4

Algebraic Cycles and Singularities of Normal Functions

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To our friend, Jacob Murre, on his 75^{th} *birthday*

Abstract

Given the data (X, L, ζ) where X is a smooth $2n$-dimensional algebraic variety, $L \to X$ is a very ample line bundle and

$$\zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}$$

is a primitive Hodge class, we shall define an analytic invariant

$$\nu_\zeta \in \Gamma(S, \tilde{\mathcal{J}}_e)$$

and algebro-geometric invariant

$$\delta\nu_\zeta \in \Gamma\left(S, \left(\mathcal{H}_e^{n,n-1} \otimes \Omega_S^1(\log D)\right)_\nabla\right)$$

where S is a blow-up of $\mathbb{P}H^0(\mathcal{O}_X(L))$ and $D \subset S$ is the quasi-local normal crossing discriminant locus (see below for definitions). We will also define the singular loci $\operatorname{sing}\nu_\zeta$ and $\operatorname{sing}\delta\nu_\zeta$ and show that, for $L \gg 0$, as subvarieties of S

$$\operatorname{sing}\nu_\zeta = \operatorname{sing}\delta\nu_\zeta$$

and that in a precise sense these loci define the algebraic cycles W on X with the property that

$$\langle \zeta, [W] \rangle \neq 0\ .$$

The Hodge conjecture (HC) is then equivalent to

$$\operatorname{sing}\nu_\zeta = \operatorname{sing}\delta\nu_\zeta \neq \varnothing$$

for $L \gg 0$. In an informal sense we may say that if the HC is true, then there is a systematic geometric procedure for producing the equations of algebraic cycles from Hodge classes.

For $L \gg 0$ an arbitrary class — not one that is rational — $\zeta \in H^n(\Omega_X^n)_{\text{prim}}$ may be localized along the locus of singularities of the universal family $\mathcal{X} \to S$. The HC is then equivalent to the condition that the integrality of the residues of $\delta\nu_\zeta$ along the descriminant locus D give the test that $\zeta \in H^{2n}(X, \mathbb{Q})$, which is an explicit form of the absolute Hodge condition.

The effective Hodge conjecture (EHC) is the statement that there is an explicit k_0 such that there is $X_s \in |L^{k_0}|$ and a subvariety $W \subset X_s$ with $\langle \zeta, [W] \rangle \neq 0$. Heuristic reasons show that in general k_0 must be bounded below by an expression whose dominant term is $(-1)^n \zeta^2$ (which is positive). The other quantities on which k_0 depends and which are independent of ζ are discussed below.

The polarizing forms on the intermediate Jacobians define line bundles, including a Poincaré line bundle P that may be pulled back to $\nu_\zeta^*(P)$ by a normal function ζ. Restricting to one dimensional families with only one ordinary node, the Chern class of $\nu_\zeta^*(P)$ evaluates to ζ^2. This again suggests the central role of ζ^2 in the study of algebraic cycles.

This is an extended research announcement of joint work in progress. The complete details of some of the results have yet to be written out. It is an expanded version of the talk given by the second author at the conference in Leiden in honor of Jacob Murre.

We would like to especially thank Mark de Cataldo, Luca Migliorini, Gregory Pearlstein, and Patrick Brassman for their interest in and comments on this work.

4.1 Introduction and Historical Perspective

4.1.1 Introduction and Statement of Results

We shall use the notations

$$
\begin{aligned}
X &= \text{smooth projective variety} \\
Z^p(X) &= \text{group of codimension-}p\text{ algebraic cycles} \\
&= \{Z = \sum_i n_i Z_i : Z_i \subset X\}
\end{aligned}
$$

where Z_i is an irreducible codimension-p subvariety, and

$$\begin{array}{ccc} Z^p(X) & \longrightarrow & \mathrm{Hg}^p(X) = H^{2p}(X,\mathbb{Z}) \cap H^{p,p}(X) \\ \Cup & & \Cup \\ Z & \longrightarrow & [Z] \end{array}$$

is the mapping given by taking the fundamental class.

Hodge's original conjecture (HC)**:** *This map is surjective.*

It is known that the HC is

- True when $p = 1$ (Lefschetz [30], c. 1924)
- False in any currently understood sense for torsion when $p \geq 2$ (Atiyah-Hirzebruch [2] and Kollár (see section 4.4.1 below))
- False in any currently understood sense for X Kähler, $p \geq 2$ (Voisin [37])

The phrase "in any currently understood sense" means this: Atiyah and Hirzebruch showed that for $p \geq 2$ there is a smooth variety X and a torsion class in $H^{2p}(X, \mathbb{Z})$, which being torsion is automatically of Hodge type (p, p), and which is not the fundamental class of an algebraic cycle. Kollár showed that there is an algebraic class

$$\left[\sum_i m_i Z_i\right] \in H^{2p}(X, \mathbb{Z})$$

where $m_i \in \mathbb{Q}$ but we cannot choose $m_i \in \mathbb{Z}$. Finally, Voisin [37] showed that there is a complex 4-torus T and $0 \neq \zeta \in \mathrm{Hg}^2(X)$ where T has no geometry — i.e., no subvarieties or coherent sheaves — other than those coming from points of T.

Conclusion: *Any general construction of codimension p cycles for $p \geq 2$ must wipe out torsion and must use the assumption that X is an algebraic variety.*

With the exception of section 4.4.1, in what follows everything is modulo torsion.

By standard techniques the HC is reduced to the case

$$\dim X = 2n, \qquad p = n, \qquad \text{primitive Hodge classes}$$

where we are given a very ample line bundle $L \to X$ with $c_1(L) = \lambda$ and where the primitive cohomology (with $\mathbb{Q}$ coefficients) is as usual defined by

$$H^{2n}(X)_{\mathrm{prim}} = \ker\{H^{2n}(X) \xrightarrow{\ \lambda\ } H^{2n+2}(X)\}.$$

If $s \in H^0(\mathcal{O}_X(L))$ and the variety X_s given by $\{s = 0\}$ is assumed to be smooth then

$$H^{2n}(X)_{\text{prim}} = \ker\{H^{2n}(X) \to H^{2n}(X_s)\}$$

which by Poincaré duality is

$$\cong \ker\{H_{2n}(X) \to H_{2n-2}(X_s)\} \ .$$

We set $S = \mathbb{P}\widetilde{H^0(\mathcal{O}_X}(L))$, where the tilde means that we have blown $\mathbb{P}H^0(\mathcal{O}_X(L))$ up so that the discriminant locus

$$D = \{s : X_s \text{ singular}\} \subset S$$

has quasi-local normal crossings (definition below). We also set

$$S^* = S \backslash D$$

so that for $s \in S^*$ the hypersurface X_s is smooth with intermediate Jacobian $J(X_s)$, and we set

$$\begin{cases} J = \bigcup\limits_{s \in S^*} J(X_s) \\ \mathcal{J} = \mathcal{O}_{S^*}(J) = \check{\mathcal{F}}^n / R_\pi^{2n-1}\mathbb{Z} \cong \mathcal{F}^n \backslash \mathcal{H}^{2n-1} / R_\pi^{2n-1}\mathbb{Z} \ . \end{cases}$$

Here we recall that

$$\begin{aligned} J(X_s) &= F^n \check{H}^{2n-1}(X_s, \mathbb{C}) / H_{2n-1}(X_s, \mathbb{Z}) \\ &\cong F^n H^{2n-1}(X_s, \mathbb{C}) \backslash H^{2n-1}(X_s, \mathbb{C}) / H^{2n-1}(X_s, \mathbb{Z}) \ . \end{aligned}$$

We consider the picture

$$\begin{array}{ccc} \mathcal{X}^* & \subset & \mathcal{X} \\ \downarrow \pi & & \downarrow \pi \\ S^* & \subset & S \end{array}$$

where $\mathcal{X} \subset X \times S$ is the smooth variety given by

$$\mathcal{X} = \{(x, s) : x \in X_s\} \ .$$

In this picture we set

$$\mathcal{H}^{2n-1} = \mathcal{O}_{S^*} \otimes R_\pi^{2n-1}\mathbb{C}$$

with the Hodge filtration

$$\mathcal{F}^p \cong R_\pi^{2n-1}\left(\Omega^{\geq p}_{\mathcal{X}^*/S^*}\right)$$

satisfying

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_{S^*}$$

where ∇ is the Gauss-Manin connection. We set

$$\mathcal{H}^{2n-1-p,p} = \mathcal{F}^{2n-1-p}/\mathcal{F}^{2n-p},$$

and the cohomology sheaf of the complex

$$\mathcal{H}^{2n-p,p-1} \xrightarrow{\nabla} \mathcal{H}^{2n-1-p,p} \otimes \Omega^1_{S^*} \xrightarrow{\nabla} \mathcal{H}^{2n-2-p,p+1} \otimes \Omega^2_{S^*} \tag{4.1}$$

will be denoted by $\left(\mathcal{H}^{2n-1-p,p} \otimes \Omega^1_{S^*}\right)_\nabla$.

Assuming for the moment that we are in the local crossing case, and the unipotency of the local monodromy operators T_i around the branches $s_i = 0$ at a point $s_0 \in S$, where in a suitable local coordinate system $s_1, \dots, s_N$

$$D = \{s_1 \cdots s_k = 0\} ,$$

it is well-known ([34]) that there are canonical extensions $\mathcal{H}_e^{2n-1}$ and $\mathcal{F}_e^p$ of $\mathcal{H}^{2n-1}$ and $\mathcal{F}^p$ with $\nabla \mathcal{F}_e^p \subset \mathcal{F}_e^{p-1} \otimes \Omega^1_S(\log D)$. We put $\mathcal{H}_e^{2n-1-p,p} = \mathcal{F}_e^{2n-1-p}/\mathcal{F}_e^{2n-p}$ leading a complex extending (4.1)

$$\begin{aligned} K^\bullet \quad &=: \quad \left\{\mathcal{H}_e^{2n-p+\bullet,p-1-\bullet} \otimes \Omega^\bullet_S(\log D), \nabla\right\} \\ H^k(K^\bullet) \quad &=: \quad \left(\mathcal{H}_e^{2n-p+k,p-1-k} \otimes \Omega^k_S(\log D\right)_\nabla . \end{aligned} \tag{4.2}$$

A general reference to background material in variation of Hodge structure is [24].

We will use an extension ([31]) of the above to the situation that we will term *quasi-local normal crossings*. This means that locally $D = \bigcup_{i\in I} D_i$ is a union of smooth divisors $D_i = (s_i)$ with the following properties:

(i) On any slice transverse to $\bigcap_{i\in I} D_i = D_I$, any subset of $q \leqq \operatorname{codim} D_I$ of the functions s_i form part of a local coordinate system in S_i, and

(ii) most importantly, the local monodromy operators T_i around $s_i = 0$ are assumed to commute and are unipotent.

We will define

- an extension $\tilde{\mathcal{J}}_e$ of $\mathcal{J}$ and the space of extended normal functions (ENF)

$$\nu \in \Gamma(S, \tilde{\mathcal{J}}_e)$$

- an infinitesimal invariant

$$\delta\nu \in \Gamma\left(\left(\mathcal{H}_e^{n,n-1} \otimes \Omega^1_S(\log D)\right)_\nabla\right)$$

- the singular sets

$$\operatorname{sing}\nu, \quad \operatorname{sing}\delta\nu \subset S$$

The main results concerning $\operatorname{sing}\nu$ and $\operatorname{sing}\delta\nu$ are

Theorem 4.1.1. *There is an isomorphism*

$$\begin{array}{ccc} \mathrm{Hg}^n(X)_{\mathrm{prim}} & \longrightarrow & \Gamma(S, \tilde{\mathfrak{J}}_e)/J(X) \\ \Cup & & \Cup \\ \zeta & \longrightarrow & \nu_\zeta \end{array}$$

Theorem 4.1.2. i) *Assume the HC in dimension* $< 2n$. *Then*

$$\operatorname{sing}\nu_\zeta = \{s \in D : \langle \zeta, [W] \rangle \neq 0 \ \textit{where } W^n \subset X_s \textit{ is a subvariety}\}\,.$$

ii) *In general*

$$\operatorname{sing}\nu_\zeta = \{s \in D : \zeta_s \neq 0 \ \textit{in } IH_{2n-2}(X_s)\}\,.$$

Corollary. *HC* $\Leftrightarrow \operatorname{sing}\nu_\zeta \neq 0$ *for* $L \gg 0$.

Theorem 4.1.3. *For* $L \gg 0$

i) $\zeta \neq 0$ mod torsion $\Rightarrow \delta\nu_\zeta \neq 0$
ii) $\operatorname{sing}\nu_\zeta = \operatorname{sing}\delta\nu_\zeta$.

Corollary. *HC* $\Leftrightarrow \operatorname{sing}\delta\nu_\zeta \neq 0$ *for* $L \gg 0$.

In (ii), $IH(X_s)$ refers to intersection homology, general references for which are [16], [17]. The definitions of $\operatorname{sing}\nu$, $\operatorname{sing}\delta\nu$ are geometric and understanding their properties makes extensive use of the theory of degenerations of VHS over arbitrary base spaces developed in recent years [9], [10], [26].

We note that for ζ a torsion class, $\langle \zeta, [W] \rangle = 0$ for all W as above, and also $\delta\nu_\zeta = 0$. Thus, in the geometry underlying Theorems 4.1.2 and 4.1.3 torsion is indeed "wiped out", as is necessary.

By the basic setting, the results stated require that we be in a projective algebraic — not just a Kähler — setting.

We remark that our definition of $\operatorname{sing}\nu_\zeta$ should be taken as provisional. Taking $S = |L|$ (not blown up) we feel that the definition is probably the correct one when the singular X_s are at most nodal, but it may well need modification in the most general case.

The above results will be explained in sections 4.2 and 4.3. In section 4.4 we will explain how all of $H^n(\Omega^n_X)_{\text{prim}}$ may be localized along the locus of singularities of the X_s, and how when this is done the HC is equivalent to being able to express the condition that the complex class $\zeta \in H^n(\Omega^n_X)_{\text{prim}}$ actually be a class in $H^{2n}(X,\mathbb{Q})$ in terms of the rationality of the residues of $\delta\nu_\zeta$, where $\delta\nu_\zeta$ may be defined even when ν_ζ cannot be.

Finally, in section 4.5 we will begin the discussion of line bundles over the family of intermediate Jacobians arising from the "polarizing forms" on the primitive cohomology groups. These "polarizations" are bilinear integral valued forms but need not be positive definite (see e.g. [20]) and hence the theory is completely standard. For this reason in section 4.5.1 we include a brief treatment of complex tori equipped with such a polarization (see also [25]). Our results here are very preliminary. They consist of an initial definition of these line bundles and a first computation of their Chern classes. Especially noteworthy is the formula for the "universal" theta line bundle M

$$c_1\left(\nu^*_{\zeta+\zeta'}(M)\right) - c_1\left(\nu^*_\zeta(M)\right) - c_1\left(\nu^*_{\zeta'}(M)\right) + c_1\left(\nu^*_0(M)\right) = \zeta\cdot\zeta'\ ,$$

where the LHS is reminiscent of the relation

$$(a\dot{+}a') - (a) - (a') + (e) \sim 0$$

on an elliptic curve E, where $\dot{+}$ is the group law, (b) is the 0-cycle associated to a point $b \in E$, e is the origin and $\sim$ is linear equivalence (see Theorem (7) in section 4.5.2).

4.1.2 Historical Perspective

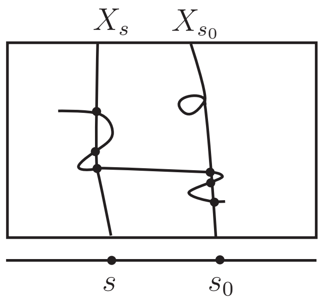
In reverse historical order the proofs of HC for $p = 1$ are

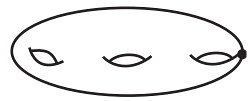
Kodaira-Spencer
- (i) $\lambda \in \mathrm{Hg}^1(X)$ gives a line bundle $L_\lambda \to X$ (Kähler fact)
- (ii) $L_\lambda \to X$ gives a divisor (GAGA-requires that X be projective)

For $p \geq 2$ the first step seems to fail in any reasonable form. In fact, as noted above, Voisin has given an example of a 4-dimensional complex torus X with $\mathrm{Hg}^2(X) \neq 0$ but where there are no coherent sheaves or subvarieties other than those arising from points.

