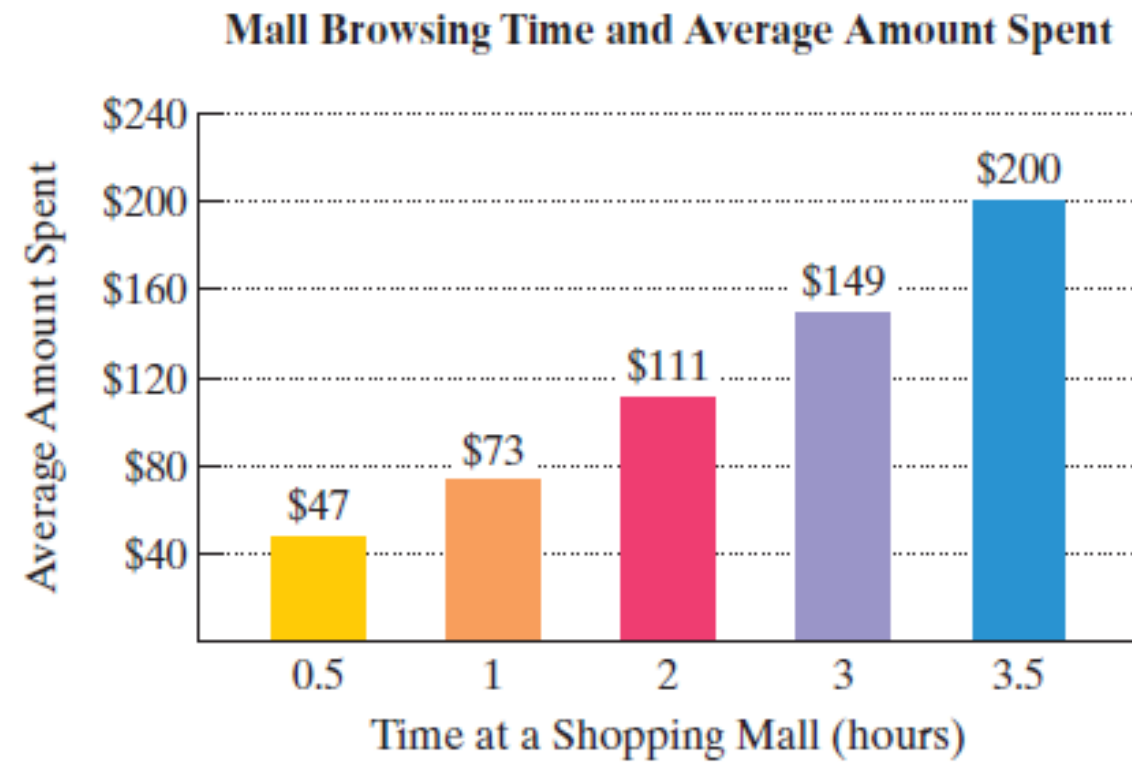


Unit 5.1 – Exponential Functions & Their Graphs

So far, this text has dealt mainly with **algebraic functions**, which include polynomial functions and rational functions. In this chapter, you will study two types of nonalgebraic functions –*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions**.

Researchers know, to the dollar, the average amount the typical consumer spends per minute at the shopping mall. And the longer you stay, the more you spend.

The data can be modeled by the function $f(x) = 42.2(1.56)^x$ where $f(x)$ is the average amount spent, in dollars, at a shopping mall after x hours. Can you see how this function is different from polynomial functions? Functions whose equations contain a variable in the exponent are called exponential functions. Many real-life situations, including population growth, growth of epidemics, radioactive decay, and other changes that involve rapid increase or decrease, can be described using exponential functions.



What's a **Geometric Sequence**? It's a sequence made by **multiplying** by some value each time. For example: 3, 6, 12, 24, 48, ... Notice that each number is 2 times the number before it. That value is called the **Common Ratio** or **Constant Ratio**.

An **Exponential Function** is similar to a Geometric Sequence.

It has the form $f(x) = ab^x$ where a is the initial value, also known as the y-intercept and b is the base, also called the Common Ratio.

Obviously, when $a = 1$, then $f(x) = b^x$

Here are some examples of exponential functions.

Each of them has a constant base and a variable exponent.

$f(x) = 2^x$	$g(x) = 10^x$	$h(x) = 3^{x+1}$	$j(x) = \left(\frac{1}{2}\right)^{x-1}$
Base is 2.	Base is 10.	Base is 3.	Base is $\frac{1}{2}$.

By contrast, the following functions are not exponential functions:

$F(x) = x^2$	$G(x) = 1^x$	$H(x) = (-1)^x$	$J(x) = x^x$
Variable is the base and not the exponent.	The base of an exponential function must be a positive constant other than 1.	The base of an exponential function must be positive.	Variable is both the base and the exponent.

EXAMPLE 1**Evaluating an Exponential Function**

$$f(x) = ab^x$$

The exponential function $f(x) = 42.2(1.56)^x$ models the average amount spent, $f(x)$, in dollars, at a shopping mall after x hours. What is the average amount spent, to the nearest dollar, after four hours?

Solution Because we are interested in the amount spent after four hours, substitute 4 for x and evaluate the function.

$$f(x) = 42.2(1.56)^x \quad \text{This is the given function.}$$

$$f(4) = 42.2(1.56)^4 \quad \text{Substitute 4 for } x.$$

Use a scientific or graphing calculator to evaluate $f(4)$. Press the following keys on your calculator to do this:

Scientific calculator: $42.2 \boxed{\times} 1.56 \boxed{y^x} 4 \boxed{=}$

OR

$$42.2 \boxed{\times} 1.56 \boxed{\wedge} 4 \boxed{=}$$

OR

$$42.2 \boxed{\times} 1.56 \boxed{\wedge} 4 \boxed{\text{Enter}}$$

The display should be approximately 249.92566.

$$f(4) = 42.2(1.56)^4 \approx 249.92566 \approx 250$$

Thus, the average amount spent after four hours at a mall is \$250.

Constructing Exponential Functions from Verbal Descriptions

You can write an **Exponential Function** of the form $f(x) = ab^x$ if you know the values of a (the initial value) and b (the Common Ratio).

When a piece of paper is folded in half, the total thickness doubles.

Suppose an unfolded piece of paper is 0.1 millimeter (mm) thick.

The total thickness $f(x)$ of the paper is an exponential function of the number of folds x .

The value of a is the original thickness of the paper before any folds are made, or 0.1 mm.

Because the thickness doubles with each fold, the value of b (the Common Ratio) is 2.

Since $f(x) = ab^x$: The equation for the function is $f(x) = 0.1(2)^x$

How thick will the paper be after 7 folds? $f(x) = 0.1(2)^7 = 0.1(128) = 12.8 \text{ mm}$

The function (name) and the exponent can be any letter. In this example, it could have been $t(n)$ where t represented the thickness, and n represented the number of folds.

a) A biologist studying ants started with a population of 500. On each successive day the population tripled. The number of ants $a(d)$ is an exponential function of the number of days d that have passed. What is the ant population on day 5?

$a =$ $b =$ $a(d) =$ $a(5) =$

b) The NCAA basketball tournament begins with 64 teams, and after each round, half the teams are eliminated. How many teams are left after 4 rounds?

$a =$ $b =$ $t(n) =$ $t(\underline{\hspace{1cm}}) =$

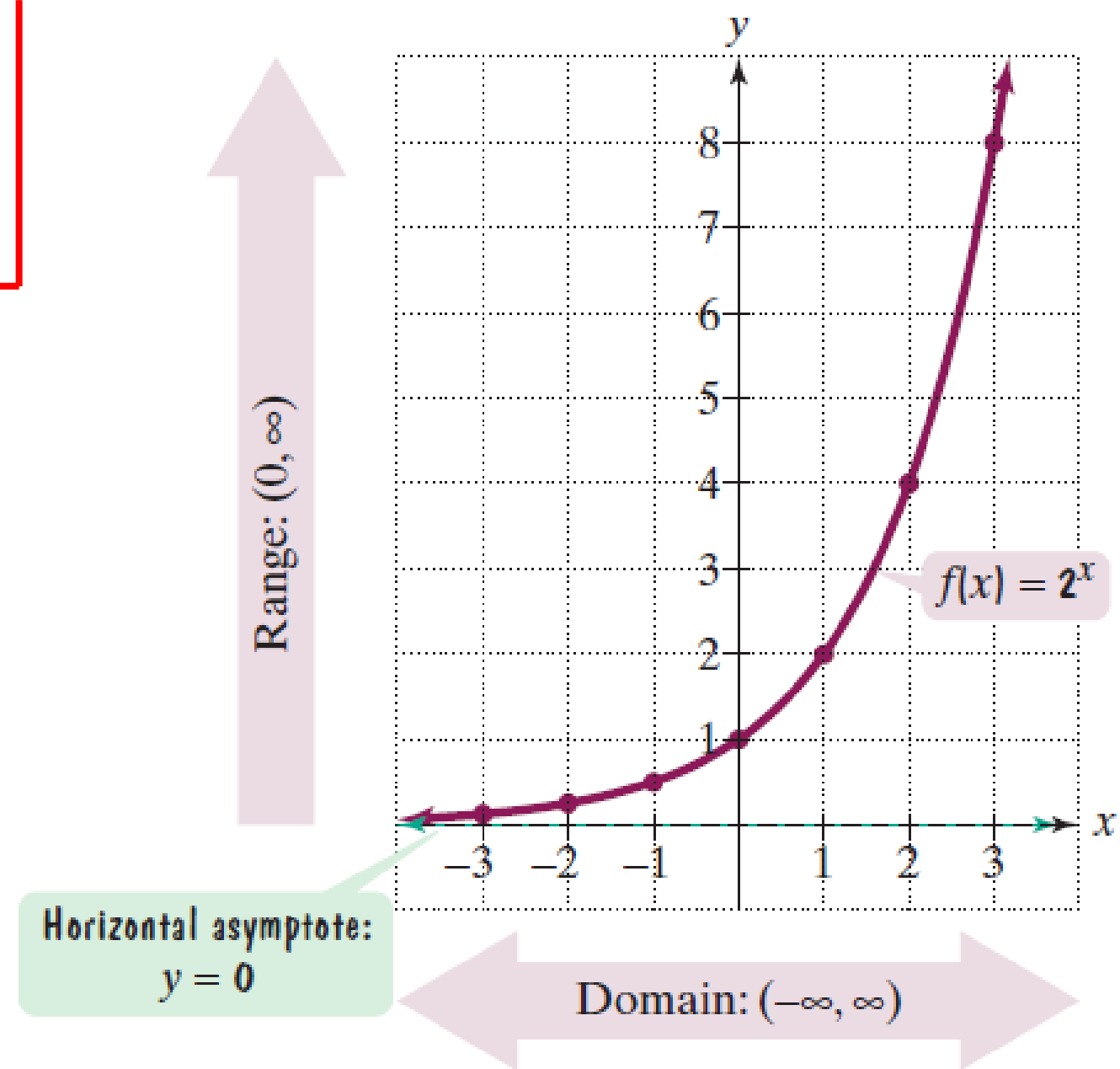
When evaluating exponential functions, you will need to use the properties of exponents, including zero and negative exponents.

For example: $8^0 = 1$ $8^{-3} = \frac{1}{8^3}$

Graphing an Exponential Function

Graph: $f(x) = 2^x$.

