ALGEBRA QUALIFYING EXAM PROBLEMS
RING THEORY

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CONTENTS

RING THEORY

General Ring Theory

1. Give an example of each of the following.
   (a) An irreducible polynomial of degree 3 in $\mathbb{Z}_3[x]$.
   (b) A polynomial in $\mathbb{Z}[x]$ that is not irreducible in $\mathbb{Z}[x]$ but is irreducible in $\mathbb{Q}[x]$.
   (c) A non-commutative ring of characteristic p, p a prime.
   (d) A ring with exactly 6 invertible elements.
   (e) An infinite non-commutative ring with only finitely many ideals.
   (f) An infinite non-commutative ring with non-zero characteristic.
   (g) An integral domain which is not a unique factorization domain.
   (h) A unique factorization domain that is not a principal ideal domain.
   (i) A principal ideal domain that is not a Euclidean domain.
   (j) A Euclidean domain other than the ring of integers or a field.
   (k) A finite non-commutative ring.
   (l) A commutative ring with a sequence $\{P_n\}_{n=1}^{\infty}$ of prime ideals such that P_n is properly contained in P_{n+1} for all n.
   (m) A non-zero prime ideal of a commutative ring that is not a maximal ideal.
   (n) An irreducible element of a commutative ring that is not a prime element.
   (o) An irreducible element of an integral domain that is not a prime element.
   (p) A commutative ring that has exactly one maximal ideal and is not a field.
   (q) A non-commutative ring with exactly two maximal ideals.

2. (a) How many units does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.
   (b) How many ideals does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.

3. How many ideals does the ring $\mathbb{Z}/90\mathbb{Z}$ have? Explain your answer.

4. Denote the set of invertible elements of the ring $\mathbb{Z}_n$ by U_n.
   Answer the following for $n = 18$, $n = 20$, $n = 24$.
   (a) List all the elements of U_n.
   (b) Is U_n a cyclic group under multiplication? Justify your answer.

5. Find all positive integers n having the property that the group of units of $\mathbb{Z}/n\mathbb{Z}$ is an elementary abelian 2-group.

6. Let $U(R)$ denote the group of units of a ring R. Prove that if m divides n, then the natural ring homomorphism $\mathbb{Z}_n \to \mathbb{Z}_m$ maps $U(\mathbb{Z}_n)$ onto $U(\mathbb{Z}_m)$.
   Give an example that shows that $U(R)$ does not have to map onto $U(S)$ under a surjective ring homomorphism $R \to S$.

7. If p is a prime satisfying $p \equiv 1 \pmod 4$, then p is a sum of two squares.

8. If $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol, prove Euler's Criterion: if p is a prime and a is any integer relatively prime to p, then $a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod p$.

9. Let R_1 and R_2 be commutative rings with identities and let $R = R_1 \times R_2$. Show that every ideal I of R is of the form $I = I_1 \times I_2$ with I_i an ideal of R_i for $i = 1, 2$.

10. Show that a non-zero ring R in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and is commutative.

11. Let R be a finite commutative ring with more than one element and no zero-divisors. Show that R is a field.

12. Determine for which integers n the ring $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of fields. Prove your answer.

13. Let R be a subring of a field F such that for each x in F either $x \in R$ or $x^{-1} \in R$. Prove that if I and J are two ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

14. The *Jacobson Radical* $J(R)$ of a ring R is defined to be the intersection of all maximal ideals of R.
    Let R be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all y in R.

15. Let R be any ring with identity, and n any positive integer. If $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in R, prove that $M_n(I)$ is an ideal of $M_n(R)$ whenever I is an ideal of R, and that every ideal of $M_n(R)$ has this form.

16. Let m, n be positive integers such that m divides n. Then the natural map $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + (n) \mapsto a + (m)$ is a surjective ring homomorphism. If U_n, U_m are the units of $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively, show that $\varphi : U_n \to U_m$ is a surjective group homomorphism.

