Common Core Georgia Performance Standards

Analytic Geometry

Student Resource Book
Unit 2

WALCH EDUCATION

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WALCH EDUCATION

Table of Contents

Introduction

Welcome to the *CCGPS Analytic Geometry Student Resource Book.* This book will help you learn how to use algebra, geometry, data analysis, and probability to solve problems. Each lesson builds on what you have already learned. As you participate in classroom activities and use this book, you will master important concepts that will help to prepare you for the EOCT and for other mathematics assessments and courses.

This book is your resource as you work your way through the Analytic Geometry course. It includes explanations of the concepts you will learn in class; math vocabulary and definitions; formulas and rules; and exercises so you can practice the math you are learning. Most of your assignments will come from your teacher, but this book will allow you to review what was covered in class, including terms, formulas, and procedures.

- In **Unit 1: Similarity, Congruence, and Proofs**, you will learn about dilations, and you will construct lines, segments, angles, polygons, and triangles. You will explore congruence and then define, apply, and prove similarity. Finally, you will prove theorems about lines, angles, triangles, and parallelograms.
- In **Unit 2: Right Triangle Trigonometry**, you will begin by exploring trigonometric ratios. Then you will go on to apply trigonometric ratios.
- In **Unit 3: Circles and Volume**, you will be introduced to circles and their angles and tangents. Then you will learn about inscribed polygons and circumscribed triangles by constructing them and proving properties of inscribed quadrilaterals. You will construct tangent lines and find arc lengths and areas of sectors. Finally, you will explain and apply area and volume formulas.
- In **Unit 4: Extending the Number System**, you will start working with the number system and rational exponents. Then you will perform operations with complex numbers and polynomials.
- In **Unit 5: Quadratic Functions**, you will begin by identifying and interpreting structures in expressions. You will use this information as you learn to create and solve quadratic equations in one variable, including taking the square root of both sides, factoring, completing the square, applying the quadratic formula, and solving quadratic inequalities. You will move on to solving quadratic equations in two or more variables, and solving systems

of equations. You will learn to analyze quadratic functions and to build and transform them. Finally, you will solve problems by fitting quadratic functions to data.

- In **Unit 6: Modeling Geometry**, you will study the links between the two math disciplines, geometry and algebra, as you derive equations of a circle and a parabola. You will use coordinates to prove geometric theorems about circles and parabolas and solve systems of linear equations and circles.
- In **Unit 7: Applications of Probability**, you will explore the idea of events, including independent events, and conditional probability.

Each lesson is made up of short sections that explain important concepts, including some completed examples. Each of these sections is followed by a few problems to help you practice what you have learned. The "Words to Know" section at the beginning of each lesson includes important terms introduced in that lesson.

As you move through your Analytic Geometry course, you will become a more confident and skilled mathematician. We hope this book will serve as a useful resource as you learn.

Lesson 1: Exploring Trigonometric Ratios

Common Core Georgia Performance Standards

MCC9–12.G.SRT.6

MCC9–12.G.SRT.7

Essential Questions

1. How are the properties of similar triangles used to create trigonometric ratios?
2. What are the relationships between the sides and angles of right triangles?
3. On what variable inputs are the ratios in trigonometry dependent?
4. What is the relationship between the sine and cosine ratios for the two acute angles in a right triangle?

WORDS TO KNOW

adjacent side the leg next to an acute angle in a right triangle that is not the hypotenuse

cofunction a trigonometric function whose ratios have the same values when applied to the two acute angles in the same right triangle. The sine of one acute angle is the cofunction of the cosine of the other acute angle.

complementary angles two angles whose sum is 90°

cosecant the reciprocal of the sine ratio; the cosecant of $\theta = \csc\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$

cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

cotangent	the reciprocal of tangent; the cotangent of θ = cot θ = $\frac{\text{length of adjacent side}}{\text{length of opposite side}}$
hypotenuse	the side opposite the vertex of the 90° angle in a right triangle
identity	an equation that is true regardless of what values are chosen for the variables
opposite side	the side across from an angle
phi (ϕ)	a Greek letter sometimes used to refer to an unknown angle measure
ratio	the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage
reciprocal	a number that, when multiplied by the original number, has a product of 1
right triangle	a triangle with one angle that measures 90°
scale factor	a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.
secant	the reciprocal of cosine; the secant of θ = sec θ = $\frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$
similar	two figures that are the same shape but not necessarily the same size. Corresponding angles must be congruent and sides must have the same ratio. The symbol for representing similarity is $\sim$.
sine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of θ = sin θ = $\frac{\text{length of opposite side}}{\text{length of hypotenuse}}$

tangent	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta =$ $\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$
theta **(θ)**	a Greek letter commonly used to refer to unknown angle measures
trigonometry	the study of triangles and the relationships between their sides and the angles between these sides

Recommended Resources

- AJ Design Software. "Triangle Equations Formulas Calculator."

  http://www.walch.com/rr/00041

  This excellent website provides links to interactive calculators that help users solve for the various attributes of different types of triangles. For the Pythagorean Theorem, users can plug in known values for the legs of a right triangle, and the site will calculate the length of the hypotenuse.

- Keisan: High Accuracy Calculation. "Solar elevation angle (for a day)."

  http://www.walch.com/rr/00042

  This website features a solar calculator that will return the angle of elevation of the sun for any time of day at any latitude. The results are shown in a table and a graph.

- MathisFun.com. "Sine, Cosine and Tangent."

  http://www.walch.com/rr/00043

  Scroll down to the bottom of this site to find a tool for changing an angle and seeing the three main ratios of trigonometry change.

- Teach Engineering. "Hands-on Activity: Stay in Shape."

  http://www.walch.com/rr/00044

  This site provides a real-life example of how the equations in basic geometry and trigonometry are used in navigation.

Lesson 2.1.1: Defining Trigonometric Ratios

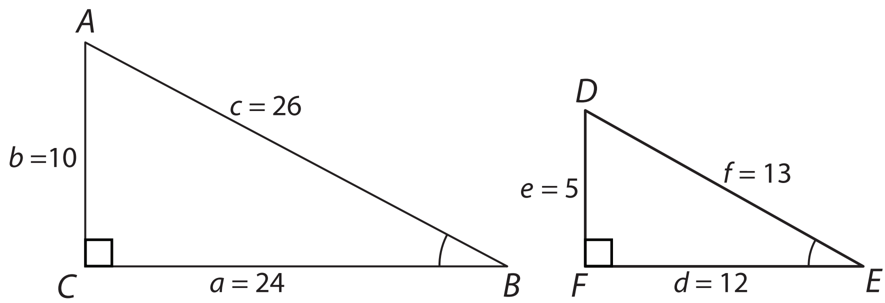
Introduction

Navigators and surveyors use the properties of similar right triangles. Designers and builders use right triangles in constructing structures and objects. Cell phones and Global Positioning Systems (GPS) use the mathematical principles of algebra, geometry, and trigonometry. **Trigonometry** is the study of triangles and the relationships between their sides and the angles between these sides. In this lesson, we will learn about the ratios between angles and side lengths in right triangles. A **ratio** is the relation between two quantities; it can be expressed in words, fractions, decimals, or as a percentage.

Key Concepts

- Two triangles are similar if they have congruent angles.
- Remember that two figures are **similar** when they are the same shape but not necessarily the same size; the symbol for representing similarity is $\sim$.
- Recall that the **hypotenuse** is the side opposite the vertex of the 90° angle in a right triangle. Every right triangle has one 90° angle.
- If two right triangles each have a second angle that is congruent with the other, the two triangles are similar.
- Similar triangles have proportional side lengths. The side lengths are related to each other by a scale factor.
- Examine the proportional relationships between similar triangles $\triangle ABC$ and $\triangle DEF$ in the diagram that follows. The scale factor is $k = 2$. Notice how the ratios of corresponding side lengths are the same as the scale factor.

Proportional Relationships in Similar Triangles



Corresponding sides

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

Side lengths

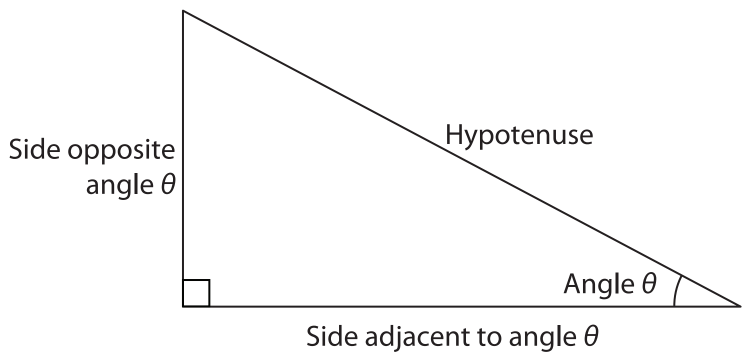
$$\frac{24}{12} = \frac{10}{5} = \frac{26}{13}$$

Examine the three ratios of side lengths in $\triangle ABC$. Notice how these ratios are equal to the same ratios in $\triangle DEF$.

Corresponding sides	Side lengths
$\frac{a}{c} = \frac{d}{f}$	$\frac{24}{26} = \frac{12}{13}$
$\frac{b}{c} = \frac{e}{f}$	$\frac{10}{26} = \frac{5}{13}$
$\frac{a}{b} = \frac{d}{e}$	$\frac{24}{10} = \frac{12}{5}$

- The ratio of the lengths of two sides of a triangle is the same as the ratio of the corresponding sides of any similar triangle.
- The three main ratios in a right triangle are the sine, the cosine, and the tangent. These ratios are based on the side lengths relative to one of the acute angles.
- The **sine** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin \theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- The **cosine** of an acute angle in a right triangle is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

- The **tangent** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of θ = tan θ = $\dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- The acute angle that is being used for the ratio can be called the angle of interest. It is commonly marked with the symbol θ (*theta*).
- ***Theta*** (θ) is a Greek letter commonly used as an unknown angle measure.



- See the following examples of the ratios for sine, cosine, and tangent.

sine of θ = sin θ = $\dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$ Abbreviation: $\dfrac{\text{opposite}}{\text{hypotenuse}}$

cosine of θ = cos θ = $\dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ Abbreviation: $\dfrac{\text{adjacent}}{\text{hypotenuse}}$

tangent of θ = tan θ = $\dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$ Abbreviation: $\dfrac{\text{opposite}}{\text{adjacent}}$

- Unknown angle measures can also be written using the Greek letter ***phi*** **(ϕ)**.
- The three main ratios can also be shown as reciprocals.
- The **reciprocal** is a number that when multiplied by the original number the product is 1.

- The reciprocal of sine is **cosecant**. The reciprocal of cosine is **secant**, and the reciprocal of tangent is **cotangent**.

  $$\text{cosecant of } \theta = \csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$$

  $$\text{secant of } \theta = \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$$

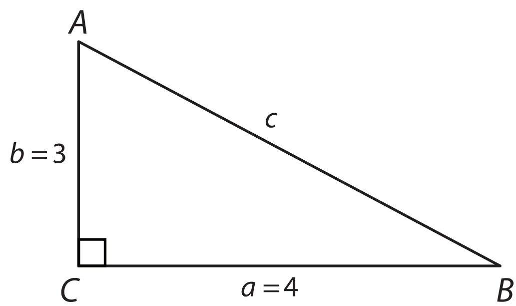
  $$\text{cotangent of } \theta = \cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$

- Each acute angle in a right triangle has different ratios of sine, cosine, and tangent.
- The length of the hypotenuse remains the same, but the sides that are opposite or adjacent for each acute angle will be different for different angles of interest.
- The two rays of each acute angle in a right triangle are made up of a leg and the hypotenuse. The leg is called the **adjacent side** to the angle. Adjacent means "next to."
- In a right triangle, the side of the triangle opposite the angle of interest is called the **opposite side**.
- Calculations in trigonometry will vary due to the variations that come from measuring angles and distances.
- A final calculation in trigonometry is frequently expressed as a decimal.
- A calculation can be made more accurate by including more decimal places.
- The context of the problem will determine the number of decimals places to which to round. Examples:
    - A surveyor usually measures tracts of land to the nearest tenth of a foot.
    - A computer manufacturer needs to measure a microchip component to a size smaller than an atom.
    - A carpenter often measures angles in whole degrees.
    - An astronomer measures angles to $\frac{1}{3600}$ of a degree or smaller.

Guided Practice 2.1.1

Example 1

Find the sine, cosine, and tangent ratios for $\angle A$ and $\angle B$ in $\triangle ABC$. Convert the ratios to decimal equivalents.