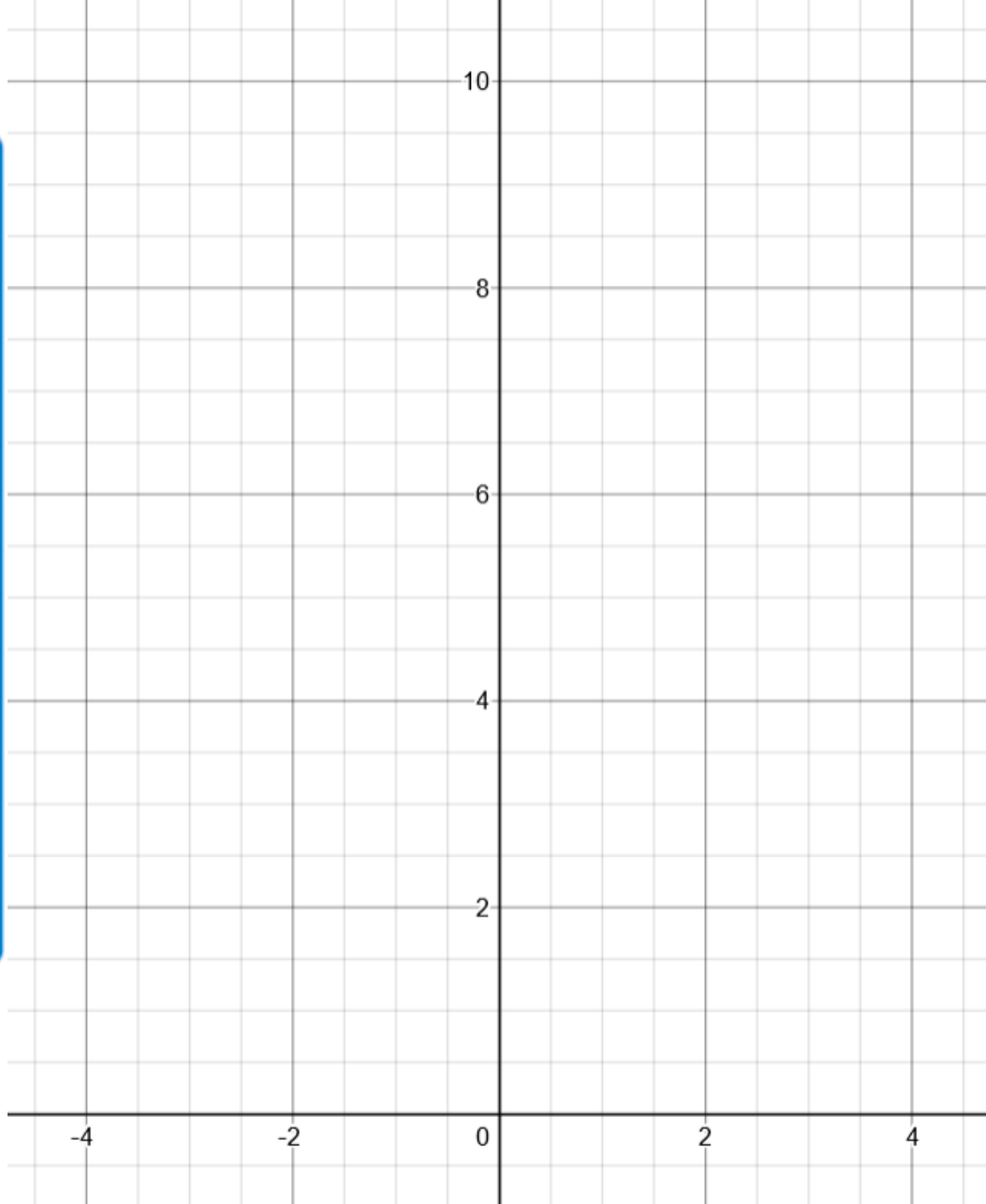
x	$f(x) = 2^x$
-3	$f(-3) = 2^{-3} = \frac{1}{8}$
-2	$f(-2) = 2^{-2} = \frac{1}{4}$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$
0	$f(0) = 2^0 = 1$
1	$f(1) = 2^1 = 2$
2	$f(2) = 2^2 = 4$
3	$f(3) = 2^3 = 8$



The Horizontal Asymptote always starts at $y = 0$, and only changes with a Vertical Translation (up or down).

Graph: $f(x) = 3^x$

x	$f(x) = 3^x$
-3	$f(-3) =$
-2	$f(-2) =$
-1	$f(-1) =$
0	$f(0) =$
1	$f(1) =$
2	$f(2) =$
3	$f(3) =$



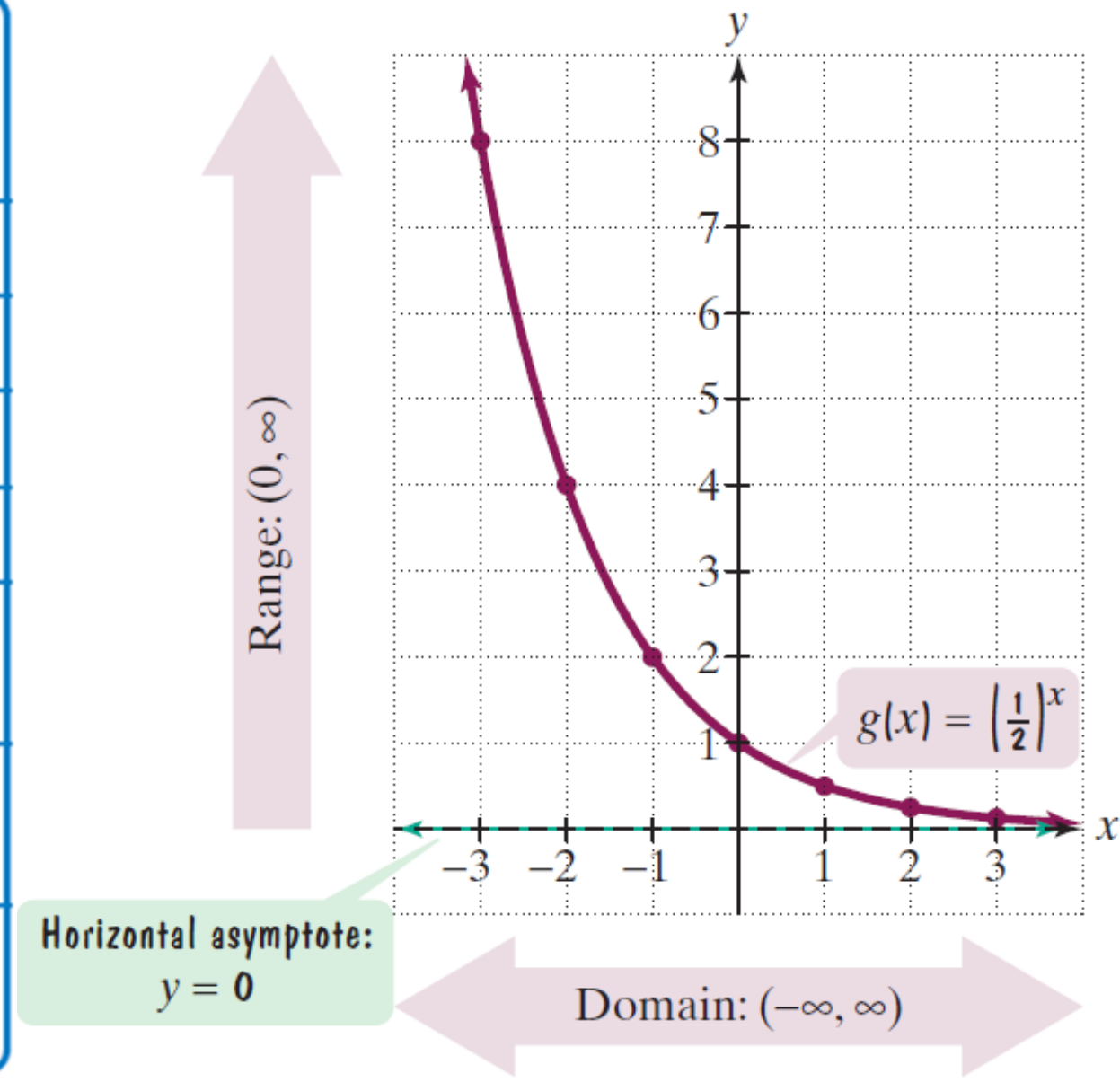
What is the Domain?

What is the Range?

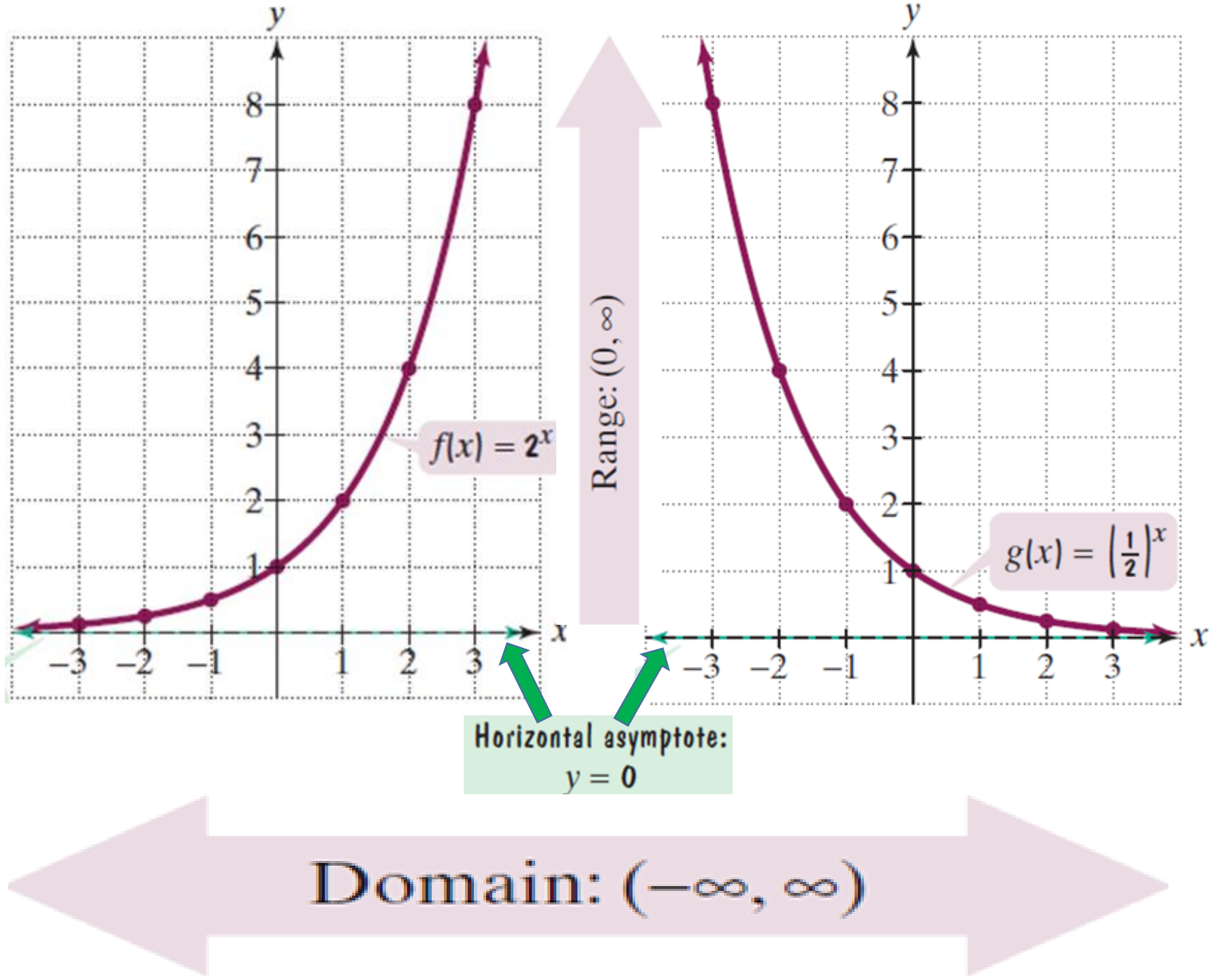
What is the Horizontal Asymptote?

Graph: $g(x) = \left(\frac{1}{2}\right)^x$

x	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$g(-3) =$
-2	$g(-2) =$
-1	$g(-1) =$
0	$g(0) =$
1	$g(1) =$
2	$g(2) =$
3	$g(3) =$



