epsilon_0 > 0$ be such that $\epsilon_0 \|U\| \|\widetilde{X}_A^{-1}\| < 1$, then, by Lemma 6.6, there exists $\delta(\epsilon_0) > 0$ such that

$$\int_0^T ||A(t) - B(t)||dt < \delta(\epsilon_0) \tag{6.47}$$

implies $\|X_A - X_B\| < \epsilon_0$. This says that whenever (6.47) holds we have

$$\|V\| = \|\widetilde{X}_A - \widetilde{X}_B\| < \epsilon_0 \|U\|, \tag{6.48}$$

which, by Lemma 6.5, implies the invertibility of the matrix $\widetilde{X}_B$. □

Now, we are ready for the following important corollaries.

Corollary 6.8. *Consider system* (S) *with* $A : [0, T] \to M_n$ *continuous. Assume further that* $\widetilde{X}_A^{-1}$ *exists. Then there exists* $\delta_1 > 0$ *such that, for every continuous* $B : [0, T] \to M_n$ *with*

$$\int_0^T ||A(t) - B(t)||dt < \delta_1, \tag{6.49}$$

the problem $((S_B), (B_3))$ *has a unique solution for every* $f \in C_n[0, T]$ *and every* $r \in \mathbb{R}^n$.

Corollary 6.9. *Let* A, $\widetilde{X}_A$ *be as in Corollary 6.8. Then there exists* $\delta_2 > 0$ *such that, for every bounded linear operator* $U_1 : C_n[0, T] \to \mathbb{R}^n$ *with* $\|U - U_1\| < \delta_2$, *the problem consisting of* (S_f) *and*

$$U_1 x = r \tag{B$_5$}$$

has a unique solution for every $f \in C_n[0, T]$ *and every* $r \in \mathbb{R}^n$.

Proof. Let U_1 be a bounded linear operator mapping $C_n[0, T]$ into $\mathbb{R}^n$ and assume that $\widetilde{X}_{1,A}$ denotes the matrix whose columns are the values of U_1 on the corresponding columns of $X_A(t)$. Moreover, let $W = \widetilde{X}_A - \widetilde{X}_{1,A} = [w_{ij}]$. Then if $X_A(t) \equiv [x_{ij}(t)]$, we have

$$\begin{aligned}
\sum_{i=1}^n |w_{ij}| &= \left\|(U - U_1)\left([x_{1j}(\cdot), x_{2j}(\cdot), \ldots, x_{nj}(\cdot)]^T\right)\right\| \\
&\leq ||U - U_1|| \left\|[x_{1j}(\cdot), x_{2j}(\cdot), \ldots, x_{nj}(\cdot)]^T\right\| \\
&= ||U - U_1|| \max_{t \in [0,T]} \sum_{i=1}^n |x_{ij}(t)|.
\end{aligned} \tag{6.50}$$

It follows that

$$
\begin{aligned}
\|W\| &= \max_j \sum_{i=1}^{n} |w_{ij}| \\
&\le \|U - U_1\| \max_{t\in[0,T]} \max_j \sum_{i=1}^{n} |x_{ij}(t)| \\
&= \|U - U_1\| \max_{t\in[0,T]} \|X(t)\| \\
&= \lambda\|U - U_1\|,
\end{aligned}
\tag{6.51}
$$

where $\lambda = \|X\|$. Thus, according to Lemma 6.5, it suffices to take $\delta_2 = \mu/\lambda$, where μ is a positive number with $\mu\|\widetilde{X}_A^{-1}\| < 1$. For such δ_2, the matrix $\widetilde{X}_{1,A}$ is nonsingular. This, by Theorem 6.1, proves our assertion. □

Combining the above corollaries, we obtain the next theorem which is the main result of this section.

Theorem 6.10. *Let $A : [0,T] \to M_n$ be continuous and let $U : C_n[0,T] \to \mathbb{R}^n$ be a bounded linear operator such that, $\widetilde{X}_A$ is nonsingular. Then there exist two positive numbers δ_1, δ_2 such that, for every continuous $B : [0,T] \to M_n$ with*

$$
\int_0^T \|A(t) - B(t)\| dt < \delta_1 \tag{6.52}
$$

and every bounded linear operator $U_1 : C_n[0,T] \to \mathbb{R}^n$ with $\|U - U_1\| < \delta_2$, the problem $((S_B), (B_5))$ has a unique solution for every $f \in C_n[0,T]$ and every $r \in \mathbb{R}^n$.

Naturally, the integral condition in Theorem 6.10 can be replaced by the condition $\|A - B\| < \delta_1/T = \delta_1^*$.

4. PERTURBED LINEAR SYSTEMS

In this section, we are interested in the solutions of the problem

$$
x' = A(t)x + F(t,x), \tag{S$_F$}
$$

$$
Ux = r, \tag{B$_3$}
$$

where A, U, r are as in Section 3 and $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We will study the same problem on the interval $\mathbb{R}_+$. As we saw in Section 1, the problem $((S_F), (B_3))$ will have a solution on $[0,T]$ if a function $x(t)$, $t \in [0,T]$, can be found satisfying the integral equation

$$
x(t) = X(t)\widetilde{X}^{-1}[r - Up(\cdot, x)] + p(t,x), \tag{6.53}
$$

where

$$p(t,x) = \int_0^t X(t)X^{-1}(s)F(s,x(s))\,ds, \quad t \in [0,T]. \tag{6.54}$$

In the following result we apply the Schauder-Tychonov theorem in order to obtain a fixed point of the operator defined by the right-hand side of (6.53).

Theorem 6.11. *Let*

$$q(t) = \max_{\|u\|\le\alpha} \{\|X^{-1}(t)F(t,u)\|\}, \quad t \in [0,T], \tag{6.55}$$

for some $\alpha > 0$, *and define the operator* K *as follows:*

$$Kf = \widetilde{X}^{-1}[r - Up(\cdot, f)] \tag{6.56}$$

for every $f \in B^\alpha$, *where* B^α *is the closed ball of* $C_n[0,T]$ *with center at zero and radius* α. *Suppose that* $L(M+N) \le \alpha$, *where*

$$L = \max_{t\in[0,T]} \{\|X(t)\|\}, \qquad M = \sup_{g\in B^\alpha} \{\|Kg\|\}, \qquad N = \int_0^T q(t)dt. \tag{6.57}$$

Then problem $((S_F), (B_3))$ *has at least one solution.*

Proof. We are going to show that the operator $V : B^\alpha \to C_n[0,T]$, defined by

$$(Vu)(t) = X(t)\left[Ku + \int_0^t X^{-1}(s)F(s,u(s))\,ds\right], \tag{6.58}$$

has a fixed point in B^α. To this end, let t, $t_1 \in [0,T]$ be given. Then we have, with some manipulation,

$$\begin{aligned}
\|(Vu)(t) - (Vu)(t_1)\| &= \bigg\|X(t)\left[Ku + \int_0^t X^{-1}(s)F(s,u(s))\,ds\right] \\
&\qquad - X(t_1)\left[Ku + \int_0^{t_1} X^{-1}(s)F(s,u(s))\,ds\right]\bigg\| \\
&\le M\|X(t) - X(t_1)\| + N\|X(t) - X(t_1)\| \\
&\qquad + \|X(t_1)\|\left|\int_t^{t_1} q(s)ds\right| \\
&\le (M+N)\|X(t) - X(t_1)\| + L\left|\int_t^{t_1} q(s)ds\right|.
\end{aligned} \tag{6.59}$$

Now, let $\epsilon > 0$ be given. Then there exists $\delta(\epsilon) > 0$ such that

$$||X(t) - X(t_1)|| < \frac{\epsilon}{2(M+N)}, \qquad \left| \int_t^{t_1} q(s)ds \right| < \frac{\epsilon}{2L} \tag{6.60}$$

for every $t, t_1 \in [0, T]$ with $|t - t_1| < \delta(\epsilon)$. This follows from the uniform continuity of the function $X(t)$ and the function

$$h(t) \equiv \int_0^t q(s)ds \tag{6.61}$$

on the interval $[0, T]$. Inequalities (6.59) and (6.60) imply the equicontinuity of the set VB^α. The fact that $VB^\alpha \subset B^\alpha$ follows easily from $L(M+N) \leq \alpha$. Thus VB^α is relatively compact (see Theorem 2.5). Now, we show that V is continuous on B^α. In fact, let $\{u_m\}_{m=1}^\infty \subset B^\alpha$, $u \in B^\alpha$ satisfy

$$||u_m - u||_\infty \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{6.62}$$

Then

$$\begin{aligned} ||Vu_m - Vu||_\infty &\leq L\Big[||Ku_m - Ku|| \\ &\qquad + \int_0^T ||X^{-1}(s)[F(s, u_m(s)) - F(s, u(s))]||ds \Big] \\ &\leq L(L||\widetilde{X}^{-1}|| \, ||U|| + 1) \\ &\qquad \times \int_0^T ||X^{-1}(s)[F(s, u_m(s)) - F(s, u(s))]||ds. \end{aligned} \tag{6.63}$$

The integrand of the second integral in (6.63) converges uniformly to zero. Thus, $\|Vu_m - Vu\|_\infty \to 0$ as $m \to \infty$. By the Schauder-Tychonov theorem, there exists a fixed point $x \in B^\alpha$ of the operator V. This function $x(t)$, $t \in [0, T]$, is a solution to the problem $((S_F), (B_3))$. □

In what follows, the symbol B_α denotes the closed ball of $\mathbb{R}^n$ with center at zero and radius $\alpha > 0$.

Theorem 6.12. *Let B^α be as in Theorem 6.11 for a fixed $\alpha > 0$. Assume that there exists a constant $k > 0$ such that, for every $x_0 \in B_\alpha$ the solution $x(t, 0, x_0)$ of* (S_F), *with $x(0) = x_0$, exists on $[0, T]$, it is unique and satisfies*

$$\sup_{\substack{t \in [0,T] \\ x_0 \in B_\alpha}} ||x(t, 0, x_0)|| \leq k. \tag{6.64}$$

Let

$$N = \int_0^T q(s)ds, \quad \textit{where } q(t) = \max_{\|u\|\le k} \left|\left|X^{-1}(t)F(t,u)\right|\right|, \tag{6.65}$$

and

$$\left|\left|\widetilde{X}^{-1}\right|\right|\left(\|r\| + \|U\|LN\right) \le \alpha, \tag{6.66}$$

where L is defined in Theorem 6.11. Then there exists a solution to the problem $((\mathrm{S}_F),(\mathrm{B}_3))$.

Proof. We are going to apply Brouwer's theorem (Corollary 2.15). To this end, consider the operator $Q : B_\alpha \to \mathbb{R}^n$ defined by

$$Qu = \widetilde{X}^{-1}\left(r - Up_1(\cdot,u)\right), \tag{6.67}$$

where

$$p_1(t,u) \equiv \int_0^t X(t)X^{-1}(s)F\left(s,x(s,0,u)\right)ds. \tag{6.68}$$

It is easy to see that our assumptions imply that $QB_\alpha \subset B_\alpha$. To show the continuity of Q, we need to show the continuity of $x(t,0,u)$ w.r.t. u. Let $u_m \in B_\alpha$, $m = 1,2,\ldots,$ $u \in B_\alpha$ be such that $\|u_m - u\| \to 0$ as $m \to \infty$ and let $x_m(t)$, $x(t)$ be the solutions of

$$\begin{aligned} x' &= A(t)x + F(t,x), \qquad x(0) = u_m, \\ x' &= A(t)x + F(t,x), \qquad x(0) = u, \end{aligned} \tag{6.69}$$

respectively. Then our assumptions imply that $\|x_m(t)\| \le k$ and $\|x(t)\| \le k$ for all $t \in [0,T]$, $m = 1,2,\ldots$. The inequality

$$\left|\left|x_m'(t)\right|\right| \le k \sup_{t\in[0,T]} \left\{\left|\left|A(t)\right|\right|\right\} + \sup_{\substack{t\in[0,T]\\ \|u\|\le k}} \left\{\left|\left|F(t,u)\right|\right|\right\} \tag{6.70}$$

proves that $\{x_m(t)\}$ is also equicontinuous. By the Arzelà-Ascoli theorem (Theorem 2.5), there exists a subsequence $\{x_j(t)\}_{j=1}^{\infty}$ of $\{x_m(t)\}$ such that $x_j(t) \to \overline{x}(t)$ as $j \to \infty$ uniformly on $[0,T]$. Here, $\bar{x}(t)$ is some function in $C_n[0,T]$. Taking limits as $j \to \infty$ in

$$x_j(t) = x_j(0) + \int_0^t A(s)x_j(s)ds + \int_0^t F\left(s,x_j(s)\right)ds, \tag{6.71}$$

we obtain

$$\bar{x}(t) = u + \int_0^t A(s)\bar{x}(s)ds + \int_0^t F\left(s,\bar{x}(s)\right)ds. \tag{6.72}$$

Thus, by uniqueness, $\bar{x}(t) \equiv x(t)$. Since we could have started with any subsequence of $\{x_m(t)\}$ instead of $\{x_m(t)\}$ itself, we have actually shown the following: every subsequence of $\{x_m(t)\}$ contains a subsequence converging uniformly to $x(t)$ on $[0,T]$. This implies the uniform convergence of $\{x_m(t)\}$ to $x(t)$ on $[0,T]$. Equivalently, if $u_m \in B_\alpha$, $m = 1,2,\dots$, $u \in B_\alpha$ satisfy $\|u_m - u\| \to 0$ as $m \to \infty$, then

$$||x(t,0,u_m) - x(t,0,u)|| \longrightarrow 0 \quad \text{uniformly on } [0,T]. \tag{6.73}$$

This yields the continuity of the function $x(t,0,x_0)$ in $x_0 \in B_\alpha$, which is uniform w.r.t. $t \in [0,T]$. Now, let $\{u_m\}$, u be as above. We have

$$||Qu_m - Qu|| \le ||\widetilde{X}^{-1}||\|U\|L\int_0^T ||X^{-1}(s)[F(s,x(s,0,u_m)) - F(s,x(s,0,u))]||ds. \tag{6.74}$$

The integrand in (6.74) tends uniformly to 0 as $m \to \infty$. Thus, $\|Qu_m - Qu\| \to 0$ as $m \to \infty$, and this implies the continuity of Q on B_α. Brouwer's theorem implies the existence of some $x_0 \in B_\alpha$ with the property $Qx_0 = x_0$. This vector x_0 is the initial value of a solution to the problem $((\mathrm{S}_F), (\mathrm{B}_3))$. □

5. PROBLEMS ON INFINITE INTERVALS

Theorem 6.11 can be extended to problems on infinite intervals. Actually, in this section we show that this can be done in two ways. If we work in the space $C_n(\mathbb{R}_+)$, we *truncate* the problem by solving an equation such as (6.53) on $[0,m]$, $m = 1,2,\dots$, and then obtain a solution on $\mathbb{R}_+$ by a suitable approximation. If the domain of the operator V is C_n^l, then we can solve (6.53) by finding directly a fixed point of V. This second method of solution is possible because we can detect the compact sets in C_n^l (see Exercise 2.5).

The following well-known lemma will be useful in the sequel. Its proof is left as an exercise.

Lemma 6.13 (Lebesgue's dominated convergence theorem). *Let $f_n : [t_0,\infty) \to \mathbb{R}^n$, $m = 1,2,\dots$, $t_0 \ge 0$, be continuous and such that the improper integrals*

$$\int_{t_0}^\infty f_m(t)dt, \quad m = 1,2,\dots, \tag{6.75}$$

are convergent. Let $f_m \to f$ as $m \to \infty$ pointwise on $[t_0,\infty)$, where $f : [t_0,\infty) \to \mathbb{R}^n$ is continuous and such that the improper integral

$$\int_{t_0}^\infty f(t)dt \tag{6.76}$$

is convergent. Furthermore, let $\|f_m(t)\| \leq g(t)$, $t \in [t_0, \infty)$, $m = 1, 2, \ldots,$ *where* $g : [t_0, \infty) \to \mathbb{R}_+$ *is continuous and such that*

$$\int_{t_0}^{\infty} g(t)dt < +\infty. \tag{6.77}$$

Then we have

$$\lim_{m\to\infty} \int_{t_0}^{\infty} f_m(t)dt = \int_{t_0}^{\infty} \lim_{m\to\infty} f_m(t)dt = \int_{t_0}^{\infty} f(t)dt. \tag{6.78}$$

In the proof of the next result the integral operator V is defined on $C_n(\mathbb{R}_+)$. As before, the symbol $\widetilde{X}$ denotes the matrix in M_n whose columns are the values of U on the corresponding columns of $X(t)$. Naturally, this pre-supposes that the columns of $X(t)$ belong to $C_n(\mathbb{R}_+)$. A similar situation will be assumed for the space C_n^l.

Theorem 6.14. *Let* $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and such that* $\|X(t)\| \leq M$, $t \in \mathbb{R}_+$, *where* M *is a positive constant. Furthermore, assume the following:*

(i) *there exist two continuous functions* $g, q : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{aligned} &\left\|X^{-1}(t)F(t,u)\right\| \leq q(t)\|u\| + g(t), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ &L = \int_0^{\infty} q(t)dt < +\infty, \qquad N = \int_0^{\infty} g(t)dt < +\infty; \end{aligned} \tag{6.79}$$

(ii) $U : C_n(\mathbb{R}_+) \to \mathbb{R}^n$ *is a bounded linear operator such that* $\widetilde{X}^{-1}$ *exists;*

(iii) $LM^2\|\widetilde{X}^{-1}\|\|U\|e^{LM} < 1$.

Then problem ((S$_F$), (B$_3$)) *has at least one solution.*

Proof. As in the proof of Theorem 6.11, it suffices to show that the equation

$$x(t) = X(t)\widetilde{X}^{-1}[r - Up(\cdot, x)] + p(t, x) \tag{6.53}$$

has a solution $x \in C_n(\mathbb{R}_+)$. To this end, we consider the spaces D_m, $m = 1, 2, \ldots,$ defined as follows:

$$D_m = \{x \in C_n(\mathbb{R}_+) : x(t) = x(m),\ t \geq m\}. \tag{6.80}$$

The spaces D_m are Banach spaces with norms

$$\|x\|_m = \sup_{t\in[0,m]} \left\|x(t)\right\|, \tag{6.81}$$

respectively. We define the operator $V_1 : D_1 \times [0,1] \to D_1$ as follows: if $(f, \mu) \in D_1 \times [0,1]$, then $V_1(f, \mu) = h \in D_1$, where the function h satisfies

$$h(t) = \mu X(t)\widetilde{X}^{-1}[r - Up(\cdot, f)] + \mu p(t, f), \quad t \in [0,1]. \tag{6.82}$$

We are planning to apply the Leray-Schauder theorem (Theorem 2.16) in order to obtain a fixed point for the operator $V_1(x,1)$ in D_1. We first show that $V_1(x,\mu)$ is continuous in x. Let $\{f_k\}_{k=1}^{\infty}$, $f \in D_1$ be given with $f_k \to f$ as $k \to \infty$ uniformly on $\mathbb{R}_+$. This is equivalent to saying that

$$||f_k - f||_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{6.83}$$

We have

$$\begin{aligned} &||V_1(f_k,\mu) - V_1(f,\mu)||_1 \\ &\quad \leq M^2||\widetilde{X}^{-1}||\|U\| \int_0^{\infty} ||X^{-1}(s)[F(s,f_k(s)) - F(s,f(s))]||ds \\ &\qquad + M\int_0^{\infty} ||X^{-1}(s)[F(s,f_k(s)) - F(s,f(s))||]||ds. \end{aligned} \tag{6.84}$$

Since $\|F(t,f_k(t)) - F(t,f(t))\| \to 0$ as $m \to \infty$ pointwise on $\mathbb{R}_+$ and

$$||X^{-1}(t)[F(t,f_k(t)) - F(t,f(t))]|| \leq q(t)[||f_k||_1 + \|f\|_1] + 2g(t), \quad t \in \mathbb{R}_+. \tag{6.85}$$

Inequality (6.84), Lemma 6.13, and our hypotheses on g, q imply that

$$||V_1(f_k,\mu) - V_1(f,\mu)||_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{6.86}$$

It follows that $V_1(\cdot,\mu)$ is continuous on D_1. Before we show the compactness of the operator $V_1(\cdot,\mu)$, we show that all possible solutions of $V_1(x,\mu) = x$ lie in a ball of D_1 which does not depend on μ (see Theorem 2.16(iii)). To this end, let $x \in D_1$ solve $V_1(x,\mu_0) = x$, for some $\mu_0 \in [0,1]$. Then

$$\begin{aligned} ||x(t)|| &\leq M||\widetilde{X}^{-1}||[\|r\| + M\|U\|(L\|x\|_1 + N)] \\ &\quad + M\int_0^t q(s)||x(s)||ds + MN, \quad t \in [0,1]. \end{aligned} \tag{6.87}$$

Letting

$$\begin{aligned} K &= LM^2||\widetilde{X}^{-1}||\|U\|, \\ Q &= M||\widetilde{X}^{-1}||(\|r\| + M\|U\|N) + MN, \end{aligned} \tag{6.88}$$

we obtain

$$||x(t)|| \leq K\|x\|_1 + Q + M\int_0^t q(s)||x(s)||ds, \quad t \in [0,1]. \tag{6.89}$$

Applying Gronwall's inequality, we arrive at

$$\begin{aligned} ||x(t)|| &\leq (K\|x\|_1 + Q) \exp\left\{M\int_0^t q(s)ds\right\} \\ &\leq (K\|x\|_1 + Q)e^{LM}, \quad t \in [0,1], \end{aligned} \tag{6.90}$$

which yields

$$\|x\|_1 \leq \left(1 - Ke^{LM}\right)^{-1} Qe^{LM}. \tag{6.91}$$

Consequently, every solution $x \in D_1$ of $V_1(x,\mu) = x$ satisfies $\|x\|_1 \leq \alpha$, where the constant α equals the right-hand side of (6.91) and does not depend on μ. The fact that $V_1(\cdot,\mu)$ maps bounded sets onto bounded sets for each $\mu \in [0,1]$, follows from

$$\left|\left|V_1(f,\mu)\right|\right|_1 \leq K\|f\|_1 + Q + M\int_0^\infty q(t)dt\|f\|_1, \tag{6.92}$$

which can be obtained as the inequality preceding (6.90). This property and the equicontinuity of the image under $V_1(\cdot,\mu)$ of any bounded subset of D_1 imply the compactness of the operator $V_1(\cdot,\mu)$ for each $\mu \in [0,1]$. The equicontinuity of $V_1(f,\cdot)$, in the sense of Theorem 2.16(i), follows from

$$\begin{aligned} &||(V_1(f,\mu) - V_1(f,\mu_0))(t)|| \\ &\quad \leq |\mu-\mu_0|\,(||X(t)||||\widetilde{X}^{-1}||[\|r\|+\|U\|||p(\cdot,f)||]+||p(t,f)||), \quad t\in[0,1], \end{aligned} \tag{6.93}$$

and the uniform boundedness of the term multiplying $|\mu - \mu_0|$ above on the set $\{(t,f) : t \in [0,1], f \in B\}$, for any bounded subset B of D_1. By the Leray-Schauder theorem (Theorem 2.16 with $t_0 = 0$ there), we obtain a solution $x_1 \in B_\alpha = \{x \in C_n(\mathbb{R}_+) : \|x\|_\infty \leq \alpha\}$ of the equation $V_1(x,1) = x$. This solution belongs to D_1 and satisfies (6.53) on $[0,1]$. Using mathematical induction, we obtain a sequence of solution $x_m \in D_m$, $m = 1,2,\ldots$, such that $x_m(t)$ satisfies (6.53) on $[0,m]$ and belongs to B_α.

It is easy to see now, using the differential equation (S_F), that the sequence $\{x_m(t)\}_{m=1}^\infty$ is equicontinuous on the interval $[0,1]$. In fact, the sequence $\{x'_m(t)\}_{m=1}^\infty$ is uniformly bounded on $[0,1]$. By Theorem 2.5, there exists a subsequence $\{x_m^1(t)\}$ of $\{x_m(t)\}$ such that $x_m^1(t) \to x^1(t)$ as $m \to \infty$ uniformly on $[0,1]$. Similarly, there exists a subsequence $\{x_m^2(t)\}$ of $\{x_m^1(t)\}$ which converges uniformly to a function $x^2(t)$, $t \in [0,2]$. We obviously have $x^2(t) = x^1(t)$, $t \in [0,1]$. Continuing this process by induction, we finally obtain a diagonal sequence $\{x_m^m(t)\}$ such that $x_m^m(t) \to x(t)$ uniformly on any interval $[0,c]$, $c > 0$. This function $x(t)$ is a continuous function on $\mathbb{R}_+$ with $\|x(t)\| \le \alpha$, $t \in \mathbb{R}_+$. Let

$$y(t) = X(t)\widetilde{X}^{-1}[r - Up(\cdot, x)] + p(t, x), \quad t \in \mathbb{R}_+. \tag{6.94}$$

Then, for any $c > 0$ and $m \ge c$, we get

$$\begin{aligned} &||x_m^m(t) - y(t)|| \\ &\quad \le M(M||\widetilde{X}^{-1}||\|U\| + 1)\int_0^\infty ||X^{-1}(s)[F(s, x_m^m(s)) - F(s, x(s))]||ds, \end{aligned} \tag{6.95}$$

for all $t \in [0, c]$. Applying once again the Lebesgue dominated convergence theorem (Lemma 6.13), we obtain that $x_m^m \to y$ as $m \to \infty$ uniformly on every interval $[0, c]$. This shows that $x(t) \equiv y(t)$, $t \in \mathbb{R}_+$. The function $x(t)$ is a solution to the problem $((S_F), (B_3))$. □

In Theorem 6.15 the space $C_n(\mathbb{R}_+)$ is replaced by the space C_n^l.

Theorem 6.15. *Along with the assumptions (i)–(iii) of Theorem 6.14, assume that*

$$\lim_{t \to \infty} X(t) = X(\infty) \tag{6.96}$$

exists as a finite matrix. Furthermore, assume that $U : C_n^l \to \mathbb{R}^n$ is a bounded linear operator such that $\widetilde{X}^{-1}$ exists. Then, for every $r \in \mathbb{R}^n$, the problem $((S_F), (B_3))$ has at least one solution.

Proof. We consider the operator $V(x, \mu) : C_n^l \times [0, 1] \to C_n^l$ defined as follows:

$$(V(f, \mu))(t) = \mu X(t)\widetilde{X}^{-1}[r - Up(\cdot, f)] + \mu p(t, f), \quad t \in \mathbb{R}_+. \tag{6.97}$$

In order to apply the Leray-Schauder theorem, we show here only the equiconvergence (see Exercise 2.5(iii)) of the set

$$\{V(f, \mu) : f \in B\} \tag{6.98}$$

for every $\mu \in [0, 1]$, where B is any bounded subset of C_n^l. This property is needed for the compactness of the operator $V(\cdot, \mu)$. The rest of the assumptions of the Leray-Schauder theorem follow as in the proof of Theorem 6.14 and are therefore omitted.

For $f \in B$, let $\|f\|_\infty \le \alpha$ and let

$$\lim_{t \to \infty} (V(f, \mu))(t) = \xi, \tag{6.99}$$

for some $\mu \in (0,1]$. Then we have

$$
\begin{aligned}
\|(V(f,\mu))(t)-\xi\| \le{}& \|X(t)-X(\infty)\|\|\widetilde{X}^{-1}\|\|r\| \\
&+ M\|X(t)-X(\infty)\|\|\widetilde{X}^{-1}\|\|U\|(L\|f\|_\infty + N) \\
&+ \|X(t)-X(\infty)\|\int_0^\infty \|X^{-1}(s)F(s,f(s))\|ds \\
&+ \|X(\infty)\|\int_t^\infty \|X^{-1}(s)F(s,f(s))\|ds \\
\le{}& \|X(t)-X(\infty)\|\|\widetilde{X}^{-1}\|\|r\| \\
&+ M\|X(t)-X(\infty)\|\|\widetilde{X}^{-1}\|\|U\|(L\alpha+N) \\
&+ \|X(t)-X(\infty)\|(L\alpha+N) \\
&+ \alpha M\int_t^\infty q(s)ds + M\int_t^\infty g(s)ds.
\end{aligned}
\tag{6.100}
$$

This shows the equiconvergence of the set in (6.98). □

Remark 6.16. Naturally, the results of the last two sections, concerning the existence of solutions of boundary value problems of perturbed linear systems, have analogues for contraction integral operators. This is the content of Exercise 6.7, where extensions are sought for Theorems 6.14 and 6.15.

Example 6.17. Consider the system

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon e^{-t}\ln\left(|x_1|+1\right) + \left[\dfrac{1}{(t^2+1)}\right]\sin x_1 \end{bmatrix}
\tag{6.101}
$$

and the boundary conditions

$$
Ux \equiv \int_0^\infty P(t)x(t)dt = \begin{bmatrix} -1 \\ \pi \end{bmatrix}
\tag{6.102}
$$

for $x \in C_n(\mathbb{R}_+)$, where

$$
P(t) \equiv \begin{bmatrix} e^{-t} & -e^{-3t} \\ 0 & 2e^{-2t} \end{bmatrix}.
\tag{6.103}
$$

Choosing, for convenience, the norm $\|x\| = |x_1| + |x_2|$ in $\mathbb{R}^2$, we have

$$
\|A\| = \sup_k \sum_i |a_{ik}|
\tag{6.104}
$$

for the 2×2 matrix A. Thus, we obtain

$$\int_0^\infty ||P(t)||dt < +\infty,$$

$$X(t) \equiv \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \qquad \widetilde{X} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}. \tag{6.105}$$

We also find

$$||X^{-1}(t)F(t,u)|| \le 2\epsilon e^{-t}\|u\| + \frac{2}{t^2+1}. \tag{6.106}$$

It follows from Theorem 6.14 that problem ((6.101), (6.102)) has at least one solution for all sufficiently small $\epsilon > 0$.

Theorem 7.14 contains another application of the Leray-Schauder theorem to b.v.p.'s on infinite intervals.

EXERCISES

6.1 (Green's function). For system (S_f), assume that $A : \mathbb{R} \to M_n$, $f : \mathbb{R} \to \mathbb{R}^n$ are continuous and T-periodic. Assume further that the only T-periodic solution of the homogeneous system (S) is the zero solution. Find a function $G(t,s)$ such that

$$x(t) = \int_0^T G(t,s)f(s)ds, \tag{6.107}$$

where $x(t)$ is the (unique) T-periodic solution of (S_f). Hint. Show that $x(t)$ is given by

$$\begin{aligned} x(t) = X(t)\big[I - X(T)\big]^{-1}X(T)\int_0^T X^{-1}(s)f(s)ds \\ + X(t)\int_0^t X^{-1}(s)f(s)ds. \end{aligned} \tag{6.108}$$

Then show that $G(t,s)$ is the function defined by

$$\begin{aligned} G(t,s) &= X(t)\big[I - X(T)\big]^{-1}X(T)X^{-1}(s) + X(t)X^{-1}(s), \quad 0 \le s \le t \le T, \\ G(t,s) &= X(t)\big[I - X(T)\big]^{-1}X(T)X^{-1}(s), \quad 0 \le t < s \le T. \end{aligned} \tag{6.109}$$

6.2. Assume that $A : \mathbb{R} \to M_n$ is continuous and T-periodic. Assume further that for every $f \in P_n(T)$ the system (S_f) has at least one T-periodic solution. Show that the only T-periodic solution of the homogeneous system (S) is the zero solution.

6.3. Consider the scalar equation

$$x'' - x = f(t), \tag{6.110}$$

where $f : [0, T] \to \mathbb{R}$ is continuous. Show that this equation has a unique solution $x(t)$, $t \in [0, T]$, such that

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{6.111}$$

Hint. Examine the system in $\mathbb{R}^2$ arising from (6.110).

6.4. Let $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous and such that $|F(t, u)| \leq M$, $t \in [0, T]$, $|u| \leq \alpha$, where M, α are positive constants. Show that if M is sufficiently small, the equation

$$x'' - x = F(t, x) \tag{6.112}$$

has at least one solution $x(t)$, $t \in [0, T]$, satisfying (6.111). Hint. Apply the Schauder-Tychonov Theorem.

6.5. Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.113}$$

and the boundary conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix}. \tag{6.114}$$

Find a constant $\delta > 0$ such that for every $B : [0, 3] \to M_2$, continuous and such that

$$\|B - A\|_\infty < \delta, \tag{6.115}$$

the problem

$$x' = B(t)x, \qquad x(0) = x(3) \tag{6.116}$$

has a unique solution. Here,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{6.117}$$

6.6. In the scalar equation

$$x'' = p(t)g(x), \tag{6.118}$$

let $p : \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}, g : \mathbb{R} \to \mathbb{R}$ be continuous. Assume further that p is T-periodic and $ug(u) > 0$ for $u \neq 0$. Show that the only T-periodic solution of (6.118) is the zero solution. Hint. Show that (6.118) has no nontrivial solutions with arbitrarily large zeros.