1. Find the length of the hypotenuse using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$4^2 + 3^2 = c^2$	Substitute values for *a* and *b*.
$16 + 9 = c^2$	Simplify.
$25 = c^2$	
$\pm\sqrt{25} = \sqrt{c^2}$	
$c = \pm 5$	

Since c is a length, use the positive value, $c = 5$.

2. Find the sine, cosine, and tangent of $\angle A$.

Set up the ratios using the lengths of the sides and hypotenuse, then convert to decimal form.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} = 1.33\overline{3}$$

3. Find the sine, cosine, and tangent of $\angle B$.

Set up the ratios using the lengths of the sides and the hypotenuse, and then convert to decimal form.

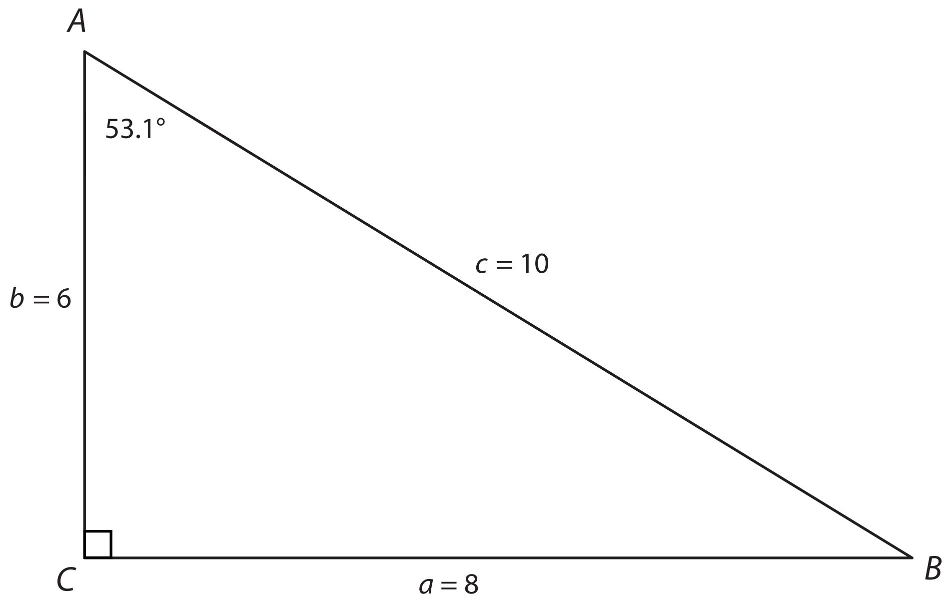
$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

Example 2

Given the triangle below, set up the three trigonometric ratios of sine, cosine, and tangent for the angle given. Compare these ratios to the trigonometric functions using your calculator.



1. Set up the ratios for sin A, cos A, and tan A and calculate the decimal equivalents.

$$\sin 53.1^\circ = \frac{8}{10} = 0.8$$

$$\cos 53.1^\circ = \frac{6}{10} = 0.6$$

$$\tan 53.1^\circ = \frac{8}{6} = 1.33\overline{3}$$

2. Use your graphing calculator to find sin 53.1°.

On a TI-83/84:

First, make sure your calculator is in Degree mode.

Step 1: Press [MODE].

Step 2: Arrow down to RADIAN.

Step 3: Arrow over to DEGREE.

Step 4: Press [ENTER]. The word DEGREE should be highlighted inside a black rectangle.

Step 5: Press [2ND].

Step 6: Press [MODE] to QUIT.

Important: You will not have to change to Degree mode again unless you have changed your calculator to Radian mode.

Now you can find the value of sin 53.1°:

Step 1: Press [SIN].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

First, make sure the calculator is in Degree mode.

Step 1: From the home screen, select 5: Settings & Status, then 2: Settings, then 2: Graphs & Geometry.

Step 2: Press [tab] twice to move to the Geometry Angle field, and then select Degree from the drop-down menu.

Step 3: Press [tab] to move to the "OK" option and select it using the center button on the navigation pad.

Next, calculate the sine of 53.1°.

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "sin," then type 53.1 and press [enter].

$\sin 53.1° \approx 0.7996846585$

3. Use your graphing calculator to find cos 53.1°.

On a TI-83/84:

Step 1: Press [COS].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "cos," then type 53.1 and press [enter].

$\cos 53.1^\circ \approx 0.6004202253$

4. Use your graphing calculator to find tan 53.1°.

On a TI-83/84:

Step 1: Press [TAN].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "tan," then type 53.1 and press [enter].

$\tan 53.1^\circ \approx 1.331874952$

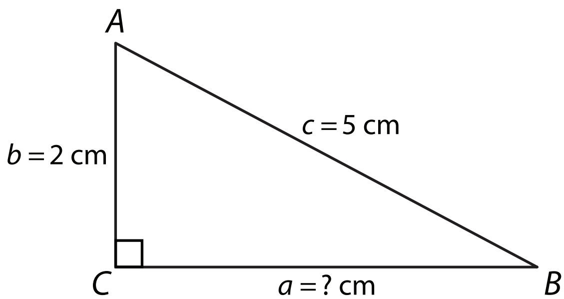
5. Compare the calculator values to the ratios. Explain the difference.

The values are very close to the ratios of the side lengths. The differences are due to the angle measurement of 53.1° being an approximation.

Example 3

A right triangle has a hypotenuse of 5 and a side length of 2. Find the angle measurements and the unknown side length. Find the sine, cosine, and tangent for both angles. Without drawing another triangle, compare the trigonometric ratios of $\triangle ABC$ with those of a triangle that has been dilated by a factor of $k = 3$.

1. First, draw the triangle with a ruler, and label the side lengths and angles.



2. Find a by using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Substitute values for b and c.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	
$a \approx 4.5826$ centimeters	

3. Use a protractor to measure one of the acute angles, and then use that measurement to find the other acute angle.

$m\angle A \approx 66.5$

We know that $m\angle C = 90$ by the definition of right angles.

The measures of the angles of a triangle sum to 180.

Subtract $m\angle A$ and $m\angle C$ from 180 to find $m\angle B$.

$m\angle B = 180 - m\angle A - m\angle C$

$m\angle B = 180 - (66.5) - (90)$

$m\angle B = 180 - 156.5$

$m\angle B \approx 23.5$

4. Find the sine, cosine, and tangent ratios for both acute angles. Express your answer in decimal form to the nearest thousandth.

$\angle A$	$\angle B$
$\sin 66.5° \approx \frac{4.5826}{5} \approx 0.916$	$\sin 23.5° \approx \frac{2}{5} \approx 0.4$
$\cos 66.5° \approx \frac{2}{5} \approx 0.4$	$\cos 23.5° \approx \frac{4.5826}{5} \approx 0.916$
$\tan 66.5° \approx \frac{4.5826}{2} \approx 2.291$	$\tan 23.5° \approx \frac{2}{4.5826} \approx 0.436$

5. Without drawing a triangle, find the sine, cosine, and tangent for a triangle that has a scale factor of 3 to $\triangle ABC$. Compare the trigonometric ratios for the two triangles.

Multiply each side length (a, b, and c) by 3 to find a', b', and c'.

$a' = 3 \bullet a = 3 \bullet (4.5826) = 13.7478$

$b' = 3 \bullet b = 3 \bullet (2) = 6$

$c' = 3 \bullet c = 3 \bullet (5) = 15$

Set up the ratios using the side lengths of the dilated triangle.

$$\sin 66.5^\circ \approx \frac{13.7478}{15} \approx 0.916 \qquad \sin 23.5^\circ \approx \frac{6}{15} \approx 0.4$$

$$\cos 66.5^\circ \approx \frac{6}{15} \approx 0.4 \qquad \cos 23.5^\circ \approx \frac{13.7478}{15} \approx 0.916$$

$$\tan 66.5^\circ \approx \frac{13.7478}{6} \approx 2.291 \qquad \tan 23.5^\circ \approx \frac{6}{13.7478} \approx 0.436$$

The sine, cosine, and tangent do not change in the larger triangle. Similar triangles have identical side length ratios and, therefore, identical trigonometric ratios.

Example 4

What are the secant (sec), cosecant (csc), and cotangent (cot) ratios for an isosceles right triangle?

1. An isosceles right triangle has two angles that measure 45°. The two side lengths have equal lengths of 1 unit.

2. Use the Pythagorean Theorem to find the hypotenuse.

$$1^2 + 1^2 = c^2$$

$$1 + 1 = c^2$$

$$2 = c^2$$

$$\sqrt{2} = c$$

3. Substitute the side lengths into the ratios.

$$\csc 45° = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{2}}{1} \approx 1.414$$

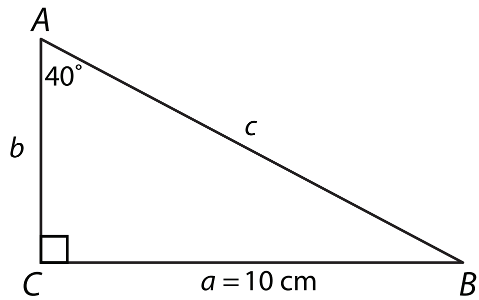
$$\sec 45° = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{2}}{1} \approx 1.414$$

$$\cot 45° = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

Example 5

Triangle ABC is a right triangle where $m\angle A = 40°$ and side a = 10 centimeters. What is the sine of $\angle A$? Check your work with the sin function on your graphing calculator.

1. Draw $\triangle ABC$ with a ruler and protractor.



2. Measure the length of hypotenuse c.

   $c \approx 15.6$ centimeters

   Substitute the side lengths into the ratios to determine the sine of $\angle A$.

   $$\sin 40° \approx \frac{10}{15.6} \approx 0.641$$

3. Use your calculator to check the answer.

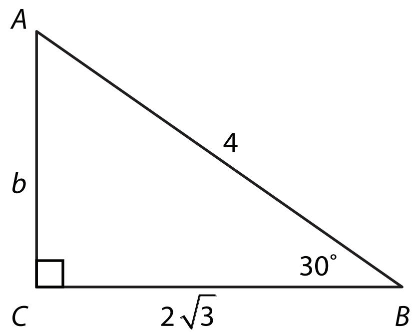
   Follow the steps outlined in Example 2.

   $\sin 40° \approx 0.6427876097$

   The two answers are fairly close. The difference is due to the imprecise nature of manually drawing and measuring a triangle.

Practice 2.1.1: Defining Trigonometric Ratios

Use $\triangle ABC$ to complete problems 1–4. Round your answers to the nearest thousandths.



1. Set up and calculate the trigonometric ratios for the sine, cosine, and tangent of $\angle A$.

2. Set up and calculate the trigonometric ratios for the cosecant, secant, and cotangent of $\angle A$.

3. Set up and calculate the trigonometric ratios for the sine, cosine, and tangent of $\angle B$.

continued

4. Set up and calculate the trigonometric ratios for cosecant, secant, and cotangent of $\angle B$.

Draw triangles using the given information to complete problems 5 and 6.

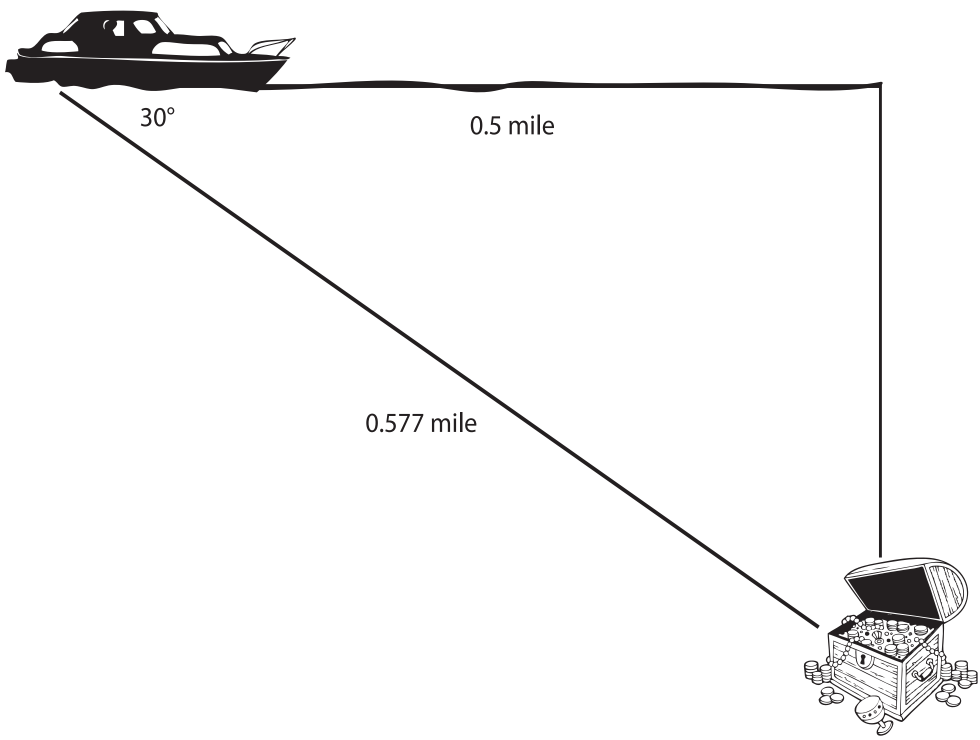
5. A right triangle has side lengths of $a = 3$ and $b = 4$. Find the angle measurements and the length of the hypotenuse. Then, find the sine, cosine, and tangent for both angles.