17. Let R be a ring with ideals A and B. Let $R/A \times R/B$ be the ring with coordinate-wise addition and multiplication. Show the following.
    (a) The map $R \to R/A \times R/B$ given by $r \mapsto (r + A, r + B)$ is a ring homomorphism.
    (b) The homomorphism in part (a) is surjective if and only if $A + B = R$.

18. Let m and n be relatively prime integers.
    (a) Show that if c and d are any integers, then there is an integer x such that $x \equiv c \pmod{m}$ and $x \equiv d \pmod{n}$.
    (b) Show that $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic as rings.

19. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I + J = R$. Show that $I \cdot J = I \cap J$.

20. [**NEW**]
    Give an example of a commutative ring R and ideals I and J in which $I \cdot J \neq I \cap J$.
    Also, prove that if $I + J = R$ then necessarily $I \cdot J = I \cap J$.

21. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

22. Let R be a commutative ring with identity and let $I_1, I_2, \ldots, I_n$ be pairwise co-maximal ideals of R (i.e., $I_i + I_j = R$ if $i \neq j$). Show that $I_i + \bigcap_{j \neq i} I_j = R$ for all i.

23. Let R be a commutative ring, not necessarily with identity, and assume there is some fixed positive integer n such that $nr = 0$ for all $r \in R$. Prove that R embeds in a ring S with identity so that R is an ideal of S and $S/R \cong \mathbb{Z}/n\mathbb{Z}$.

24. Let R be a ring with identity 1 and $a, b \in R$ such that $ab = 1$. Denote $X = \{x \in R \mid ax = 1\}$. Show the following.
    (a) If $x \in X$, then $b + (1 - xa) \in X$.
    (b) If $\varphi : X \to X$ is the mapping given by $\varphi(x) = b + (1 - xa)$, then φ is one-to-one.
    (c) If X has more than one element, then X is an infinite set.

25. Let R be a commutative ring with identity and define $U_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in R \right\}$. Prove that every R-automorphism of $U_2(R)$ is inner.

26. Let $\mathbb{R}$ be the field of real numbers and let F be the set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -3b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$. Show that F is a field under the usual matrix operations.

27. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are real numbers. Prove that R is isomorphic to $\mathbb{C}$, the field of complex numbers.

28. Let p be a prime and let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ pb & a \end{bmatrix}$, where $a, b \in \mathbb{Z}$. Prove that R is isomorphic to $\mathbb{Z}[\sqrt{p}]$.

29. Let p be a prime and F_p the set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where $a, b \in \mathbb{Z}_p$.
    (a) Show that F_p is a commutative ring with identity.
    (b) Show that F_7 is a field.
    (c) Show that F_{13} is not a field.

30. Let $I \subseteq J$ be right ideals of a ring R such that $J/I \cong R$ as right R-modules. Prove that there exists a right ideal K such that $I \cap K = (0)$ and $I + K = J$.

31. A ring R is called simple if $R^2 \neq 0$ and 0 and R are its only ideals. Show that the center of a simple ring is 0 or a field.

32. Give an example of a field F and a one-to-one ring homomorphism $\varphi : F \to F$ which is not onto. Verify your example.

33. Let D be an integral domain and let $D[x_1, x_2, \ldots, x_n]$ be the polynomial ring over D in the n indeterminates $x_1, x_2, \ldots, x_n$. Let

$$V = \begin{bmatrix} x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & \cdots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ x_n^{n-1} & \cdots & x_n^2 & x_n & 1 \end{bmatrix}.$$

Prove that the determinant of V is $\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$.

34. Let $R = C[0,1]$ be the set of all continuous real-valued functions on $[0,1]$. Define addition and multiplication on R as follows. For $f, g \in R$ and $x \in [0,1]$,

$$(f+g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x).$$

(a) Show that R with these operations is a commutative ring with identity.

(b) Find the units of R.

(c) If $f \in R$ and $f^2 = f$, then $f = 0_R$ or $f = 1_R$.

