ADVANCED PROGRAMME MATHEMATICS: PAPER II

Time: 1 hour   100 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1.   This question paper consists of 14 pages, a Diagram Sheet and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.

2.   This question paper consists of THREE modules:

      Choose **ONE** of the **THREE** modules:

      MODULE 2:   STATISTICS (100 marks) OR
      MODULE 3:   FINANCE AND MODELLING (100 marks) OR
      MODULE 4:   MATRICES AND GRAPH THEORY (100 marks)

3.   Non-programmable and non-graphical calculators may be used.

4.   All necessary calculations must be clearly shown and writing must be legible.

5.   Diagrams have not been drawn to scale.

6.   **Rounding of final answers.**

      MODULE 2:   Four decimal places, unless otherwise stated.

      MODULE 3:   Two decimal places, unless otherwise stated.

      MODULE 4:   Two decimal places, unless otherwise stated.

MODULE 2   STATISTICS

QUESTION 1

1.1   A vegetable bowl contains 4 carrots and 7 green beans. Riyaadh randomly takes out three vegetables and eats them. Find the probability that

    (a)   2 green beans and 1 carrot are eaten, in any order.   (6)

    (b)   the third vegetable eaten is a green bean.   (7)

1.2   From a survey done at her school, Kate found that 60% of the students wore a watch on their left wrist, 30% wore a watch on their right wrist and 10% did not wear a watch.

    (a)   From a random sample of 20, how many students can Kate expect not to be wearing a watch?   (2)

    (b)   Given a random sample of 5 students, find the probability that at most 3 students wear a watch on their **right** wrist.   (7)

    (c)   A random sample of 200 students was taken. Using the normal approximation, find the probability that more than 125 wore a watch on their **left** wrist.   (7)

[29]

QUESTION 2

2.1   The number of eggs laid by a sample of 90 female sea gulls are shown in the table.

Number of eggs	1	2	3	4
Frequency	15	45	20	10

    (a)   Find the mean and standard deviation, to two decimal places, of the number of eggs laid per sea gull.   (7)

    (b)   Seth noticed that the sample did not include female sea gulls that laid no eggs. How would the mean and standard deviation change if these sea gulls were included?   (2)

2.2   When Nicola is stung by a bee she always develops an allergic reaction. The time taken in minutes for Nicola to develop the reaction can be modelled using the probability density function given by

$$f(x) = \begin{cases} \dfrac{k}{x+1} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

    (a)   Show that $k = \dfrac{1}{\ln 5}$.   (6)

    (b)   Find the median time for Nicola to develop a reaction.   (6)

[21]

QUESTION 3

3.1   The random variable $Z \sim N(0,1)$ {i.e. with mean, 0 and variance, 1}

- R is the event $Z > 1,1$
- Q is the event $-1,8 < Z < 1,8$

Determine:

(a) $P(R)$ (3)

(b) $P(R \cup Q)$ (6)

3.2   The random variable X has a normal distribution with a mean of 200 and a standard deviation 50. Find the value of c if it is given that
$P(X > c \mid X > 280) = 0,625$ (8)

                                                                                                                        [17]

QUESTION 4

4.1   When the council published a plan for a new road, only 15% of local residents accepted the plan. The council then published a revised plan and, out of a random sample of 300 local residents, 60 accepted the revised plan.

(a)   Determine a 98% confidence interval for the proportion of all the local residents who accepted the **revised** plan.   (6)

(b)   Using the confidence interval in Question 4.1 (a), is there evidence to support the claim that the proportion of local residents who accepted the revised plan is greater than the support for the original plan?   (2)

4.2   A pharmaceutical manufacturer purchased two medicine bottle-filling machines. In order to compare the performance of the two machines, a random sample of 60 bottles filled by the first machine and a random sample of 50 bottles filled by the second machine were checked. The volumes of the contents from the first machine (x), and from the second machine (y), are summarised as follows:

$n_x = 60$	$\bar{x} = 30,06$ ml	$\sigma_x^2 = 0,0784$
$n_y = 50$	$\bar{y} = 29,84$ ml	$\sigma_y^2 = 0,168$

(a)   Test, at the 2% level of significance, whether the mean volume content for the first machine is greater than the mean volume of the second machine.   (10)

(b)   Find the set of values of α for which there would be evidence at the $\alpha\%$ significance level that $\mu_x - \mu_y > 0,1$.   (5)

                                                                                                                        [23]

QUESTION 5

Consider the word **CEASELESS**.