6. A right triangle has a side length of $a = 5$ and a hypotenuse of 6. Find the angle measurements and the other side length. Then, find the sine, cosine, and tangent for both acute angles.

continued

Use the given information to complete problems 7–10.

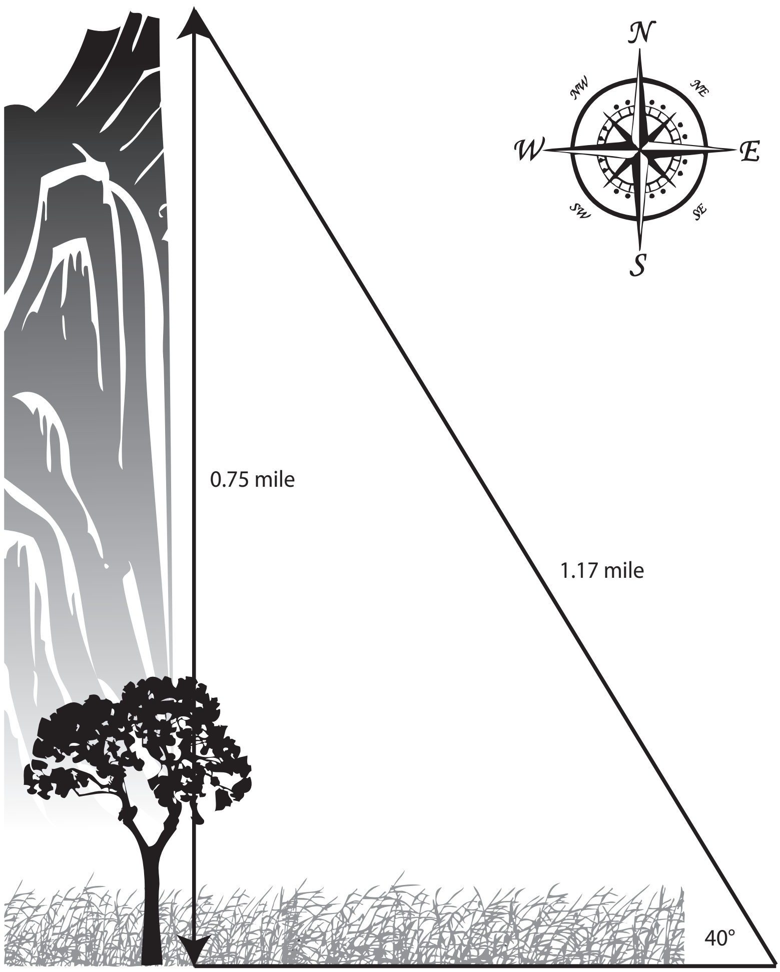
7. A ship's sonar radio tracking system detected a mysterious object on the ocean floor. The object is straight ahead of the ship and at a 30° angle beneath the ocean's surface. The distance from the ship to the object is 0.577 mile. The ship moves 0.5 mile toward the object. The object is now directly beneath the ship. How deep is the water? Use your graphing calculator to find the tangent of 30°.



8. Students are planning a flower garden around the perimeter of the triangular base of a statue. They know the following information: The statue's base is a right triangle; the lengths of the two sides are each 5.657 meters; and both of the acute angles are 45°. Find the hypotenuse of the triangular base. What is the perimeter of the base of the statue? Use your graphing calculator to find the cosine of 45°.

continued

9. A surveyor is mapping a large marsh in a state park. He walks and measures a straight line south from a cliff for 0.75 mile to a large tree. He walks east, but he has to make a detour around the marsh. He gets around the marsh to a spot where he is perpendicular to the large tree and the trail he originally walked south on. He looks northwest and measures a 40° angle northwest to the cliff from his line of sight to the large tree. He measures the distance back to the cliff as 1.17 miles. What is the straight-line distance from the large tree east to $\angle A$? What is the tangent of $\angle B$?



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10. A ramp needs to be built to the front door of an office building. The ramp will start 40 feet from the door and rise 5 feet. The ramp will rise at angle of 7.2°. How long will the ramp be? Use your calculator to find the sine of 7.2°.

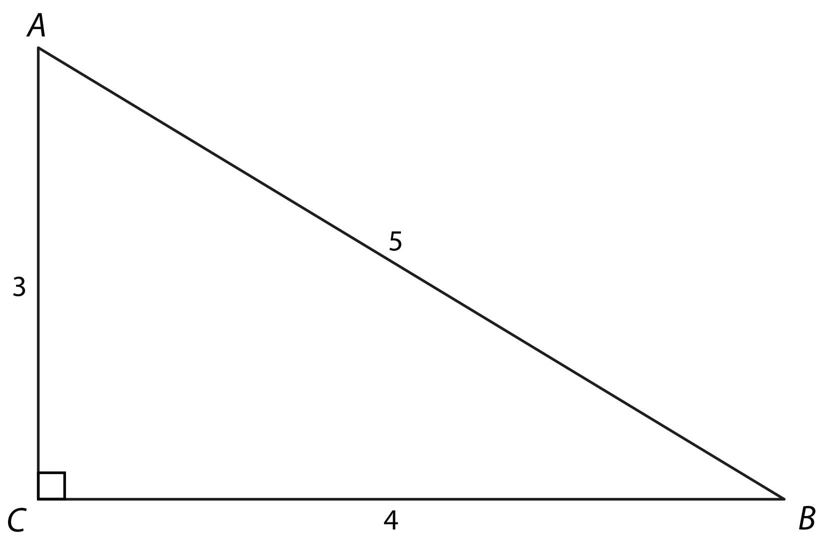
Lesson 2.1.2: Exploring Sine and Cosine As Complements

Introduction

In the previous lesson, we applied the properties of similar triangles to find unknown side lengths. We discovered that the side ratios of similar triangles are always the same. As a preparation to using trigonometry to solve problems, we will look more deeply into the relationship between sine and cosine in this lesson.

Key Concepts

- Sine and cosine are side length ratios in right triangles.
- The ratio for the sine of an angle is as follows: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$.
- The ratio for the cosine of an angle is as follows: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.
- Examine $\triangle ABC$.



- Determine the sine of $\angle A$.

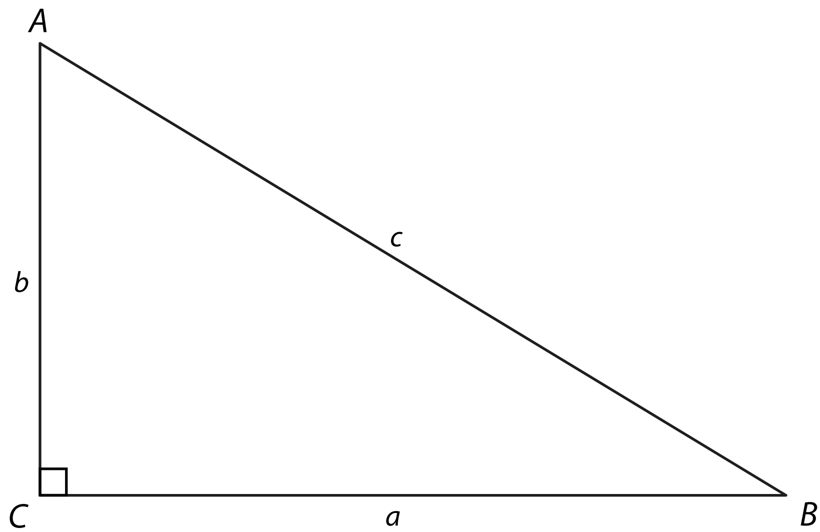
$$\sin A = \frac{4}{5}$$

- Determine the cosine of $\angle B$.

$$\cos B = \frac{4}{5}$$

- This shows $\sin A = \cos B$.

- You can also see from the diagram that $\sin B = \frac{3}{5} = \cos A$.
- Show that this relationship will work for any right triangle.



$$\sin A = \frac{a}{c} = \cos B \qquad \sin B = \frac{b}{c} = \cos A$$

- In $\triangle ABC$, $\sin A = \cos B$, and $\sin B = \cos A$.
- This relationship between sine and cosine is known as an identity. An equation is an **identity** if it is true for every value that is used in the equation.
- Sine and cosine are called **cofunctions** because the value of one ratio for one angle is the same as the value of the other ratio for the other angle.
- The two acute angles in a right triangle have a sum of 90°. They are **complementary angles**. If one acute angle has a measure of x, the other angle has a measure of $90° - x$.
- For example, if one acute angle x has a measure of 70°, the other acute angle must measure $90 - x$.

  $90 - x = 20$ or $20°$

- The sine-cosine cofunction can be written as:

  $\sin \theta = \cos (90° - \theta)$

  $\cos \theta = \sin (90° - \theta)$

- In other words, you can use the sine of one acute angle to find the cosine of its complementary angle.
- Also, you can use the cosine of one acute angle to find the sine of its complementary angle.
- This identity relationship makes sense because the same side lengths are being used in the ratios for the different angles.
- Cofunctions such as sine-cosine give you flexibility in solving problems, particularly if several ratios of trigonometry are used in the same problem.

Postulate
Sine and cosine are cofunction identities. $\sin \theta = \cos (90° - \theta)$ $\cos \theta = \sin (90° - \theta)$

Guided Practice 2.1.2

Example 1

Find sin 28° if cos 62° $\approx$ 0.469.

1. Set up the identity.

   $\sin\theta = \cos(90° - \theta)$

2. Substitute the values of the angles into the identity and simplify.

   $\sin 28° = \cos(90° - 28°)$

   $\sin 28° = \cos 62°$

3. Verify the identity by calculating the sine of 28° and the cosine of 62° using a scientific calculator.

   $\sin 28° \approx 0.469$

   $\cos 62° \approx 0.469$

Example 2

Complete the table below using the sine and cosine identities.

Angle	Sine	Cosine
10°	0.174	0.985
80°		

1. Determine the relationship between the two given angles.

   10° and 80° are complementary angles.

   $10° + 80° = 90°$

2. Apply the sine identity.

   $\sin \theta = \cos (90° - \theta)$

   $\sin 10° = \cos 80°$

3. Use the given value of sin 10° from the table to find sin 80°.

   The sine of 10° = 0.174; therefore, the cosine of 80° = 0.174.

4. Apply the cosine identity.

   $\cos \theta = \sin (90° - \theta)$

   $\cos 10° = \sin 80°$

5. Use the given value of cos 10° from the table to find sin 80°.

   The cosine of 10° = 0.985; therefore, the sine of 80° = 0.985.

6. Fill in the table.

Angle	Sine	Cosine
10°	0.174	0.985
80°	0.985	0.174

Example 3

Find a value of θ for which $\sin \theta = \cos 15^\circ$ is true.

1. Determine which identity to use.

   The cosine was given, so use the cosine identity.

   Since θ is used as the variable in the problem, use the variable *phi* (ϕ) for the identity.

   $\cos \phi = \sin (90^\circ - \phi)$

   $\phi = 15^\circ$

   $\cos 15^\circ = \sin (90^\circ - 15^\circ)$

   The cosine of 15° is equal to the sine of its complement.

2. Find the complement of 15°.

   $90^\circ - 15^\circ = 75^\circ$

   The complement of 15° is 75°.

3. Substitute the complement of 15° into the identity.

   $\cos 15^\circ = \sin 75^\circ$ or $\sin 75^\circ = \cos 15^\circ$

4. Write the value of θ.

   $\theta = 75^\circ$

Example 4

Complete the table below using the sine and cosine identities.