(d) If n is a positive integer and $f \in R$ is such that $f^n = 0_R$, then $f = 0_R$.

35. Let S be the ring of all bounded, continuous functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Let I be the set of functions f in S such that $f(t) \to 0$ as $|t| \to \infty$.

(a) Show that I is an ideal of S.

(b) Suppose $x \in S$ is such that there is an $i \in I$ with $ix = x$. Show that $x(t) = 0$ for all sufficiently large $|t|$.

36. Let $\mathbb{Q}$ be the field of rational numbers and $D = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

(a) Show that D is a subring of the field of real numbers.

(b) Show that D is a principal ideal domain.

(c) Show that $\sqrt{3}$ is not an element of D.

37. Show that if p is a prime such that $p \equiv 1 \pmod 4$, then $x^2 + 1$ is not irreducible in $\mathbb{Z}_p[x]$.

38. Show that if p is a prime such that $p \equiv 3 \pmod 4$, then $x^2 + 1$ is irreducible in $\mathbb{Z}_p[x]$.

39. Show that if p is a prime such that $p \equiv 1 \pmod 6$, then $x^3 + 1$ splits in $\mathbb{Z}_p[x]$.

Prime, Maximal, and Primary Ideals

40. Let R be a non-zero commutative ring with 1. Show that an ideal M of R is maximal if and only if R/M is a field.

41. Let R be a commutative ring with 1. Show that an ideal P of R is prime if and only if R/P is an integral domain.

42. (a) Let R be a commutative ring with 1. Show that if M is a maximal ideal of R then M is a prime ideal of R.

(b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.

43. Let R be a non-zero ring with identity. Show that every proper ideal of R is contained in a maximal ideal.

44. Let R be a commutative ring with 1 and P a prime ideal of R. Show that if I and J are ideals of R such that $I \cap J \subseteq P$ and $J \not\subseteq P$, then $I \subseteq P$.

45. Let $M_1 \neq M_2$ be two maximal ideals in the commutative ring R and let $I = M_1 \cap M_2$. Prove that R/I is isomorphic to the direct sum of two fields.

46. Let R be a non-zero commutative ring with 1. Show that if I is an ideal of R such that $1+a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R.

47. Let R be a commutative ring with identity. Suppose R contains an idempotent element a other than 0 or 1. Show that every prime ideal in R contains an idempotent element other than 0 or 1. (An element $a \in R$ is idempotent if $a^2 = a$.)

48. Let R be a commutative ring with 1.
    (a) Prove that (x) is a prime ideal in $R[x]$ if and only if R is an integral domain.
    (b) Prove that (x) is a maximal ideal in $R[x]$ if and only if R is a field.

49. Find all values of a in $\mathbb{Z}_3$ such that the quotient ring

$$\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)$$

is a field. Justify your answer.

50. Find all values of a in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

is a field. Justify your answer.

51. Let R be a commutative ring with identity and let U be maximal among non-finitely generated ideals of R. Prove U is a prime ideal.

52. Let R be a commutative ring with identity and let U be maximal among non-principal ideals of R. Prove U is a prime ideal.

53. Let R be a non-zero commutative ring with 1 and S a multiplicative subset of R not containing 0. Show that if P is maximal in the set of ideals of R not intersecting S, then P is a prime ideal.

54. Prove that the set of nilpotent elements of a commutative ring R is contained in the intersection of all prime ideals of R.

55. **[NEW]**
    Let R be a ring in which there are no non-zero nilpotent elements. Prove that every idempotent is central.

56. Let R be a non-zero commutative ring with 1.
    (a) Let S be a multiplicative subset of R not containing 0 and let P be maximal in the set of ideals of R not intersecting S. Show that P is a prime ideal.
    (b) Show that the set of nilpotent elements of R is the intersection of all prime ideals.