5.1   How many ways can the letters of the word **CEASELESS** be arranged?   (4)

5.2   Find the number of different ways in which the 9 letters of the word **CEASELESS** can be arranged if exactly two of the Es are next to each other.   (6)

[10]

Total for Module 2: 100 marks

MODULE 3   FINANCE AND MODELLING

QUESTION 1

In each of the questions a financial scenario is described, with three formulae given. Choose the formula best suited to the scenario and write down just the letter of the formula.

1.1   Archie's parents invest an amount with the Bank of England for a period of 21 years. The interest rate is i% per annum, compounded annually. At the end of this period, the investment is worth four times the original amount.

A $1 = 4(1 + i)^{21}$

B $4 = (1 + i)^{21}$

C $21 = (1 + i)^4$ (2)

1.2   A company pays quarterly contributions into a sinking fund, starting on 1 March 2020 with its final payment on 1 March 2026, at which time the fund is closed.

A $F_v = \dfrac{x\left[\left(1+\dfrac{i}{4}\right)^{24} - 1\right]}{\dfrac{i}{4}}$

B $F_v = \dfrac{x\left[\left(1+\dfrac{i}{4}\right)^{24} - 1\right]\left(1+\dfrac{i}{4}\right)}{\dfrac{i}{4}}$

C $F_v = \dfrac{x\left[\left(1+\dfrac{i}{4}\right)^{25} - 1\right]}{\dfrac{i}{4}}$ (3)

1.3   A once-off deposit of x earns interest at a rate of 4% per annum, compounded monthly for the first eight months. After that, the interest rate changes to 4,6% per annum, compounded monthly for the next 16 months. The account accrues to a value of y.

   A $x\left(1+\dfrac{0,04}{12}\right)^{8} = y\left(1+\dfrac{0,046}{12}\right)^{-16}$

   B $x = y\left(1+\dfrac{0,04}{12}\right)^{8}\left(1+\dfrac{0,046}{12}\right)^{16}$

   C $x\left(1+\dfrac{0,04}{12}\right)^{-8}\left(1+\dfrac{0,046}{12}\right)^{-16} = y$ (3)

1.4   A loan P is taken out, at an interest rate of i% per annum, compounded monthly. The first repayment on the loan occurs four months after the loan commences. The loan is to be paid off, in monthly instalments of x at the end of each month, three years after the start of the loan. The final payment y is smaller than the other payments.

   A $P\left(1+\dfrac{i}{12}\right)^{4} - y\left(1+\dfrac{i}{12}\right)^{33} = \dfrac{x\left[1-\left(1+\dfrac{i}{12}\right)^{-32}\right]}{\dfrac{i}{12}}$

   B $P\left(1+\dfrac{i}{12}\right)^{3} = \dfrac{x\left[1-\left(1+\dfrac{i}{12}\right)^{-32}\right]}{\dfrac{i}{12}} + y\left(1+\dfrac{i}{12}\right)^{-33}$

   C $P\left(1+\dfrac{i}{12}\right)^{3} = \dfrac{x\left[1-\left(1+\dfrac{i}{12}\right)^{-35}\right]}{\dfrac{i}{12}} - y\left(1+\dfrac{i}{12}\right)^{-36}$ (4)

                                                                                                [12]

QUESTION 2

Four years ago, Clarence's Car Company purchased a bus for R1 850 000; it is now worth R920 000. To purchase a new bus now would cost the company R2 680 000.

2.1   Calculate the annual rate of depreciation of the value of the bus, expressed as a percentage correct to two decimal places, assuming the rate is based on a reducing balance.   (4)

2.2   Four years ago, Clarence set up a sinking fund to cover the purchase of a new bus now. The fund earned interest at 4,2% per annum, compounded monthly. The old bus is traded in to purchase the new bus and his final payment to the fund is made three months before the end of the four years.

Calculate what his monthly contribution to the fund should have been if he started paying exactly four years ago.   (8)

[12]

QUESTION 3

3.1   Calculate the value of a single deposit Tshepang must invest at an annual interest rate of 8,2%, compounded quarterly, so that the account generates R1 000 interest in one year.   (6)

3.2   Lohini invested R2 600 and R1 800 in two separate accounts at the same time. Both accounts earn simple interest rates, but the R2 600 account earns interest at a rate that is 2,5% per annum more than the interest rate on the R1 800 account. After a year, the two accounts together have earned R274 interest. Calculate the larger of the two interest rates, as a percentage, correct to two decimal places.   (8)

3.3   Calculate to the nearest month how long it will take for R10 000 invested at 7,2% per annum, compounded monthly, to yield the same value as R12 000 invested at 6,4% per annum, compounded monthly.   (8)

[22]

QUESTION 4

The natural lifespan of a rhino is 50 years. Females, who form 65% of the population, have only one calf **every four years**. The natural survival rate of a calf is 82%. The territorial space of rhinos is large, and hence South Africa's carrying capacity for rhinos is estimated to be only 40 000. There are currently about 18 000 rhinos in South Africa.