Angle	Sine	Cosine
28°	0.470	
	0.883	0.470

1. Use a graphing calculator to find the cosine of 28°.

   cos 28° = 0.883

2. Analyze the two sets of values for the sine and cosine.

   The values are switched, suggesting that the second angle of interest is the complement of the first angle of interest, 28°.

   Find the complement of the first angle of interest.

   90° – 28° = 62°

   The complement of 28° is 62°.

   The second angle must be 62°. Therefore, sin 62° = cos 28°.

3. Verify that the second angle of interest is correct using a scientific calculator.

   sin 62° = 0.883

   cos 28° = 0.883

   The second angle of interest is correct.

4. Use your findings to fill in the table.

Angle	Sine	Cosine
28°	0.470	0.883
62°	0.883	0.470

Practice 2.1.2: Exploring Sine and Cosine As Complements

Use what you have learned about the sine-cosine identity relationship to complete the following problems.

1. $\cos 45^\circ \approx 0.707$. What is $\sin 45^\circ$?

2. $\sin 85^\circ \approx 0.996$. Find the cosine of the complementary angle.

3. Find a value of θ for which $\cos \theta = \sin 52^\circ$ is true.

4. Find a value of θ for which $\sin \theta = \cos 2^\circ$ is true.

5. $\cos 21^\circ \approx 0.934$. Use this information to write an equation for the sine.

6. $\sin 74^\circ \approx 0.961$. Use this information to write an equation for the cosine.

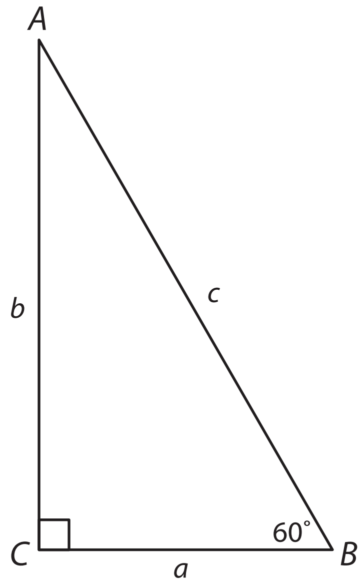
continued

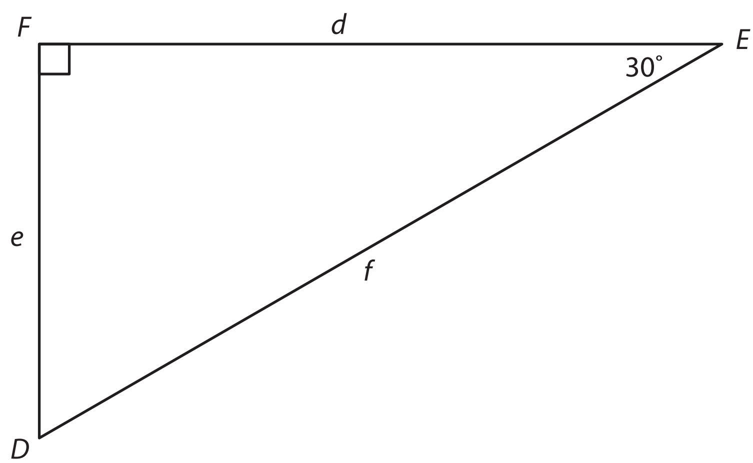
7. $\cos 30^\circ - \cos 50^\circ \approx 0.223$. Find θ if $\sin \theta - \sin 40^\circ \approx 0.223$.

8. Use the diagrams of $\triangle ABC$ and $\triangle DEF$ to fill in the empty boxes for the following equations.

$$\sin A = \frac{\square}{c} = \cos \square = \frac{\square}{\square}$$

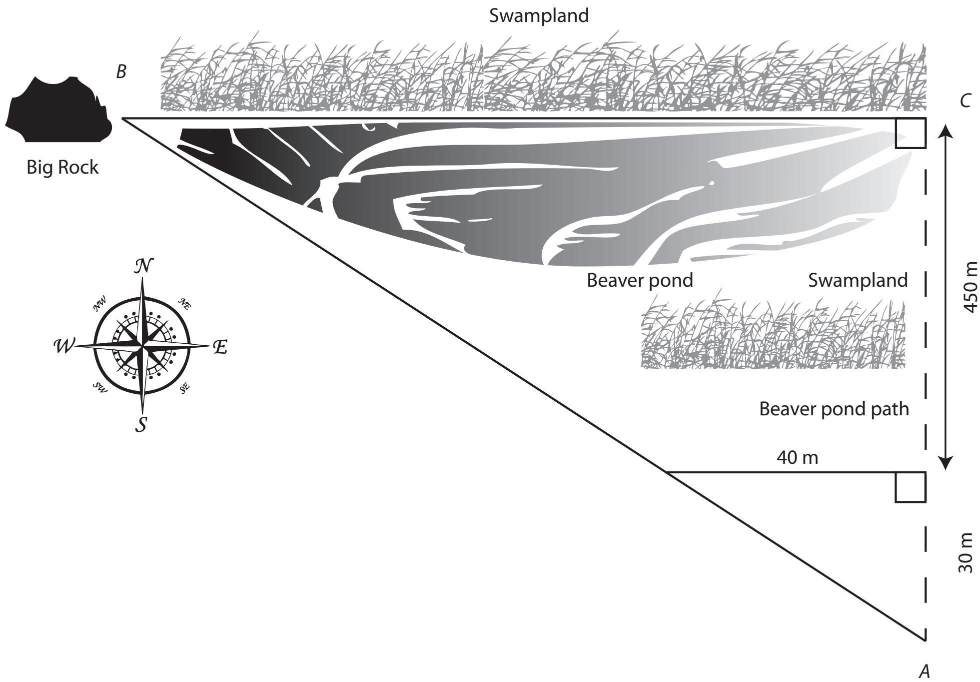
$$\sin B = \frac{\square}{c} = \cos \square = \frac{\square}{\square}$$





continued

9. Students in a biology class measured the length of a beaver pond to see if it is large enough to support other types of water-dependent mammals and birds. Most of the pond is surrounded by swampland, making it impractical to measure the pond directly. The following diagram shows the measurements the students were able to make. First, they put a two-meter-tall stake in the ground at $\angle A$. Next, they measured north on the walking path 30 meters, and west 40 meters to a spot that is in line with the stake and the Big Rock landmark on the west side of the pond. Going north on the path to the east end of the pond, they walked 450 meters to $\angle C$ using a rolling measuring wheel. Approximately how long is the pond?



10. Use the information you found in problem 9 to calculate the sine and cosine values for angles A and B.

Lesson 2: Applying Trigonometric Ratios

Common Core Georgia Performance Standard

MCC9–12.G.SRT.8

Essential Questions

1. How could you use sine, cosine, tangent, and the Pythagorean Theorem to find missing angles and side lengths of a right triangle?
2. How could you use cosecant, secant, cotangent, and the Pythagorean Theorem to find missing angles and side lengths of a right triangle?
3. How are the angle of depression and angle of elevation used to solve real-world right triangle problems?
4. How do you determine which trigonometric function to use when solving real-world triangle problems?

WORDS TO KNOW

altitude the perpendicular line from a vertex of a figure to its opposite side; height

angle of depression the angle created by a horizontal line and a downward line of sight to an object that is below the observer

angle of elevation the angle created by a horizontal line and an upward line of sight to an object that is above the observer

arccosine the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$

arcsine the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$

arctangent the inverse of the tangent function, written $\tan^{-1}\theta$ or $\arctan\theta$

cosecant the reciprocal of the sine ratio; $\csc\theta = \dfrac{1}{\sin\theta}$

cotangent the reciprocal of the tangent ratio; $\cot\theta = \frac{1}{\tan\theta}$

secant the reciprocal of the cosine ratio; $\sec\theta = \frac{1}{\cos\theta}$

Recommended Resources

- MathIsFun.com. "Sine, Cosine, and Tangent."

  http://www.walch.com/rr/00064

  This website gives a brief overview of three trigonometric functions and shows some applications of the uses of the ratios. Toward the bottom of the page, a manipulative allows the user to drag the vertex of a triangle to change the acute angle and observe the relationships between the legs of the triangle and the trigonometric ratios when the hypotenuse is one unit.

- Purplemath. "Angles of Elevation / Inclination and Angles of Depression / Declination."

  http://www.walch.com/rr/00045

  This website boasts a variety of resources regarding basic trigonometry. Specifically, the angles of depression and elevation links offer common misconceptions and additional tricks to mastering proper trigonometric equation setups.

- Wolfram MathWorld. "Trigonometry."

  http://www.walch.com/rr/00046

  This online math encyclopedia contains definitions for terms as well as visuals to support continued terminology mastery.

Lesson 2.2.1: Calculating Sine, Cosine, and Tangent

Introduction

In the real world, if you needed to verify the size of a television, you could get some measuring tools and hold them up to the television to determine that the TV was advertised at the correct size. Imagine, however, that you are a fact checker for *The Guinness Book of World Records*. It is your job to verify that the tallest building in the world is in fact Burj Khalifa, located in Dubai. Could you use measuring tools to determine the size of a building so large? It would be extremely difficult and impractical to do so. You can use measuring tools for direct measurements of distance, but you can use trigonometry to find indirect measurements. First, though, you must be able to calculate the three basic trigonometric functions that will be applied to finding those larger distances. Specifically, we are going study and practice calculating the sine, cosine, and tangent functions of right triangles as preparation for measuring indirect distances.

Key Concepts

- The three basic trigonometric ratios are ratios of the side lengths of a right triangle with respect to one of its acute angles.
- As you learned previously:
  - Given the angle θ, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$.
  - Given the angle θ, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.
  - Given the angle θ, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.
- Notice that the trigonometric ratios contain three unknowns: the angle measure and two side lengths.
- Given an acute angle of a right triangle and the measure of one of its side lengths, use sine, cosine, or tangent to find another side.
- If you know the value of a trigonometric function, you can use the inverse trigonometric function to find the measure of the angle.
- The inverse trigonometric functions are arcsine, arccosine, and arctangent.
- **Arcsine** is the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$.
- **Arccosine** is the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$.

- **Arctangent** is the inverse of the tangent function, written $\tan^{-1}\theta$ or $\arctan\theta$.
- There are two different types of notation for these functions:
  - $\sin^{-1}\theta = \arcsin\theta$; if $\sin\theta = \frac{2}{3}$, then $\arcsin\left(\frac{2}{3}\right) = \theta$.
  - $\cos^{-1}\theta = \arccos\theta$; if $\cos\theta = \frac{2}{3}$, then $\arccos\left(\frac{2}{3}\right) = \theta$.
  - $\tan^{-1}\theta = \arctan\theta$; if $\tan\theta = \frac{2}{3}$, then $\arctan\left(\frac{2}{3}\right) = \theta$.
- Note that "–1" is not an exponent; it is simply the notation for an inverse trigonometric function. Because this notation can lead to confusion, the "arc-" notation is frequently used instead.

  $\sin^{-1}\theta = \arcsin\theta$ but $(\sin\theta)^{-1} = \left(\sin\theta\right)^{-1} = \frac{1}{\sin\theta}$
- To calculate inverse trigonometric functions on your graphing calculator:

On a TI-83/84:

Step 1: Press [2ND][SIN].

Step 2: Type in the ratio.

Step 3: Press [ENTER].

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$."

Step 2: Type in the ratio.

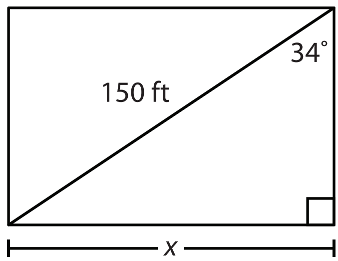
Step 3: Press [enter].

- Use the inverses of the trigonometric functions to find the acute angle measures given two sides of a right triangle.

Guided Practice 2.2.1

Example 1

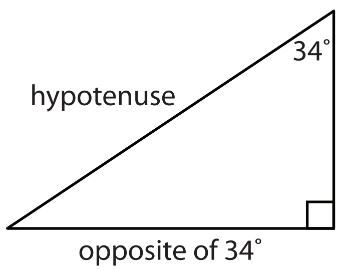
Leo is building a concrete pathway 150 feet long across a rectangular courtyard, as shown below. What is the length of the courtyard, *x*, to the nearest thousandth?



1. Determine which trigonometric function to use by identifying the given information.

   Given an angle of 34°, the length of the courtyard, *x*, is opposite the angle.

   The pathway is the hypotenuse since it is opposite the right angle of the triangle.

