Radicals

We'll open this section with the definition of the radical. If n is a positive integer that is greater than 1 and a is a real number then,

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

where n is called the **index**, a is called the **radicand**, and the symbol $\sqrt{\ }$ is called the **radical**. The left side of this equation is often called the radical form and the right side is often called the exponent form.

From this definition we can see that a radical is simply another notation for the first rational exponent that we looked at in the rational exponents section.

Note as well that the index is required in these to make sure that we correctly evaluate the radical. There is one exception to this rule and that is square root. For square roots we have,

$$\sqrt[2]{a} = \sqrt{a}$$

In other words, for square roots we typically drop the index.

Let's do a couple of examples to familiarize us with this new notation.

Example 1 Write each of the following radicals in exponent form.

(a) $\sqrt[4]{16}$

(b) $\sqrt[10]{8x}$

(c) $\sqrt{x^2 + y^2}$

Solution

(a) $\sqrt[4]{16} = 16^{\frac{1}{4}}$

(b) $\sqrt[10]{8x} = (8x)^{\frac{1}{10}}$

(c) $\sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{\frac{1}{2}}$

As seen in the last two parts of this example we need to be careful with parenthesis. When we convert to exponent form and the radicand consists of more than one term then we need to enclose the whole radicand in parenthesis as we did with these two parts. To see why this is consider the following,

$$8x^{\frac{1}{10}}$$

From our discussion of exponents in the previous sections we know that only the term immediately to the left of the exponent actually gets the exponent. Therefore, the radical form of this is,

$$8x^{\frac{1}{10}} = 8\sqrt[10]{x} \neq \sqrt[10]{8x}$$

So, we once again see that parenthesis are very important in this class. Be careful with them.

Since we know how to evaluate rational exponents we also know how to evaluate radicals as the following set of examples shows.

Example 2 Evaluate each of the following.

(a) $\sqrt{16}$ and $\sqrt[4]{16}$ [Solution]

(b) $\sqrt[5]{243}$ [Solution]

(c) $\sqrt[4]{1296}$ [Solution]

(d) $\sqrt[3]{-125}$ [Solution]

(e) $\sqrt[4]{-16}$ [Solution]

Solution

To evaluate these we will first convert them to exponent form and then evaluate that since we already know how to do that.

(a) These are together to make a point about the importance of the index in this notation. Let's take a look at both of these.

$$\sqrt{16} = 16^{\frac{1}{2}} = 4 \qquad \text{because } 4^2 = 16$$

$$\sqrt[4]{16} = 16^{\frac{1}{4}} = 2 \qquad \text{because } 2^4 = 16$$

So, the index is important. Different indexes will give different evaluations so make sure that you don't drop the index unless it is a 2 (and hence we're using square roots).

(b) $\sqrt[5]{243} = 243^{\frac{1}{5}} = 3$ because $3^5 = 243$

(c) $\sqrt[4]{1296} = 1296^{\frac{1}{4}} = 6$ because $6^4 = 1296$

(d) $\sqrt[3]{-125} = \left(-125\right)^{\frac{1}{3}} = -5$ because $\left(-5\right)^3 = -125$

(e) $\sqrt[4]{-16} = \left(-16\right)^{\frac{1}{4}}$

As we saw in the integer exponent section this does not have a real answer and so we can't evaluate the radical of a negative number if the index is even. Note however that we can evaluate the radical of a negative number if the index is odd as the previous part shows.

Let's briefly discuss the answer to the first part in the above example. In this part we made the claim that $\sqrt{16} = 4$ because $4^2 = 16$. However, 4 isn't the only number that we can square to get 16. We also have $\left(-4\right)^2 = 16$. So, why didn't we use -4 instead? There is a general rule about evaluating square roots (or more generally radicals with even indexes). When evaluating square roots we ALWAYS take the positive answer. If we want the negative answer we will do the following.

$$-\sqrt{16} = -4$$

This may not seem to be all that important, but in later topics this can be very important. Following this convention means that we will always get predictable values when evaluating roots.