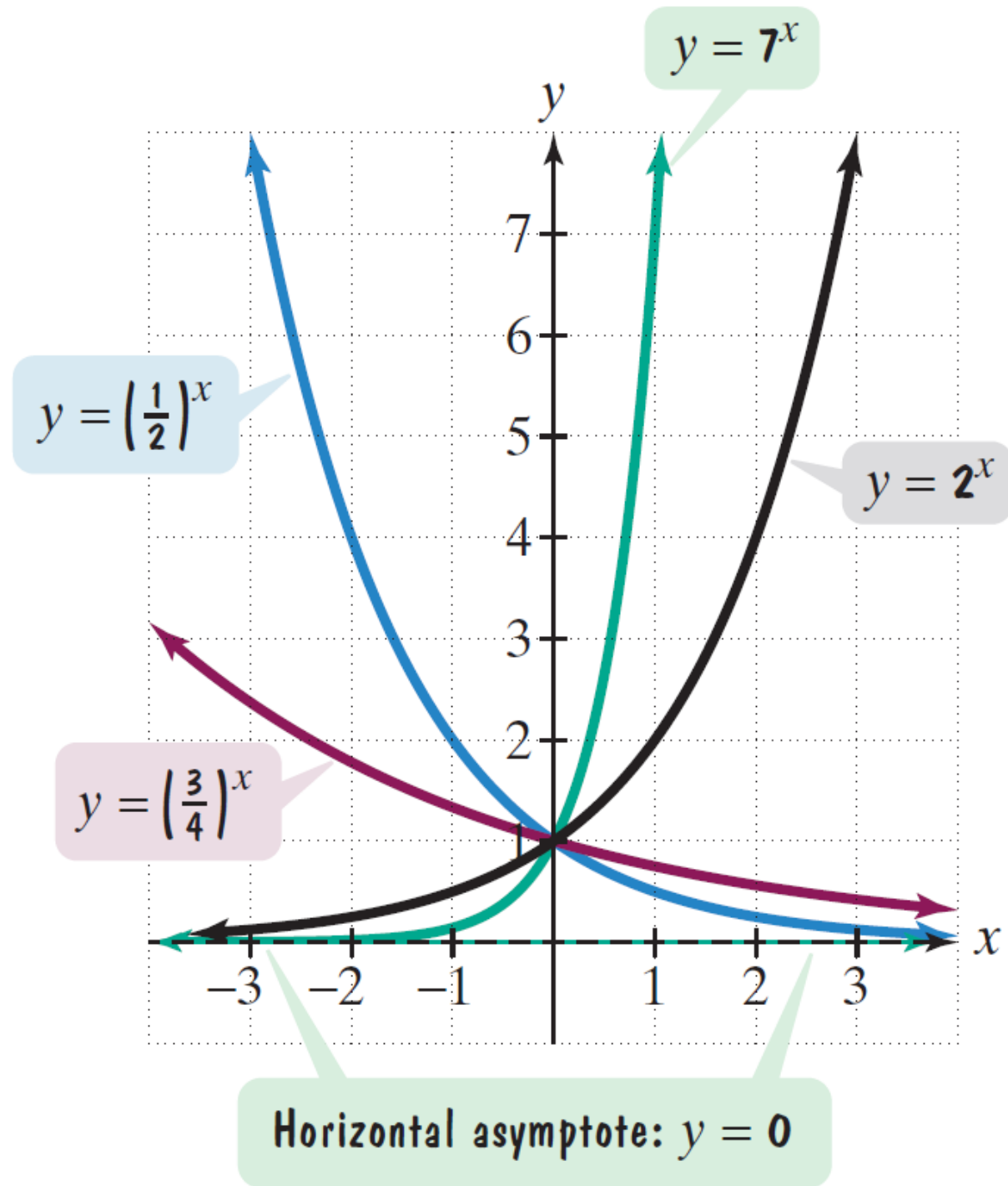
x	$f(x) = 2^x$	$g(x) = (\frac{1}{2})^x$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$



What's common between them?

What do you notice?

- 1) The domain consists of all real numbers.
- 2) The range consists of all real numbers > 0 .
- 3) All graphs pass through the point $(0,1)$ because all of them:
 - * are of the form $f(x) = ab^x$
 - * have an a value of 1And a is the y-intercept.
- 4) If $b > 1$, the graph goes up to the right and down to the left.
The larger the value of b , the steeper it is.
- 5) If b is between 0 and 1, the graph goes up to the left and down to the right.
The smaller the value of b , the steeper it is.
- 6) The graphs approach but never touch the x-axis. The x-axis, or $y = 0$, is a horizontal asymptote to all functions.



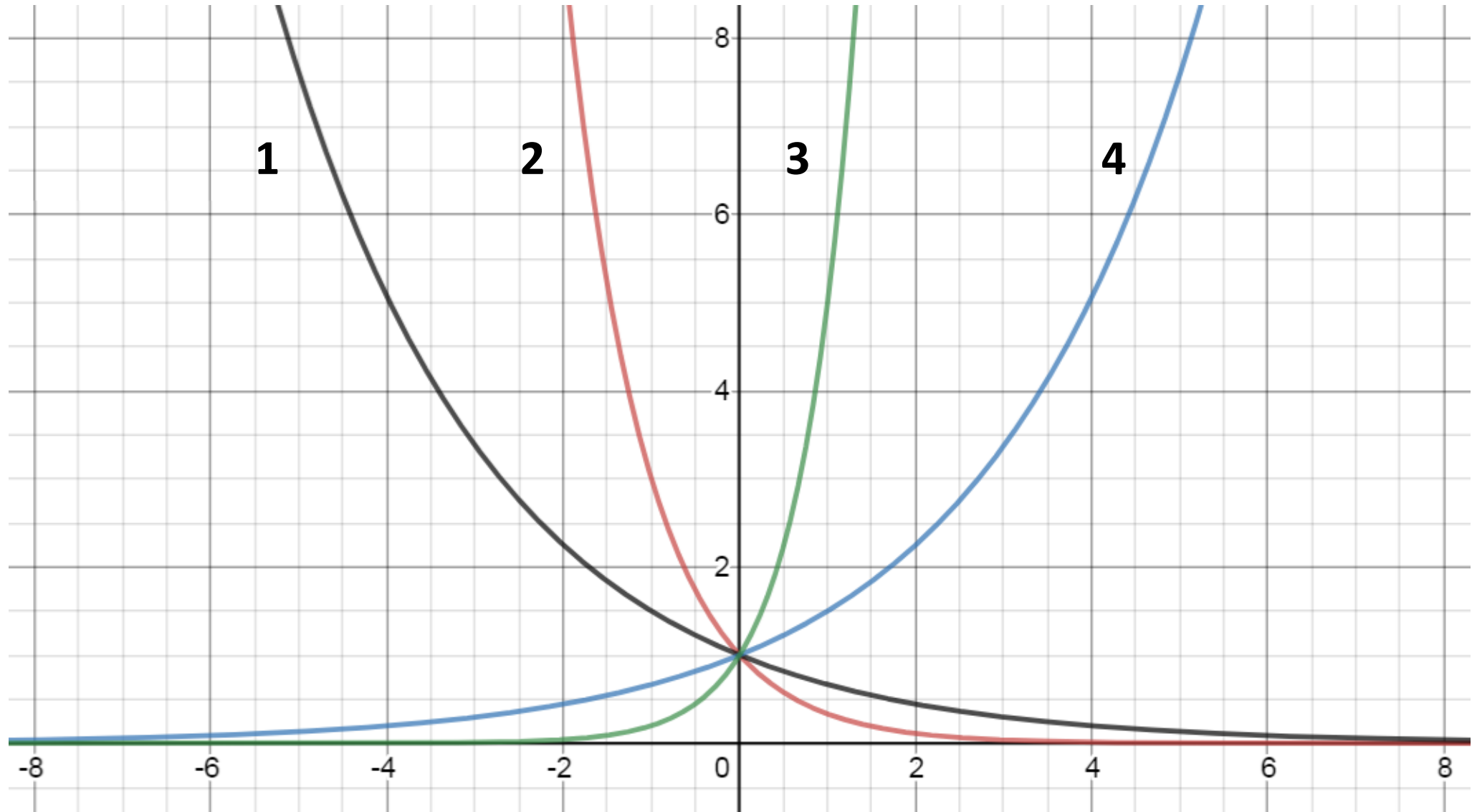
Match the
function to the
graph number.

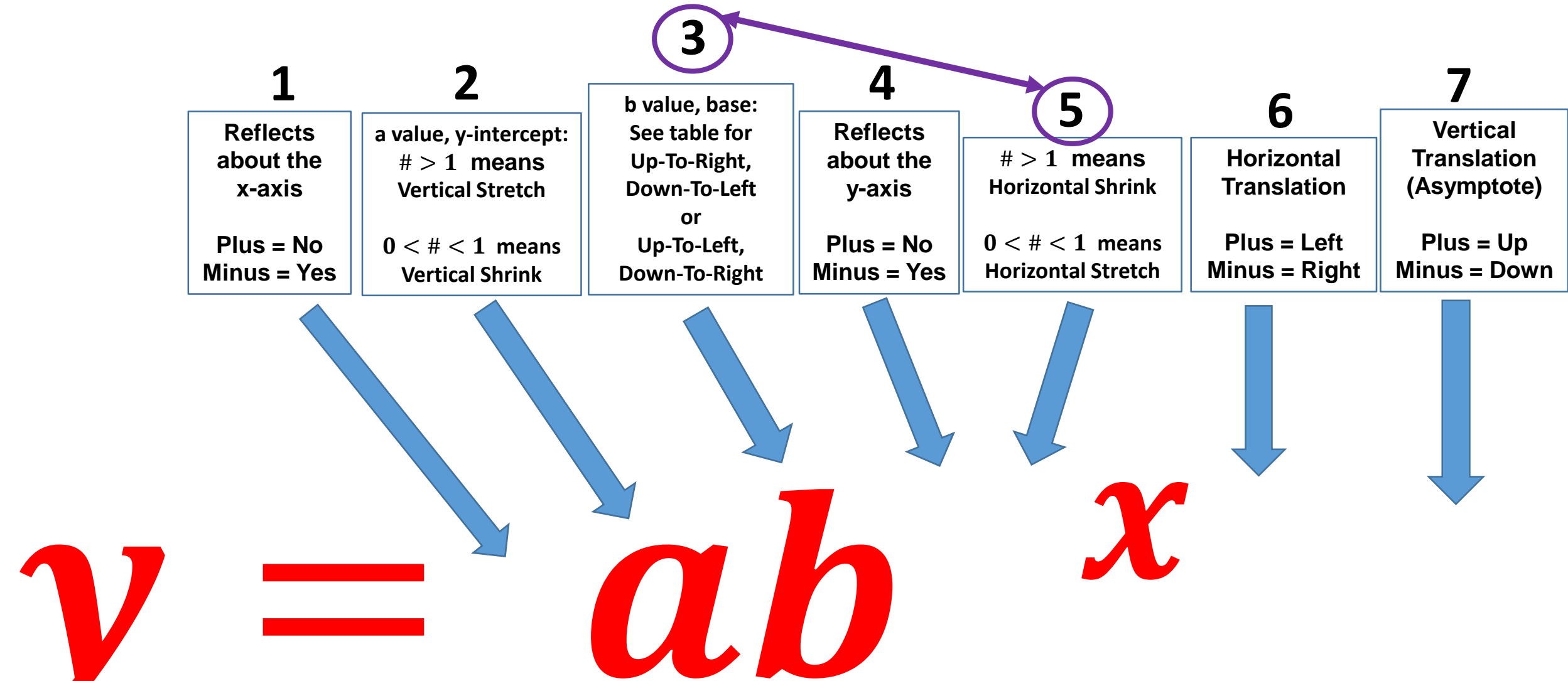
— $y = 5^x$

— $y = 1.5^x$

— $y = \left(\frac{1}{3}\right)^x$

— $y = \left(\frac{2}{3}\right)^x$





There are 4 transformations:

- * **Translation** (Horizontal & Vertical)
- * **Stretch** (Horizontal & Vertical)
- * **Shrink** (Horizontal & Vertical) – aka Compression
- * **Reflection** (Across the x & y axes)

3 and **5** work together. Consider:

$$2^{2x} = (2^2)^x = 4^x$$

Incidentally – How many possibilities are there?

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

TRANSFORMATIONS

(1a) Vertical Translation

1	2	3	4	5	6	7
Reflects about the x-axis	a value, y-intercept: # > 1 means Vertical Stretch 0 < # < 1 means Vertical Shrink	b value, base: See table for Up-To-Right, Down-To-Left or Up-To-Left, Down-To-Right	Reflects about the y-axis	# > 1 means Horizontal Shrink 0 < # < 1 means Horizontal Stretch	Horizontal Translation	Vertical Translation (Asymptote)
Plus = No Minus = Yes			Plus = No Minus = Yes		Plus = Left Minus = Right	Plus = Up Minus = Down