6.7. Using the contraction mapping principle, obtain unique solutions to the boundary value problems considered in Theorems 6.14 and 6.15. Naturally, suitable Lipschitz conditions are needed here for the function $F(t,u)$.

6.8. Consider the scalar equation

$$x' = x + q(t), \tag{6.119}$$

where $q : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. Show that the unique T-periodic solution $x(t)$ of (6.119) is given by the formula

$$x(t) = \left[\frac{e^T}{(1-e^T)}\right]\int_0^T e^{t-s}q(s)ds + \int_0^t e^{t-s}q(s)ds, \quad t \in [0,T]. \tag{6.120}$$

Let $T = 2\pi$ and determine the size of the constant $k > 0$ so that the equation

$$x' = x + k(\sin t)\big(|x| + 1\big) \tag{6.121}$$

has at least one 2π-periodic solution. Extend this result to the equation

$$x' = x + f(t,x), \tag{6.122}$$

for a suitable function $f(t,u)$ which is T-periodic in t.

6.9. Prove Lemma 6.5.

6.10. Prove Lebesgue's dominated convergence theorem (Lemma 6.13).

6.11. Assume that the linear system (S_f), with $A : [0,T] \to M_n$, $f : [0,T] \to \mathbb{R}^n$ continuous, has a unique solution satisfying the *boundary conditions*

$$Ux = r, \tag{6.123}$$

where $Ux = x(0)$, for any $x \in C_n[0,T]$, and $r \in \mathbb{R}^n$ is fixed. Consider the boundary conditions

$$U_1x = x(0) - Nx(T) = r, \tag{U_1}$$

where N is a matrix in M_n. Show that if $\|N\|$ is sufficiently small, the problem $((\mathrm{S}_f), (\mathrm{U}_1))$ has a unique solution.

6.12. Find constants a, b, c, d, not all zero, such that the problem

$$\begin{gathered}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{gathered} \tag{6.124}$$

has a unique solution $x(t)$, $t \in [0,1]$. Hint. Use the result of Exercise 6.11.

6.13. Assume that $A : \mathbb{R} \to M_n$, $f : \mathbb{R} \to \mathbb{R}^n$ are continuous and T-periodic. Assume further that the fundamental matrix $X(t)$ ($X(0) = I$) of system (S) satisfies

$$||X(t)|| \leq e^{-\lambda t}, \quad t \in [0,T], \tag{6.125}$$

where λ is a positive constant. Show that the system (S_f) has a unique T-periodic solution.

6.14. Show that results like Theorems 6.1 and 6.10 are true, under suitable assumptions, on infinite intervals. The b.v.p. now is the problem ((S_f), (B_3)), where $U : C_n^l \to \mathbb{R}^n$ is a bounded linear operator.

6.15. Consider the system

$$x' = A(t)x + F(t,x), \tag{6.126}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad F(t,u) = \epsilon e^{-3t} \begin{bmatrix} \sin u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-3t} \sin u_1 \end{bmatrix}, \tag{6.127}$$

and the boundary conditions

$$x(0) - x(\infty) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{B}$$

Show that the problem ((S_F), (B)) has at least one solution for all sufficiently small $\epsilon > 0$.

6.16 (Massera). Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume further that $F(t,u)$ is T-periodic in t and that for every compact set $K \subset \mathbb{R}^2$ there exists a constant $L_K > 0$ such that

$$|F(t,u_1) - F(t,u_2)| \leq L_K |u_1 - u_2| \tag{6.128}$$

for every $(t,u_1),(t,u_2)\in K$. Let $x(t)$, $t\in\mathbb{R}_+$, be a solution of

$$x' = F(t,x) \tag{E}$$

such that $\|x\|_\infty \le M$, where M is a positive constant. Show that (E) has at least one T-periodic solution. Hint. Assume that $x(t)$ is not T-periodic. (1) Show that the functions $x_m(t)\equiv x(t+mT)$, $m=1,2,\dots$, are also solutions of (E). (2) We may assume that $y_m \neq y_{m+1}$ for all $m=1,2,\dots$, where $y_m = x(mT)$. In fact, if $y_m = y_{m+1}$, for some m, then $x(mT)=x((m+1)T)$ and uniqueness imply that $x_m(t)\equiv x(t+mT)$ is a T-periodic solution, and we are done. Let $x(0)<x_1(0)$. Then, by uniqueness, $x(t)<x_1(t)$. Thus, for $t=mT$, $y_m<y_{m+1}$, which implies $x_m(t)<x_{m+1}(t)$. We have

$$\lim_{m\to\infty} x_m(t) = \bar{x}(t), \quad t\in\mathbb{R}_+, \tag{6.129}$$

where $\bar{x}(t)$ is some function. Since $\{x_m\}$ is actually an equicontinuous and uniformly bounded sequence on any compact subinterval of $\mathbb{R}_+$, the limit in (6.129) is uniform on any such interval by the Arzelà-Ascoli theorem. Thus $\bar{x}(t)$ is continuous. Let $\xi = \lim_{m\to\infty} y_m$. Then

$$\begin{aligned}\bar{x}(T) &= \lim_{m\to\infty} x_m(T) = \lim_{m\to\infty} y_{m+1} = \xi,\\ \bar{x}(0) &= \lim_{m\to\infty} x_m(0) = \lim_{m\to\infty} y_m = \xi\end{aligned} \tag{6.130}$$

and uniqueness say that $\bar{x}(t+T)\equiv\bar{x}(t)$.

6.17. Is every solution $x(t)$, $t\in\mathbb{R}_+$, of (E) in Exercise 6.16 necessarily bounded? Why?

6.18. Assume that the eigenvalues of the matrix $A\in M_n$ have negative real parts. Let $f:\mathbb{R}\to\mathbb{R}^n$ be continuous and T-periodic. Show that the function

$$x(t)\equiv\int_{-\infty}^{t} e^{(t-s)A} f(s)\,ds \tag{6.131}$$

is the unique T-periodic solution of the system

$$x' = Ax + f(t). \tag{6.132}$$

Hint. Let $u = t-s$ to see the periodicity.

6.19. Assume that the eigenvalues of the matrix $A\in M_n$ have positive real parts. Let $f:\mathbb{R}\to\mathbb{R}^n$ be continuous and T-periodic. What does the unique T-periodic solution of the system (6.132) look like?

CHAPTER 7

MONOTONICITY

This chapter is devoted to the study of systems of the form

$$x' = A(t)x + F(t,x) \tag{S_F}$$

under *monotonicity* assumptions on the matrices $A(t)$ and/or the functions $F(t,u)$. By *monotonicity assumptions* we mean conditions that include inner products involving $A(t)$ or $F(t,u)$. Although there is some overlapping between this chapter and Chapter 5, the present development is preferred because it constitutes a good step toward the corresponding theory in Banach and Hilbert spaces. This theory encompasses a substantial part of the modern theory of ordinary and partial differential equations.

In Section 1, we introduce a new norm for $\mathbb{R}^n$ which depends on a positive definite matrix. We also establish some fundamental properties of this norm.

In Section 2, we examine the stability properties of solutions of (S_F) via monotonicity conditions.

Stability regions are introduced in Section 3. These regions have to do with stability properties of solutions with further restricted initial conditions.

Section 4 is concerned with periodic solutions of differential systems, while Section 5 indicates the applicability of monotonicity methods to a certain boundary value problem. This problem has boundary conditions which do not contain periodicity conditions as a special case.

1. A MORE GENERAL INNER PRODUCT

Using a positive definite matrix $V \in M_n$, we introduce another inner product for $\mathbb{R}^n$ which is reduced to the usual one if $V = I$. Lemma 7.1 contains this introduction as well as some fundamental properties of such an inner product.

Lemma 7.1. *Let $V \in M_n$ be positive definite and define $\langle \cdot, \cdot \rangle_V$ by*

$$\langle x, y \rangle_V = \langle Vx, y \rangle, \quad x, y \in \mathbb{R}^n. \tag{7.1}$$

Then $\langle \cdot, \cdot \rangle_V$ *has the following properties:*

(i) $\langle x, y\rangle_V = \langle y, x\rangle_V$, $x, y \in \mathbb{R}^n$;
(ii) $\langle x, \alpha y + \beta z\rangle_V = \alpha\langle x, y\rangle_V + \beta\langle x, z\rangle_V$, $\alpha, \beta \in \mathbb{R}$, $x, y, z \in \mathbb{R}^n$;
(iii) $\langle x, x\rangle_V \geq 0$, $x \in \mathbb{R}^n$, *and* $\langle x, x\rangle_V = 0$ *if and only if* $x = 0$;
(iv) $|\langle x, y\rangle_V| \leq \|V\|\|x\|\|y\|$, $x, y \in \mathbb{R}^n$.

The proof is left as an exercise.

The following lemma is needed for the establishment of a certain monotonicity property of a matrix whose eigenvalues have negative real parts. This property is the content of Lemma 7.3.

Lemma 7.2. *Let* $A, B \in M_n$ *be given, with* A *having all of its eigenvalues with negative real parts and* B *positive definite. Then there exists a positive definite* $V \in M_n$ *such that*

$$A^T V + VA = -B. \tag{7.2}$$

Proof. From Theorem 4.6 we know that the system $x' = Ax$ is asymptotically stable. Since, for autonomous systems, asymptotic stability is equivalent to uniform asymptotic stability, Inequality (4.5) implies that $\|e^{tA}\| \leq Ke^{-\alpha t}$, $t \in [0, \infty)$, where α, K are positive constants. Similarly, since e^{tA^T} satisfies the system

$$X' = XA^T, \tag{7.3}$$

we also obtain $\|e^{tA^T}\| \leq K_1 e^{-\alpha_1 t}$ for some positive constants α_1, K_1. Consequently, the matrix

$$V = \int_0^\infty e^{tA^T} B e^{tA} dt \tag{7.4}$$

is well defined. Let $W = e^{tA^T} B e^{tA}$, $t \in \mathbb{R}_+$. Then $W(0) = B$ and

$$W' = A^T W + WA. \tag{7.5}$$

Integrating (7.5) from 0 to ∞, taking into consideration that $W(t) \to 0$ as $t \to +\infty$, we obtain

$$-B = A^T V + VA. \tag{7.6}$$

The positive definiteness of V follows easily from (7.4) and the positive definiteness of the matrix B. $\square$

It should be remarked here that V is the unique solution of (7.2), but this fact will not be needed in the sequel.

LEMMA 7.3. *Let A, B, V be as in Lemma 7.2. Then*

$$\langle Ax, x\rangle_V \leq -\left(\frac{\lambda}{(2\mu)}\right)\langle x, x\rangle_V, \tag{7.7}$$

where λ is the smallest eigenvalue of B and μ is the largest eigenvalue of V.

PROOF. We have

$$\begin{aligned}\langle Ax, x\rangle_V &= \langle VAx, x\rangle = \langle Ax, V^T x\rangle \\ &= \langle Ax, Vx\rangle = \langle x, A^T Vx\rangle \\ &= \langle A^T Vx, x\rangle = -\langle Bx, x\rangle - \langle VAx, x\rangle \\ &= -\langle Bx, x\rangle - \langle Ax, x\rangle_V\end{aligned} \tag{7.8}$$

for every $x \in \mathbb{R}^n$, which implies

$$\langle Ax, x\rangle_V = -\left(\frac{1}{2}\right)\langle Bx, x\rangle \leq -\left(\frac{\lambda}{2}\right)\langle x, x\rangle \leq -\left(\frac{\lambda}{(2\mu)}\right)\langle x, x\rangle_V. \tag{7.9}$$

Here, we have used Theorem 1.14. □

The inner product $\langle\cdot,\cdot\rangle_V$ induces a norm $\|\cdot\|_V$ on $\mathbb{R}^n$ with $\|u\|_V^2 = \langle Vu, u\rangle$. Since all norms of $\mathbb{R}^n$ are equivalent, it is easy to see that a vector-valued function $(x_1(t),\dots,x_n(t))$ is continuous (differentiable) w.r.t. the Euclidean norm if and only if it is continuous (differentiable) w.r.t. the norm $\|\cdot\|_V$.

LEMMA 7.4. *Let $x : \mathbb{R}_+ \to \mathbb{R}^n$ be continuously differentiable on $[0, T)$, $T \in (0, +\infty]$. Then $\|x(t)\|_V^2$ is also continuously differentiable on $[0, T)$, and*

$$\left(\frac{d}{dt}\right)\|x(t)\|_V^2 = 2\langle x'(t), x(t)\rangle_V, \quad t \in [0, T). \tag{7.10}$$

Here, V is any positive definite matrix in M_n.

PROOF. We have

$$\begin{aligned}\left(\frac{d}{dt}\right)\langle x(t), x(t)\rangle_V &= \langle (Vx(t))', x(t)\rangle + \langle Vx(t), x'(t)\rangle \\ &= \langle Vx'(t), x(t)\rangle + \langle Vx(t), x'(t)\rangle \\ &= 2\langle x'(t), x(t)\rangle_V.\end{aligned} \tag{7.11}$$

□

The next theorem provides an upper bound for $|\lambda - \lambda'|$, where λ is an eigenvalue of a matrix $A \in M_n$ and λ' is a corresponding eigenvalue of another matrix $B \in M_n$. The proof of this theorem can be found in Ostrowski [44, page 334].

Theorem 7.5. *Let* $A = [a_{ij}]$, $B = [b_{ij}]$, $i, j, = 1, 2, \dots, n$, *be two matrices in* M_n *with eigenvalues denoted by* λ, λ', *respectively. Let*

$$M = \max_{i,j=1,2,\dots,n} \{ |a_{ij}|, |b_{ij}| \},$$
$$r = \frac{1}{nM} \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|. \tag{7.12}$$

Then to every eigenvalue λ' *corresponds to an eigenvalue* λ *such that*

$$|\lambda' - \lambda| \leq (n+2)Mr^{1/n}. \tag{7.13}$$

2. STABILITY OF DIFFERENTIAL SYSTEMS

In our first stability result we need the following existence, uniqueness, and continuation theorem.

Theorem 7.6. *Assume that* $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuous. Furthermore, assume the existence of a function* $p : \mathbb{R}_+ \to \mathbb{R}$ *which is continuous and such that*

$$\langle F(t,x) - F(t,y), x - y \rangle \leq p(t)\|x - y\|^2 \tag{7.14}$$

for every $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$. *Then for every* $x_0 \in \mathbb{R}^n$ *there exists a unique solution* $x(t)$, $t \in \mathbb{R}_+$, *of* (S_F) *such that* $x(0) = x_0$. *Moreover, if* $x(t)$, $y(t)$, $t \in \mathbb{R}_+$, *are two solutions of* (S_F) *such that* $x(0) = x_0$, $y(0) = y_0$, *then*

$$||x(t) - y(t)|| \leq ||x_0 - y_0|| \exp\left\{ \int_0^t [p(s) + q(s)]ds \right\} \tag{7.15}$$

for every $t \in \mathbb{R}_+$, *where* $q(t)$ *is the largest eigenvalue of* $(1/2)[A(t) + A^T(t)]$.

Proof. Let $u(t) = x(t) - y(t)$, $t \in [0, T)$, where $x(t)$, $y(t)$ are two solutions of (S_F) such that $x(0) = x_0$, $y(0) = y_0$, and T is some positive number. Then we have

$$u'(t) = A(t)u(t) + F(t, x(t)) - F(t, y(t)) \tag{7.16}$$

for $t \in [0, T)$, and $u(0) = x_0 - y_0$. Applying Lemma 7.4 for $V = I$, we get

$$\left(\frac{d}{dt}\right)||u(t)||^2 = 2\langle A(t)u(t) + F(t, x(t)) - F(t, y(t)), u(t)\rangle. \tag{7.17}$$

It is easy to see that

$$2\langle A(t)u(t), u(t)\rangle = \langle [A(t) + A^T(t)]u(t), u(t)\rangle. \tag{7.18}$$

The continuity of the symmetric matrix $(1/2)[A(t)+A^T(t)]$ implies the continuity of its largest eigenvalue $q(t)$ (see Exercise 1.23), and we have

$$2\langle A(t)u(t), u(t)\rangle \le 2q(t)||u(t)||^2. \tag{7.19}$$

This inequality, combined with (7.17), yields

$$\begin{aligned}\left(\frac{d}{dt}\right)||u(t)||^2 &\le 2q(t)||u(t)||^2 + 2p(t)||u(t)||^2\\ &= 2[p(t)+q(t)]||u(t)||^2\end{aligned} \tag{7.20}$$

for every $t \in [0, T)$. Applying Lemma 4.11 to (7.20), we obtain

$$||u(t)||^2 \le ||u(0)||^2 \exp\left\{2\int_0^t [p(s)+q(s)]ds\right\}, \quad t \in [0, T). \tag{7.21}$$

If $y(t)$ is a solution of (S_F) with $y(0) = 0$, defined on a right neighborhood of 0, then

$$\begin{aligned}\left(\frac{d}{dt}\right)||y(t)||^2 &= 2\langle A(t)y(t), y(t)\rangle + 2\langle F(t,y(t)) - F(t,0), y(t)\rangle\\ &\quad + 2\langle F(t,0), y(t)\rangle\\ &\le 2p(t)||y(t)||^2 + 2q(t)||y(t)||^2 + 2||F(t,0)||\,||y(t)||\\ &\le 2[p(t)+q(t)]||y(t)||^2 + 2||F(t,0)||\,||y(t)||.\end{aligned} \tag{7.22}$$

Using

$$v' = 2\,|\,p(t)+q(t)\,|\,v + 2||F(t,0)|||v|^{1/2} \tag{7.23}$$

as a comparison scalar equation (see Exercise 7.5), we find that $y(t)$ is continuable to $+\infty$ (see Corollary 5.17). If we let this function $y(t)$ in (7.21), we obtain

$$||x(t)|| - ||y(t)|| \le ||x(t)-y(t)|| \le ||x(0)|| \exp\left\{\int_0^t [p(s)+q(s)]ds\right\}. \tag{7.24}$$

It follows that

$$\limsup_{t\to T^-} ||x(t)|| \le ||y(T)|| + ||x(0)|| \exp\left\{\int_0^T [p(s)+q(s)]ds\right\}, \tag{7.25}$$

which, by Theorem 5.15, implies the continuability of every solution $x(t)$ to $+\infty$. Inequality (7.21) implies of course the uniqueness of the solutions of (S_F) w.r.t. the initial conditions at $t = 0$. A similar argument from t_0 to $t \ge t_0$ proves the uniqueness of solutions with respect to any initial conditions. □

We note here that (7.15) with $y(t) \equiv 0$ ensures the continuability (but not necessarily the uniqueness) of all local solutions of (S_F) to $+\infty$, if $A(t)$ and $F(t,x)$ satisfy the following condition.

Condition (M). $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, $F(t,0) = 0$, $t \in \mathbb{R}_+$, and

$$\langle F(t,x), x\rangle \le p(t)\|x\|^2, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^n, \tag{7.26}$$

where $p : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function.

Condition (M) is important in the following stability result.

Theorem 7.7. *Let Condition (M) be satisfied and assume that $q(t)$ is as in Theorem 7.6. Then the zero solution of system (S_F) is stable if*

$$\sup_{t\in\mathbb{R}_+} \int_0^t [p(s) + q(s)]ds < +\infty. \tag{7.27}$$

It is asymptotically stable if

$$\lim_{t\to\infty} \int_0^t [p(s) + q(s)]ds = -\infty. \tag{7.28}$$

It is uniformly stable if

$$p(t) + q(t) \le 0, \quad t \in \mathbb{R}_+. \tag{7.29}$$

It is uniformly asymptotically stable if

$$p(t) + q(t) \le -c, \quad t \in \mathbb{R}_+, \tag{7.30}$$

where c is a positive constant.

Proof. The proof is almost identical to the proof of Theorem 4.12 and it is therefore left as an exercise. □

In the following stability result we make use of the new inner product.

Theorem 7.8. *Let $A : \mathbb{R}_+ \to M_n$ be continuous and such that $A(t) \to A_0$ as $t \to +\infty$, where A_0 has all of its eigenvalues with negative real parts. Let B be a fixed positive definite matrix and V as in Lemma 7.2, where $A = A_0$. Let the rest of the assumptions of Condition (M) be satisfied with $\langle F(t,x), x\rangle$ replaced by $\langle F(t,x), Vx\rangle$. Let q be the smallest eigenvalue of $(1/2)B$ and assume that, for some $T \ge 0$,*

$$p = \sup_{t\ge T} p(t) < q. \tag{7.31}$$

Then the zero solution of (S_F) is asymptotically stable.

Proof. From Lemma 7.4 we obtain

$$\left(\frac{d}{dt}\right)||x(t)||_V^2 = 2\langle A(t)x(t), Vx(t)\rangle + 2\langle F(t,x(t)), Vx(t)\rangle, \tag{7.32}$$

where $x(t)$ is a solution of (S_F) with initial condition $x(0) = x_0$. Letting $B(t) \equiv -[A^T(t)V + VA(t)]$, we see that $B(t)$ is continuous and that $B(t) \to B$ as $t \to +\infty$, where B is the matrix in the statement of the theorem. If $q(t)$ is the smallest eigenvalue of $(1/2)B(t)$, then, as in (7.8), we have

$$\begin{aligned} 2\langle A(t)x(t), Vx(t)\rangle &= -\langle x(t), B(t)x(t)\rangle \\ &\le -2q(t)||x(t)||^2, \quad t \in \mathbb{R}_+. \end{aligned} \tag{7.33}$$

This inequality is now combined with (7.32) to give

$$\left(\frac{d}{dt}\right)||x(t)||_V^2 \le -2[q(t) - p(t)]||x(t)||^2, \quad t \in \mathbb{R}_+. \tag{7.34}$$

Naturally, the extendability of the solution $x(t)$ to $+\infty$ follows as in Theorem 7.6.

Theorem 7.5 implies that to every eigenvalue $\lambda(t)$ of $B(t) = [b_{ij}(t)]$ corresponds to an eigenvalue λ' of $B = [b_{ij}]$ such that

$$|\lambda(t) - \lambda'| \le (n+2)M(t)r^{1/n}(t), \quad t \in \mathbb{R}_+, \tag{7.35}$$

where

$$\begin{aligned} M(t) &= \max_{i,j=1,2,\ldots,n} \{|b_{ij}(t)|, |b_{ij}|\}, \\ r(t) &= \frac{1}{nM(t)} \sum_{i,j=1}^{n} |b_{ij}(t) - b_{ij}|. \end{aligned} \tag{7.36}$$

From the continuity of $B(t)$, its convergence to B as $t \to +\infty$, and

$$M(t) \ge \max_{i,j} \{|b_{ij}|\} > 0, \tag{7.37}$$

we conclude that

$$\lim_{t\to\infty} r(t) = 0. \tag{7.38}$$

It follows that, given a constant $\epsilon > 0$, there exists a constant $T(\epsilon) > 0$ such that

$$|\lambda(t) - \lambda'| < \epsilon, \quad t \in [T(\epsilon), \infty). \tag{7.39}$$

If $\lambda(t) \equiv 2q(t)$, then there exists an eigenvalue $2\tilde{q}$ of B such that

$$2\,|\,q(t) - \tilde{q}\,| < \epsilon, \quad t \in [T(\epsilon), \infty). \tag{7.40}$$

It should be noted that the eigenvalues λ', $2\tilde{q}$ above actually depend on t. Since B is positive definite, we may choose ϵ so that $\epsilon \in (0, 2q)$, where $2q$ is the smallest eigenvalue of B. Then (7.40) implies

$$2q(t) > 2\tilde{q} - \epsilon \geq 2q - \epsilon > 0, \quad t \in [T(\epsilon), \infty). \tag{7.41}$$

Thus, (7.34) and (7.41) give, for $\epsilon_1 = \epsilon/2$,

$$\left(\frac{d}{dt}\right)||x(t)||_V^2 \leq -2[q - \epsilon_1 - p(t)]||x(t)||^2, \quad t \in [T(\epsilon), \infty). \tag{7.42}$$

We now choose a new $\epsilon < q - p$ and $T = T(\epsilon)$ large enough, taking into consideration the number T in the statement of the theorem, to arrive at

$$\begin{aligned} \left(\frac{d}{dt}\right)||x(t)||_V^2 &\leq -2(q - p - \epsilon)||x(t)||^2 \\ &\leq -2Q||x(t)||_V^2, \quad t \in [T, \infty), \end{aligned} \tag{7.43}$$

where $Q = (q - p - \epsilon)/\mu$. Here, μ is the largest eigenvalue of V. Applying Lemma 4.11, we obtain

$$||x(t)||_V \leq e^{-Q(t-T)}||x(T)||_V, \quad t \in [T, \infty). \tag{7.44}$$

Similarly, (7.34) leads to

$$||x(t)||_V \leq \exp\left\{\frac{1}{\mu}\int_0^t |\,p(s) - q(s)\,|\,ds\right\}||x_0||_V \leq N_1||x_0||_V, \tag{7.45}$$

for all $t \in [0, T]$, where

$$N_1 = \exp\left\{\frac{1}{\mu}\int_0^T |\,p(s) - q(s)\,|\,ds\right\}. \tag{7.46}$$

Combining (7.44) and (7.45), we arrive at

$$||x(t)|| \leq N||x_0||, \quad t \in [0, T], \tag{7.47}$$

$$||x(t)|| \leq Ne^{-Q(t-T)}||x_0||, \quad t \in [T, \infty), \tag{7.48}$$

for a new positive constant N. Here, we have used the equivalence of the two norms $\|\cdot\|$, $\|\cdot\|_V$. It follows that the zero solution of (S_F) is asymptotically stable. □

3. STABILITY REGIONS

Assume that $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Assume further that there exists $\Omega \subset \mathbb{R}^n$ and $D \subset \Omega$ such that every solution $x(t)$ of (S_F) with $x(0) = x_0 \in D$ remains in Ω for all $t \geq 0$. Then it is possible to introduce a concept of stability for (S_F) according to which the initial conditions are restricted to the set D. The reason for doing this is that the system (S_F) possesses stability properties that stem from conditions on $A(t)$, $F(t,x)$ holding only on a subset Ω of $\mathbb{R}^n$. We now define the concept of a stability (or asymptotic stability) region (D,Ω). Similarly one defines regions (D,Ω) for the other stability types.

Definition 7.9. Consider the system (S_F) with $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ continuous and such that $F(t,0) = 0$, $t \in \mathbb{R}_+$. Assume further the existence of two subsets D, Ω of $\mathbb{R}^n$ such that $D \subset \Omega$, $0 \in D$, and every solution $x(t)$ of (S_F) with $x(0) = x_0 \in D$ remains in Ω for as long as it exists. Then the pair (D,Ω) is called a *region of stability* (*asymptotic stability*) for (S_F) if the zero solution of (S_F) is stable (asymptotically stable) w.r.t. initial conditions $x(0) = x_0 \in D$, that is, if in the definition of stability (asymptotic stability) (Definition 4.1 with $x_0(t) \equiv 0$) the initial conditions $x(0)$ are assumed to lie in D.

It is easy to establish results like Theorems 7.7 and 7.8, but with assumptions taking into consideration the sets D,Ω as above. In this connection we have Theorem 7.10.

Theorem 7.10. *Let the system* (S_F) *satisfy the conditions on A, F, D,* Ω *of Definition 7.9. Furthermore, let* $p : \mathbb{R}_+ \to \mathbb{R}$ *be continuous and such that*

$$\langle F(t,x), Vx\rangle \leq p(t)\|x\|^2, \quad t \in \mathbb{R}_+,\ x \in \Omega, \tag{7.49}$$

where V is as in Lemma 7.2. Let q be the smallest eigenvalue of $(1/2)B$*, where B is the matrix in* (7.2)*. Then if*

$$\sup_{t\in\mathbb{R}_+} p(t) < q, \tag{7.50}$$

the pair (D,Ω) *is a region of asymptotic stability for the system* (S_F)*.*

The proof is left as an exercise.

Example 7.11. In this example we search for an asymptotic stability region for the system

$$x' = Ax + F(t,x), \tag{S_1}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad F(t,u) = \begin{bmatrix} 0 \\ -\left(\dfrac{1}{5}\right)u_1^2 \end{bmatrix} \tag{7.51}$$

with

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{7.52}$$

This system arises from the scalar equation

$$y'' + y' + y + \left(\frac{1}{5}\right) y^2 = 0. \tag{7.53}$$

We note that the equation

$$A^T V + VA = -2I \tag{7.54}$$

has the (unique) solution $V = [\begin{smallmatrix} 3 & 1 \\ 1 & 2 \end{smallmatrix}]$. We shall determine a region Ω and a constant $\beta > 0$ such that

$$\left(\frac{d}{dt}\right) ||x(t)||_V^2 \leq -2\beta ||x(t)||_V^2, \quad t \in \mathbb{R}_+, \tag{7.55}$$

for any solution $x(t)$ of (S_1) lying in Ω.

To this end, we first observe that Lemma 7.4 implies

$$\begin{aligned} \left(\frac{d}{dt}\right) ||x(t)||_V^2 &= 2\langle A(t)x(t) + F(t, x(t)), Vx(t)\rangle \\ &= -2\left[x_1^2(t) + x_2^2(t)\right] - \left(\frac{2}{5}\right) x_1^2(t)\left[x_1(t) + 2x_2(t)\right]. \end{aligned} \tag{7.56}$$

Thus, (7.55) will hold if the solution $x(t)$ satisfies

$$\begin{aligned} x_1^2(t) + x_2^2(t) &+ \left(\frac{1}{5}\right) x_1^2(t)\left[x_1(t) + 2x_2(t)\right] \\ &\geq \beta ||x(t)||_V^2 = \beta\left[3x_1^2(t) + 2x_1(t)x_2(t) + 2x_2^2(t)\right]. \end{aligned} \tag{7.57}$$

To determine Ω, we consider first the vectors $x \in \mathbb{R}^2$ such that

$$x_1^2 + x_2^2 + \left(\frac{1}{5}\right) x_1^2 (x_1 + 2x_2) \geq \beta(3x_1^2 + 2x_1x_2 + 2x_2^2). \tag{7.58}$$

Since $4x_1^2 + 3x_2^2 \geq 3x_1^2 + 2x_1x_2 + 2x_2^2$, Inequality (7.58) is satisfied for all $x \in \mathbb{R}^2$ with

$$x_1^2 + x_2^2 + \left(\frac{1}{5}\right) x_1^2 (x_1 + 2x_2) \geq \beta(4x_1^2 + 3x_2^2) \tag{7.59}$$

or

$$x_1^2\left[1 - 4\beta + \left(\frac{1}{5}\right)(x_1 + 2x_2)\right] + (1 - 3\beta)x_2^2 \geq 0. \tag{7.60}$$

It follows that if we take $\beta = 1/5$, Inequality (7.58) is satisfied for all $x \in \Omega$, where

$$\Omega = \{x \in R^2 : x_1 + 2x_2 \geq -1\}. \tag{7.61}$$

Now, we find a set $D \subset \Omega$ such that whenever a solution $x(t)$ of (S_1) starts inside D, it remains in Ω and satisfies (7.55). In this particular example, we can take D to be inside a V-ball D_1, that is,

$$D_1 = \{x \in R^2 : \|x\|_V \leq \sqrt{\alpha}\}, \tag{7.62}$$

for some $\alpha > 0$. In fact, we first notice that the set D_1 consists of the interior of the ellipse

$$3x_1^2 + 2x_1x_2 + 2x_2^2 = \alpha \tag{7.63}$$

along with the ellipse itself. We also notice that Ω consists of all $x \in R^2$ which lie above the line $x_1 + 2x_2 = -1$. Thus, we can determine α so that D_1 has this line as a tangent line and lies in Ω. In fact, if we take $\alpha = 1/2$, we find that $(0, -1/2)$ is the tangent point in question. Letting $D = \{x \in \mathbb{R}^n : \|x\|_V < \sqrt{\mu}\}$, for some $\mu \in (0, \alpha)$, it suffices to show that every solution of (S_1) emanating from a point inside D exists for all $t \in \mathbb{R}_+$ and lies in D.

Let $x(t)$ be a solution of (S_1) starting at $x(0) = x_0$ with $x_0 \in D$. Then, as long as $x(t)$ lies in Ω, (7.55) is satisfied. Applying Lemma 4.11, we see that

$$||x(t)||_V^2 \leq e^{-(2/5)t}||x(0)||_V^2 \leq ||x(0)||_V^2. \tag{7.64}$$

Assume now that $x(t)$ leaves the set D at some $t = t_0$. Then it is easy to see that we can take t_0 to be the first time at which this happens. Then, there exists a neighborhood $[t_0, t_1)$ such that $x(t) \notin D$ and $x(t) \in D_1$ for all $t \in (t_0, t_1)$. Thus, again from (7.55), we obtain

$$\left(\frac{d}{dt}\right)||x(t)||_V^2\,|_{t_0} \leq -\left(\frac{2}{5}\right)||x(t_0)||_V^2 = -\left(\frac{2}{5}\right)\mu < 0. \tag{7.65}$$

Since $(d/dt)\|x(t)\|_V^2$ is continuous, (7.65) says that $\|x(t)\|_V$ is strictly decreasing in a right neighborhood of t_0. Thus, the solution $x(t)$ cannot penetrate the boundary of the V-ball D, because its V-norm does not increase to the right of the point t_0. It follows that (7.64) holds for as long as $x(t)$ exists. This fact implies that $x(t)$ is continuable to $+\infty$. Moreover, (7.64) also implies easily that the pair (D, Ω) is a region of asymptotic stability for (S_1) (see (7.43), (7.44), (7.45), (7.47), and (7.48)). Actually, here (D, D) is an even better region of asymptotic stability.