Note that we don't have a similar rule for radicals with odd indexes such as the cube root in part (d) above. This is because there will never be more than one possible answer for a radical with an odd index.

We can also write the general rational exponent in terms of radicals as follows.

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \qquad \text{OR} \qquad a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

We now need to talk about some properties of radicals.

Properties

If n is a positive integer greater than 1 and both a and b are positive real numbers then,

1. $\sqrt[n]{a^n} = a$
2. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
3. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Note that on occasion we can allow a or b to be negative and still have these properties work. When we run across those situations we will acknowledge them. However, for the remainder of this section we will assume that a and b must be positive.

Also note that while we can "break up" products and quotients under a radical we can't do the same thing for sums or differences. In other words,

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b} \qquad \text{AND} \qquad \sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$$

If you aren't sure that you believe this consider the following quick number example.

$$5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

If we "break up" the root into the sum of the two pieces we clearly get different answers! So, be careful to not make this very common mistake!

We are going to be simplifying radicals shortly so we should next define **simplified radical form**. A radical is said to be in simplified radical form (or just simplified form) if each of the following are true.

1. All exponents in the radicand must be less than the index.
2. Any exponents in the radicand can have no factors in common with the index.
3. No fractions appear under a radical.
4. No radicals appear in the denominator of a fraction.

In our first set of simplification examples we will only look at the first two. We will need to do a little more work before we can deal with the last two.

Example 3 Simplify each of the following.

(a) $\sqrt{y^7}$ [Solution]

(b) $\sqrt[9]{x^6}$ [Solution]

(c) $\sqrt{18x^6y^{11}}$ [Solution]

(d) $\sqrt[4]{32x^9y^5z^{12}}$ [Solution]

(e) $\sqrt[5]{x^{12}y^4z^{24}}$ [Solution]

(f) $\sqrt[3]{9x^2}\,\sqrt[3]{6x^2}$ [Solution]

Solution

(a) $\sqrt{y^7}$

In this case the exponent (7) is larger than the index (2) and so the first rule for simplification is violated. To fix this we will use the first and second properties of radicals above. So, let's note that we can write the radicand as follows.

$$y^7 = y^6 y = \left(y^3\right)^2 y$$

So, we've got the radicand written as a perfect square times a term whose exponent is smaller than the index. The radical then becomes,

$$\sqrt{y^7} = \sqrt{\left(y^3\right)^2 y}$$

Now use the second property of radicals to break up the radical and then use the first property of radicals on the first term.

$$\sqrt{y^7} = \sqrt{\left(y^3\right)^2}\,\sqrt{y} = y^3\sqrt{y}$$

This now satisfies the rules for simplification and so we are done.

Before moving on let's briefly discuss how we figured out how to break up the exponent as we did. To do this we noted that the index was 2. We then determined the largest multiple of 2 that is less than 7, the exponent on the radicand. This is 6. Next, we noticed that 7=6+1.

Finally, remembering several rules of exponents we can rewrite the radicand as,

$$y^7 = y^6 y = y^{(3)(2)} y = \left(y^3\right)^2 y$$

In the remaining examples we will typically jump straight to the final form of this and leave the details to you to check.

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(b) $\sqrt[9]{x^6}$

This radical violates the second simplification rule since both the index and the exponent have a common factor of 3. To fix this all we need to do is convert the radical to exponent form do some simplification and then convert back to radical form.

$$\sqrt[9]{x^6} = \left(x^6\right)^{\frac{1}{9}} = x^{\frac{6}{9}} = x^{\frac{2}{3}} = \left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$$

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(c) $\sqrt{18x^6y^{11}}$

Now that we've got a couple of basic problems out of the way let's work some harder ones. Although, with that said, this one is really nothing more than an extension of the first example.