Lefschetz-Poincaré: For $n = 1$ we take a Lefschetz pencil $|X_s|_{s\in\mathbb{P}^1}$ to have the classic picture, where $\tilde{X}$ is the blow-up of X along the base locus

$$\begin{array}{ccc} X & \leftarrow & \tilde{X} \\ & & \downarrow \pi \\ & & \mathbb{P}^1 \end{array}$$



$X_{s_0} =$ 

A primitive algebraic cycle Z on X gives

$$\begin{aligned} Z_s &= Z \cdot X_s \in \mathrm{Div}^0(X_s) \\ \nu(Z_s) &\in J(X_s) \\ Z \to \nu_Z &\in \Gamma(\mathbb{P}^1, \mathcal{J}_e) \end{aligned}$$

where we have

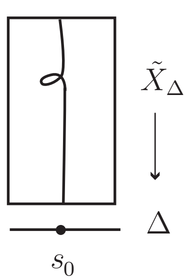
$$0 \to R^1_\pi \mathbb{Z} \to R^1_\pi \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{J}_e \to 0$$

$$J(X_{s_0}) = H^1\left(\mathcal{O}_{X_{s_0}}\right)/H^1\left(\tilde{X}_{s_0}, \mathbb{Z}\right) \cong \begin{pmatrix} \text{fibre of} \\ R^1_\pi \mathcal{O}_{\tilde{X}} \\ \text{at } s_0 \end{pmatrix} / \left(R^1_\pi \mathbb{Z}\right)_{s_0}.$$

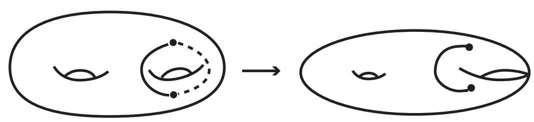
(By moving Z in a rational equivalence we may assume that its support misses the nodes on the singular fibres.) Poincaré's definition of a *normal function* was a section of $\mathcal{J}_e$. Equivalently, setting $\mathbb{P}^{1*} = \mathbb{P}^1$-$\{s_0 : X_{s_0}$ has a node$\}$, $\mathcal{J} = \mathcal{J}_e | \mathbb{P}^{1*}$, he formulated a normal function as a section of $\mathcal{J}$ with the properties

— over Δ^* it lifts to a section of $R^1_\pi \mathcal{O}_{\tilde{X}_\Delta}$ (i.e. no monodromy)
— it extends across s_0 to $(R^1_\pi \mathcal{O}_{\tilde{X}})_{s_0}$ (moderate growth).

Here, $\Delta \subset \mathbb{P}^1$ is a disc with origin s_0 and $\Delta^* = \Delta \backslash \{s_0\}$.

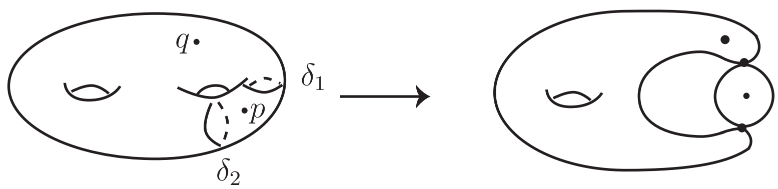


Geometrically we think of

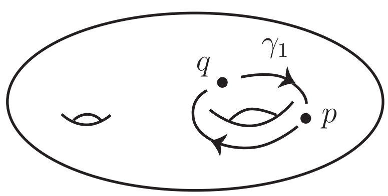


where we choose $\int_{\substack{\text{solid}\\ \text{arc}}} \omega$ rather than $\int_{\substack{\text{dotted}\\ \text{arc}}} \omega$ for the abelian sums.

Ruled out is a picture (which is *not* a Lefschetz pencil)



Here, any path γ joining p to q has monodromy, while we may choose a path $\tilde{\gamma}$ with $\partial\tilde{\gamma} = 2(p-q)$ that has no monodromy.



$\gamma =$ closed loop around the hole

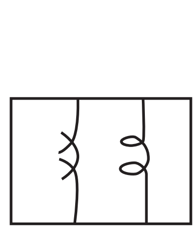
$$\partial\gamma_1 = p - q$$
$$\gamma \in H_1(X_s, \mathbb{Z})$$

Proof With $T = T_1$ the *monodromy operator* we have:

$$\left.\begin{aligned} \partial\gamma_1 &= p - q \\ (T-I)\gamma_1 &= \delta_1 \\ (T-I)\gamma &= 2\delta_1 \end{aligned}\right\} \Rightarrow (T-I)(2\gamma_1 - \gamma) = 0 \ \text{ in } \ H_1(X_s, \mathbb{Z}).$$

□

Moral: For any family



with a one dimensional base and Z with $\deg Z_s = 0$, for some non-zero $m \in \mathbb{Z}$

$$m\nu_Z \textit{ gives a normal function.}$$

As will be seen below this is a consequence of the local invariant cycle theorem.

Proof of HC: $0 \to R^1_\pi \mathbb{Z} \to R^1_\pi \mathcal{O}_{\tilde{X}} \to \mathcal{J}_e \to 0$ gives

$$\begin{array}{ccccccc} 0 \to \mathrm{Pic}^\circ(X) \to \Gamma(\mathbb{P}^1, \mathcal{J}_e) & \xrightarrow{\delta} & H^1(R^1_\pi \mathbb{Z}) & \to & H^1\left(R^1_\pi \mathcal{O}_{\tilde{X}}\right) \\ & & \wr\| & & \wr\| \\ & & H^2(X, \mathbb{Z})_{\mathrm{prim}} & \to & H^2\left(\mathcal{O}_X\right) \end{array}$$

There are then two steps:

(1) $\zeta \in \mathrm{Hg}^1(X)_{\mathrm{prim}} \cong \ker\{H^1(R^1_\pi \mathbb{Z}) \to H^1(R^1_\pi \mathcal{O}_{\tilde{X}})\} \Rightarrow \zeta = \delta\nu_\zeta$

(2) ν_ζ arises from an algebraic cycle Z (Jacobi inversion with dependence on parameters)

□

Extensions of (1): $\dim X = 2n$, $L \to X$ very ample.

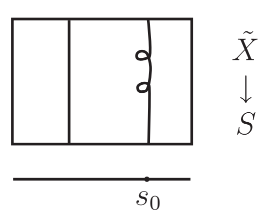
The first was the general Lefschetz pencil case (Bloch-Griffiths unpublished notes from 1972), where for a section ν with lifting $\tilde{\nu}$ as in the following diagram

$$\begin{array}{ccc} \mathcal{H}^{2n-1} & \longrightarrow & \mathcal{F}^n \backslash \mathcal{H}^{2n-1} / R^{2n-1}_\pi \mathbb{Z} \\ \Cup & & \Cup \\ \tilde{\nu} & \longrightarrow & \nu \end{array}$$

we have to add the condition

$$\nabla \tilde{\nu} \in \mathcal{F}^{n-1}.$$

The next was the definitive extension by Zucker [38] and El Zein-Zucker (cf. [14] and the references cited therein) to a general one parameter family of generically smooth hypersurface sections



with the assumption on $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}$ that we should have Poincaré's first condition

$$\zeta = 0 \text{ in } H^{2n}\left(\tilde{X}_\Delta, \mathbb{Z}\right)$$

Their result is now generally referred to as the *Theorem on Normal Functions.*

Now we discuss the *Clemens-Schmid exact sequence* (cf. Chapter VI in [24]). It implicitely uses the *Monodromy Theorem* which states that the eigenvalues of T are all roots of unity and so T is quasi-unipotent, i.e. in the decomposition $T = T_s T_u$ in semi-simple and unipotent parts T_s is of finite order k; after base changing via $z \mapsto z^k$ the monodromy operator T becomes unipotent. We may and do assume that this is the case and put

$$N := \log(T) = \sum_{k \geq 1} (-1)^{k+1} \frac{(T-I)^k}{k},$$

the left-hand side of which is a finite sum with $\mathbb{Q}$-coefficients. This explains we need $\mathbb{Q}$-coefficients in the sequence

$$\begin{array}{ccc} H^p\left(\tilde{X}_\Delta, \partial\tilde{X}_\Delta\right) & \to & H^p\left(\tilde{X}_\Delta\right) \to H^p(X_s) \xrightarrow{N} H^p(X_s) \to \\ \wr\| & & \wr\| \\ H_{4n+2-p}(\tilde{X}_\Delta) & & H^p(X_{s_0}) \\ \wr\| & & \wr\| \\ H_{4n+2-p}(X_{s_0}) & & \left(R^p_\pi \mathbb{Z}\right)_{s_0} \end{array} \quad \left\{ \begin{array}{rl} \ker N & = \text{invariant cycles} \\ \ker N^\perp & = \text{vanishing cycles} \end{array} \right.$$

With the additional assumption

$$(T-I)^2 = 0 \Rightarrow G = \ker(T-I)^\perp / \mathrm{im}\ (T-I) \text{ is a finite group}$$

we have a Néron model $\bar{J}_e$ with an exact sequence

$$0 \to \mathcal{O}(J_e) \to \mathcal{O}(\bar{J}_e) \to G \to 0$$

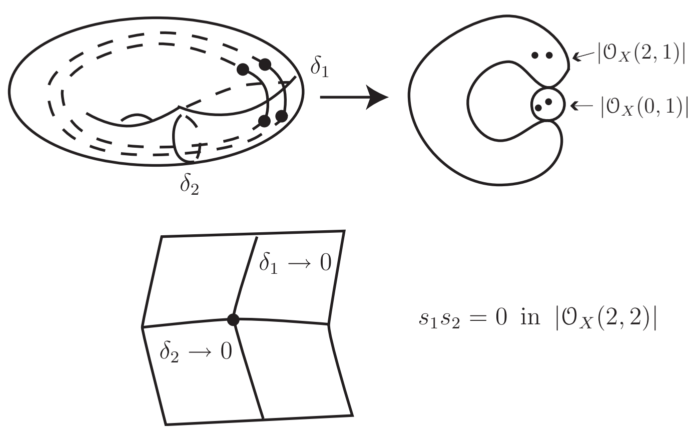
and Clemens [12] and M. Saito [33] showed that (1) extends using $\bar{J}_e$ ($G = \mathbb{Z}_2$ in the above example).

Issues. Due to the failure in general of Jacobi inversion the above method, at least as it has been applied, fails in general to lead to the construction of cycles (cf. [33]). Among the issues that have arisen in this study are:

— the need $\dim S$ arbitrary to see non-torsion phenomena and to have $\delta\nu_\xi$ non-trivial
— the assumption $(T-I)^2 = 0$ is too restrictive.

Of these the first may be the more significant, since the second is satisfied when the singularities are nodal and as discussed below, these seem to be sufficient to capture much of the geometry.† However, it is only when the base is higher dimensional that the full richness of the theory of degenerations of Hodge structures and the use of arguments requiring $L \gg 0$ and all of $H^0(\mathcal{O}_X(L))$ can be brought to bear on the problem. It is also only when $L \gg 0$ and the full $H^0(\mathcal{O}_X(L))$ is used that the infinitesimal invariant $\delta\nu$ captures the information in ν.

Example 4.1.4. $X = Q \subset \mathbb{P}^3$, $L = \mathcal{O}_X(2,2)$, $g(X_s) = 1$ and $Z = L_1 - L_2$ where the L_i are lines from the two rulings on Q. We then have the following picture



With ν an extended normal function as defined below we have

— $\nu(s_1, s_2) \equiv n_1 \log s_1 + n_2 \log s_2$ modulo (periods and holomorphic terms)
— ν extends to $\mathcal{J}_e \Leftrightarrow n_1 = n_2$
— $\tilde{\mathcal{J}}_{e,s_0}/\mathcal{J}_{e,s_0} \cong \mathbb{Z} \qquad (\nu \to n_1 - n_2)$
— $\nu_Z(s_1, s_2) \equiv \underbrace{2 \log s_1}_{\text{integrate over } —} - \underbrace{2 \log s_2}_{\text{integrate over } - - - -}$ modulo (periods and holomorphic terms)

Here $\tilde{\mathcal{J}}_e \supset \mathcal{J}_e$ is the sheaf of extended normal functions.

† Although from the physicists work on mirror symmetry we see that the "most singular" degenerations may also be very useful.

4.2 Extended Normal Functions and their Singularities

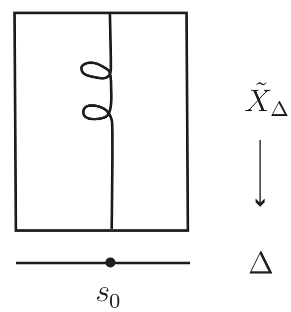
4.2.1 Geometric Motivation

Given $W^n \subset X^{2n}$, by a rational equivalence and working modulo torsion and complete intersection cycles, we may assume that W is smooth, and then for $L \gg 0$ there will be $s_0 \in S$ such that $W \subset X_{s_0}$; we may even assume that X_{s_0} is nodal (cf. section 4.4.1 below). If

$$\langle \zeta, [W] \rangle \neq 0 \qquad\qquad (\Rightarrow s_0 \in D \text{ and } [W]_{\text{prim}} \neq 0)$$

then ζ does *not* satisfy the analogue of Poincaré's first condition

$$\zeta = 0 \text{ in } H^{2n}(\tilde{X}_\Delta)$$



This suggests studying the behaviour of $\nu_\zeta(s)$, defined initially over $S^* = \{s \in S : X_s \text{ smooth}\}$, as $s \to s_0$. Such a study was attempted in [22] and [23], but this was inconclusive as the understanding of degenerations of Hodge structures over higher dimensional base spaces was not yet in place.

4.2.2 Definition of Extended Normal Functions (ENF)

Near $s_0 \in D$ where we have quasi-local normal crossings, for $\omega \in \mathcal{F}^n_{e,s_0}$ we have

$$\langle \nu, \omega \rangle (s) = P(\log s_1, \dots, \log s_k) + \{\text{meromorphic function} f(s)\}$$

for some polynomial P. By definition, *moderate growth* is the condition that $f(s)$ be holomorphic; we assume this analogue of Poincaré's second condition. In $\mathcal{U}^*$ choose a (multi-valued) lift $\tilde{\nu}$ to $\tilde{\mathcal{F}}^n_e$; then modulo homomorphic functions

$$\langle (T_i - I)\tilde{\nu}, \omega \rangle (s) \equiv \int_{\delta_{i,s}} \omega(s), \qquad \delta_{i,s} \in H_{2n-1}(X_s)$$

where $(T_i - I)\tilde{\nu}$ is the change in $\tilde{\nu}$ by analytic continuation around the puncture in the disk $|s_i| < 1$, $s_j = \text{constant}$ for $j \neq i$. The condition that ν

can be extended to $\mathcal{J}_e$ is

$$\delta_{i,s} = (T_i - I)\lambda_s \qquad \lambda_s \in H_{2n-1}(X_s, \mathbb{Z}) , \tag{4.3}$$

and then $m\nu$ extends to $\mathcal{J}_e$ if, and only if, (4.3) holds for $m\delta_{i,s}$.

Definition 4.2.1. An ENF is given by the sections ν of $\mathcal{J} \to S^*$ that near a point of the descriminant locus have moderate growth and satisfy

$$m\delta_{i,s} = (T_i - I)\lambda_{i,s} \quad \text{for some integer } m. \tag{4.4}$$

Thinking of $\mathcal{J}_e = \mathcal{F}^n_e \backslash \mathcal{H}^{2n-1}_e / R^{2n-1}_\pi \mathbb{Z}$ this is equivalent to

$$m\tilde{\nu} \equiv \sum_i e^{(N_i \log s_i)\lambda_{i,s}} \mod \mathcal{H}^{2n-1}_e$$

where $\tilde{\nu}$ is a lift of ν to $\mathcal{H}^{2n-1}$ over the punctured polycylinder $\mathcal{U}^*$ and $e^{(N_i \log s_i)\lambda_{i,s}}$ is a multi-valued section of $\mathcal{H}^{2n-1}$ over $\mathcal{U}^*$.

Notation: $\tilde{\mathcal{J}}_e$ is the sheaf of ENF's.

Theorem 4.2.2. ν_ζ *gives an ENF.*

When the base has dimension one this condition to be an ENF is equivalent to

$$m\nu \in \mathcal{J}_{e,s_0} \quad \text{for some integer } m.$$

The proof of Theorem 4.2.2 uses the full strength of the Clemens-Schmid exact sequence to show that (4.3) holds.

Note. We are indebted to the authors of [15] for pointing out to use the close relationship between our notion of an ENF and M. Saito's concept of an extended normal function [33]. Briefly, over S^* a normal function may be thought of as arising from a variation of mixed Hodge structure (VMHS). Along the discriminant locus $D = S\backslash S^*$ the condition of *admissibility* for a VMHS assumes a simple form for 2-step adjacent mixed Hodge structures; i.e., those for which the weight filtration has only two non-trivial adjacent terms. This is the case for normal functions and, the condition (2) above is essentially equivalent to admissibility as explained in the preprint [15].

4.2.3 Singularities of ENF's

By definition there is over S an exact sheaf sequence

$$0 \to \mathcal{J}_e \to \tilde{\mathcal{J}}_e \to \mathcal{G} \to 0 .$$

Definition 4.2.3. $\operatorname{sing} \nu$ is given by the support of the image of ν in $H^0(S, \mathcal{G})$.

From the works of Cattani-Kaplan-Schmid ([10]) one is led to consider the complex given by

$$\begin{aligned} V &= H^{2n-1}(X_s, \mathbb{Q}), \qquad s \in \mathcal{U}^* \\ B^p &= \bigoplus_{i_1 < \cdots < i_p} N_{i_1} \cdots N_{i_p} V \ , \end{aligned}$$

and with a Koszul-type boundary operator.

Theorem. i) *There is an injective map*

$$\mathcal{G}_{s_0} \otimes \mathbb{Q} \to H^1(B^\bullet) \ , \ \textit{and}$$

ii) *There is an isomorphism*

$$H^1(B^\circ) \approx IH^1(R_\pi^{2n-1}\mathbb{Q}) \ .$$

In the local normal crossing case the second isomorphism is based on the work of Cattani-Kaplan-Schmid [9] and Beilinson-Bernstein-Deligne-Gabber [3]. A proof that works also in the quasi-local normal crossing case has been shown to us by Mark de Cataldo and Luca Migliorini using their theory developed in [13]. The "purity" result of Gabber implies that the weights of $\mathcal{G}_{s_0} \otimes \mathbb{Q}$ are non-positive. This theorem will follow from Theorem 3 in section 4.3.2 below.

