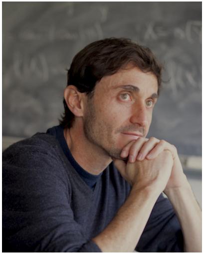
Algebraic and geometric structures in combinatorics

Federico Ardila
San Francisco State University
Mathematical Sciences Research Institute
Universidad de Los Andes

AMS Invited Address
Joint Math Meetings
January 11, 2018

Ongoing joint work (08-18) with
Marcelo Aguiar (Cornell).



Part 1:
Hopf monoids and generalized permutahedra,
`arXiv:1709.07504`

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I'm very grateful to be here, and to finally get this work out.

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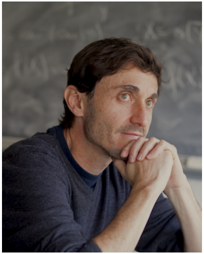

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Why did we take so long?

WIRED

MATHEMATICS

NOVEMBER 28, 2017 | ERICA KLARREICH

This Mathematician Lost Five Years of Work—And Reconstructed It All

Federico Ardila opens up about his journey as a mathematician, teacher, Colombian transplant, DJ and creator of mathematical spaces.

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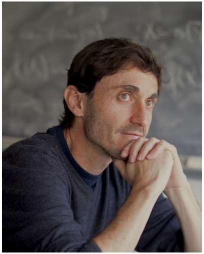

Part 1:
Hopf monoids and generalized permutahedra,
arXiv:1709.07504

This is closely related to work of:

Carolina Benedetti, Nantel Bergeron, Lou Billera, Eric Bucher, Harm Derksen, Alex Fink, Rafael González D'León, Vladimir Grujić, Joshua Hallam, Brandon Humpert, Ning Jia, Carly Klivans, John Machacek, Swapneel Mahajan, Jeremy Martin, Vic Reiner, Bruce Sagan, Tanja Stojadinović, Jacob White...

What is combinatorics about? A personal view.

Is it about counting a set of objects?

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(Toy example: How many chairs are there in this room?)

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Is it about counting a set of objects? We do count, but not

$$1, 2, 3, \ldots.$$

(Toy example: How many chairs are there in this room?)

We usually:

1. Study the structure of the individual objects or the set.
2. Use this structure to count them.

Main objective:
Understanding the underlying structure of discrete objects.
(Often this structure is algebraic, geometric, topological.)

What is alg + geom + top combinatorics about?

Understanding the underlying structure of discrete objects.
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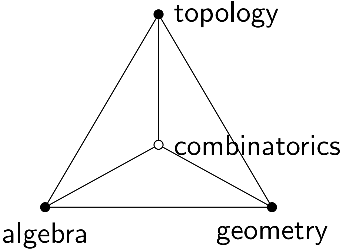

1.1. A tale of two polytopes: Permutations

$\{1, 2, \ldots, n\}$ has $n!$ permutations. How are they structured?

$n = 3$: 123, 132, 213, 231, 312, 321

What does the "space of permutations" look like?

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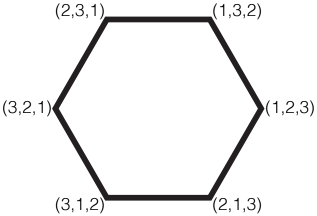

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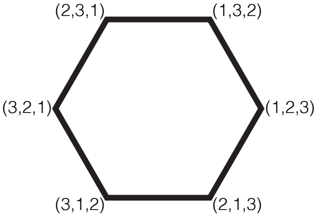

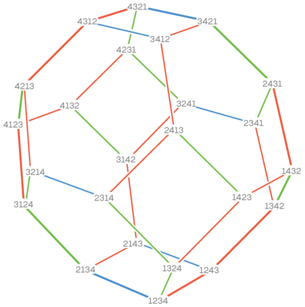

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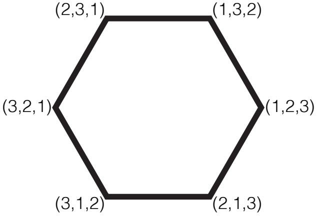

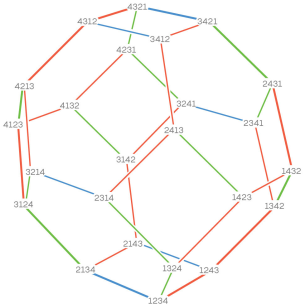

A convex polytope!

The permutahedron. Schoute 11, Bruhat/Verma 68, Stanley 80

1.2. A tale of two polytopes: Associations

$x_1 x_2 \cdots x_n$ has $\frac{1}{n+1}\binom{2n}{n}$ associations. How are they structured?