There is more than one term here but everything works in exactly the same fashion. We will break the radicand up into perfect squares times terms whose exponents are less than 2 (*i.e.* 1).

$$18x^6y^{11} = 9x^6y^{10}\left(2y\right) = 9\left(x^3\right)^2\left(y^5\right)^2\left(2y\right)$$

Don't forget to look for perfect squares in the number as well.

Now, go back to the radical and then use the second and first property of radicals as we did in the first example.

$$\sqrt{18x^6y^{11}} = \sqrt{9\left(x^3\right)^2\left(y^5\right)^2\left(2y\right)} = \sqrt{9}\sqrt{\left(x^3\right)^2}\sqrt{\left(y^5\right)^2}\sqrt{2y} = 3x^3y^5\sqrt{2y}$$

Note that we used the fact that the second property can be expanded out to as many terms as we have in the product under the radical. Also, don't get excited that there are no x's under the radical in the final answer. This will happen on occasion.

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(d) $\sqrt[4]{32x^9y^5z^{12}}$

This one is similar to the previous part except the index is now a 4. So, instead of get perfect squares we want powers of 4. This time we will combine the work in the previous part into one step.

$$\sqrt[4]{32x^9y^5z^{12}} = \sqrt[4]{16x^8y^4z^{12}\left(2xy\right)} = \sqrt[4]{16}\sqrt[4]{\left(x^2\right)^4}\sqrt[4]{y^4}\sqrt[4]{\left(z^3\right)^4}\sqrt[4]{2xy} = 2x^2y\,z^3\sqrt[4]{2xy}$$

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(e) $\sqrt[5]{x^{12}y^4z^{24}}$

Again this one is similar to the previous two parts.

$$\sqrt[5]{x^{12}y^4z^{24}} = \sqrt[5]{x^{10}z^{20}\left(x^2y^4z^4\right)} = \sqrt[5]{\left(x^2\right)^5}\sqrt[5]{\left(z^4\right)^5}\sqrt[5]{x^2y^4z^4} = x^2z^4\sqrt[5]{x^2y^4z^4}$$

In this case don't get excited about the fact that all the y's stayed under the radical. That will happen on occasion.

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(f) $\sqrt[3]{9x^2}\,\sqrt[3]{6x^2}$

This last part seems a little tricky. Individually both of the radicals are in simplified form. However, there is often an unspoken rule for simplification. The unspoken rule is that we should have as few radicals in the problem as possible. In this case that means that we can use the second property of radicals to combine the two radicals into one radical and then we'll see if there is any simplification that needs to be done.

$$\sqrt[3]{9x^2}\,\sqrt[3]{6x^2} = \sqrt[3]{\left(9x^2\right)\left(6x^2\right)} = \sqrt[3]{54x^4}$$

Now that it's in this form we can do some simplification.

$$\sqrt[3]{9x^2}\,\sqrt[3]{6x^2} = \sqrt[3]{27x^3\left(2x\right)} = \sqrt[3]{27x^3}\,\sqrt[3]{2x} = 3x\sqrt[3]{2x}$$

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Before moving into a set of examples illustrating the last two simplification rules we need to talk briefly about adding/subtracting/multiplying radicals. Performing these operations with radicals is much the same as performing these operations with polynomials. If you don't remember how to add/subtract/multiply polynomials we will give a quick reminder here and then give a more in depth set of examples the next section.

Recall that to add/subtract terms with x in them all we need to do is add/subtract the coefficients of the x. For example,

$$4x + 9x = \left(4+9\right)x = 13x \qquad\qquad 3x - 11x = \left(3-11\right)x = -8x$$

Adding/subtracting radicals works in exactly the same manner. For instance,

$$4\sqrt{x} + 9\sqrt{x} = \left(4+9\right)\sqrt{x} = 13\sqrt{x} \qquad\qquad 3\,\sqrt[10]{5} - 11\,\sqrt[10]{5} = \left(3-11\right)\sqrt[10]{5} = -8\,\sqrt[10]{5}$$

We've already seen some multiplication of radicals in the last part of the previous example. If we are looking at the product of two radicals with the same index then all we need to do is use the second property of radicals to combine them then simplify. What we need to look at now are problems like the following set of examples.