Theorem 4.2.4. i) *Assuming the HC in dimension* $< 2n$*,*

$$\operatorname{sing} \nu_\zeta = \{ s_0 \in D : \langle \zeta, [W] \rangle \neq 0 \ \textit{ where } \ W^n \subset X_{s_0} \} \ .$$

ii) *In general*

$$\operatorname{sing} \nu_\zeta = \{ s_0 \in D : \zeta_{s_0} \neq 0 \ \textit{ in } \ IH_{2n-2}(X_{s_0}) \} \, .$$

Corollary. *HC* $\Leftrightarrow \operatorname{sing} \nu_\zeta \neq 0$ *for* $L \gg 0$.

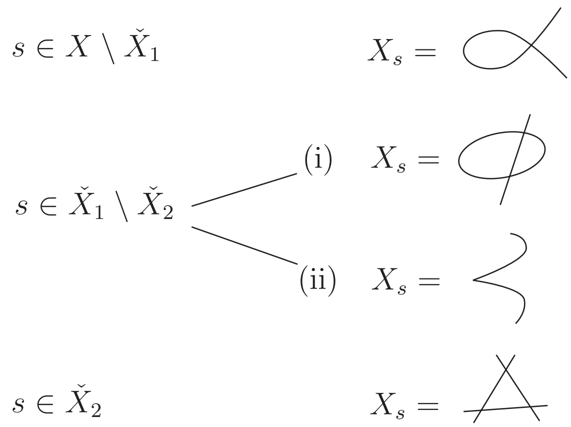
Example 4.2.5. Perhaps the simplest non-trivial example that illustrates how the singularities of a normal function are captured by the locus where $H^1(B^\bullet) \neq 0$ in the dual variety is given by a smooth cubic surface

$$X \subset \mathbb{P}^3 \ .$$

The dual has a stratification

$$\check{\mathbb{P}}^3 \supset \check{X} \supset \check{X}_1 \supset \check{X}_2$$

where $\check{X}_1 = \check{X}_{\text{sing}}$, $\check{X}_2 = (\check{X}_1)_{\text{sing}}$ and where the pictures are as follows:



As will be seen from the general result quoted in the next section

$$\begin{cases} H^1(B_s^\bullet)_{\mathbb{Q}} \cong \mathbb{Q} & s \in \check{X}_1 - \check{X}_2 \text{ of type (i)} \\ H^2(B_s^\bullet)_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q} & s \in \check{X}_2 \ . \end{cases} \tag{4.5}$$

We need not consider the locus $s \in \check{X}_1 - \check{X}_2$ of type (ii), since there the local monodromy is finite and we are working modulo torsion.

We will think of X as the blow up of 6 points $P_1, \dots, P_6 \in \mathbb{P}^2$ that are in general position with respect to lines and conics. The mapping

$$X \to \mathbb{P}^3$$

is given by the cubics in $\mathbb{P}^3$ that pass through $P_1, \dots, P_6$ — thus

$$H^0(\mathcal{O}_X(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}(3)(-P_1 - \cdots - P_6))$$

and we take the line bundle L on X to be $\mathcal{O}_X(1)$. The 27 lines on X are given classically by

$$\begin{cases} E_i = \text{blow up of } P_i \\ F_{ij} = \text{image of the line through } P_i \text{ and } P_j \\ G_i = \text{image of the conic through } P_1, \dots \check{P}_i, \dots, P_6. \end{cases}$$

The table of intersection numbers is straightforward to write down. A piece of it is

	E_j	F_{jk}	G_j
E_i	$-\delta_{ij}$	$\delta_{ij}+\delta_{1k}$	$1-\delta_{ij}$
E_2	$-\delta_{2j}$	$\delta_{2j}+\delta_{2k}$	$1-\delta_{2j}$
G_2	$1-\delta_{2j}$	$\delta_{2j}+\delta_{2k}$	$-\delta_{2j}$

For a line $\Lambda\subset X$ we will denote by $\Lambda^\perp\subset\check{X}$ the corresponding line in the dual projective space, so that with the obvious notation

$$\check{X}_1(i)=\bigcup_i E_i^\perp\bigcup_{ij}F_{ij}^\perp\bigcup_i G_i^1\ .$$

For purposes of illustration to get started we consider the two Hodge classes

$$\begin{cases}\zeta=[E_1-E_2] & E_1\cdot E_2=0\\ \zeta'=[E_1-G_2] & E_1\cdot G_2\neq 1.\end{cases}$$

The singular loci of $\nu_\zeta,\nu_{\zeta'}$ are unions of the $E_i^\perp,F_{ij}^\perp,G_j^\perp$. We evidently have

$$\begin{aligned}\Lambda^\perp\subset\operatorname{sing}\nu_\zeta &\quad\Leftrightarrow\quad E_1\cdot\Lambda\neq E_2\cdot\Lambda\\ \Lambda^\perp\subset\operatorname{sing}\nu_{\zeta'} &\quad\Leftrightarrow\quad E_1\cdot\Lambda\neq G_2\cdot\Lambda\ .\end{aligned}$$

From the above table we have

$$\begin{cases}\operatorname{sing}\nu_\zeta=E_1^\perp\cup E_2^\perp\cup F_{13}^\perp\cup\cdots\cup F_{16}^\perp\cup F_{23}^\perp\cup\cdots\cup F_{26}^\perp\cup G_1^\perp\cup G_2^\perp\\ \operatorname{sing}\nu_{\zeta'}=E_1^\perp\cup E_\zeta^\perp\cup\cdots\cup E_6^\perp\cup F_{13}^\perp\cup\cdots\cup F_{16}^\perp F_{23}^\perp\cup\cdots\\ \qquad\qquad\cdots\cup F_{26}^\perp\cup G_2^\perp\cup\cdots\cup G_6^\perp\end{cases}$$

which have degrees 12 and 18 respectively. In particular, ζ and ζ' are distinguished by their singular sets.

To formalize this we let in general

$$D\supset D_1\supset\cdots\supset D_N$$

be the stratification of the discriminant locus, and we set

$$\begin{cases}D_I=\bigcap_{i\in I}D_i\\ D_I^0=\text{non-singular part of }D_I\\ D_{I,\lambda}^0=\text{irreducible (= connected) components of }D_{I,\lambda}^0.\end{cases}$$

Then

$$H^1(B_s^\bullet)_{\mathbb{Q}}\ \textit{has constant dimension for}\ s\in D_{I,\lambda}^0\ .$$

Thus we may think of $H^1(B^\bullet_s)_{\mathbb{Q}}$ as a local system $V_{I,\lambda}$ on $D^0_{I,\lambda}$ and, as described above, a Hodge class ζ induces a section

$$(\operatorname{sing}\nu_\zeta)_{I,\lambda} \in H^0(D_{I,\lambda}, V_{I,\lambda}) .$$

We think of this as a map

$$\zeta \to \operatorname{sing}\nu_\zeta = \sum_{I,\lambda} (\operatorname{sing}\nu_\zeta)_{I,\lambda} D_{I,\lambda} \tag{4.6}$$

which assigns to each Hodge class the formal $\{V_{I,\lambda}\}$-valued cycle as above. The HC is equivalent to the assertion that the mapping (4.6) is injective for $L \gg 0$ (just how ample L must be will be discussed in section 4.4.1 below).

Returning to the cubic surface, it is easy to see that in this case there is no new information in the components of $\check{X}_2$. Moreover, for each line component $\Lambda^\perp$ of $\check{X}_1$ we may, by (4.5), canonically identify $H^0(\Lambda^\perp, V_{\Lambda^\perp})$ with $\mathbb{Q}$. When this is done, we have

$$\operatorname{sing}\nu_\zeta = \sum_\Lambda (\zeta \cdot \Lambda)\Lambda^\perp$$

where Λ runs over the lines on X and we have only summed over the codimension one components of $\check{X}$. Since any primitive Hodge class is uniquely specified by its intersection numbers with the lines, we see that for $L = \mathcal{O}_X(1)$ the map (4.6) is injective.

4.2.4 Nodal Hypersurface Sections †

.

As $s \to s_0$ we have vanishing cycles $\delta_\lambda \to p_\lambda \in \Delta_{s_0}$. The following numerical invariants reflect the topology, algebraic geometry and Hodge theory associated to the degeneration $X_s \to X_{s_0}$:

$$\begin{aligned}
\rho(\text{i}) &= \dim\{\text{space of relations among } \delta_\lambda\text{'s}\}\\
\rho(\text{ii}) &= \dim\{\text{image of } (H_{2n}(X_{s_0}) \to H_{2n}(X)_{\text{prim}})\}\\
\rho((\text{iii}) &= \dim\left\{\begin{array}{l}\text{failure of } p_\lambda \text{ to impose independent}\\ \text{conditions on } H^0(K_X\otimes L^n)\end{array}\right\}\\
&= h^1\left(\mathcal{I}_{\Delta_{s_0}} \otimes K_X \otimes L^n\right), \quad L \gg 0\\
\rho(\text{iv}) &= h^{n,n-1}(\tilde{X}_{s_0}) - \left(h^{n,n-1}(X_s) - \#\text{ double points}\right)\\
\rho(\text{v}) &= \dim\left\{\operatorname{Hg}^{n-1}(\tilde{X}_{s_0})/\operatorname{im}\operatorname{Hg}^{n-1}(X)\right\}\\
\rho(\text{vi}) &= \dim H^1(B^\bullet)
\end{aligned}$$

† This section is based in part on correspondence with Herb Clemens and Richard Thomas; cf. [35]

Theorem. $\rho(\mathrm{i}) = \rho(\mathrm{ii}) = \rho(\mathrm{iii}) = \rho(\mathrm{iv}) = \rho(\mathrm{v}) = \rho(\mathrm{vi})$.

Given a smooth codimension-n subvariety $W \subset X$, for $L \gg 0$ there exist nodal hypersurfaces X_{s_0} passing through W. Generically all of the nodes p_λ will be on W (Bertini) and there is a Chern class formula for the quantities

(a) $h^0(\mathcal{I}_W(L))$
(b) number of nodes of X_{s_0}.

Theorem. *For X_{s_0} general among hypersurfaces containing W, the subvariety W is uniquely determined by the fundamental class $[\tilde{W}] \in H^{2n-2}(\tilde{X}_{s_0})$ of the proper transform $\tilde{W}$ in the canonical desingularization $\tilde{X}_{s_0}$ of X_{s_0}.*

A consequence of this result is that, for $L \gg 0$, a component of the Hodge-theoretically defined variety $\operatorname{sing} \nu_\zeta$ is equal to the locus

$$\{s_0 \in D : \text{there exists a unique } W \subset X_{s_0} \text{ with } \langle \zeta, [W] \rangle \neq 0\} .$$

It is in this precise sense that a Hodge class gives the equations of the dual algebraic cycles.

Theorem (Clemens). *For $L \gg 0$ the monodromy action on the nodes $p_\lambda \in X_{s_0}$, where W is fixed and $X_{s_0} \supset W$ varies, is doubly transitive.*

A consequence is that for $L \gg 0$ and X_{s_0} a general nodal hypersurface containing W

$$\rho(\mathrm{i}) = 1;$$

in fact, the generating relation is

$$\sum_\lambda \pm \delta_\lambda = 0$$

where the $\pm$ reflects a choice of orientation. From

$$\rho(\mathrm{i}) = \rho(\mathrm{vi})$$

in the theorem above we conclude that

$$\dim IH^1 \left(R_\pi^{2n-1} \mathbb{Q} \right) = 1$$

where the intersection homology of the local system $R_\pi^{2n-1}\mathbb{Q}$ is taken over a neighborhood of s_0.

It is easy to check that if X_{s_0} has nodes $p_1, \ldots, p_m$ that impose independent conditions on the linear system $|X_s|$, so that locally

$$D = D_1 \cup \cdots \cup D_m$$

where $\delta_\lambda \to 0$ along D_λ, then

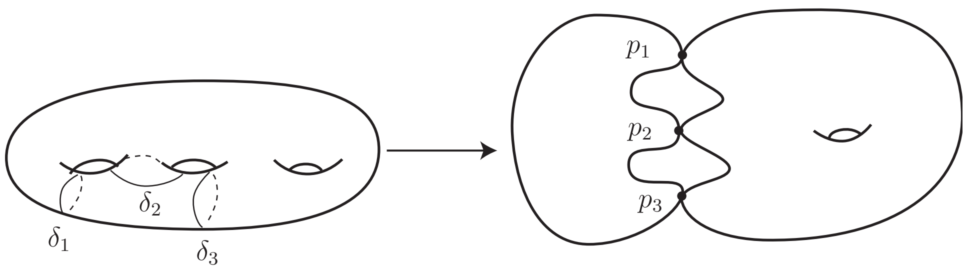
$$H^1(B^\bullet_{s_0}) \cong \{\text{relations along the } \delta_\lambda\} \ .$$

This is the case in the above theorem of Clemens. However, this is a very special circumstance, as illustrated by the following

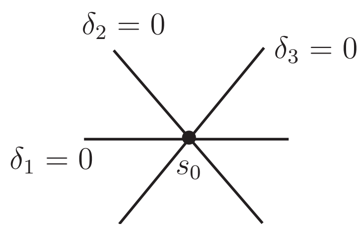
Example. Let $X \subset \mathbb{P}^3$ be a smooth quartic surface containing a line Λ but otherwise general. The planes containing Λ give a line

$$\Lambda^\perp \subset \check{X} \subset \check{\mathbb{P}}^3 \ .$$

For s_0 a general point of $\Lambda^\perp$, the picture of X_s as $s \to s_0$ is



where $\delta_i \to p_i$ as $s \to s_0$. A transverse plane slice of $\check{X}$ locally looks like



This means that locally $\check{X}$ is the union of three smooth hypersurfaces $\check{X}_i$ that intersect pairwise transversely along a smooth curve, and where $\delta_i = 0$ on $\check{X}_i$. This is a situation where one has quasi-local normal crossings; to obtain the local normal crossing picture we must blow up $\check{X}$ along $\Lambda^\perp$.

In this case the condition $(*)$ in section 4.1.1 is satisfied. For the complex $B^\bullet$ we have

$$V \xrightarrow{\alpha} N_1V \oplus N_2V \oplus N_3V \xrightarrow{\beta} N_1N_2V \oplus N_1N_3V \oplus N_2N_3V.$$

Since there is one relation among the δ_i we have

$$\begin{aligned} \dim\ker\alpha &= 4 \\ \Rightarrow \dim\operatorname{coker}\alpha &= 1 \\ \Rightarrow H^1(B^\bullet) &\cong \mathbb{Q} \end{aligned}$$

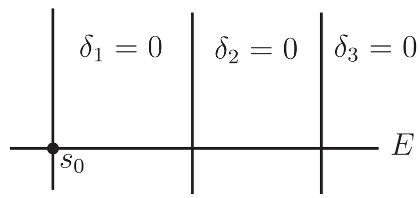
since $\beta = 0$. For $H \subset X$ a general plane section and

$$\zeta = [4\Lambda - H] \in \mathrm{Hg}^1(X)_{\mathrm{prim}} \ ,$$

one may check that in a neighborhood of s_0 as above

$$\operatorname{sing} \nu_\zeta = \Lambda^\perp \ .$$

Now suppose we blow up $\Lambda^\perp$ so as to obtain



where along the exceptional divisor E we have $\delta_1 = \delta_2 = \delta_3 = 0$. Then, say, around s_0 we have

$$\begin{cases} T_1 = T_{s_1} & \text{(around } \delta_1 = 0) \\ T_2 = T_{\delta_1} + T_{\delta_2} + T_{\delta_3} = 0 & \text{(around } E), \end{cases}$$

where in the second relation attention must be paid to signs.

The complex $B^\bullet$ is now

$$V \xrightarrow{\alpha} N_1 V \oplus N_2 V \ .$$

With suitable choice of bases and signs we will have

$$N_1 \gamma_1 = \delta_1$$
$$\begin{cases} N_2 \gamma_1 = 2\delta_1 + \delta_2 \\ N_2 \gamma_2 = \delta_1 + 2\delta_2 \end{cases}$$

and all other $N_i \gamma_j = 0$. Then

$$\begin{cases} \alpha(\gamma_1) = (\delta_1, 2\delta_1 + \delta_2) \\ \alpha(\gamma_2) = (0, \delta_1 + 2\delta_2) \end{cases}$$

so that

$$\operatorname{coker} \alpha = H^1(B^\bullet)$$

has generator $(\delta_1, 0)$ and thus is of dimension one.

Note: We had originally thought that under blowing up the singularities of ν_ζ might disappear, and are grateful to the authors of [**?**] for pointing out to us that this is not the case.

This example is typical in that when we have the situation

$$W \subset X_{s_0} \in |L|$$

as above, then in a neighborhood $\mathcal{U}$ in $\mathbb{P}H^0(\mathcal{O}_X(L))$ of $s_0 \in \check{X}$ we will have that

$$\check{X} \cap \mathcal{U} = D_1 \cap \cdots \cap D_m$$

where the conditions in $(*)$ in 4.1.1 are generally satisfied. More precisely, if we consider the universal local deformation space (Kuraniski space) $\mathcal{M}$ for X_{s_0}, then in many circumstances it will be the case that the nodes p_i may be independently smoothed. Denoting by $\mathcal{M}_i \subset \mathcal{M}$ the hypersurface where the node p_i remains, $\mathcal{M}_{s_0} = \underset{i}{\cup} \mathcal{M}_i$ forms a normal crossing divisor in $\mathcal{M}$. If $\mathcal{U} \subset |L|$ is a neighborhood of s_0 and we set $D_i = \mathcal{U} \cap \mathcal{M}_i$, then we may again generally expect that $\mathcal{U}$ meets each $\mathcal{M}_I$, so that the quasi-normal condition in 4.1.1 will be satisfied.