$n = 4$: a((bc)d), a(b(cd)),(ab)(cd), ((ab)c)d, (a(bc))d

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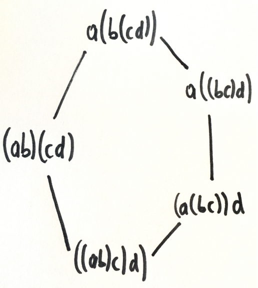

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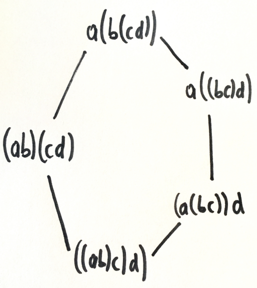

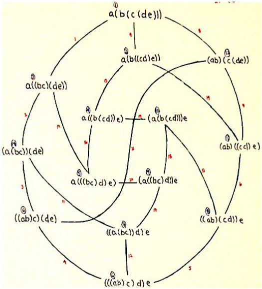

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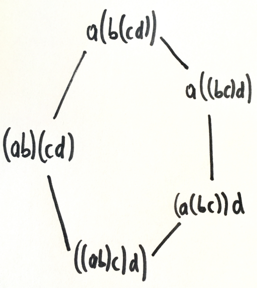

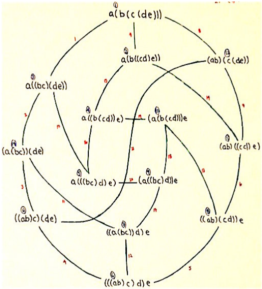

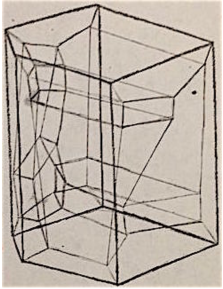

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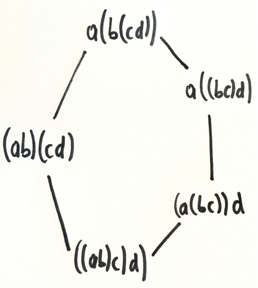
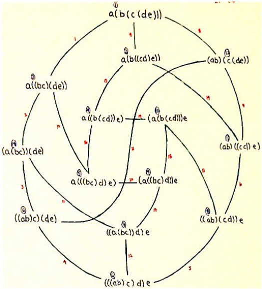
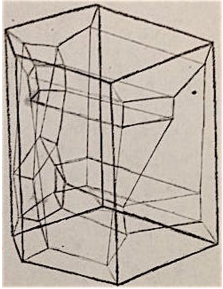






A convex polytope!

The associahedron. Stasheff 63, Haiman 84, Loday 04, Escobar 14

1.3. Motivating application: Inverting power series

A new take on an old question:

How do we invert a power series under **multiplication**?

Let $A(x) = \sum a_n \frac{x^n}{n!}$ and $B(x) = \sum b_n \frac{x^n}{n!}$ be multiplicative inverses.
Assume $a_0 = b_0 = 1$. (Ex: $\sec x = 1/\cos x$.)

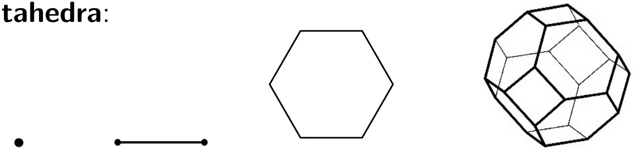
Then $B(x) = 1/A(x)$ is given by

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\ b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 24a_1^4 \\ &\vdots \end{aligned}$$

How to make sense of these numbers?

Inverting power series: Multiplication.

Permutahedra:



π_1: point, π_2: segment, π_3: hexagon, π_4: truncated octahedron...

For exponential generating functions, $B(x) = 1/A(x)$ is given by

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Faces of π_4:

- 1 truncated octahedron π_4
- 8 hexagons $\pi_3 \times \pi_1$ and 6 squares $\pi_2 \times \pi_2$
- 36 segments $\pi_2 \times \pi_1 \times \pi_1$
- 24 points $\pi_1 \times \pi_1 \times \pi_1 \times \pi_1$

Inverting power series: Composition.

A new take on an old question:

How do we invert a power series under **composition**?

$A(x) = \sum a_{n-1}x^n$, $B(x) = \sum b_{n-1}x^n$: **compositional** inverses.
Assume $a_0 = b_0 = 1$. (Ex: $\arcsin x = 1/\sin x$.)

Then $B(x) = A(x)^{\langle -1 \rangle}$ is given by Lagrange inversion:

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\ b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4 \\ &\vdots \end{aligned}$$

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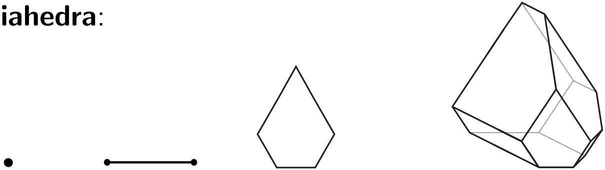
Inverting power series: Composition.

Associahedra:



$\mathfrak{a}_1$: point, $\mathfrak{a}_2$: segment, $\mathfrak{a}_3$: pentagon, $\mathfrak{a}_4$: 3-associahedron...