57. Let R be a commutative ring with identity and let $x \in R$ be a non-nilpotent element. Prove that there exists a prime ideal P of R such that $x \notin P$.

58. Let R be a commutative ring with identity and let S be the set of all elements of R that are *not* zero-divisors. Show that there is a prime ideal P such that $P \cap S$ is empty. (Hint: Use Zorn's Lemma.)

59. Let R be a commutative ring with identity and let $\mathcal{C}$ be a chain of prime ideals of R. Show that $\bigcup_{P\in\mathcal{C}} P$ and $\bigcap_{P\in\mathcal{C}} P$ are prime ideals of R.

60. Let R be a commutative ring and P a prime ideal of R. Show that there is a prime ideal $P_0 \subseteq P$ that does not properly contain any prime ideal.

61. Let R be a commutative ring with 1 such that for every x in R there is an integer $n > 1$ (depending on x) such that $x^n = x$. Show that every prime ideal of R is maximal.

62. Let R be a commutative ring with 1 in which every ideal is a prime ideal. Prove that R is a field. (Hint: For $a \neq 0$ consider the ideals (a) and (a^2).)

63. Let D be a principal ideal domain. Prove that every nonzero prime ideal of D is a maximal ideal.

64. Show that if R is a finite commutative ring with identity, then every prime ideal of R is a maximal ideal.

65. Let $R = C[0,1]$ be the ring of all continuous real-valued functions on $[0,1]$, with addition and multiplication defined as follows. For $f, g \in R$ and $x \in [0,1]$,
$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x).$$
Prove that if M is a maximal ideal of R, then there is a real number $x_0 \in [0,1]$ such that $M = \{f \in R \mid f(x_0) = 0\}$.

66. Let R be a commutative ring with identity, and let $P \subset Q$ be prime ideals of R. Prove that there exist prime ideals P^*,Q^* satisfying $P \subseteq P^* \subset Q^* \subseteq Q$, such that there are no prime ideals strictly between P^* and Q^*. HINT: Fix $x \in Q - P$ and show that there exists a prime ideal P^* containing P, contained in Q and maximal with respect to not containing x.

67. Let R be a commutative ring with 1. An ideal I of R is called a *primary* ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geqslant 1$.
   (a) Show that an ideal I of R is primary if and only if $R/I \neq 0$ and every zero-divisor in R/I is nilpotent.
   (b) Show that if I is a primary ideal of R then the radical $\mathrm{Rad}(I)$ of I is a prime ideal. (Recall that $\mathrm{Rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n\}$.)

Commutative Rings

68. Let R be a commutative ring with identity. Show that R is an integral domain if and only if R is a subring of a field.

69. Let R be a commutative ring with identity. Show that if x and y are nilpotent elements of R then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in R.

70. Let R be a commutative ring with identity. An ideal I of R is *irreducible* if it cannot be expressed as the intersection of two ideals of R neither of which is contained in the other. Show the following.
   (a) If P is a prime ideal then P is irreducible.
   (b) If x is a non-zero element of R, then there is an ideal I_x, maximal with respect to the property that $x \notin I_x$, and I_x is irreducible.
   (c) If every irreducible ideal of R is a prime ideal, then 0 is the only nilpotent element of R.

71. Let R be a commutative ring with 1 and let I be an ideal of R satisfying $I^2 = \{0\}$. Show that if $a + I \in R/I$ is an idempotent element of R/I, then the coset $a + I$ contains an idempotent element of R.

72. Let R be a commutative ring with identity that has exactly one prime ideal P. Prove the following.
   (a) R/P is a field.
   (b) R is isomorphic to R_P, the ring of quotients of R with respect to the multiplicative set $R - P = \{s \in R \mid s \notin P\}$.

73. Let R be a commutative ring with identity and $\sigma : R \to R$ a ring automorphism.
   (a) Show that $F = \{r \in R \mid \sigma(r) = r\}$ is a subring of R and the identity of R is in F.
   (b) Show that if σ^2 is the identity map on R, then each element of R is the root of a monic polynomial of degree two in $F[x]$.