4.1 Is this a Malthusian or Logistic model? Explain your answer. (2)

4.2 Two different graphs, (a) and (b), of this population model are to be sketched. Describe the shape of the discrete graphs by choosing one word from the list below:

Linear, S-shaped, Exponential, Hyperbolic

(a) The rhino population P (on the *y*-axis) is sketched against time *n* (on the *x*-axis). (2)

(b) The proportional growth rate $\frac{\Delta P}{P}$ (on the *y*-axis) is sketched against the rhino population P (on the *x*-axis). (2)

4.3 Calculate, correct to three decimal places, the intrinsic growth rate of rhinos per four-year cycle. (4)

4.4 South Africa loses 4 000 rhinos to poaching every four years. Taking this new information into account, recalculate the intrinsic growth rate per four-year cycle required for the rhino population to at least remain stable. (8)

4.5 Assume that the intrinsic growth rate per four-year cycle is 0,4. Determine an equivalent effective **annual** growth rate, and hence calculate the rhino population after 6 years. (8)

[26]

QUESTION 5

ANSWER THIS ENTIRE QUESTION ON THE DIAGRAM SHEET PROVIDED.
The phase plot represents a predator-prey relationship between two species, according to the Lotka-Volterra model.

PREDATOR-PREY LOTKA-VOLTERRA MODEL
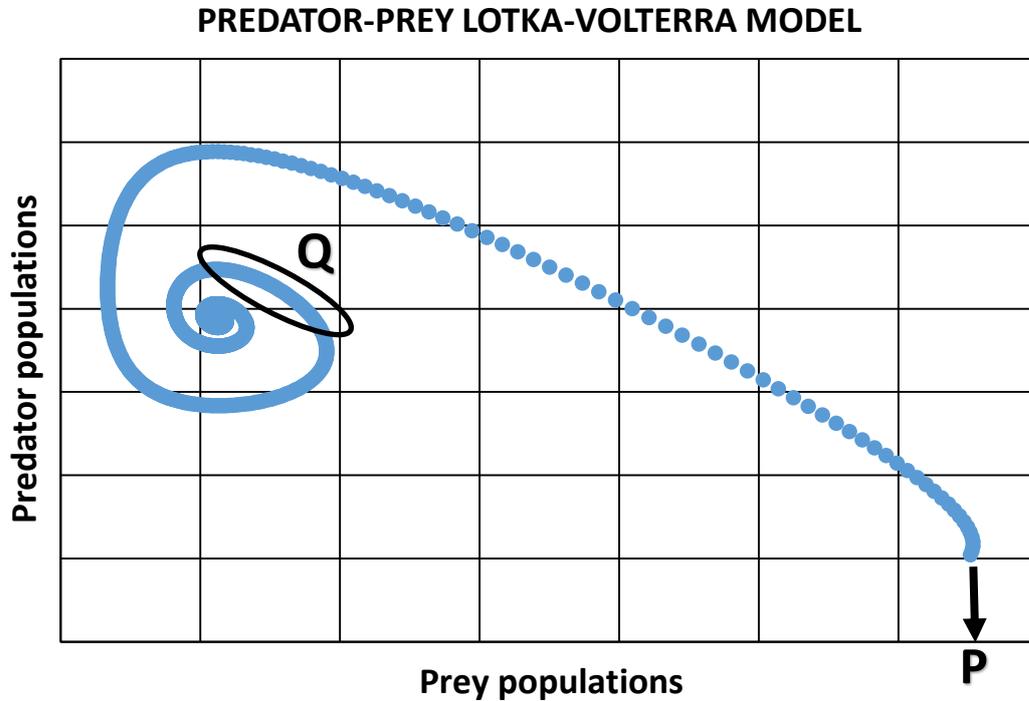
5.1 The letter **P** and the arrow indicate where **on the axes** to read off the initial population of the prey. Do the following in a similar manner:

    (a) Indicate with arrows and the letter **A**, where on the axes to read off the equilibrium population of each species. (4)

    (b) Indicate with an arrow and the letter **B**, where on the axis to read off the maximum population of the predators. (2)