Example 4 Multiply each of the following.

(a) $\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right)$ [Solution]

(b) $\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right)$ [Solution]

(c) $\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)$ [Solution]

Solution

In all of these problems all we need to do is recall how to FOIL binomials. Recall,

$$\left(3x-5\right)\left(x+2\right) = 3x\left(x\right) + 3x\left(2\right) - 5\left(x\right) - 5\left(2\right) = 3x^2 + 6x - 5x - 10 = 3x^2 + x - 10$$

With radicals we multiply in exactly the same manner. The main difference is that on occasion we'll need to do some simplification after doing the multiplication

(a) $\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right)$

$$\begin{aligned}\left(\sqrt{x}+2\right)\left(\sqrt{x}-5\right) &= \sqrt{x}\left(\sqrt{x}\right) - 5\sqrt{x} + 2\sqrt{x} - 10\\ &= \sqrt{x^2} - 3\sqrt{x} - 10\\ &= x - 3\sqrt{x} - 10\end{aligned}$$

As noted above we did need to do a little simplification on the first term after doing the multiplication.

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(b) $\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right)$

Don't get excited about the fact that there are two variables here. It works the same way!

$$\begin{aligned}\left(3\sqrt{x}-\sqrt{y}\right)\left(2\sqrt{x}-5\sqrt{y}\right) &= 6\sqrt{x^2}-15\sqrt{x}\sqrt{y}-2\sqrt{x}\sqrt{y}+5\sqrt{y^2}\\ &= 6x-15\sqrt{xy}-2\sqrt{xy}+5y\\ &= 6x-17\sqrt{xy}+5y\end{aligned}$$

Again, notice that we combined up the terms with two radicals in them.

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(c) $\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)$

Not much to do with this one.

$$\left(5\sqrt{x}+2\right)\left(5\sqrt{x}-2\right)=25\sqrt{x^2}-10\sqrt{x}+10\sqrt{x}-4=25x-4$$

Notice that, in this case, the answer has no radicals. That will happen on occasion so don't get excited about it when it happens.

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The last part of the previous example really used the fact that

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

If you don't recall this formula we will look at it in a little more detail in the next section.

Okay, we are now ready to take a look at some simplification examples illustrating the final two rules. Note as well that the fourth rule says that we shouldn't have any radicals in the denominator. To get rid of them we will use some of the multiplication ideas that we looked at above and the process of getting rid of the radicals in the denominator is called **rationalizing the denominator**. In fact, that is really what this next set of examples is about. They are really more examples of rationalizing the denominator rather than simplification examples.

Example 5 Rationalize the denominator for each of the following.

(a) $\dfrac{4}{\sqrt{x}}$ [Solution]

(b) $\sqrt[5]{\dfrac{2}{x^3}}$ [Solution]

(c) $\dfrac{1}{3-\sqrt{x}}$ [Solution]

(d) $\dfrac{5}{4\sqrt{x}+\sqrt{3}}$ [Solution]

Solution

There are really two different types of problems that we'll be seeing here. The first two parts illustrate the first type of problem and the final two parts illustrate the second type of problem.

Both types are worked differently.

(a) $\dfrac{4}{\sqrt{x}}$

In this case we are going to make use of the fact that $\sqrt[n]{a^n} = a$. We need to determine what to multiply the denominator by so that this will show up in the denominator. Once we figure this out we will multiply the numerator and denominator by this term.

Here is the work for this part.

$$\frac{4}{\sqrt{x}} = \frac{4}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{4\sqrt{x}}{\sqrt{x^2}} = \frac{4\sqrt{x}}{x}$$

Remember that if we multiply the denominator by a term we must also multiply the numerator by the same term. In this way we are really multiplying the term by 1 (since $\frac{a}{a} = 1$) and so aren't changing its value in any way.