For ordinary generating functions, $B(x) = A(x)^{\langle -1 \rangle}$ is given by

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 5a_2a_1 - 5a_1^3 \\ b_4 &= -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4 \end{aligned}$$

Faces of $\mathfrak{a}_4$:

- 1 3-associahedron $\mathfrak{a}_4$
- 6 pentagons $\mathfrak{a}_3 \times \mathfrak{a}_1$ and 3 squares $\mathfrak{a}_2 \times \mathfrak{a}_2$
- 21 segments $\mathfrak{a}_2 \times \mathfrak{a}_1 \times \mathfrak{a}_1$
- 14 points $\mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1$

Inverting power series: Composition.

So

- permutahedra “know” how to compute multiplicative inverses
- associahedra “know” how to compute compositional inverses

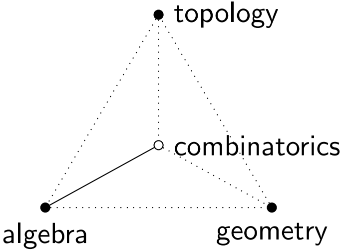
Why?

We discovered this as a very unexpected consequence of our Hopf monoid of generalized permutahedra.

Let’s

1. build a bit of abstract architecture, and
2. reap the(se and other) concrete benefits.

2. Hopf monoids.



2. Hopf monoids.

- Do you know what a Hopf algebra is?

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I didn't know until I started on this project.

Hopf monoids refine Hopf algebras. They are a bit more abstract but better suited for many combinatorial purposes.

There is a Fock functor

$$\text{Hopf monoids} \longrightarrow \text{Hopf algebras}$$

so there are Hopf algebra analogs of all of our results.

2.1. Hopf monoids: Definition.

(Joni-Rota, *Coalgebras and bialgebras in combinatorics.*)
(Aguiar-Mahajan, *Monoidal functors, species, and Hopf algebras.*)

Think:

- A family of combinatorial structures. (graphs, posets, matroids, ...)
- Rules for "merging" and "breaking" those structures.

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A Hopf monoid $(\mathrm{H}, \mu, \Delta)$ consists of:

- For each finite set I, a vector space $\mathrm{H}[I]$.
- For each partition $I = S \sqcup T$, maps

product $\mu_{S,T} : \mathrm{H}[S] \otimes \mathrm{H}[T] \to \mathrm{H}[I]$
coproduct $\Delta_{S,T} : \mathrm{H}[I] \to \mathrm{H}[S] \otimes \mathrm{H}[T]$.

satisfying various axioms.

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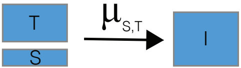
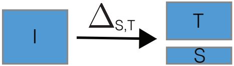
satisfying various axioms.

For us, $\mathrm{H}[I] = \mathrm{span}\{$combinatorial structures of type H on $I\}$

product: 

coproduct: 

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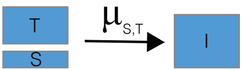
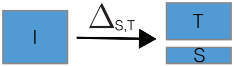
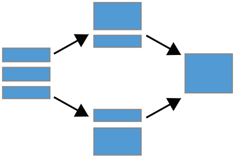
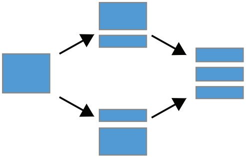
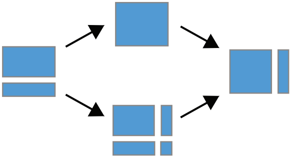
product 

coproduct 

Axioms:

μ is associative.	Δ is coassociative.	μ and Δ are compatible.
		
can merge several structures into one	can break one structure into several	merge, then break = break, then merge

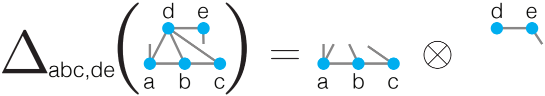
Example 1: The Hopf monoid of graphs.

$G[I] :=$ span{graphs (with half edges) on vertex set I}.

Product: $g_1 \cdot g_2 = g_1 \sqcup g_2$ (disjoint union)

Coproduct: $\Delta_{S,T}(g) = g|_S \otimes g/_S$ where

$$g|_S = \text{keep everything incident to } S,$$
$$g/_S = \text{remove everything incident to } S.$$

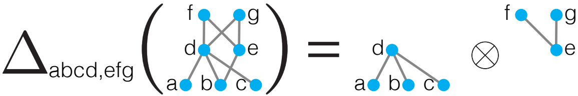


Example 2: The Hopf monoid of posets.

$\mathrm{P}[I] := \mathrm{span}\{\text{posets on } I\}$.

Product: $p_1 \cdot p_2 = p_1 \sqcup p_2$ (disjoint union)

Coproduct: $$\Delta_{S,T}(p) = \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p \\ 0 & \text{otherwise} \end{cases}$$



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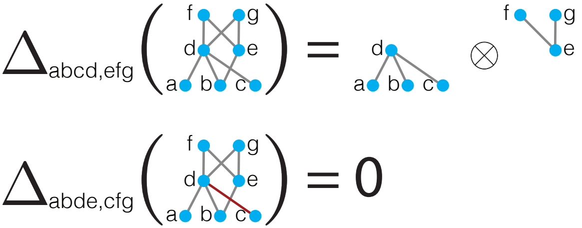
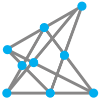


Example 3: The Hopf monoid of matroids.