4. PERIODIC SOLUTIONS

In this section, we study systems of the form

$$x' = F(t, x), \tag{E}$$

where $F(t+T, u) = F(t, u)$, $(t, u) \in \mathbb{R} \times \mathbb{R}^n$. Here, T is a fixed positive constant. We are looking for T-periodic solutions of (E), that is, solutions $x(t)$ which exist on $\mathbb{R}$ and satisfy $x(t+T) = x(t)$, $t \in \mathbb{R}$.

Our first existence result is contained in Theorem 7.12.

THEOREM 7.12. *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that $F(t+T, u) = F(t, u)$, $(t, u) \in \mathbb{R} \times \mathbb{R}^n$, where T is a positive constant. Moreover, let*

$$\begin{gathered} \langle F(t, u_1) - F(t, u_2), u_1 - u_2 \rangle \le 0, \quad (t, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \\ \langle F(t, u), u \rangle < 0, \end{gathered} \tag{7.66}$$

for every $t \in [0, T]$ and every $u \in \mathbb{R}^n$ with $\|u\| = r$, where r is a positive constant. Then (E) *has at least one T-periodic solution.*

PROOF. We first note that Theorem 7.6 implies the existence and uniqueness w.r.t. initial conditions of solutions of (E) on $\mathbb{R}_+$. Assume that $x(t)$ is a solution of (E) defined on $[0, T]$ and such that $x(0) = x(T)$. Then, by the T-periodicity of $F(t, u)$ w.r.t. t, $x(t)$ can be easily defined on the entire real line so that it is T-periodic (see also Exercise 3.7). Thus, it suffices to show that (E) has a solution $x(t)$ defined on $[0, T]$ and such that $x(0) = x(T)$. To this end, let $U : \overline{B_r(0)} \to \mathbb{R}^n$ map every $u \in \overline{B_r(0)}$ to the value at T of the unique solution of the problem

$$x' = F(t, x), \quad x(0) = u, \ t \in \mathbb{R}_+. \tag{7.67}$$

Then if $Ux_0 = x_0$, for some $x_0 \in \overline{B_r(0)}$, we have $x_0 = x(0) = x(T)$ for some solution $x(t)$ of (E), and the proof is complete. In order to apply the Brouwer Theorem (Corollary 2.15), we have to show that $U\overline{B_r(0)} \subset \overline{B_r(0)}$ and that U is continuous. If $x(t)$, $y(t)$ are two solutions of (E) on $[0, T]$, then (7.15) implies

$$||x(T) - y(T)|| \le ||x(0) - y(0)||, \tag{7.68}$$

which shows the continuity of U. Now, let a solution $x(t)$ of (E) satisfy $\|x(0)\| \le r$ and let

$$D = \{t \in [0, T] : ||x(s)|| \le r \text{ for } s \in [0, t)\}. \tag{7.69}$$

We intend to prove that the number $c = \sup D$ is equal to T. In fact, let $c < T$ and suppose that $\|x(c)\| < r$. Then, since $x(t)$ is continuous, there exists an interval $[c, t_1) \subset [c, T)$ such that $\|x(s)\| < r$, $s \in [c, t_1)$. However, this contradicts the definition of c. It follows that $\|x(c)\| = r$. Hence, by Lemma 7.4, we have

$$\begin{aligned}\left(\frac{d}{dt}\right)\|x(t)\|^2\big|_{t=c} &= 2\langle x'(c), x(c)\rangle \\ &= 2\langle F(c, x(c)), x(c)\rangle \\ &< 0.\end{aligned} \tag{7.70}$$

Since $(d/dt)\|x(t)\|^2$ is continuous, there exists an interval $[c, t_2) \subset [c, T)$ such that $(d/dt)\|x(t)\|^2 < 0$ there. Integrating this last inequality once, we obtain $\|x(s)\| < \|x(c)\| = r$ for all s in this interval. This is a contradiction to the definition of c. It follows that $U\overline{B_r(0)} \subset \overline{B_r(0)}$ and that U has a fixed point in $\overline{B_r(0)}$. □

In our second existence result the operator U above is a contraction on $\mathbb{R}^n$.

Theorem 7.13. *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that $F(t+T, u) = F(t, u)$, $(t, u) \in \mathbb{R} \times \mathbb{R}^n$, where T is a positive constant. Moreover, let M be a negative constant such that*

$$\langle F(t, u_1) - F(t, u_2), u_1 - u_2\rangle \leq M\|u_1 - u_2\|^2 \tag{7.71}$$

for all $(t, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$. Then there exists a unique T-periodic solution of the system (E).

Proof. Continuation and uniqueness w.r.t. initial conditions follow from Theorem 7.6. It suffices to show that the operator U, defined in the proof of Theorem 7.12, has a unique fixed point in $\mathbb{R}^n$. To this end, we observe that if $x(t)$, $y(t)$ are two solutions of (E) on $[0, T]$, we have

$$\begin{aligned}\left(\frac{d}{dt}\right)\left[e^{-2Mt}\|x(t) - y(t)\|^2\right] &= -2Me^{-2Mt}\|x(t) - y(t)\|^2 \\ &\quad + 2e^{-2Mt}\langle F(t, x(t)) - F(t, y(t)), x(t) - y(t)\rangle \\ &\leq \left(-2Me^{-2Mt} + 2Me^{-2Mt}\right)\|x(t) - y(t)\|^2 \\ &= 0.\end{aligned} \tag{7.72}$$

One integration above gives

$$e^{-2Mt}\|x(t) - y(t)\|^2 \leq \|x(0) - y(0)\|^2, \tag{7.73}$$

which implies

$$\|x(T) - y(T)\| \leq e^{MT}\|x(0) - y(0)\|. \tag{7.74}$$

Since $M < 0$, the operator U is a contraction mapping on $\mathbb{R}^n$ and, as such, it has a unique fixed point. □

5. BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

Consider the problem

$$x' = A(t)x + F(t,x), \tag{S_F}$$

$$Ux = r, \tag{B}$$

where U is a bounded linear operator on $C_n(\mathbb{R}_+)$ and r is a fixed vector in $\mathbb{R}^n$. As we saw in Chapter 6, there is a large family of boundary value problems that can be written in the form $((S_F), (B))$. Our intention here is to show the existence of solutions of such problems under monotonicity assumptions on $A(t)$, $F(t,x)$, and for boundary conditions (B) which do not include periodicity conditions. The relevant result is contained in the following theorem.

Theorem 7.14. *Let $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $F(t,0) = 0$, $t \in \mathbb{R}_+$. Moreover, assume the following:*

(i) *for every $(t,u,\mu) \in \mathbb{R}_+ \times \mathbb{R}^n \times [0,1]$,*

$$\langle A(t)u + \mu F(t,u), u\rangle \leq 0. \tag{7.75}$$

(ii) $\|X(t)\| \leq M$, $t \in \mathbb{R}_+$, *where M is a positive constant, and there exist two functions $q, g : \mathbb{R}_+ \to \mathbb{R}_+$, continuous, and such that*

$$\left\|X^{-1}(t)F(t,u)\right\| \leq q(t)\|u\| + g(t), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{7.76}$$

Here, $X(t)$ denotes the fundamental matrix of the system

$$x' = A(t)x \tag{S}$$

with $X(0) = I$. Moreover,

$$\int_0^\infty q(t)dt < +\infty, \qquad \int_0^\infty g(t)dt < +\infty. \tag{7.77}$$

(iii) *$U : C_n(\mathbb{R}_+) \to \mathbb{R}^n$ is a bounded linear operator such that $\|Uu\| \geq \phi(\|u(0)\|)$ for every $u \in C_n(\mathbb{R}_+)$ with $\|u(t)\| \leq \|u(0)\|$, $t \in \mathbb{R}_+$. Here, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, surjective, and such that $\phi(0) = 0$.*

(iv) *$\widetilde{X}^{-1}$ exists, where $\widetilde{X}$ is defined in Chapter 6, Section 1.*

Then the problem $((S_F), (B))$ has at least one solution.

Proof. We proceed as in Theorem 6.14. To this end, we need to prove the existence of a fixed point for the operator W, where

$$(Wf)(t) = X(t)\widetilde{X}^{-1}[r - Up(\cdot, f)] + p(t,f), \quad t \in \mathbb{R}_+. \tag{7.78}$$

Let $D_m, V_1 : D_1 \to D_1$ be as in the proof of Theorem 6.14. There we showed that the operator $V_1(x,\mu)$ is compact in x and continuous in μ uniformly w.r.t. x in bounded subsets of D_1. In order to apply the Leray-Schauder theorem (Theorem 2.16), we need to show that all possible solutions of the problem $x - V_1(x,\mu) = 0$ in D_1 lie inside a ball of D_1 which does not depend on $\mu \in [0,1]$. To prove this, let $x \in D_1$ solve $x - V_1(x,\mu_0) = 0$ for some $\mu_0 \in [0,1]$. Then the function $x(t)$ satisfies the integral equation

$$x(t) = \mu(X(t)\widetilde{X}^{-1}[r - Up(\cdot,x)] + p(t,x)) \tag{7.79}$$

for $\mu = \mu_0$ and for every $t \in [0,1]$. It follows that $Ux = \mu_0 r$ and that

$$x'(t) = A(t)x(t) + \mu_0 F(t,x(t)), \quad t \in [0,1]. \tag{7.80}$$

Now, we apply (7.10) for $V = I$ to get

$$\left(\frac{d}{dt}\right)||x(t)||^2 = 2\langle A(t)x(t) + \mu_0 F(t,x(t)), x(t)\rangle \le 0, \quad t \in [0,1]. \tag{7.81}$$

Integrating this inequality, we get

$$||x(t)|| \le ||x(0)||, \quad t \in [0,1]. \tag{7.82}$$

Consequently, by hypothesis (iii), we obtain

$$\phi(||x(t)||) \le \phi(||x(0)||) \le \|Ux\| = \mu_0\|r\| \le \|r\|. \tag{7.83}$$

Thus, we have a bound for $x(t) : \|x(t)\| \le \phi^{-1}(\|r\|)$, $t \in [0,1]$. The number $\phi^{-1}(\|r\|)$ is a uniform bound for all solutions of $x - V_1(x,\mu) = 0$ in D_1, independently of $\mu \in [0,1]$. Similarly, by induction, we obtain a sequence of functions $x_m \in D_m$ such that

$$||x_m(t)|| \le \phi^{-1}(\|r\|), \quad t \in [0,m], \tag{7.84}$$

and each $x_m(t)$ satisfies (7.79), for $\mu = 1$, on the interval $[0,m]$. The proof now follows as the proof of Theorem 6.14, and it is therefore omitted. □

Example 7.15. Consider the system

$$x' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x + F(t,x) \tag{7.85}$$

and the boundary condition

$$Ux = x(0) + \int_0^\infty V(t)x(t)dt = r, \tag{7.86}$$

where $V : \mathbb{R}_+ \to M_n$ is continuous and such that

$$\int_0^{\infty} ||V(t)||dt < 1. \tag{7.87}$$

Here, we may take

$$F(t,x) = \begin{bmatrix} f_1(t)x_1 \\ \dfrac{f_2(t)x_2}{x_1^2 + x_2^2 + 1} \end{bmatrix}, \tag{7.88}$$

where $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_-$ are continuous and such that

$$-\int_0^{\infty} ||X^{-1}(t)||f_1(t)dt < +\infty, \qquad -\int_0^{\infty} ||X^{-1}(t)||f_2(t)dt < +\infty. \tag{7.89}$$

Then $q(t) \equiv -\|X^{-1}(t)\| f_1(t)$, $g(t) \equiv -\|X^{-1}(t)\| f_2(t)$, and

$$\phi(\mu) \equiv \left(1 - \int_0^{\infty} ||V(t)||dt\right)\mu. \tag{7.90}$$

In fact, if $u \in C_n(\mathbb{R}_+)$ with $\|u(t)\| \leq \|u(0)\|$, we have

$$\begin{aligned} \|Uu\| &= \left\|u(0) + \int_0^{\infty} V(t)u(t)dt\right\| \geq ||u(0)|| - \int_0^{\infty} ||V(t)||\,||u(t)||dt \\ &\geq ||u(0)|| - \int_0^{\infty} ||V(t)||dt||u(0)|| = \phi(||u(0)||). \end{aligned} \tag{7.91}$$

The rest of the assumptions of Theorem 7.14 are also satisfied. Thus, the problem ((7.85), (7.86)) has at least one solution for every $r \in \mathbb{R}^n$.

EXERCISES

7.1. Show that the solution $V \in M_n$ of (7.2), given by (7.4), is unique and positive definite.

7.2. Show that the zero solution of the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} e^{-t} & 1 \\ -2 & -3 + \left[\dfrac{1}{t+1}\right] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ |\sin(tx_2)|(x_1 - x_2) \end{bmatrix} \tag{7.92}$$

is asymptotically stable by using Theorem 7.8.

7.3. Show that the first order scalar equation

$$x' + |\sin t|x + x^3 = \cos t \tag{7.93}$$

has at least one 2π-periodic solution.

7.4. Show that the system (in $\mathbb{R}^n$)

$$x' = -Lx + F(t,x) \tag{7.94}$$

has a unique T-periodic solution if $L > 0$ is a constant and F is continuous, T-periodic in t, and satisfies the Lipschitz condition

$$||F(t,u_1) - F(t,u_2)|| \leq K||u_1 - u_2||, \tag{7.95}$$

for every $(t,u_1,u_2) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$. Here, $K \in (0,L)$ is another constant.

7.5. Let $a : \mathbb{R}_+ \to \mathbb{R}$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Show that all solutions to the problem

$$u' = a(t)u + b(t)|u|^{1/2}, \qquad u(0) = u_0 \geq 0 \tag{7.96}$$

are continuable to $+\infty$.

7.6. Let $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $K \subset \mathbb{R}^n$ be compact. Show that there exists a sequence of continuous functions $\{F_m\}_{m=1}^{\infty}$, $F_m : [a,b] \times K \to \mathbb{R}^n$ such that

(i) $\lim_{m\to\infty} \|F_m(t,u) - F(t,u)\| = 0$ uniformly on $[a,b] \times K$.

(ii) $\|F_m(t,u_1) - F_m(t,u_2)\| \leq L_m\|u_1 - u_2\|$ for all $(t,u_1,u_2) \in [a,b] \times K \times K$, $m = 1,2,\ldots$, where L_m is a positive constant.

7.7. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be T-periodic in its first variable and continuous. Show that system (E) has at least one T-periodic solution under the mere assumption

$$\langle F(t,u), u\rangle \leq 0 \tag{7.97}$$

for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$ with $\|u\| = r$. Here, r is a positive constant. This is a considerable improvement over Theorem 7.12. Hint. Consider the systems

$$x' = F_m(t,x) - \delta_m x. \tag{*}$$

Here, $\{F_m\}$ is the sequence in Exercise 7.6 defined on $[0,T] \times \overline{B_{2r}(0)}$, and $\{\delta_m\}_{m=1}^{\infty}$ is a decreasing sequence of positive constants such that $\delta_m \to 0$ as $m \to \infty$. Show that it is possible to choose a subsequence $\{\bar{F}_m\}$ of $\{F_m\}$ such that

$$\langle \bar{F}_m(t,u) - \delta_m u, u\rangle \leq -\frac{\delta_m r^2}{2} \tag{7.98}$$

for $\|u\| = r$ and $m \geq 1$. Using the method of Theorem 7.12, obtain a T-periodic solution $x_m(t)$, $t \in [0,T]$, of (*) with F_m replaced by $\bar{F}_m$. Show that a subsequence $\{x_{m_k}(t)\}_{k=1}^{\infty}$ of $\{x_m(t)\}$ converges uniformly on $[0,T]$ to a T-periodic solution of the system (E).

7.8. Using Exercise 7.7, show that the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = -\begin{bmatrix} (\sin^2 t)(\|x\|^3 - 1) \\ (1+x_1)^2 x_2 \\ x_2^4 x_3 \end{bmatrix} \tag{7.99}$$

has at least one 2π-periodic solution.

7.9. Assume that $A : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and T-periodic in their first variable t. Assume further that

$$\langle A(t,u), u\rangle \le -\lambda \|u\|^2, \qquad ||B(t,u)|| \le \mu \tag{7.100}$$

for every $(t,u) \in R \times \mathbb{R}^n$, where λ, μ are positive constants. Show that the system

$$x' = A(t,x) + B(t,x) \tag{7.101}$$

has at least one T-periodic solution. Hint. Base your argument on the closed ball $\overline{B_r(0)}$, where r is any number in $(\mu/\lambda, \infty)$. Use Exercise 7.7.

7.10. Find a region of asymptotic stability (D, Ω) for the system (S_1) in the text, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad F(t,u) = \begin{bmatrix} 0 \\ -\lambda(t)u_1^2 \end{bmatrix}. \tag{7.102}$$

Here, $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that $\lambda(t) \le L$, $t \in \mathbb{R}_+$, where L is a number in $[0,1)$.

7.11 (*asymptotic equilibrium*). Consider the system

$$x' = F(x), \tag{E_1}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$ continuous and such that

$$\langle F(u_1) - F(u_2), u_1 - u_2\rangle_V \le -\lambda ||u_1 - u_2||^2 \tag{7.103}$$

for every $(u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Here, V is a positive definite matrix and λ is a positive constant. Show that (E_1) has a unique equilibrium solution, that is, a vector $\bar{x} \in \mathbb{R}^n$ such that $F(\bar{x}) = 0$. Hint. Pick an arbitrary solution $x(t)$, $t \in \mathbb{R}_+$, of (E_1) and let $u(t) = x(t+h) - x(t)$, $t \in \mathbb{R}_+$, where h is a fixed positive number. Show first that

$$\left(\frac{d}{dt}\right) ||u(t)||_V^2 \le -2\lambda ||u(t)||^2 \tag{7.104}$$

and conclude, after appraising $\|u(t)\|$, that $x(t) \to \bar{x}$ as $t \to \infty$, where $\bar{x}$ is a finite vector. Then, from an inequality of the form

$$||x'(t)|| \leq M \exp\{-\lambda_1(t-t_0)\}||x'(t_0)||, \tag{7.105}$$

where λ_1, M are positive constants, conclude that $x'(t) \to 0$ as $t \to \infty$.

7.12. Show that the problem

$$x' = Ax + F(t, x), \qquad x(0) - Nx(\infty) = r, \tag{7.106}$$

with

$$\begin{aligned} A &= \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}, \\ F(t,u) &= \begin{bmatrix} -\dfrac{e^{-t}u_1}{1+u_1^2} \\ -e^{-2t}\,|\sin u_1\,|\,u_2 \end{bmatrix}, \\ r &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned} \tag{7.107}$$

has at least one solution if the norm of the matrix $N \in M_2$ is sufficiently small. Here,

$$x(\infty) = \lim_{t\to\infty} x(t). \tag{7.108}$$

CHAPTER 8

BOUNDED SOLUTIONS ON THE REAL LINE; QUASILINEAR SYSTEMS; APPLICATIONS OF THE INVERSE FUNCTION THEOREM

In Chapter 1, we saw that if P is a projection matrix, then $I - P$ is also a projection matrix and $\mathbb{R}^n = \mathbb{R}_1 \oplus \mathbb{R}_2$, where $\mathbb{R}_1 = P\mathbb{R}^n$ and $\mathbb{R}_2 = (I - P)\mathbb{R}^n$. In this chapter, we show that the projection P may induce a *splitting* in the space of solutions of the system

$$x' = A(t)x. \tag{S}$$

This means that the solutions of (S) with initial conditions $x_0 \neq 0$ in $\mathbb{R}_1$ tend to zero as $t \to +\infty$, whereas the solutions with initial conditions x_0 such that $(I - P)x_0 \neq 0$ have norms that tend to $+\infty$ as $t \to +\infty$. A corresponding situation, with the roles of $\mathbb{R}_1$, $\mathbb{R}_2$ reversed, holds in $\mathbb{R}_-$. This splitting occurs if the system (S) possesses a so-called *exponential dichotomy*.

Exponential dichotomies for systems of the type (S) and their effect on the existence of bounded solutions on $\mathbb{R}$ of the system

$$x' = A(t)x + F(t,x) \tag{S_F}$$

are mainly the subject of Sections 1 and 2.

Section 3 contains some results for the so-called *quasilinear systems*

$$x' = A(t,x)x + F(t,x), \tag{S_Q}$$

where $A(t,u)$ is an $n \times n$ matrix. These results are intimately dependent upon the behaviour of the associated linear systems

$$x' = A\big(t, f(t)\big)x + F\big(t, f(t)\big), \tag{S^f}$$

where the functions f belong to suitable classes.

Several results concerning further asymptotic properties of nonlinear systems are included in the exercises at the end of this chapter.

1. EXPONENTIAL DICHOTOMIES

In what follows, $P \in M_n$ is a projection matrix. For convenience, we write $P_1 = P$ and $P_2 = I - P$. We then have $P_1\mathbb{R}^n = \mathbb{R}_1$ and $P_2\mathbb{R}^n = \mathbb{R}_2$. Here, of course, we have identified the projections P_1, P_2 with the corresponding bounded linear operators.

We now define the concept of the *angular distance* between the subspaces $\mathbb{R}_1$ and $\mathbb{R}_2$.

Definition 8.1. The *angular distance* $\alpha(\mathbb{R}_1, \mathbb{R}_2)$ between the subspaces $\mathbb{R}_1 \neq \{0\}$, $\mathbb{R}_2 \neq \{0\}$ is defined as follows:

$$\alpha(\mathbb{R}_1, \mathbb{R}_2) = \inf\{||u_1 + u_2||; u_k \in R_k,\ ||u_k|| = 1,\ k = 1, 2\}. \tag{8.1}$$

The following lemma establishes the basic relationship between the norms of the projections P_1, P_2 and the function $\alpha(\mathbb{R}_1, \mathbb{R}_2)$.

Lemma 8.2. *Let $\mathbb{R}^n = \mathbb{R}_1 \oplus \mathbb{R}_2$ and let P_1, P_2 be the projections of $\mathbb{R}_1$, $\mathbb{R}_2$, respectively. Then if $\|P_k\| > 0$, $k = 1, 2$, we have*

$$\frac{1}{||P_k||} \leq \alpha(\mathbb{R}_1, \mathbb{R}_2) \leq \frac{2}{||P_k||}, \quad k = 1, 2. \tag{8.2}$$

Proof. Let $\mu > \alpha(\mathbb{R}_1, \mathbb{R}_2)$, where μ is a positive constant, and let $u_1 \in \mathbb{R}_1$, $u_2 \in \mathbb{R}_2$ be such that $\|u_k\| = 1$, $k = 1, 2$, and $\|u_1 + u_2\| < \mu$. Then if $u = u_1 + u_2$, we have $P_k u = u_k$, $k = 1, 2$, and

$$1 = ||u_k|| = ||P_k u|| \leq ||P_k|| \|u\| < \mu ||P_k||. \tag{8.3}$$

Consequently, we find

$$\frac{1}{||P_k||} < \mu, \tag{8.4}$$

which implies $1/\|P_k\| \leq \alpha(\mathbb{R}_1, \mathbb{R}_2)$.

Now, let $u \in \mathbb{R}^n$ be such that $P_k u \neq 0$, $k = 1, 2$. Then we have

$$\begin{aligned}
\alpha(\mathbb{R}_1, \mathbb{R}_2) &\leq \left|\left| \frac{P_1 u}{||P_1 u||} + \frac{P_2 u}{||P_2 u||} \right|\right| \\
&= \frac{1}{||P_1 u||} \left|\left| P_1 u + \frac{||P_1 u||}{||P_2 u||} P_2 u \right|\right| \\
&= \frac{1}{||P_1 u||} \left|\left| u + \frac{||P_1 u|| - ||P_2 u||}{||P_2 u||} P_2 u \right|\right| \\
&\leq \frac{1}{||P_1 u||} \left(\|u\| + \frac{||P_1 u + P_2 u||}{||P_2 u||} ||P_2 u|| \right) \\
&= \frac{1}{||P_1 u||} (\|u\| + \|u\|) = \frac{2\|u\|}{||P_1 u||}.
\end{aligned} \tag{8.5}$$

This implies that

$$\sup \left\{ \frac{||P_1 u||}{\|u\|} \alpha(\mathbb{R}_1, \mathbb{R}_2); P_k u \neq 0,\ k = 1, 2 \right\} \leq 2 \tag{8.6}$$

or (see Exercise 8.1)

$$||P_1|| \alpha(\mathbb{R}_1, \mathbb{R}_2) \leq 2. \tag{8.7}$$

Similarly, $\|P_2\| \alpha(\mathbb{R}_1, \mathbb{R}_2) \leq 2$. □

We define below the *splitting* that was referred to in the introduction. The symbol $X(t)$ denotes again the fundamental matrix of (S) with $X(0) = I$.

Definition 8.3. Let $A : \mathbb{R} \to M_n$ be continuous. We say that the system (S) possesses an *exponential splitting* if there exist two positive numbers H, m_0 and a projection matrix P with the following properties:

(i) every solution $x(t) \equiv X(t)x_0$ of (S) with $x_0 \in \mathbb{R}_1$ satisfies

$$||x(t)|| \leq H \exp\{-m_0(t - s)\} ||x(s)||, \quad t \geq s; \tag{8.8}$$

(ii) every solution $x(t) \equiv X(t)x_0$ of (S) with $x_0 \in \mathbb{R}_2$ satisfies

$$||x(t)|| \leq H \exp\{-m_0(s - t)\} ||x(s)||, \quad s \geq t; \tag{8.9}$$

(iii) $\|P_k\| \neq 0$, $k = 1, 2$, and if we denote by $P_1(t)$, $P_2(t)$ the projections $X(t)P_1X^{-1}(t)$, $X(t)P_2X^{-1}(t)$, respectively, and let $\mathbb{R}_1(t) \equiv P_1(t)\mathbb{R}^n$, $\mathbb{R}_2(t) \equiv P_2(t)\mathbb{R}^n$, then there exists a constant $\beta > 0$ such that

$$\alpha(\mathbb{R}_1(t), \mathbb{R}_2(t)) \geq \beta, \quad t \in \mathbb{R}. \tag{8.10}$$

In the following definition we introduce the concept of an exponential dichotomy. The existence of an exponential dichotomy for the system (S) (with $P_1 \neq 0, I$) is equivalent to the existence of an exponential splitting. This is shown in Theorem 8.5.

Definition 8.4. Let $A : \mathbb{R} \to M_n$ be continuous. We say that the system (S) possesses an *exponential dichotomy* if there exist two positive constants H_1, m_0 and a projection matrix P with the following properties:

$$\begin{aligned} ||X(t)P_1X^{-1}(s)|| &\leq H_1 \exp\{-m_0(t-s)\}, \quad t \geq s, \\ ||X(t)P_2X^{-1}(s)|| &\leq H_1 \exp\{-m_0(s-t)\}, \quad s \geq t. \end{aligned} \tag{8.11}$$

Theorem 8.5. *The system* (S) *possesses an exponential dichotomy with $P_1 \neq 0$, I if and only if* (S) *possesses an exponential splitting.*

Proof. Assume that the system (S) possesses an exponential dichotomy with $P_1 \neq 0, I$ and constants H_1, m_0 as in Definition 8.4. Let $x(t)$ be a solution of (S) with $x(0) = x_0 = P_1x_0 \in \mathbb{R}_1$. Then we have $x(s) = X(s)x_0$ and $x_0 = X^{-1}(s)x(s)$. Thus,

$$\begin{aligned} ||x(t)|| = ||X(t)P_1x_0|| &= ||X(t)P_1X^{-1}(s)x(s)|| \\ &\leq H_1 \exp\{-m_0(t-s)\}||x(s)||, \quad t \geq s. \end{aligned} \tag{8.12}$$

Inequality (8.9) is proved the same way.

To show that $\alpha(\mathbb{R}_1(t), \mathbb{R}_2(t))$ has a positive lower bound, it suffices to observe, by virtue of Lemma 8.2, that $\|P_k(t)\| \leq H_1$, $t \in \mathbb{R}$.

Conversely, assume that the system (S) possesses an exponential splitting with H, m_0 as in Definition 8.3. Let $x(t)$ be a solution of (S) with $x(0) = P_1X^{-1}(s)u$, for a fixed $(s,u) \in \mathbb{R} \times \mathbb{R}^n$. Then from property (8.8) we obtain

$$\begin{aligned} ||X(t)P_1X^{-1}(s)u|| &= ||x(t)|| \\ &\leq H \exp\{-m_0(t-s)\}||x(s)|| \\ &= H \exp\{-m_0(t-s)\}||X(s)P_1X^{-1}(s)u|| \\ &\leq MH \exp\{-m_0(t-s)\}\|u\|, \quad t \geq s, \end{aligned} \tag{8.13}$$

where the constant M is an upper bound for $\|P_1(t)\|$.

Here we have used the fact that the existence of a positive lower bound for $\alpha(\mathbb{R}_1(t), \mathbb{R}_2(t))$ is equivalent to the boundedness of the projections $P_1(t), P_2(t)$ on $\mathbb{R}$. Thus, the first inequality in (8.11) is true.

The second inequality in (8.11) is proved in a similar way. □

Now, assume that (S) possesses an exponential splitting. It is easy to see that every solution $x(t)$ of (S) with $x(0) = P_1x(0)$ satisfies

$$\lim_{t\to\infty} ||x(t)|| = 0. \tag{8.14}$$

Let $x(t)$ be a solution of (S) such that $x(0) \in \mathbb{R}_2$, $x(0) \neq 0$. Then from Definition 8.3(ii) we obtain

$$||x(s)|| \geq \left(\frac{1}{H}\right) \exp\{m_0(s-t)\}||x(t)||, \quad s \geq t, \tag{8.15}$$

which implies

$$\lim_{s\to\infty} ||x(s)|| = +\infty. \tag{8.16}$$

If $x(t)$ is now any solution of (S) with $P_2x(0) \neq 0$, then $x(0) = x_1(0) + x_2(0)$ with $x_1(0) \in \mathbb{R}_1$ and $x_2(0) \in \mathbb{R}_2$. It follows that

$$x(t) = X(t)x(0) = X(t)x_1(0) + X(t)x_2(0) \equiv x_1(t) + x_2(t), \tag{8.17}$$

with $x_1(t), x_2(t)$ solutions of (S). From the above considerations and

$$||x(t)|| \geq ||x_2(t)|| - ||x_1(t)||, \tag{8.18}$$

it follows that $\|x(t)\| \to +\infty$ as $t \to \infty$. Consequently, $\mathbb{R}_1$ is precisely the space of all initial conditions of solutions of (S) which are bounded on $\mathbb{R}_+$. The situation is reversed on the interval $\mathbb{R}_-$.

Example 8.6. The system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8.19}$$

possesses an exponential dichotomy. In fact, here we have

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \tag{8.20}$$

and general solution

$$x(t) = X(t)x(0) = \begin{bmatrix} e^{-t}x_1(0) \\ e^t x_2(0) \end{bmatrix} = X(t)P_1x(0) + X(t)P_2x(0), \tag{8.21}$$

where

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{8.22}$$

It is easy to see that

$$\begin{aligned} X(t)P_1X^{-1}(s) &= \begin{bmatrix} e^{-(t-s)} & 0 \\ 0 & 0 \end{bmatrix}, \quad t \geq s, \\ X(t)P_2X^{-1}(s) &= \begin{bmatrix} 0 & 0 \\ 0 & e^{-(s-t)} \end{bmatrix}, \quad s \geq t. \end{aligned} \tag{8.23}$$

2. BOUNDED SOLUTIONS ON THE REAL LINE

In Section 1 we established the fact that in the presence of an exponential splitting the system (S) can have only one bounded solution on $\mathbb{R}$-the zero solution. It is easy to see that this situation prevails even in the case of an exponential dichotomy with $P_1 = I$. Thus, if (S) possesses either one of these properties, the system

$$x' = A(t)x + f(t) \tag{S_f}$$

can have at most one bounded solution on $\mathbb{R}$. Here, f is any continuous function on $\mathbb{R}$.

The following theorem ensures the existence of a bounded solution on $\mathbb{R}$ of the system (S_F). This solution has some interesting stability properties. We need the following definition.