$$y = ab^x$$

Horizontal Asymptote $y = 3$

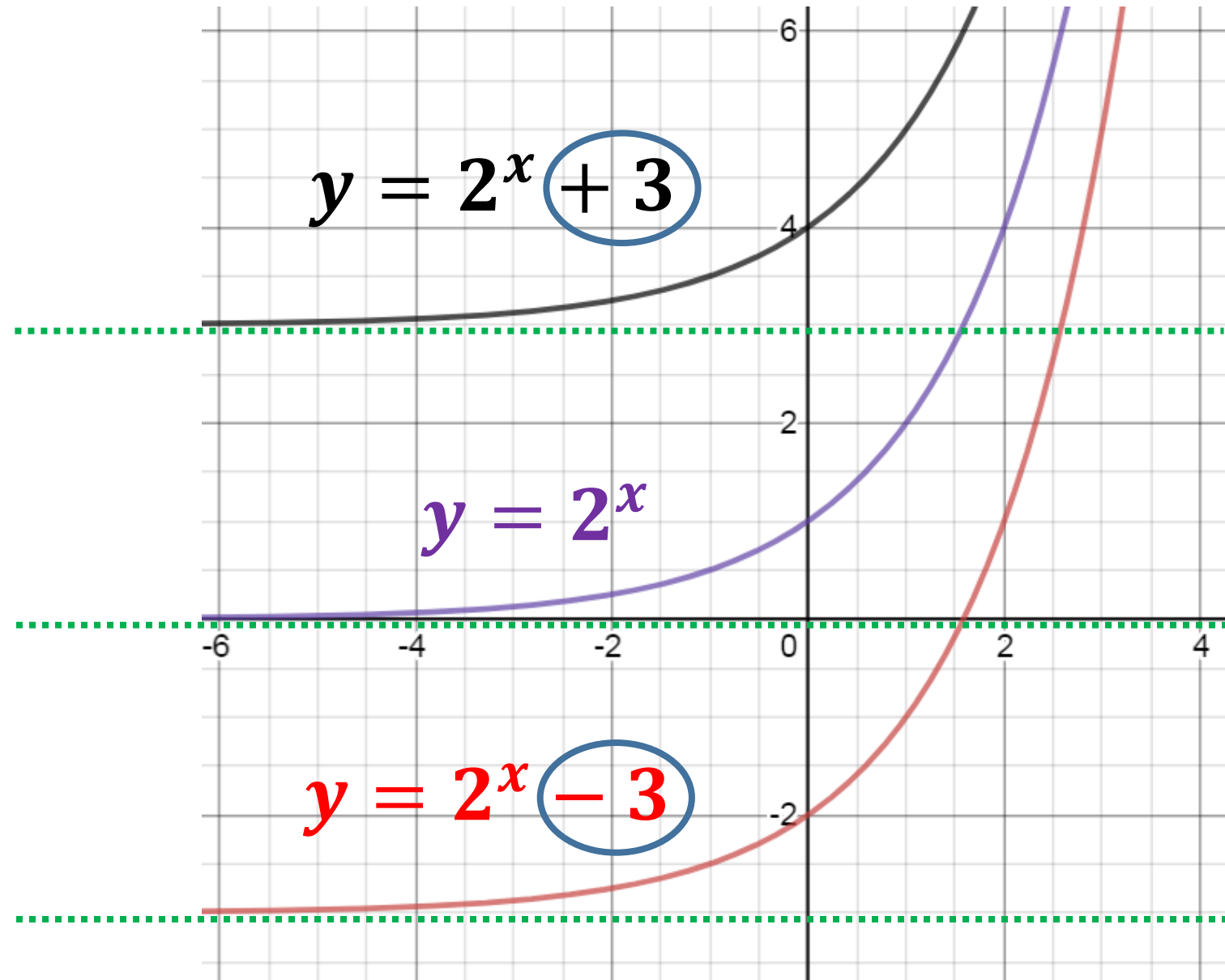
Range: $y > 3$

Horizontal Asymptote $y = 0$

Range: $y > 0$

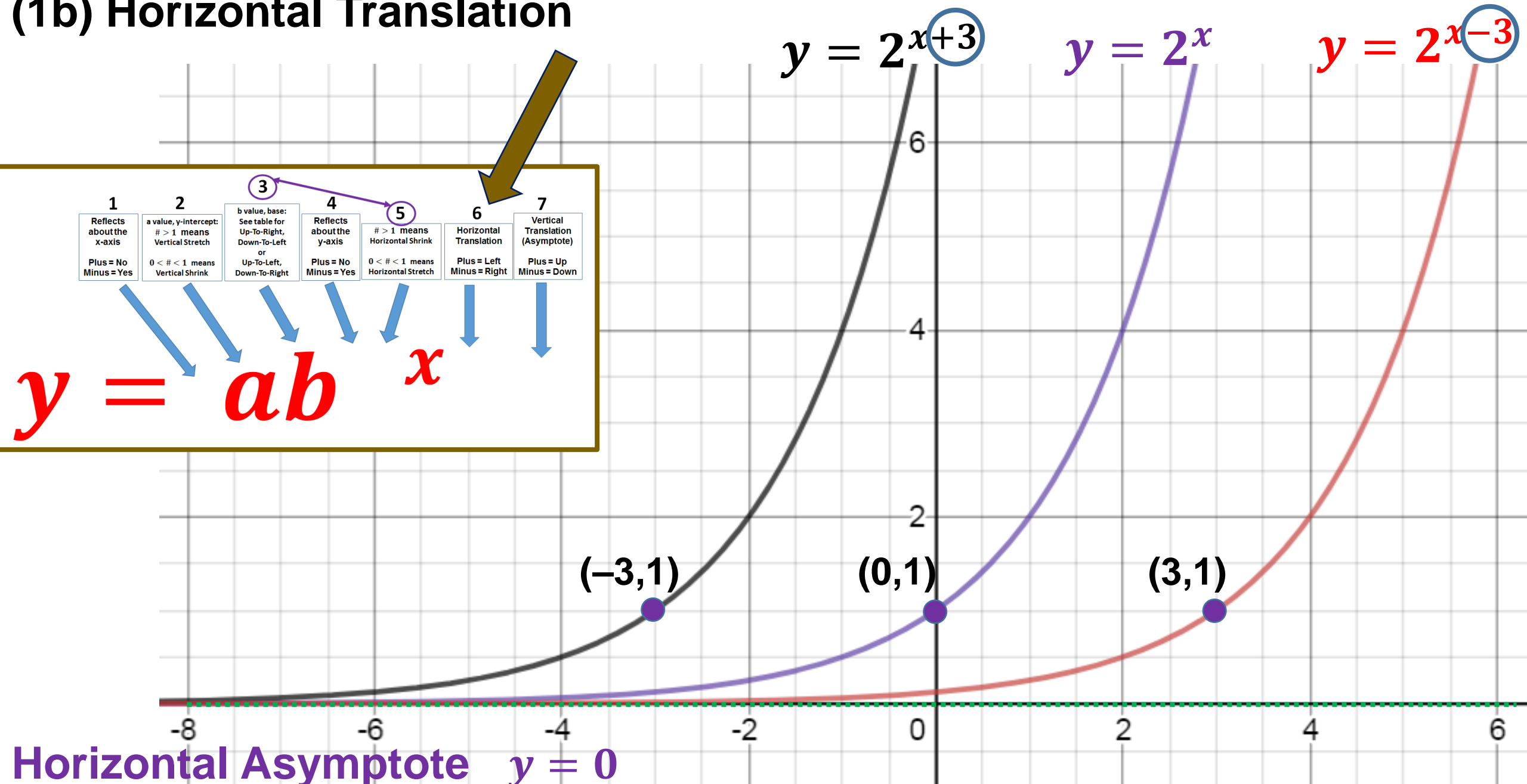
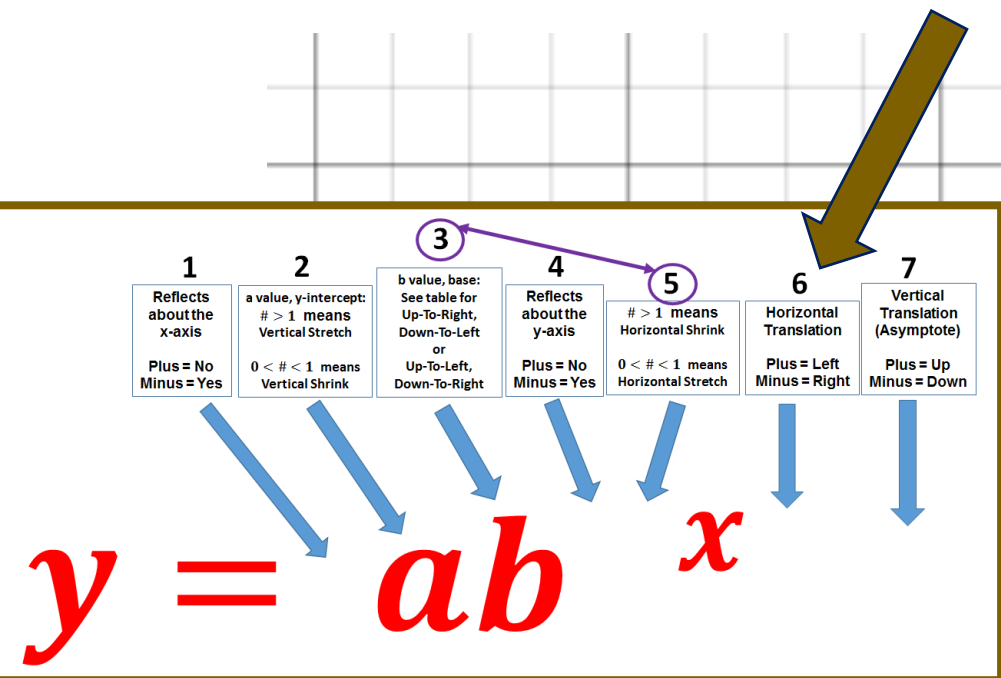
Horizontal Asymptote $y = -3$

Range: $y > -3$



TRANSFORMATIONS

(1b) Horizontal Translation



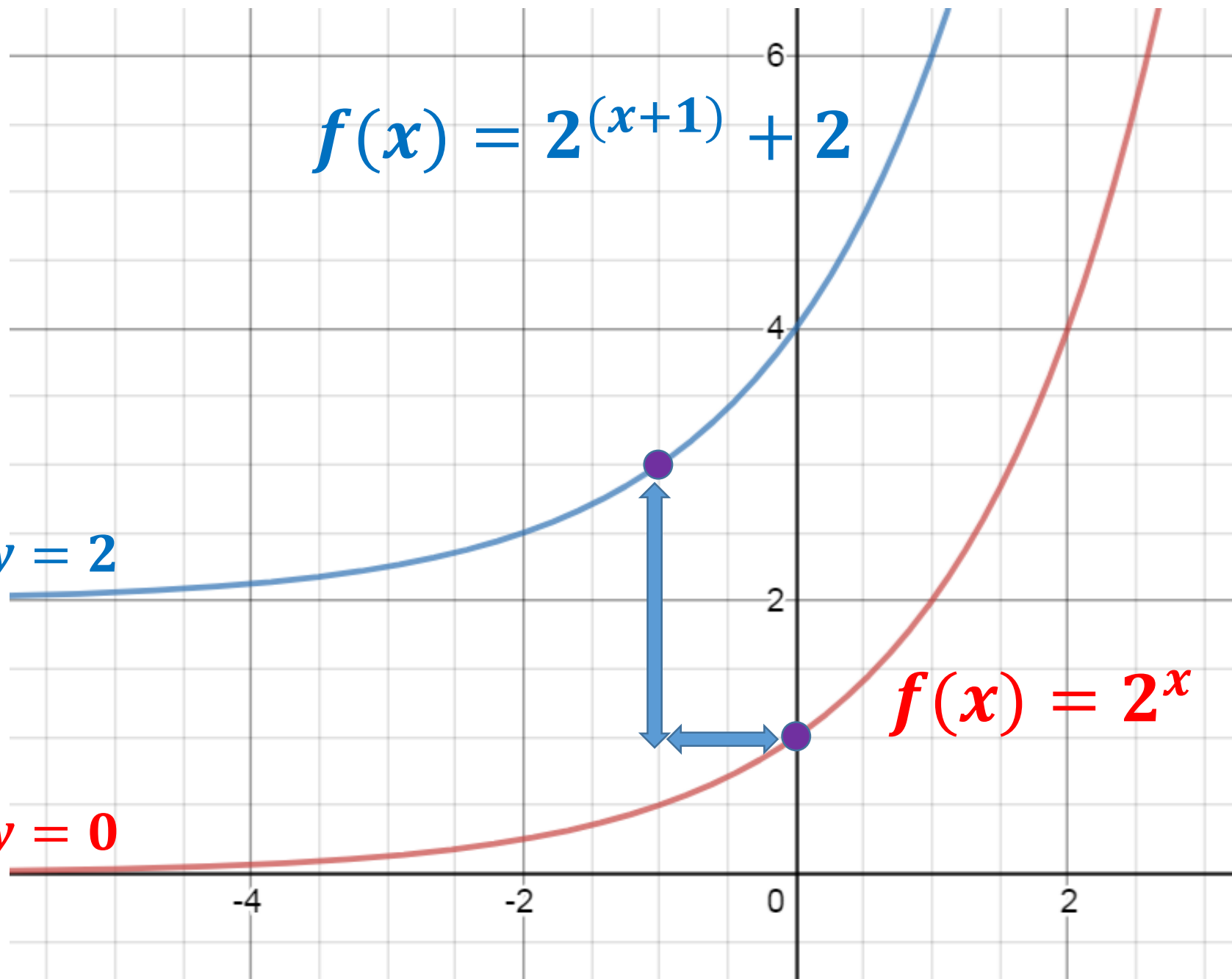
Combining Horizontal & Vertical Translations

Horizontal Asymptote

$$y = 2$$

Horizontal Asymptote

$$y = 0$$



Here is the graph of $f(x) = 2^x$

Label all 3 graphs

CW # 4

1) What is its Horizontal Asymptote? _____

2) What is its y-intercept? (set x to 0) _____

Graph a vertical translation of 4 units down.