74. Let R be a commutative ring with identity that has exactly three ideals, $\{0\}$, I, and R.
   (a) Show that if $a \notin I$, then a is a unit of R.
   (b) Show that if $a, b \in I$ then $ab = 0$.

75. Let R be a commutative ring with 1. Show that if u is a unit in R and n is nilpotent, then $u + n$ is a unit.

76. Let R be a commutative ring with identity. Suppose that for every $a \in R$, either a or $1 - a$ is invertible. Prove that $N = \{a \in R \mid a \text{ is not invertible}\}$ is an ideal of R.

77. Let R be a commutative ring in which any two ideals are comparable (that is, either $I \subseteq J$ or $J \subseteq I$). Prove that every finitely generated ideal of R is principal.

78. Let R be a commutative ring with 1. Show that the sum of any two principal ideals of R is principal if and only if every finitely generated ideal of R is principal.

79. Let R be an integral domain. Show that if all prime ideals of R are principal, then R is a Principal Ideal Domain.

80. Let R be a commutative ring with identity such that not every ideal is a principal ideal.
   (a) Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.
   (b) If I is the ideal of part (a), show that R/I is a principal ideal ring.

81. Recall that if $R \subseteq S$ is an inclusion of commutative rings (with the same identity) then an element $s \in S$ is *integral over* R if s satisfies some monic polynomial with coefficients in R. Prove the equivalence of the following statements.
   (i) s is integral over R.
   (ii) $R[s]$ is finitely generated as an R-module.
   (iii) There exists a faithful $R[s]$ module which is finitely generated as an R-module.

82. Recall that if $R \subseteq S$ is an inclusion of commutative rings (with the same identity) then S is an *integral* extension of R if every element of S satisfies some monic polynomial with coefficients in R. Prove that if $R \subseteq S \subseteq T$ are commutative rings with the same identity, then S is integral over R and T is integral over S if and only if T is integral over R.

83. Let $R \subseteq S$ be commutative domains with the same identity, and assume that S is an integral extension of R. Let I be a nonzero ideal of S. Prove that $I \cap R$ is a nonzero ideal of R.

Domains

84. Suppose R is a domain and I and J are ideals of R such that IJ is principal. Show that I (and by symmetry J) is finitely generated.

    [Hint: If $IJ = (a)$, then $a = \sum_{i=1}^{n} x_i y_i$ for some $x_i \in I$ and $y_i \in J$. Show the x_i generate I.]

85. Prove that if D is a Euclidean Domain, then D is a Principal Ideal Domain.

86. Show that if p is a prime such that there is an integer b with $p = b^2 + 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

87. Show that if p is a prime such that $p \equiv 1 \pmod 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

88. Let $D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\}$ — a subring of the field of real numbers and necessarily an integral domain (you need not show this) — and $F = \mathbb{Q}(\sqrt{5})$ its field of fractions. Show the following:
    (a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
    (b) D is not a unique factorization domain.

89. Let $D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of D. Show the following:
    (a) $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$.
    (b) D is not a unique factorization domain.

90. Let $D = \mathbb{Z}(\sqrt{-11}) = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{-11})$ its field of fractions. Show the following:
    (a) $x^2 - x + 3$ is irreducible in $D[x]$ but not in $F[x]$.
    (b) D is not a unique factorization domain.

91. Let $D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{13})$ its field of fractions. Show the following:
    (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
    (b) D is not a unique factorization domain.

92. Let D be an integral domain and F a subring of D that is a field. Show that if each element of D is algebraic over F, then D is a field.

93. Let R be an integral domain containing the subfield F and assume that R is finite dimensional over F when viewed as a vector space over F. Prove that R is a field.

94. Let D be an integral domain.
    (a) For $a, b \in D$ define a *greatest common divisor* of a and b.
    (b) For $x \in D$ denote $(x) = \{dx \mid d \in D\}$. Prove that if $(a) + (b) = (d)$, then d is a greatest common divisor of a and b.