$\mathrm{M}[I] := \mathrm{span}\{\text{matroids on } I\}$.

- Do you know what a matroid is?



Matroids are a combinatorial model of independence.

They capture the properties of (linear,algebraic,graph,matching,...) independence.

Product: $m_1 \cdot m_2 \;=\; m_1 \oplus m_2$ (direct sum)

Coproduct: $\Delta_{S,T}(m) = m|_S \otimes m/_S$ where

$$m|_S \quad = \text{restriction to } S \text{ of } m,$$
$$m/_S \quad = \text{contraction of } S \text{ from } m.$$

Other Hopf monoids.

There are many interesting Hopf monoids in combinatorics, algebra, and representation theory.

A few of them:

- graphs G
- posets P
- matroids M

- set partitions Π (symmetric functions)
- paths A (Faá di Bruno)

- simplicial complexes SC
- hypergraphs HG
- building sets BS

2.2. The antipode of a Hopf monoid.

Think: groups $\rightsquigarrow$ inverses
Hopf monoids $\rightsquigarrow$ antipodes

$(\mathrm{s}^2 = \mathrm{id})$

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Takeuchi: The antipode of a connected Hopf monoid H is :

$$\mathrm{S}_I(h) = \sum_{\substack{I = S_1 \sqcup \cdots \sqcup S_k \\ k \geq 1}} (-1)^k \, \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(h).$$

summing over all ordered set partitions $I = S_1 \sqcup \cdots \sqcup S_k$. $(S_i \neq \emptyset)$

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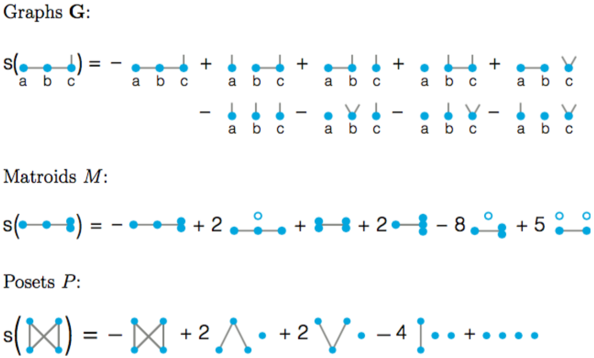
General problem. Find the simplest possible formula
for the antipode of a Hopf monoid.

(Usually there is **much** cancellation in the definition above.)

Examples: The antipode of a graph, matroid, poset.

Ex. Takeuchi: $\mathrm{s}_I \;=\; \sum_{I=S_1\sqcup\cdots\sqcup S_k} (-1)^k\, \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}.$

For $n = 3, 4, 4$ this sum has 13, 73, 73 terms. However,



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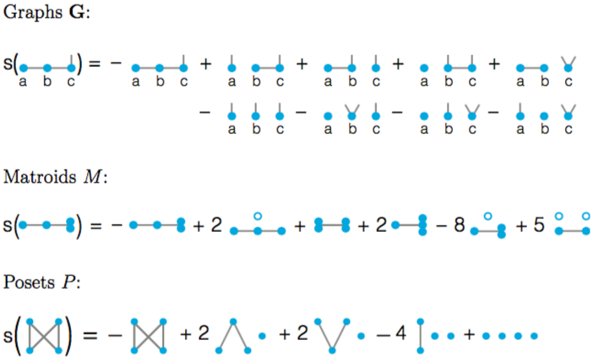

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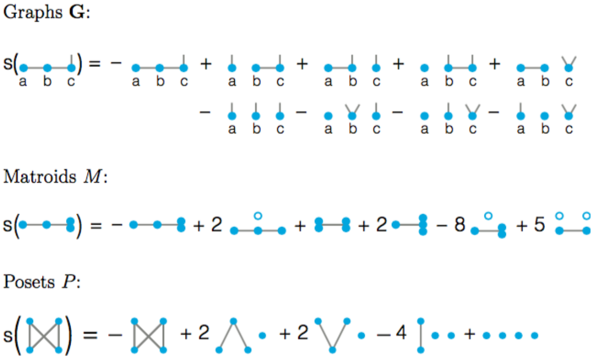


How do we explain (and predict) the simplification?

- In the past: clever combinatorics.
- Our approach: geometry + topology: Euler characteristics.

Some antipodes of interest.

There are many other Hopf monoids of interest.
Very few of their (optimal) antipodes were known.

- graphs G: ?, Humpert–Martin 10
- posets P: ?
- matroids M: ?

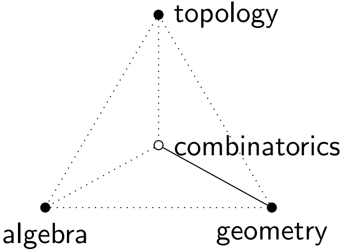
- set partitions / symm fns. Π: Aguiar–Mahajan 10
- paths A: ?