   Sine is the trigonometric function that uses opposite and hypotenuse, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, so we will use it to calculate the length of the courtyard.

2. Set up an equation using the sine function and the given measurements.

   $\theta = 34°$

   opposite side = x

   hypotenuse = 150 ft

   Therefore, $\sin 34° = \frac{x}{150}$.

3. Solve for x by multiplying both sides of the equation by 150.

   $150 \bullet \sin 34° = x$

4. Use a calculator to determine the value of x.

   On a TI-83/84:

   First, make sure your calculator is in Degree mode.

   Step 1: Press [MODE].
   Step 2: Arrow down twice to RADIAN.
   Step 3: Arrow right to DEGREE.
   Step 4: Press [ENTER]. The word "DEGREE" should be highlighted inside a black rectangle.
   Step 5: Press [2ND].
   Step 6: Press [MODE] to QUIT.
   Note: You will not have to change to Degree mode again unless you have changed your calculator to Radian mode.

   Next, perform the calculation.

   Step 1: Enter [150][×][SIN][34][)].
   Step 2: Press [ENTER].

   $x = 83.879$

(*continued*)

On a TI-Nspire:

First, make sure the calculator is in Degree mode.

Step 1: Choose 5: Settings & Status, then 2: Settings, and 2: Graphs and Geometry.
Step 2: Move to the Geometry Angle field and choose "Degree".
Step 3: Press [tab] to "ok" and press [enter].

Then, set the Scratchpad in Degree mode.

Step 1: In the calculate window from the home screen, press [doc].

Step 2: Select 7: Settings and Status, then 2: Settings, and 1: General.

Step 3: Move to the Angle field and choose "Degree".

Step 4: Press [tab] to "Make Default" and press [enter] twice to apply this as the new default setting.

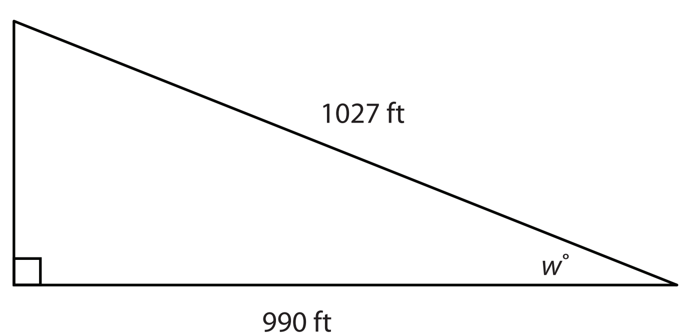
Next, perform the calculation.

Step 1: In the calculate window from the home screen, enter (150), then press [×][trig]. Use the keypad to select "sin," then type 34.
Step 2: Press [enter].
$x = 83.879$

The length of Leo's courtyard is about 84 feet.

Example 2

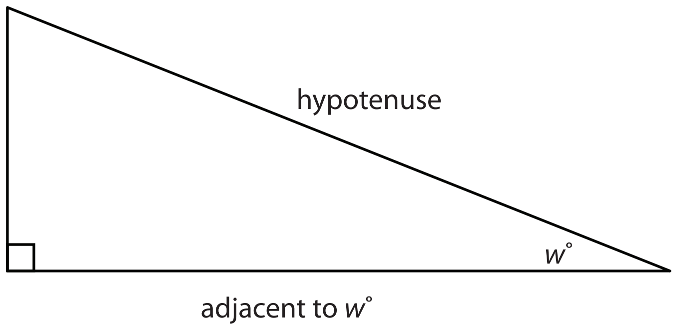
A trucker drives 1,027 feet up a hill that has a constant slope. When the trucker reaches the top of the hill, he has traveled a horizontal distance of 990 feet. At what angle did the trucker drive to reach the top? Round your answer to the nearest degree.



1. Determine which trigonometric function to use by identifying the given information.

   Given an angle of $w°$, the horizontal distance, 990 feet, is adjacent to the angle.

   The distance traveled by the trucker is the hypotenuse since it is opposite the right angle of the triangle.

   Cosine is the trigonometric function that uses adjacent and hypotenuse, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, so we will use it to calculate the angle the truck drove to reach the bottom of the road.

   Set up an equation using the cosine function and the given measurements.

   $\theta = w°$

   adjacent leg = 990 ft

   hypotenuse = 1027 ft

   Therefore, $\cos w = \frac{990}{1027}$.

   Solve for w.

   Solve for w by using the inverse cosine since we are finding an angle instead of a side length.

   $$\cos^{-1}\left(\frac{990}{1027}\right) = w$$

2. Use a calculator to calculate the value of w.

On a TI-83/84:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: Press [2ND][COS][990][÷][1027][)].
Step 2: Press [ENTER].
$w = 15.426$, or 15°.

On a TI-Nspire:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

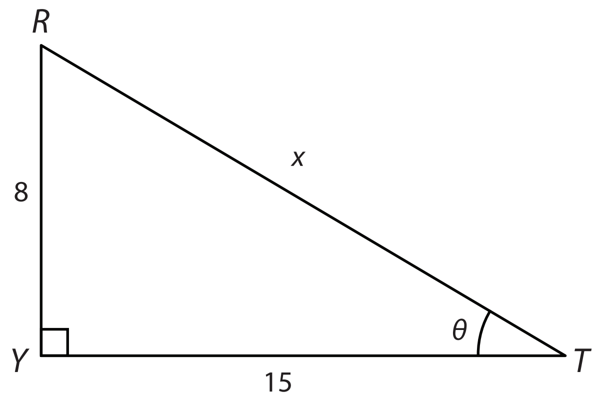
Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\cos^{-1}$." Enter 990, then press [÷] and enter 1027.
Step 2: Press [enter].
$w = 15.426$, or 15°.

The trucker drove at an angle of 15° to the top of the hill.

Example 3

In $\triangle TRY$, $\angle Y$ is a right angle and $\tan T = \dfrac{8}{15}$. What is $\sin R$? Express the answer as a fraction and as a decimal.

1. Draw a diagram.



2. Use the Pythagorean Theorem to find the hypotenuse.

$8^2 + 15^2 = x^2$

$64 + 225 = x^2$

$\sqrt{289} = x$

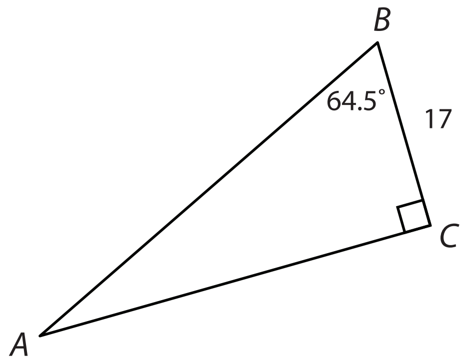
$x = 17$

3. Calculate $\sin R$.

$$\sin R = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{15}{x} = \frac{15}{17} \approx 0.882$$

Example 4

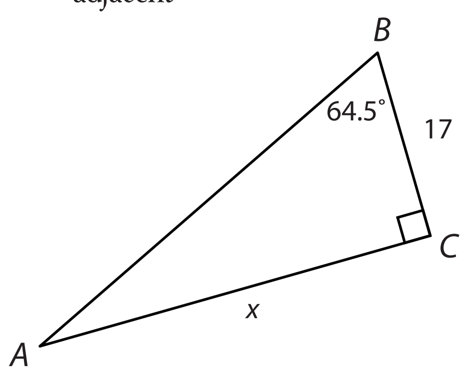
Solve the right triangle below. Round sides to the nearest thousandth.



1. Find the measures of $\overline{AC}$ and $\overline{AB}$.

   Solving the right triangle means to find all the missing angle measures and all the missing side lengths. The given angle is 64.5° and 17 is the length of the adjacent side. With this information, we could either use cosine or tangent since both functions' ratios include the adjacent side of a right triangle. Start by using the tangent function to find $\overline{AC}$.

   Recall that $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.

   $$\tan 64.5^\circ = \frac{x}{17}$$

   $$17 \bullet \tan 64.5^\circ = x$$

   (*continued*)

On a TI-83/84:

Step 1: Press [17][TAN][64.5][)].
Step 2: Press [ENTER].
$x = 35.641$

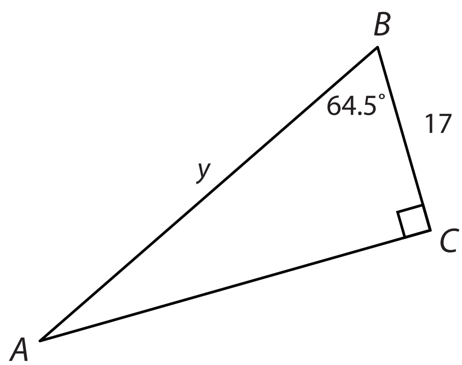
On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 17, then press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "tan," then enter 64.5.
Step 2: Press [enter].
$x = 35.641$

The measure of $AC = 35.641$.

To find the measure of $\overline{AB}$, either acute angle may be used as an angle of interest. Since two side lengths are known, the Pythagorean Theorem may be used as well.

Note: It is more precise to use the given values instead of approximated values.



2. Use the cosine function based on the given information.

Recall that $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.

$\theta = 64.5°$

adjacent leg = 17

hypotenuse = y

$\cos 64.5° = \frac{17}{y}$

$y \bullet \cos 64.5° = 17$

$y = \frac{17}{\cos 64.5°}$

On a TI-83/84:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: Press [17][÷][COS][64.5][)].
Step 2: Press [ENTER].
$y = 39.488$

On a TI-Nspire:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: In the calculate window from the home screen, enter 17, then press [÷][trig]. Use the keypad to select "cos," and then enter 64.5.
Step 2: Press [enter].
$y = 39.488$

The measure of $AB = 39.488$.

3. Use the Pythagorean Theorem to check your trigonometry calculations.

$17^2 + 35.641^2 = y^2$

$289 + 1267.36 = y^2$

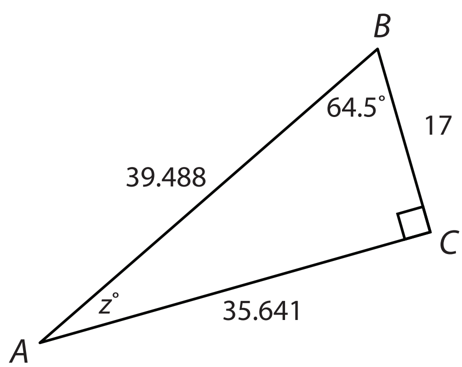
$1559.281 = y^2$

$\sqrt{1559.281} = y$

$y = 39.488$ The answer checks out.

$AC = 35.641$ and $AB = 39.488$.

4. Find the value of $\angle A$.



Using trigonometry, you could choose any of the three functions since you have solved for all three side lengths. In an attempt to be as precise as possible, let's choose the given side length and one of the approximate side lengths.

$$\sin z = \frac{17}{39.488}$$

5. Use the inverse trigonometric function since you are solving for an angle measure.

$$z=\arcsin\left(\frac{17}{39.488}\right)$$

On a TI-83/84:

Step 1: Press [2ND][SIN][17][÷][39.488][)].
Step 2: Press [ENTER].
$z = 25.500°$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$," and then enter 17, press [÷], and enter 39.488.
Step 2: Press [enter].
$z = 25.500°$

Check your angle measure by using the Triangle Sum Theorem.

$$m\angle A+64.5+90=180$$

$$m\angle A+154.5=180$$

$m\angle A=25.5$ The answer checks out.

$\angle A$ is 25.5°.

Practice 2.2.1: Calculating Sine, Cosine, and Tangent

Draw a sketch of the triangle relationship for all problems that do not have a diagram. Unless otherwise stated, round all side lengths to the nearest thousandth and round angle measures to the nearest degree.

1. In $\triangle MAD$, $\angle D$ is a right angle and $\sin M = \frac{20}{29}$. What are the cosine and tangent of $\angle A$? Write your answers as fractions and as decimals.

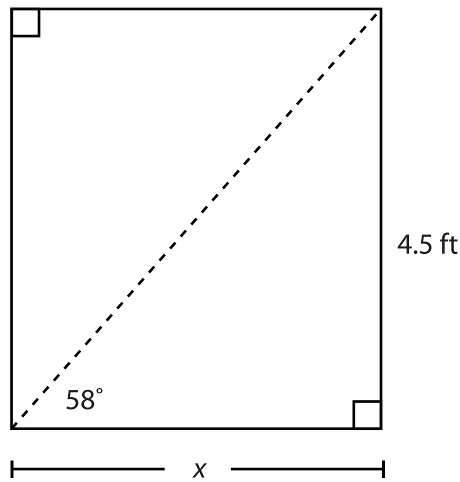
2. In $\triangle TID$, $\angle D$ is a right angle and $\cos T = \frac{4}{5}$. What are the sine and tangent of $\angle I$? Write your answers as fractions and as decimals.