3) What is its Horizontal Asymptote? _____

4) What is its equation? _____

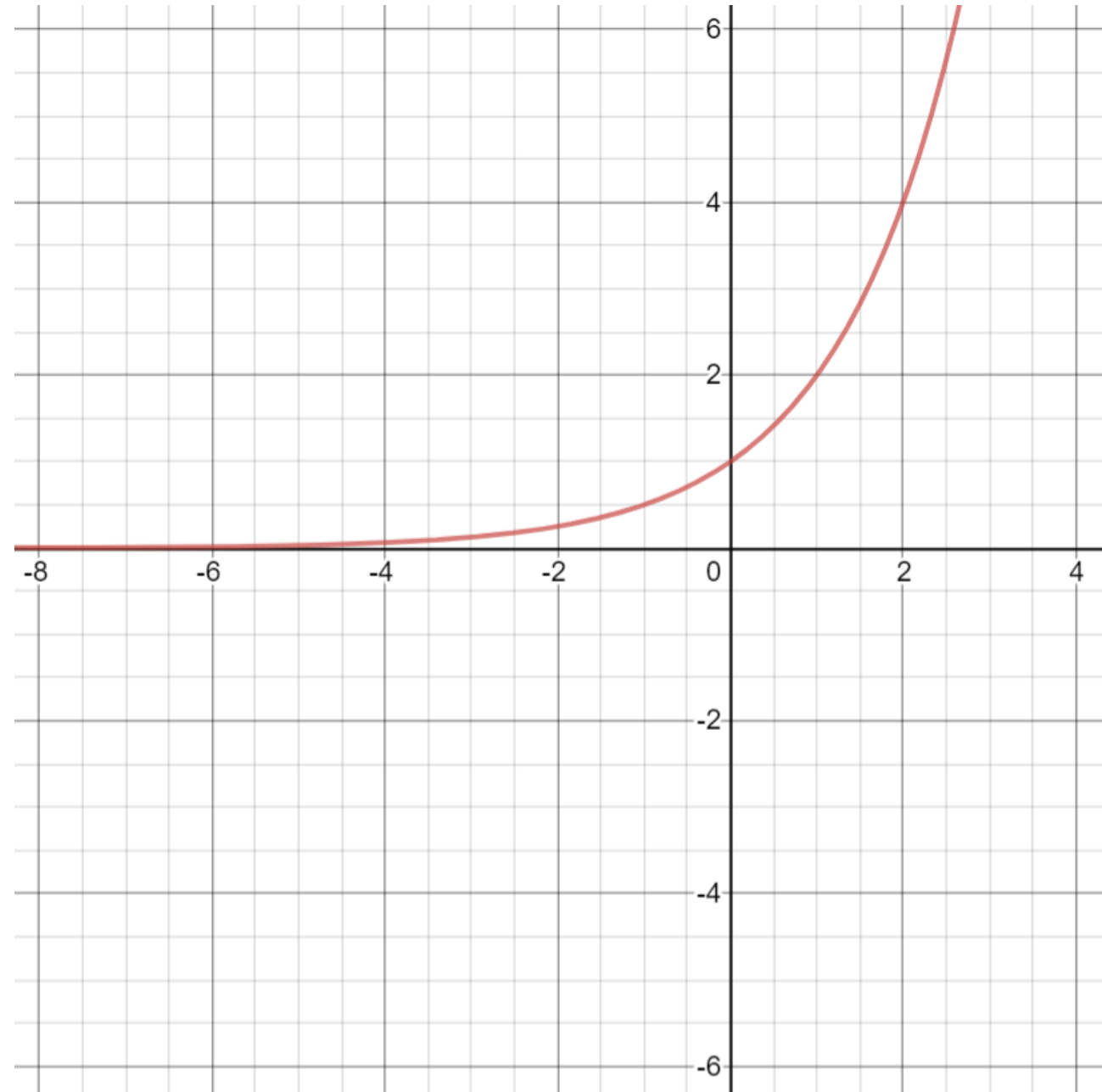
5) What is its y-intercept? _____

From that second one, graph a horizontal translation of 4 units to the left.

6) What is its Horizontal Asymptote? _____

7) What is its equation? _____

8) What is its y-intercept? _____



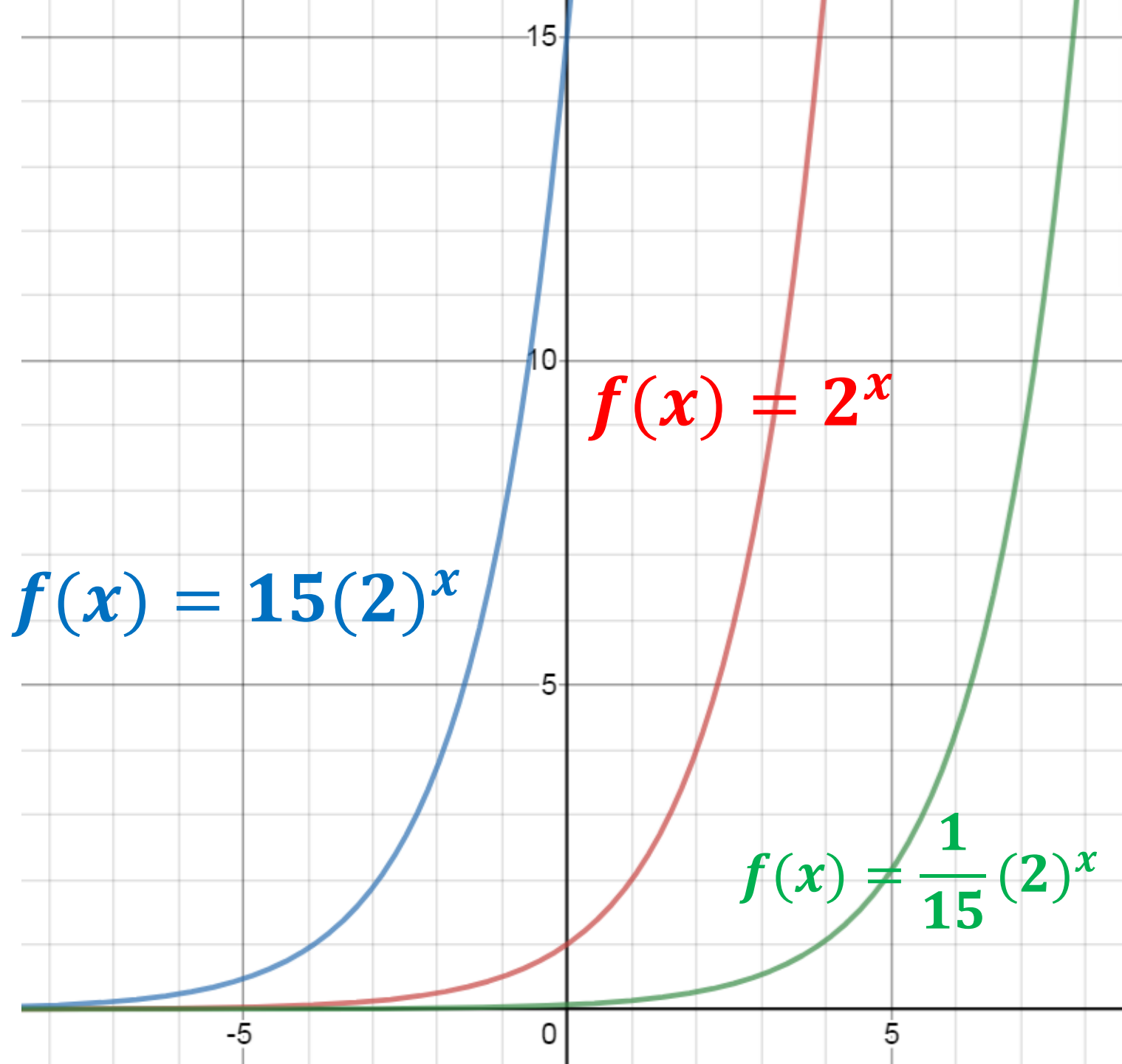
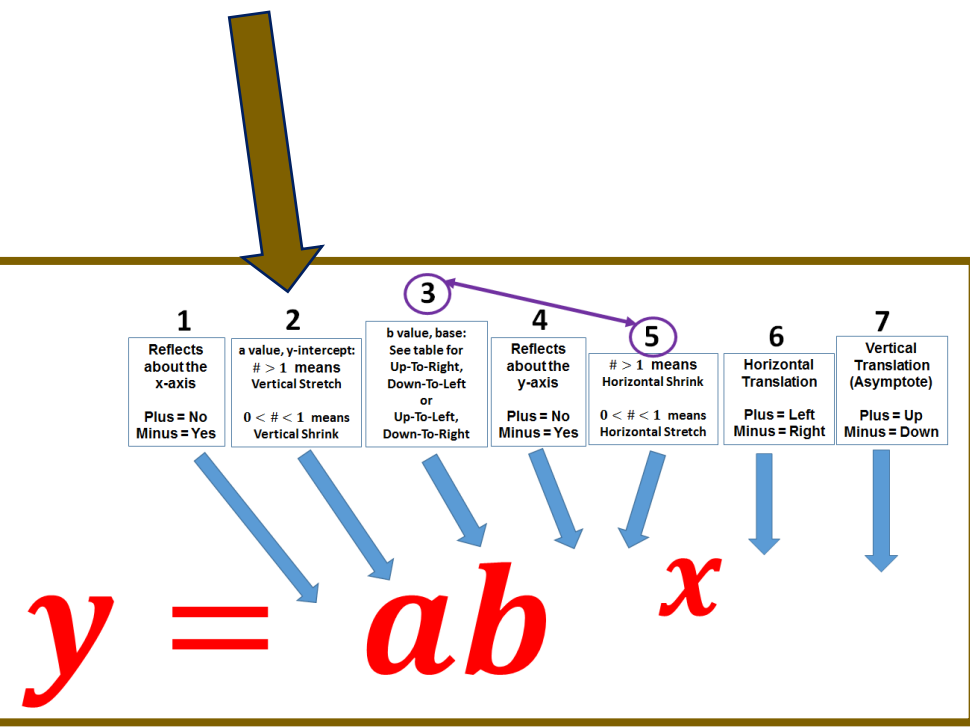
TRANSFORMATIONS

(2 & 3) Vertical Stretch vs. Vertical Shrink

a is the y-intercept.

When $a > 1$ it stretches.

When $0 < a < 1$ it shrinks.

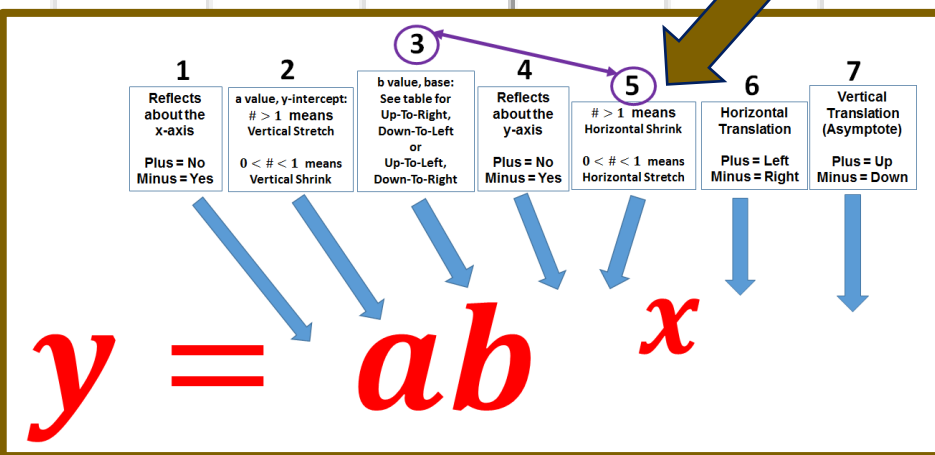


TRANSFORMATIONS

(2 & 3) Horizontal Stretch vs. Horizontal Shrink

When $\# > 1$ it shrinks.

When $0 < \# < 1$ it stretches.