The general result is:

Let $I = J \cup K$ be a set of nodes with $J \cap K = \varnothing$ and the nodes in each of J, K independent. Then

$$H^1(B^\bullet) \cong \{\text{relations among the nodes in } I\}\ .$$

4.3 Infinitesimal Invariant and its Singularities

4.3.1 Definition of the Infinitesimal Invariant

Recalling (4.2) we have:

Definition. $\delta\nu = [\nabla\tilde{\nu}] \in H^0\left(\mathcal{H}_e^{n-1,n} \otimes \Omega_S^1(\log D)\right)_\nabla$.

The basic properties of $\delta\nu$ (cf. [18] and [36]) are as follows.

i) $\delta\nu = 0$ if, and only if, ν lifts to a locally constant section $\tilde{\nu}$ of $\mathcal{H}^{2n-1}$. This requires $L \gg 0$, which also implies that
ii) the vanishing cycles are of finite index in

$$H^{2n-1}(X_s, \mathbb{Z})/H^{2n-1}(X, \mathbb{Z})\ .$$

From this an argument using the Picard-Lefschetz formula gives
iii) $\tilde{\nu}$ locally constant $\Rightarrow \nu$ is torsion in $\mathcal{J}(X_s)/\mathcal{J}(X)$. It then follows that
iv) ν_ζ torsion $\Rightarrow \zeta$ is torsion.

This finally implies the mapping

$$\zeta \to \delta\nu_\zeta$$

is injective modulo torsion, which is (ii) in Theorem 4.1.3.

4.3.2 Singularities of $\delta\nu$

Theorem 4.3.1. i) *There is a canonical map*

$$\left(\mathcal{H}_e^{n-1,n} \otimes \Omega_S^p(\log D)\right)_{\nabla,s_0} \to H^p(B^\bullet)$$

which then allows us to define

$$\operatorname{sing}\delta\nu_\zeta = \textit{image of } \delta\nu_\zeta \in H^1(B^\bullet)\ ,$$

and with this definition we have for $L \gg 0$

$$\operatorname{sing}\nu_\zeta = \operatorname{sing}\delta\nu_\zeta .$$

ii) *For* $L \gg 0$

$$\zeta \neq 0 \mod \text{torsion} \ \Rightarrow \delta\nu_\zeta \neq 0\ .$$

Corollary. $HC \Leftrightarrow \operatorname{sing}\delta\nu_\zeta \neq 0$ *for* $L \gg 0$.

Whereas ν_ζ is an analytic invariant, $\delta\nu_\zeta$ is an algebro-geometric invariant; by (ii) the information in ζ is, for $L \gg 0$, captured by $\delta\nu_\zeta$.

To sketch the basic idea of the proof, if D has quasi-local normal crossings there is a map

$$\operatorname{Res}_{s_0}\Omega_S^p(\log D) \to \bigoplus_I \mathbb{C}_I$$

where $I = \{i_1 < \cdots < i_p\}$ and $\mathbb{C}_I$ is the constant sheaf supported on D_I. This gives

$$\operatorname{Res}_{s_0}\Omega_S^p(\log D) \otimes \mathcal{H}_e \to \bigoplus_I (\mathcal{H}_{e,s_0})_I$$

where $(\mathcal{H}_{e,s_0})_I = \mathcal{H}_{e,s_0} \otimes_{\mathbb{C}} \mathbb{C}_I$. If $\tilde{\nu}$ is a local multi-valued lifting on ν, then by definition of an ENF, for some integer m we have

$$m(T_i - I)\tilde{\nu} \in \operatorname{Im}\left\{(T_i - I)H^{2n-1}(X_{s_0}, \mathbb{Z})\right\}$$

where $T_i - I$ is "analytic continuation around D_i". This relation translates into

$$m\operatorname{Res}_{s_0}(\nabla\tilde{\nu}) \in \bigoplus_i (T_i - I)\mathcal{H}_{\mathbb{Z},e,s_0}\ .$$

so that from the complex

$$(*) \qquad \mathcal{H}_{\mathbb{Z},e;s_0} \to \oplus(T_i - I)\mathcal{H}_{\mathbb{Z},e;s_0} \to \underset{i<j}{\oplus}(T_i - I)(T_j - I)\mathcal{H}_{\mathbb{Z},e;s_0}$$

we obtain

$$m\mathrm{Res}_{\,s_0}(\nabla\tilde{\nu}) \in H^1(*)\ .$$

Now $\tilde{\nu}$ is well-defined up to

$$\tilde{\nu} \to \tilde{\nu} + f + \lambda \tag{4.7}$$

where $f \in \mathcal{F}_e^n$ and $\lambda \in \mathcal{H}_{\mathbb{Z},e;s_0}$. The contribution of λ is $\sum(T_i - I)\lambda$ which gives a coboundary which This disposes of the ambiguity λ in (4.7). The ambiguity f disappears because $\nabla f = 0$ in the complex (4.2).

Next, we need to replace $(T_i - I)$ by $N_i = (T_i - I)A_i$ where A_i is an invertible matrix defined over $\mathbb{Q}$ and where all the A_i commute among each other. So over $\mathbb{Q}$ the new complex has the same cohomology as the complex $(*)$. At this stage we have, over $\mathbb{Q}$, essentially described the definition of the map

$$\mathrm{sing}\,\nu \to H^1(B^0_{s_0})_{\mathbb{Q}}\ .$$

Next we define the subcomplex

$$\Omega^\bullet_S\left(\frac{df_1}{f_1}N_1\mathcal{H}_e,\dots,\frac{df_k}{f_k}N_k\mathcal{H}_e\right) \subset \Omega^\bullet_S(\log D)\otimes\mathcal{H}_e\ ,$$

where the differential on the subcomplex is given by

$$\left(\wedge\sum_i \frac{df_i}{f_i}N_i\right)\ .$$

There is then a map of complexes

$$\begin{array}{ccccc}
\mathcal{H}_e & \longrightarrow & \Omega^1_S & \longrightarrow & \Omega^2_S\left(\frac{df_1}{f_1}N_1\mathcal{H}_e,\dots,\frac{df_k}{k_k}N_k\mathcal{H}_e\right) \\
\big\downarrow{\scriptstyle \mathrm{Res}_{\,s_0}} & & \big\downarrow{\scriptstyle \mathrm{Res}_{\,s_0}} & & \big\downarrow{\scriptstyle \mathrm{Res}_{\,s_0}} \\
\mathcal{H}_{e,s_0} & \longrightarrow & \underset{i}{\oplus} N_i\mathcal{H}_{e,s_0} & \longrightarrow & \underset{i<j}{\oplus} N_iN_j\mathcal{H}_{e,s_0}
\end{array}$$

under which $\delta\nu$ maps to $\mathrm{sing}\,\nu$ in $H^1(B^\bullet_{s_0})_{\mathbb{Q}}$. This is the construction of Theorem 3.

4.4 Issues and Deeper Structures

4.4.1 The Effective Hodge Conjecture (EHC)

Let X be a smooth variety, of any dimension for the moment, $L \to X$ a very ample line bundle with Chern class $\lambda = c_1(L)$, and

$$\zeta \in \mathrm{Hg}^p(X) .$$

Since torsion consideration will be important in this section we shall use $\mathbb{Z}$ coefficients. The (HC) is equivalent to the statement

(1) *There exist integers k_0, m_0 such that for $m \geqq m_0$*

$$k_0\zeta + m(\lambda)^p = [Z] \tag{4.8}$$

where Z is an effective, integral algebraic cycle.

Indeed, if

$$k_0\zeta = [Z' - Z'']$$

where Z', Z'' are effective cycles, then writing

$$Z'' = \sum n_i Z_i, \qquad n_i \in \mathbb{Z} ,$$

where the Z_i are irreducible of codimension p, by passing hypersurfaces of high degree through each Z_i we will have

$$Z_i + W_i = H_1 \cap \cdots \cap H_p$$

for a subvariety W_i, which then gives

$$[-Z_i] = l\lambda^p + [W_i]$$

for some integer l, from which (1) follows. We note that if (1) holds for $m = m_0$, then it also holds for $m \geqq m_0$. The (EHC), in various forms to be discussed below, asks for (1) with estimates on k_0, m_0.

We are grateful to the referee for pointing out to us the following result of Kollár, which illustrates the care that must be taken in consideration of the torsion coefficient k_0.

We consider the space $\mathcal{M}$ of smooth hypersurfaces

$$X \subset \mathbb{P}^{n+1}, \qquad n \geqq 3$$

of degree $d \geqq n+1$. Then

$$\mathrm{Hg}^{n-1}(X) \cong H^{2n-2}(X, \mathbb{Z}) \cong \mathbb{Z} .$$

Denoting by

$$\mathcal{M}_k \subset \mathcal{M}$$

the subvariety of X's containing a curve of degree k, Kollár [29, 32] showed that if $(k,d)=1$ and d is divisible by a prime power p^3 where $p>n$, then $\mathcal{M}_k$ is a proper subvariety; and Soulé-Voisin [32, § 2] further remark that $\bigcup_{(k,d)=1}\mathcal{M}_k$ is dense in $\mathcal{M}$. Of course, for any integer l

$$\mathcal{M}_{dl}=\mathcal{M}\,.$$

One conclusion is that the torsion coefficient k_0 in (4.8) could have subtle dependence on X.

Now we return to the main situation in this paper where $\dim X=2n$ and we are considering a primitive Hodge class ζ. If

$$\zeta=[Z]$$

for an integral algebraic cycle Z, then it follows from results of Kleiman [28] that we will have

$$(n-1)!\zeta+m\lambda^m=[W], \qquad m\geqq m_0\,, \tag{4.9}$$

where W is a smooth, codimension n subvariety. In fact, we may take W to be the degeneracy classes of general sections $\sigma_1,\dots,\sigma_{r-1}$ of a very ample rank r vector bundle $F\to X$. This implies that the normal bundle $N_{W/X}$ is ample, and by a result of Fulton-Lazarsfeld (*Annals of Math.* **118** (1983), 35–60)

$$c_n(N_{W/X})>0\,. \tag{4.10}$$

When n is odd, so that

$$-\zeta^2>0$$

by the Hodge-Riemann bilinear relation, (4.9) and (4.10) give

$$m_0\geqq\frac{1}{(n-1)!}\sqrt[2n]{\frac{-\zeta^2}{\lambda^{2n}}},\quad n\geq 2. \tag{4.11}$$

This suggests that any estimate on m_0 in (4.8) must involve ζ^2.

In fact, for $n=1$, where the above relation does not make sense, it follows from known results that

The (EHC) (4.8) *holds for* $k_0=1$ *and where we may take*

$$m_0=-\zeta^2+C(\zeta c_1(X),\lambda c_1(X),c_1^2(X),c_2(X)) \tag{4.12}$$

where C is a universal linear combination of the constants listed. Moreover, we shall give an heuristic argument that there exist divisors in surfaces X for which a lower bound (4.11) *holds, up to constants.*

The last statement follows by considering the case of quartic surfaces $X \subset \mathbb{P}^3$. Letting H be a hyperplane class, we first note that there is no uniform m_0 such that for every X and $\zeta \in \mathrm{Hg}^1(X)_{\mathrm{prim}}$

$$\zeta + m_0 H \text{ is effective.} \tag{4.13}$$

In fact, by considering X for which $NS(X)$ is generated by H plus a non-complete intersection curve of degree d we see that

There is no uniform m such that for every X there are curves $W_1, \dots, W_\rho$ in X that span $NS(X)$ and have degree $\leqq m$.

Thus there is no uniform m for which (4.13) holds for all X.

Next, to see why the lower bound should hold we consider the following statement.

> *Any estimate on m_0 in* (4.13) *must in general involve the lengths of the shortest spanning vectors in* $\mathrm{Hg}^1(X, \mathbb{Z})_{\mathrm{prim}}$. (4.14)

The heuristic argument for (4.14) is based on the following quite plausible (and possibly known) statements: Let $\Lambda = H^2(X, \mathbb{Z})$ and $\Lambda_0 = \{\zeta \in \Lambda : \zeta \cdot H = 0\}$. Let $P \subset \Lambda$ be the vectors that are not divisible by any $n \in \mathbb{Z}$, $n \neq \pm 1$ (these are primitive in a different sense of the term). Then

i) there exists $\zeta_n \in P \cap \Lambda$, with $\zeta_n^2 \to -\infty$
ii) there exists a polarized Hodge structure H_n on $\Lambda_{\mathbb{C}}$ with $\zeta_n \in \mathrm{Hg}^1(H_n)$
iii) in (ii) we may arrange that the Picard number $\rho(H_n) = 2$
(iv) there exists a (possibly singular) $X_n \subset \mathbb{P}^3$ such that H_n is a direct summand of the Hodge structure on a desingularization $\tilde{X}_n$ of X.

Let $\lambda = [H]$ and Z be an irreducible curve on $\tilde{X}_n$ with

$$[Z] = a\lambda + b\zeta_n \qquad a, b \in \mathbb{Z}\ .$$

Then $\deg Z = a$. By adjunction, since Z is irreducible the arithmetic genus $\pi(Z)$ satisfies

$$0 \leqq \pi(Z) = \frac{Z \cdot Z}{2} + 1$$

while

$$Z \cdot Z = 4a^2 + b^2 \zeta_n^2$$

which implies

$$a \geqq \frac{\sqrt{-\zeta_n^2}}{2}\ .$$

Returning to (4.11) which only gives information for odd n, for the first even case $n = 2$ for purposes of illustration we assume that there exists a rank-two vector bundle $E \to X$ with

$$\begin{cases} c_2(E) = \zeta \\ c_1(E) = a\lambda \ . \end{cases}$$

Setting as usual $E(m) = E \otimes L^m$, let m_0 be such that there is $\sigma \in H^0(\mathcal{O}_X(E(m_0)))$ with $(\sigma) = W$ where $N^\vee_{W/X} \cong E(m_0)|_W$ is ample. Then we claim that

$$m_0^2 + bm_0 > c\sqrt{\zeta^2} \tag{4.15}$$

where b, c are constants depending only on X, E and L.

Proof By another result due to Fulton-Lazarsfeld (loc. cit.), since $N_{W/X}$ is ample we have

$$c_1(N_{W/X})^2 > c_2(N_{W/X}) \ ,$$

from which(4.15) follows. □

This again suggests the possibility of there being, in general, a lower bound on m_0 for which (4.8) holds in terms of $|\zeta|^2$.

This possibility is reinforced by the following considerations: Let $\mathcal{M}$ be a quasi-projective algebraic variety parametrizing a family of smooth projective X's with reference variety $X_0 \in \mathcal{M}$. For example, $\mathcal{M}$ could be a moduli space if such exists. Letting $\mathcal{U}$ be a sufficiently small neighborhood of X_0 and $\zeta \in \mathrm{Hg}^n(X_c)_{\mathrm{prim}}$, the locus

$$\mathcal{U}_\zeta = \mathcal{U} \cap \{X \in \mathcal{U} : \zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}\}$$

of nearby points where ζ remains a Hodge class is an analytic variety. By a theorem of Cattani-Deligne-Kaplan [8] it is part of an *algebraic* subvariety

$$\mathcal{M}_\zeta \subset \mathcal{M} \ .$$

We shall write points of M_ζ as (X, ζ) to signify that there is a Hodge class ζ extending the one defined over $\mathcal{U}_\zeta$, where we may have to go to a finite covering to make ζ single-valued. For each k, m with $m > 0$, $k \neq 0$ we consider the subvarieties

$$\mathcal{M}_{k,m} = \{(X, \zeta) \in \mathcal{M}_\zeta : k\zeta + m\lambda^n = [Z]\}$$

where Z is an effective algebraic cycle.

Assuming the (HC) we have

$$\bigcup_{k,m} \mathcal{M}_{k,m} = \mathcal{M}_\zeta \ .$$

It follows that the LHS is a *finite* union; thus

$$\mathcal{M}_\zeta = \bigcup_{i=1}^{N} \mathcal{M}_{k_i,m_i} \ .$$

Letting k_0, m_0 be multiples of all the k_i, m_i respectively, and using that for a positive integer a

$$\mathcal{M}_{k,m} \subset \mathcal{M}_{ak,am}$$

we have that

$$\mathcal{M}_{k_0,m_0} = \mathcal{M}_\zeta \tag{4.16}$$

from which we conclude:

If the (HC) is true, then (4.8) *holds for a uniform* k_0, m_0 *when* (X, ζ) *varies in an algebraic family.*

Now suppose we now let the Hodge class ζ vary. Then on the one hand, for each positive constant c we shall give an heuristic argument that

$$\bigcup_{|\zeta|^2 \leqq c} \mathcal{M}_\zeta = \mathcal{M}_c \text{ is an } \textit{algebraic} \text{ subvariety of } \mathcal{M}. \tag{4.17}$$

On the other hand, typically

$$\bigcup_{\zeta} \mathcal{M}_\zeta \ \textit{ is dense in } \mathcal{M} \ . \tag{4.18}$$

Letting k_ζ, m_ζ be integers such that (4.16) holds with $k_0 = k_\zeta, m_0 = m_\zeta$ we will then have

$$\left\{ \begin{array}{l} k_\zeta, m_\zeta \ \text{ are bounded if } \ |\zeta|^2 < c \\ k_\zeta, m_\zeta \ \text{ are not bounded for all } \ \zeta \ . \end{array} \right. \tag{4.19}$$

This again suggests the possibility of a lower bound on m_ζ in terms of $|\zeta|^2$. A proof of (4.17) follows from [8]. Here we give a slightly different way of proceeding, anticipating some possible consequences of the recent work [27].