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(b) $\sqrt[5]{\dfrac{2}{x^3}}$

We'll need to start this one off with first using the third property of radicals to eliminate the fraction from underneath the radical as is required for simplification.

$$\sqrt[5]{\frac{2}{x^3}} = \frac{\sqrt[5]{2}}{\sqrt[5]{x^3}}$$

Now, in order to get rid of the radical in the denominator we need the exponent on the x to be a 5. This means that we need to multiply by $\sqrt[5]{x^2}$ so let's do that.

$$\sqrt[5]{\frac{2}{x^3}} = \frac{\sqrt[5]{2}}{\sqrt[5]{x^3}}\frac{\sqrt[5]{x^2}}{\sqrt[5]{x^2}} = \frac{\sqrt[5]{2x^2}}{\sqrt[5]{x^5}} = \frac{\sqrt[5]{2x^2}}{x}$$

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(c) $\dfrac{1}{3-\sqrt{x}}$

In this case we can't do the same thing that we did in the previous two parts. To do this one we will need to instead to make use of the fact that

$$(a+b)(a-b) = a^2 - b^2$$

When the denominator consists of two terms with at least one of the terms involving a radical we will do the following to get rid of the radical.

$$\frac{1}{3-\sqrt{x}} = \frac{1}{\left(3-\sqrt{x}\right)}\frac{3+\sqrt{x}}{\left(3+\sqrt{x}\right)} = \frac{3+\sqrt{x}}{\left(3-\sqrt{x}\right)\left(3+\sqrt{x}\right)} = \frac{3+\sqrt{x}}{9-x}$$

So, we took the original denominator and changed the sign on the second term and multiplied the numerator and denominator by this new term. By doing this we were able to eliminate the radical in the denominator when we then multiplied out.

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(d) $\dfrac{5}{4\sqrt{x}+\sqrt{3}}$

This one works exactly the same as the previous example. The only difference is that both terms in the denominator now have radicals. The process is the same however.

$$\frac{5}{4\sqrt{x}+\sqrt{3}}=\frac{5}{\left(4\sqrt{x}+\sqrt{3}\right)}\frac{\left(4\sqrt{x}-\sqrt{3}\right)}{\left(4\sqrt{x}-\sqrt{3}\right)}=\frac{5\left(4\sqrt{x}-\sqrt{3}\right)}{\left(4\sqrt{x}+\sqrt{3}\right)\left(4\sqrt{x}-\sqrt{3}\right)}=\frac{5\left(4\sqrt{x}-\sqrt{3}\right)}{16x-3}$$

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Rationalizing the denominator may seem to have no real uses and to be honest we won't see many uses in an Algebra class. However, if you are on a track that will take you into a Calculus class you will find that rationalizing is useful on occasion at that level.

We will close out this section with a more general version of the first property of radicals. Recall that when we first wrote down the properties of radicals we required that a be a positive number. This was done to make the work in this section a little easier. However, with the first property that doesn't necessarily need to be the case.

Here is the property for a general a (*i.e.* positive or negative)

$$\sqrt[n]{a^n}=\begin{cases}|a| & \text{if } n \text{ is even}\\ a & \text{if } n \text{ is odd}\end{cases}$$

where $|a|$ is the absolute value of a. If you don't recall absolute value we will cover that in detail in a section in the next chapter. All that you need to do is know at this point is that absolute value always makes a a positive number.

So, as a quick example this means that,

$$\sqrt[8]{x^8}=|x| \qquad \text{AND} \qquad \sqrt[11]{x^{11}}=x$$

For square roots this is,

$$\sqrt{x^2}=|x|$$

This will not be something we need to worry all that much about here, but again there are topics in courses after an Algebra course for which this is an important idea so we needed to at least acknowledge it.