95. Let D be a principal ideal domain.
   (a) For $a, b \in D$, define a *least common multiple* of a and b.
   (b) Show that $d \in D$ is a least common multiple of a and b if and only if $(a) \cap (b) = (d)$.

96. Let D be a principal ideal domain and let $a, b \in D$.
   (a) Show that there is an element $d \in D$ that satisfies the properties
      i. $d|a$ and $d|b$ and
      ii. if $e|a$ and $e|b$ then $e|d$.
   (b) Show that there is an element $m \in D$ that satisfies the properties
      i. $a|m$ and $b|m$ and
      ii. if $a|e$ and $b|e$ then $m|e$.

97. Let R be a principal ideal domain. Show that if (a) is a nonzero ideal in R, then there are only finitely many ideals in R containing (a).

98. Let D be a unique factorization domain and F its field of fractions. Prove that if d is an irreducible element in D, then there is no $x \in F$ such that $x^2 = d$.

99. Let D be a Euclidean domain. Prove that every non-zero prime ideal is a maximal ideal.

100. Let π be an irreducible element of a principal ideal domain R. Prove that π is a prime element (that is, $\pi \mid ab$ implies $\pi \mid a$ or $\pi \mid b$).

101. Let D with $\varphi : D - \{0\} \to \mathbb{N}$ be a Euclidean domain. Suppose $\varphi(a+b) \leqslant \max\{\varphi(a), \varphi(b)\}$ for all $a, b \in D$. Prove that D is either a field or isomorphic to a polynomial ring over a field.

102. Let D be an integral domain and F its field of fractions. Show that if g is an isomorphism of D onto itself, then there is a unique isomorphism h of F onto F such that $h(d) = g(d)$ for all d in D.

103. Let D be a unique factorization domain such that if p and q are irreducible elements of D, then p and q are associates. Show that if A and B are ideals of D, then either $A \subseteq B$ or $B \subseteq A$.

104. Let D be a unique factorization domain and p a fixed irreducible element of D such that if q is any irreducible element of D, then q is an associate of p. Show the following.
   (a) If d is a nonzero element of D, then d is uniquely expressible in the form up^n, where u is a unit of D and n is a non-negative integer.
   (b) D is a Euclidean domain.

105. Prove that $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain.

106. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean ring and compute the greatest common divisor of $5 + i$ and 13 using the Euclidean algorithm.

Polynomial Rings

107. Show that the polynomial $f(x) = x^4 + 5x^2 + 3x + 2$ is irreducible over the field of rational numbers.

108. Let D be an integral domain and $D[x]$ the polynomial ring over D. Suppose $\varphi : D[x] \to D[x]$ is an isomorphism such that $\varphi(d) = d$ for all $d \in D$. Show that $\varphi(x) = ax + b$ for some $a, b \in D$ and that a is a unit of D.

109. Let $f(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_n x^n \in \mathbb{Z}[x]$ and p a prime such that $p|a_i$ for $i = 1, \ldots, k-1$, $p \nmid a_k$, $p \nmid a_n$, and $p^2 \nmid a_0$. Show that $f(x)$ has an irreducible factor in $\mathbb{Z}[x]$ of degree at least k.

110. Let D be an integral domain and $D[x]$ the polynomial ring over D in the indeterminate x. Show that if every nonzero prime ideal of $D[x]$ is a maximal ideal, then D is a field.

111. Let R be a commutative ring with 1 and let $f(x) \in R[x]$ be nilpotent. Show that the coefficients of f are nilpotent.

112. Show that if R is an integral domain and $f(x)$ is a unit in the polynomial ring $R[x]$, then $f(x)$ is in R.

113. Let D be a unique factorization domain and F its field of fractions. Prove that if $f(x)$ is a monic polynomial in $D[x]$ and $\alpha \in F$ is a root of f, then $\alpha \in D$.