Definition 8.7. The zero solution of the system (S_F) is *negatively unstable* if there exists a number $r > 0$ with the following property: every other solution $x(t)$ of (S_F) defined on an interval $(-\infty, a]$, for any number a, satisfies

$$\sup_{t \leq a} ||x(t)|| > r. \tag{8.24}$$

Theorem 8.8. *Consider the system* (S_F) *under the following assumptions:*

(i) $A : \mathbb{R} \to M_n$ *is continuous and such that the system* (S) *possesses an exponential dichotomy given by* (8.11)*;*
(ii) $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and, for some constant* $r > 0$*, there exists a function* $\beta : \mathbb{R} \to \mathbb{R}_+$*, continuous and such that*

$$||F(t, u) - F(t, v)|| \leq \beta(t)\|u - v\| \tag{8.25}$$

for every $(t, u, v) \in \mathbb{R} \times \overline{B_r(0)} \times \overline{B_r(0)}$*;*

(iii) *we have*

$$\rho = H_1 \sup_{t\in R}\left\{\int_{-\infty}^{0} \exp\{m_0 s\}\beta(s+t)ds + \int_{0}^{\infty} \exp\{-m_0 s\}\beta(s+t)ds\right\} < 1; \tag{8.26}$$

(iv) *we have*

$$\begin{aligned}&\left\|\int_{-\infty}^{t} X(t)P_1X^{-1}(s)F(s,0)ds - \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,0)ds\right\| \\ &\qquad < \frac{r(1-\rho)}{2}, \quad t\in\mathbb{R}.\end{aligned} \tag{8.27}$$

Then

(1) *there exists a unique solution $x(t)$, $t \in \mathbb{R}$, of the system* (S_F) *such that* $\|x\|_\infty \le r$;

(2) *if $P = I$, $F(t,0) \equiv 0$, and (iii) holds without necessarily the second integral, then the zero solution of* (S_F) *is negatively unstable;*

(3) *let $P = I$ and let (iii) hold without necessarily the second integral. Let*

$$\lambda = \limsup_{t\to\infty}\left\{\left(\frac{1}{t}\right)\int_0^t \beta(s)ds\right\} < \frac{m_0}{H_1}. \tag{8.28}$$

Then there exists a constant $\delta > 0$ with the following property: if $x(t)$ is the solution of conclusion (1) and $y(t)$, $t \in [0, T)$, $T \in (0,\infty)$, is another solution of (S_F) *with*

$$\|x(0) - y(0)\| \le \delta, \tag{8.29}$$

then $y(t)$ exists on $\mathbb{R}_+$,

$$\begin{aligned}&\|y(t)\| \le r, \quad t\in\mathbb{R}_+, \\ &\lim_{t\to\infty}\|x(t) - y(t)\| = 0.\end{aligned} \tag{8.30}$$

Moreover, if $F(t,0) \equiv 0$, then the zero solution of (S_F) *is asymptotically stable.*

Proof. (1) We consider the operator U defined as follows:

$$(Uf)(t) = \int_{-\infty}^{t} X(t)P_1X^{-1}F(s,f(s))ds - \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,f(s))ds. \tag{8.31}$$

This operator satisfies $UB^r \subset B^r$, where

$$B^r = \{f \in C_n(\mathbb{R}) : \|f\|_\infty \le r\}. \tag{8.32}$$

Actually,

$$\|Uf\|_\infty \leq \frac{r(1+\rho)}{2} < r. \tag{8.33}$$

Also,

$$\|Uf_1 - Uf_2\|_\infty \leq \rho\|f_1 - f_2\|_\infty, \quad f_1, f_2 \in B^r \tag{8.34}$$

(see Exercise 8.3).

By the Banach contraction principle, U has a unique fixed point x in the ball B^r. It is easy to see that the function $x(t)$ is a solution to the system (S_F) on $\mathbb{R}$.

Let $y(t)$, $t \in \mathbb{R}$, be another solution of the system (S_F) with $\|y\|_\infty \leq r$, and let $g(t) \equiv (Uy)(t)$. Then the function $g(t)$ satisfies the equation

$$x' = A(t)x + F(t, y(t)). \tag{8.35}$$

However, by the discussion at the beginning of this section, (8.11) implies that (8.35) can have only one bounded solution on $\mathbb{R}$. Hence, $g(t) = y(t)$, $t \in \mathbb{R}$, and y is a fixed point of U in B^r. This says that $y(t) \equiv x(t)$. Thus, $x(t)$ is unique.

(2) Now, let $P = I$, $F(t,0) \equiv 0$, and let (iii) hold without necessarily the second integral. Then, by what has been shown above, zero is the only solution of (S_F) in the ball B^r. Assume that $y(t)$, $t \in (-\infty, a]$, is some solution of (S_F) such that

$$\sup_{t \leq a} \|y(t)\| \leq r. \tag{8.36}$$

It is easy to see, as above, that the operator U^a, defined by

$$(U^a f)(t) \equiv \int_{-\infty}^{t} X(t)X^{-1}(s)F(s, f(s))\,ds, \tag{8.37}$$

has a unique fixed point x in the ball

$$B_a^r = \{f \in C(-\infty, a] : \|f\|_\infty \leq r\}. \tag{8.38}$$

We must have $x(t) = 0$, $t \in (-\infty, a]$. The function $y(t)$ satisfies the equation

$$y(t) = X(t)\left[X^{-1}(t_0)y(t_0) + \int_{t_0}^{t} X^{-1}(s)F(s, y(s))\,ds\right] \tag{8.39}$$

for any t_0, $t \in (-\infty, a]$ with $t_0 \leq t$. We fix t in (8.39) and take the limit of the right-hand side as $t_0 \to -\infty$. This limit exists as a finite vector because

$$\|X(t)X^{-1}(t_0)y(t_0)\| \leq rH \exp\{-m_0(t - t_0)\}, \quad t \geq t_0. \tag{8.40}$$

We find

$$y(t) = X(t)\int_{-\infty}^{t} X^{-1}(s)F(s,y(s))ds. \tag{8.41}$$

Consequently, y is a fixed point for the operator U^a in B_a^r. This says that $y(t) = 0$, $t \in (-\infty, a]$.

It follows that every solution $y(t)$ of (S_F), defined on an interval $(-\infty, a]$, must be such that $\|y(t_m)\| > r$, for a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \to -\infty$ as $m \to \infty$. Therefore, the zero solution of (S_F) is negatively unstable.

(3) Let the assumptions in conclusion (3) be satisfied (without necessarily $F(t,0) \equiv 0$) and let the positive number $\delta < r(1-\rho)/2$ be such that

$$||x(0) - y(0)|| < \delta, \tag{8.42}$$

where $x(t)$ is the solution in conclusion (1) and $y(t)$ is another solution of (S_F) defined on the interval $[0, T)$, $T \in (0, \infty)$. Then there exists a sufficiently small neighborhood $[0, T_1) \subset [0, T)$ such that $\|y(t)\| < r$ for all $t \in [0, T_1)$. Here we have used the fact that $\|x\|_\infty < r(1+\rho)/2$ from (8.33). For such values of t, we use the Variation of Constants Formula to obtain

$$\begin{aligned} ||x(t) - y(t)|| \le{}& ||X(t)X^{-1}(0)||||x(0) - y(0)|| \\ &+ \int_0^t ||X(t)X^{-1}(s)||\beta(s)||x(s) - y(s)||ds. \end{aligned} \tag{8.43}$$

Using the dichotomy (8.11), we further obtain

$$\begin{aligned} &\exp\{m_0 t\}||x(t) - y(t)|| \\ &\quad \le H_1\Big[||x(0) - y(0)|| + \int_0^t \exp\{m_0 s\}\beta(s)||x(s) - y(s)||ds\Big]. \end{aligned} \tag{8.44}$$

An application of Gronwall's inequality to (8.44) yields

$$||x(t) - y(t)|| \le H_1||x(0) - y(0)|| \exp\Big\{ -m_0 t + H_1\int_0^t \beta(s)ds\Big\} \tag{8.45}$$

for all $t \in [0, T_1)$. From the definition of λ in conclusion (3), we obtain that if $\epsilon > 0$ is such that $\lambda + 2\epsilon < m_0/H_1$, then there exists an interval $[t_0, \infty)$, $t_0 > 0$, such that

$$\Big(\frac{1}{t}\Big)\int_0^t \beta(s)ds < \lambda + \epsilon < \Big(\frac{m_0}{H_1}\Big) - \epsilon, \quad t \ge t_0. \tag{8.46}$$

This immediately implies

$$H_1\int_0^t \beta(s)ds - m_0 t < -\epsilon H_1 t, \quad t \ge t_0. \tag{8.47}$$

Thus, we may choose

$$\delta < \min\left\{\frac{r(1-\rho)}{(2H_1M)}, \frac{r(1-\rho)}{2}\right\}, \tag{8.48}$$

where

$$M = \sup_{t\geq 0}\left\{\exp\left\{-m_0t + H_1\int_0^t \beta(s)ds\right\}\right\}. \tag{8.49}$$

For such δ, (8.45) implies

$$\|x(t) - y(t)\| < \frac{r(1-\rho)}{2}, \quad t \in [0, T). \tag{8.50}$$

In fact, if (8.50) is assumed false, then, letting

$$T_2 = \inf\left\{t \in [0, T) : \|x(t) - y(t)\| = \frac{r(1-\rho)}{2}\right\}, \tag{8.51}$$

we obtain from (8.45)

$$\|x(T_2) - y(T_2)\| < \frac{r(1-\rho)}{2}, \tag{8.52}$$

that is, a contradiction. It follows that $y(t)$ is continuable to $+\infty$ and that $\|y\|_\infty < r$. If we further assume that $F(t, 0) = 0$, $t \in \mathbb{R}_+$, then we must take $x(t) = 0$, $t \in \mathbb{R}_+$. In this case (8.45) implies the asymptotic stability of the zero solution of (S_F). □

In the next theorem the solution of conclusion (1), Theorem 8.8, is shown to be T-periodic, or almost periodic, provided that the functions A, F have similar properties w.r.t. the variable t. We need the following definition.

Definition 8.9. The continuous function $F : \mathbb{R} \times S \to \mathbb{R}^n$ ($S \subset \mathbb{R}^n$) is said to be S-*almost periodic* if for every $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains at least one number τ with

$$\|F(t+\tau, u) - F(t, u)\| < \epsilon, \quad (t, u) \in \mathbb{R} \times S. \tag{8.53}$$

In Chapter 1, we defined the concept of an $\mathbb{R}^n$-valued almost periodic function. Naturally, a corresponding definition can be given for an M_n-valued function. It will be used in the following theorem.

Theorem 8.10. *Let the assumptions (i)–(iv) of Theorem 8.8 be satisfied and let $x(t)$ be the solution in conclusion (1) there. Then we have the following:*

(i) *if $A(t)$, $F(t, u)$ are T-periodic in t, then $x(t)$ is T-periodic;*

(ii) *let $P_1 = I$, and let (iii) of Theorem 8.8 hold without necessarily the second integral. Let $A(t)$, $F(t,u)$ be almost periodic and $\overline{B_r(0)}$-almost periodic, respectively. Then $x(t)$ is almost periodic.*

Proof. (i) The function $z(t) = x(t+T)$, $t \in \mathbb{R}$, is also a solution of (S_F) with $\|z\|_\infty \le r$. In fact,

$$z'(t) = A(t+T)z(t) + F\big(t+T, z(t)\big) = A(t)z(t) + F\big(t, z(t)\big). \tag{8.54}$$

Since $x(t)$ is unique in B^r (see Theorem 8.8 and its proof), we must have $x(t) = x(t+T)$, $t \in \mathbb{R}$. Hence, $x(t)$ is periodic with period T.

(ii) Given $\epsilon > 0$ we can find a number $l(\epsilon) > 0$ such that in each interval of length $l(\epsilon)$ there exists at least one number τ with

$$||A(t+\tau) - A(t)|| < \epsilon, \qquad ||F(t+\tau, u) - F(t,u)|| < \epsilon \tag{8.55}$$

for every $(t,u) \in \mathbb{R} \times \overline{B_r(0)}$ (see also Exercise 1.20). We fix τ, ϵ and we let

$$\phi(t) = x(t+\tau) - x(t), \quad t \in \mathbb{R}. \tag{8.56}$$

Then $\phi(t)$ satisfies the equation

$$\phi' = A(t+\tau)\phi + Q(t), \tag{8.57}$$

where

$$Q(t) \equiv \big[A(t+\tau) - A(t)\big]x(t) + F\big(t+\tau, x(t+\tau)\big) - F\big(t, x(t)\big). \tag{8.58}$$

Using (8.55), we obtain

$$\begin{aligned} ||Q(t)|| &\le \epsilon r + \big|\big|F\big(t+\tau, x(t+\tau)\big) - F\big(t, x(t+\tau)\big)\big|\big| \\ &\quad + \big|\big|F\big(t, x(t+\tau)\big) - F\big(t, x(t)\big)\big|\big| \\ &\le \epsilon r + \epsilon + \beta(t)||\phi(t)|| \\ &= \beta(t)||\phi(t)|| + \epsilon(r+1), \quad t \in \mathbb{R}. \end{aligned} \tag{8.59}$$

It is easy to see now that

$$Y(t) \equiv X(t+\tau)X^{-1}(\tau) \tag{8.60}$$

is the fundamental matrix of the system

$$y' = A(t+\tau)y \tag{8.61}$$

with $Y(0) = I$, and that this system possesses an exponential dichotomy given by (8.11), where X is replaced by Y, and $P_2 = 0$. Thus, for $\phi(t)$ we have the expression

$$\phi(t) = \int_{-\infty}^{t} Y(t)Y^{-1}(s)Q(s)ds, \quad t \in \mathbb{R}. \tag{8.62}$$

In fact, the function on the right-hand side of (8.62) is a bounded solution of the system (8.57). However, this system has a unique bounded solution on $\mathbb{R}$. Using the estimate on $\|Q(t)\|$ in (8.59), we obtain

$$\begin{aligned} ||\phi(t)|| &\le \int_{-\infty}^{t} ||Y(t)Y^{-1}(s)||\,||Q(s)||ds \\ &\le H_1 \int_{-\infty}^{t} \exp\{-m_0(t-s)\}[\beta(s)||\phi(s)|| + (r+1)\epsilon]ds \\ &< \rho\|\phi\|_\infty + \left(\frac{H_1}{m_0}\right)(r+1)\epsilon = \rho\|\phi\|_\infty + \sigma\epsilon, \end{aligned} \tag{8.63}$$

where σ is another positive constant. It follows that

$$||\phi(t)|| = ||x(t+\tau) - x(t)|| \le \left[\frac{\sigma}{(1-\rho)}\right]\epsilon, \quad t \in \mathbb{R}. \tag{8.64}$$

We summarize the situation as follows: for every $\epsilon_1 > 0$ there exists $l_1(\epsilon_1) \equiv l((1-\rho)\epsilon_1/\sigma)$ such that every interval of length $l_1(\epsilon_1)$ contains at least one number τ such that

$$||x(t+\tau) - x(t)|| \le \left[\frac{\sigma}{(1-\rho)}\right]\left[\frac{(1-\rho)\epsilon_1}{\sigma}\right] = \epsilon_1, \quad t \in \mathbb{R}. \tag{8.65}$$

We have shown that $x(t)$ is almost periodic. □

As it is expected, fixed points of the operator U in the proof of Theorem 8.8 can be obtained via the use of the Schauder-Tychonov theorem (Theorem 2.13). Actually, it suffices to show the existence of a sequence $\{x_m(t)\}_{m=1}^{\infty}$ such that $x_m : [-m,m] \to \mathbb{R}^n$, $\|x_m\|_\infty \le K$ (for some constant $K > 0$), and

$$\begin{aligned} x_m(t) = &\int_{-m}^{t} X(t)P_1X^{-1}(s)F(s,x_m(s))ds \\ &- \int_{t}^{m} X(t)P_2X^{-1}(s)F(s,x_m(s))ds \end{aligned} \tag{E_m}$$

for every $m = 1,2,\ldots$, $t \in [-m,m]$. Since each function x_m satisfies the system (S_F) on $[-m,m]$, the existence of a solution $x(t)$ of (S_F) on $\mathbb{R}$ will follow from Theorem 3.9. This process is followed in the following theorem.

THEOREM 8.11. *Assume that* $A : \mathbb{R} \to M_n$ *is continuous and such that the system* (S) *possesses an exponential dichotomy given by* (8.11). *Furthermore, assume that* $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and such that*

$$\liminf_{m\to\infty} \left\{ \frac{1}{m} \sup_{\substack{t\in\mathbb{R} \\ \|u\|\le m}} \{ ||F(t,u)|| \} \right\} = 0. \tag{8.66}$$

Then system (S_F) *has at least one bounded solution* $x(t)$ *on* $\mathbb{R}$.

PROOF. We consider first the functional $q : C_n(\mathbb{R}) \to \mathbb{R}_+$ defined as follows:

$$q(f) = \frac{2H_1}{m_0} ||F(\cdot, f(\cdot))||_\infty. \tag{8.67}$$

We show that there exists $r > 0$ such that $q(B^r) \subset [0, r]$, where

$$B^r = \{ f \in C_n(\mathbb{R}); \|f\|_\infty \le r \}. \tag{8.68}$$

In fact, assume that the contrary is true and let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers such that

$$\lim_{k\to\infty} \left\{ \frac{1}{n_k} \sup_{\substack{t\in\mathbb{R} \\ \|u\|\le n_k}} \{ ||F(t,u)|| \} \right\} = 0 \quad \text{as } k \longrightarrow \infty. \tag{8.69}$$

This is possible by virtue of (8.66). Then we have that $q(B^{n_k}) \not\subset [0, n_k]$, $k = 1, 2, \ldots$. Consequently, each ball B^{n_k} contains at least one function f_{n_k} such that

$$q(f_{n_k}) = \frac{2H_1}{m_0} ||F(\cdot, f_{n_k}(\cdot))||_\infty > n_k. \tag{8.70}$$

This implies

$$1 < \frac{2H_1}{m_0 n_k} ||F(\cdot, f_{n_k}(\cdot))||_\infty \le \frac{2H_1}{m_0 n_k} \sup_{\substack{t\in\mathbb{R} \\ \|u\|\le n_k}} \{ ||F(t,u)|| \}, \tag{8.71}$$

contradicting (8.69). Let $r > 0$ be such that $q(B^r) \subset [0, r]$ and let $U_1 : C_n[-1,1] \to C_n[-1,1]$ be defined as follows:

$$\begin{aligned}(U_1 f)(t) \equiv & \int_{-1}^{t} X(t) P_1 X^{-1}(s) F(s, f(s))\, ds \\ & - \int_{t}^{1} X(t) P_2 X^{-1}(s) F(s, f(s))\, ds.\end{aligned} \tag{8.72}$$

Then $U_1 B_1^r \subset B_1^r$, where

$$B_1^r = \{f \in C_n[-1,1] : \|f\|_\infty \le r\}. \tag{8.73}$$

In fact, given $f \in B_1^r$, let

$$\bar{f}(t) \equiv \begin{cases} f(1), & t \in [1,\infty), \\ f(t), & t \in [-1,1], \\ f(-1), & t \in (-\infty,-1]. \end{cases} \tag{8.74}$$

Then we have, for $t \in [-1,1]$,

$$\begin{aligned} ||(U_1 f)(t)|| &\le \int_{-1}^{t} ||X(t)P_1X^{-1}(s)||\,||F(s,\bar{f}(s))||ds \\ &\quad + \int_t^1 ||X(t)P_2X^{-1}(s)||\,||F(s,\bar{f}(s))||ds \\ &\le H_1\Big[\int_{-1}^{t} \exp\{-m_0(t-s)\}ds + \int_t^1 \exp\{-m_0(s-t)\}ds\Big] \\ &\quad \times ||F(\cdot,\bar{f}(\cdot))||_\infty \\ &= H_1\Big[\int_{-t-1}^{0} \exp\{m_0 s\}ds + \int_0^{1-t} \exp\{-m_0 s\}ds\Big] \\ &\quad \times ||F(\cdot,\bar{f}(\cdot))||_\infty \\ &\le H_1\Big[\int_{-\infty}^{0} \exp\{m_0 s\}ds + \int_0^{\infty} \exp\{-m_0 s\}ds\Big] \\ &\quad \times ||F(\cdot,\bar{f}(\cdot))||_\infty \\ &= \frac{2H_1}{m_0}||F(\cdot,\bar{f}(\cdot))||_\infty = q(\bar{f}) \le r. \end{aligned} \tag{8.75}$$

It is easy to see that the set $U_1 B_1^r$ is relatively compact in $C_n[-1,1]$ and that U_1 is continuous on B_1^r. By the Schauder-Tychonov theorem, U_1 has at least one fixed point $x_1 \in B_1^r$. Similarly, we obtain that each operator U_m, $m = 2,\dots$, defined on $C_n[-m,m]$ by the right-hand side of (E_m), has a fixed point x_m, $t \in [-m,m]$, such that $\|x_m\|_\infty \le r$ and (E_m) is satisfied on $[-m,m]$. Theorem 3.9 implies now the existence of a solution $x(t)$, $t \in \mathbb{R}$, of (S_F) such that $\|x\|_\infty \le r$. □

3. QUASILINEAR SYSTEMS

The most important property of a quasilinear system

$$x' = A(t,x)x + F(t,x) \tag{S_Q}$$

with $A : J \times \mathbb{R}^n \to M_n$, $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ (with J a real interval), is that the system

$$x' = A(t, f(t))x + F(t, f(t)) \tag{S^f}$$

is linear for any $\mathbb{R}^n$-valued function f on J. Linear systems of the type (S^f) have been extensively studied in the preceding chapters. It is therefore natural to ask whether information about (S_Q) can be obtained by somehow exploiting the properties of the system (S^f), where f belongs to a certain class $A(J)$ of continuous functions on J.

It is shown here that some of the properties of the system (S^f) can be carried over to the system (S_Q) via fixed point theory. In fact, if U denotes the operator which maps the function $f \in A(J)$ into the (unique) solution $x_f \in A(J)$ of (S^f), then the fixed points of U are solutions in $A(J)$ of the system (S_Q).

This procedure is followed here in order to obtain some stability and periodicity properties of the system (S_Q).

It should be noted that the quasilinear systems constitute quite a large class. To see this, it suffices to observe that if $B : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable w.r.t. its second variable, then there exists a matrix $A(t,x)$ such that

$$B(t,x) \equiv A(t,x)x + B(t,0), \quad (t,x) \in J \times \mathbb{R}^n. \tag{8.76}$$

This assertion follows from the next lemma.

Lemma 8.12. *Let D be an open, convex subset of $\mathbb{R}^n$. Let $F : D \to \mathbb{R}^n$ be continuously differentiable on D. Let $x_1, x_2 \in D$ be given. Then*

$$F(x_2) - F(x_1) = \int_0^1 F_x(sx_2 + (1-s)x_1)\,ds(x_2 - x_1), \tag{8.77}$$

where $F_x(u)$ is the Jacobian matrix $[\partial F_i(u)/\partial x_j]$, $i, j = 1, 2, \ldots, n$, of F at u.

Proof. Consider the function

$$g(s) = F(sx_2 + (1-s)x_1), \quad s \in [0,1]. \tag{8.78}$$

This function is well defined because the set D is convex. Using the chain rule for vector-valued functions, we have

$$g'(s) \equiv F_x(sx_2 + (1-s)x_1)(x_2 - x_1). \tag{8.79}$$

Integrating (8.79) from $s = 0$ to $s = 1$ and recalling that $g(0) = F(x_1)$, $g(1) = F(x_2)$, we get (8.77). □

We notice that if the function B in (8.76) satisfies $B(t,0) \equiv 0$, then the system

$$x' = B(t,x) \tag{8.80}$$

becomes

$$x' = A(t,x)x. \tag{8.81}$$

Before we state and prove the main stability result of this section, it is convenient to establish some definitions. In what follows, $A : \mathbb{R}_+ \times \mathbb{R}^n \to M_n$ will be assumed to be continuous on its domain.

Definition 8.13. The zero solution of the system (8.81) is called *weakly stable* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with the following property: for every $\xi \in \mathbb{R}^n$ with $\|\xi\| \le \delta(\epsilon)$ there exists at least one solution $x \in C_n(\mathbb{R}_+)$ of (8.81) satisfying $x(0) = \xi$ and

$$\|x\|_\infty \le \epsilon. \tag{8.82}$$

This definition coincides with the usual definition of Chapter 4 if the solutions of (8.81) are unique w.r.t. initial conditions at 0.

Definition 8.14. The systems

$$x' = A(t, f(t))x \tag{E_f}$$

are called *iso-stable* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with the property: for every $f \in C_n(\mathbb{R}_+)$ with $\|f\|_\infty \le \epsilon$ and every solution $y_f(t)$, $t \in \mathbb{R}_+$, of (E_f) with $\|y_f(0)\| \le \delta(\epsilon)$, we have $\|y_f\|_\infty \le \epsilon$.

The iso-stability of (E_f) is of course guaranteed if for every $\epsilon > 0$ there exists $K = K(\epsilon) > 0$ such that

$$\limsup_{t\to\infty} \int_0^t \mu(A(s, f(s)))ds \le K < +\infty \tag{8.83}$$

for any $f \in C_n(\mathbb{R}_+)$ with $\|f\|_\infty \le \epsilon$. Here, μ is the measure of Definition 4.7. This fact follows from Theorem 4.12.

The following theorem provides conditions for the weak stability of the zero solution of (8.81).

Theorem 8.15. *Let $A : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that the systems (E_f) are iso-stable. Then the zero solution of the system (8.81) is weakly stable. If, moreover, the solutions of the system (8.81) are unique w.r.t initial conditions at zero, then the zero solution of the system (8.81) is stable.*

Proof. Let $X_f(t)$ be the fundamental matrix of (E_f) with $X_f(0) = I$. Fix $\epsilon > 0$ and let $\delta(\epsilon) > 0$ be such that $\|X_f(t)\xi\| \leq \epsilon$ for every $t \in \mathbb{R}_+$, $f \in C_n(\mathbb{R}_+)$ with $\|f\|_\infty \leq \epsilon$, and $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta(\epsilon)$. This is possible by virtue of the iso-stability of the systems (E_f). Now, fix $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta(\epsilon)$ and consider the set S consisting of all functions $f \in C_n[0,1]$ such that $f(0) = \xi$ and $\|f\|_\infty \leq \epsilon$. Obviously, S is a closed, convex and bounded set in the Banach space $C_n[0,1]$. We define the operator $T : S \to C_n[0,1]$ as follows: given a function $f \in S$, $y = Tf$ is the solution of the system (E_f) with $y(0) = \xi$, restricted on the interval $[0,1]$. From our assumptions above, we obtain that $\|y\|_\infty \leq \epsilon$ because $\|\xi\| \leq \delta(\epsilon)$. It follows that $TS \subset S$. Let

$$p = \sup_{\substack{t\in[0,1] \\ \|u\|\leq\epsilon}} ||A(t,u)||. \tag{8.84}$$

Then $f \in S$ and $y = Tf$ imply

$$\|y'\|_\infty \leq ||A(\cdot, f(\cdot))||_\infty \|y\|_\infty \leq p\epsilon. \tag{8.85}$$

This says that the set TS is equicontinuous. Since it is also uniformly bounded, it is relatively compact by virtue of Theorem 2.5. In order to apply the Schauder-Tychonov Theorem, it remains to show that T is continuous on S. Let $\{f_m\}_{m=1}^\infty \subset S$, $f \in S$ be such that

$$\lim_{m\to\infty} ||f_m - f||_\infty = 0. \tag{8.86}$$

Then, since $\{Tf_m\}_{m=1}^\infty$ is uniformly bounded and equicontinuous, there exists a subsequence $\{Tf_{m_k}\}_{k=1}^\infty$ such that $u_k = Tf_{m_k} \to y \in S$ uniformly as $k \to \infty$. Let

$$u(t) \equiv \xi + \int_0^t A(s, f(s))y(s)ds. \tag{8.87}$$

The sequence $\{u_k\}$ satisfies

$$u_k(t) \equiv \xi + \int_0^t A(s, f_{m_k}(s))u_k(s)ds. \tag{8.88}$$

Subtracting the last two equations we get

$$||u_k - u||_\infty \leq \int_0^1 ||A(s, f_{m_k}(s))u_k(s) - A(s, f(s))y(s)||ds, \tag{8.89}$$

which implies $\|u_k - u\|_\infty$ as $k \to \infty$. Thus, $u(t) = y(t)$, $t \in [0,1]$. It follows that $y(t)$ satisfies the system (E_f) and that $Tf_{m_k} \to Tf$ as $k \to \infty$ uniformly on $[0,1]$. Since we could have started with an arbitrary subsequence of $\{f_m\}$ instead of $\{f_m\}$ itself, we have actually shown that every subsequence of $\{Tf_m\}$ contains a further subsequence converging uniformly to Tf on $[0,1]$. This is equivalent to saying that $Tf_m \to Tf$ as $m \to \infty$ uniformly on $[0,1]$. Proceeding similarly, we obtain by induction a sequence $\{x_m\}_{m=1}^{\infty}$ of functions with the property: $x_m \in C_n[0,m]$, $x_m(0) = \xi$, $\|x_m(t)\| \le \epsilon$ for $t \in [0,m]$, and each x_m satisfies the system (8.81) on the interval $[0,m]$. As in Theorem 3.9, we obtain now the existence of a solution $x(t)$, $t \in \mathbb{R}_+$, of (8.81) such that $x(0) = \xi$ and $\|x\|_\infty \le \epsilon$. This proves the weak stability of the zero solution of (8.81).

If, in addition, the solutions of (8.81) are unique w.r.t. initial conditions at zero, then the solution $x(t)$ above is the only solution of (8.81) with $x(0) = \xi$. This proves the stability of the zero solution of (8.81). □

Example 8.16. To illustrate Theorem 8.15, consider the system (8.81) with

$$A(t,u) \equiv \begin{bmatrix} \exp\{-(t-u_1^2)\} & \dfrac{u_2}{(t+1)^2} \\ -\dfrac{u_2}{(t+2)^2} & \exp\{-t^2\} \end{bmatrix} \tag{8.90}$$

for any $(t,u_1,u_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. We choose, for convenience, the norm

$$\|x\|_1 = \max\{|x_1|, |x_2|\}. \tag{8.91}$$

Then, using Table 1, we have

$$\begin{aligned} \mu(A(t,u)) &= \max\left\{\exp\{-(t-u_1^2)\} + \frac{|u_2|}{(t+1)^2}, \frac{|u_2|}{(t+2)^2} + \exp\{-t^2\}\right\} \\ &= \exp\{-(t-u_1^2)\} + \frac{|u_2|}{(t+1)^2}, \end{aligned} \tag{8.92}$$

for all $t \ge 0$. Given $f \in C_2(\mathbb{R}_+)$ with $\|f\|_\infty \le \epsilon$, we obtain

$$\begin{aligned} \mu(A(t,f(t))) &= \exp\{-(t-f_1^2(t))\} + \frac{|f_2(t)|}{(t+1)^2} \\ &\le \exp\{\epsilon^2 - t\} + \frac{\epsilon}{(t+1)^2}, \quad t \ge 1. \end{aligned} \tag{8.93}$$

Thus, (8.83) is satisfied. Since $A(t,u)u$ satisfies a Lipschitz condition w.r.t. u on any compact subset of $\mathbb{R}_+ \times \mathbb{R}^n$, the zero solution of the system (8.81) is stable.

Example 8.17. Similarly, the zero solution of the system $x' = B(t,x)$ with

$$B(t,u) \equiv \begin{bmatrix} e^{-t}(u_1^2 + u_2) + e^{-2t}u_2^2 \\ te^{-t}u_1^2 + (1+t^2)^{-1}u_2 \sin(u_2^2) \end{bmatrix} \tag{8.94}$$

is stable. The proof is left as an exercise (see Exercise 8.4).

Other stability properties of the system (8.81) can be studied by means of the preceding method. Since the corresponding statements and methods of proof are very similar to our considerations above, they are omitted.

As the reader might have expected, it is sometimes more convenient to transfer the properties of the system (S^f) to the system (S_Q) without the use of fixed point theory. In fact, it might be advisable to pick a local solution $x(t)$ of (S_Q) and apply the assumed uniform conditions on (S^f) (like iso-stability) to the system

$$u' = A(t,\tilde{x}(t))u + F(t,\tilde{x}(t)), \tag{8.95}$$

or the system

$$u' = A(t,\tilde{x}(t))u + F(t,u), \tag{8.96}$$

where $\tilde{x}(t)$ coincides with $x(t)$ on some interval and is constant everywhere else. To illustrate this procedure, we extend below Theorem 4.14 to quasilinear systems.