Heuristic argument for (4.17)*:* Let D be the classifying space for polarized Hodge structures of the same type as $H^{2n}(X_0, \mathbb{Z})_{\text{prim}}/\text{mod torsion}$. Thus we are given a lattice with integral non-degenerate quadratic form $(H_\mathbb{Z}, Q)$ and

D consists of all Hodge-type filtrations $\{F^p\}$ on $H_{\mathbb{C}}$ satisfying the Hodge-Riemann linear relations. There is a period mapping

$$\varphi : \mathcal{M} \to \Gamma\backslash D$$

where $\Gamma = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$. By the work of Kato-Usui [27], it is reasonable to expect that there will be a partial compactification $(\overline{\Gamma\backslash D})_\Sigma$ such that φ extends to

$$\overline{\varphi} : \overline{\mathcal{M}} \to \overline{(\Gamma\backslash D)_\Sigma} \tag{4.20}$$

for a suitable compactification of $\overline{\mathcal{M}}$ of $\mathcal{M}$. Here, Σ stands for a set of *fans* that arise in the work of Kato-Usui (loc. cit.)

Now let $H_{\mathbb{Z}}^{\mathrm{prim}}$ be the lattice vectors ζ that are primitive in the arithmetic sense; i.e., ζ is only divisible by ± 1 in $H_{\mathbb{Z}}$. Then it is known that

There are only finitely many Γ *orbits in* $H_{\mathbb{Z}}^{\mathrm{prim}}$ *with fixed* $Q(\zeta, \zeta)$. (4.21)

For each $\zeta \in H_{\mathbb{Z}}^{\mathrm{prim}}$ we let

$$D_\zeta = \left\{ \{F^0\} \in D : \zeta \in F^n \text{ is a Hodge class} \right\} .$$

Then, by (4.21), D_ζ projects to a closed analytic subvariety

$$(\Gamma\backslash D)_\zeta \subset \Gamma\backslash D .$$

Analysis similar to that in Cattani-Deligne-Kaplan (loc. cit.) suggests that $(\Gamma\backslash D)_\zeta$ extends to a closed log-subvariety

$$\overline{(\Gamma\backslash D)_\zeta} \subset \overline{(\Gamma\backslash D)_\Sigma} .$$

Then

$$\overline{\mathcal{M}_\zeta} = \overline{\varphi}^{-1}(\overline{(\Gamma\backslash D)}_\zeta)$$

will be an algebraic subvariety (which, as noted above, we know by Cattani-Deligne-Kaplan) and essentially because of (4.21), there are only *finitely* many such $\overline{\mathcal{M}}_\zeta$'s with $|\zeta|^2 \leqq c$.

Summary. i) *There is heuristic evidence that any bounds on* k_0, m_0 *such that* (4.8) *holds will depend on* $|\zeta|^2$*, together with quantities* $a|\zeta| + b$*, where* a, b *are constants independent of* ζ*.*

ii) *For* $n = 1$ *we may take* $k_0 = 1$ *and there is an upper bound* (4.12) *on* m_0*. For a general surface* X*, this bound is sharp.*

In a subsequent work, we shall show that obtaining an estimate on codim(sing ν_ζ) requires that we let X vary in its moduli space and consider the Noether-Lefschetz loci. Heuristic reasoning then suggest the following formulation of an effective (HC)

(EHC): *There is a relation* (4.8) *where the constants* k_0, m_0 *depend on* $|\zeta^2|, a\zeta + b$, *and universal characters constructed from* $H^*(X)$ *and* $H^*(\mathcal{M})$ *that do not depend on* ζ.

4.4.2 Localization of Primitive Cohomology along the Singular Locus

The central issue is that $\delta\nu_\zeta$ — but *not* ν_ζ — can be defined for any class

$$\zeta \in H^{n,n}(X)_{\mathrm{prim}} \subset H^n\left(\Omega^n_X\right) \ , \tag{4.22}$$

and then as a consequence of Theorem 3

$$\begin{aligned} & \zeta \in \mathrm{Hg}^n(X)^\perp \cap H^{n,n}(X)_{\mathrm{prim}} \\ \Rightarrow \quad & \mathrm{sing}\,(\delta\nu_\zeta) = 0 \quad \text{for } L \gg 0 \ . \end{aligned}$$

Thus, any existence result involving $\delta\nu_\zeta$ must involve the condition that ζ be an integral class, or equivalently that ν_ζ exist. Roughly speaking the *residues* of $\delta\nu_\zeta$ must be integral in order to be able to "integrate" and enable us to define

$$\nu_\zeta = \text{“}\textstyle\int \delta\nu_\zeta\text{”}.$$

This brings us to the

Question 4.4.1. Given ζ as in (4.53), how can we tell if $\zeta \in H^{2n}(X,\mathbb{Q})$ — i.e.

$$\int_\Gamma \zeta \in \mathbb{Q} \ , \qquad \Gamma \in H_{2n}(X,\mathbb{Z}) \ ?$$

It turns out that there is a very nice geometric structure underlying this question. It is based on two principles

1st Principle: *Denoting by* $\check{X}$ *the dual variety of* X *and by* $H \to \check{X}$ *the hyperplane bundle, the group* $H^n(\Omega^n_X)_{\mathrm{prim}}$ *may be expressed globally along the singular locus*

$$\Delta \subset X \times \check{X}$$

by the failure collectively of the $\Delta_s = \Delta \cap X_s \times \{s\}$, $s \in D$, *to impose independent conditions on* $|K_X \otimes L^n \otimes H^n|$.

Here we are thinking algebraically with $H^n(\Omega^n_X)$ being defined in the Zariski topology.

2nd Principle: *For $p \in \Delta_s$ there is a map*

$$K_{X,p} \otimes L_p^n \to \mathbb{C} \ ,$$

well-defined up to sign and given by

$$\omega \to \pm \mathrm{Res}_p \left(\frac{\omega}{s^n} \right) , \qquad \omega \in K_{X,p} \otimes L_p^n \ ,$$

where $\tilde{\omega}$ is any extension of ω to a neighborhood and $s \in H^0(\mathcal{O}_X(L))$ defines X_s.

Remark. This leads to an integral structure expressed by (4.27) below.

The injection (4.22) also arises from a canonical section

$$\eta \in H^0 \left(\mathcal{O}_\Delta \left(K_X^2 \otimes L^{2n} \otimes H^{2n} \right) \right)$$

and we may think of the image of $\mathbb{Z}\zeta$ in (4.17) as being

$$\mathbb{Z}\sqrt{\eta} \subset \mathcal{O}_\Delta \left(K_X \otimes L^n \otimes H^n \right) \ .$$

The section η is constructed as follows: At points $p \in \Delta$, the universal section

$$s = \text{quadratic } + \text{ (higher order terms)}$$

and the quadratic terms give a canonical symmetric map

$$T_{X,p} \to T_{X,p}^* \otimes L_p \otimes H_p$$

which by exterior algebra induces (recalling that $\dim X = 2n$)

$$\Lambda^{2n} T_{X,p} \to \Lambda^{2n} T_{X,p}^* \otimes L_p^{2n} \otimes H_p^{2n}$$

and then we obtain

$$\eta(p) \in \left(\Lambda^{2n} T_{X,p}^* \right)^2 \otimes L_p^{2n} \otimes H_p^{2n}$$

with the property that

$$\eta(p) \neq 0 \Leftrightarrow p \text{ is a node.}$$

Globalizing over $X \times S$, this map gives an injection of sheaves

$$\mathbb{Z}\sqrt{\eta} \to \mathcal{O}_\Delta \left(K_X \otimes L^n \otimes H^n \right) \ . \tag{4.23}$$

Combining these two principles leads to a commutative diagram for $L \gg 0$

$$\begin{array}{ccc} & & H^n(\Omega^n_X)_{\text{prim}} \\ & & \downarrow\cong \\ \dfrac{H^0\left(\mathcal{O}_\Delta(K_X \otimes L^n \otimes H^n)\right)}{H^0\left(\mathcal{O}_{X\times S}(K_X \otimes L^n \otimes H^n)\right)} & \xrightarrow{\approx} & H^1\left(\mathcal{I}_\Delta \otimes K_X \otimes L^n \otimes H^n\right) \\ & & \bigcup \\ & & \Lambda \end{array} \tag{4.24}$$

The horizontal isomorphism is the standard long exact cohomology sequence arising from $0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X\times S} \to \mathcal{O}_\Delta \to 0$ and using $L \gg 0$, and Λ is the subgroup arising from (4.23) and the numerator in (4.24) under the horizontal isomorphism there.

The vertical isomorphism is more interesting. It uses the Koszul complex associated to

$$ds \in H^0\left(\mathcal{O}_{X\times S}(\Sigma^* \otimes L)\right)$$

where the prolongation bundle Σ arises from

$$0 \to \Omega^1_{X\times S} \to \mathcal{O}_{X\times S}(\Sigma) \to \mathcal{O}_{X\times S} \to 0$$

with extension class $c_1(L \otimes H)$, and the vanishing theorems necessary to have the isomorphism require $L \gg 0$ to ensure Castelnuovo-Mumford type of regularity. This isomorphism is constructed purely algebraically. The Leray spectral sequence applied to the universal family

$$\begin{array}{ccc} & \mathcal{X} & \subset X \times S \\ \swarrow & & \searrow^{\pi} \\ X & & S \end{array}$$

lead to a spectral sequence which, when combined with (4.24), gives a diagram

$$\begin{array}{ccc} H^1\left(\mathcal{I}_\Delta \otimes (K_X \otimes L^n \otimes H^n)\right) & \xrightarrow{\alpha} & H^0\left(R^1_\pi \mathcal{I}_\Delta \otimes (K_X \otimes L^n \otimes H^n)\right) \\ \wr\Vert & & \\ H^n(\Omega^n)_{\text{prim}} & & \\ \bigcup & & \\ \text{Hg}^n(X)_{\text{prim}} & & \end{array}$$

Theorem. *Combining* (4.23) *and* (4.24) *we have*

$$\text{Hg}^n(X) \to \left\{\text{image of } \Lambda \text{ in } H^0\left(R^1_\pi \mathcal{I}_\Delta \otimes K_X \otimes L^n \otimes H^n\right)\right\}\ ; \tag{4.25}$$

i.e., Hodge classes have integral residues.

Then it may be shown that (this is essentially $\rho(\text{iii}) = \rho(\text{v})$ in section 4.2.4 above)

$$HC \Rightarrow \alpha \textit{ is injective on } \mathrm{Hg}^n(X)_{\mathrm{prim}} \, . \tag{4.26}$$

From (4.24) a consequence is that

$$\textit{a class } \zeta \in H^n(\Omega^n_X)_{\mathrm{prim}} \textit{ is integral} \Leftrightarrow \textit{the residues of } \zeta \textit{ are integral} \tag{4.27}$$

The spectral sequence argument also gives

$$\begin{aligned} H^1\left(R^0_\pi \mathcal{I}_\Delta \otimes (K_X \otimes L^n \otimes H^n)\right) = 0 \ \textit{for } L \gg 0 \\ \Rightarrow \alpha \text{ is injective} \\ \Rightarrow HC. \end{aligned} \tag{4.28}$$

The statements (4.26) and (4.28) give precise meaning to the general principle:

The HC may be reduced to (in fact, is equivalent to) a statement about the global geometry of

$$\Delta \subset X \times S \, . \tag{4.29}$$

We thus have:

$$HC \Leftrightarrow \text{geometric property of (4.29) when } L \gg 0.$$

Above we have discussed the question: *Can we a priori estimate how positive L must be?* The condition $L \gg 0$ in this section requires sufficient positivity to have vanishing of cohomology plus Castelnuovo-Mumford regularity. Above, we gave a heuristic argument to the effect that for each ζ the condition $L \gg 0$ must also involve ζ^2.

Discussion: Denote by $\check{X}_k$ the dual variety to the image of

$$X \to \mathbb{P}\check{H}^0(\mathcal{O}_X(L^k)) \, .$$

One may ask the question

What are the properties of the singular set $\check{X}_{k,\mathrm{sing}}$ of $\check{X}_k$ for $k \gg 0$?

Although we shall not try to make it precise, one may imagine two types of singularities: (i) Ones that are present for a general $X \subset \mathbb{P}\check{H}^0(\mathcal{O}_X(L))$ having the same numerical characters as X; in particular, they should be invariant as X varies in moduli. (In this regard, one may assume that $L \to X$ is already sufficiently ample so as to have those vanishing theorems that will ensure that $\dim \check{X}_k$ can be computed from the numerical characters of X_1). (ii) Ones that are only present for special X. What our study shows is that:

If the HC is true, then non-generic singularities of type (ii) are necessarily present when $\mathrm{Hg}^n(X)_{\mathrm{prim}} \neq 0$.

One may of course ask if singularities of type (ii) are caused by anything other than Hodge classes?

4.4.3 Remarks on Absolute Hodge Theory

Recent works [19] on Hodge-theoretic invariants of algebraic cycles have shown that in codimension $\geqq 2$ arithmetic aspects of the geometry — specifically the *spread* of both X and of cycles in X — must be taken into account and this might be a consideration for an effective HC. In considering spreads a central issue is that one does not know that

$$\textit{A Hodge class is an absolute Hodge class.} \tag{4.30}$$

That is, for X defined over a field k of characteristic zero, a class in $H^n(\Omega^n_{X(k)/k})$ (sheaf cohomology computed algebraically in the Zariski topology) that is a Hodge class for one embedding $k \subset \mathbb{C}$ using

$$H^n\left(\Omega^n_{X(k)/k}\right) \otimes \mathbb{C} \cong H^n\left(\Omega^n_{X(\mathbb{C})}\right)$$

(GAGA) is a Hodge class for any embedding of k in $\mathbb{C}$ (here we also assume that $k = \bar{k}$). We shall refer to the statement (4.30) as *absolute Hodge* (AH).

We close by remarking that the above geometric story works over any algebraically closed field of characteristic zero — in particular one has the diagram (4.24) and integrality conditions on $H^n(\Omega^n_{X(k)/k})$ given by the image of Λ in $H^0(R^1_\pi \mathcal{I}_\Delta \otimes K_X \otimes L^n \otimes H^n)$. For any embedding $k \subset \mathbb{C}$ such that the (well-defined) map

$$\mathrm{Hg}^n(X)_{\mathrm{prim}} \to \Lambda$$

is injective (which is implied by the HC), one has a direct geometric "test" for when a class in $H^n\left(\Omega^n_{X(k)/k}\right)$ is in $H^{2n}(X(\mathbb{C}), \mathbb{Q})$.

Remark. We shall give a very heuristic argument to suggest that

$$\mathrm{AH} \Rightarrow \mathrm{HC}\ . \tag{4.31}$$

The reasoning is as follows.

i) Assuming AH, the statement of HC is purely algebraic;
ii) When $p = 1$ the HC is true, and although the existing proofs both

use transcendental arguments, by model theory there will be a purely algebraic proof of the algebraic statement

$$\operatorname{sing}\nu_\zeta \neq \varnothing \text{ for } L \gg 0 , \tag{4.32}$$

which is equivalent to the HC;

iii) Finally (and most "heuristically"), because the geometric picture of the structure of $\operatorname{sing}\nu_\zeta$ is "uniform" for all n — in contrast, for example, to Jacobi inversion — any purely algebraic proof of (4.32) that works for $n = 1$ will work for all n.

4.5 The Poincaré Line Bundle

Given a Hodge class $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}$ there is an associated analytic invariant $\nu_\zeta \in H^0(S, \widetilde{\mathfrak{J}_E})$ and its singular locus

$$\operatorname{sing}\nu_\zeta \subset D .$$

Although the local behaviour of ν_ζ and subsequent local structure of $\operatorname{sing}\nu_\zeta$ can perhaps be understood, the direct study of the global behaviour of ν_ζ and of $\operatorname{sing}\nu_\zeta$ — e.g., is $\operatorname{sing}\nu_\zeta \neq \varnothing$ for $L \gg 0$ — seems of course to be more difficult. In this section we will begin the study of potentially important global invariants of ν_ζ obtained by pulling back canonical line bundles (or rather line bundle stacks) that arise from the polarizations on the intermediate Jacobians $J(X_s)$. We shall do this only in the simplest non-trivial case and there we shall find, among other things, that

$$c_1\left(\nu^*_{\zeta\times\zeta}(\text{Poincaré line bundle})\right) = \zeta^2 .$$

This is perhaps significant since as we have given in section 4.4.1 an heuristic argument to the effect that any lower bound estimate required for an EHC will involve ζ^2.

4.5.1 Polarized Complex Tori and the Associated Poincaré Line Bundle

The material in this section is rather standard; see for instance [25, Ch. 2]. We shall use the notations

- V is a complex vector space of dimension b,
- $\Lambda \subset V$ is a lattice of rank $2b$.
- $T = V/\Lambda$ is the associated complex torus.