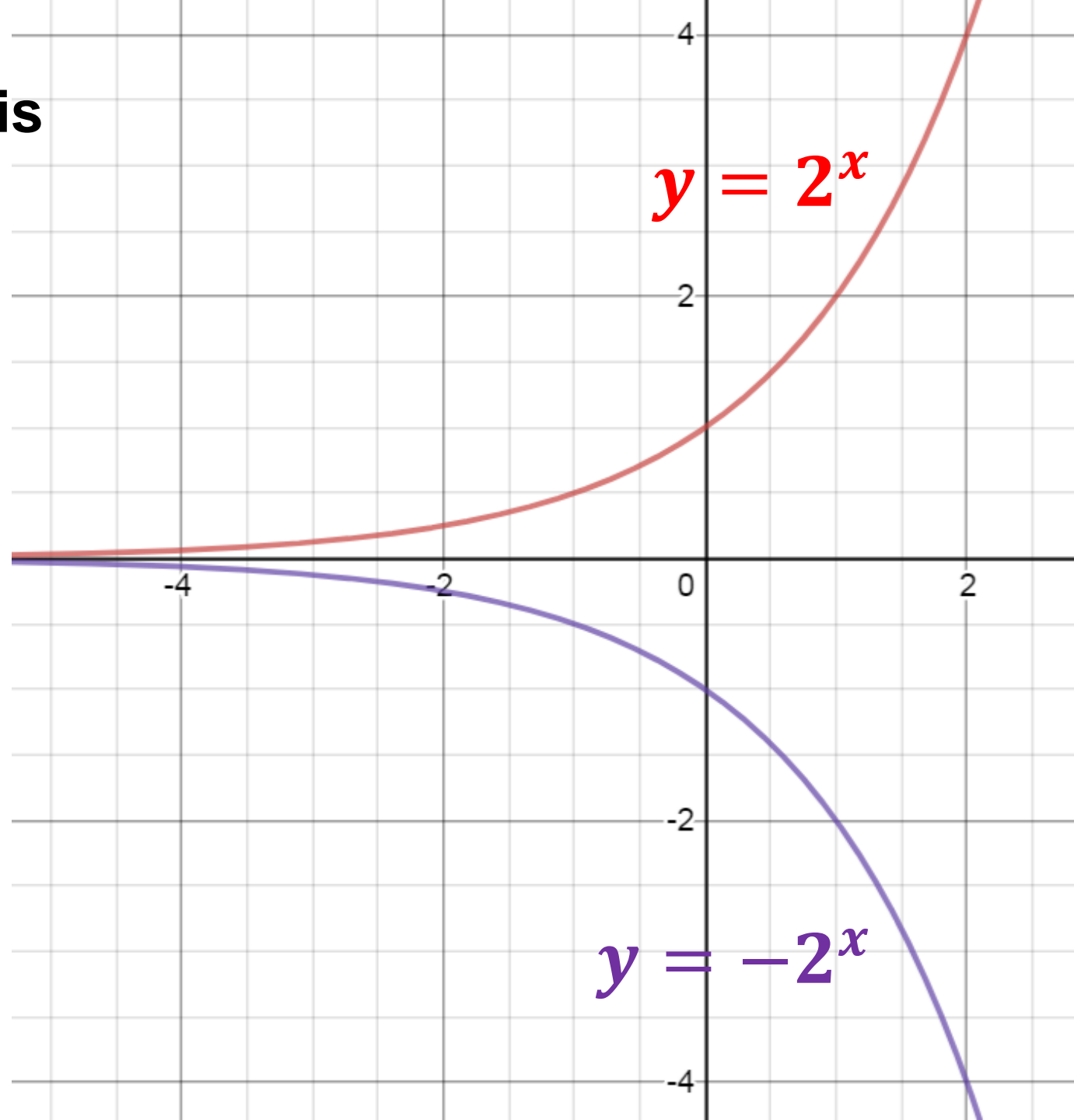
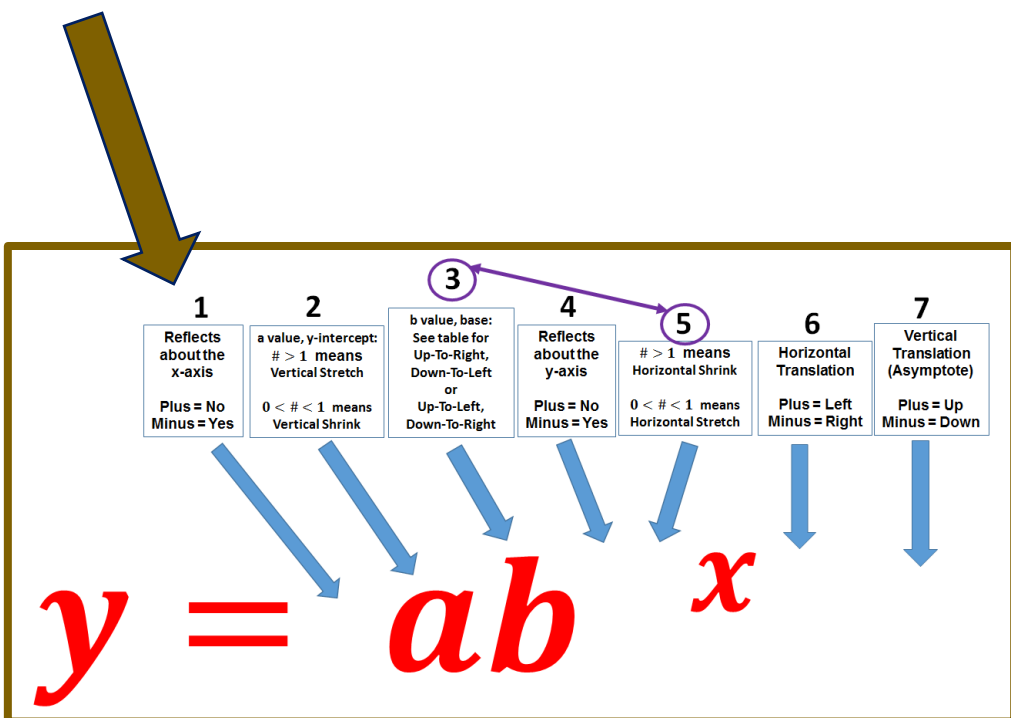
$$y = 2^{3x} = 8^x$$

$$y = 2^x$$

$$y = 2^{0.33x}$$

TRANSFORMATIONS

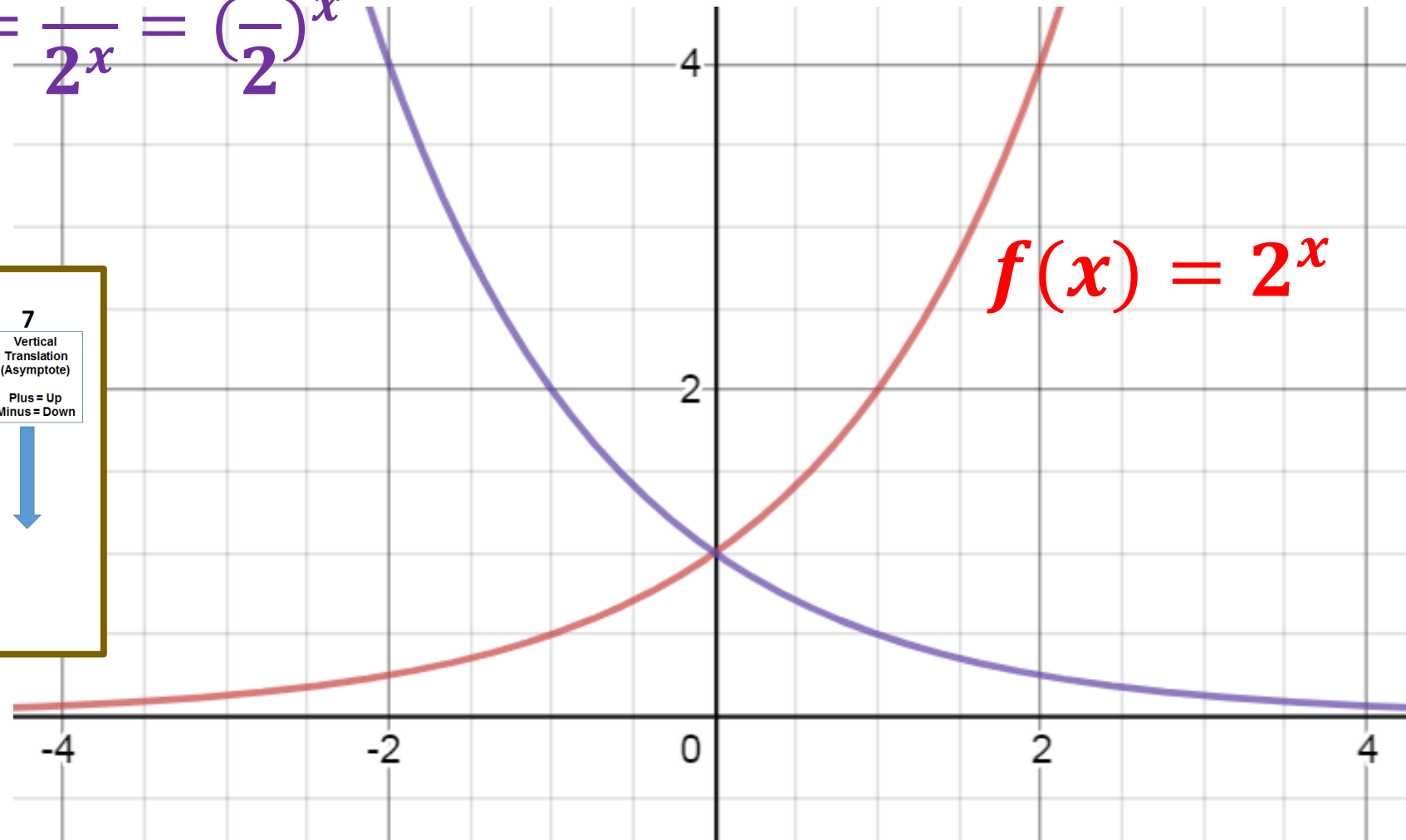
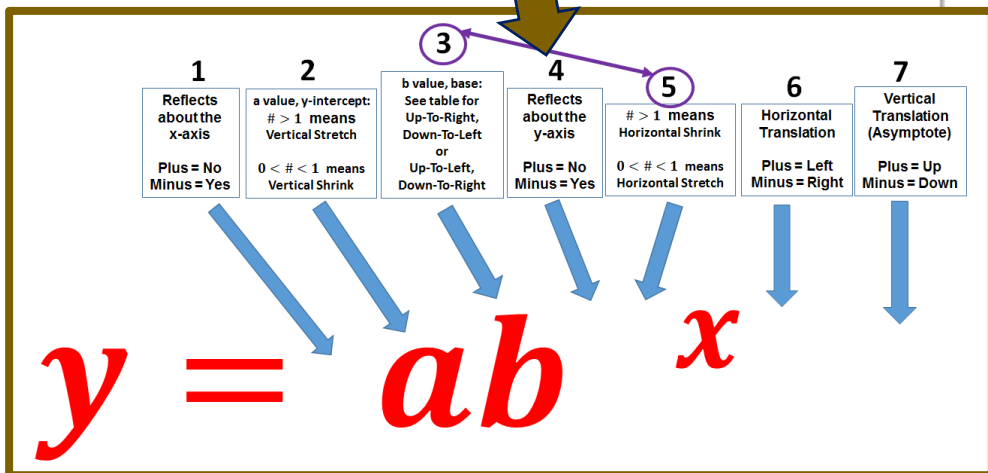
(4a) Reflection – Across the x-axis



TRANSFORMATIONS

(4b) Reflection – Across the y-axis

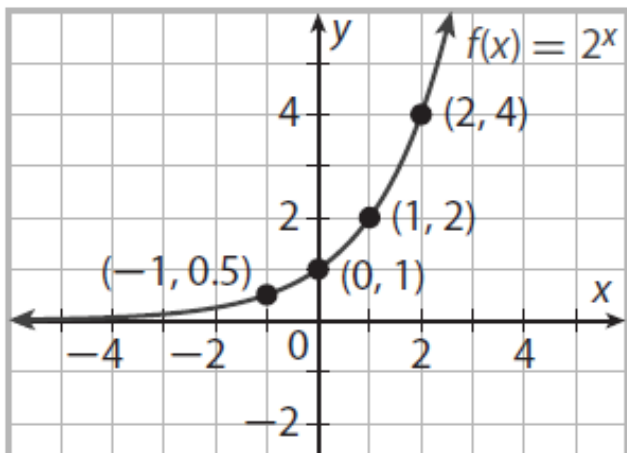
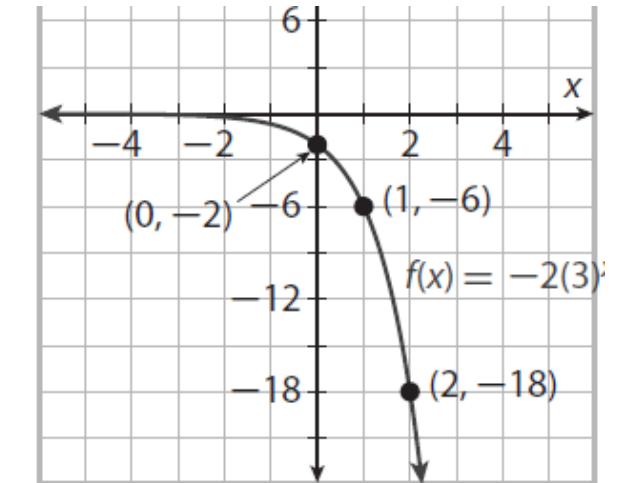
$$f(x) = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$

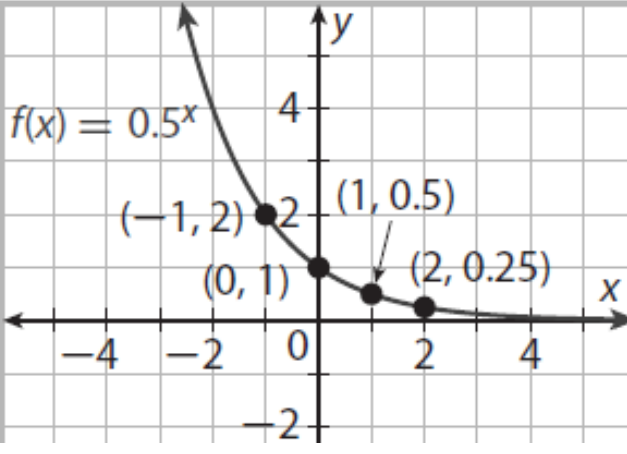
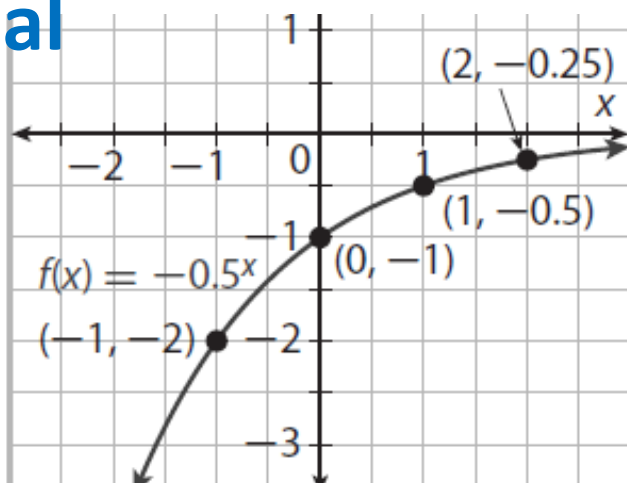


Note: Reflecting across both x and y axes means across the origin.