Polynomials

In this section we will start looking at polynomials. Polynomials will show up in pretty much every section of every chapter in the remainder of this material and so it is important that you understand them.

We will start off with **polynomials in one variable**. Polynomials in one variable are algebraic expressions that consist of terms in the form ax^n where n is a non-negative (*i.e.* positive or zero) integer and a is a real number and is called the **coefficient** of the term. The **degree** of a polynomial in one variable is the largest exponent in the polynomial.

Note that we will often drop the "in one variable" part and just say polynomial.

Here are examples of polynomials and their degrees.

$$
\begin{array}{ll}
5x^{12} - 2x^6 + x^5 - 198x + 1 & \text{degree : 12} \\
x^4 - x^3 + x^2 - x + 1 & \text{degree : 4} \\
56x^{23} & \text{degree : 23} \\
5x - 7 & \text{degree : 1} \\
-8 & \text{degree : 0}
\end{array}
$$

So, a polynomial doesn't have to contain all powers of x as we see in the first example. Also, polynomials can consist of a single term as we see in the third and fifth example.

We should probably discuss the final example a little more. This really is a polynomial even it may not look like one. Remember that a polynomial is any algebraic expression that consists of terms in the form ax^n. Another way to write the last example is

$$-8x^0$$

Written in this way makes it clear that the exponent on the x is a zero (this also explains the degree…) and so we can see that it really is a polynomial in one variable.

Here are some examples of things that aren't polynomials.

$$
\begin{array}{l}
4x^6 + 15x^{-8} + 1 \\
5\sqrt{x} - x + x^2 \\
\dfrac{2}{x} + x^3 - 2
\end{array}
$$

The first one isn't a polynomial because it has a negative exponent and all exponents in a polynomial must be positive.

To see why the second one isn't a polynomial let's rewrite it a little.

$$5\sqrt{x} - x + x^2 = 5x^{\frac{1}{2}} - x + x^2$$

By converting the root to exponent form we see that there is a rational root in the algebraic expression. All the exponents in the algebraic expression must be integers in order for the algebraic expression to be a polynomial. As a general rule of thumb if an algebraic expression has a radical in it then it isn't a polynomial.

Let's also rewrite the third one to see why it isn't a polynomial.

$$\frac{2}{x}+x^3-2=2x^{-1}+x^3-2$$

So, this algebraic expression really has a negative exponent in it and we know that isn't allowed. Another rule of thumb is if there are any variables in the denominator of a fraction then the algebraic expression isn't a polynomial.

Note that this doesn't mean that radicals and fractions aren't allowed in polynomials. They just can't involve the variables. For instance, the following is a polynomial

$$\sqrt[3]{5}\,x^4-\frac{7}{12}x^2+\frac{1}{\sqrt{8}}x-5\,\sqrt[14]{113}$$

There are lots of radicals and fractions in this algebraic expression, but the denominators of the fractions are only numbers and the radicands of each radical are only a numbers. Each x in the algebraic expression appears in the numerator and the exponent is a positive (or zero) integer. Therefore this is a polynomial.

Next, let's take a quick look at **polynomials in two variables**. Polynomials in two variables are algebraic expressions consisting of terms in the form $ax^n y^m$. The degree of each term in a polynomial in two variables is the sum of the exponents in each term and the **degree** of the polynomial is the largest such sum.

Here are some examples of polynomials in two variables and their degrees.

$x^2y-6x^3y^{12}+10x^2-7y+1$	degree : 15
$6x^4+8y^4-xy^2$	degree : 4
$x^4y^2-x^3y^3-xy+x^4$	degree : 6
$6x^{14}-10y^3+3x-11y$	degree : 14

In these kinds of polynomials not every term needs to have both x's and y's in them, in fact as we see in the last example they don't need to have any terms that contain both x's and y's. Also, the degree of the polynomial may come from terms involving only one variable. Note as well that multiple terms may have the same degree.