We then have, setting $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$,

$$\Lambda_{\mathbb{C}} = V \oplus \bar{V}$$

where the conjugation is relative to the real structure on $\Lambda_{\mathbb{C}}$. There are canonical identifications

$$\begin{aligned} &\text{(i)} && H^{p,q}(T) \cong \Lambda^p V^* \otimes \Lambda^q \bar{V}^* \\ &\text{(ii)} && (\Lambda^p V^* \otimes \Lambda^q \bar{V}^*)^* \cong \Lambda^{b-p} V^* \otimes \Lambda^{b-q} \bar{V}^* \end{aligned} \tag{4.33}$$

where (ii) is given by

$$\varphi \otimes \psi \to \int_T \varphi \wedge \psi \ .$$

Definition. A polarization on T is given by a non-degenerate, alternating bilinear form

$$Q : \Lambda \otimes \Lambda \to \mathbb{Z}$$

which, when extended to $\Lambda_{\mathbb{C}}$, satisfies

$$Q(V, V) = 0 \ . \tag{4.34}$$

The polarization is principal in case $\det Q = \pm 1$.

We shall see that a polarization gives a holomorphic line bundle

$$M \to T \ ,$$

well-defined up to translation, and satisfying

$$c_1(M)^b[T] \neq 0 \ .$$

We shall also see that Q defines an Hermitian metric in M whose Chern form is expressed as

$$c_1(M) = \frac{\sqrt{-1}}{2\pi} \sum h_{i\bar{j}} dv^i \wedge d\bar{v}^j$$

where $v_1, \ldots, v_b \in V^*$ gives a basis and

$$\begin{cases} h_{i\bar{j}} = \bar{h}_{j\bar{i}} \\ \det \|h_{i\bar{j}}\| \neq 0 \ . \end{cases}$$

Thus the Hermitian matrix $\|h_{i\bar{j}}\|$ is non-degenerate but, *in contrast to the usual terminology, we do not require that it be positive or negative definite.*

We shall give two constructions of M. For the first, assuming as we shall always do that the polarization is principal, we may choose a Q-symplectic basis $x^1, \dots, x^b; y^1, \dots, y^b$ for $\Lambda^*_{\mathbb{Z}}$ so that

$$Q = \sum_i dx^i \wedge dy^i \ .$$

Thinking of Λ as $H_1(T, \mathbb{Z})$ and with the canonical identification

$$H^2(T, \Lambda) = \mathrm{Hom}(\Lambda^2 H_1(T, \mathbb{Z}), \mathbb{Z})$$

we have

$$Q \in H^2(T, \mathbb{Z}) \ .$$

By (4.34), when expressed in terms of $dv^1, \dots, dv^b, d\bar{v}^1, \dots, d\bar{v}^b$ we have that

$$Q = \frac{\sqrt{-1}}{2\pi} \sum h_{i\bar{j}} dv^i \wedge d\bar{v}^j$$

where the matrix $\|h_{i\bar{j}}\|$ is Hermitian and non-singular. Thus

$$Q \in \mathrm{Hg}^1(T) \ ,$$

and since T is a compact Kähler manifold it is well-known that there exists a holomorphic line bundle $M \to T$ with a Hermitian metric and with

$$Q = c_1(M)$$

being the resulting Chern form.

It is also well-known that the subgroup $\mathrm{Pic}^0(T)$ of line bundles with trivial Chern class has a canonical identification

$$\mathrm{Pic}^0(T) \cong \bar{V}^* / \Lambda^* \tag{4.35}$$

and that the action of T on $\mathrm{Pic}^0(T)$ by translation is the natural linear algebra one using the above identification. Thus, M is uniquely determined by $c_1(M)$ up to translation.

Before doing that we want to recall the construction of the Poincaré line bundle

$$P \to T \times \mathrm{Pic}^0(T) \ .$$

For this we have the canonical identification

$$H_1(T \times \mathrm{Pic}^0(T), \mathbb{Z}) \cong \Lambda \otimes \Lambda^*$$

and from this the canonical inclusion

$$\mathrm{Hom}(\Lambda^*, \Lambda^*) \subset H^2(T \times \mathrm{Pic}^0(T), \mathbb{Z}) \ .$$

The class $\mathbb{T}$ in $H^2(T\times\mathrm{Pic}^0(T),\mathbb{Z})$ corresponding to the identity in $\mathrm{Hom}(\Lambda^*,\Lambda^*)$ is easily seen to lie in $\mathrm{Hg}^1(T\times\mathrm{Pic}^0(T))$ and therefore defines a class of line bundles

$$P\to T\times\mathrm{Pic}^0(T)\ .$$

We may uniquely specify P by requiring that both

$$\begin{aligned}&P\mid T\times\{e^\circ\}\\&P\mid\{e\}\times\mathrm{Pic}^0(T)\end{aligned}$$

are trivial, where e,e° are the respective origins in $T,\mathrm{Pic}^0(T)$. The map

$$\mathrm{Pic}^0(T)\to\mathrm{Pic}^0(T)$$

given by

$$a^\circ\to P\mid T\times\{a^\circ\},\qquad a^\circ\in\mathrm{Pic}^0(T)$$

is the identity.

The above construction gives what is usually called the Poincaré line bundle. However, for the purposes of this work we assume given a principal polarization Q in T and will canonically define a line bundle

$$P_Q\to T\times T \tag{4.36}$$

which will induce an isomorphism

$$T\cong\mathrm{Pic}^0(T)$$

by

$$a\to P_Q\mid T\times\{a\},\qquad a\in T$$

and via this isomorphism the identification

$$P\cong P_Q\ .$$

Definition. Denoting by

$$\mu:T\times T\to T$$

the group law, the Poincaré line bundle (4.36) is defined by

$$P_Q=\mu^*M\otimes p_1^*M^*\otimes p_2^*M^*\otimes M_e \tag{4.37}$$

where the $p_i:T\times T\to T$ are the coordinate projections of $M\to T$ in any line bundle with $c_1(M)=Q$ and M_e is the fibre of M over the identity.

Proof that (4.37) *is well defined:* Denote by P_M the RHS of (4.37) . Since

$$P_{M\otimes R} = P_M \otimes P_R$$

we have to show that for a line bundle $R \to T$

$$P_R = 0 \ \text{ if } \ c_1(R) = 0 \ . \tag{4.38}$$

In fact we will show that P_R is *canonically* trivial. We will check that

$$c_1(R) = 0 \Rightarrow c_1(P_R) = 0 \ . \tag{4.39}$$

Assuming this we have

$$P_R \in \mathrm{Pic}^0(T \times T) \cong \mathrm{Pic}^0(T) \oplus \mathrm{Pic}^0(T) \ .$$

Then, by definition, for $a, a' \in T$

$$\begin{aligned} (P_R)_{(a,e)} &= R_e^* \\ (P_R)_{(e,a')} &= R_e^* \end{aligned}$$

so that the two "coordinates" of P_R are zero, hence P_R is trivial. To make the trivialization canonical we need to show independence of scaling, and this is the role of the M_e factor.

To verify (4.39), in general we may choose coordinates $x^i, y^i \in \Lambda_{\mathbb{Z}}^*$ so that any line bundle R has $c_1(R)$ represented by

$$\sum_i \lambda_i dx^i \wedge dy^i \ .$$

Using coordinates (x^i, y^i, x'^j, y'^j) on $\Lambda_{\mathbb{R}} \oplus \Lambda_{\mathbb{R}}$ and using that

$$\mu^i((x,y)\dot{+}(x',y')) = (x^i + x'^i, y^i + y'^i)$$

where $\dot{+}$ is the group law on T, it follows that $c_1(P_R)$ is represented by

$$\sum_i \lambda_i (dx^i \wedge dy'^i + dx'^i \wedge dy^i) \ .$$

In particular, if the $\lambda_i = 0$ then (4.39) follows. □

Remark. For later use we note for Q as above

$$c_1(P_Q) = \sum_i dx^i \wedge dy'^i + dx'^i \wedge dy^i \ .$$

In particular

$$c_1(P_Q)^{2b}[T \times T] = 2^b \ . \tag{4.40}$$

4.5.2 Topological Properties of the Poincaré Line Bundle in Smooth Families

We let X and

$$\begin{array}{l} \mathcal{X} \subset X \times S \\ \downarrow p \\ S \end{array}$$

be as above. For simplicity of exposition we assume that for a general point $s \in S$ all of $H^{2n-1}(X_s)$ is primitive, so that the intermediate Jacobian

$$\begin{aligned} J(X_s) &= F^n H^{2n-1}(X_s)\backslash H^{2n-1}(X_s)/H^{2n-1}(X_s, \mathbb{Z}) \\ &\cong F^n \check{H}^{2n-1}(X_s)/H_{2n-1}(X_s, \mathbb{Z}) \end{aligned}$$

has a principal polarization as discussed in the preceding section. In this section we will assume the existence of a smooth curve $B \subset S$ such that all the X_s, $s \in B$, are smooth. This is a very rare circumstance, but one that will help to prepare the way for the treatments below of the case when B is a general curve in S. We set

$$\mathcal{X}_B = p^{-1}(B)$$

and denote by

$$J_B \to B$$

the smooth analytic fibre space of complex tori with fibres $J(X_s)$, $s \in B$. Then

(1) *There exists a complex line bundle stack $\mathcal{M}_B \to J_B$ whose restriction to each fibre gives a line bundle, defined up to translation by a point of finite order, and whose Chern class is the polarizing form.*

The meaning of the term "stack" in the present context will be explained below.

(2) *There exists a complex line bundle $\mathcal{P}_B \to J_B \times_B J_B$ whose restriction to each fibre of $J_B \times_B J_B \to B$ is the Poincaré line bundle.*

The point is that *in each case the Chern classes*

$$c_1(\mathcal{M}_B), c_1(\mathcal{P}_B) \in H^2(B, \mathbb{Q})$$

may be defined.

We let D be the classifying space for polarized complex tori and

$$\mathcal{T} \to D$$

the universal family of these tori. Then there is a line bundle

$$\begin{array}{c} \mathcal{M} \to \mathcal{T} \times_D \mathcal{T} \\ \downarrow \\ D \end{array} \tag{4.43}$$

such that over each $T \times T \cong T \times \mathrm{Pic}^0(T)$ the line bundle $\mathcal{M}$ restricts to be the family of all principally polarized line bundles whose Chern class is the polarizing form, as explained in the preceding section. We denote by $\mathcal{M}_e$ the restriction of $\mathcal{M}$ to $\mathcal{T} \times_D \{e\} \cong \mathcal{T}$.

Let $\Gamma_{\mathbb{Z}} \cong Sp(b, \mathbb{Z})$ be the arithmetic group associated to the above situation. Then $D/\Gamma_{\mathbb{Z}}$ is an analytic variety whose points are in one-to-one correspondence with equivalence classes of principally polarized complex tori. As usual in the theory of stacks, there is no universal family of complex tori over all of $D/\Gamma_{\mathbb{Z}}$, although the quotient

$$\mathbb{C}^b \times D/\mathbb{Z}^{2b} \times \Gamma_{\mathbb{Z}}$$

exists as a family of complex tori over the automorphism-free ones. Given $\mathcal{X}_B \to B$ as above, letting $\Gamma \subset \Gamma_{\mathbb{Z}}$ be the monodromy group we have the picture

$$\begin{array}{l} J_B \\ \downarrow \\ B \xrightarrow{\varphi} D/\Gamma \end{array} \tag{4.44}$$

which one thinks of as the family of complex tori that would be induced by pulling back the universal family if the latter existed.

Turning to (4.56), there is an action of $\mathbb{Z}^{2b} \times \Gamma_{\mathbb{Z}}$ on $D \times \mathbb{C}^b \times \mathbb{C}^b \times \mathbb{C}$ that would represent the descent of (4.56) to $D/\Gamma_{\mathbb{Z}}$ were it not for the presence of automorphisms. In addition, it can be shown that for $\gamma \in \Gamma_{\mathbb{Z}}$, $\mathbb{Z}^{2b} \times \{\gamma\}$ maps $\mathcal{M}_e$ to $\mathcal{M}_{a(\gamma)}$ where

$$\mathcal{M}_{a(\gamma)} \mid T = \mathcal{M} \mid T \times \{a(\gamma)\}$$

where $a(\gamma)$ is a division point in T. (This is well-known phenomenon for principally polarized abelian varieties.) Turning to (4.44), there will be a subgroup $\Gamma_0 \subset \Gamma$ of finite index and such that $a(\gamma) = e$ for $\gamma \in \Gamma_0$. Let $\pi : \tilde{B} \to B$ be a finite covering such that (4.44) lifts to

$$\begin{array}{l} J_{\tilde{B}} \\ \downarrow \\ \tilde{B} \xrightarrow{\tilde{\varphi}} D/\Gamma_0 \,. \end{array}$$

Then $\mathcal{M}_e$ is invariant under Γ_0 and induces a line bundle

$$\begin{array}{ccc} \tilde{\varphi}^*(\mathcal{M}_e) & \longrightarrow & J_{\tilde{B}} \\ & & \downarrow \\ & & \tilde{B} \end{array}$$

that on each fibre induces a line bundle whose Chern class is the polarizing form. If d is the degree of $\tilde{B} \xrightarrow{\pi} B$, then even though $\mathcal{M}_B$ is not defined, we may set

$$c_1(\mathcal{M}_B) = \frac{1}{d} c_1\left(\pi_*(\tilde{\varphi}^*(\mathcal{M}_e)\right) \in H^2(B,\mathbb{Q}) \,. \tag{4.45}$$

One may check that this is independent of the choice of covering $\tilde{B} \to B$.

The discussion of the Poincaré line bundle is similar but easier since it is uniquely characterized by the properties discussed in the preceding section.

Having defined $\mathcal{M}_B$ and $\mathcal{P}_B$ we now consider normal functions ν_ζ viewed as cross-sections of

$$\begin{array}{c} \mathcal{J}_B \\ \downarrow \curvearrowright \nu_\zeta \\ B \end{array}$$

and ask:

What is the dependence of $c_1(\nu_\zeta^*(\mathcal{M}_B))$ *and* $c_1(\nu_{\zeta\times\zeta'}^*(\mathcal{P}_B))$ *on* ζ, ζ'*?*

Here, the Chern classes are considered as rational numbers using $H^2(B,\mathbb{Q}) \cong \mathbb{Q}$. To discuss this question we set

$$Z_\zeta = \nu_\zeta(B) - \nu_0(B)$$

and define the quantities

$$\left.\begin{array}{lll} \text{(i)} & Q_1(\zeta,\zeta') = c_1(\nu_{\zeta+\zeta'}^*(\mathcal{M}_B)) - c_1(\nu_\zeta^*(\mathcal{M}_B)) \\ & \qquad\qquad - c_1(\nu_{\zeta'}^*(\mathcal{M}_B)) + c_1(\nu_0^*(\mathcal{M}_B)) \\ \text{(ii)} & Q_2(\zeta,\zeta') = c_1(\nu_{\zeta\times\zeta'}^*(\mathcal{P}_B)) - c_1(\nu_{0\times 0}^*(\mathcal{P}_B)) \\ \text{(iii)} & Q_3(\zeta,\zeta') = p_1^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}] \cup c_1(\mu^*(\mathcal{M}_B)) \,. \end{array}\right\} \tag{4.46}$$

Here in (iii), we are working in the cohomology ring of $J_B \times_B J_B$ and the p_i are the projections onto the two coordinate factors. We remark that:

The definition of $Q_1(\zeta,\zeta')$ *is motivated by the property*

$$(a \dot{+} a') - (a) - (a') + (e) \sim 0$$

on an elliptic curve E, where $\dot{+}$ is the group law on E, (b) is the 0*-cycle associated to a point $b \in E$ and $\sim$ is linear equivalence.*

Theorem 4.5.1. *$Q_1(\zeta, \zeta') = Q_2(\zeta, \zeta') = Q_3(\zeta, \zeta') = \zeta\zeta'$. The last term is to be taken as the value of the $\zeta \cup \zeta'$ on $[X]$.*

Since for $\zeta \neq 0$ primitive,

$$(-1)^n \zeta^2 > 0$$

we have the following curious

Corollary 4.5.2. *Setting $\mathcal{P}_{B,\zeta} = \nu^*_{\zeta\times\zeta}(\mathcal{P}_B)$, we have for $m \in \mathbb{Z}$*

$$h^0(\mathcal{P}^{(\pm)}_{B,m\zeta}) = m^2(\zeta^2) + (\textit{terms not depending on } m)\ .$$

where $\pm$ is the parity of n.