End Behavior describes what happens to **y**

The Four Basic Exponential Graph Shapes

Graph	End Behavior	
	As $x \rightarrow \infty$	As $x \rightarrow -\infty$
 <p>A coordinate plane showing the graph of the exponential function $f(x) = 2^x$. The curve passes through the points $(-1, 0.5)$, $(0, 1)$, $(1, 2)$, and $(2, 4)$. The x-axis ranges from -4 to 4, and the y-axis ranges from -2 to 4.</p>	$y \rightarrow \infty$	$y \rightarrow 0$
 <p>A coordinate plane showing the graph of the exponential function $f(x) = -2(3)^x$. The curve passes through the points $(0, -2)$, $(1, -6)$, and $(2, -18)$. The x-axis ranges from -4 to 4, and the y-axis ranges from -18 to 6.</p>	$y \rightarrow -\infty$	$y \rightarrow 0$

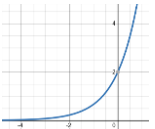
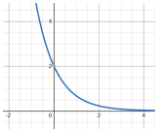
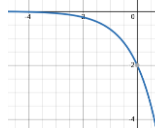
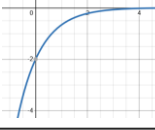
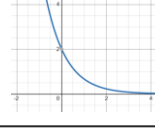
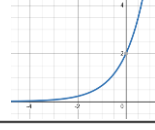
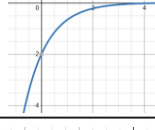
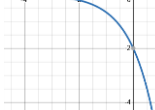
Graph	End Behavior	
	As $x \rightarrow \infty$	As $x \rightarrow -\infty$
 <p>A coordinate plane showing the graph of the exponential function $f(x) = 0.5^x$. The curve passes through the points $(-1, 2)$, $(0, 1)$, $(1, 0.5)$, and $(2, 0.25)$. The x-axis ranges from -4 to 4, and the y-axis ranges from -2 to 4.</p>	$y \rightarrow 0$	$y \rightarrow \infty$
 <p>A coordinate plane showing the graph of the exponential function $f(x) = -0.5^x$. The curve passes through the points $(-1, -2)$, $(0, -1)$, $(1, -0.5)$, and $(2, -0.25)$. The x-axis ranges from -2 to 2, and the y-axis ranges from -3 to 1.</p>	$y \rightarrow 0$	$y \rightarrow -\infty$

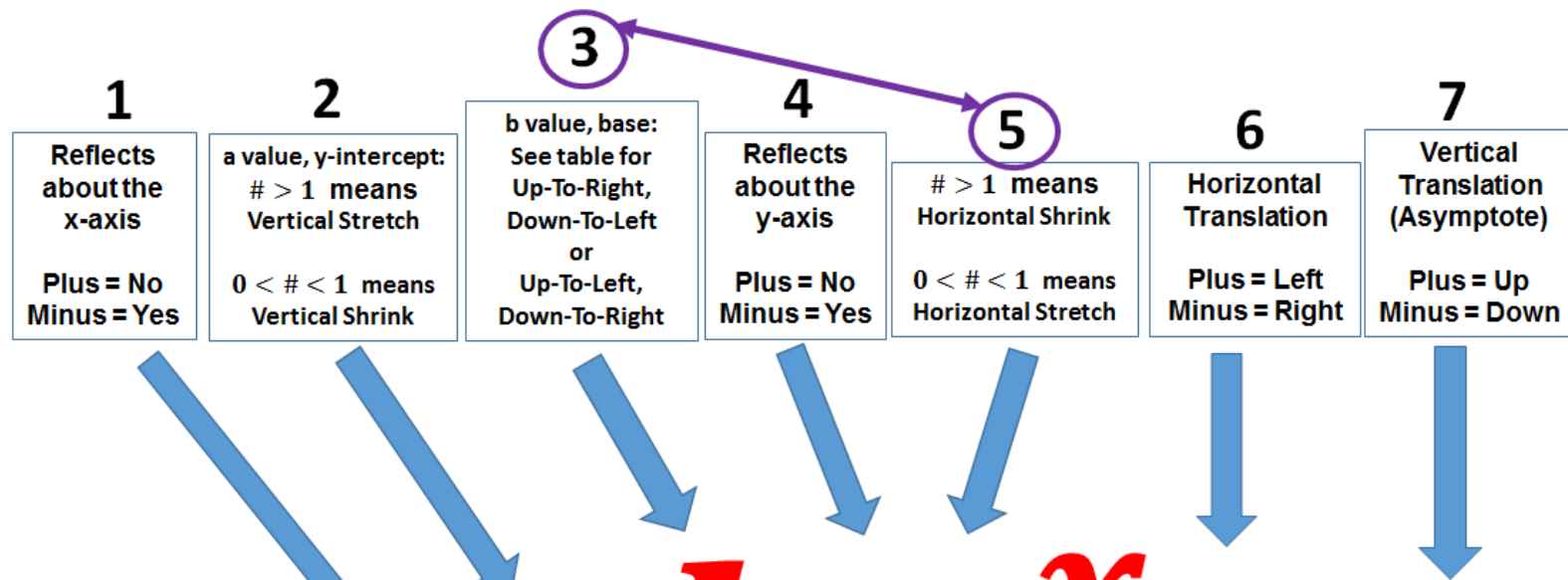
$y = 2^x$

$y = -2(3)^x$

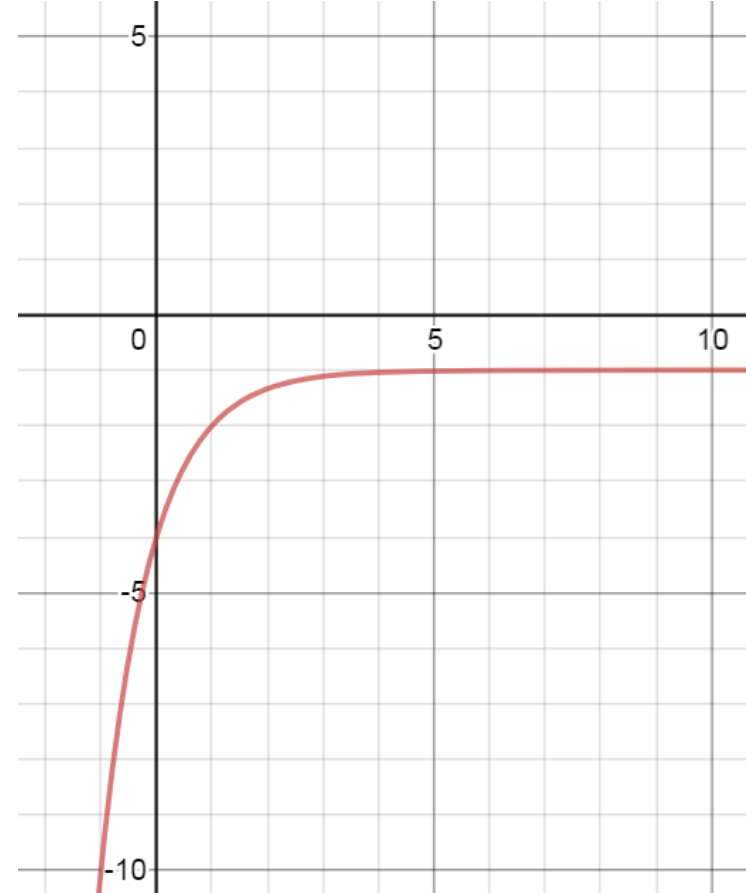
$y = 0.5^x$
 $y = \frac{1}{2}^x$

$y = -0.5^x$
 $y = -\frac{1}{2}^x$

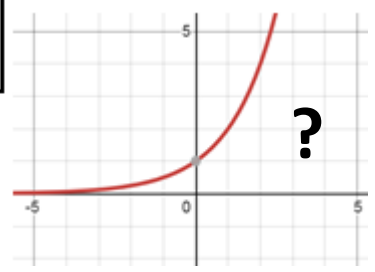
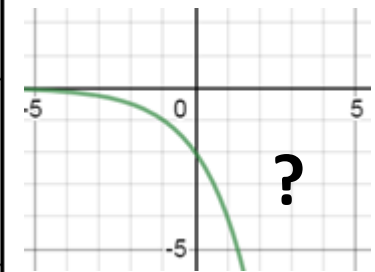
b	Example	Reflection about x-axis?	Reflection about y-axis?	Up-to-Left, Down-to-Right or Up-to-Right, Down-to-Left	Graph
$b > 1$	$2(3)^x$	N	N	Down-to-Left Up-to-Right	
	$2(3)^{-x}$	N	Y	Up-to-Left, Down-To-Right	
	$-2(3)^x$	Y	N	Up-to-Left, Down-To-Right	
	$-2(3)^{-x}$	Y	Y	Down-to-Left Up-to-Right	
$0 \leq b \leq 1$	$2\left(\frac{1}{2}\right)^x$	N	N	Up-to-Left, Down-To-Right	
	$2\left(\frac{1}{2}\right)^{-x}$	N	Y	Down-to-Left Up-to-Right	
	$-2\left(\frac{1}{2}\right)^x$	Y	N	Down-to-Left Up-to-Right	
	$-2\left(\frac{1}{2}\right)^{-x}$	Y	Y	Up-to-Left, Down-To-Right	



$$y = ab^x$$



Box	1	2	4	1,3,4 See Table	6	7	Based On 7			
Exponential Function	Reflect About x-axis?	Vertical Stretch? Shrink?	Reflect About y-axis?	U-to-R D-to-L or U-to-L D-to-R	Horizontal Translation To Left? To Right? How Many?	Vertical Translation Up? Down? How Many?	Horizontal Asymptote	y-int (set x to 0)	As $x \rightarrow +\infty$	As $x \rightarrow -\infty$
$y = (-3)3^{-x} - 1$	Yes	Stretch	Yes	U-to-R D-to-L	No	Down 1	$y = -1$	-4	$y \rightarrow +\infty$	$y \rightarrow 0$



You have evaluated a^x for integer and rational values of x . For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x , you need to interpret forms with *irrational* exponents.

Use a calculator to evaluate each function at the indicated value of x .

Note: It may be necessary to enclose fractional exponents in parentheses.

Keystrokes: Graphing vs. Windows Calculator				
Function	Value	Function Value		Display
a. $f(x) = 2^x$	$x = -3.1$	$f(-3.1) = 2^{-3.1}$	2 \wedge $(-)$ 3.1 ENTER 2 x^y $()$ $(-)$ 3.1 $()$ $=$	0.1166291
b. $f(x) = 2^{-x}$	$x = \pi$	$f(\pi) = 2^{-\pi}$	2 \wedge $(-)$ π ENTER 2 x^y $()$ $(-)$ π $()$ $=$	0.1133147
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$	$f(\frac{3}{2}) = 0.6^{3/2}$.6 \wedge $()$ 3 \div 2 $()$ ENTER .6 x^y $()$ 3 \div 2 $()$ $=$	0.4647580

Use a calculator to evaluate

$$f(x) = 8^{-x} \text{ at } x = \sqrt{2}$$

$$f(x) = \left(\frac{2}{3}\right)^{5x} \text{ at } x = \frac{3}{10}$$

Notice that the graph of an exponential function is always increasing or always decreasing. As a result, the graphs pass the Horizontal Line Test, and therefore the functions are one-to-one functions. You can use the following **One-to-One Property** to solve simple exponential equations.