We can also talk about polynomials in three variables, or four variables or as many variables as we need. The vast majority of the polynomials that we'll see in this course are polynomials in one variable and so most of the examples in the remainder of this section will be polynomials in one variable.

Next we need to get some terminology out of the way. A **monomial** is a polynomial that consists of exactly one term. A **binomial** is a polynomial that consists of exactly two terms. Finally, a **trinomial** is a polynomial that consists of exactly three terms. We will use these terms off and on so you should probably be at least somewhat familiar with them.

Now we need to talk about adding, subtracting and multiplying polynomials. You'll note that we left out division of polynomials. That will be discussed in a later section where we will use division of polynomials quite often.

Before actually starting this discussion we need to recall the distributive law. This will be used repeatedly in the remainder of this section. Here is the distributive law.

$$a\left(b+c\right)=ab+ac$$

We will start with adding and subtracting polynomials. This is probably best done with a couple of examples.

Example 1 Perform the indicated operation for each of the following.

(a) Add $6x^5-10x^2+x-45$ to $13x^2-9x+4$. [Solution]

(b) Subtract $5x^3-9x^2+x-3$ from x^2+x+1. [Solution]

Solution

(a) Add $6x^5-10x^2+x-45$ to $13x^2-9x+4$.

The first thing that we should do is actually write down the operation that we are being asked to do.

$$\left(6x^5-10x^2+x-45\right)+\left(13x^2-9x+4\right)$$

In this case the parenthesis are not required since are adding the two polynomials. They are there simply to make clear the operation that we are performing. To add two polynomials all that we do is **combine like terms**. This means that for each term with the same exponent we will add or subtract the coefficient of that term.

In this case this is,

$$\begin{aligned}\left(6x^5-10x^2+x-45\right)+\left(13x^2-9x+4\right)&=6x^5+\left(-10+13\right)x^2+\left(1-9\right)x-45+4\\&=6x^5+3x^2-8x-41\end{aligned}$$

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(b) Subtract $5x^3-9x^2+x-3$ from x^2+x+1.

Again, let's write down the operation we are doing here. We will also need to be very careful with the order that we write things down in. Here is the operation

$$x^2+x+1-\left(5x^3-9x^2+x-3\right)$$

This time the parentheses around the second term are absolutely required. We are subtracting the whole polynomial and the parenthesis must be there to make sure we are in fact subtracting the whole polynomial.

In doing the subtraction the first thing that we'll do is **distribute the minus sign** through the parenthesis. This means that we will change the sign on every term in the second polynomial. Note that all we are really doing here is multiplying a "-1" through the second polynomial using the distributive law. After distributing the minus through the parenthesis we again combine like terms.

Here is the work for this problem.

$$\begin{aligned}x^2+x+1-\left(5x^3-9x^2+x-3\right)&=x^2+x+1-5x^3+9x^2-x+3\\&=-5x^3+10x^2+4\end{aligned}$$

Note that sometimes a term will completely drop out after combing like terms as the x did here. This will happen on occasion so don't get excited about it when it does happen.

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Now let's move onto multiplying polynomials. Again, it's best to do these in an example.

Example 2 Multiply each of the following.

(a) $4x^2\left(x^2-6x+2\right)$ [Solution]

(b) $\left(3x+5\right)\left(x-10\right)$ [Solution]

(c) $\left(4x^2-x\right)\left(6-3x\right)$ [Solution]

(d) $\left(3x+7y\right)\left(x-2y\right)$ [Solution]

(e) $\left(2x+3\right)\left(x^2-x+1\right)$ [Solution]

Solution

(a) $4x^2\left(x^2-6x+2\right)$

This one is nothing more than a quick application of the distributive law.

$$4x^2\left(x^2-6x+2\right)=4x^4-24x^3+8x^2$$

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(b)

$\left(3x+5\right)\left(x-10\right)$ This one will use the FOIL method for multiplying these two binomials.

$$\left(3x+5\right)\left(x-10\right)=\underbrace{3x^2}_{\text{First Terms}}-\underbrace{30x}_{\text{Outer Terms}}+\underbrace{5x}_{\text{Inner Terms}}-\underbrace{50}_{\text{Last Terms}}=3x^2-25x-50$$

Recall that the FOIL method will only work when multiplying two binomials. If either of the polynomials isn't a binomial then the FOIL method won't work.