114. Explain why $F = \mathbb{Z}_3[x]/\langle x^3 + x^2 + 2 \rangle$ is a field and find the multiplicative inverse of $x^2 + 1$ in F.

115. (a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.
    (b) Show that $x^4 + 1$ is not irreducible in $\mathbb{Z}_3[x]$.

116. Let $F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

117. Let F be a field. Prove that the polynomial ring $F[x]$ is a PID and that $F[x, y]$ is not a PID.

118. Let D be an integral domain and let c be an irreducible element in D. Show that the ideal (x, c) generated by x and c in the polynomial ring $D[x]$ is not a principal ideal.

119. Show that if R is a commutative ring with 1 that is not a field, then $R[x]$ is not a principal ideal domain.

120. (a) Let $\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{a}{2^n} \;\middle|\; a, n \in \mathbb{Z},\, n \geqslant 0 \right\}$, the smallest subring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\frac{1}{2}$. Let $(2x - 1)$ be the ideal of $\mathbb{Z}[x]$ generated by the polynomial $2x - 1$. Show that $\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}\left[\frac{1}{2}\right]$.
    (b) Find an ideal I of $\mathbb{Z}[x]$ such that $(2x - 1) \subsetneq I \subsetneq \mathbb{Z}[x]$.

Non-commutative Rings

121. Let R be a ring with identity such that the identity map is the only ring automorphism of R. Prove that the set N of all nilpotent elements of R is an ideal of R.

122. Let p be a prime. A ring S is called a *p-ring* if the characteristic of S is a power of p. Show that if R is a ring with identity of finite characteristic, then R is isomorphic to a finite direct product of p-rings for distinct primes.

123. If R is any ring with identity, let $J(R)$ denote the Jacobson radical of R. Show that if e is any idempotent of R, then $J(eRe) = eJ(R)e$.

124. If n is a positive integer and F is any field, let $M_n(F)$ denote the ring of $n \times n$ matrices with entries in F. Prove that $M_n(F)$ is a simple ring. Equivalently, $\mathrm{End}_F(V)$ is a simple ring if V is a finite dimensional vector space over F.

125. Let R be a ring.
   (a) Show that there is a unique smallest (with respect to inclusion) ideal A such that R/A is a commutative ring.
   (b) Give an example of a ring R such that for every proper ideal I, R/I is not commutative. Verify your example.
   (c) For the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ with the usual matrix operations, find the ideal A of part (a).

126. A ring R is *nilpotent-free* if $a^n = 0$ for $a \in R$ and some positive integer n implies $a = 0$.
   (a) Suppose there is an ideal I such that R/I is nilpotent-free. Show there is a unique smallest (with respect to inclusion) ideal A such that R/A is nilpotent-free.
   (b) Give an example of a ring R such that for every proper ideal I, R/I is not nilpotent-free. Verify your example.
   (c) Show that if R is a commutative ring with identity, then there is a proper ideal I of R such that R/I is nilpotent-free, and find the ideal A of part (a).

Local Rings, Localization, Rings of Fractions

127. Let R be an integral domain. Construct the field of fractions F of R by defining the set F and the two binary operations, and show that the two operations are well-defined. Show that F has a multiplicative identity element and that every nonzero element of F has a multiplicative inverse.

128. A *local* ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring R is local if and only if the set of non-units in R is an ideal.

129. Let R be a commutative ring with $1 \neq 0$ in which the set of nonunits is closed under addition. Prove that R is local, i.e., has a unique maximal ideal.

130. Let D be an integral domain and F its field of fractions. Let P be a prime ideal in D and $D_P = \{ab^{-1} \mid a, b \in D,\ b \notin P\} \subseteq F$. Show that D_P has a unique maximal ideal.