- simplicial complexes SC: Benedetti–Hallam–Michalak 16
- hypergraphs HG: ?
- building sets BS: ?

Goal 1. a unified approach to compute these and other antipodes.

(We do this).

3. Generalized permutahedra.



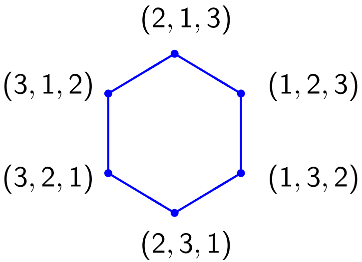
3.1. Permutahedra.

The standard permutahedron is

$$\pi_n \; := \; \text{Convex Hull}\{\text{permutations of } \{1, 2, \ldots, n\}\} \subseteq \mathbb{R}^n$$

$\{x_1 + x_2 + x_3 = 6\}$

$\pi_3 \; =$



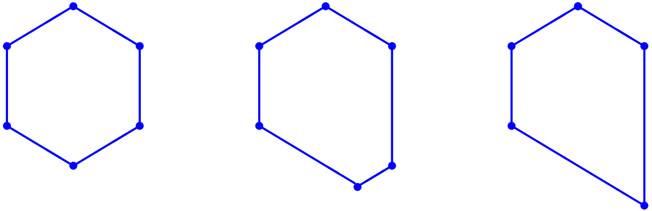
3.2. Generalized permutahedra.

Edmonds (70), Postnikov (05), Postnikov–Reiner–Williams (07),...

Equivalent formulations:

- Move the facets of the permutahedron without passing vertices.
- Move the vertices while preserving edge directions.

Generalized permutahedra:



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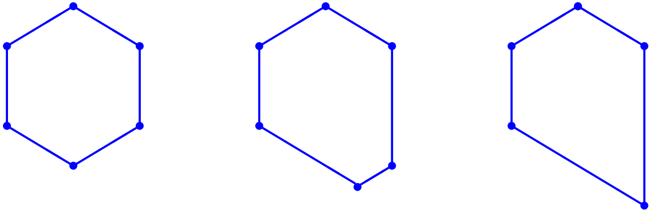
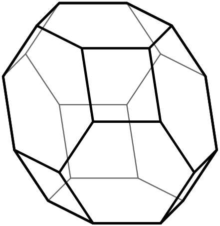


Gen. permutahedra = "polymatroids" = "submodular functions".

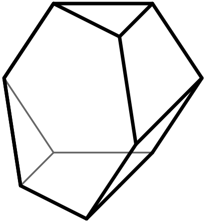
Many natural gen. permutahedra! Especially in optimization.

Generalized permutahedra in 3-D.

The permutahedron π_4.

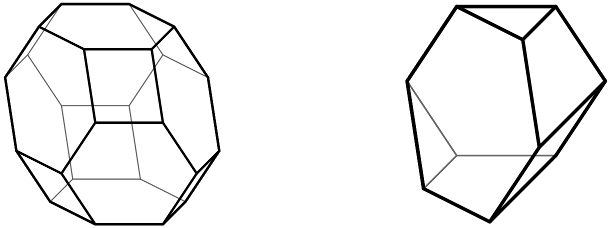


A generalized permutahedron.

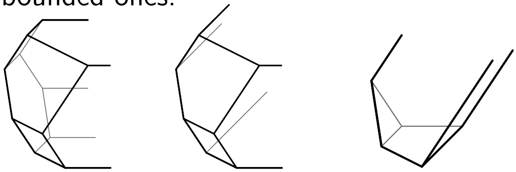


Generalized permutahedra in 3-D.

The permutahedron π_4. A generalized permutahedron.



We allow unbounded ones:



Goal 0. A Hopf monoid of generalized permutahedra.

How do we merge gen. permutahedra? How do we split them?

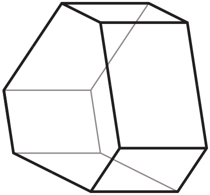
3.3. The Hopf monoid GP: Product.

Key Lemma. If P, Q are generalized permutahedra in $\mathbb{R}^S$ and $\mathbb{R}^T$ and $I = S \sqcup T$, then

$$P \times Q = \{(p, q) : p \in \mathbb{R}^S, q \in \mathbb{R}^T\}$$

is a generalized permutahedron in $\mathbb{R}^{S \sqcup T} = \mathbb{R}^I$.

Example: hexagon $\times$ segment $=$



Hopf product of P and Q:

$$P \cdot Q := P \times Q.$$

3.3. The Hopf monoid GP: Coproduct.

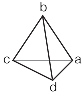
Given a polytope $P \subseteq \mathbb{R}^I$ and $I = S \sqcup T$, let

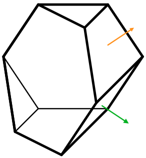
$$P_{e_{S,T}} := \text{ face of } P \text{ where } \textstyle\sum_{s \in S} x_s \text{ is maximum.}$$

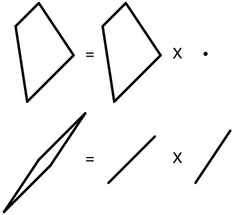
Key Lemma. If P is a generalized permutahedron and $I = S \sqcup T$,

$$P_{e_{S,T}} = P|_S \times P/_S$$

for generalized permutahedra $P|_S \subseteq \mathbb{R}^S$ and $P/_S \subseteq \mathbb{R}^T$.