Theorem 8.18. *For the system* (S_Q) *assume the following:*

(i) $A : \mathbb{R}_+ \times \mathbb{R}^n \to M_n$ *is continuous;*

(ii) $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and such that*

$$\|F(t,x)\| \le \lambda\|x\|, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{8.97}$$

where λ *is a positive constant;*

(iii) *there exists a positive constant* $K \in (0, 1/\lambda)$ *with the property: for every* $f \in C_n(\mathbb{R}_+)$ *there exists a fundamental matrix* $X_f(t)$ *of the system* (E_f) *such that*

$$\int_0^t \|X_f(t)X_f^{-1}(s)\|ds \le K, \quad t \in \mathbb{R}_+. \tag{8.98}$$

Then the zero solution of the system (S_Q) *is asymptotically stable.*

Proof. Let $x_0 \in \mathbb{R}^n$ be given and let $x(t)$ be a local solution of (S_Q) on the interval $[0, p)$, for some $p > 0$. Let q be a constant in $(0, p)$ and consider the function

$$\tilde{x}(t) = \begin{cases} x(t), & t \in [0,q], \\ x(q), & t \in [q,\infty). \end{cases} \tag{8.99}$$

Then $\tilde{x}(t) \in C_n(\mathbb{R}_+)$ and the function $x(t)$ satisfies the system (8.96) on the interval $[0, q]$. Now, we can follow the steps of the proof of Theorem 4.14 to show that $x(t)$ satisfies the inequality

$$\|x(t)\| \le (1 - \lambda K)^{-1}\|x(0)\| \tag{8.100}$$

on the interval $[0, q]$. Since $q \in [0, p)$ is arbitrary, and the right-hand side of (8.100) does not depend on q, it follows that $x(t)$ satisfies (8.100) on the entire interval $[0, p)$. Thus, by Theorem 3.8, $x(t)$ is continuable to the point $t = p$. It follows that $x(t)$ is continuable to $+\infty$ and that (8.100) holds on $\mathbb{R}_+$. This shows that the zero solution of the system (S_Q) is stable. The proof of the fact that the zero solution of (S_Q) is asymptotically stable follows again as in Theorem 4.14 because we have

$$\int_0^t ||X_x(s)X_x^{-1}(s)||ds \le K, \quad t \ge 0, \tag{8.101}$$

for every solution $x(t)$, $t \in \mathbb{R}_+$, of (S_Q). □

The existence of T-periodic solutions of the system (S_Q) is the content of the following theorem.

THEOREM 8.19. *Assume that $A : \mathbb{R}\times\mathbb{R}^n \to M_n$ is a symmetric matrix, T-periodic in its first variable, and such that its largest eigenvalue $\lambda_M(t, u)$ is bounded above by a negative constant $-q$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, T-periodic in t, and such that*

$$\liminf_{m\to\infty} \left\{ \frac{1}{m} \int_0^T \sup_{||u||\le m} ||F(t, u)||dt \right\} = 0. \tag{8.102}$$

Then the system (S_Q) has at least one T-periodic solution.

PROOF. Assume for the moment that for every continuous T-periodic $f \in C_n(\mathbb{R})$ the system (S^f) has a unique T-periodic solution $x_f(t)$. Then, following the theory developed in Chapter 6, it is easy to see that

$$\begin{aligned} x_f(t) \equiv X_f(t)[I - X_f(T)]^{-1} X_f(T) \int_0^T X_f^{-1}(s)F(s, f(s))ds \\ + \int_0^t X_f(t)X_f^{-1}(s)F(s, f(s))ds, \end{aligned} \tag{8.103}$$

where $X_f(t)$ is the fundamental matrix of (E_f) with $X_f(0) = I$. Obviously, the fixed points of the operator $U : f \to x_f$ are the T-periodic solutions of the system (S_Q). In order to apply the Schauder-Tychonov theorem, we first show that the system (E_f) has indeed a unique T-periodic solution for every T-periodic function f. Fix $f \in P_n(T)$, and let $x_f(t)$, $t \in \mathbb{R}_+$, be a solution of the system (E_f). Then we have

$$\left(\frac{d}{dt}\right)\left(\exp\{2qt\}\|x_f(t)\|^2\right) = 2q\exp\{2qt\}\|x_f(t)\|^2 + 2\exp\{2qt\}\langle A(t,f(t))x_f(t), x_f(t)\rangle \le 2q\exp\{2qt\}\|x_f(t)\|^2 - 2q\exp\{2qt\}\|x_f(t)\|^2 = 0. \tag{8.104}$$

Integrating (8.104) from s to $t \ge s$, we get

$$\exp\{2qt\}\|x_f(t)\|^2 \le \exp\{2qs\}\|x_f(s)\|^2. \tag{8.105}$$

Since $x(t) \equiv X_f(t)X_f^{-1}(s)x_f(s)$, (8.105) implies

$$\|X_f(t)X_f^{-1}(s)x_f(s)\|^2 \le \exp\{-2q(t-s)\}\|x_f(s)\|^2. \tag{8.106}$$

Since $x_f(s)$ is an arbitrary vector in $\mathbb{R}^n$, (8.106) yields

$$\|X_f(t)X_f^{-1}(s)\| \le \exp\{-q(t-s)\}, \quad t \ge s. \tag{8.107}$$

Letting $s = 0$ in (8.107), we obtain $\|X_f(t)\| \to 0$ as $t \to \infty$. This says that every solution $x(t)$, $t \in \mathbb{R}_+$, of the system (E_f) tends to zero as $t \to \infty$. It follows that the only possible T-periodic solution of (E_f) is the zero solution.

Now, we employ Theorem 6.1 to conclude that for each $f \in P_n(T)$ the system (S^f) has a unique T-periodic solution.

We need to show that $[I - X_f(T)]^{-1}$ is bounded uniformly w.r.t. $f \in P_n(T)$. We are considering only $t \in [0,T]$, because, as it was established in Chapter 6, the existence of an $x \in C_n[0,T]$, which is T-periodic, is sufficient for the existence of a T-periodic solution of the system (S_Q). Since (8.107) implies

$$\|X_f(T)\| \le \exp\{-qT\}, \tag{8.108}$$

we have

$$\|[I - X_f(T)]\xi\| \ge \|\xi\| - \|X_f(T)\|\|\xi\| \ge (1 - \exp\{-qT\})\|\xi\|, \quad \xi \in \mathbb{R}^n. \tag{8.109}$$

Thus, we have shown that

$$\|[I - X_f(T)]^{-1}\| \le (1 - \exp\{-qT\})^{-1}, \quad f \in P_n(T). \tag{8.110}$$

The rest of the proof follows as in Theorem 8.11 in order to show that U has a fixed point in some closed ball of $P_n(T)$ with center at zero. It is therefore omitted. □

Theorem 8.19 is well illustrated by the following example.

Example 8.20. Consider the system

$$x' = \begin{bmatrix} -p(t,x) & 0 & 0 \\ 0 & -q(t,x) & 0 \\ 0 & 0 & -r(t,x) \end{bmatrix} x + F(t,x), \tag{8.111}$$

where $p, q, r : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are continuous, 2π-periodic in t and such that

$$p(t,u) \geq 1, \qquad q(t,u) \geq 2, \qquad r(t,u) \geq 3 \tag{8.112}$$

for every $(t,u) \in [0,2\pi] \times \mathbb{R}^3$. Moreover, $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is continuous, 2π-periodic in t and such that

$$||F(t,u)|| \leq \lambda \|u\|^{\sigma} + \mu, \quad t \in [0,2\pi],\ u \in \mathbb{R}^3, \tag{8.113}$$

where λ, μ, σ are positive constants with $\sigma \in (0,1)$. Then $\lambda_M(t,u) \leq -1$ and the rest of the assumptions of Theorem 8.19 are satisfied.

4. APPLICATIONS OF THE INVERSE FUNCTION THEOREM

Our applications of the inverse function theorem are concerned with boundary value problems

$$x' = A(t)x + F(t,x), \tag{S_F}$$

$$Ux = r \tag{B}$$

of the type considered in Chapter 6. We recall again that if the homogeneous problem ($F \equiv 0$, $r = 0$) has only the zero solution, then the problem ((S_F), (B)) is equivalent to the problem

$$x(t) = X(t)\widetilde{X}^{-1}[r - Up(\cdot,x)] + p(t,x), \tag{8.114}$$

where $\widetilde{X}$ is the matrix whose columns are the values of U on the corresponding columns of $X(t)$ and

$$p(t,x) \equiv \int_0^t X(t)X^{-1}(s)F(s,x(s))\,ds. \tag{8.115}$$

Here, and in what follows, we assume for convenience that F is defined on all of $\mathbb{R}^n$ w.r.t. its second variable, and that $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $A : [0,T] \to M_n$ are continuous. Let x_0 be a fixed element of $C_n[0,T]$, and let

$$B^r = \{x \in C_n[0,T] : ||x - x_0||_\infty < r\}, \tag{8.116}$$

where r is a positive constant. Then the operator $T_0 : B^r \to C_n[0,T]$, given by

$$(T_0 f)(t) \equiv F(t, f(t)), \tag{8.117}$$

is continuous on B^r. This follows as in Example 2.28 because F is uniformly continuous on the set $[0,T] \times S$, where

$$S = \{u \in \mathbb{R}^n : \|u\| < r + \|x_0\|_\infty\}. \tag{8.118}$$

If F has a continuous Jacobian matrix $F_x(t,u)$ on $[0,T] \times S$, then it is also easy to see, again as in Example 2.28, that the operator T_0 is Fréchet differentiable at x_0 with Fréchet derivative $T_0'(x_0)$ given by

$$[T_0'(x_0)h](t) = F_x(t, x_0(t))h(t) \tag{8.119}$$

for every $h \in C_n[0,T]$ and every $t \in [0,T]$.

We now state the main result on finite intervals.

Theorem 8.21. *For the equation* (8.114) *assume the following:*

(i) *$A : [0,T] \to M_n$, $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous;*
(ii) *the Jacobian matrix $F_x(t,u)$ is defined and continuous on the set $[0,T] \times \mathbb{R}^n$;*
(iii) *fix $x_0 \in C_n[0,T]$ and let $f_0 \in C_n[0,T]$ be given by*

$$x_0(t) = f_0(t) - X(t)\widetilde{X}^{-1}Up(\cdot, x_0) + p(t, x_0). \tag{8.120}$$

Assume that the equation

$$x(t) = f(t) - X(t)\widetilde{X}^{-1}Uq(\cdot, x_0, x) + q(t, x_0, x) \tag{8.121}$$

has a unique solution $x \in C_n[0,T]$ for every $f \in C_n[0,T]$, where

$$q(t, x_0, x) \equiv \int_0^t X(t)X^{-1}(s)F_x(s, x_0(s))x(s)ds. \tag{8.122}$$

Then there exist two constants $\alpha > 0$, $\beta > 0$ with the property: for every $f \in C_n[0,T]$ with $\|f - f_0\|_\infty \le \beta$ there exists a unique solution $x(t)$ to the equation

$$x(t) = f(t) - X(t)\widetilde{X}^{-1}Up(\cdot, x) + p(t, x) \tag{8.123}$$

such that $\|x - x_0\|_\infty \le \alpha$.

Proof. Consider the operator $V : C_n[0,T] \to C_n[0,T]$ given by

$$(Vx)(t) = x(t) + X(t)\widetilde{X}^{-1}Up(\cdot,x) - p(t,x), \quad t \in [0,T]. \tag{8.124}$$

It is easy to see that V is continuous on $C_n[0,T]$ and Fréchet differentiable there (see Example 2.28 and Exercise 2.6) with Fréchet derivative $V'(x_0)$ given by

$$[V'(x_0)h](t) \equiv h(t) + X(t)\widetilde{X}^{-1}Uq(\cdot,x_0,h) - q(t,x_0,h). \tag{8.125}$$

Now, fix $f \in C_n[0,T]$ and consider the equation $V'(x_0)h = f$. Our assumption (iii) implies that h is the unique solution of the linear equation (8.121). Thus, $V'(x_0)$ is one-to-one, and onto. To show that $V'(x)$ is continuous in x, let, for some constant $r > 0$,

$$\begin{aligned} D &= \{u \in \mathbb{R}^n : \|u\| < r + \|x_0\|_\infty\}, \\ D_1 &= \{x \in C_n[0,T] : \|x\|_\infty < r + \|x_0\|_\infty\}. \end{aligned} \tag{8.126}$$

Then given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|F_x(\cdot,u_1) - F_x(\cdot,u_2)\|_\infty \le \frac{\epsilon}{2\mu} \tag{8.127}$$

for any $u_1, u_2 \in D$ with $\|u_1 - u_2\| \le \delta(\epsilon)$. Here, $\mu = \max\{\mu_1, \mu_2\}$ with

$$\begin{aligned} \mu_1 &= \max_{t\in[0,T]} \left\{ \int_0^t \|X(t)X^{-1}(s)\| ds \right\}, \\ \mu_2 &= \|X\|_\infty \|\widetilde{X}^{-1}\| \|U\| \mu_1. \end{aligned} \tag{8.128}$$

It follows that for every $x_1, x_2 \in D_1$ with $\|x_1 - x_2\|_\infty \le \delta(\epsilon)$ we have

$$\|V'(x_1)h - V'(x_2)h\|_\infty \le \epsilon \|h\|_\infty \tag{8.129}$$

for any $h \in C_n[0,T]$, or

$$\|V'(x_1) - V'(x_2)\| \le \epsilon. \tag{8.130}$$

Our assertion follows now from the inverse function theorem (Theorem 2.27). □

The preceding theorem has the following important and applicable corollary.

Corollary 8.22. *Let the assumptions of Theorem 8.21 be satisfied with* $\|f_0\|_\infty < \beta$. *Then there exist positive numbers* α, μ *such that for each* $r \in \mathbb{R}^n$ *with* $\|r\| \le \mu$ *there exists a unique solution* $x(t)$ *of the problem* ((S_F), (B)) *satisfying* $\|x - x_0\|_\infty \le \alpha$.

PROOF. Let $\epsilon > 0$ be such that $\|f_0\|_\infty + \epsilon \le \beta$. Then since

$$\lim_{\|r\|\to 0} \sup_{t\in[0,T]} \|X(t)\widetilde{X}^{-1}r - f_0(t)\| = \|f_0\|_\infty, \tag{8.131}$$

there exists $q(\epsilon) > 0$ such that

$$\sup_{t\in[0,T]} \|X(t)\widetilde{X}^{-1}r - f_0(t)\| \le \|f_0\|_\infty + \epsilon \le \beta \tag{8.132}$$

whenever $\|r\| \le \mu = q(\epsilon)$. Thus, for every $r \in \mathbb{R}^n$ with $\|r\| \le \mu$, equation (8.114) has a unique solution in the set $\{x \in C_n[0,T] : \|x - x_0\| \le \alpha\}$. □

The next theorem solves the problem $((S_F), (B))$ on the interval $\mathbb{R}_+$. The solutions actually belong to $C_n(\mathbb{R}_+)$, which consists of the bounded continuous functions on $\mathbb{R}_+$.

THEOREM 8.23. *For the problem* $((S_F), (B))$ *assume the following:*

(i) $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuous;*
(ii) *the fundamental matrix* $X(t)$ $(X(0) = I)$ *satisfies*

$$\sup_{t\in\mathbb{R}_+} \int_0^t \|X(t)X^{-1}(s)\| ds < +\infty. \tag{8.133}$$

Moreover, $U : C_n(\mathbb{R}_+) \to \mathbb{R}^n$ *is a bounded linear operator such that* $\widetilde{X}^{-1}$ *exists;*
(iii) *the Jacobian matrix* $F_x(t,u)$ *exists and is continuous on* $\mathbb{R}_+ \times \mathbb{R}^n$. *Given a bounded set* $M \subset \mathbb{R}^n$, *the sets* $F(\mathbb{R}_+ \times M)$, $F_x(\mathbb{R}_+ \times M)$ *are bounded and for every* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that*

$$\|F_x(t,u_1) - F_x(t,u_2)\| < \epsilon \tag{8.134}$$

for all $(t,u_1,u_2) \in \mathbb{R}_+ \times M \times M$ *such that* $\|u_1 - u_2\| \le \delta(\epsilon)$*;*
(iv) *for every* $f \in C_n(\mathbb{R}_+)$ *the equation* (8.121) *has a unique solution* $x \in C_n(\mathbb{R}_+)$, *where the function* q *is given by* (8.122).

Given $x_0 \in C_n(\mathbb{R}_+)$, *let* $f_0(t)$ *be defined by* (8.120). *Then there exist two constants* $\alpha > 0$, $\beta > 0$ *with the property: for every* $f \in C_n(\mathbb{R}_+)$ *with* $\|f - f_0\|_\infty \le \beta$ *there exists a unique solution* $x \in C_n(\mathbb{R}_+)$ *of the equation* (8.123) *such that* $\|x - x_0\|_\infty \le \alpha$. *If, moreover,* $\|f_0\|_\infty < \beta$, *then the problem* $((S_F), (B))$ *has a unique solution for all* $r \in \mathbb{R}^n$ *with* $\|r\|$ *sufficiently small.*

PROOF. The key element in the proof is again the continuity of the operator $V'(x)$ in (8.124) in x, which follows from an inequality like (8.127) as in the proof of Theorem 8.21. The details are omitted. □

The reader will have no difficulty in applying the above considerations to equations of the type

$$\begin{aligned} x(t) = f(t) + \int_{-\infty}^{t} X(t)P_1X^{-1}(s)F(s,x(s))\,ds \\ - \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,x(s))\,ds \end{aligned} \tag{8.135}$$

under suitable assumptions, where P_1 is a projection matrix in M_n. The conclusion in this case would be that the system (S_F) has bounded solutions on $\mathbb{R}$ if $f_0 = Vx_0$ has a sufficiently small norm $\|f_0\|_\infty$ (so that f can be taken identically equal to zero in (8.135)). Here, V is the operator defined by (8.135) written as $Vx = f$.

Now, we examine the problem

$$x' = F(t,x), \tag{E}$$

$$Ux = 0 \tag{B}$$

from a different point of view. We first notice that if U is a nonlinear operator, then we cannot in general reduce the problem ((E), (B)) to an integral equation of the type (8.114). We also observe that the problem ((E), (B)) is equivalent to the problem

$$Vx \equiv [Nx, Ux] = [0,0], \tag{8.136}$$

where

$$(Nx)(t) \equiv x'(t) - F(t,x(t)). \tag{8.137}$$

Thus, solutions to the problem ((E), (B)) can actually be obtained from an application of the inverse function theorem to the operator V in (8.136). This is accomplished below for boundary conditions (B), where Ux has a Fréchet derivative at any $x_0 \in C_n^1(\mathbb{R}_+)$. Thus, we obtain solutions of ((E), (B)) in $C_n^1(\mathbb{R}_+)$. Another theorem (Theorem 8.25) is given extending this result to problems

$$x' = F(t,x) + G(t,x), \tag{8.138}$$

$$Ux = Wx, \tag{8.139}$$

with no differentiability assumptions on the function G and the nonlinear operator W. Two interesting corollaries cover the case of perturbations depending on a small parameter $\epsilon > 0$. Extensions to problems on $\mathbb{R}$ can be similarly treated, and they are therefore omitted. We let $C_l^1 = C_n^1(\mathbb{R}_+) \cap C_n^l$. The space C_l^1 is a closed subspace of $C_n^1(\mathbb{R}_+)$. Thus, it is a Banach space with norm

$$\|f\|_1 = \|f\|_\infty + \|f'\|_\infty. \tag{8.140}$$

We also note that $C_n(\mathbb{R}_+) \times \mathbb{R}^n$ (with addition and multiplication by real scalars defined in the obvious way) is a Banach space with norm

$$||[f,r]|| = \|f\|_\infty + \|r\|. \tag{8.141}$$

The following condition on F will be needed in the sequel.

Condition (F). (i) $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $F(\mathbb{R}_+ \times M)$ is bounded for every bounded set $M \subset \mathbb{R}^n$. Moreover, the Jacobian matrix $F_x(t,u)$ exists and is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$.

(ii) For every bounded set $M \subset \mathbb{R}^n$, $F_x(\mathbb{R}_+ \times M)$ is bounded and for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$||F_x(t,u_1) - F_x(t,u_2)|| < \epsilon, \quad (t,u_1,u_2) \in \mathbb{R}_+ \times M \times M. \tag{8.142}$$

If Condition (F) holds, then the operator $N : C_n^1(\mathbb{R}_+) \to C_n(\mathbb{R}_+)$ satisfies

$$(N(x+h))(t) - (Nx)(t) = h'(t) - F(t,x(t)+h(t)) + F(t,x(t)) \tag{8.143}$$

for all $(t,h) \in \mathbb{R}_+ \times C_n^1(\mathbb{R}_+)$, and it is easy to see that N is Fréchet differentiable at each $x_0 \in C_n^1(\mathbb{R}_+)$ with Fréchet derivative $N'(x_0)$ given by

$$(N'(x_0)h)(t) = h'(t) - F_x(t,x_0(t))h(t), \quad (t,h) \in \mathbb{R}_+ \times C_n^1(\mathbb{R}_+). \tag{8.144}$$

Theorem 8.24. *Assume that the function F satisfies Condition (F) and that the operator $U : C_n^1(\mathbb{R}_+) \supset S \to \mathbb{R}^n$ is continuous and Fréchet differentiable on the open and bounded set S. Let $U'(x)$ be continuous on S, that is, for every $x_0 \in S$ we have: for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with*

$$||[U'(x) - U'(x_0)]h|| \le \epsilon \|h\|_\infty \tag{8.145}$$

for every $x \in S$ with $\|x - x_0\|_1 < \epsilon$ and every $h \in C_n^1(\mathbb{R}_+)$.

Fix $x_0 \in S$ and assume that the linear problem

$$x' - F_x(t,x_0(t))x = 0, \tag{8.146}$$

$$U'(x_0)x = 0 \tag{8.147}$$

has only the zero solution in $C_n^1(\mathbb{R}_+)$. Assume further that

$$\sup_{t\in\mathbb{R}_+}\left\{\int_0^t ||X(t)X^{-1}(s)||ds\right\} < +\infty, \tag{8.148}$$

where $X(t)$ is the fundamental matrix of (8.146) *with $X(0) = I$.*

Let $f_0(t) = x_0'(t) - F(t, x_0(t))$, $t \in \mathbb{R}_+$, $r_0 = Ux_0$. *Then there exist numbers* $\alpha > 0$, $\beta > 0$ *such that for every* $[f, r] \in C_n(\mathbb{R}_+) \times \mathbb{R}^n$ *with* $\|[f - f_0, r - r_0]\| \le \beta$, *there exists a unique solution* $x \in C_n^1(\mathbb{R}_+)$ *of the problem*

$$\begin{aligned} x' &= F(t, x) + f(t), \\ Ux &= r \end{aligned} \tag{8.149}$$

such that $\|x\|_1 \le \alpha$. *If, in addition,* $\|[f_0, r_0]\| \le \beta$, *then the problem* ((E), (B)) *has a unique solution* $x(t)$ *with* $\|x\|_1 \le \alpha$.

Proof. It is easy to see that the operator V is Fréchet differentiable on $C_n^1(\mathbb{R}_+)$. The Fréchet derivative $V'(x)$ is given by

$$\begin{aligned} [V'(x)h](t) &= [h'(t) - F_x(t, x(t))h(t), U'(x)h] \\ &= [(N'(x)h)(t), U'(x)h] \end{aligned} \tag{8.150}$$

for every $t \in \mathbb{R}_+$, $h \in C_n^1(\mathbb{R}_+)$. In view of the inequality preceding (8.146), the continuity of $V'(x)$ in x follows from that of $N'(x)$. The continuity of $N'(x)$ follows from Condition (F) and the identity

$$||(N'(x_1)h)(t) - (N'(x_2)h)(t)|| \equiv ||[F_x(t, x_1(t)) - F_x(t, x_2(t))]h(t)|| \tag{8.151}$$

for all $x_1, x_2, h \in C_n^1(\mathbb{R}_+)$. The operator $V'(x_0)$ in one-to-one and onto because of our assumptions on the problem ((8.146), (8.147)) in connection with the remark following Theorem 6.1. Indeed, since the problem ((8.146), (8.147)) has only the zero solution in $C_n^1(\mathbb{R}_+)$, the problem

$$\begin{aligned} x' &= F_x(x_0(t))x + f(t), \\ U'(x_0)x &= r \end{aligned} \tag{8.152}$$

has a unique solution for every $[f, r] \in C_n(\mathbb{R}_+) \times \mathbb{R}^n$ given by

$$x(t) \equiv X(t)\widetilde{X}^{-1}[r - U'(x_0)q(\cdot, f)] + q(t, f), \tag{8.153}$$

where

$$q(t, f) \equiv \int_0^t X(t)X^{-1}(s)f(s)ds. \tag{8.154}$$

The inverse function theorem (Theorem 2.27) implies our first assertion. Our second assertion follows from the first because we are now allowed to choose $f \equiv 0$ and $r = 0$. $\square$

Now, we examine the problem ((8.138), (8.139)), where no differentiability conditions are placed on G, W. We assume, for convenience, that $x_0 \equiv 0$, $f_0 \equiv 0$, and $F(\cdot, 0) \equiv 0$.

THEOREM 8.25. *Let the assumptions of Theorem 8.24 be satisfied. Furthermore, assume the following:*

(i) *the function*

$$q(t) = \sup_{u \in \overline{B_\alpha(0)}} ||F(t,u)||, \tag{8.155}$$

where α is given in the conclusion of Theorem 8.24 (see proof below), satisfies

$$\int_0^\infty q(t)dt < +\infty; \tag{8.156}$$

(ii) *W is defined and continuous on the ball $B^\alpha = \{u \in C_n(\mathbb{R}_+); \|u\|_\infty \leq \alpha\}$ with values in $\mathbb{R}^n$;*

(iii) *$G : \mathbb{R}_+ \times \overline{B_\alpha(0)} \to \mathbb{R}^n$ is continuous, and for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$||G(t,u) - G(t,v)|| < \epsilon \tag{8.157}$$

for all $t \in \mathbb{R}_+$ and all $u, v \in \overline{B_\alpha(0)}$ with $\|u - v\| < \delta(\epsilon)$. Moreover, $\|\sigma\|_\infty + \|Wu\| \leq \beta$, $u \in B^\alpha$, where

$$\sigma(t) = \sup_{u \in \overline{B_\alpha(0)}} ||G(t,u)|| \tag{8.158}$$

with

$$\int_0^\infty \sigma(t)dt < +\infty. \tag{8.159}$$

Then the problem ((8.138), (8.139)) *has at least one solution $x \in C_l^1$.*

PROOF. Consider the operator $V : B^\alpha \to B^\alpha$ which assigns to each function $u \in B^\alpha$ the unique solution x_u of the problem

$$x' = F(t,x) + G(t,u(t)), \tag{8.160}$$

$$Ux = Wu \tag{8.161}$$

belonging to the ball $B^l = \{x \in C_l^1 : \|x\|_1 \leq \alpha\} \subset B^\alpha$. The existence of a solution $x_u \in \{x \in C_n^1(\mathbb{R}_+) : \|x\|_1 \leq \alpha\}$ is guaranteed by Theorem 8.24. The convergence of x_u to a finite limit $x_u(\infty)$ follows from

$$x_u(t) = x_u(0) + \int_0^t F(s, x_u(s))\,ds + \int_0^t G(s, u(s))\,ds, \tag{8.162}$$

which, taking limits as $t \to \infty$, implies

$$x_u(\infty) = x_u(0) + \int_0^\infty F(s, x_u(s))\,ds + \int_0^\infty G(s, u(s))\,ds. \tag{8.163}$$

The limits on the right-hand side exist by virtue of the integral assumptions on q, σ.

To show that V is continuous, let $\{u_m\}_{m=1}^\infty \subset B^\alpha$ be given with $\|u_m - u\|_\infty \to 0$ and let $x_m = V u_m$, $m = 1, 2, \ldots$. Then we have

$$\begin{aligned} x_m'(t) &= F(t, x_m(t)) + G(t, u_m(t)), \\ U x_m &= W u_m. \end{aligned} \tag{8.164}$$

Since

$$||x_m(t) - x_m(t')|| \le \left| \int_t^{t'} q(s)ds \right| + \left| \int_t^{t'} \sigma(s)ds \right| \tag{8.165}$$

for every $m = 1, 2, \ldots$, it follows that $\{x_m\}$ is equicontinuous on $\mathbb{R}_+$. Since it is also equiconvergent (Exercise 2.5(iii)), there exists a subsequence $\{u_m^1(t)\}_{m=1}^\infty$ of $\{u_m(t)\}$ such that, for $x_m^1 = V u_m^1$, we have

$$||x_m^1 - y||_\infty = ||V u_m^1 - y||_\infty \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{8.166}$$

where y is some element of C_n^l. Letting

$$v_m(t) = \left(\frac{d}{dt}\right) x_m^1(t), \quad t \in \mathbb{R}_+, \tag{8.167}$$

we also obtain

$$\begin{aligned} ||v_m(t) - v_k(t)|| \le{}& ||F(t, x_m^1(t)) - F(t, x_k^1(t))|| \\ &+ ||G(t, u_m^1(t)) - G(t, u_k^1(t))||. \end{aligned} \tag{8.168}$$

It is easy to see now that this inequality, the boundedness of $F_x(t, u)$ on $\mathbb{R}_+ \times \overline{B_\alpha(0)}$ and (8.157) imply that $\{v_m(t)\}$ is a Cauchy sequence of functions. Thus, by a well-known theorem of Advanced Calculus, we have

$$\lim_{m\to\infty} v_m(t) = y'(t) \quad \text{uniformly on } \mathbb{R}_+. \tag{8.169}$$

We let

$$z(t) = y(0) + \int_0^t F(s, y(s))\,ds + \int_0^t G(s, u(s))\,ds \tag{8.170}$$

and observe that

$$x_m^1(t) = x_m^1(0) + \int_0^t F(s, x_m^1(s))\,ds + \int_0^t G(s, u_m^1(s))\,ds. \tag{8.171}$$

Subtracting these two equations, we obtain, eventually,

$$\begin{aligned} ||x_m^1(t) - z(t)|| \le ||x_m^1(0) - y(0)|| &+ \int_0^\infty ||F(s, x_m^1(s)) - F(s, y(s))||\,ds \\ &+ \int_0^\infty ||G(s, u_m^1(s)) - G(s, u(s))||\,ds. \end{aligned} \tag{8.172}$$

Using Lebesgue's dominated convergence theorem, along with the integral conditions on q, σ, we obtain

$$\lim_{m\to\infty} ||x_m^1 - z||_\infty = 0. \tag{8.173}$$

Thus, $z(t) = y(t)$, $t \in \mathbb{R}_+$. This shows that $y(t)$ solves (8.160) on $\mathbb{R}_+$. From (8.169) we also obtain that $y' \in C_n(\mathbb{R}_+)$. Consequently, $y \in C_l^1$. The continuity of U on $C_n^1(\mathbb{R}_+)$ and W on $C_n(\mathbb{R}_+)$ imply that $Ux_m^1 \to Uy$ and $Wu_m^1 \to Wu$. Thus, $Uy = Wu$. Hence $y(t)$ is the unique solution of the problem ((8.160), (8.161)) in B^l. Since we could have started with any subsequence of $\{u_m\}$ instead of $\{u_m\}$ itself, we have actually proven the following statement: every subsequence $\{V\tilde{u}_m\}$ of $\{Vu_m\}$ contains a further subsequence $\{V\tilde{u}_{m_k}\}$ which converges to the same function $y(t)$ in the norm of $C_n(\mathbb{R}_+)$ as $k \to \infty$. This proves the continuity of V. The relative compactness of VB^α in C_n^l (hence in $C_n(\mathbb{R}_+)$) follows from the equicontinuity, the equiconvergence, and the boundedness ($VB^\alpha \subset B^\alpha$) of VB^α. The Schauder-Tychonov theorem implies now the existence of a solution $x(t)$ of the problem ((8.138), (8.139)) which belongs to B^l. □

Corollary 8.26. *Assume that the hypotheses of the first part of Theorem 8.24 are satisfied, and let η be a positive constant. Let $f : \mathbb{R}_+ \times (0, \eta) \to \mathbb{R}^n$ be continuous and such that*

$$\lim_{\epsilon\to 0^+} ||f(\cdot, \epsilon) - f_0||_\infty = 0. \tag{8.174}$$

Let $r : (0, \eta) \to \mathbb{R}^n$ satisfy

$$\lim_{\epsilon\to 0^+} ||r(\epsilon) - r_0|| = 0. \tag{8.175}$$

Then there exists $\epsilon_0 > 0$ *such that, for each* $\epsilon \in (0, \epsilon_0)$ *the problem*

$$\begin{aligned} x' &= F(t,x) + f(t,\epsilon) \\ Ux &= r(\epsilon) \end{aligned} \tag{8.176}$$

has a unique solution x_ϵ *such that* $\|x_\epsilon\|_1 \leq \alpha$.