131. Let R be a commutative ring with identity and M a maximal ideal of R. Let R_M be the ring of quotients of R with respect to the multiplicative set $R - M = \{s \in R \mid s \notin M\}$. Show the following.
   (a) $M_M = \{\frac{a}{s} \mid a \in M,\ s \notin M\}$ is the unique maximal ideal of R_M.
   (b) The fields R/M and R_M/M_M are isomorphic.

132. Let R be an integral domain, S a multiplicative set, and let $S^{-1}R = \{\frac{r}{s} \mid r \in R,\ s \in S\}$ (contained in the field of fractions of R). Show that if P is a prime ideal of R, then $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

133. Let R be a commutative ring with identity and P a prime ideal of R. Let R_P be the ring of quotients of R with respect to the set $R - P = \{s \in R \mid s \notin P\}$. Show that R_P/P_P is the field of fractions of the integral domain R/P.

134. Let D be an integral domain and F its field of fractions. Denote by $\mathcal{M}$ the set of all maximal ideals of D. For $M \in \mathcal{M}$, let $D_M = \{\frac{a}{s} \mid a, s \in D,\ s \notin M\} \subset F$. Show that $\bigcap_{M \in \mathcal{M}} D_M = D$.

135. Let R be a commutative ring with 1 and D a multiplicative subset of R containing 1. Let J be an ideal in the ring of fractions $D^{-1}R$ and let

$$I = \left\{ a \in R \;\middle|\; \frac{a}{d} \in J \text{ for some } d \in D \right\}.$$

Show that I is an ideal of R.

136. Let D be a principal ideal domain and let P be a non-zero prime ideal. Show that D_P, the localization of D at P, is a principal ideal domain and has a unique irreducible element, up to associates.

Chains and Chain Conditions

137. Let R be a commutative ring with identity. Prove that any non-empty set of prime ideals of R contains maximal *and* minimal elements.

138. Let R be an integral domain that satisfies the descending chain condition; i.e., whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a descending chain of ideals of R, there exists $m \in \mathbb{N}$ such that $I_k = I_m$ for all $k \geqslant m$. Prove that R is a field.

139. Let R be a ring satisfying the descending chain condition on right ideals. Prove that if R is not a division ring, then R contains non-trivial zero divisors.

140. Let R be a commutative ring with 1. We say R satisfies the *ascending chain condition* if whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an ascending chain of ideals, there is an integer N such that $I_k = I_N$ for all $k \geqslant N$. Show that R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.

141. Define *Noetherian ring* and prove that if R is Noetherian, then $R[x]$ is Noetherian.

142. Let R be a commutative Noetherian ring with identity. Prove that there are only finitely many *minimal* prime ideals of R.

143. Let R be a commutative Noetherian ring in which every 2-generated ideal is principal. Prove that R is a Principal Ideal Domain.

144. Let R be a commutative Noetherian ring with identity and let I be an ideal in R. Let $J = \text{Rad}(I)$. Prove that there exists a positive integer n such that $j^n \in I$ for all $j \in J$.

145. Let R be a commutative Noetherian domain with identity. Prove that every nonzero ideal of R contains a product of nonzero *prime* ideals of R.

146. Let R be a ring satisfying the *descending chain condition* on right ideals. If $J(R)$ denotes the Jacobson radical of R, prove that $J(R)$ is nilpotent.

147. Show that if R is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

148. Let P be a nonzero prime ideal of the commutative Noetherian domain R. Assume P is principal. Prove that there does not exist a prime ideal Q satisfying $(0) < Q < P$.

149. Let R be a commutative Noetherian ring. Prove that every nonzero ideal A of R contains a product of prime ideals (not necessarily distinct) each of which contains A.

150. Let R be a commutative ring with 1 and let M be an R-module that is not Artinian (Noetherian, of finite composition length). Let $\mathcal{I}$ be the set of ideals I of R such that there exists an R-submodule N of M with the property that N/NI is not Artinian (Noetherian, of finite composition length, respectively). Show that if $A \in \mathcal{I}$ is a maximal element of $\mathcal{I}$, then A is a prime ideal of R.