$abcd = abd \sqcup c$

$abcd = ad \sqcup bc$

Hopf coproduct of P:

$$\Delta_{S,T}(P) := P|_S \otimes P/_S$$

3.3. The Hopf monoid of generalized permutahedra.

$\mathrm{GP}[I] :=$ span $\{$generalized permutahedra in $\mathbb{R}^I\}$.

Product: $P_1 \cdot P_2 = P_1 \times P_2$

Coproduct: $\Delta_{S,T}(P) = P|_S \otimes P/_S$

Theorem. (Aguiar–A. 08, Derksen–Fink 10)
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GP is the universal Hopf monoid of polytopes with these operations.

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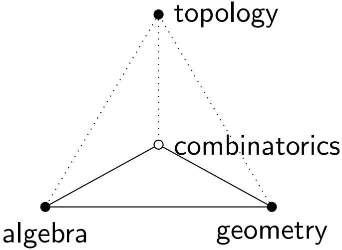

3.4. Generalized permutahedra: Posets, graphs, matroids.

There is a long tradition of modeling combinatorics geometrically.
There are polyhedral models:

G graph $\rightarrow$ graphic zonotope $Z(G)$ (Stanley 73)

$Z(G) = \sum_{ij \in G}(e_i - e_j)$

P poset $\rightarrow$ poset cone C_P (Geissinger 81)

$C_P : \text{cone}\{e_i - e_j \, : \, i < j \text{ in } P\}$.

M matroid $\rightarrow$ matroid polytope P_M (Edmonds 70, GGMS 87)

$P_M = \text{conv}\{e_{i_1} + \cdots + e_{i_k} \mid \{i_1, \ldots, i_k\} \text{ is a basis of } M\}$.

Proposition. (Aguiar–A. 08) These are inclusions of Hopf monoids:

$$\text{G} \hookrightarrow \text{GP}, \qquad \text{M} \hookrightarrow \text{GP}, \qquad \text{P} \hookrightarrow \text{GP}.$$

3.5. The antipode of GP.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{s}(P) \;=\; \sum_{Q \leq P} (-1)^{\operatorname{codim} Q}\, Q.$$

The sum is over all faces Q of P.

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The sum is over all faces Q of P.

Proof. Takeuchi:

$$\begin{aligned} \mathrm{s}(P) &= \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k \mu_{S_1,\ldots,S_k} \otimes \Delta_{S_1,\ldots,S_k}(P) \\ &= \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k P_{S_1,\ldots,S_k} \end{aligned}$$

where $P_{S_1,\ldots,S_k}$ = face of P minimizing $x_{S_1} + 2x_{S_2} + \cdots + kx_{S_k}$.
Coeff. of a face Q: huge sum of 1s and -1s. How to simplify it?
It is the reduced **Euler characteristic** of a sphere! □

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Coeff. of a face Q: huge sum of 1s and -1s. How to simplify it?
It is the reduced **Euler characteristic** of a sphere! □

This is the best possible formula. No cancellation or grouping.
(One advantage of working with Hopf monoids!)

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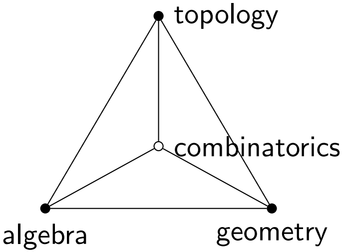
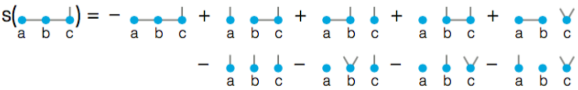
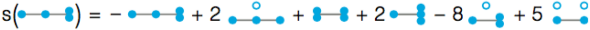


The antipodes of graphs, matroids, posets.

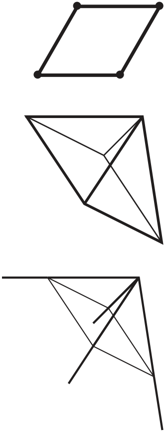
Graphs **G**:



Matroids M:



Posets P:



$$\mathrm{s}(P) \;=\; \sum_{Q \leq P} (-1)^{\operatorname{codim} Q}\, Q.$$

Many antipode formulas.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{s}(P) = \sum_{Q \leq P} (-1)^{\operatorname{codim} Q} \, Q.$$

As a consequence, we get best possible formulas for:

objects	polytopes	Hopf algebra	antipode
set partitions	permutahedra	Joni-Rota	Joni-Rota
paths	associahedra	Joni-Rota, new	Haiman-Schmitt, new
graphs	graphic zonotopes	Schmitt	new, Humpert-Martin
matroids	matroid polytopes	Schmitt	new
posets	braid cones	Schmitt	new
submodular fns	polymatroids	Derksen-Fink, new	new
hypergraphs	hg-polytopes	new	new
simplicial cxes	new: sc-polytopes	Benedetti et al	Benedetti et al
building sets	nestohedra	new, Grujić et al	new
simple graphs	graph associahedra	new	new

Lots of interesting algebra and combinatorics.