Sketch of the proof: The *real* dimension of $\mathcal{J}_B$ is $4b+2$, and denoting the Leray filtration by F^p_L we have that the fundamental class

$$[Z_\zeta] \in F^1_L H^{2b}(J_B) \to H^1(B, R^{2b-1}\mathbb{Z}_{J_B})$$

where we use $\mathbb{Z}$ coefficients throughout and $R^{2b-1}\mathbb{Z}_{J_B}$ is the $(2b-1)^{\text{st}}$ direct image of $\mathbb{Z}$ under the map $J_B \to B$. Here and below the notation means that $[Z_\zeta] \in F^1_L H^{2b}(J_B)$ which then maps to $Gr^1_L H^{2b}(J_B) \cong H^1(B, R^{2b-1}\mathbb{Z}_{J_B})$. Denoting by $\mathcal{J}_B$ the sheaf of holomorphic sections of $J_B \to B$ we have

$$\mathcal{J}_B = \frac{\mathcal{R}^{2n-1}\mathbb{C}_{\mathcal{X}_B}}{\mathcal{F}^n\mathcal{R}^{2n-1}\mathbb{C}_{\mathcal{X}_B} + R^{2n-1}\mathbb{Z}_{\mathcal{X}_B}} \cong \frac{\mathcal{R}^{2b-1}\mathbb{C}_{J_B}}{\mathcal{F}^b\mathcal{R}^{2b-1}\mathbb{C}_{J_B} + R^{2b-1}\mathbb{Z}_{J_B}}$$

where $\mathcal{R}^{2n-1}\mathbb{C}_{\mathcal{X}_B}$ is understood to be

$$\mathcal{O}_B \otimes R^{2n-1}\mathbb{C}_{\mathcal{X}_B}$$

and $R^{2n-1}\mathbb{C}_{\mathcal{X}_B}$ is $R^{2n-1}\mathbb{C}$ for the projection $p : \mathcal{X}_B \to B$. Now $\zeta \to \nu_\zeta \in H^0(B, \mathcal{J}_B)$ is linear in ζ, and we have

$$\begin{array}{ccc} H^0(B, \mathcal{J}_B) & \to & H^1(B, R^{2n-1}\mathbb{Z}_{\mathcal{X}_B}) \\ & & \wr\| \\ & & H^1(B, R^{2b-1}\mathbb{Z}_{J_B}) \end{array}$$

where in the top row

$$\nu_\zeta \to \zeta \in F^1_L H^{2n-1}(\mathcal{X}_B) \to H^1(B, R^{2n-1}\mathbb{Z}_{\mathcal{X}_B})$$

and under the vertical isomorphism

$$\zeta \to [Z_\zeta] \in F^1_L H^{2b}(J_B) \to H^1(B, R^{2b-1}\mathbb{Z}_{J_B})\ .$$

Thus this mapping is linear in ζ.

For simplicity of notation we set

$$\begin{aligned} M_\zeta &= \nu_\zeta^*(\mathcal{M}_B) \\ P_{\zeta\times\zeta'} &= \nu_{\zeta\times\zeta'}^*(\mathcal{P}_B) \\ &= \nu_{\zeta\times\zeta'}^*\left(\mu^*\mathcal{M}_B \otimes p_1^*\mathcal{M}_B^* \otimes p_2^*\mathcal{M}_B^* \otimes \mathcal{M}_0\right) \end{aligned}$$

where $\mathcal{M}_{B,0} \to B$ is the line bundle stack whose fibres are the fibres of $\mathcal{M}_B$ over the 0-section and μ is the fiberwise addition map. Then we have

$$c_1(P_{\zeta\times\zeta'}) = c_1(M_{\zeta+\zeta'}) - c_1(M_\zeta) - c_1(M_{\zeta'}) + c_1(M_0)\ . \tag{4.47}$$

The first step is to analyze

$$p_1^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}]\ . \tag{4.48}$$

Since cup product is Poincaré dual to intersection on a smooth manifold, and since $p_1^*[Z_\zeta]$ is the cycle traced by

$$\{\nu_\zeta(s) \times J_s - \nu_0(s) \times J_s\}_{s\in S}$$

and similarly for ζ', we see that (4.48) is Poincaré dual to the cycle traced out by

$$\{\nu_\zeta(s) \times \nu_{\zeta'}(s) - \nu_\zeta(s) \times \nu_0(s) - \nu_0(s) \times \nu_{\zeta'}(s) + \nu_0(s) \times \nu_0(s)\}_{s\in S}\ .$$

Call this cycle $Z_{\zeta\times\zeta'}$, so that

$$p_1^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}] = [Z_{\zeta\times\zeta'}]\ .$$

For the second step, since

$$[Z_{\zeta\times\zeta'}] \cup c_1(\mu^*\mathcal{M}_B) = \int_{Z_{\zeta\times\zeta'}} \mu^* c_1(\mathcal{M}_B)$$

where the RHS is the sum with signs of the values of $\mu^*c_1(\mathcal{M}_B)$ on the four curves in the definition of the cycle $Z_{\zeta\times\zeta'}$, we have from (4.47) that

$$p_1^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}] \cup c_1(\mu^* M_B) = c_1(P_{\zeta\times\zeta'})\ . \tag{4.49}$$

For the next step, since ζ and ζ' are primitive and thus live in $F_L^1 H^{2n}(\mathcal{X}_B, \mathbb{Z})$ where F_L is the Leray filtration, they define classes

$$[\zeta], [\zeta'] \in H^1(B, R^{2n-1}\mathbb{Z}_{\mathcal{X}_B})$$

in $Gr_L^1 = F_L^1/F_L^2$. As above, the notation $R^{2n-1}\mathbb{Z}_{\mathcal{X}_B}$ means $R_p^{2n-1}\mathbb{Z}$ for the projection $p : \mathcal{X}_B \to B$. We then have

$$H^1(B, R^{2n-1}\mathbb{Z}_{\mathcal{X}_B}) \otimes H^1(B, R^{2n-1}\mathbb{Z}_{\mathcal{X}_B}) \to H^2(B, R^{4n-2}\mathbb{Z}_{\mathcal{X}_B}) \cong \mathbb{Z}\ , \tag{4.50}$$

where the last isomorphism uses $R^{4n-2}\mathbb{Z}_{\mathfrak{X}_B} \cong \mathbb{Z}$ and $H^2(B,\mathbb{Z}) \cong \mathbb{Z}$. It is known that

$$\textit{Under the mapping } (4.50),\ [\zeta]\otimes[\zeta'] \to \zeta\cdot\zeta' \tag{4.51}$$

Passing to $J_B \to B$, we have by definition

$$R^{2b-1}\mathbb{Z}_{J_B} \cong R^{2n-1}\mathbb{Z}_{\mathfrak{X}_B}$$

and that the image of $[Z_\zeta]$ in $H^{2b}(J_B,\mathbb{Z}) \cong H^1(B, R^{2n-1}\mathbb{Z}_{J_B})$ corresponds to the image of ζ in $Gr^1 H^{2b}(\mathfrak{X}_B,\mathbb{Z}) \cong H^1(B, R^{2n-1}\mathbb{Z}_{\mathfrak{X}_B})$. Moreover, *since the polarization is principal* we have

$$Q^{b-1} : R^1\mathbb{Z}_{J_B} \cong R^{2b-1}\mathbb{Z}_{J_B}\ .$$

Thus $[Z_\zeta]$ defines a class

$$[Z_\zeta]_Q \in H^1(B, R^1\mathbb{Z}_{J_B})\ ,$$

and it may be shown from (4.50) that under the pairing

$$H^1(B,R^1\mathbb{Z}_{J_B}) \otimes H^1(B,R^{2b-1}\mathbb{Z}_{J_B}) \to H^2(B,R^{2b}\mathbb{Z}_{J_B}) \cong \mathbb{Z}\ , \tag{4.52}$$

where the last isomorphism results from $R^{2b}\mathbb{Z}_{J_B} \cong \mathbb{Z}$ and $H^2(B,\mathbb{Z}) \cong \mathbb{Z}$, we have in (4.52)

$$[Z_\zeta]_Q \otimes [Z_{\zeta'}]\ \textit{ maps to }\ \zeta\cdot\zeta'\ . \tag{4.53}$$

For the final step, for a torus $T = V \mid \Lambda$ with principal polarization $Q \in \Lambda^2\Lambda^*$ we have

i) $\Lambda^{2b}\Lambda^* \cong \mathbb{Z}$ (using [35])

ii) $\Lambda^* \otimes \Lambda^{2b-1}\Lambda^* \to \mathbb{Z}$ (cup product)

where we have

iii) $Q^{b-1}\otimes$ identity: $\Lambda^*\otimes\Lambda^{2b-1}\Lambda^* \simeq \Lambda^{2b-1}\Lambda^*\otimes\Lambda^{2b-1}\Lambda^*$,

and

iv) the diagram

$$\begin{array}{ccc}
H^1(T,\mathbb{Z})\otimes H^{2b-1}(T,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\
\wr\| & & \| \\
H^{2b-1}(T,\mathbb{Z})\otimes H^{2b-1}(T,\mathbb{Z}) & \xrightarrow{Q} & \mathbb{Z} \\
\downarrow & & \| \\
H^{4b-2}(T\times T,\mathbb{Z}) & \xrightarrow{c_1(P_Q)} & Z
\end{array}$$

commutes where the top vertical isomorphism is (iii).

This gives the conclusion

$$p_1^*[Z_\zeta] \otimes p_2^*[Z_{\zeta'}] \otimes c_1(\mu^*\mathcal{M}_B) \to \mathbb{Z} \text{ computes } \zeta \cdot \zeta',$$

where the LHS is in

$$H^{2b-1}(J_B \times_B J_B, \mathbb{Z}) \otimes H^{2b-1}(J_B \times_B J_B, \mathbb{Z}) \otimes H^2(J_B \times_B, \mathbb{Z})$$

and the mapping is cup product. This completes the sketch of the proof of Theorem 4.5.1. □

4.5.3 Generalized Complex Tori and Their Compactifications

For the purposes of this work one needs the construction and properties of the Poincaré line bundle in families in which there are singular fibres. In fact, heuristic reasoning suggests that this line bundle may have some sort of "topological discontinuity" along the locus $H^1(B^\bullet_{s_0}) \neq 0$. What we are able to do here is only to take some first steps in this program. Specifically, for smooth curves $B \subset S$ such that the fibres of $\mathcal{X}_B \to B$ have at most one ordinary node as singularities we shall

i) construct an analytic fibre space

$$J_B \to B$$

of complex Lie groups whose fibre over $s \in B$ is $J(X_s)$ when X_s is smooth and is the generalized intermediate Jacobian $J_e(X_s)$ when X_s has a node; and where

$$\mathcal{O}_B(J_B) = \mathcal{J}_e$$

as defined in sections 4.2.1, 4.5.2 above,

ii) construct a compactification

$$\begin{array}{ccc} \bar{J}_B & \supset & J_B \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

where $\bar{J}_B$ is a smooth compact complex manifold and for X_{s_0} nodal

$$(\bar{J}_{B,s_0})_{\text{sing}} \stackrel{\text{defn}}{=} J_{\infty,s_0}$$

has dimension $b-1$ and is smooth,

(iii) contruct a desingularlization

$$\begin{array}{ccc} \widetilde{\bar J_B \times_B \bar J_B} & \to & \bar J_B \times_B \bar J_B \\ \downarrow & & \downarrow \\ B & = & B\,; \end{array}$$

where

$$(\bar J_B \times_B \bar J_B)_{\text{sing}} = \bar J_\infty \times_B \bar J_\infty\ ,$$

iii) although we shall not construct the line bundle stack $\mathcal{M}_B \to \bar J_B$ and Poincaré line bundle $\mathcal{P}_B \to \widetilde{\bar J_B \times_B \bar J_B}$, we will show that their Chern classes

$$\begin{aligned} c_1(\mathcal{M}_B) &\in H^2(\bar J_B, \mathbb{Q}) \\ c_1(\mathcal{P}_B) &\in H^2(\widetilde{\bar J_B \times_B \bar J_B}, \mathbb{Q}) \end{aligned}$$

can be defined, and

iv) finally, we shall show that the arguments in the preceding section can be extended to give the main result Theorem 4.5.1 in this context.

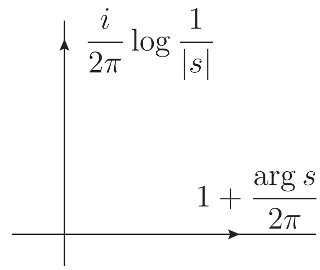
Remark. There is a substantial literature on compactification of quasi-abelian varieties and of generalized Jacobians of singular curves, both singly and in families. Although we shall not get into it here, for our study the paper [4] by Lucia Caporaso and its sequel [5] together with [1] are especially relevant. In those papers there is an extensive bibliography to other work on the compactifications referred to above. In addition the papers [7] and [6] have been useful in that they directly relate Hodge theory to compactifications.

We now realize our program outlined above.

i) We begin by recalling the construction for a family of elliptic curves. The question is local over a disc $\Delta = \{s : |s| < 1\}$, where X_s is smooth for $s \neq 0$ and X_s has a node p. It is well-known that the normalized period matrix of X_s is

$$\left(1, \frac{\log s}{2\pi i} + a(s)\right)\ ,$$

where $a(s)$ is a holomorphic function at $s = 0$. It represents an inessential peturbation term and for simplicity of exposition will be assumed to be zero. The period lattice of X_s, $s \neq 0$, thus has generators



We let $\mathbb{Z}^2$ act on $\mathbb{C} \times \Delta$ by

$$\begin{aligned} e_1 \cdot (z, s) &= (z + 1 + \frac{\arg s}{2\pi}, s) \\ e_1 \cdot (z, s) &= \left(\left\{ \begin{array}{cc} z + \frac{i}{2\pi} \log \frac{1}{|s|}, & s \neq 0 \\ z, & s = 0 \end{array} \right\}, s \right) . \end{aligned}$$

The quotient by this action is $J_\Delta \to \Delta$.

To see that it is an analytic fibre space of complex Lie groups, we first restrict to the axis Im $s = 0$ and factor out the action of e_1 by setting

$$w = e^{2\pi i z} \in \mathbb{C}^* .$$

Then e_2 acts on $\mathbb{C}^* \times \Delta$ by

$$e_2 \cdot (w, s) = \left\{ \begin{array}{cc} |s| \cdot w & s \in 0 \\ w & s = 0 \end{array} \right\} .$$

By similar but more complicated expressions one may extend this to all s, and when this is done the resulting action is visibly properly discontinuous and exhibits $J_\Delta \to \Delta$ as an analytic fibre space of complex Lie groups.

For a curve of genus g the normalized period matrix is

$$(I_g, Z(s))$$

where $Z(s) \in \mathcal{H}_g$, the Siegel generalized upper-half-plane, is given by

$$Z(s) = \begin{pmatrix} \frac{\log s}{2\pi i} + a(s) & {}^{\mathrm{T}}b(s) \\ b(s) & \tilde{Z}(s) \end{pmatrix} \tag{4.54}$$

where $a(s)$ is holomorphic at $s = 0$, and $b(s) \in \mathbb{C}^{g-1}$ and $\tilde{Z}(s) \in \mathcal{H}_{g-1}$ are holomorphic at $s = 0$. The above discussion extends to define an analytic fibre space $J_\Delta \to \Delta$ of complex Lie groups. The fibre $J_{\Delta,s}$ over $s \neq 0$ is the Jacobian $J(X_s)$, and over $s = 0$ we have

$$1 \to \mathbb{C}^* \to J_{\Delta,o} \to J(\tilde{X}_o) \to 0 \tag{4.55}$$

where $\tilde{X}_o \to X_o$ is the normalization. The extension class of (4.55) is represented by $b(0)$. Locally over a point of $J(\tilde{X}_0)$, J_Δ is a product

$$\mathbb{C}^* \times \mathcal{U}$$

where $\mathcal{U} \subset J(\tilde{X}_0)$ is an open set. This local splitting is as complex manifolds, not as complex Lie groups, and locally refers to the strong property of holding outside a compact set in the $\mathbb{C}^*$ factor.

In general, for $X_s \subset X^{2n}$ as above and for L sufficiently ample so that $h^{n,n-1}(X_s) \neq 0$ for $s \neq 0$, it is known (cf. [21]) that the period matrix will have the form (4.33) where now $\tilde{Z}(s)$ represents the period matrix of a family of polarized complex tori with $h^{n,n-1} = h^{n,n-1}(X_s) - 1$, $s \neq 0$. Thus the same conclusion — that $J_\Delta \to \Delta$ may be constructed as an analytic fibre space of complex Lie groups — holds. Moreover, we have (4.55) where now $J(\tilde{X}_0)$ is the intermediate Jacobian of the standard desingularization $\tilde{X}_0 \to X_0$ obtained by blowing up the node $p \in X_0$. We shall refer to $J_{\Delta,0}$ as the *generalized intermediate Jacobian of* X_0.

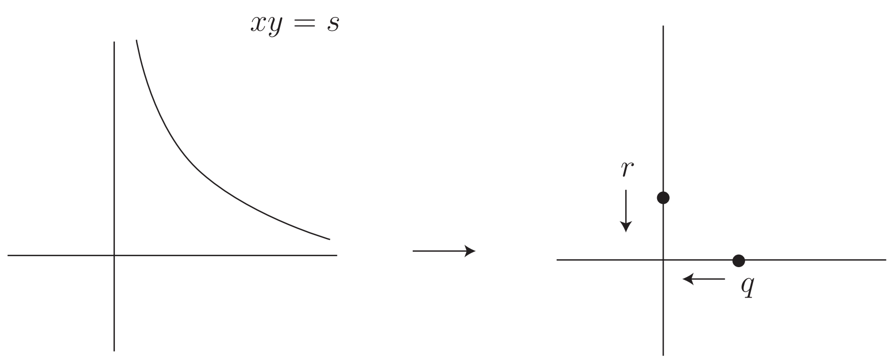
We may summarize as follows:

The analytic fibre space of complex Lie groups $J_B \to B$ is locally biholomorphic to the product of a smooth fibre space and an elliptic curve acquiring a node across a disc. (4.56)

Here, as noted above, locally has the strong meaning of “outside a compact set in the $\mathbb{C}^*$ factor”.

ii) Because of (4.56) does not work !! it will suffice to analyze the elliptic curve picture in a way that will extend to the local product situation as described above. Here we may be guided by the geometry. Namely, locally in the analytic topology around a nodal elliptic curve X_{s_0} there are local coordinates x, y on $\mathcal{X}_B$ and s on B such that s_0 is the origin and the map $\mathcal{X}_B \to B$ is given by

(4.57)



Then

$$\frac{dx}{x} \equiv -\frac{dy}{y} \quad \bmod ds \ ,$$

and using the above notation, on $\mathbb{C}^*$ with coordinate w we have

$$\begin{cases} \dfrac{dw}{w} \equiv \dfrac{dx}{x} & \text{near} \quad w = 0 \\ \dfrac{dw}{w} \equiv -\dfrac{dy}{y} & \text{near} \quad w = \infty \end{cases}$$

where $\equiv$ denotes congruence modulo holomorphic terms. Then we compactify $\mathbb{C}^*$ by adding one ideal point p with

$$\lim_{q \to p} \int^q \frac{dw}{w} = \lim_{r \to p} - \int^r \frac{dw}{w}$$

in the above figure.