Proof. It suffices to observe that $\|[f(\cdot,\epsilon) - f_0, r(\epsilon) - r_0]\| \leq \beta$ for all sufficiently small ϵ. □

Corollary 8.27. *Assume that the hypotheses of Theorem 8.25 are satisfied with* $G(t,x)$, Wx *replaced by* $G(t,x,\epsilon)$, $W(x,\epsilon)$, *respectively, for every* ϵ *in the interval* $(0,\eta)$, *where* η *is a positive constant. Let*

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \sup_{\|x\|_\infty \leq \alpha} \{||W(x,\epsilon)||\} = 0, \\ &\lim_{\epsilon\to 0^+} \sup_{\substack{t\in\mathbb{R}_+ \\ \|x\|\leq\alpha}} \{\|G(t,x,\epsilon)||\} = 0. \end{aligned} \tag{8.177}$$

Then there exists $\epsilon_0 > 0$ *such that, for every* $\epsilon \in (0,\epsilon_0)$ *the problem*

$$\begin{aligned} x' &= F(t,x) + G(t,x,\epsilon), \\ Ux &= W(x,\epsilon) \end{aligned} \tag{8.178}$$

has at least one solution $x_\epsilon \in C_l^1$ *such that* $\|x_\epsilon\|_1 \leq \alpha$.

EXERCISES

8.1. Show that in the setting of Lemma 8.2 we have

$$\sup_{\substack{\|u\|=1 \\ u\in R_l}} ||P_1 u|| \leq \sup_{\substack{\|u\|=1 \\ P_k u \neq 0, k=1,2}} ||P_1 u||, \quad l = 1,2. \tag{8.179}$$

8.2. Let $A : \mathbb{R} \to M_n$ be continuous and let the system (S) possess an exponential splitting. Show that if $x(t)$ is a solution of (S) with $x(0) \in \mathbb{R}_1$, $x(0) \neq 0$, then $x(t) \notin \mathbb{R}_2(t)$ for any $t \in \mathbb{R}$.

8.3. Show that $UB^r \subset B^r$ and $\|Uf_1 - Uf_2\| \leq \rho$, for $f_1, f_2 \in B^r$, in the proof of Theorem 8.8(1).

8.4. Show the stability of the zero solution of the system $x' = B(t,x)$, where the vector $B(t,u)$ is given in Example 8.17.

8.5. Consider the quasilinear system

$$x' = A(t,x)x + F(t,x), \tag{S_Q}$$

where $A : \mathbb{R}_+ \times \mathbb{R}^n \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Assume that there exist two functions $p, q : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and such that

$$||A(t,u)|| \le p(t), \quad ||F(t,u)|| \le q(t), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{8.180}$$

Show that every local solution $x(t)$ of (S_Q) is continuable to $+\infty$.

8.6. Show that the scalar quasilinear equation

$$x' = \left(\cos^2 x - 2\right)x + (\sin t)x^{1/3} + \sin(2t) \tag{8.181}$$

has a 2π-periodic solution.

8.7. Suppose that $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Let the constant $r > 0$ be such that

$$\langle F(t,u), u \rangle \le 0 \tag{8.182}$$

for every $u \in \mathbb{R}^n$ with $\|u\| = r$. Then the system (E) has at least one solution $x(t)$, $t \in \mathbb{R}$, such that $\|x\|_\infty \le r$. Hint. Consider the systems

$$x' = F(t,x) - \epsilon x, \tag{S_ϵ}$$

where $\epsilon > 0$. Obtain solutions $x_\epsilon(t)$, $t \in \mathbb{R}$, of (S_ϵ) such that $\|x_\epsilon\| \le r$ (see proof of Theorem 7.12).

8.8. Consider the system

$$x' = A(t,x)x + F(t,x), \tag{S_Q}$$

where A, F, p, q are as in Exercise 8.5. Assume further that

$$\int_0^\infty p(t)dt < +\infty, \qquad \int_0^\infty q(t)dt < +\infty. \tag{8.183}$$

Show that for every $\xi \in \mathbb{R}^n$ there exists at least one solution $x(t)$ of the system (S_Q) which is defined for all large t and converges to ξ as $t \to \infty$.

8.9. Assume that $A : \mathbb{R}_+ \to M_n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ are continuous. Assume that the system (S) possesses a dichotomy given by (8.11), but for $t, s \ge 0$ and $m_0 = 0$. Let the function F satisfy

$$||F(t,u_1) - F(t,u_2)|| \le \lambda(t)||u_1 - u_2||, \qquad \int_0^\infty ||F(t,0)||dt < +\infty \tag{8.184}$$

for every $u_1, u_2 \in \mathbb{R}^n$, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that

$$\int_0^\infty \lambda(t)dt < +\infty. \tag{8.185}$$

Show that for every bounded solution $y(t)$, $t \in \mathbb{R}_+$, of the system

$$x' = A(t)x + f(t) \tag{S_f}$$

there exists a unique bounded solution $x(t)$, $t \in \mathbb{R}_+$, of the system

$$x' = A(t)x + f(t) + F(t,x) \tag{S_1}$$

such that the operator $T : x \to y$ is one-to-one, onto, and bicontinuous (T, T^{-1} are continuous). Hint. Choose t_1 so that

$$H_1 \int_{t_1}^{\infty} \lambda(t)dt < 1 \tag{8.186}$$

and find a unique fixed point $x(t)$ for the operator $V : C_n[t_1,\infty) \to C_n[t_1,\infty)$ with

$$\begin{aligned}(Vf)(t) = y(t) &+ \int_{t_1}^{t} X(t)P_1X^{-1}(s)F(s,f(s))ds \\ &- \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,f(s))ds.\end{aligned} \tag{8.187}$$

Show that $x(t)$ is continuable (in a unique way) to $t = 0$. Study the correspondence $T : x \to y$ with $y = x - Vx$, where $x \in C_n(\mathbb{R}_+)$ is a solution of (S_1).

8.10. Let $A : \mathbb{R}_+ \times \mathbb{R}^n \to M_n$ be continuous. Using the method of Theorem 8.15, show that the zero solution of the system (8.81), $x' = A(t,x)x$, is uniformly asymptotically stable, provided that suitable stability properties are assumed for the systems $x' = A(t,f(t))x$. In addition, examine the problem of strong stability for the system (8.81) via the same method.

8.11. Let $B : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with $B(t,u)$ continuously differentiable w.r.t. u on $\mathbb{R}_+ \times \mathbb{R}^n$. Assume further that

$$||B_x(t,u)|| \le p(t), \qquad ||F(t,u)|| \le q(t) \tag{8.188}$$

for every $(t,u) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $p,q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and such that

$$\int_0^{\infty} p(t)dt < +\infty, \qquad \int_0^{\infty} q(t)dt < +\infty. \tag{8.189}$$

Show that for every solution $y(t)$, $t \in \mathbb{R}_+$, of the system

$$y' = B(t,y) \tag{8.190}$$

there exists at least one solution $x(t)$, $t \in [t_1,\infty)$ (for some $t_1 \ge 0$), of the system

$$x' = B(t,x) + F(t,x) \tag{8.191}$$

such that $x(t) - y(t) \to 0$ as $t \to \infty$. Hint. Consider the integral equation

$$u(t) = -\int_t^\infty A(s, u(s))\,ds - \int_t^\infty F_0(s, u(s))\,ds, \tag{8.192}$$

where $A(t,u) \equiv B(t, u + y(t)) - B(t, y(t))$, $F_0(t,u) \equiv F(t, u + y(t))$, and $u(t) \equiv x(t) - y(t)$.

8.12. Consider the problem

$$\begin{gathered}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -x_1^2 + x_2 \\ x_1 - x_2^2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} - \begin{bmatrix} x_1(2\pi) \\ x_2(2\pi) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \end{gathered} \tag{S$_2$}$$

Show that there exists a number $\delta > 0$ such that whenever $|r_1|, |r_2| < \delta$ and $\|f_1\|_\infty, \|f_2\|_\infty < \delta$ ($f_1, f_2 \in C_1[0, 2\pi]$), the problem (S$_2$) has at least one solution $x \in C_2[0, 2\pi]$.

8.13. Prove that the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} (|\sin x_1| - 3)x_1 + x_2 \\ x_1 + (\exp\{-|x_2|\} - 2)x_2 \end{bmatrix} + \begin{bmatrix} x_1^{1/3} \\ \sin^2 t \end{bmatrix} \tag{8.193}$$

has at least one 2π-periodic solution. Hint. Examine the eigenvalues of the matrix

$$\begin{bmatrix} |\sin f_1(t)| - 3 & 1 \\ 1 & \exp\{-|f_2(t)|\} - 2 \end{bmatrix} \tag{8.194}$$

for every $f \in C_2[0, 2\pi]$ which is 2π-periodic.

8.14. Consider the quasilinear system

$$x' = A(t,x)x + f(t), \tag{8.195}$$

where $A : \mathbb{R} \times \mathbb{R}^n \to M_n$, $f : \mathbb{R} \to \mathbb{R}^n$ are continuous, T-periodic in t and such that

$$\|A(t,u) - A_1\| \le K \tag{8.196}$$

for every $(t,u) \in \mathbb{R} \times \mathbb{R}^n$, where $A_1 \in M_n$ is fixed. Show that if

$$x' = A_1 x, \qquad x(0) = x(T), \tag{8.197}$$

has only the zero solution and K is sufficiently small, then the system (8.81) has at least one T-periodic solution.

8.15. Show that the function

$$x(t) \equiv \left[\frac{1}{(a-b)}\right]\left[(b-t)\int_a^t (s-a)f(s)ds+(t-a)\int_t^b (b-s)f(s)ds\right], \tag{8.198}$$

where $f \in C_1[a,b]$, is the unique solution to the scalar boundary value problem

$$x'' = f(t), \qquad x(a) = x(b) = 0. \tag{8.199}$$

Using the inverse function theorem, impose conditions on the function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ that ensure the solvability of the problem

$$x'' = F(t,x), \qquad x(0) = x(1) = 0. \tag{8.200}$$

8.16. Complete the proof of Theorem 8.19.

8.17. Assume that the system (S) possesses an exponential dichotomy as in Definition 8.4. Then for each $f \in C_n(\mathbb{R}_+)$ there exists at least one solution $x \in C_n(\mathbb{R}_+)$ of the system (S_f). This solution is given by

$$x(t) \equiv \int_0^\infty G(t,s)f(s)ds \tag{8.201}$$

as in Exercise 3.23. Show that

$$\|x\|_\infty \le K\|f\|_\infty, \tag{8.202}$$

where K is some positive constant. Notice that

$$x(0) = \int_0^\infty G(0,s)f(s)ds = -P_2\int_0^\infty X^{-1}(s)f(s)ds \tag{8.203}$$

says that $x(0) \in \mathbb{R}_2$. Show that all bounded solutions of (S_f) on $\mathbb{R}_+$ are given by the formula

$$x(t) \equiv X(t)x_0 + \int_0^\infty G(t,s)f(s)ds, \tag{8.204}$$

where x_0 is any vector in $\mathbb{R}_1$.

8.18. Improve Theorems 8.24 and 8.25 by imposing local conditions on the function $F(t,u)$. The properties of this function were actually used in connection with a neighborhood S of the point $x_0 \in C_n^1(\mathbb{R}_+)$.

CHAPTER 9

INTRODUCTION TO DEGREE THEORY

Degree Theory is a very applicable branch of Nonlinear Analysis. One of its main concerns is the development of methods in order to solve equations of the type $f(x) = p$. Here, f maps a subset A of a Banach space X into X and p is a point in X. The solvability of this equation is often achieved by estimating the *degree* $d(f, D, p)$, always an integer or zero, where D is an open, bounded set with $\bar{D} \subset A$ and $p \in X \setminus f(\partial D)$. It turns out that if this degree is not zero, then $f(x) = p$ has a solution $x \in D$.

As we will see later, we cannot define an adequate concept of degree for the class of all continuous functions f mapping suitable subsets of X into X, unless X is finite dimensional.

Our purpose in this chapter is to introduce the concept and the fundamental properties of a degree which is defined for quite a large class of continuous functions, namely, the *compact displacements of the identity*. This concept of degree is of fundamental nature and is due to Leray and Schauder [36].

1. PRELIMINARIES

Unless otherwise stated, the symbol D denotes an open bounded subset of $\mathbb{R}^n$. We use the symbol $C(\overline{D})$ to denote the set $\{f : \overline{D} \to \mathbb{R}^n : f \text{ is continuous on } \overline{D}\}$. We define the space $C^1(\overline{D})$ as follows: $C^1(\overline{D}) = \{f \in C(\overline{D}) : f$ can be extended to a function $\bar{f}$ on an open set $D_1 \supset \bar{D}$ in such a way that $\bar{f}$ has continuous first-order partial derivatives on $D_1\}$.

The symbol $f'(x)$ denotes the Jacobian matrix of $f \in C^1(\overline{D})$ at $x \in \bar{D}$. For such functions f, we denote by $J_f(x)$ the (Jacobian) determinant of $f'(x)$. The symbol I denotes the $n \times n$ identity matrix in M_n as well as the identity operator on the real normed space under consideration.

Let $f : \mathbb{R}^n \supset D(f) \to \mathbb{R}^k$ be given. The *support* of f (supp f) is defined to be the closure of the set $\{x \in D(f) : f(x) \neq 0\}$. We denote by $C^k(A, B)$ the space of all k times continuously differentiable functions $f : A \to B$.

We use the term *homotopy* for any continuous function $H(t,x)$ of the real variable t and the vector x. The term *homotopy of compact operators* has a more specific meaning (see Definition 9.30). In this chapter, we let $B_r(x_0)$ denote the open ball of the underlying normed space with center at x_0 and radius $r > 0$.

2. DEGREE FOR FUNCTIONS IN $C^1(\overline{D})$

In this section, we introduce the degree for functions in $C^1(\overline{D})$. The fact that every function in $C(\overline{D})$ is the uniform limit of a sequence $\{f_n\} \subset C^1(\overline{D})$ allows us to extend this concept of degree to arbitrary functions in $C(\overline{D})$ (see Section 3).

Given $f \in C^1(\overline{D})$, we call the point $x \in \overline{D}$ a *critical point* of f if $J_f(x) = 0$. The set of all critical points of f in $\overline{D}$ is denoted by $Q_f(\overline{D})$, or, simply, Q_f.

Theorem 9.1. *Let $f \in C^1(\overline{D})$, $p \in \mathbb{R}^n$ be given with $p \notin f(Q_f)$. Then the set $f^{-1}(p)$ is either finite or empty.*

Proof. Let $f^{-1}(p) \neq \varnothing$. The set $\overline{D}$ is compact. Our conclusion will follow from this, if we show that the set $f^{-1}(p)$ consists of isolated points. Let $\{x_n\} \subset \bar{D}$ be an infinite sequence with $f(x_n) = p$. Then since $\bar{D}$ is compact, there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $x_n \to x_0 \in \bar{D}$. Since f is continuous,

$$\lim_{n\to\infty} f(x_n) = f(x_0) = p. \tag{9.1}$$

Now, we observe that

$$0 = f(x_n) - f(x_0) = f'(x_0)(x_n - x_0) + w(x_n - x_0), \tag{9.2}$$

where $w(u)/\|u\| \to 0$ as $u \to 0$. This follows from the fact that f' is the Fréchet derivative of f and Definition 2.18. Since $p \notin f(Q_f)$, the matrix $f'(x_0)$ is nonsingular. As such, it maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ and has an inverse $[f'(x_0)]^{-1}$ which defines a bounded linear operator on $\mathbb{R}^n$. If we denote this operator also by $[f'(x_0)]^{-1}$, we obtain

$$\left|\left|[f'(x_0)]^{-1}u\right|\right| \le m\|u\|, \quad u \in \mathbb{R}^n, \tag{9.3}$$

where m is a positive constant. Letting $v = [f'(x_0)]^{-1}u$, we find

$$\left|\left|f'(x_0)v\right|\right| \ge \frac{1}{m}\|v\|, \quad v \in \mathbb{R}^n. \tag{9.4}$$

It is easy to see from (9.2) that there exists a positive integer n_0 such that

$$\left|\left|f'(x_0)(x_n - x_0)\right|\right| \le \frac{1}{2m}\left|\left|x_n - x_0\right|\right|, \quad n \ge n_0. \tag{9.5}$$

This contradicts (9.4) unless $x_n = x_0$ for all large n. Consequently, $f^{-1}(p)$ is finite. □

We are now ready to define the degree of a function $f \in C^1(\overline{D})$.

DEFINITION 9.2. Let $f \in C^1(\overline{D})$, $p \in \mathbb{R}^n$ be given with $p \notin f(\partial D)$ and $p \notin f(Q_f)$. The *degree* of f at p with respect to D, $d(f, D, p)$, is defined by

$$d(f, D, p) = \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x). \tag{9.6}$$

We set $\sum_{x \in \varnothing} \operatorname{sgn} J_f(x) = 0$.

THEOREM 9.3. *If $p \in D$, then $d(I, D, p) = 1$. If $p \notin \overline{D}$, then $d(I, D, p) = 0$.*
The proof follows easily from Definition 9.2.

EXAMPLE 9.4. Let

$$f(x) = \left(x_1^2 - 1, x_2 - x_3, (x_3 - 3)(x_2^2 + 1)\right), \quad x \in \mathbb{R}^3. \tag{9.7}$$

Then

$$f'(x) = \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2x_2(x_3 - 3) & x_2^2 + 1 \end{bmatrix}. \tag{9.8}$$

We find $f^{-1}((0,0,0)) = \{(1,3,3), (-1,3,3)\}$. We also find $\operatorname{sgn} J_f((1,3,3)) = 1$ and $\operatorname{sgn} J_f((-1,3,3)) = -1$. Thus, we have $d(f, B_5(0), 0) = 0$.

EXAMPLE 9.5. Consider the function

$$f(x) = \left(x_1^3 - 1, x_1 + 3x_2\right), \quad x \in \mathbb{R}^2. \tag{9.9}$$

For this function we have $f^{-1}((0,0)) = \{(1, -1/3)\}$ and

$$f'(x) = \begin{bmatrix} 3x_1^2 & 0 \\ 1 & 3 \end{bmatrix}. \tag{9.10}$$

Thus, $\operatorname{sgn} J_f((1, -1/3)) = 1 = d(f, B_2(0), 0)$.

EXAMPLE 9.6. For the function

$$f(x) = \left(x_1^3 - 6x_1^2 + 11x_1 - 6, x_2 + 1, x_3 + x_2\right), \quad x \in \mathbb{R}^3, \tag{9.11}$$

we have $f^{-1}((0,2,3)) = \{(1,1,2), (2,1,2), (3,1,2)\}$ (the roots of the first component are 1, 2, and 3) and $J_f(x) = 3x_1^2 - 12x_1 + 11$. This gives $J_f((1,1,2)) = 2$, $J_f((2,1,2)) = -1$ and $J_f((3,1,2)) = 2$. It follows that $d(f, B_4(0), (0,2,3)) = 1 - 1 + 1 = 1$.

Theorem 9.7. *Assume that* $D = \bigcup_{i=1}^{m} D_i$*, where* D_i*,* $i = 1, 2, \ldots, m$*, are mutually disjoint open sets. Let* $f \in C^1(\overline{D})$ *and* $p \in \mathbb{R}^n$ *with* $p \notin f(\partial D)$*,* $p \notin f(Q_f)$*. Then*

$$d(f, D, p) = \sum_{1 \le i \le m} d(f, D_i, p). \tag{9.12}$$

Proof. The proof follows easily from Definition 9.2 because we have that $p \notin f(\partial D_i)$ and $p \notin f(Q_f(\bar{D}_i))$, $i = 1, 2, \ldots, m$. □

3. DEGREE FOR FUNCTIONS IN $C(\overline{D})$

The proof of the following lemma can be found in Sard's paper [50].

Lemma 9.8 (Sard). *Let* $f \in C^1(\overline{D})$ *be given. Then the set* $f(Q_f)$ *has measure zero in* $\mathbb{R}^n$*.*

The next lemma provides an integral representation for the degree of a function in $C^1(\overline{D})$.

Lemma 9.9. *Let* $f \in C^1(\overline{D})$*,* $p \in \mathbb{R}^n$ *with* $p \notin f(\partial D)$ *and* $p \notin f(Q_f)$*. Given* $\epsilon > 0$*, let* $f_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous function such that*

$$\operatorname{supp} f_\epsilon \subset [0, \epsilon], \qquad \int_{\mathbb{R}^n} f_\epsilon(\|x\|)\,dx = 1. \tag{9.13}$$

Then there exists $\epsilon_0 = \epsilon_0(f, D, p) > 0$ *such that*

$$d(f, D, p) = \int_D f_\epsilon(\|f(x) - p\|) J_f(x)\,dx, \quad \epsilon \in (0, \epsilon_0). \tag{9.14}$$

Proof. Since $p \notin f(Q_f)$, the set $f^{-1}(p)$ is either finite or empty by Theorem 9.1. If it is empty, then $d(f, D, p) = 0$, by definition. Also, the integral in (9.14) equals zero, for all small $\epsilon > 0$, because $\|f(x) - p\| \ge$ (some) ϵ, $x \in D$. Let $f^{-1}(p) = \{q_1, q_2, \ldots, q_m\}$ ($\subset D$). The inverse function theorem and Exercise 1.14 imply the existence of open neighborhoods $A_i \subset D$ of the points q_i, $i = 1, 2, \ldots, m$, respectively, such that $f : A_i \to f(A_i)$ is a homeomorphism onto, $J_f(x) \ne 0$ for $x \in \cup A_i$, J_f has fixed sign on each A_i, and $A_i \cap A_j = \emptyset$ for $i \ne j$. Consider the mapping $g(x) \equiv f(x) - p$. Then $g(A_i)$ is an open neighborhood of zero. This implies that there is $\delta_1 > 0$ such that $\bigcap_{i=1}^{m} g(A_i) \supset B_{\delta_1}(0)$. Moreover, for some $\delta_2 > 0$, we have $\|g(x)\| \ge \delta_2$ for $x \in \bar{D} \setminus \bigcup_{i=1}^{m} A_i$. If we pick $\epsilon_0 \in (0, \min\{\delta_1, \delta_2\})$ and let $\epsilon \in (0, \epsilon_0)$, we have $\operatorname{supp} f_\epsilon(\|f(\cdot) - p\|) \subset \bigcup A_i$ because $\|f(x) - p\| > \epsilon$ for $x \notin \bigcup A_i$, and $\operatorname{supp} f_\epsilon \subset [0, \epsilon]$. Thus,

$$\int_D f_\epsilon(\|f(x) - p\|) J_f(x)\,dx = \sum_{i=1}^{m} \int_{A_i} f_\epsilon(\|f(x) - p\|) J_f(x)\,dx. \tag{9.15}$$

Since $J_f(x)$ has fixed sign in each set A_i, we have

$$J_f(x) = |J_f(x)| \operatorname{sgn} J_f(q_i), \quad x \in A_i,$$

$$\begin{aligned}\int_{A_i} f_\epsilon(\|f(x)-p\|)J_f(x)dx &= \operatorname{sgn} J_f(q_i)\int_{A_i} f_\epsilon(\|f(x)-p\|)\,|J_{f-p}(x)|\,dx \\ &= \operatorname{sgn} J_f(q_i)\int_{\mathbb{R}^n} f_\epsilon(\|y\|)dy = \operatorname{sgn} J_f(q_i).\end{aligned} \tag{9.16}$$

Here, $f-p$ denotes the function $f(\cdot)-p$. It follows that

$$\int_D f_\epsilon(\|f(x)-p\|)J_f(x)dx = \sum_{i=1}^m \operatorname{sgn} J_f(q_i) = d(f,D,p). \tag{9.17}$$

□

The next two lemmas will be used to obtain a more concrete integral representation of $d(f,D,p)$. This is done in Theorem 9.12.

Lemma 9.10. *Let $f \in C^1(\overline{D})$ be such that $\|f(x)\| > \epsilon > 0$ on ∂D. Moreover, let $g \in C_1(\mathbb{R}_+)$ vanish in the set $[\epsilon,\infty)$ and in a neighborhood of zero. Assume further that*

$$\int_0^\infty u^{n-1}g(u)du = 0. \tag{9.18}$$

Then

$$\int_D g(\|f(x)\|)J_f(x)dx = 0. \tag{9.19}$$

Proof. It suffices to prove the lemma under the assumption that $f \in C^2(\overline{D}) = \{h \in C^1(\overline{D}) : h$ has an extension on an open set $D_1 \supset \overline{D}$ with continuous second partials on $D_1\}$. If this is shown, then Weierstrass' approximation theorem ensures the validity of the lemma for functions $f \in C^1(\overline{D})$.

Consider the function

$$h(u) = \begin{cases} u^{-n}\displaystyle\int_0^u t^{n-1}g(t)dt, & u \in (0,\infty), \\ 0, & u = 0. \end{cases} \tag{9.20}$$

It is easy to see that h is continuously differentiable on $\mathbb{R}_+$ and identically equal to zero in $[\epsilon,\infty)$ and in a neighborhood of zero. Moreover,

$$uh'(u) + nh(u) = g(u), \quad u \geq 0. \tag{9.21}$$

It follows that the functions $k_j(y) \equiv h(\|y\|)y_j$, $j = 1, 2, \ldots, n$, are continuously differentiable on $\mathbb{R}^n$ and such that $k_j(y) = 0$ for all $y \in \mathbb{R}^n$ with $\|y\| \geq \epsilon$. Consequently, $k_j(f(\cdot)) \in C^1(\overline{D})$ and vanishes identically in a neighborhood of ∂D. Using the well-known equations

$$\sum_{i=1}^{n} \left[\frac{\partial A_{ij}(x)}{\partial x_i} \right] = 0, \quad j = 1, 2, \ldots, n, \ x \in D, \tag{9.22}$$

where $A_{ij}(x)$ is the cofactor of the number $\partial f_j(x)/\partial x_i$ in the determinant $J_f(x)$, we obtain

$$\begin{aligned} z(x) &\equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} A_{ij}(x) k_j(f(x)) \\ &= J_f(x) \sum_{j=1}^{n} \frac{\partial k_j(f(x))}{\partial y_j} \\ &= J_f(x) \left[uh'(u) + nh(u) \right]_{u = \|f(x)\|} \\ &= g(||f(x)||) J_f(x), \quad x \in D, \end{aligned} \tag{9.23}$$

which, after one integration, gives (9.19). In fact, the function $z(x)$ above is the divergence of a function vanishing identically in a neighborhood of ∂D. Thus,

$$\int_D z(x)dx = 0. \tag{9.24}$$

□

Lemma 9.11. *Let $f \in C^1(\overline{D})$ be such that $\|f(x) - p\| > \epsilon$ on ∂D, for some $p \in \mathbb{R}^n$, $\epsilon > 0$. Then the value of the integral*

$$\int_D g(||f(x) - p||) J_f(x) dx \tag{9.25}$$

is independent of the function $g \in C_1(\mathbb{R}_+)$ which satisfies the following conditions:

(i) *$g(u) = 0$, $u \in [\epsilon, \infty)$, and g vanishes identically in a neighborhood of zero;*

(ii) *we have*

$$\int_{\mathbb{R}^n} g(\|x\|) dx = 1. \tag{9.26}$$

Proof. Let G denote the vector space of all functions $g \in C_1(\mathbb{R}_+)$ satisfying (i), and let

$$\begin{aligned} Lg &= \int_0^\infty u^{n-1} g(u)\,du, \\ Mg &= \int_{\mathbb{R}^n} g(\|x\|)\,dx, \\ Ng &= \int_D g(\|f(x)-p\|) J_f(x)\,dx. \end{aligned} \tag{9.27}$$

Applying Lemma 9.10 to the mapping $f_1(x) \equiv x$, with $D = B_{2\epsilon}(0)$ there, and the mapping $f_2(x) \equiv f(x) - p$, $x \in D$, we obtain that if $Lg = 0$, for some $g \in G$, then $Mg = 0$ and $Ng = 0$. Let $g_1, g_2 \in G$ satisfy $Mg_1 = Mg_2 = 1$. The equation

$$L\big((Lg_2)g_1 - (Lg_1)g_2\big) = 0 \tag{9.28}$$

implies

$$\begin{aligned} M\big((Lg_2)g_1 - (Lg_1)g_2\big) &= Lg_2 Mg_1 - Lg_1 Mg_2 \\ &= Lg_2 - Lg_1 \\ &= L(g_2 - g_1) = 0. \end{aligned} \tag{9.29}$$

This yields $N(g_1 - g_2) = 0$, or $Ng_1 = Ng_2$. □

As it is expected, Theorem 9.12 shows that the integral in (9.25) is actually the degree $d(f, D, p)$ provided that $p \notin f(Q_f)$.

Theorem 9.12. *Let $f \in C^1(\overline{D})$, $p \in \mathbb{R}^n$ be given with $\|f(x) - p\| > \epsilon$, $x \in \partial D$, and $p \notin f(Q_f)$. Let $g \in C_1(\mathbb{R}_+)$ be a function satisfying (i), (ii) of Lemma 9.11. Then*

$$d(f, D, p) = \int_D g(\|f(x)-p\|) J_f(x)\,dx. \tag{9.30}$$

Proof. Let ϵ_0 be as in the proof of Lemma 9.9 and choose a number $\epsilon_1 \in (0, \min\{\epsilon, \epsilon_0\})$. Pick a function $g_1 \in C_1(\mathbb{R}_+)$ which vanishes in a neighborhood of zero and in the interval $[\epsilon_1, \infty)$, and is such that

$$\int_{\mathbb{R}^n} g_1(\|x\|)\,dx = 1. \tag{9.31}$$

Then, by Lemma 9.9,

$$d(f, D, p) = \int_D g_1(\|f(x)-p\|) J_f(x)\,dx \tag{9.32}$$

and, by Lemma 9.11,

$$\int_D g(||f(x)-p||)J_f(x)dx = \int_D g_1(||f(x)-p||)J_f(x)dx, \tag{9.33}$$

because g_1 has all the properties of g. □

Lemmas 9.13 and 9.14 below allow us to extend the notion of degree to points $p \in f(Q_f)$.

Lemma 9.13. *Let $f_i \in C^1(\overline{D})$, $i = 1,2$, $p \in \mathbb{R}^n$, $\epsilon > 0$ be such that $p \notin f_i(Q_{f_i})$ and*

$$\begin{gathered} ||f_i(x)-p|| \geq 7\epsilon, \quad i = 1,2,\ x \in \partial D, \\ ||f_1(x)-f_2(x)|| < \epsilon, \quad x \in \overline{D}. \end{gathered} \tag{9.34}$$

Then $d(f_1, D, p) = d(f_2, D, p)$.

Proof. By Theorem 9.12, $d(f, D, p) = d(f-p, D, 0)$. Thus, we may assume, without loss of generality, that $p = 0$. Let $g \in C_1^2(\mathbb{R}_+)$ be such that

$$\begin{cases} g(u) = 1, & u \in [0, 2\epsilon], \\ g(u) = 0, & u \in [3\epsilon, \infty), \\ g(u) \in [0,1], & u \in (2\epsilon, 3\epsilon). \end{cases} \tag{9.35}$$

Let

$$f_3(x) = (1 - g(||f_1(x)||))f_1(x) + g(||f_1(x)||)f_2(x). \tag{9.36}$$

It is easy to see that $f_3 \in C^1(\overline{D})$. Moreover, we have

$$\begin{gathered} ||f_i(x)-f_3(x)|| < \epsilon, \quad i = 1,2,\ x \in \overline{D}, \\ ||f_3(x)|| > 6\epsilon, \quad x \in \partial D, \end{gathered} \tag{9.37}$$

$$f_3(x) = \begin{cases} f_1(x) & \text{if } ||f_1(x)|| > 3\epsilon, \\ f_2(x) & \text{if } ||f_1(x)|| < 2\epsilon. \end{cases} \tag{9.38}$$

In order to show (9.37), we observe first that

$$\begin{aligned} ||f_1(x)-f_3(x)|| &\leq g(||f_1(x)||)||f_1(x)-f_2(x)|| < 1 \cdot \epsilon = \epsilon, \\ ||f_2(x)-f_3(x)|| &\leq ||f_2(x)-f_1(x)+g(||f_1(x)||)[f_1(x)-f_2(x)]|| \\ &\leq (1-g(||f_1(x)||))||f_1(x)-f_2(x)|| < \epsilon, \quad x \in \overline{D} \end{aligned} \tag{9.39}$$

and $\|f_3(x)\| > \|f_1(x)\| - \epsilon \geq 7\epsilon - \epsilon = 6\epsilon$, $x \in \partial D$.