4.1. Characters of Hopf monoids.

Think: character = multiplicative function on our objects

Let H be a Hopf monoid. A character ζ consists of maps

$$\zeta_I : \mathrm{H}[I] \to \mathbb{k}$$

which are multiplicative: for each $I = S \sqcup T$, $s \in H[S], t \in H[T]$:

$$\zeta(s)\zeta(t) = \zeta(s \cdot t).$$

4.2. The group of characters.

The group of characters of a Hopf monoid H:

Identity:

$$u(h) = \begin{cases} 1 & \text{if } h = 1 \in \mathrm{H}[\emptyset] \\ 0 & \text{otherwise.} \end{cases}$$

Operation: Convolution across coproduct. For $h \in \mathrm{H}[I]$

$$\zeta_1 * \zeta_2(h) = \sum_{I = S \sqcup T} \zeta_1(h|_S)\zeta_2(h/_S)$$

Inverse: Antipode.

$$\zeta^{-1} = \zeta \circ \mathrm{S}$$

- This group is hard to describe in general.
- To understand it, we must understand the antipode.

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Let's study two special cases: permutahedra and associahedra.

(Non-)Goal 2. a unified approach to inversion of power series.

4.3. The group of characters for permutahedra.

Theorem. The group of characters of Π is the group of exponential generating functions $1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots$ under multiplication.

inversion of egfs $\longleftrightarrow$ antipode of permutahedra

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inversion of egfs $\longleftrightarrow$ antipode of permutahedra

For exponential generating functions, $B(x) = 1/A(x)$ is given by

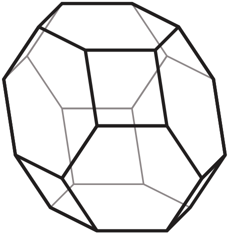
$b_1 = -a_1$

$b_2 = -a_2 + 2a_1^2$

$b_3 = -a_3 + 6a_2 a_1 - 6a_1^3$

$b_4 = -a_4 + 8a_3 a_1 + 6a_2^2 - 36a_2 a_1^2 + 24a_1^4$

$\vdots$



These numbers come from the antipode of the permutahedron:

$\mathrm{S}(\pi_4) = -\pi_4 + 8\pi_3\pi_1 + 6\pi_2^2 - 36\pi_2\pi_1^2 + 24\pi_1^4$

(1 perm., 8 hexagons and 6 squares, 36 segments, 24 points.)

4.4. The group of characters of associahedra.

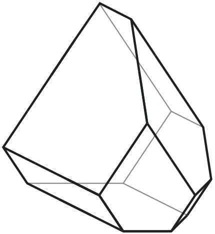
Theorem. The group of characters of Π is the group of generating functions $x + a_1x^2 + a_2x^3 + \cdots$ under composition.

compositional inversion of gfs $\longleftrightarrow$ antipode of associahedra

For ordinary generating functions, $B(x) = A(x)^{\langle -1 \rangle}$ is given by

$b_1 = -a_1$
$b_2 = -a_2 + 2a_1^2$
$b_3 = -a_3 + 5a_2a_1 - 5a_1^3$
$b_4 = -a_4 + 6a_3a_1 + 3a_2^2 - 21a_2a_1^2 + 14a_1^4$
$\vdots$

These numbers come from the antipode of the associahedron:



$\mathrm{S}(\mathfrak{a}_4) = -\mathfrak{a}_4 + 6\mathfrak{a}_3\mathfrak{a}_1 + 3\mathfrak{a}_2^2 - 21\mathfrak{a}_2\mathfrak{a}_1^2 + 14\mathfrak{a}_1^4$

(1 assoc., 6 pentagons and 3 squares, 21 segments, 14 points.)

4.4. The group of characters of associahedra.

This reformulation of the Lagrange inversion formula for

$$B(x) = A(x)^{\langle -1 \rangle}$$

answers Loday's question:

> *"...it would be interesting to find [a proof of the Lagrange inversion formula] which involves the topological structure of the associahedron."* *(Loday, 2005)*

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Note. This only works for Loday's $\mathfrak{a}_4$! (A. – Escobar – Klivans)

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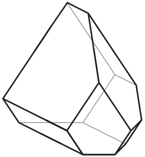
Note. This only works for Loday's $\mathfrak{a}_4$! (A. – Escobar – Klivans)

Project.
(A. – Benedetti – González d'León – Supina.)
Compute the group of characters and reciprocity rules for other interesting submonoids of GP.



5. Polynomial invariants and reciprocity.

Each character ζ of a Hopf monoid H gives a polynomial χ.

Think. $\chi_h(n)$ = Split object h into n parts, apply ζ to each part.