3. The sides of a rectangle are 30 cm and 10 cm. What is the measure of the angle formed by the short side and the diagonal of the rectangle?

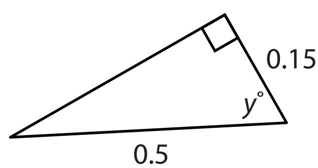
4. A ladder rests against the side of a building. The ladder is 13 feet long and forms an angle of 70.5° with the ground. How far is the base of the ladder from the base of the building?

continued

5. A family is covering the windows in their home in anticipation of a coming storm. Each window in the house has the same size and shape, rectangular with a height of 4.5 feet. What is the length of each window?



6. What is the value of y?



7. You are standing on a riverbank. An observation tower on the other side of the river is known to be 150 feet tall. An imaginary line from the top of the observation tower to your feet makes an angle of 13° with the ground. How far are you from the base of the tower?

continued

8. Solve the triangle ABC, given that $\angle C$ is a right angle, $m\angle A = 50°$, and $BC = 54$ centimeters.

9. Solve the triangle XYZ, given that $\angle Y$ is a right angle, $XZ = 9$ inches, and $XY = 6$ inches.

10. Keiko is flying a kite. The kite string makes a 69° angle with the horizontal, and she has let out 250 feet of string. She holds the string 5 feet off the ground. How high is the kite?

Lesson 2.2.2: Calculating Cosecant, Secant, and Cotangent

Introduction

In previous lessons, you defined and calculated using the three basic trigonometric functions, sine, cosine, and tangent. In this lesson, you will extend your working definitions of their reciprocal functions and use them to determine the unknown sides and angles of triangles.

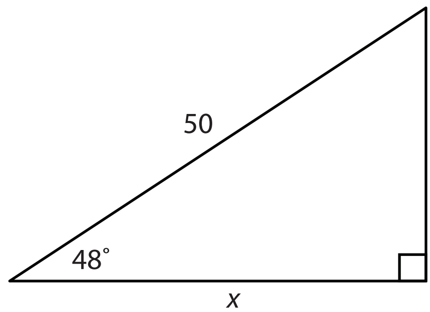
Key Concepts

- Remember:
  - Cosecant is the reciprocal of sine. Given the angle θ, $\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$.
  - Secant is the reciprocal of cosine. Given the angle θ, $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$.
  - Cotangent is the reciprocal of tangent. Given the angle θ, $\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$.

Guided Practice 2.2.2

Example 1

Determine the correct reciprocal trigonometric function to solve for x in the triangle below. Write the value of x to the nearest thousandth.



1. Identify the given information.

   θ = 48°, the hypotenuse is 50, and we are trying to calculate the adjacent side, x.

2. Set up the correct reciprocal function.

   Recall that:

   $$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

   Based on the given information, we must use the secant function, the reciprocal of cosine.

   $$\sec 48° = \frac{50}{x}$$

   $$x \bullet \sec 48° = 50$$

   $$x = \frac{50}{\sec 48°}$$

3. Use the calculator to determine the value of x.

Secant is the reciprocal of cosine. This means that $\sec 48^\circ = \dfrac{1}{\cos 48^\circ}$.

Since most calculators do not have buttons for the reciprocal functions, you will have to substitute this value in the expression in order to correctly calculate the value of x.

$$x = \frac{50}{\sec 48^\circ}$$

$$x = \frac{50}{\dfrac{1}{\cos 48^\circ}}$$

On a TI-83/84:

Be sure that your calculator is in Degree mode.

Step 1: Press [50][÷][(][1][÷][COS][48][)][)].
Step 2: Press [ENTER].
$x \approx 33.457$

On a TI-Nspire:

Be sure that your calculator is in Degree mode.

Step 1: In the calculate window from the home screen, enter 50, then press [÷][(][1][÷][trig]. Use the keypad to select "cos," then enter 48 and press [)] twice.
Step 2: Press [enter].
$x \approx 33.457$

4. Check your solution by using the reciprocal function to the function you chose earlier.

   We solved for x by using the secant function, which is the reciprocal of cosine. To check the answer, use cosine to see if you get the same solution.

   $$\cos 48° = \frac{x}{50}$$

   $$50 \bullet \cos 48° = x$$

   On a TI-83/84:

   Step 1: Press [50][COS][48][)].
   Step 2: Press [ENTER].
   $x \approx 33.457$ The answer checks out.

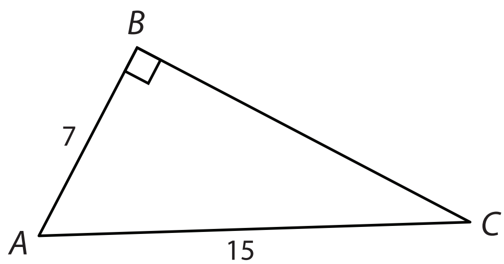
   On a TI-Nspire:

   Step 1: In the calculate window from the home screen, enter 50, then press [×][trig]. Use the keypad to select "cos," and then enter 48.
   Step 2: Press [enter].
   $x \approx 33.457$ The answer checks out.

   Note: You can check your solutions when using reciprocal functions by following the steps in this example.

Example 2

Use cosecant, secant, or cotangent to find $m\angle C$.



1. Identify the given information.

   The opposite side is given, $AB = 7$, as well as the hypotenuse, $AC = 15$. We are solving for $m\angle C$.

2. Set up the correct reciprocal function.

   Based on the given information, use the cosecant function, the reciprocal of sine.

   $$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

   $$\csc C = \frac{15}{7}$$

   Use the inverse of this function to find the desired angle measure.

   $$C = \csc^{-1}\left(\frac{15}{7}\right)$$

3. Use the calculator to calculate the measure of $\angle C$.

Cosecant is the reciprocal of sine. This means that

$\csc^{-1}\left(\frac{15}{7}\right)=\sin^{-1}\left(\frac{7}{15}\right)$. Because of calculator errors with domain, there is no other way to enter the reciprocal's inverse.

On a TI-83/84:

Step 1: Press [2ND][SIN][7][÷][15][)].
Step 2: Press [ENTER].
$m\angle C \approx 28°$

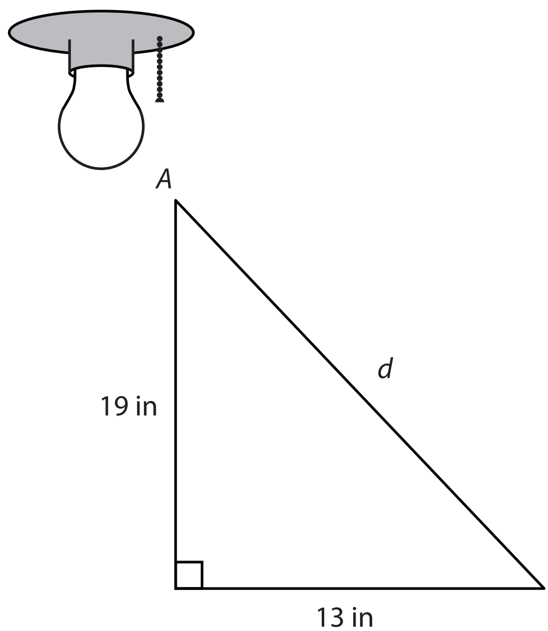
On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$," and then enter 7, press [÷], and enter 15.
Step 2: Press [enter].
$m\angle C \approx 28°$

Example 3

The light from a 19-inch-tall egg incubator casts a 13-inch shadow across a shelf. Use reciprocal functions and the Pythagorean Theorem as necessary to determine the distance from the top of the incubator light to the farthest part of the shadow. What is the angle at which the incubator light casts its shadow?

1. Create a drawing of the scenario. Label the distance d and the angle A.



2. Identify the given information.

   Two side lengths are given: 19 inches and 13 inches.

3. Determine which trigonometric functions are necessary to find the unknown values.

   Since the given information includes two side lengths, use the Pythagorean Theorem.

   $19^2 + 13^2 = d^2$

   $361 + 169 = d^2$

   $530 = d^2$

   $\sqrt{530} = d$

   $d \approx 23.022$

   The distance from the top of the incubator to the farthest tip of the shadow is about 23.022 inches.

   Find the measure of $\angle A$.

   To find the measure of $\angle A$, use the two side lengths that were given to produce the most precise answer possible. Since those sides are adjacent to the angle and opposite from the angle, use the cotangent function, which is the reciprocal of tangent.

   $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

   $\cot A = \dfrac{19}{13}$

   $A = \cot^{-1}\left(\dfrac{19}{13}\right)$

(*continued*)

To enter this into the calculator, first write it in its reciprocal format, $A = \tan^{-1}\left(\frac{13}{19}\right)$.

On a TI-83/84:

Step 1: Press [2ND][TAN][13][÷][19][)].
Step 2: Press [ENTER].
$m\angle A \approx 34°$

On a TI-Nspire:

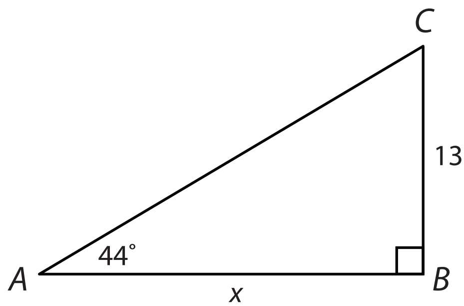
Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 13, press [÷], and enter 19.
Step 2: Press [enter].
$m\angle A \approx 34°$

The incubator casts a shadow at an angle of $34°$.

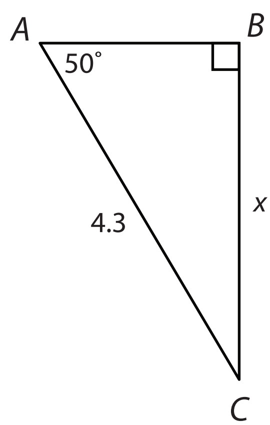
Practice 2.2.2: Calculating Cosecant, Secant, and Cotangent

Use what you've learned about cosecant, secant, and cotangent to solve the following problems. Unless otherwise stated, round your answers to the nearest thousandth.

1. Use cosecant, secant, or cotangent to find the value of *x*.

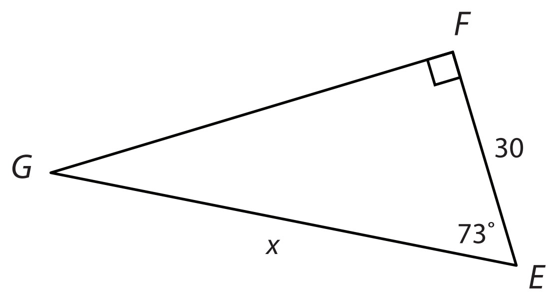


2. Use cosecant, secant, or cotangent to find the value of *x*.

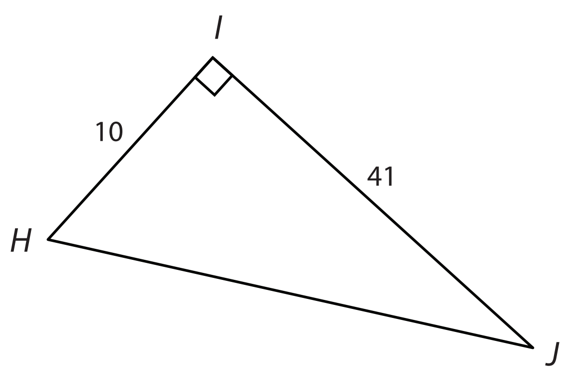


continued

3. Use cosecant, secant, or cotangent to find the value of x.



4. Use cosecant, secant, or cotangent to find $m\angle J$.



continued

In $\triangle EFG$, $\angle F$ is a right angle. Given the following trigonometric ratios for problems 5 and 6, find the cosecant, secant, and cotangent of $\angle E$. Write your answers as fractions and as decimals.

5. $\sin E = \dfrac{3}{5}$

6. $\cos E = \dfrac{8}{10}$

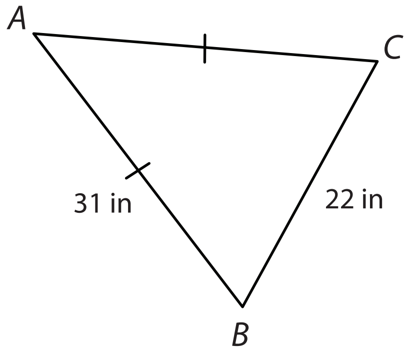
Use your knowledge of trigonometric ratios to complete the problems that follow.