Also note that all we are really doing here is multiplying every term in the second polynomial by every term in the first polynomial. The FOIL acronym is simply a convenient way to remember this.

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(c) $\left(4x^2-x\right)\left(6-3x\right)$

Again we will just FOIL this one out.

$$\left(4x^2-x\right)\left(6-3x\right)=24x^2-12x^3-6x+3x^2=-12x^3+27x^2-6x$$

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(d) $\left(3x+7y\right)\left(x-2y\right)$

We can still FOIL binomials that involve more than one variable so don't get excited about these kinds of problems when they arise.

$$\left(3x+7y\right)\left(x-2y\right)=3x^2-6xy+7xy-14y^2=3x^2+xy-14y^2$$

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(e) $(2x+3)(x^2-x+1)$

In this case the FOIL method won't work since the second polynomial isn't a binomial. Recall however that the FOIL acronym was just a way to remember that we multiply every term in the second polynomial by every term in the first polynomial.

That is all that we need to do here.

$$(2x+3)(x^2-x+1)=2x^3-2x^2+2x+3x^2-3x+3=2x^3+x^2-x+3$$

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Let's work another set of examples that will illustrate some nice formulas for some special products. We will give the formulas after the example.

Example 3 Multiply each of the following.

(a) $(3x+5)(3x-5)$ [Solution]

(b) $(2x+6)^2$ [Solution]

(c) $(1-7x)^2$ [Solution]

(d) $4(x+3)^2$ [Solution]

Solution

(a) $(3x+5)(3x-5)$

We can use FOIL on this one so let's do that.

$$(3x+5)(3x-5)=9x^2-15x+15x-25=9x^2-25$$

In this case the middle terms drop out.

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(b) $(2x+6)^2$

Now recall that $4^2=(4)(4)=16$. Squaring with polynomials works the same way. So in this case we have,

$$(2x+6)^2=(2x+6)(2x+6)=4x^2+12x+12x+36=4x^2+24x+36$$

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(c) $(1-7x)^2$

This one is nearly identical to the previous part.

$$(1-7x)^2=(1-7x)(1-7x)=1-7x-7x+49x^2=1-14x+49x^2$$

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(d) $4\left(x+3\right)^2$

This part is here to remind us that we need to be careful with coefficients. When we've got a coefficient we MUST do the exponentiation first and then multiply the coefficient.

$$4\left(x+3\right)^2 = 4\left(x+3\right)\left(x+3\right) = 4\left(x^2+6x+9\right) = 4x^2+24x+36$$

You can only multiply a coefficient through a set of parenthesis if there is an exponent of "1" on the parenthesis. If there is any other exponent then you CAN'T multiply the coefficient through the parenthesis.

Just to illustrate the point.

$$4\left(x+3\right)^2 \neq \left(4x+12\right)^2 = \left(4x+12\right)\left(4x+12\right) = 16x^2+96x+144$$

This is clearly not the same as the correct answer so be careful!

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The parts of this example all use one of the following special products.

$$\left(a+b\right)\left(a-b\right) = a^2-b^2$$

$$\left(a+b\right)^2 = a^2+2ab+b^2$$

$$\left(a-b\right)^2 = a^2-2ab+b^2$$

Be careful to not make the following mistakes!

$$\left(a+b\right)^2 \neq a^2+b^2$$

$$\left(a-b\right)^2 \neq a^2-b^2$$

These are very common mistakes that students often make when they first start learning how to multiply polynomials.