5.2 The letter **Q** and the encircled region indicate where **on the phase plot** a decreasing prey population and an increasing predator population occurs for the second time. Do the following in a similar manner:

    (a) Indicate with an encircled region and the letter **C**, where on the phase plot the predator population is decreasing most rapidly for the first time. (2)

    (b) Indicate with an encircled region and the letter **D**, where on the phase plot the change in the populations is the greatest from one time period to the next. (2)

5.3 Use dotted lines and draw in the axes that divide the phase plot into the four quadrants that indicate the different ways in which the populations increase or decrease. (4)

[14]

QUESTION 6

At the start of the new year, Collin makes a once-off deposit of R20 000 into an account that earns 4,8% interest per annum, compounded monthly. He is determined to make further deposits at the end of each month for the next year into the same account. His first monthly deposit will be R400, and thereafter he will increase each monthly deposit by 0,5% of the previous month's deposit.

6.1   Calculate the balance in Collin's account at the end of each of the first three months, just after his monthly deposit has been made. (8)

6.2   Design a recursive formula that will determine the balance in Collin's account at the end of each month, just after his monthly deposit has been made. (6)

[14]

Total for Module 3: 100 marks

MODULE 4   MATRICES AND GRAPH THEORY

QUESTION 1

1.1   It is given that matrix $A = \begin{pmatrix} -5 & 4 \\ 3 & -1 \end{pmatrix}$

Write down A^{-1}, the inverse of A, with integer elements in the matrix.   (4)

1.2   Given matrices $C = \begin{pmatrix} 2 & 6 & 3 \\ 1 & y & 4 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 4 & z \\ 1 & 1 & -2 \end{pmatrix}$

Calculate the elements $x,\ y$ and z if $C - 3D = \begin{pmatrix} 8 & -6 & 12 \\ x & 0 & 10 \end{pmatrix}$ (6)

1.3   Given $det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = k$, express the determinant of the following in terms of k:

(a) $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

(b) $\begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$

(c) $\begin{pmatrix} a & b & c \\ d & e & f \\ -3g & -3h & -3i \end{pmatrix}$

(d) $\begin{pmatrix} a-d & b-e & c-f \\ d & e & f \\ g & h & i \end{pmatrix}$ (8)

[18]

QUESTION 2

2.1   In the sketch below, Figure T has been translated to form Figure T' (not shown on the sketch), which has then been enlarged to create Figure T".
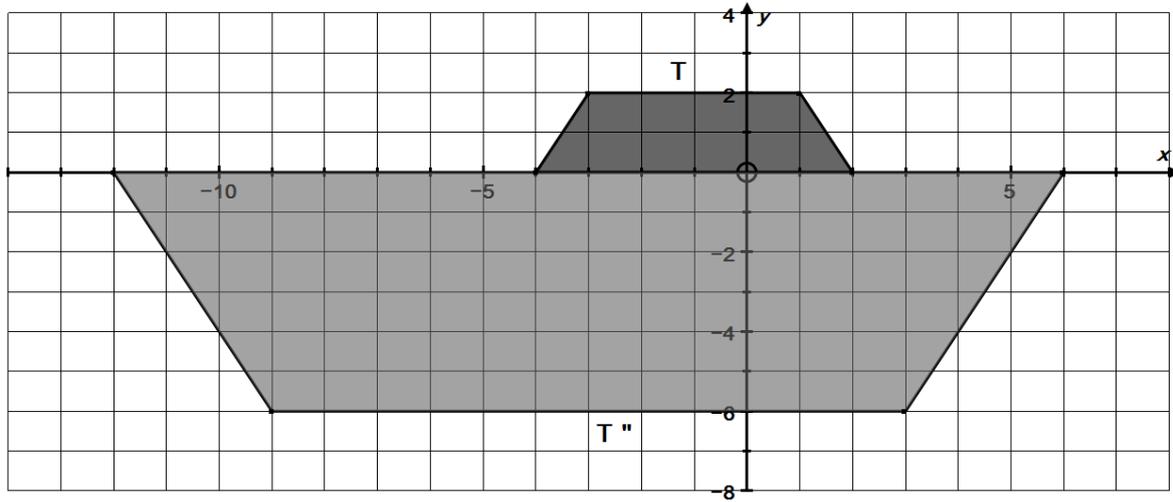
(a)   Describe the translation of Figure T to Figure T' in words.   (2)

(b)   Give the factor by which Figure T' was enlarged to create Figure T".   (2)

2.2   The point (3; –2) is mapped onto (3,232; –1,598) by a reflection across the line $y = mx$. Determine the inclination of the line of reflection.   (12)

2.3   A vertical line segment with endpoints *(t; v)* and *(t, r)* is sheared parallel to the *x*-axis by a factor of *k*.