7. A tourist in Washington, D.C., is sitting in the grass gazing up at the Washington Monument. The angle of her line of sight from the ground to the top of the monument is 25°. Given that the Washington Monument is 555 feet tall, find her approximate distance from the base of the monument.

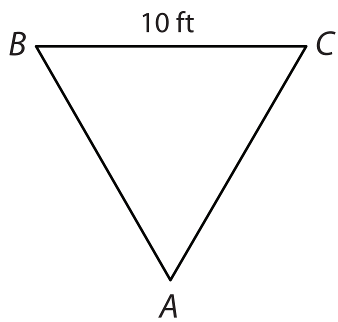
8. An accessible ramp must be constructed so that the slope rises no more than 1 inch for every 1 foot of run. What is the maximum angle that the ramp can make with the ground?

continued

9. The triangle below is isosceles. Calculate the cosecant, secant, and tangent of the measure of $\angle B$.



10. The triangle below is equilateral. Calculate the cosecant, secant, and tangent of the measure of $\angle B$.

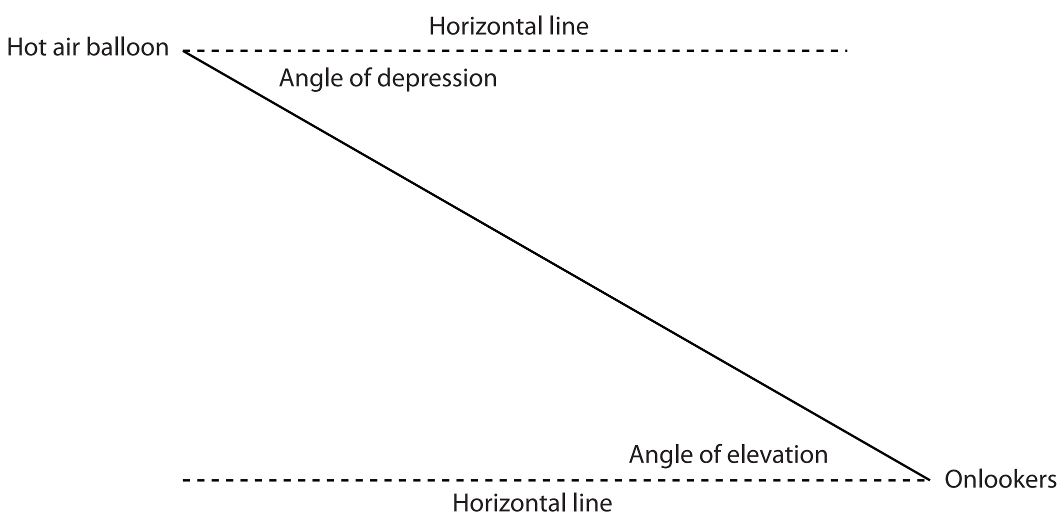


Lesson 2.2.3: Problem Solving with the Pythagorean Theorem and Trigonometry

Introduction

Imagine it is a beautiful afternoon in northern Georgia. Your class has raised enough money to take a ride in a hot air balloon that overlooks Lake Lanier, Stone Mountain, Kennesaw Mountain, and the Atlanta skyline, among other sights. As you fly high in the sky, you observe the geography below, and onlookers below marvel at the beauty of the hot air balloon. The angles of observation from both parties, downward toward the sights and upward toward the balloon, have specific names.

As you look down toward the onlookers, you view the landscape at an **angle of depression**, which is the angle created by a horizontal line and a downward line of sight. The onlookers view the hot air balloon at an **angle of elevation**, which is the angle created by a horizontal line and an upward line of sight. In the following diagram, notice the labeled angles given the horizontals.



These two angles are very important to understand and will be the basis for our study and practice during this lesson. Specifically, you will study and practice calculating angles of depression and elevation and use these angles to calculate distances that are not easily measured by common devices.

It is important to note that, in this example, the horizontal lines are parallel to each other, and therefore the line of sight behaves as a transversal to the parallel lines. As such, the angle of elevation and angle of depression are alternate interior angles and, therefore, their angle measures are congruent.

Key Concepts

- The angle of depression is the angle created by a horizontal line and a downward line of sight to an object below the observer.
- The angle of elevation is the angle created by a horizontal line and an upward line of sight to an object above the observer.
- Remember:
    - Given the angle θ, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$.
    - Given the angle θ, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.
    - Given the angle θ, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.
    - Given right triangle ABC, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

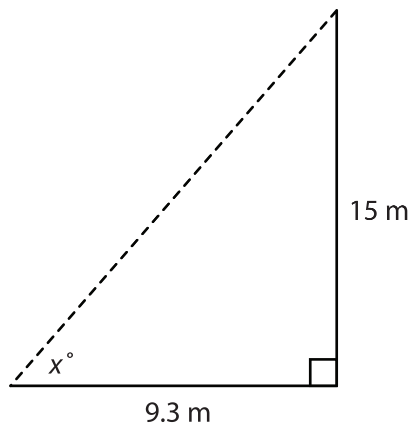
Guided Practice 2.2.3

Example 1

The height of a tree is 15 meters. To the nearest whole degree, what is the angle of elevation of the sun when the tree casts a shadow that is 9.3 meters long on level ground?

1. Make a drawing of the scenario.

   Make sure to correctly identify the angle of elevation as the measure from the ground, above the horizontal.



2. Identify the trigonometric function that you must use in order to find the angle of elevation.

   In this case, the height of the tree represents the side that is opposite the angle of elevation, *x*. The length of the shadow represents the adjacent side to *x*. Therefore, use the tangent ratio.

   Use the inverse since you are finding an angle and not a side length.

   $\tan x = \dfrac{\text{opposite}}{\text{adjacent}}$; opposite leg = 15; adjacent leg = 9.3

   $$\tan x = \frac{15}{9.3}$$

   $$x = \tan^{-1}\left(\frac{15}{9.3}\right)$$

(*continued*)

On a TI-83/84:

Step 1: Press [2ND][TAN][15][÷][9.3].
Step 2: Press [ENTER].
$x \approx 58.2°$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 15, press [÷], and enter 9.3.
Step 2: Press [enter].
$x \approx 58.2°$

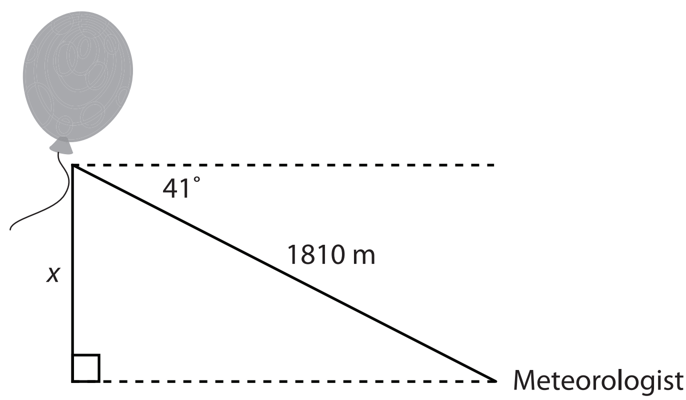
The angle of elevation of the sun is about 58°.

Example 2

A meteorologist reads radio signals to get information from a weather balloon. The last alert indicated that the angle of depression of the weather balloon to the meteorologist was 41° and the balloon was 1,810 meters away from his location on the diagonal. To the nearest meter, how high above the ground was the balloon?

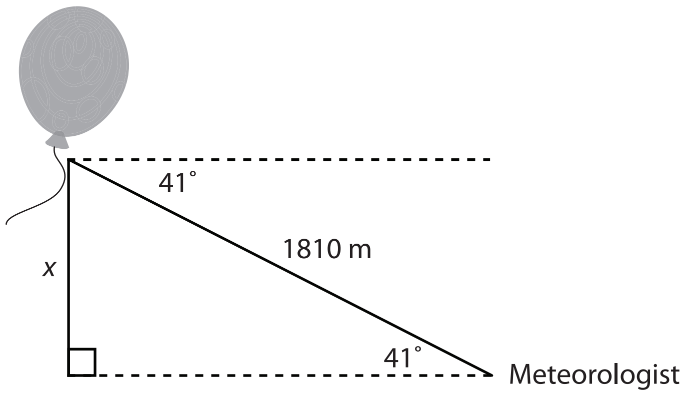
1. Make a drawing of the scenario.

   Remember, the angle of depression is above the diagonal line of sight and below the horizontal.



2. Identify the correct trigonometric function to use given the angle of depression.

   Recall that the angles of depression and elevation are congruent, so we can use this to determine the trigonometric function.



   Since x is opposite the angle of elevation (which yields the same information as the angle of depression), and the distance from the weather balloon to the meteorologist is represented by the hypotenuse of the right triangle, use sine to find the vertical height.

3. Set up the function and solve for the height.

$$\sin 41° = \frac{x}{1810}$$

$$1810 \sin 41° = x$$

On a TI-83/84:

Step 1: Press [1810][SIN][41].
Step 2: Press [ENTER].
$x \approx 1187.467$ m

On a TI-Nspire:

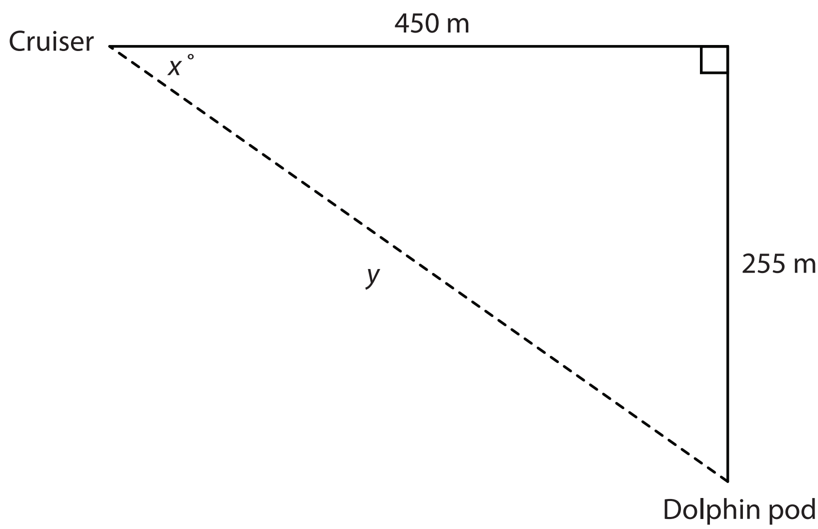
Step 1: In the calculate window from the home screen, enter 1810 and then press [trig]. Use the keypad to select "sin," then enter 41.
Step 2: Press [enter].
$x \approx 1187.467$ m

The weather balloon was, vertically, about 1,188 meters from the meteorologist's location.

Example 3

A sonar operator on an anchored cruiser detects a pod of dolphins feeding at a depth of about 255 meters directly below. If the cruiser travels 450 meters west and the dolphins remain at the same depth to feed, what is the angle of depression, *x*, from the cruiser to the pod? What is the distance, *y*, between the cruiser and the pod? Round your answers to the nearest whole number.

1. Make a drawing of the scenario.



2. Find the angle of depression. Identify the given information to determine which trigonometric function to use.

   Since you are calculating an angle, use the inverse of the trigonometric function. Notice that because of the orientation of the triangle and the horizontal side, the angle of depression lies above the diagonal.

   We are given the distances opposite and adjacent to the angle of depression. Therefore, use the tangent function.

$$\tan x = \frac{255}{450}$$

$$x = \tan^{-1}\left(\frac{255}{450}\right)$$

(*continued*)

On a TI-83/84:

Step 1: Press [2ND][TAN][255][÷][450].
Step 2: Press [ENTER].
$x \approx 29.539$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "tan," then enter 255, press [÷], and enter 450.
Step 2: Press [enter].
$x \approx 29.539$

The angle of depression from the cruiser is about 30°.

3. Determine the distance, y.

Since two side lengths were given, to determine the distance between the cruiser and the pod, there is the option of using either a trigonometric ratio or the Pythagorean Theorem. However, since the value of the angle of depression was not given and had to be approximated, using the Pythagorean Theorem given the two distances will yield a more precise answer.

$$450^2 + 255^2 = y^2$$

$$202{,}500 + 65{,}025 = y^2$$

$$267{,}525 = y^2$$

$$\pm\sqrt{267{,}525} = y$$

$$y \approx 517$$

The distance from the cruiser to the dolphin pod after travelling 450 meters west of the vertical is about 517 meters.

