Algebraic Geometry of Matrices I

Lek-Heng Lim

University of Chicago

July 2, 2013

objectives

- give a taste of algebraic geometry
- with minimum prerequisites
- provide pointers for a more serious study
- not intended to be a formal introduction
- tailored specially for this audience:
    - assumes familiarity with linear algebra, matrix analysis
    - maybe even some operator theory, differential geometry
    - but less comfortable with (abstract) algebra

promise: *we shall see lots of matrices and linear algebra*

Overview

why algebraic geometry

- possibly the most potent tool in modern mathematics
- applications to other areas of mathematics

  number theory: Fermat's last theorem

  partial differential equations: soliton solutions of KdV

  many more . . . , but not so surprising
- applications to other areas *outside* of mathematics

  biology: phylogenetic invariants

  chemistry: chemical reaction networks

  physics: mirror symmetry

  statistics: Markov bases

  optimization: sum-of-squares polynomial optimization

  computer science: geometric complexity theory

  communication: Goppa code

  cryptography: elliptic curve cryptosystem

  control theory: pole placement

  machine learning: learning Gaussian mixtures
- why should folks in linear algebra/matrix theory care?

solves long standing conjectures

Horn: $A, B \in \mathbb{C}^{n\times n}$ Hermitian, $I, J, K \subsetneq \{1, \dots, n\}$,

$$\sum_{k\in K} \lambda_k(A+B) \leq \sum_{i\in I} \lambda_i(A) + \sum_{j\in J} \lambda_j(B)$$

holds iff Schubert cycle s_K is component of $s_I \cdot s_J$ [Klyachko, 1998], [Knutson-Tao, 1999]

Strassen: no approximate algorithm for 2×2 matrix product in fewer than 7 multiplications [Landsberg, 2006]

- involve Schubert varieties and secant varieties respectively
- for now, variety = affine variety = zero loci of polynomials

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : F_j(x_1, \dots, x_n) = 0 \text{ for all } j \in J\}$$

$F_j \in \mathbb{C}[x_1, \dots, x_n]$, J arbitrary index set

view familiar objects in new light

linear affine variety: solutions to linear equation

$$\{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{b}\}$$

where $A \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^m$

determinantal variety: rank-r matrices

$$\{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) \le r\}$$

Segre variety: rank-1 matrices

$$\{X \in \mathbb{C}^{m\times n} : X = \mathbf{u}\mathbf{v}^{\mathsf{T}}\}$$

Veronese variety: rank-1 symmetric matrices

$$\{X \in \mathbb{C}^{n\times n} : X = \mathbf{v}\mathbf{v}^{\mathsf{T}}\}$$

Grassmann variety: n-dimensional subspaces in $\mathbb{C}^m$

$$\{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) = n\}/\operatorname{GL}_n(\mathbb{C})$$

gain new insights

secant variety: rank-r matrices

lines through r points on $\{X \in \mathbb{P}^{m \times n} : \mathrm{rank}(X) = 1\}$

dual variety: singular matrices

$\{X \in \mathbb{P}^{n \times n} : \mathrm{rank}(X) = 1\}^\vee = \{X \in \mathbb{P}^{n \times n} : \det(X) = 0\}$

Fano variety: vector spaces of matrices of low rank

set of k-planes in $\{X \in \mathbb{P}^{m \times n} : \mathrm{rank}(X) \leq r\}$

projective n-space: $\mathbb{P}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\sim$ with equivalence relation $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for $\lambda \in \mathbb{C}^\times$

encouraging observation

last two slides: if you know linear algebra/matrix theory, you have seen many examples in algebraic geometry

next three slides: more such examples

moral: you have already encountered quite a bit of algebraic geometry

zero loci of matrices

twisted cubic: 2×3 rank-1 Hankel matrices

$$\left\{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : \operatorname{rank}\left(\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}\right) = 1\right\}$$

rational normal curve: $2 \times d$ rank-1 Hankel matrices

$$\left\{[x_0 : x_1 : \cdots : x_d] \in \mathbb{P}^d : \operatorname{rank}\left(\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{bmatrix}\right) = 1\right\}$$

rational normal scroll: $(d-k+1) \times (k+1)$ rank-1 Hankel matrices

$$\left\{[x_0 : x_1 : \cdots : x_d] \in \mathbb{P}^d : \operatorname{rank}\left(\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_k \\ x_1 & x_2 & \cdots & \cdots & x_{k+1} \\ x_2 & \cdots & \cdots & \cdots & x_{k+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_{d-1} \\ x_{d-k} & \cdots & \cdots & x_{d-1} & x_d \end{bmatrix}\right) = 1\right\}$$

discriminant hypersurface of singular quadrics in $\mathbb{P}^n$:

$$\left\{[x_{00} : x_{01} : \cdots : x_{nn}] \in \mathbb{P}^{n(n+3)/2} : \det\left(\begin{bmatrix} x_{00} & x_{01} & \cdots & x_{0n} \\ x_{01} & x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{0n} & x_{1n} & \cdots & x_{nn} \end{bmatrix}\right) = 0\right\}$$

algebraic groups

- elliptic curve: $y^2 = (x-a)(x-b)(x-c)$,

$$E = \left\{ (x,y) \in \mathbb{C}^2 : \det\left(\begin{bmatrix} x-a & 0 & y \\ 0 & 1 & \frac{1}{2}(b+c)+x \\ y & \frac{1}{2}(b+c)-x & -\frac{1}{4}(b-c)^2 \end{bmatrix} \right) = 0 \right\}$$