Now, let h_1, h_2 be two functions in $C_1(\mathbb{R}_+)$ which vanish in a neighborhood of zero and are such that

$$\begin{gathered} h_1(u) = 0, \quad u \in [0, 4\epsilon] \cup [5\epsilon, \infty), \\ h_2(u) = 0, \quad u \in [\epsilon, \infty), \\ \int_{\mathbb{R}^n} h_i(\|x\|)dx = 1, \quad i = 1, 2. \end{gathered} \tag{9.40}$$

Then we have

$$\begin{aligned} h_1(||f_3(x)||)J_{f_3}(x) &= h_1(||f_1(x)||)J_{f_1}(x), \\ h_2(||f_3(x)||)J_{f_3}(x) &= h_2(||f_2(x)||)J_{f_2}(x), \quad x \in D. \end{aligned} \tag{9.41}$$

In order to show the first of (9.41), assume first that $\|f_1(x)\| > 3\epsilon$. Then $f_3(x) = f_1(x)$ and we are done. If $\|f_1(x)\| \le 3\epsilon$, then $\|f_3(x) - f_1(x)\| < \epsilon$ implies $\|f_3(x)\| < \|f_1(x)\| + \epsilon \le 4\epsilon$. This gives $h_1(\|f_3(x)\|) = 0$ and $h_1(\|f_1(x)\|) = 0$, which completes the argument. To show the second of (9.41), let $\|f_1(x)\| \le 2\epsilon$. Then $f_3(x) = f_2(x)$ and we are done. Let $\|f_1(x)\| > 2\epsilon$. Then $\|f_3(x)\| \ge \|f_1(x)\| - \|f_3(x) - f_1(x)\| > 2\epsilon - \epsilon = \epsilon$, and $\|f_2(x)\| \ge \|f_1(x)\| - \|f_2(x) - f_1(x)\| > 2\epsilon - \epsilon = \epsilon$. Thus, $h_2(\|f_3(x)\|) = h_2(\|f_2(x)\|) = 0$.

Theorem 9.12 implies now our conclusion from

$$\begin{aligned} d(f_3, D, 0) &= \int_D h_1(||f_3(x)||)J_{f_3}(x)dx \\ &= \int_D h_1(||f_1(x)||)J_{f_1}(x)dx \\ &= d(f_1, D, 0), \\ d(f_3, D, 0) &= \int_D h_2(||f_3(x)||)J_{f_3}(x)dx \\ &= \int_D h_2(||f_2(x)||)J_{f_2}(x)dx \\ &= d(f_2, D, 0). \end{aligned} \tag{9.42}$$

□

LEMMA 9.14. *Let* $f_i \in C^1(\overline{D})$, $p_i \in \mathbb{R}^n$, $i = 1, 2$, $\epsilon > 0$ *be such that* $p_j \notin f_i(Q_{f_i})$, $i, j = 1, 2$, *and*

$$\begin{gathered} ||f_i(x) - p_j|| \ge 7\epsilon, \quad i, j = 1, 2, \ x \in \partial D, \\ ||f_1(x) - f_2(x)|| < \epsilon, \quad x \in \overline{D}, \\ ||p_1 - p_2|| < \epsilon. \end{gathered} \tag{9.43}$$

Then $d(f_1, D, p_1) = d(f_2, D, p_2)$.

Proof. We first note that the first inequality above implies that $p_j \notin f_i(\partial D)$ for any $i, j = 1, 2$. From Lemma 9.13 we obtain

$$d(f_1, D, p_1) = d(f_2, D, p_1). \tag{9.44}$$

We also obtain

$$d(f_2, D, p_1) = d(f_2 + (p_1 - p_2), D, p_1). \tag{9.45}$$

From the definition of the degree, letting $\bar{f}(x) \equiv f_2(x) + p_1 - p_2$, we have

$$\begin{aligned} d(\bar{f}, D, p_1) &= \sum_{x \in \bar{f}^{-1}(p_1)} \operatorname{sgn} J_{\bar{f}}(x) \\ &= \sum_{\{x; \bar{f}(x) = p_1\}} \operatorname{sgn} J_{\bar{f}}(x) \\ &= \sum_{\{x; f_2(x) = p_2\}} \operatorname{sgn} J_{f_2}(x) \\ &= \sum_{x \in f_2^{-1}(p_2)} \operatorname{sgn} J_{f_2}(x) \\ &= d(f_2, D, p_2). \end{aligned} \tag{9.46}$$

Our conclusion follows from (9.44), (9.45), and (9.46). □

We are now ready to define $d(f, D, p)$ at points $p \in f(Q_f)$. Let $f \in C^1(\overline{D})$, $p \in \mathbb{R}^n$ be given with $p \notin f(\partial D)$ and $p \in f(Q_f)$. Then since, by Lemma 9.8, $f(Q_f)$ does not contain any open set, the point p can be approximated by a sequence $\{p_n\}$ such that $p_n \notin f(Q_f)$. Since $p \notin f(\partial D)$, (which is a compact set), we may assume that $p_n \notin f(\partial D)$, $n = 1, 2, \ldots$. Actually, there exists $\epsilon > 0$ such that

$$||f(x) - p_n|| \geq 7\epsilon, \quad n = 1, 2, \ldots, \; x \in \partial D. \tag{9.47}$$

To see this, assume that (9.47) is false. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{x_{n_k}\} \subset \partial D$ such that $f(x_{n_k}) - p_{n_k} \to 0$ as $k \to \infty$. Since ∂D is compact, we may assume the $x_{n_k} \to x_0 \in \partial D$. Then $f(x_{n_k}) - p_{n_k} \to f(x_0) - p = 0$, that is, a contradiction.

Applying Lemma 9.14, we see now that $d(f, D, p_n) = c$ (constant) for all large n. Since for any other sequence $\{q_n\}$ having the same properties as $\{p_n\}$ we have

$$||p_n - q_n|| \leq ||p_n - p|| + ||q_n - p|| < \epsilon \tag{9.48}$$

for all large n, we see that this constant c is independent of the particular sequence $\{p_n\}$, that is,

$$\lim_{n\to\infty} d(f, D, p_n) = \lim_{n\to\infty} d(f, D, q_n). \tag{9.49}$$

This approach justifies the following definition.

Definition 9.15. The degree $d(f, D, p)$ is defined for every $f \in C^1(\overline{D})$ and every $p \in \mathbb{R}^n$ such that $p \notin f(\partial D)$ as follows:

$$d(f, D, p) = \lim_{n\to\infty} d(f, D, p_n). \tag{9.50}$$

The following lemma will allow us to define the degree for functions in $C(\overline{D})$.

Lemma 9.16. *Let $f_i \in C^1(\overline{D})$, $i = 1, 2$, $p \in \mathbb{R}^n$, $\epsilon > 0$ be such that*

$$\begin{gathered} ||f_i(x) - p|| \geq 8\epsilon, \quad i = 1, 2,\ x \in \partial D, \\ ||f_1(x) - f_2(x)|| < \epsilon, \quad x \in \overline{D}. \end{gathered} \tag{9.51}$$

Then $d(f_1, D, p) = d(f_2, D, p)$.

Proof. Let the sequence $\{p_n\}$ be as in the discussion preceding Definition 9.15 for $f = f_1$ and $f = f_2$ (i.e., $p_n \notin f_i(\partial D)$ and $p_n \notin f_i(Q_{f_i})$, $i = 1, 2$, $n = 1, 2, \ldots$). Then Lemma 9.13 implies that

$$d(f_1, D, p_n) = d(f_2, D, p_n), \quad n \geq n_0, \tag{9.52}$$

where $\|p_n - p\| < \epsilon$ for $n \geq n_0$. In fact,

$$||f_i(x) - p_n|| \geq ||f_i(x) - p|| - ||p_n - p|| > 8\epsilon - \epsilon = 7\epsilon, \quad n \geq n_0. \tag{9.53}$$

Letting $n \to \infty$ in the above equation, we obtain our conclusion. □

Now, let $f \in C(\overline{D})$, $p \notin f(\partial D)$. Let $\{f_n\}$ be a sequence of functions in $C^1(\overline{D})$ such that $f_n \to f$ uniformly on $\overline{D}$ as $n \to \infty$. Then, for some $\epsilon > 0$, some index n_0 and all $m, n \geq n_0$, we have $\|f_n(x) - p\| \geq 8\epsilon$, $x \in \partial D$, and $\|f_n(x) - f_m(x)\| < \epsilon$, $x \in \bar{D}$. In fact, we assume that $\|f_{n_k}(x_{n_k}) - p\| \to 0$ as $k \to \infty$, where $\{x_{n_k}\} \subset \partial D$. We may also assume that $x_{n_k} \to x_0 \in \partial D$ as $k \to \infty$, because ∂D is compact. Then we obtain $\|f_{n_k}(x_{n_k}) - p\| \to \|f(x_0) - p\| = 0$, that is, a contradiction. From Lemma 9.16 we see that

$$d(f_n, D, p) = d(f_{n_0}, D, p), \quad n \geq n_0. \tag{9.54}$$

Also, by Lemma 9.16, any two such degrees are eventually independent of the sequence $\{f_n\}$ under consideration. Thus, we have the following definition.

Definition 9.17. The degree $d(f, D, p)$ is defined for every $f \in C(\overline{D})$, $p \in \mathbb{R}^n$ such that $p \notin f(\partial D)$ as follows:

$$d(f, D, p) = \lim_{n \to \infty} d(f_n, D, p). \tag{9.55}$$

As an example, we consider the function

$$f(x_1, x_2) = (2|x_2| - x_1, x_1^2 + 4x_1 + 4x_2^2), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{9.56}$$

Obviously, $f \notin C^1(\overline{B_1(0)})$. In order to compute the degree $d(f, \overline{B_1(0)}, 0)$, we observe first that each

$$f_\epsilon(x_1, x_2) \equiv (2|x_2| - x_1 + \epsilon, x_1^2 + 4x_1 + 4x_2^2), \quad \epsilon \in (0, 1), \tag{9.57}$$

is the uniform limit of a sequence of functions $f_{\epsilon,n} \in C^1(\overline{B_1(0)})$ such that $f_{\epsilon,n}(x) \neq 0$, $x \in \overline{B_1(0)}$. This follows from the fact that since $\|f_\epsilon(x_1, x_2)\|$ is positive on $\overline{B_1(0)}$, it attains a positive minimum on the compact set $\overline{B_1(0)}$. Consequently, since $f_{\epsilon,n}^{-1}(0) = \varnothing$, we have that $d(f_{\epsilon,n}, B_1(0), 0) = 0$ (see also Theorem 9.19 below) and

$$d(f_\epsilon, B_1(0), 0) = \lim_{n \to \infty} d(f_{\epsilon,n}, B_1(0), 0). \tag{9.58}$$

Since $f_\epsilon \to f$ uniformly on $\overline{B_1(0)}$, it follows that $d(f, B_1(0), 0) = 0$.

4. PROPERTIES OF THE FINITE-DIMENSIONAL DEGREE

Theorem 9.18. *Let $D_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots, m$, be mutually disjoint, open and bounded. Let $f \in C(\overline{\bigcup D_i}, \mathbb{R}^n)$, $p \in \mathbb{R}^n$ be such that $p \notin f(\partial(\bigcup D_i))$. Then*

$$d\Big(f, \bigcup D_i, p\Big) = \sum_{i=1}^{m} d(f, D_i, p). \tag{9.59}$$

Proof. Obvious by virtue of Theorem 9.7. □

The result in Theorem 9.19 below contains one of the most important properties of the degree in $\mathbb{R}^n$. It provides conditions for the solvability of the equation $f(x) = p$.

Theorem 9.19. *Let $f \in C(\overline{D})$, $p \in \mathbb{R}^n$ be such that $p \notin f(\partial D)$. Let $d(f, D, p) \neq 0$. Then there exists a point $x_0 \in D$ such that $f(x_0) = p$.*

Proof. Assume that the conclusion is false. Then there exists $\epsilon > 0$ such that $\|f(x) - p\| > 2\epsilon$, $x \in \overline{D}$. This is a consequence of the fact that $\overline{D}$ is compact and that the continuous function $\|f(x) - p\|$ attains its (positive) minimum on $\overline{D}$. Let $\{f_n\} \subset C^1(\overline{D})$ be such that $f_n \to f$ uniformly on $\overline{D}$ and let $\{p_n\}$ be a sequence in $\mathbb{R}^n$ with $p_n \to p$ and $p_n \notin \bigcup_{k=1}^{\infty} f_k(Q_{f_k})$. This is possible because this union has measure zero, and thus it contains no open set. Furthermore,

$$||f_n(x) - p_k|| > \epsilon, \quad k, n \geq n_0, \ x \in \overline{D}, \tag{9.60}$$

for some n_0. At this point we apply the formula of Theorem 9.12. Let $g \in C_1(\mathbb{R}_+)$ be such that $g(x) = 0$ in a neighborhood of zero and in the set $[\epsilon, \infty)$. Assume further that

$$\int_{\mathbb{R}^n} g(\|x\|)dx = 1. \tag{9.61}$$

Then

$$d(f_n, D, p_k) = \int_D g(||f_n(x) - p_k||)J_{f_n}(x)dx = 0, \quad k, n \geq n_0. \tag{9.62}$$

Our conclusion follows now from this equality and Definitions 9.15 and 9.17. □

Theorem 9.20 (invariance under homotopies). *Let $H : [a, b] \times \overline{D} \to \mathbb{R}^n$ be continuous and such that $H(t, x) \neq p$, $t \in [a, b]$, $x \in \partial D$, where $p \in \mathbb{R}^n$ is a given point. Then $d(H(t, \cdot), D, p)$ is constant on $[a, b]$.*

Proof. Our assumptions imply the existence of some $\epsilon > 0$ such that

$$||H(t, x) - p|| > 9\epsilon, \quad t \in [a, b], \ x \in \partial D. \tag{9.63}$$

Moreover, we can find a number $\delta(\epsilon) > 0$ such that

$$||H(t_1, x) - H(t_2, x)|| < \frac{\epsilon}{3} \tag{9.64}$$

for all $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta(\epsilon)$ and all $x \in \overline{D}$. Fix two such points t_1, t_2. Let $\{f_{1,n}\}, \{f_{2,n}\} \subset C^1(\overline{D})$ approximate the functions $H(t_1, x)$, $H(t_2, x)$, respectively, uniformly on $\overline{D}$. Then there exists n_0 such that

$$\begin{aligned} &||f_{i,n}(x) - p|| > 8\epsilon, \quad i = 1, 2, \ x \in \partial D, \\ &||f_{1,n}(x) - f_{2,n}(x)|| < \epsilon, \quad x \in \overline{D}, \end{aligned} \tag{9.65}$$

for every $n \geq n_0$. Applying Lemma 9.16, we see that $d(f_{1,n}, D, p) = d(f_{2,n}, D, p)$, $n \geq n_0$. Thus,

$$d(H(t_1, \cdot), D, p) = d(H(t_2, \cdot), D, p). \tag{9.66}$$

Since this equality actually holds for any $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta(\epsilon)$, a simple covering argument yields our conclusion. □

As a consequence of the above result, we obtain the following improvement of Lemma 9.16.

Theorem 9.21 (Rouché). *Let $f_i \in C(\overline{D})$, $i = 1, 2$, $p \in \mathbb{R}^n$ be such that $p \notin f_i(\partial D)$, $i = 1, 2$, and*

$$\|f_1(x) - f_2(x)\| < \|f_1(x) - p\|, \quad x \in \partial D. \tag{9.67}$$

Then $d(f_1, D, p) = d(f_2, D, p)$.

Proof. We let

$$H(t, x) \equiv f_1(x) + t\big(f_2(x) - f_1(x)\big), \quad t \in [0, 1],\ x \in \overline{D}. \tag{9.68}$$

It is easy to see that $p \notin H(t, \partial D)$ for any $t \in [0, 1]$. The result follows from Theorem 9.20. □

Before we state and prove Borsuk's theorem, which concerns itself with the fact that "odd mappings have odd degrees at $p = 0 \in D$," we should observe first that the degree of a mapping $f \in C(\overline{D})$ with respect to a point $p \notin f(\partial D)$ depends only on the values of f on ∂D. In fact, let $g \in C(\overline{D})$ be such that $g(x) = f(x)$, $x \in \partial D$. Then $p \notin g(\partial D) = f(\partial D)$. Consider the homotopy $H(t, x) \equiv tf(x) + (1 - t)g(x)$, $t \in [0, 1]$, $x \in \overline{D}$. If we assume that $H(t_0, x_0) = p$, for some $t_0 \in [0, 1]$, $x_0 \in \partial D$, then $f(x_0) = g(x_0) = p$, which is a contradiction. Thus, by Theorem 9.20, $d(f, D, p) = d(g, D, p)$. At this point we need the following result which is a special case of the well-known Dugundji Theorem (cf. [14]). We denote by co A the *convex hull* of the set A, that is, the smallest convex set containing A.

Theorem 9.22 (Dugundji). *Let S be a closed subset of X, and let Y be another real Banach space. Then every continuous mapping $f : S \to Y$ has a continuous extension $F : X \to Y$ such that $R(F) \subset \operatorname{co} f(S)$.*

On the basis of this theorem, it is evident that every continuous function $f : \partial D \to \mathbb{R}^n$ can be extended to a continuous function $\bar{f}$ on all of $\overline{D}$. Since the degree of any such extension $\bar{f}$ with respect to $p \notin f(\partial D) = \bar{f}(\partial D)$ depends only on the set $f(\partial D)$, it is appropriate to define $d(f, D, p) = d(\bar{f}, D, p)$.

The proof of Borsuk's theorem is based on the following two lemmas. For a proof of Lemma 9.23 the reader is referred to Schwartz [52, page 79]. A set $D \subset X$ is called *symmetric* if $x \in D$ implies $-x \in D$.

Lemma 9.23. *Let D be symmetric with $0 \notin \overline{D}$. Let $f : \partial D \to \mathbb{R}^n$ be continuous, odd and such that $0 \notin f(\partial D)$. Then f can be extended to a function $\bar{f} : \overline{D} \to \mathbb{R}^n$ which is continuous, odd and nonvanishing on $\overline{D} \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1}$ is identified with the space $\{x \in \mathbb{R}^n : x = (0, x_2, \ldots, x_n)\}$.*

Lemma 9.24. *Let D be symmetric and such that $0 \notin \overline{D}$. Assume that $f : \partial D \to \mathbb{R}^n$ is continuous, odd and such that $0 \notin f(\partial D)$. Then $d(f, D, 0)$ is an even integer.*

Proof. By Lemma 9.23, we can extend the function f to all of $\overline{D}$ so that its extension (denoted again by f) is an odd continuous function never vanishing on $\overline{D} \cap R^{n-1}$. Now, let $g \in C^1(\overline{D})$ be odd and so close to f (in the sup-norm) that

$$0 \notin g(\partial D), \qquad 0 \notin g\Big(\overline{D} \bigcap R^{n-1}\Big), \qquad d(g, D, 0) = d(f, D, 0). \tag{9.69}$$

This can be done by approximating f sufficiently close by a function $g \in C^1(\overline{D})$ so that the above three conditions are satisfied and then replacing g, if necessary, by the odd function $(1/2)(g(x) - g(-x))$. In order to estimate $d(g, D, 0)$, we set $D^+ = \mathbb{R}^{n+} \cap D$, $D^- = \mathbb{R}^{n-} \cap D$, where $\mathbb{R}^{n+} = \{x \in \mathbb{R}^n : x_1 > 0\}$ and $R^{n-} = \{x \in \mathbb{R}^n : x_1 < 0\}$. Since the function g never vanishes on $\overline{D} \cap \mathbb{R}^{n-1}$, and $D = D^+ \cup D^- \cup (D \cap \mathbb{R}^{n-1})$, we may eliminate the set $D \cap \mathbb{R}^{n-1}$ below (this actually follows from the so-called "excision" property of the degree (see Exercise 9.23)) to obtain

$$d(g, D, 0) = d\Big(g, D^+ \cup D^-, 0\Big) = d(g, D^+, 0) + d(g, D^-, 0). \tag{9.70}$$

Let $p \in \mathbb{R}^n$ be so close to zero that $p \notin g(Q_g)$ and $p \notin g(\partial D)$. This is possible because $g(Q_g)$ contains no open set (Lemma 9.8). Since the function g is odd, J_g is an even mapping on $\overline{D}$. Thus, we also have $-p \notin g(Q_g)$. In view of this, choosing the point p further, so that

$$d(g, D^+, 0) = d(g, D^+, p), \qquad d(g, D^-, 0) = d(g, D^-, -p), \tag{9.71}$$

we obtain

$$\begin{aligned} d(g, D^+, 0) &= \sum_{\substack{g(x)=p \\ x\in D^+}} \operatorname{sgn} J_g(x), \\ d(g, D^-, 0) &= \sum_{\substack{g(y)=-p \\ y\in D^-}} \operatorname{sgn} J_g(y). \end{aligned} \tag{9.72}$$

Since the set $\{x \in D^+ : g(x) = p\}$ equals minus the set $\{y \in D^- : g(y) = -p\}$ and $J_g(x) = J_g(-x)$, we obtain

$$d(g, D^+, 0) = d(g, D^-, 0). \tag{9.73}$$

This and (9.70) imply that $d(g, D, 0)$ is an even integer. □

Theorem 9.25 (Borsuk). *Let D be symmetric with $0 \in D$. Let $f \in C(\overline{D})$ be odd and such that $0 \notin f(\partial D)$. Then $d(f, D, 0)$ is an odd integer.*

Proof. Let $\bar{U} \subset D$, where U is an open ball centered at zero, and let $g : \overline{D} \to \mathbb{R}^n$ be a continuous function with the properties: $g|_{\partial D} = f|_{\partial D}$, $g|_{\bar{U}} = I$ (= the identity function). Such a function g exists by Dugundji's theorem (Theorem 9.22). If we let $h(x) = (1/2)(g(x) - g(-x))$, $x \in \overline{D}$, we obtain that h is an odd function which has the properties of $g : h|_{\partial D} = f|_{\partial D}$, $h|_{\bar{U}} = I$. Since $0 \notin h(\partial U)$, we have

$$\begin{aligned} d(h, D, 0) &= d(h, U \cup (D \setminus \bar{U}), 0) \\ &= d(h, U, 0) + d(h, D \setminus \bar{U}, 0) \\ &= 1 + d(h, D \setminus \bar{U}, 0). \end{aligned} \tag{9.74}$$

However, since $\overline{D \setminus \bar{U}}$ is symmetric and does not contain zero, and $h|_{\overline{D \setminus \bar{U}}}$ is continuous, odd, and such that $0 \notin h(\partial(D \setminus \bar{U}))$, we have that $d(h, D \setminus \bar{U}, 0)$ is even by Lemma 9.24. It follows that $d(h, D, 0) = d(f, D, 0)$ is an odd integer. □

5. DEGREE THEORY IN BANACH SPACES

In this section, we extend the notion of degree developed in Sections 1, 2, 3, and 4 to mappings of the form $I - T$, where T is a compact operator. Before we do this, we need to make some observations about the degree in the space $\mathbb{R}^n$. We first note that this degree is independent of the basis that we have chosen for $\mathbb{R}^n$. In fact, let $h \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be one-to-one, onto and such that the Jacobian determinant J_h is never zero. We view this mapping h as a representation of a change in the coordinate system of $\mathbb{R}^n$. We let $f \in C^1(\overline{D})$, $p \in \mathbb{R}^n$ be such that $p \notin f(\partial D)$ and $p \notin f(Q_f)$. Then we have

$$d(h \circ f \circ h^{-1}, h(D), h(p)) = d(f, D, p). \tag{9.75}$$

Here, of course, D is a bounded open subset of $\mathbb{R}^n$. In fact, if x is a point in the old coordinate system and $y = h(x)$, then the function $f^*(y) \equiv h \circ f \circ h^{-1}(y)$ is the function f in the new coordinate system. We have

$$J_{f^*}(y) = J_h(f \circ h^{-1}(y)) J_f(h^{-1}(y)) J_{h^{-1}}(y), \tag{9.76}$$

and since $J_h(h^{-1}(y)) J_{h^{-1}}(y) = 1$ and J_h never vanishes, the Jacobians J_h, $J_{h^{-1}}$ have the same sign. Thus,

$$\operatorname{sgn} J_{f^*(y)} = \operatorname{sgn} J_f(h^{-1}(y)) = \operatorname{sgn} J_f(x). \tag{9.77}$$

Our assertion follows, for such functions f, from noting that

$$y \in f^{*-1}(h(p)) \quad \text{if and only if } x \in f^{-1}(p), \text{ where } y = h(x). \tag{9.78}$$

By using suitable approximations, we conclude that the degree is independent of the basis chosen for all $f \in C(\overline{D})$ and all points $p \notin f(\partial D)$.

It is now rather evident that we can define an appropriate degree function for a continuous mapping from the closure of a bounded open subset of one n-dimensional normed space into another. This process, which requires some caution involving the orientation of the spaces involved, is well-described in the book of Rothe [47, pages 117–118]. The relevant degrees for mappings between such spaces will be assumed to be known in the sequel. In what follows, we assume that X is a real Banach space.

6. DEGREE FOR COMPACT DISPLACEMENTS OF THE IDENTITY

We start with an auxiliary result which shows that the degree of certain mappings $g : \mathbb{R}^n \supset \overline{D} \to \mathbb{R}^n$ is actually equal to the degree of the restrictions $\bar{g} : \mathbb{R}^m \cap \overline{D} \to \mathbb{R}^n$, where $m < n$ (we identify the vector $(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ with the vector $(x_1, x_2, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n$).

Theorem 9.26. *Let $D \subset \mathbb{R}^n$ be open and bounded and let $m < n$. Let $f : \overline{D} \to \mathbb{R}^m$ be continuous and set $g(x) = x + f(x)$, $x \in \overline{D}$. Then if $p \in \mathbb{R}^m$ does not belong to $g(\partial D)$, we have*

$$d(g, D, p) = d\left(g|_{\mathbb{R}^m \cap \overline{D}}, \mathbb{R}^m \cap D, p\right). \tag{9.79}$$

Proof. We assume that $\mathbb{R}^m \cap D \neq \emptyset$. Otherwise, $p \notin g(\overline{D})$ and the two degrees in the statement equal zero. We note first that $g(\mathbb{R}^m \cap \overline{D}) \subset \mathbb{R}^m + \mathbb{R}^m = \mathbb{R}^m$. We also have $p \notin g|_{\mathbb{R}^m \cap \overline{D}}(\partial(\mathbb{R}^m \cap D))$ because $\partial(\mathbb{R}^m \cap D) \subset \mathbb{R}^m \cap \partial D \subset \partial D$. Thus, the degree $d(g|_{\mathbb{R}^m \cap \overline{D}}, \mathbb{R}^m \cap D, p)$ is well defined. We only need to prove the statement for $f \in C^1(\overline{D}, \mathbb{R}^m)$ and $p \notin g(Q_g)$. The rest follows by approximation.

If $g(y) = y + f(y) = p$, then $y = p - f(y) \in \mathbb{R}^m$. Hence $g^{-1}(p) \subset \mathbb{R}^m \cap D$. It follows that the points which are to be computed for $g : \overline{D} \to \mathbb{R}^n$ are also the points to be computed for $G = g|_{\mathbb{R}^m \cap \overline{D}} : \mathbb{R}^m \cap \overline{D} \to \mathbb{R}^m$, that is, $G^{-1}(p) = g^{-1}(p)$. However, there still could be a difference in the degrees of these two mappings due to differences in the signs of the respective Jacobian determinants at these points. As it turns out, this does not happen because

$$J_G(x) \equiv \det \begin{bmatrix} A(x) & B(x) \\ O^m_{n-m} & I_{n-m} \end{bmatrix}, \quad \text{with } A(x) \equiv I_m + \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{i,j=1}^m, \tag{9.80}$$

where $B(x) \equiv [(\partial f_i/\partial x_j)(x)]$, $i = 1, 2, \ldots, m$, $j = m+1, m+2, \ldots, n$, O^m_{n-m} is the $(n-m) \times m$ matrix of zeros, and I_k is the identity matrix in $\mathbb{R}^k$. Since $J_G(x) = J_g(x) \equiv \det A(x)$, $x \in g^{-1}(p)$, the proof is complete. □

Theorem 9.26 is one of the two key results that make possible the extendability of the notion of degree to infinite-dimensional spaces. The other key result is that "a compact operator can be approximated by compact operators of finite dimensional range." The latter is the content of Lemma 9.27 below.

Given a relatively compact set $K \subset X$ and an ϵ-net $\{v_1, v_2, \dots, v_r\}$ of $\bar{K}$ (see Definition 2.6), we define

$$F_\epsilon(x) = \frac{\sum_{i=1}^r q_i(x) v_i}{\sum_{i=1}^r q_i(x)}, \quad x \in \bar{K}, \tag{9.81}$$

where

$$q_i(x) = \begin{cases} \epsilon - ||x - v_i|| & \text{if } ||x - v_i|| < \epsilon, \\ 0 & \text{if } ||x - v_i|| \geq \epsilon. \end{cases} \tag{9.82}$$

Note that the denominator in (9.81) cannot vanish at any $x \in \bar{K}$ because $q_i(x) = \epsilon - \|x - v_i\| > 0$ for some i.

Lemma 9.27. *Let $T : X \supset M \to X$ be compact, where M is a bounded subset of X. Let $K = T(M)$. Let F_ϵ be defined on $\bar{K}$ as above. If $x \in M$, then*

$$||Tx - F_\epsilon(Tx)|| < \epsilon. \tag{9.83}$$

Proof. For every $x \in M$ we have

$$\begin{aligned} ||Tx - F_\epsilon(Tx)|| &= \frac{\left|\left| \sum_{i=1}^r q_i(Tx)Tx - \sum_{i=1}^r q_i(Tx)v_i \right|\right|}{\sum_{i=1}^r q_i(Tx)} \\ &\leq \frac{\sum_{i=1}^r \left[q_i(Tx)||Tx - v_i||\right]}{\sum_{i=1}^r q_i(Tx)} < \epsilon. \end{aligned} \tag{9.84}$$

□

Now, let $D \subset X$ be bounded, open, and let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers such that $\epsilon_n \to 0$. Let $T : \overline{D} \to X$ be a compact operator and set $K = T(\overline{D})$ and $T_n \equiv F_{\epsilon_n} \circ T$, where F_ϵ is the mapping defined on $\bar{K}$ by (9.81). Assume that $p \notin (I - T)(\partial D)$. We are going to give a meaning to the symbol $d(I - T, D, p)$.

We show first that there exists a number $r > 0$ such that

$$\inf_{x \in \partial D} \|x - Tx - p\| \geq r. \tag{9.85}$$

In fact, suppose that there exists a sequence $\{y_m\} \subset \partial D$ with

$$\lim_{m \to \infty} ||y_m - Ty_m - p|| = 0. \tag{9.86}$$

Since T is compact, we may assume that $Ty_m \to y_0$. Then

$$\lim_{m\to\infty} Ty_m = \lim_{m\to\infty} y_m - p = y_0. \tag{9.87}$$

Thus, $T(y_0+p) = y_0$. Since $\{y_m\} \subset \partial D$, we have $u = y_0+p \in \partial D$ and $u - Tu = p$. This contradicts $p \notin (I-T)(\partial D)$.

Now, recall that $T_n(\overline{D})$ lies in a finite-dimensional subspace of X and that

$$\lim_{n\to\infty} ||T_n x - Tx|| = 0 \quad \text{uniformly on } \bar{D}. \tag{9.88}$$

Since $\|x - Tx - p\| \geq r$, $x \in \partial D$, there exists some n_0 such that

$$||x - T_n x - p|| \geq \frac{r}{2}, \quad x \in \partial D,\ n \geq n_0. \tag{9.89}$$

Let $X_n = \operatorname{span}\{T_n(\bar{D}), p\}$. The space X_n is a real normed space, associated with the norm of X, and $D_n = X_n \cap D$ is a bounded open subset of X_n with $\partial D_n \subset X_n \cap \partial D \subset \partial D$. It follows that $p \notin (I-T)(\partial D_n)$. Since $(I-T_n)(\bar{D}_n) \subset X_n$ and

$$\inf_{x\in\partial D_n} ||x - T_n x - p|| \geq \frac{r}{2}, \tag{9.90}$$

the degree $d_{X_n}(I - T_n, D_n, p) \equiv d(I - T_n, D_n, p)$ is defined for $n \geq n_0$.

We prove that $d(I - T_n, D_n, p)$ is constant for all large n and independent of the particular sequence $\{T_n\}$ of compact operators of finite-dimensional range, which converges uniformly to the operator T on $\bar{D}$ as above. To this end, let T_{n_1}, $T_{n_2} : \bar{D} \to X$ be two compact operators with ranges in the finite-dimensional subspaces X_{n_1}, X_{n_2}, respectively. These two spaces are spans as above. Let these mappings satisfy

$$||Tx - T_{n_i}x|| < \frac{r}{2}, \quad i = 1,2,\ x \in \overline{D}. \tag{9.91}$$

It suffices to show that $d(I - T_{n_1}, D_{n_1}, p) = d(I - T_{n_2}, D_{n_2}, p)$, where $D_{n_i} = X_{n_i} \cap D$.