(a)   Calculate the coordinates of the sheared endpoints in terms of the variables *k, t, v,* and *r*.   (6)

(b)   Express as an equation the relationship between *m* (the gradient of the line segment after it has been sheared) and *k*.   (4)

[26]

QUESTION 3

Consider the matrices $M = \begin{pmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{pmatrix}$ and $N = \begin{pmatrix} -192 & -100 & -64 & -60 \\ -32 & 4 & 72 & 52 \\ 42 & 49 & 14 & 17 \\ 6 & 7 & 2 & -33 \end{pmatrix}$

3.1 Explain in words why we should use the second row or the fourth column of *M*, rather than the first row, to calculate the determinant of *M*. (2)

3.2 Hence, or otherwise, show through calculation that the determinant of M is −248. (6)

3.3 It is now given that *N* is the matrix of minors of *M*. Hence write down the inverse of *M*. (6)

[14]

QUESTION 4

4.1 A complete graph has *n* vertices. State in terms of *n*:

(a) the number of edges present in a spanning tree of the graph. (2)

(b) the number of edges present in the graph. (2)

(c) the sum of the degrees of the vertices of the graph. (2)

4.2 Answer the questions with reference to the sketch.
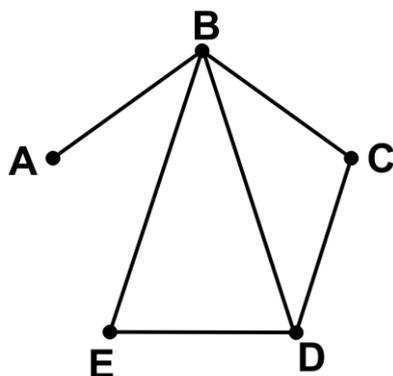
(a) Design an Eulerian path on the graph. Clearly state the order of the edges chosen, as well as the start and end vertices. (4)

(b) Draw the complement of the graph. Label the vertices carefully. (4)

[14]

QUESTION 5

Four graphs with the same order and size are given below.
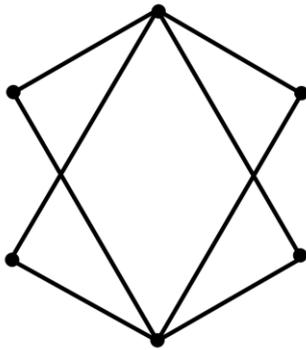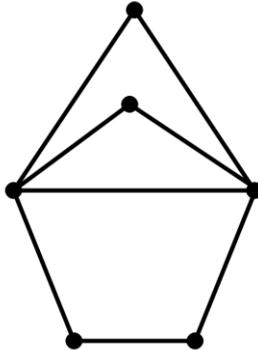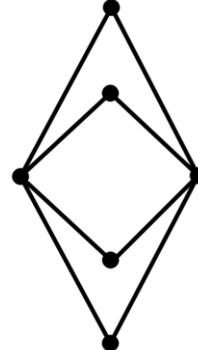
GRAPH A   GRAPH B   GRAPH C   GRAPH D
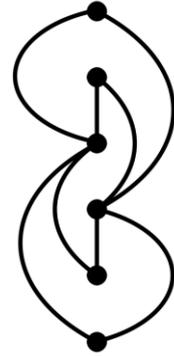
5.1   None of these graphs are regular. Briefly explain why.   (2)

5.2   State which graphs (if any) have Hamiltonian circuits.   (2)

5.3   List the graphs that are isomorphic to each other.   (4)

[8]

QUESTION 6

A graph is represented by the following adjacency matrix:

	A	B	C	D	E	F	G	H	J
A		5				4		8	6
B	5								3
C								9	5
D					8	10			7
E				8					6
F	4			10			7		
G						7		4	
H	8		9				4		
J	6	3	5	7	6				

6.1   Use the principles of Kruskal's Algorithm to find the spanning tree of **maximum** length. Clearly state the order in which you choose the edges, as well as the length of the tree.   (10)

6.2   **ANSWER THIS QUESTION ON THE DIAGRAM SHEET PROVIDED.**
        Use Dijkstra's Algorithm to find the **shortest** route from vertex E to vertex G. Show clear evidence of your working, e.g. a tree diagram, table record, in-out box, etc. Be sure to state your final route, as well as its length.   (10)

[20]

Total for Module 4: 100 marks