Of course, in this case the compactification of $J_{B,o} \cong \mathbb{C}^*$ is just the original elliptic curve X_{s_0}. But using (4.43) and the above coordinate description enables us to infer the general case from the particular case.

Remark. One obvious but slightly subtle point is that we are *not* saying that a general family $\mathcal{X}_B \to B$ has around a node the local coordinate description (4.57). Rather, for $n \geqq 2$ that is

$$x_1^2 + \cdots + x_{2n}^2 = s \ .$$

What we *are* saying is that in the family $J_B \to B$, the "$\mathbb{C}^*$ direction" has the coordinate description (4.57).

iii) We now turn to the study of the singularities of $\bar{J}_B \times_B \bar{J}_B$. Again, locally in the sense explained above the situation is a product of the elliptic curve picture with some parameters. Around a point on a smooth fibre, respectively a node on a singular fibre, the map $\bar{J}_B \to B$ is

$$\begin{cases} (x,y) & \longrightarrow \quad s = y \qquad \text{(smooth case)} \\ (x,y) & \longrightarrow \quad s = xy \qquad \text{(nodal case)} . \end{cases}$$

From this it follows directly that

$$\left(\bar{J}_B \times_B \bar{J}_B\right)_{\text{sing}} \subseteq \bar{J}_\infty \times_B \bar{J}_\infty \tag{4.58}$$

where $\bar{J}_{B,\infty} \subset \bar{J}_B$ is the set of singular points on fibres. Moreover, in coordinates $(x,y,x',y',s) \in \mathbb{C}^5$ such that $\bar{J}_{B,\infty} \times_B \bar{J}_{B,\infty}$ is locally given by

$$\begin{cases} f = xy - s = 0 \\ f' = x'y' - s = 0 \end{cases} , \tag{4.59}$$

from

$$df \wedge df' = 0 \Leftrightarrow x = y = x' = y' = 0$$

we see that we have equality in (4.58). Moreover, for the Jacobian of (f, f') we have that

$$\operatorname{rank}(J(f,f')) = 1$$

along $\bar{J}_{B,\infty}$. Finally, (4.59) gives

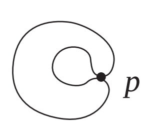
$$xy = x'y'$$

which is a quadric cone in $\mathbb{C}^4$ and has a canonical desingularization.

Remark 4.5.3. For later reference we note that

a) the 0-section of $J_B \subset \bar{J}_B$ is a smooth section not meeting $\bar{J}_\infty$;
b) for the group law $\mu : J_B \times_B J_B \to J_B$ we have that $\mu^{-1}(0) = W$, and in $\bar{J}_B \times_B \bar{J}_B$ we have for the closure $\overline{W} = \bar{J}_\infty \times_B \bar{J}_\infty$. The model here is

$$\overline{\mathbb{C}}^* = (\mathbb{P}^1, \{0,\infty\}) =$$



$$\mu : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \text{ is multiplication}$$

$$\Rightarrow (p,p') = \overline{\mu^{-1}(1)} \cap \left((P^1, \{0,\infty\})\backslash\mathbb{C}^*\right) \times \left((\mathbb{P}^1, \{0,\infty\})\backslash\mathbb{C}^*\right)$$

4.5.4 Topological Properties of the Poincaré Line Bundle in some Families with Singular Fibres

The objective of this section is to show that, using $\bar{J}_B \to B$, the argument sketched above for the proof of Theorem 4.5.1 may be extended to the case in which there are singular fibres as in the preceding section.

First we shall explain why $J_B \to B$ is *not* the right object. We give three reasons.

i) Although J_B is a smooth manifold it is non-compact; in particular, it does not have a fundamental class and Poincaré duality does not hold (both of which were used in the proof of Theorem 4.5.1.

ii) The *local invariant cycle* theorem does not hold for $J_B \to B$, whereas it does hold for $\bar{J}_B \to B$. Thus, for X_{s_0} having a simple node and s close to s_0 with (as usual) T representing monodromy, we have

$$(R^{2b-1}\mathbb{Z}_{\bar{J}_B})_{s_0} \cong \ker\{T - I : H^{2b-1}(J_s, \mathbb{Z}) \to H^{2b-1}(J_s, \mathbb{Z})\} \quad (4.60)$$

but

$$(R^{2b-1}\mathbb{Z}_{J_B})_{s_0} \neq \ker\{T - I : H^{2b-1}(J_s, \mathbb{Z}) \to H^{2b-1}(J_s, \mathbb{Z})\} . \quad (4.61)$$

Note: This is related to the fact that for Δ a disc around s_0 and with $\bar{J}_\Delta = \bar{p}^{-1}(\Delta)$

$$\bar{J}_\Delta \ \textit{retracts onto}\ \bar{J}_{s_0}$$

while this fails to be the case for J_Δ.

iii) Relatedly, the Leray spectral sequence for $\bar{p} : \bar{J}_B \to B$ degenerates at E_2 while this fails to be the case for $J_B \to B$.

Example. Let $\dim J_s = 1$ so that $\bar{J}_B \to B$ is an elliptic surface whose singular fibres $\bar{J}_{s_i}$ are all nodal elliptic curves while

$$J_{s_i} = \bar{J}_{s_i} \backslash \{p_i\} \cong \mathbb{C}^* . \quad (4.62)$$

Then

$$(R^q\mathbb{Z}_{\bar{J}_B})_s \cong (R^q\mathbb{Z}_{J_B})_s$$

for all points $s \in B$ and all q, except that

$$(R^3\mathbb{Z}_{\bar{J}_B})_{s_i} = 0, \quad (R^3\mathbb{Z}_{J_B})_{s_i} \cong \mathbb{Z} . \quad (4.63)$$

This follows from localizing (4.62) over a disc Δ_i around s_i. Then

$$\mathbb{Z} \cong H^2(B, R^2\mathbb{Z}_{\bar{J}_B}) \cong H^2(B, R^2\mathbb{Z}_{J_B}) \quad (4.64)$$

but whereas the Leray spectral sequence for $\bar{J}_B \to B$ degenerates at E_2 and the first isomorphism in (4.64) gives

$$H^4(\bar{J}_B, \mathbb{Z}) \cong \mathbb{Z}\ ,$$

the Leray spectral sequence for $J_B \to B$ has by (4.63)

$$\begin{array}{ccc} H^0(B, R^3\mathbb{Z}_{J_B}) & \xrightarrow{d_2} & H^2(B, R^2\mathbb{Z}_{J_B}) \\ \wr\| & & \\ \bigoplus_i \mathbb{Z} & & \end{array}$$

so that the right term in (4.64) is killed by d_2.

Before begining the argument we remark that we are *not* claiming that the line bundle stack $\mathcal{M}_B \to \bar{J}_B$ and Poincaré line bundle $\mathcal{P}_B \to \widetilde{\bar{J}_B \times_B \bar{J}_B}$ exist, although this may well be true. What we shall use is that what would be images of their Chern classes

$$\begin{cases} c_1(\mathcal{M}_B) \in H^0(B, R^2\mathbb{Z}_{\bar{J}_B}) \\ c_1(\mathcal{P}_B) \in H^0(B, R^2\mathbb{Z}_{\widetilde{\bar{J}_B \times_B \bar{J}_B}}) \end{cases}$$

do exist, and their pullbacks under ν_ζ and $\nu_{\zeta \times \zeta'}$ are all that is really required for the argument. Thus we are able to proceed pretending that $\mathcal{M}_B$ and $\mathcal{P}_B$ exist as in the case treated in section 4.5.2.

We think that the issue of defining $\mathcal{M}$ and $\mathcal{P}$ over the family of *all* $J(X_s)$, $s \in S$ is a very attractive and probably important geometric problem.

Referring to the proof of Theorem 4.5.1 in section 4.5.2, we note that both ν_ζ and Z_ζ avoid the singularities in the fibres of $\mathcal{X}_B \to B$ and $\bar{J}_B \to B$, respectively. Moreover, the argument that

$$[Z_\zeta] \to H^1(B, R^{2b-1}\mathbb{Z}_{\bar{J}_B})$$

is defined and is linear in ζ carries over verbatim.

The next step, which uses Poincaré duality on $\bar{J}_B$ and $\widetilde{\bar{J}_B \times_B \bar{J}_B}$, also carries over to give

$$p^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}] = [Z_{\zeta \times \zeta'}]$$
$$p_1^*[Z_\zeta] \cup p_2^*[Z_{\zeta'}] \cup c_1(\mu^*\mathcal{M}_B) = c_1\ (P)$$

as before. Additionally, (4.50) and the discussion just under remain as stated there, with $\bar{J}_B$ replacing J_B.

Next comes the main somewhat subtle point; namely, that

$$Q^{b-1} : R^1\mathbb{Z}_{\bar{J}_B} \cong R^{2b-1}\mathbb{Z}_{\bar{J}_B} \tag{4.65}$$

continues to hold. Essentially this is because of (ii) above. Namely, we have

$$Q^{b-1} : H^1(\bar{J}_{B,s}, \mathbb{Z}) \cong H^{2b-1}(\bar{J}_{B,s}, \mathbb{Z}) \tag{4.66}$$

for s near s_i and where we have set $\bar{J}_{B,s} = \bar{p}^{-1}(s)$. Moreover since

$$TQ = Q \tag{4.67}$$

and

$$(R^1\mathbb{Z}_{\bar{J}_B})_{s_i} \cong \ker\{T - I : H^q(\bar{J}_{B,s}, \mathbb{Z}) \to H^1(\bar{J}_B, \mathbb{Z})\} \tag{4.68}$$

we may infer (4.65) from (4.66)–(4.68).

The final step is essentially the same as before, where over s_i we replace Λ by

$$\begin{aligned} (R^{2b-1}\mathbb{Z}_{\bar{J}_B})_{s_i} &\cong \text{RHS of (4.68)} \\ &\cong (R^1\mathbb{Z}_{\bar{J}_B})_{s_i} \end{aligned}$$

of (4.65), and then the pairing

$$(R^1\mathbb{Z}_{\bar{J}_B})_{s_i} \otimes (R^{2b-1}\mathbb{Z}_{\bar{J}_B})_{s_i} \to \mathbb{Z}$$

follows from the fact that the *compact* analytic variety $\bar{J}_{B,s_i}$ has a fundamental class.

Note: The condition to be able to fill in a family of intermediate Jacobian

$$\{\mathcal{J}_s\}_{s\in\Delta^*}$$

with a *compactification* $\overline{\mathcal{J}}_0$ of the generalized intermediate Jacobian over the origin is probably very special to the case $n = 1$. Namely, first recall that for $s \neq 0$

$$H^1(\mathcal{J}_s, \mathbb{Z}) \cong H_{2n-1}(X_s, \mathbb{Z}) \,. \tag{4.69}$$

Suppose that we can compactify the family

$$\mathcal{J}_\Delta \xrightarrow{\ \pi\ } \Delta$$

where $\pi^{-1}(s) = \mathcal{J}_s$ to have

$$\overline{\mathcal{J}}_\Delta \to \Delta \,.$$

It is reasonable to expect that the total space $\overline{\mathcal{J}}_\Delta$ will be a Kähler manifold, and Clemens [11] has shown that in this situation the local monodromy theorem holds, so that after after passing to a finite covering the monodromy

$$T : H^1(\mathcal{J}_\eta, \mathbb{Z}) \to H^1(\mathcal{J}_\eta, \mathbb{Z}) \qquad (\eta \neq 0)$$

will satisfy

$$(T - I)^2 = 0\ . \tag{4.70}$$

But by (4.70) all we can expect in general is

$$(T - I)^{2n} = 0\ .$$

In other words, (4.70) (which is satisfied in the model case) is perhaps a necessary condition to be able to compactify $\mathcal{J}_0$ in a family. More plausible is that $\mathcal{J}_\Delta \to \Delta$ will have a *partial compactification* of the sort appearing in the work of Kato-Usui [27].

4.6 Conclusions

The theory discussed above is, we feel, only part of what could be a rather beautiful story of the geometry associated to a Hodge class $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}$ through its normal function $\nu_\zeta \in H^0(S, \mathcal{J}_E)$ where S is either $\mathbb{P}H^0(X, L^k)$, or is a suitable blowup of that space If one wants to use the theory to construct algebraic cycles, i.e. to show that

$$\operatorname{sing} \nu_\zeta \neq \varnothing\ ,$$

then the following four assumptions must enter:

i) ζ is an integral class in $H^{2n}(X, \mathbb{Z})$
ii) ζ is of Hodge type (n, n)
iii) a) $k \geqq k_0(\zeta)$
  b) where the ζ-dependence of k_0 is at least $|\zeta^2|$;
and
iii) all of $H^0(X, L^k)$ is used.

In our work above, there are two main approaches to studying the geometry associated to ζ

A) *the "capturing" of ζ along the singular locus* $\Delta \subset \mathcal{X}$ (cf. section 4.4.2); and
B) *the (as yet only partially defined) line bundles* $\nu_\zeta^*(M)$ *and* $\nu_{\zeta\times\zeta'}(P)$.

In A) we have used the assumptions (iii), (iiia), (iv) in order to have the necessary vanishing theorems so as to have the isomorphism

$$H^n(\Omega^n_X)_{\text{prim}} \cong H^1(\mathcal{I}_\Delta \otimes K_X \otimes L^n \otimes H^n) \tag{4.71}$$

with the resulting conclusion

$$\text{HC} \Leftrightarrow \text{Hg}^n(X)_{\text{prim}} \hookrightarrow H^0(R^1_p \mathcal{I}_\Delta \otimes K \otimes L^n \otimes H^n) \tag{4.72}$$

where $p : \Delta \to D$ is the projection. We refer to section 4.4.2 for a discussion of how the assumption (i) *should* enter, and in fact *will* enter if the HC is true.

We remark that, based on the heuristics discussed in section 4.4.1, one may reasonably expect that the stronger assumption (iiib) must be used. In this regard, the condition (iiia) needed to have (4.33) is locally uniform in the moduli space of X, whereas the stronger assumption (iiib) cannot have this local uniformity.

In B) we have used from the very outset the assumptions (i) and (ii), and moreover the quantity ζ^2 appears naturally in $c_1(\nu^*_{\zeta\times\zeta}(P))$. However, the assumptions (iiib), (iv) have as yet to appear, even heuristically, in the geometry of $\nu^*_\zeta(M)$ and $\nu^*_{\zeta\times\zeta'}(P)$.

In closing we would like to suggest three examples whose understanding would, we feel, shed light on the question of existence of singularities of ν_ζ. These are all examples in the case $n = 1$ of curves on an algebraic surface, where of course the HC is known. However, one should ignore this and seek to analyze $\operatorname{sing}\nu_\zeta$ in the context of this paper.

Example 4.6.1. (i) $X = \mathbb{P}^1 \times \mathbb{P}^2$, $L = \mathcal{O}_X(2,2)$ and ζ is the class of $L_1 - L_2$ where the L_i are lines from different rulings of X realized as a quadric in $\mathbb{P}$.

(ii) X is a general smooth quartic surface in $\mathbb{P}^3$ containing a line Λ, $L = \mathcal{O}_X(1)$ and $\zeta = [H - 4\Lambda]$ where H is a hyperplane.

(iii) X is a general smooth surface of degree $d \geqq 4$ in $\mathbb{P}^3$ containing a twisted curve C, $L = \mathcal{O}_X(1)$ and $\zeta = [H - dC]$.

In example 4.6.1.1 the general fibre X_s is an elliptic curve where degenerations are well understood, although in this case the base space is 8 dimensional and the non-torsion phenomena in our extened Néron-type model $\tilde{\mathcal{J}}_E$ is what is of interest.

In example 4.6.1.2 we have the situation where the nodes do not impose independent conditions on $|L|$, which must then be blown up so that the discriminant locus D has local normal crossings. This example has the advantage that $\dim S = 3$ so that the analysis of, e.g., the "singularities" of the Poincaré line bundle should be easier to do directly.

Example 4.6.1.3 exhibits the phenomenon that ν_ζ has no singularities on $\mathbb{P}H^0(\mathcal{O}_X(L))$; one must pass to L^2 to have $\operatorname{sing}\nu_\zeta \neq 0$. This will of course be the general case.

Of course these examples could be extended, e.g. to smooth hypersurfaces in $\mathbb{P}^5$ where in example 4.6.1.3 the condition is to contain a Veronese surface. As explained in section 4.4.3, we see no a priori reason why the geometric picture as regards $\operatorname{sing}\nu_\zeta$ should be significantly different from the $n = 1$ case, although analyzing the geometry will of course be technically much more involved.

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