Define, for each object $h \in \mathrm{H}[I]$ and $n \in \mathbb{N}$,

$$\chi_h(n) := \sum_{S_1 \sqcup \cdots \sqcup S_n = I} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(h),$$

summing over all weak ordered set partitions $I = S_1 \sqcup \cdots \sqcup S_n$

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Proposition.

1. $\chi_h(n)$ is a polynomial function of n.
2. $\chi_h(-n) = \chi_{\mathrm{S}(h)}(n)$. (antipode $\to$ reciprocity thms)

Three reciprocity theorems.

1. Graphs: χ_g = chromatic polynomial of g.

For $n \in \mathbb{N}$, it counts proper colorings of g with $[n]$. (Birkhoff, 12).
For -1, it counts acyclic orientations of g. (Stanley, 73).
For $-n$, it counts... (Stanley, 73).

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2. Posets: χ_p = strict order polynomial of p.
For $n \in \mathbb{N}$ it counts order-preserving n-labelings of p. (Stanley, 70).
For -1 it equals ± 1
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3. Matroids: χ_m = Billera-Jia-Reiner polynomial of m.
For $n \in \mathbb{N}$, it counts m-generic functions $f : I \to [n]$. (BJR 06).
For -1 it counts bases of m (BJR 06).
For $-n$ it counts... (BJR 06).

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1. Graphs: χ_g = chromatic polynomial of g.
For $n \in \mathbb{N}$, it counts proper colorings of g with $[n]$. (Birkhoff, 12).
For -1, it counts acyclic orientations of g. (Stanley, 73).
For $-n$, it counts... (Stanley, 73).

2. Posets: χ_p = strict order polynomial of p.
For $n \in \mathbb{N}$ it counts order-preserving n-labelings of p. (Stanley, 70).
For -1 it equals ± 1
For $-n$ it counts weakly order-preserving n-labelings of p. (Stanley, 70).

3. Matroids: χ_m = Billera-Jia-Reiner polynomial of m.
For $n \in \mathbb{N}$, it counts m-generic functions $f : I \to [n]$. (BJR 06).
For -1 it counts bases of m (BJR 06).
For $-n$ it counts... (BJR 06).

Goal 3. A unified approach to these and other reciprocity results.

A GP polynomial for posets, graphs, matroids.

Theorem. (Aguiar–A., Billera–Jia–Reiner) The basic character of GP:

$$\zeta(P) := \begin{cases} 1 & \text{if the polytope } P \text{ is a point,} \\ 0 & \text{otherwise.} \end{cases}$$

gives the basic polynomial

$$\chi_P(n) := \# \text{ of } P\text{-generic vectors in } [n]^I \subset \mathbb{R}^I.$$

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Under the inclusions $\mathrm{P}, \mathrm{G}, \mathrm{M} \hookrightarrow \mathrm{GP}$ polynomial $\chi_P(n)$ specializes to:

$$\chi_q(\mathrm{n}) \text{ (posets)}, \quad \chi_g(\mathrm{n}) \text{ (graphs)}, \quad \chi_m(\mathrm{n}) \text{ (matroids)}$$

Their reciprocity theorems are really the same theorem.
The Hopf monoid GP **discovers and proves** them directly.

6.1. Current direction 1: the polytope algebra.

Theorem. (A.–Aguiar, Derksen–Fink) The Hopf structure GP descends to $\overline{GP}$ in McMullen's polytope algebra. There we have

$$\mathrm{s}(P) = \sum_{Q \leq P} (-1)^{\operatorname{codim} Q} \, Q = (-1)^{\operatorname{codim} P} \operatorname{interior}(P)$$

The Euler involution! (McMullen)

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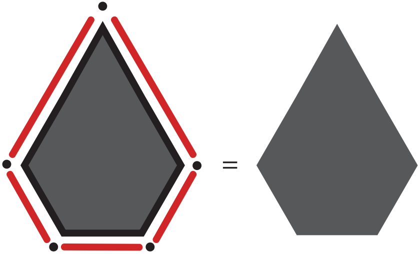
Example: $s($  $) =$



Many interesting consequences (with M. Aguiar, M. Sanchez)

6.2. Current direction 2: The Coxeter-Hopf monoid GP_W.

Symmetric group	Coxeter group	
graphs	sets of roots	Zaslavsky
posets	Coxeter cones	Reiner
matroids	Coxeter matroids	Gelfand, Serganova

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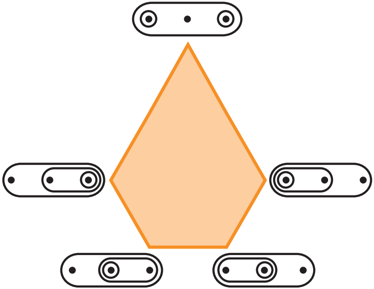
Hopf monoid	Coxeter–Hopf monoid?	A., Aguiar, Bastidas,
GP	GP?	Mahajan, Rodríguez

¡Muchas gracias!



Federico Ardila and Marcelo Aguiar
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