First we observe that if X_m is the vector space spanned by $X_{n_1} \cup X_{n_2}$, then

$$d(I - T_{n_i}, D_{n_i}, p) = d(I - T_{n_i}, D_m, p), \tag{9.92}$$

where $D_m = X_m \cap D$. Indeed, this follows from Theorem 9.26. We consider the homotopy

$$H(t,x) = t(I - T_{n_1})x + (1-t)(I - T_{n_2})x, \quad t \in [0,1],\ x \in \bar{D}_m. \tag{9.93}$$

For $x \in \bar{D}_m$, we obtain

$$\begin{aligned}||H(t,x)-(I-T)x|| &= ||t(I-T_{n_1})x+(1-t)(I-T_{n_2})x-(I-T)x|| \\ &\leq ||t(I-T_{n_1})x-t(I-T)x|| \\ &\quad + ||(1-t)(I-T_{n_2})x-(1-t)(I-T)x|| \\ &< \frac{tr}{2}+\frac{(1-t)r}{2}=\frac{r}{2}.\end{aligned} \tag{9.94}$$

Hence, for $x \in \partial D_m$ and $t \in [0,1]$, we find

$$\begin{aligned}||H(t,x)-p|| &= ||(I-T)x-p+H(t,x)-(I-T)x|| \\ &\geq ||(I-T)x-p||-||H(t,x)-(I-T)x|| \\ &> r-\frac{r}{2}=\frac{r}{2}.\end{aligned} \tag{9.95}$$

Consequently, Theorem 9.20 implies that

$$d(I-T_{n_1},D_m,p)=d(I-T_{n_2},D_m,p). \tag{9.96}$$

Our assertion follows from (9.92) and (9.96).

In view of the above construction, we are ready for the following definition.

Definition 9.28. Let $T:\overline{D}\to X$ be compact and let $p \notin (I-T)(\partial D)$. Then the degree $d(I-T,D,p)$ is defined by

$$d(I-T,D,p)=\lim_{n\to\infty} d_{X_n}(I-T_n,D_n,p), \tag{9.97}$$

where $X_n=\operatorname{span}\{T_n(\overline{D}),p\}$, $D_n=X_n\cap D$.

7. PROPERTIES OF THE GENERAL DEGREE FUNCTION

We establish some properties of $d(I-T,D,p)$ analogous to those of Section 4. We always assume that X is a real Banach space and D an open, bounded subset of X. We make free use of the symbols T_n, D_n, X_n from Section 6.

Theorem 9.29. *Let $p \notin (I-T)(\partial D)$. If $d(I-T,D,p)\neq 0$, then there exists $x_0 \in D$ such that $(I-T)x_0=p$.*

Proof. Theorem 9.19 implies the existence of a point $x_n \in D_n = X_n \cap D$ such that $(I-T_n)x_n=p$. Hence,

$$||x_n-Tx_n-p||=||x_n-Tx_n-T_nx_n+T_nx_n-p||=||Tx_n-T_nx_n||<\epsilon_n\ (\downarrow 0). \tag{9.98}$$

Since T is compact, we may assume that $Tx_n \to y \in X$. This implies that $x_n \to y+p$ and

$$||x_n - Tx_n - p|| \longrightarrow ||y + p - T(y + p) - p|| = ||x_0 - Tx_0 - p|| = 0 \tag{9.99}$$

with $x_0 = y + p \in D$ because $p \notin (I - T)(\partial D)$. □

Definition 9.30. For $t \in [0,1]$, let $T(t) : X \supset E \to X$ be a compact operator with E closed. Assume that for every $\epsilon > 0$ and every bounded $M \subset E$ there exists $\delta(\epsilon, M) > 0$ such that

$$||T(t_1)x - T(t_2)x|| < \epsilon \tag{9.100}$$

for all $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta(\epsilon, M)$ and all $x \in M$. Then we call $T(t)$ or $H(t,x) \equiv T(t)x$ a *homotopy of compact operators on E*.

If $H(t,x)$ is a homotopy of compact operators on E, then H is continuous on $[0,1] \times E$ (see also Exercise 9.16).

Theorem 9.31 (invariance under homotopies). *Let $p \in X$ be given. If $T(t)$ is a homotopy of compact operators on $\bar{D}$ and if $(I - T(t))x \neq p$ for all $x \in \partial D$ and all $t \in [0,1]$, then $d(I - T(t), D, p)$ is constant on* $[0,1]$.

Proof. We show first that there exists a number $r > 0$ such that $x \in \partial D$ and $t \in [0,1]$ imply

$$||(I - T(t))x - p|| \geq r. \tag{9.101}$$

In fact, assume the contrary. Then there exist sequences $\{x_n\} \subset \partial D$ and $t_n \in [0,1]$ such that $\| y_n \| < 1/n$, where

$$y_n = (I - T(t_n))x_n - p. \tag{9.102}$$

We may assume that $t_n \to t_0 \in [0,1]$. Since $\{T(t_0)x_n\}$ lies in a compact set, we may assume that $T(t_0)x_n \to y \in X$. Then

$$||T(t_n)x_n - y|| \leq ||T(t_n)x_n - T(t_0)x_n|| + ||T(t_0)x_n - y|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{9.103}$$

This implies that $x_n = T(t_n)x_n + y_n + p \to y + p$. Since $T(t_n)x_n \to T(t_0)(y + p)$, we have $T(t_0)(y + p) + p = y + p$ or $p = y + p - T(t_0)(y + p) = u - T(t_0)u$, where $u = y + p \in \partial D$. This is a contradiction.

Now, fix $t_1 \in [0,1]$ and assume that $T_n : \bar{D} \to X_n$ approximates $T(t_1)$ in such a way that

$$||T_n x - T(t_1)x|| < \frac{r}{4}, \quad x \in \bar{D}. \tag{9.104}$$

All values of t in the rest of the proof lie in the interval $[0,1]$. Since $T(t)$ is a homotopy of compact operators, there exists $\delta > 0$ such that $|t - t_1| < \delta$, $x \in \overline{D}$ imply

$$||T(t)x - T(t_1)x|| < \frac{r}{4}. \tag{9.105}$$

Consequently, for the same t, x,

$$\begin{aligned} ||T(t)x - T_n x|| &\le ||T(t)x - T(t_1)x|| + ||T(t_1)x - T_n x|| \\ &< \frac{r}{4} + \frac{r}{4} = \frac{r}{2} \end{aligned} \tag{9.106}$$

and, for $x \in \partial D$,

$$\begin{aligned} ||x - T_n x - p|| &\ge ||x - T(t)x - p|| - ||T_n x - T(t)x|| \\ &> r - \frac{r}{2} = \frac{r}{2} > 0. \end{aligned} \tag{9.107}$$

By the definition of the degree, we have

$$d(I - T(t), D, p) = d(I - T_n, D_n, p) \tag{9.108}$$

for $|t - t_1| < \delta$. Since t_1 is arbitrary, our conclusion follows from an easy covering argument. To see this, note first that the interval $[0,1]$ can be covered with a finite number, say k, of overlapping intervals of the type $(t_m - \delta, t_m + \delta)$, where $t_m \in (0,1)$ and δ does not depend on m and is such that

$$||T(t)x - T(t_m)x|| < \frac{r}{4}, \quad |t - t_m| < \delta,\ x \in \bar{D},\ m = 1, 2, \ldots, k. \tag{9.109}$$

Thus, we may choose the mapping $T_m : \bar{D} \to X_m$, as above, to obtain

$$\begin{aligned} ||T(t)x - T_m x|| < \frac{r}{2}, \quad |t - t_m| < \delta,\ x \in \bar{D}, \\ d(I - T(t), D, p) = d(I - T_m, D_m, p), \quad t \in (t_m - \delta, t_m + \delta). \end{aligned} \tag{9.110}$$

Since the intervals are overlapping,

$$d(I - T(t), D, p) = d(I - T_m, D, p) = \text{const}, \quad m = 1, 2, \ldots, k. \tag{9.111}$$

□

We can now state one of the most powerful tools of Nonlinear Analysis.

Theorem 9.32 (Schauder-Tychonov). *Let K be a closed, convex and bounded subset of X. Let $T : K \to K$ be compact. Then there exists a fixed point of T in K, that is, a point $x_0 \in K$ such that $Tx_0 = x_0$.*

The following two lemmas will be used in the proof of Theorem 9.32.

LEMMA 9.33. *If $p \in D$, then $d(I, D, p) = 1$. If $p \notin \bar{D}$, then $d(I, D, p) = 0$.*

PROOF. The proof follows from Theorem 9.3 and the approximation preceding Definition 9.28. □

LEMMA 9.34. *Let S be an open, convex, and bounded subset of X containing zero. Let $T : \bar{S} \to \bar{S}$ be compact. Then T has a fixed point in $\bar{S}$.*

PROOF. Consider the homotopy

$$H(t,x) = x - tTx, \quad t \in [0,1], \ x \in \bar{S}. \tag{9.112}$$

Assume that $Tx \neq x$, $x \in \partial S$. Then $H(1, \partial S) \not\ni 0$. Let $x \in \partial S$ be given. Then since the line segment $[0, Tx] \subset \bar{S}$, the points tTx, $t \in [0,1)$, belong to $\operatorname{int} S$. Thus, $H(t,x) \neq 0$, $(t,x) \in [0,1) \times \partial S$. Theorem 9.31 implies now that $d(I - T, S, 0) = d(I, S, 0)$ $(= 1$ by Lemma 9.33). This and Theorem 9.29 imply that $Tx_0 = x_0$ for some $x_0 \in S$. □

PROOF OF THEOREM 9.32. Let $B_r(0) \supset K$ for some $r > 0$. By Dugundji's theorem (Theorem 9.22), there exists a continuous function $g : X \to K$ such that $g = I$ on K. Let $\widetilde{T} = T \circ g$. Then $\widetilde{T}$ maps $\overline{B_r(0)}$ into itself and is compact. By Lemma 9.34, $\widetilde{T}$ has a fixed point $x_0 \in \overline{B_r(0)}$. It follows that $\widetilde{T}x_0 = T(g(x_0)) = x_0$. Since $\widetilde{T}x_0 \in K$, we have $x_0 \in K$ and $T(g(x_0)) = Tx_0 = x_0$. □

The abstract analogue of Borsuk's theorem (Theorem 9.25), is given in the following result.

THEOREM 9.35 (Borsuk). *Let D be a bounded, open, and symmetric subset of X with $0 \in D$. Let $T : \bar{D} \to X$ be an odd compact operator. Let $Tx \neq x$, $x \in \partial D$. Then $d(I - T, D, 0)$ is an odd integer.*

PROOF. Let $\{v_1, v_2, \ldots, v_r\}$ be an ϵ-net of $\overline{T(\bar{D})}$ and set $v_{r+1} = -v_1$, $v_{r+2} = -v_2, \ldots, v_{2r} = -v_r$. For $x \in \overline{T(\bar{D})}$, define $F_\epsilon(x)$ as in (9.81), but with r replaced by $2r$. Let X_n be the space spanned by the vectors $v_1, v_2, \ldots, v_r, -v_1, -v_2, \ldots, -v_r$. Then the operators $T_n = F_{\epsilon_n} \circ T$ are odd mappings on the symmetric sets $\bar{D}_n$, and our assertion follows from Theorem 9.25. □

For the sake of a more comprehensive and illuminating set of exercises and Example 9.37, we introduce below the concept of a real Hilbert space.

DEFINITION 9.36 (real Hilbert space). A *real Hilbert space* X is a real Banach space associated with an *inner product* $\langle \cdot, \cdot \rangle : X \times X \to R$ with the following two properties: (a) $\langle x, y \rangle$ is a bilinear form on H, that is, it is linear in x and in y; (b) $\|x\| = \sqrt{\langle x, x \rangle}$.

The space $X = l^2$ of all real square summable sequences $x = (x_1, x_2, \ldots)$ is a real Hilbert space with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \tag{9.113}$$

and norm

$$\|x\| = \left[\sum_{i=1}^{\infty} x_i^2\right]^{1/2}. \tag{9.114}$$

The space $X = L^2(\Omega)$ of all real square integrable (in the Lebesgue sense) functions $f : \Omega \to \mathbb{R}$ is a real Hilbert space with inner product

$$\langle f, g\rangle = \int_{\Omega} f(x)g(x)d\mu \tag{9.115}$$

and norm

$$\|f\| = \left[\int_{\Omega} [f(x)]^2 d\mu\right]^{1/2}, \tag{9.116}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$ and μ is the Lebesgue measure of $\mathbb{R}^n$.

The inner product of a real Hilbert space satisfies the Cauchy-Schwarz inequality

$$\left|\langle x, y\rangle\right| \le \|x\|\|y\|. \tag{9.117}$$

We now show that we cannot define an adequate notion of degree for all continuous functions mapping $\bar{D} \subset X$ into X. This assertion follows from the fact that the Schauder-Tychonov theorem (Theorem 9.32) fails, in general, if we replace the assumption of compactness on T by the mere assumption of continuity. To this end, we give the following example.

Example 9.37. Let $X = l^2$ and let $T : \overline{B_1(0)} \to X$ be defined as follows:

$$Tx = \left((1 - \|x\|^2)^{1/2}, x_1, x_2, \ldots\right). \tag{9.118}$$

Then T is continuous on $\overline{B_1(0)}$. Moreover, T maps $\overline{B_1(0)}$ into itself. In fact, if $\|x\| \le 1$, then

$$\|Tx\|^2 = 1 - \|x\|^2 + \sum_{i=1}^{\infty} x_i^2 = 1 - \|x\|^2 + \|x\|^2 = 1. \tag{9.119}$$

If we assume that x_0 is a fixed point of T in $\overline{B_1(0)}$, then we must have $\|Tx_0\| = \|x_0\| = 1$. In addition,

$$Tx_0 = (0, x_{01}, x_{02}, \dots) = (x_{01}, x_{02}, \dots) \tag{9.120}$$

implies that $x_{0i} = 0$, $i = 1, 2, \dots$. It follows that $x_0 = 0$, that is, a contradiction to $\|x_0\| = 1$.

EXERCISES

9.1. Let $K \subset X$ be nonempty, closed, bounded and $T_0 : K \to X$ compact. Show that there exists a compact extension $T : X \to X$ of T_0 such that $TX \subset$ co T_0K. Hint. Let T be an extension of T_0 by Dugundji's theorem (Theorem 9.22). Use the fact that $\overline{\text{co}}\,\overline{T_0K}$ is compact.

9.2 (Schaefer). Let $T : X \to X$ be compact. Assume further that the set

$$S = \{x \in X : x = tTx \text{ for some } t \in [0, 1)\} \tag{9.121}$$

is bounded. Show that T has fixed point. Hint. Let $S \subset D = B_r(0)$, for some $r > 0$. Let $H : [0, 1] \times \overline{D} \to X$ be the homotopy defined by $H(t, x) \equiv x - tTx$. Apply Theorem 9.31 with $p = 0$ and use the fact that $d(I, D, 0) = 1$.

9.3. Let $D \subset X$ be open, bounded and $T : \overline{D} \to X$ compact. Assume that $p \notin (I - T)(\partial D)$ and $q \in \mathbb{R}^n$. Show that

$$d(I - T - q, D, p - q) = d(I - T, D, p). \tag{9.122}$$

Hint. (1) Let $f \in C^1(\overline{D})$, $p \notin f(Q_f)$, $p \notin f(\partial D)$. Using the definition of the degree function, show that $d(f - q, D, p - q) = d(f, D, p)$. This amounts to showing that

$$J_{f_0}(x)\big|_{x \in f_0^{-1}(p-q)} = J_f(x)\big|_{x \in f^{-1}(p)}, \tag{9.123}$$

where $f_0(x) \equiv f(x) - q$.

(2) Generalize the above to functions $f \in C(\overline{D})$ and points $p \notin f(\partial D)$.

(3) In the case of a Banach space, show that

$$\begin{aligned} d(I - T, D, p) &= d(I - T_n, D_n, p), \\ d(I - T - q, D, p - q) &= d(I - T_n - q, D_n, p - q), \end{aligned} \tag{9.124}$$

where $T_n = F_{\epsilon_n} \circ T$ and $D_n = X_n \cap D$, for some suitable finite-dimensional space X_n. The right-hand sides of these equalities are equal by step (2).

9.4. Let $S \subset X$ be closed, bounded and have nonempty interior. Let $T : S \to X$ be compact and $v \in D = \operatorname{int} S$. Assume that

$$Tx - v \neq t(x - v) \tag{9.125}$$

for all $t > 1$ and all $x \in \partial D$. Show that T has a fixed point in $\overline{D}$. Hint. Consider the homotopy

$$H(t,x) = x - v - t(Tx - v), \quad t \in [0,1],\ x \in \overline{D}. \tag{9.126}$$

Assume that T has no fixed point in ∂D and apply Theorem 9.31 along with $d(I - T, D, 0) = d(I - v, D, 0) = 1$, because $v \in D$ and $d(I - v, D, 0) = d(I, D, v) = 1$ (see Exercise 9.3 and Lemma 9.33).

9.5 (Rouché). Let D be nonempty, open, bounded and $T_0, T_1 : \overline{D} \to X$ compact. Let $p \notin (I - T_0)(\partial D)$ and assume that

$$||T_0x - T_1x|| < ||(I - T_0)x - p||, \quad x \in \partial D. \tag{9.127}$$

Show that $d(I - T_1, D, p)$ is defined and

$$d(I - T_1, D, p) = d(I - T_0, D, p). \tag{9.128}$$

Hint. Consider the homotopy

$$H(t,x) = t(I - T_1)x + (1 - t)(I - T_0)x, \quad t \in [0,1],\ x \in \overline{D}. \tag{9.129}$$

By our assumption, $H(0,x) \neq p$, $x \in \partial D$. Show that $x \in \partial D$ implies $\|H(t,x) - p\| > 0$, $t \in (0,1]$. The rest follows from Theorem 9.31.

9.6 (invariance of domain). Let $D \subset X$ be open and $T : \overline{D} \to X$ compact and such that $g = I - T$ is one-to-one. Show that $g(D)$ is open. Hint. Let $p \in g(D)$. Since D is open, there exists an open ball $B \subset D$ such that $g^{-1}(p) \in B$. Also, if $q \in X$ is such that

$$\|p - q\| < r = \inf_{x \in \partial B} \{||g(x) - p||\}, \tag{9.130}$$

then $d(g, B, p) = d(g, B, q)$. To see this, observe that Exercise 9.3 gives

$$\begin{aligned} d(g, B, p) &= d(g - p, B, 0), \\ d(g, B, q) &= d(g - q, B, 0) \end{aligned} \tag{9.131}$$

and Exercise 9.5 (with $T_0x \equiv Tx + p$, $T_1x \equiv Tx + q$ and $p = 0$) implies

$$d(g - p, B, 0) = d(g - q, B, 0). \tag{9.132}$$

Thus, if we show that $d(g,B,p) \neq 0$, then $q \in g(B)$ for all $q \in B_r(p)$, proving that $g(D)$ is open.

We may take $p = 0$, $g(0) = 0$. In fact, $p \in g(B)$ if and only if $0 \in \tilde{g}(\widetilde{B})$ with $\tilde{g}(x) = g(x + x_0) - p$, $x \in \widetilde{B} = B - x_0$. Here, $x_0 \in B$ is such that $g(x_0) = p$ and $B - x_0 = \{x - x_0;\ x \in B\}$. We see that $\tilde{g}(0) = 0$.

Consider the homotopy

$$H(t,x) = g\left(\frac{x}{1+t}\right) - g\left(\frac{-tx}{1+t}\right), \quad t \in [0,1],\ x \in \bar{B}. \tag{9.133}$$

Show that $H(t,x) \equiv I - H_0(t,x)$, where $H_0(t,x)$ is a homotopy of compact operators. Show that $H(t,x) = 0$ implies $x = 0$. Use Theorem 9.31 to conclude that $d(g,B,0) = d(H(1,\cdot),B,0)$. Then use Borsuk's theorem (Theorem 9.35).

9.7 (Schauder-Tychonov theorem for starlike domains). An open, bounded set $D \subset X$ is called a *starlike domain* if there exists $x_0 \in D$ such that every ray starting at x_0 intersects ∂D at exactly one point. Such a ray is given by $\{x \in X : x = (1-t)x_0 + tx_1$ for some $t \geq 0\}$, where x_1 is another point in X. An open, bounded, and convex set D is a starlike domain. Prove the following improvement of the Schauder-Tychonov theorem.

Let $D \subset X$ be a starlike domain. Let $T : \bar{D} \to X$ be compact and such that $T(\partial D) \subset \bar{D}$. Then T has fixed point in $\bar{D}$. Hint. If T has a fixed point in ∂D, we are done. Assume that $Tx \neq x$, $x \in \partial D$, and apply Exercise 9.4 as follows: assume that $x_0 \in D$ is such that D is a starlike domain with respect to x_0. Let

$$Tx - x_0 = t(x - x_0) \tag{9.134}$$

for some $t > 1$, $x \in \partial D$, and show that $Tx = (1-t)x_0 + tx$ is a contradiction to $T(\partial D) \subset \bar{D}$.

9.8 (degree for generalized Leray-Schauder operators). Let $D \subset X$ be open, bounded and let

$$Ux \equiv \lambda(x)x - T_0x, \quad x \in \overline{D}, \tag{9.135}$$

where $T_0 : \bar{D} \to X$ is compact and $\lambda : \bar{D} \to [m,M]$ is continuous. Here, m, M are positive constants. Let $p \notin U(\partial D)$ and consider the operator

$$Tx \equiv \left(\frac{1}{\lambda(x)} - 1\right)p + \left(\frac{1}{\lambda(x)}\right)T_0x. \tag{9.136}$$

Show that $(I-T)x = p$ if and only if $Ux = p$. Then show that $p \notin (I-T)(\partial D)$ and that T is compact. Thus, $d(I-T,D,p)$ is well defined. Actually, one may define

$$d(U,D,p) \equiv d(I-T,D,p). \tag{9.137}$$

This definition extends the notion of degree to large classes of operators U.

9.9. Let $H_0 : [0,1] \times \bar{D} \to X$, $\lambda : [0,1] \times \bar{D} \to [m,M]$ be homotopies of compact operators, where $D \subset X$ is open, bounded and m, M are positive constants. Let

$$H_1(t,x) \equiv \lambda(t,x)x - H_0(t,x) \tag{9.138}$$

and let $p \in X$ be such that $p \notin H_1(t,\partial D)$, $t \in [0,1]$. Show that $d(H_1(t,\cdot),D,p)$ is independent of $t \in [0,1]$. This degree function is defined, for each $t \in [0,1]$, in Exercise 9.8. Hint. Consider the mapping

$$H(t,x) \equiv x - \left[\left(\frac{1}{\lambda(t,x)} - 1\right)p + \left(\frac{1}{\lambda(t,x)}\right)H_0(t,x)\right]. \tag{9.139}$$

9.10 (Krasnosel'skii's fixed point theorem). Let X be a real Hilbert space and $T : \overline{B_r(0)} \to X$ compact and such that

$$\langle Tx, x\rangle \le \|x\|^2, \quad x \in \partial B_r(0). \tag{9.140}$$

Prove that T has a fixed point in $\overline{B_r(0)}$. Hint. Let $B = B_r(0)$ and consider the homotopy

$$H(t,x) = x - tTx, \quad t \in [0,1],\ x \in \bar{B}. \tag{9.141}$$

Assume that $x - Tx \neq 0$ for $x \in \partial B$ and show that $H(t,x) = 0$ for some $t \in [0,1)$ implies $x = 0$. Then $d(I - T, B, 0) = d(I, B, 0) = 1$, which shows that $Tx = x$ for some $x \in B$.

9.11 (Altman). Let $T : \overline{B_r(0)} \to X$ be compact and such that

$$\|x - Tx\|^2 \ge \|Tx\|^2 - \|x\|^2, \quad x \in \partial B_r(0). \tag{9.142}$$

Then T has a fixed point in $\overline{B_r(0)}$. Hint. Let $B = B_r(0)$ and consider the operator

$$T_0 x = \begin{cases} Tx & \text{for } \|Tx\| \le r, \\ \dfrac{rTx}{\|Tx\|} & \text{for } \|Tx\| > r. \end{cases} \tag{9.143}$$

Show that T_0 is compact and that $T_0\bar{B} \subset \bar{B}$. By the Schauder-Tychonov theorem (Theorem 9.32), there exists $x_0 \in \bar{B}$ with $T_0 x_0 = x_0$. Show that $Tx_0 = x_0$. In fact,

$$x_0 = T_0 x_0 = \begin{cases} Tx_0 & \text{for } \|Tx_0\| \le r, \\ \dfrac{rTx_0}{\|Tx_0\|} & \text{for } \|Tx_0\| > r. \end{cases} \tag{9.144}$$

If $\|Tx_0\| \le r$, we are done. If $\|Tx_0\| > r$, then $\|x_0\| = r$ and

$$Tx_0 = \frac{\|Tx_0\|}{r}x_0 = \alpha x_0 \tag{9.145}$$

with $\alpha > 1$. Use the condition (9.142) to find a contradiction.

9.12. Show that (9.140) and (9.142) are equivalent when X is a real Hilbert space. Show that

$$\|Tx\| \le \|x\|, \quad x \in \partial B_r(0), \tag{9.146}$$

implies (9.142) in a real Banach space X.

9.13. Prove the Cauchy-Schwarz inequality (9.117).

9.14. Let $D \subset X$ be open, bounded, and symmetric. Assume that $0 \in D$ and that $T : \overline{D} \to X$ is compact. Assume further that the function $g = I - T$ is such that $0 \notin g(\partial D)$ and

$$\frac{g(x)}{\|g(x)\|} \neq \frac{g(-x)}{\|g(-x)\|}, \quad x \in \partial D. \tag{9.147}$$

Then $d(I - T, D, 0)$ is an odd integer. Hint. Consider the homotopy

$$H(t,x) = \frac{1}{1+t}g(x) - \frac{t}{1+t}g(-x), \quad t \in [0,1],\ x \in \overline{D}. \tag{9.148}$$

As in Exercise 9.6, show that $H(t,x) \equiv I - H_0(t,x)$, where $H_0(t,x)$ is a homotopy of compact operators. Then show that $H(t,x) \neq 0$ for all $(t,x) \in [0,1] \times \partial D$. Conclude that $d(H(0,\cdot),D,0) = d(H(1,\cdot),D,0) =$ an odd integer.

9.15. Let $T : X \to X$ be compact and $I - T$ one-to-one. Assume further that

$$\lim_{\|x\| \to \infty} \|x - Tx\| = +\infty. \tag{9.149}$$

Show that $R(I - T) = X$. Hint. Use Exercise 9.6 to show that $R(I - T)$ is open. Then show that $R(I - T)$ is closed. In fact, if $y \in \overline{R(I-T)}$, then $x_n - Tx_n \to y$ for some $\{x_n\} \subset X$. Show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in X$ with $x_0 - Tx_0 = y$. Note that the only open and closed subsets of a Banach space X are X and $\emptyset$.

9.16. Let $D \subset X$ be open and bounded and let $H : [0,1] \times \overline{D} \to X$ be a homotopy of compact operators on $\overline{D}$. Show that H is a compact operator (on $[0,1] \times \overline{D}$).

9.17. Let $D \subset X$ be bounded and $T : D \to X$ compact. Set $g = I - T$. Then

(i) $g(D)$ is bounded;
(ii) if D is closed, then $g(D)$ is closed;
(iii) if $C \subset X$ is compact, then $g^{-1}(C)$ is compact;
(iv) if D is closed and g is one-to-one, then $g^{-1} : g(D) \to D$ can be written as $I - T_1$ with T_1 compact.

9.18. Let $D \subset X$ be open, bounded and let $T : \overline{D} \to X$ be compact. Assume that $g : I - T : \overline{D} \to X$ is one-to-one. Show that $g(\overline{D}) = \overline{g(D)}$ and $g(\partial D) = \partial g(D)$. Hint. Use Exercise 9.6 for the openness of $g(D)$. Then use Exercise 9.17(ii), for the closedness of $g(\overline{D})$.

9.19. Let $D \subset X$ be open and bounded. Let $T_1, T_2 : \overline{D} \to X$ be compact and set $g_1 = I - T_1$, $g_2 = I - T_2$. Assume that $y_0 \notin g_i(\partial D)$, $i = 1, 2$. Assume that there is no $x \in \partial D$ such that $g_1(x) - y_0$, $g_2(x) - y_0$ have opposite direction, that is, there are no $\lambda > 0$, $x \in \partial D$ with

$$g_2(x) - y_0 = -\lambda(g_1(x) - y_0). \tag{9.150}$$

Then $d(g_1, D, y_0) = d(g_2, D, y_0)$. Hint. Consider the homotopy

$$H(t, x) \equiv (1 - t)g_1(x) + tg_2(x). \tag{9.151}$$

Show that $y_0 \notin H(t, \partial D)$ for $t \in [0, 1]$.

9.20. Let $D \subset X$ be open, bounded, where X is a real Hilbert space. Let $T_i : \overline{D} \to X$ be compact and $g_i \equiv I - T_i$, $i = 1, 2$. Assume further that $y_0 \in X$ is such that $y_0 \notin g_i(\partial D)$, $i = 1, 2$, and

$$\langle g_1(x) - y_0, g_2(x) - y_0 \rangle > 0, \quad x \in \partial D. \tag{9.152}$$

Use Exercise 9.19 to show that $d(g_1, D, y_0) = d(g_2, D, y_0)$. If, moreover, $g_2 = I$ and $y_0 \in D$, then $g_1(x) = y_0$ has a solution $x \in D$.

9.21. Let $T : X \to X$ be compact. Assume that there exists a compact linear operator S such that, for some constant λ,

$$\|Tx - \lambda Sx\| < \|x - \lambda Sx\|, \quad x \in \partial B_r(0). \tag{9.153}$$

Show that the operator T has a fixed point in $\overline{B_r(0)}$. Hint. Consider the operator $g(x) \equiv x - Tx$. This operator satisfies the assumptions of Exercise 9.14 with $D = B_r(0)$. In fact, if the inequality in Exercise 9.14 is not satisfied, then there exists some $\alpha > 0$ and points $x_0, x_1 = -x_0 \in \partial B_r(0)$ such that $g(x_0) = \alpha g(x_1)$. Thus,

$$(\alpha + 1)x_0 = Tx_0 - \alpha Tx_1. \tag{9.154}$$

It follows that

$$(\alpha+1)(x_0-\lambda Sx_0) = Tx_0-\lambda Sx_0-\alpha(Tx_1-\lambda Sx_1). \tag{9.155}$$

Use our assumption to contradict this equality.

9.22. Let $D = B_r(0)$ and let $H : [0,1]\times\bar{D} \to X$ be a homotopy of compact operators with $H(0,\partial D)\subset\bar{D}$ and

$$x-H(t,x)\neq 0 \quad \text{for } (t,x)\in[0,1]\times\partial D. \tag{9.156}$$

Show that $H(1,\cdot)$ has a fixed point in D. Hint. Consider the homotopy $h(t,x)\equiv x-H(t,x)$, $(t,x)\in[0,1]\times\bar{D}$. Show that $d(h(1,\cdot),D,0) = d(h(0,\cdot),D,0)$. Then show that $d(h(0,\cdot),D,0) = 1$ by considering the new homotopy $m_\tau \equiv x-\tau H(0,x)$, $(\tau,x)\in[0,1]\times\bar{D}$. Show that $0\notin m_\tau(\partial D)$ for any $\tau\in[0,1]$, and that $d(m_0,D,0) = d(m_1,D,0)$ $(= d(h(0,\cdot),D,0))$.

9.23 (excision property of the degree). Let $D\subset X$ be open and bounded. Let $D_1\subset X$ be open and such that $\overline{D}_1\subset D$. Let $T:\overline{D}\to X$ be compact and $p\in X$ with $p\notin(I-T)(\overline{D}_1\cup\partial D)$. Show that $d(I-T,D,p) = d(I-T,D\setminus\overline{D}_1,p)$. Hint. It suffices to show this statement for $X = R^n$, a continuous function $f:\overline{D}\to X$ in place of $I-T$ and a compact set $K\subset D$ in place of the open set D_1, that is, it has to be shown that

$$d(f,D,p) = d(f,D\setminus K,p), \tag{9.157}$$

where $p\notin f(K\cup\partial D)$. Show first that

$$\inf_{x\in K\cup\partial D}\|f(x)-p\|>0 \tag{9.158}$$

implies

$$\inf_{x\in K\cup\partial D}\|f_n(x)-p\|>0 \tag{9.159}$$

for any uniformly approximating sequence $\{f_n\}$ and all large n. Thus, from Definition 9.17 we obtain

$$d(f,D\setminus K,p) = \lim_{n\to\infty}d(f_n,D\setminus K,p) \qquad\text{and}\qquad d(f,D,p) = \lim_{n\to\infty}d(f_n,D,p), \tag{9.160}$$

where $f_n\in C^1(\overline{D})$. It remains to show that

$$d(f_n,D,p) = d(f_n,D\setminus K,p) \tag{9.161}$$

for all large n. This can be done by approximating the point $p\notin f_n(K\cup\partial D)$ by points $q_n\notin f_n(Q_{f_n}\cup K\cup\partial D)$. Fill in the details.

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