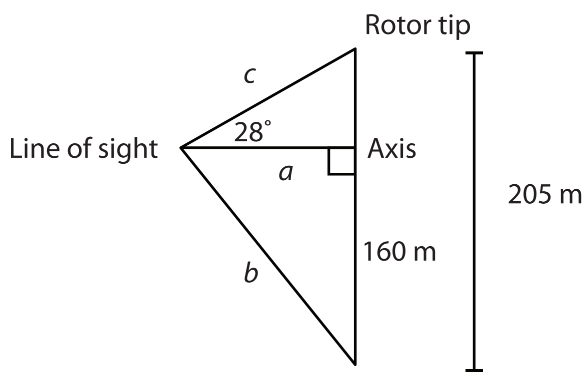
Example 4

You are on a steep hillside directly across from the axis of a wind turbine. The information post on the hill indicates that the wind turbines are 160 meters to their axis (or center of rotation) and that the rotor tips reach a height of 205 meters when they are in line, or at 180°, with the wind turbine pole. The information post also says that for an observer who is 6 feet tall, the angle of elevation to the rotor tip is 28° and the observer's eyes are level with the axis of the wind turbine. Use this information to answer the following questions.

- If an observer is 6 feet tall, to the nearest meter, what is the distance from the observer's eye level to the wind turbine's axis?
- What is the distance, to the nearest thousandth of a meter, from the observer's eye level to the base of the wind turbine?
- What is the distance, to the nearest meter, from the observer's eye level to the tip of a rotor when it reaches its highest point?

1. Make a drawing of the scenario.

   Label the given information accordingly.



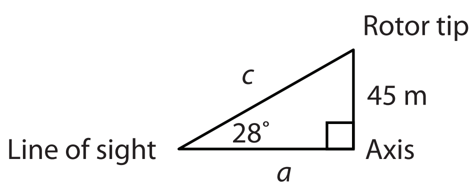
2. Use the given information to solve for the base in the top triangle to determine the distance from the observer's eye level to the wind turbine's axis.

   Given an angle of elevation in the top triangle, subtract 160 meters from 205 meters to get the height from the axis to the rotor tip, which is also a leg of the triangle. Calculate the distance from the axis to the tip of a rotor at its highest point. Knowing this side will help you to find the distance, *a*, from the observer's eye level to the wind turbine's axis.

   $205 - 160 = 45$

   The vertical distance from the axis to the rotor tip is 45 meters.

   Now use the angle of elevation and the distance from the axis to the rotor tip at its highest point to find *a*.



   Given that the opposite leg is 45 meters and the adjacent leg is the distance *a* from the given angle 28°, use the tangent function.

   $$\tan 28^\circ = \frac{45}{a}$$

   $$a \tan 28^\circ = 45$$

   $$a = \frac{45}{\tan 28^\circ}$$

(*continued*)

On a TI-83/84:

Step 1: Press [45][÷][TAN][28][)].
Step 2: Press [ENTER].
$a \approx 84.633$

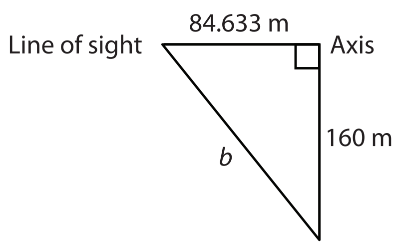
On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 45, then press [÷][trig]. Use the keypad to select "tan," then enter 28.
Step 2: Press [enter].
$a \approx 84.633$

The distance from the observer's eye level to the axis of the wind turbine is about 85 meters.

3. Find the length of the diagonal in the bottom triangle, *b*, to determine the distance from the observer's eye level to the base of the wind turbine.

   Now that we know *a*, and since *a* represents the length of the same segment for both triangles, we can use *a* and the distance from the ground to the axis to find the distance from the observer's line of sight to the base of the wind turbine.



$84.633^2 + 160^2 = b^2$

$7162.692 + 25{,}600 = b^2$

$32{,}762.692 = b^2$

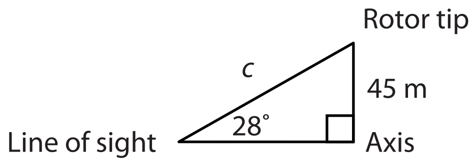
$\pm\sqrt{32762.692} = b$

$b \approx 181.004$

The distance from the observer's eye level to the base of the wind turbine is about 181 meters.

4. Solve for the diagonal of the top triangle, c, to determine the distance from the observer's eye level to the tallest tip of a rotor.

   Use the top triangle to find c.



   Use sine to find c.

$$\sin 28^\circ = \frac{45}{c}$$

$$c \sin 28^\circ = 45$$

$$c = \frac{45}{\sin 28^\circ}$$

$$c \approx 95.852$$

   The distance from the observer's line of sight to the tallest tip of a rotor is about 96 meters.

Practice 2.2.3: Problem Solving with the Pythagorean Theorem and Trigonometry

Unless otherwise specified, round all final answers to the nearest whole number.

1. Brianna is hiking on a mountain trail. She hikes 345 feet uphill but a horizontal distance of 295 feet. To the nearest degree, what is the angle of elevation of the trail?

2. A building is 100 meters tall and, at a certain time of day, casts a shadow from the sun that is 56 meters long. What is the angle of elevation of the sun at that time?

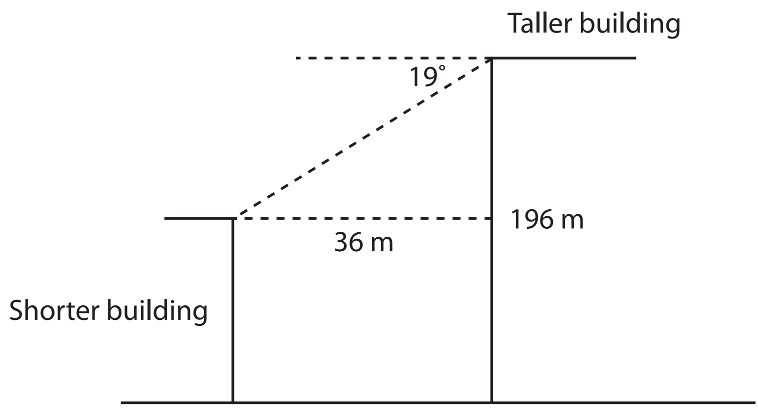
3. A blimp provides aerial footage of a football game at an altitude, or vertical height, of 400 meters. The television crew estimates the distance of their line of sight to the stadium to be 3,282.2 meters. What is the television crew's angle of depression from inside the blimp?

4. It is estimated that 20,000 to 25,000 homes get their water through a pipeline from the Lake Lanier reservoir. One section of the pipeline slopes down with a 21° angle of depression for a horizontal distance of 4,000 feet. To the nearest foot, how long is that section of the pipeline?

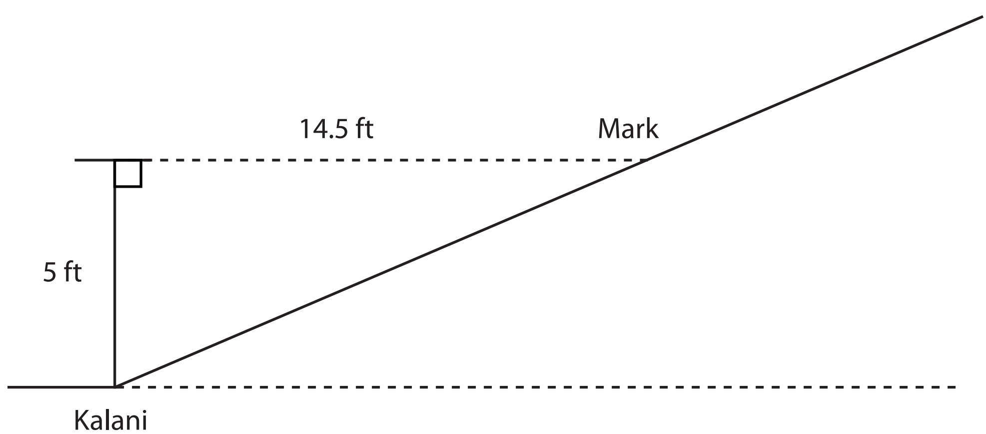
5. A parasailing company uses a 50-foot cable to connect the parasail to the back of a boat. About how far is the parasail from the water when the cable has a 35° angle of elevation? What is the horizontal distance from the boat to the parasail at the same angle of elevation?

continued

6. Two office buildings are 36 meters apart. The height of the taller building is 196 meters. The angle of depression from the top of the taller building to the top of the shorter building is 19°. What is the height of the shorter building? Refer to the diagram below.



7. Kalani is standing at the bottom of a hill. Mark is standing on the hill so that when Kalani's line of sight is perpendicular to her body, she is looking at Mark's shoes. If Kalani's eyes are 5 feet above the ground and 14.5 feet from Mark's shoes, what is the angle of elevation of the hill to the nearest degree? How far are Kalani's shoes from Mark's shoes, to the nearest foot? Refer to the diagram below.



continued

Use the following scenario to complete problems 8 and 9.

A salvage ship's sonar locates wreckage at a 12° angle of depression east of the ship. A diver attached to a thick cord is lowered 45 meters to the ocean floor.

8. How far east does the diver have to swim to reach the wreckage? What is the length of the rope once the diver reaches the wreckage?

9. A sudden wind moves the boat 10 meters west of its starting location. What is the angle of elevation from the wreckage and how long is the rope extended now?

Read the scenario that follows and use the information to answer the questions.

10. You and your friend are standing on a steep hill directly across from the axis of a wind turbine. You know that this wind turbine is 160 meters to its axis (or center of rotation) and that the rotor tips reach a height of 205 meters when they are in line, or at 180°, with the wind turbine pole. Your friend is 6 feet tall, and says that the angle of elevation to the rotor tip is 32° and that his eyes are level with the axis of the wind turbine.

    a. What is the distance from your friend's line of sight to the axis?

    b. What is the distance from your friend's line of sight to the base of the wind turbine?

    c. What is the distance from your friend's line of sight to the tip of a rotor at its tallest point?

Answer Key

Lesson 1: Exploring Trigonometric Ratios

Practice 2.1.1: Defining Trigonometric Ratios, pp. 18–22

Answers will vary due to the variation that comes when drawing and measuring or rounding.

1. $\sin A \approx 0.866$; $\cos A = 0.5$; $\tan A \approx 1.732$
3. $\sin B = 0.5$; $\cos B \approx 0.866$; $\tan B \approx 0.577$
5. hypotenuse = 5; $m\angle A \approx 37°$; $m\angle B \approx 53°$; $\sin A = 0.6$; $\cos A = 0.8$; $\tan A = 0.75$; $\sin B = 0.8$; $\cos B \approx 0.6$; $\tan B \approx 1.333$
7. The object is 0.288 miles deep; $\tan 30° \approx 0.577$.
9. distance $\approx$ 0.898 miles; $\tan 50° \approx 1.192$

Practice 2.1.2: Exploring Sine and Cosine As Complements, pp. 30–32

1. approximately 0.707
3. $\theta = 38°$
5. $\sin 69° \approx 0.934$
7. $\theta = 60°$
9. 640 meters

Lesson 2: Applying Trigonometric Ratios

Practice 2.2.1: Calculating Sine, Cosine, and Tangent, pp. 48–50

1. $\cos M = \frac{21}{29} \approx 0.724$; $\tan M = \frac{20}{21} \approx 0.952$
3. 72°
5. 2.812 ft
7. 649.721 ft
9. $YZ = 6.708$ in; $m\angle X = 48.2°$; $m\angle Z = 41.8°$

Practice 2.2.2: Calculating Cosecant, Secant, and Cotangent, pp. 60–63

1. 13.462
3. 102.609
5. $\csc E = \frac{5}{3} \approx 1.667$; $\sec E = \frac{5}{4} \approx 1.25$; $\cot E = \frac{4}{3} \approx 1.333$
7. 1190.201 ft
9. $\csc B = \frac{31}{29} \approx 1.069$; $\sec B = \frac{29}{11} \approx 2.636$; $\cot E = \frac{11}{29} \approx 0.379$

Practice 2.2.3: Problem Solving with the Pythagorean Theorem and Trigonometry, pp. 75–77

1. 31°
3. 7°
5. 29 ft; 41 ft
7. 19°; 15 ft
9. 79°; 226 m