Advanced Quantitative Research Methodology, Lecture Notes: Models for Time Varying Events Data[1]

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Case-Control Data Collection Designs

Readings:

1. King, Gary and Langche Zeng. "Logistic Regression in Rare Events Data," *Political Analysis*, Vol. 9, No. 2 (Spring, 2001): Pp. 137–163.
2. King, Gary and Langche Zeng. "Explaining Rare Events in International Relations," *International Organization*, Vol. 55, No. 3 (Summer, 2001): Pp. 693–715. [a less technical version of the PA article.]
3. King, Gary and Langche Zeng. 2001. "Estimating Risk and Rate Levels, Ratios, and Differences in Case-Control Studies," *Statistics in Medicine*, in press.
4. Tomz, Michael; Gary King; and Langche Zeng. *ReLogit: Rare Events Logistic Regression software*. for Gauss and Stata.
5. Copies of all are at `http://GKing.Harvard.edu`.

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   (f) Think of it as *the rate of event occurrence*.

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Common practice: model the hazard rate directly and compute the density and then log-likelihood function from it.

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6. Censoring could be added as described previously.

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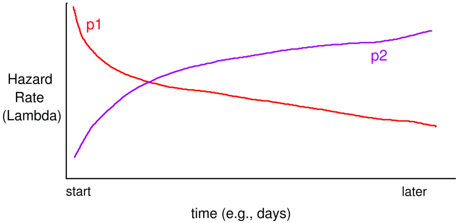

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Then substitute in for the systematic component $\lambda_i = e^{-X_i\beta}$ and take logs to get the log-likelihood.

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5. Cox also devised an estimation strategy that made estimating the baseline hazard (the constant term) unnecessary. We will discuss this shortly.

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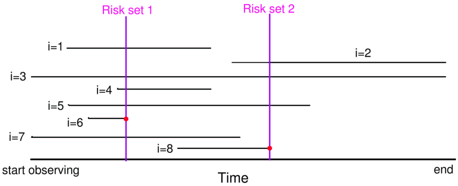

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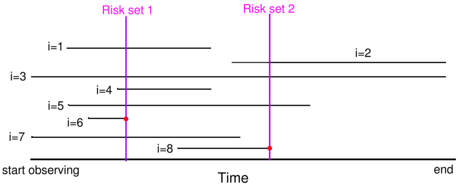

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   (c) For each unit within a risk set, compute the probability of being an event, *conditional* on the total number of events in that risk set.

(d) To do this, we use almost the same logic as Chamberlain's clogit or multinomial logit (see notes Part 3). Suppose we have exactly 1 event in each risk set: $\sum_i Y_{it} = 1$, where $i = 1, \ldots, n_t$ indexes events in the risk set, and t indexes risk sets. This is typical, since events happen at discrete points in a continuum of time.

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10. When a risk set includes multiple cases, because of timing ties, the conditional probability expression is more complicated, but the approach remains the same.