For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.

$$9 = 3^{x+1} \quad \text{Original equation}$$

$$3^2 = 3^{x+1} \quad 9 = 3^2$$

$$2 = x + 1 \quad \text{One-to-One Property}$$

$$1 = x \quad \text{Solve for } x.$$

$$\left(\frac{1}{2}\right)^x = 8 \quad \text{Original equation}$$

$$\underbrace{\left(\frac{1}{2}\right)^x = 8}_{= \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x} = 8} \quad \text{Rewritten}$$

$$2^{-x} = 2^3 \quad 2^3 = 8$$

$$-x = 3 \quad \text{One-to-One Property}$$

$$x = -3 \quad \text{Solve for } x.$$

CW # 7

Use the One-to-One Property to solve the equation for x .

a. $8 = 2^{2x-1}$

b. $\left(\frac{1}{3}\right)^{-x} = 27$

The Natural Base e

An irrational number, symbolized by the letter e , appears as the base in many applied exponential functions. The number e is defined as the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as n gets larger and larger. The **Table** shows values of $\left(1 + \frac{1}{n}\right)^n$ for increasingly large values of n . As $n \rightarrow \infty$, the approximate value of e to nine decimal places is $e \approx 2.718281827$.

The irrational number e , approximately 2.72, is called the **natural base**. The function $f(x) = e^x$ is called the **natural exponential function**.

Use a scientific or graphing calculator with an e^x key to evaluate e to various powers. For example, to find e^2 , press the following keys on most calculators:

Scientific calculator: 2 e^x

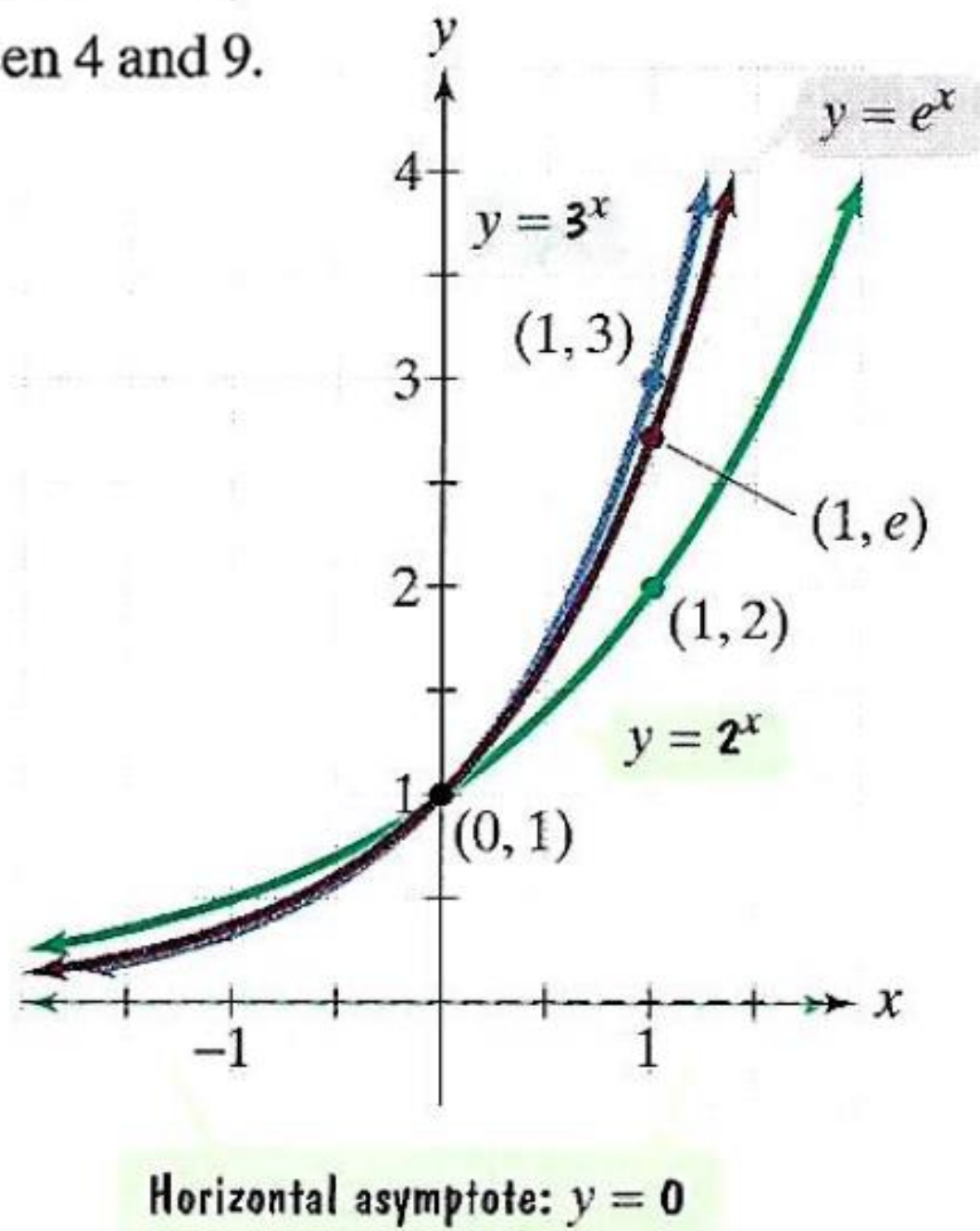
Graphing calculator: e^x 2 ENTER.

The display should be approximately 7.389. $e^2 \approx 7.389$

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829
1000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
1,000,000,000	2.718281827
As $n \rightarrow \infty$, $\left(1 + \frac{1}{n}\right)^n \rightarrow e$.	

The number e lies between 2 and 3. Because $2^2 = 4$ and $3^2 = 9$, it makes sense that e^2 , approximately 7.389, lies between 4 and 9.

Because $2 < e < 3$, the graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$.



Use a calculator to evaluate the function $f(x) = e^x$ at each value of x .

Value	Function Value	Keystrokes: Graphing vs. Windows Calculator	Display
$x = -2$	$f(-2) = e^{-2}$	e^x $(-)$ 2 ENTER $()$ $(-)$ 2 $()$ e^x	0.1353353
$x = 0.25$	$f(0.25) = e^{0.25}$	e^x 0.25 ENTER 0.25 e^x	1.2840254

CW # 8 Use a calculator to evaluate the function $f(x) = e^x$ at each value of x .

a) $x = -1.2$

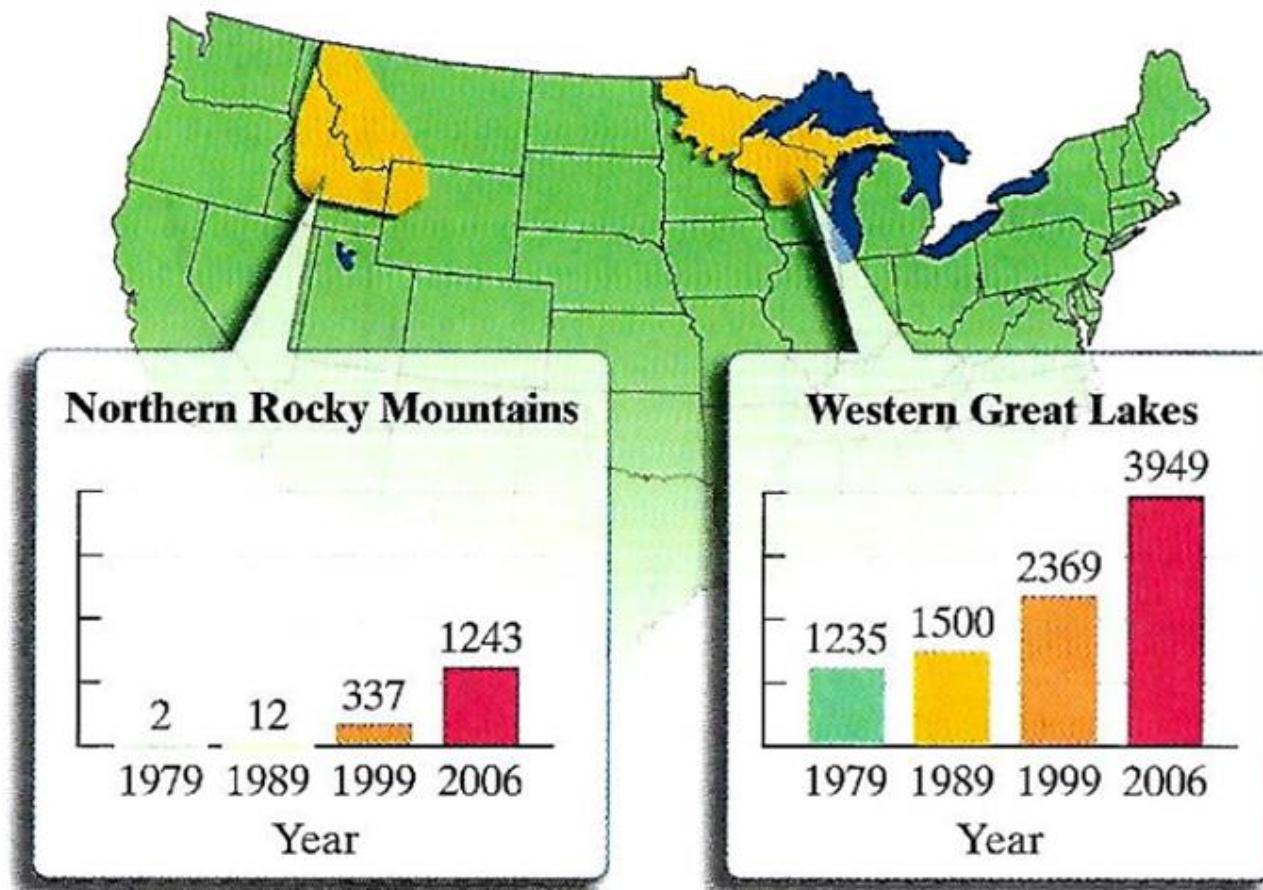
b) $x = 6.2$

Use the One-to-One Property
to solve the equation for x .

c) $e^{x^2-3} = e^{2x}$

Insatiable killer. That's the reputation the gray wolf acquired in the United States in the nineteenth and early twentieth centuries. Although the label was undeserved, an estimated two million wolves were shot, trapped, or poisoned. By 1960, the population was reduced to 800 wolves. **Figure 4.6** shows the rebounding population in two recovery areas after the gray wolf was declared an endangered species and received federal protection. The exponential function $f(x) = 1.26e^{0.247x}$

Gray Wolf Population in Two Recovery Areas for Selected Years



models the gray wolf population of the Northern Rocky Mountains, $f(x)$, x years after 1978. If the wolf is not removed from the endangered species list and trends shown continue, project its population in the recovery area in 2010.

Solution Because 2010 is 32 years after 1978, we substitute 32 for x in the given function.

$$f(x) = 1.26e^{0.247x} \quad \text{This is the given function.}$$

$$f(32) = 1.26e^{0.247(32)} \quad \text{Substitute 32 for } x.$$

Perform this computation on your calculator.

The display should be approximately 3412.1973.

This indicates that the gray wolf population of the Northern Rocky Mountains in the year 2010 is projected to be 3412.

CW # 9

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, x years after 1978.

If trends continue, project the gray wolf's population in the recovery area in 2012.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$



You invest \$12,000 at an annual rate of 3%. Find the balance after 5 years when the interest is compounded (a) quarterly (b) monthly (c) continuously.

(a) For quarterly compounding, you have $n = 4$. So, in 5 years at 3%, the balance is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Formula for compound interest

$$= 12,000\left(1 + \frac{0.03}{4}\right)^{4(5)}$$

Substitute for P , r , n , and t .

$$\approx \$13,934.21.$$

Use a calculator.

(b) For monthly compounding, you have $n = 12$. So, in 5 years at 3%, the balance is:

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

Formula for compound interest

$$= 12,000 \left(1 + \frac{0.03}{12} \right)^{12(5)}$$

Substitute for P , r , n , and t .

$$\approx \$13,939.40.$$

Use a calculator.

NOTE: For a given principal, interest rate, and time, continuous compounding will always yield a larger balance than compounding n times per year.

(c) For continuous compounding, the balance is

$$A = Pe^{rt}$$

Formula for continuous compounding

$$= 12,000e^{0.03(5)}$$

Substitute for P , r , and t .

$$\approx \$13,942.01$$

Use a calculator.

CW # 10

You invest \$6,000 at an annual rate of 4%. Find the balance after 7 years for each type of compounding: (a) Quarterly (b) monthly (c) continuously.