- E is abelian variety, i.e., variety that is abelian group
- generalization: algebraic groups
- multiplication/inversion defined locally by rational functions
- two most important classes:

  projective: abelian varieties
  affine: linear algebraic groups

- examples:

  general linear group: $\mathrm{GL}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n\times n} : \det(X) \neq 0\}$
  special linear group: $\mathrm{SL}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n\times n} : \det(X) = 1\}$
  projective linear group: $\mathrm{PGL}_n(\mathbb{F}) = \mathrm{GL}_n(\mathbb{F})/\{\lambda I : \lambda \in \mathbb{F}^\times\}$

linear algebraic groups

char($\mathbb{F}$) $\neq 2$

orthogonal goup: q symmetric nondegenerate bilinear

$$\mathsf{O}_n(\mathbb{F}, q) = \{X \in \mathsf{GL}_n(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

special orthogonal goup: q symmetric nondegenerate bilinear

$$\mathsf{SO}_n(\mathbb{F}, q) = \{X \in \mathsf{SL}_n(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

symplectic goup: q skew-symmetric nondegenerate bilinear

$$\mathsf{Sp}_{2n}(\mathbb{F}, q) = \{X \in \mathsf{SL}_n(\mathbb{F}) : q(X\mathbf{v}, X\mathbf{w}) = q(\mathbf{v}, \mathbf{w})\}$$

special case: $q(\mathbf{v}, \mathbf{w}) = \mathbf{v}^\mathsf{T}\mathbf{w}$, get $\mathsf{O}_n(\mathbb{F})$, $\mathsf{SO}_n(\mathbb{F})$, $\mathsf{Sp}_{2n}(\mathbb{F})$,

$$\mathsf{PO}_n(\mathbb{F}) = \mathsf{O}_n(\mathbb{F})/\{\pm I\}, \quad \mathsf{PSO}_{2n}(\mathbb{F}) = \mathsf{SO}_{2n}(\mathbb{F})/\{\pm I\}$$

comes in different flavors

complex algebraic geometry: varieties over $\mathbb{C}$

real algebraic geometry: semialgebraic sets & varieties over $\mathbb{R}$, e.g. hyperbolic cone, $A \succ 0$, $\mathbf{b} \in \mathbb{R}^n$,

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\mathsf{T} A \mathbf{x} \le (\mathbf{b}^\mathsf{T}\mathbf{x})^2,\ \mathbf{b}^\mathsf{T}\mathbf{x} \ge 0\}$$

convex algebraic geometry: convex sets with algebraic structure, e.g. spectrahedron, $A_0, \dots, A_n \in \mathbb{S}^{m \times m}$,

$$\{A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0 : \mathbf{x} \in \mathbb{R}^n\}$$

tropical algebraic geometry: varieties over $(\mathbb{R} \cup \{\infty\}, \min, +)$, e.g. tropical linear space, tropical polytope, tropical eigenspace, tropical Grassmannian

many others: diophantine geometry (over $\mathbb{Q}, \mathbb{Q}_p, \mathbb{F}_q, \mathbb{F}_q((t)), \mathbb{Z}$, etc), noncommutative algebraic geometry, etc

going beyond matrices

provides groundwork to go beyond linear algebra and matrices

linear to multilinear: $f : V_1 \times \cdots \times V_d \to W$,

$$f(\mathbf{v}_1, \dots, \alpha\mathbf{u}_k + \beta\mathbf{w}_k, \dots, \mathbf{v}_d) = \alpha f(\mathbf{v}_1, \dots, \mathbf{u}_k, \dots, \mathbf{v}_d) + \beta f(\mathbf{v}_1, \dots, \mathbf{w}_k, \dots, \mathbf{v}_d)$$

matrices to hypermatrices:

$$(a_{ij}) \in \mathbb{C}^{m\times n}, \quad (a_{ijk}) \in \mathbb{C}^{l\times m\times n}, \quad (a_{ijkl}) \in \mathbb{C}^{l\times m\times n\times p}, \dots$$

linear/quadratic to polynomial: $\mathbf{a}^{\mathsf{T}}\mathbf{x}$, $A\mathbf{x}$, $\mathbf{x}^{\mathsf{T}}A\mathbf{x}$, $\mathbf{x}^{\mathsf{T}}A\mathbf{y}$, to

$$f(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i,j=1}^{n} c_{ij} x_i x_j + \sum_{i,j,k=1}^{n} d_{ijk} x_i x_j x_k + \sum_{i,j,k,l=1}^{n} e_{ijkl} x_i x_j x_k x_l + \cdots$$

vector spaces to vector bundles: family of vector spaces (later)

advertisement

L.-H. Lim, *Lectures on Tensors and Hypermatrices*

basic notions: tensor, multilinear functions, hypermatrices, tensor fields, covariance & contravariance, symmetric hypermatrices & homogeneous polynomials, skew-symmetric hypermatrices & exterior forms, hypermatrices with partial skew-symmetry/symmetry & Schur functors, Dirac & Einstein notations

ranks & decompositions: tensor rank, multilinear rank & multilinear nullity, rank-retaining decompositions, border rank, generic & typical rank, maximal rank, nonexistence of canonical forms, symmetric rank, nonnegative rank, Waring rank, Segre, Veronese, & Segre-Veronese varieties, secant varieties

eigenvalues & singular values: symmetric eigenvalues & eigenvectors, eigenvalues & eigenvectors, singular values & singular vectors, nonnegative hypermatrices, Perron-Frobenius theorem, positive seimidefinite & Gram hypermatrices

norms, hyperdeterminants, & other loose ends spectral norm, nuclear norm, Holder *p*-norms, geometric hyperdeterminant, combinatorial hyperdeterminant, tensor products of other objects: modules, Hilbert space, Banach space, matrices, operators, representations, operator spaces, computational complexity

advertisement

- biology: phylogenetic invariants
- chemistry: fluorescence spectroscopy, matrix product state DMRG
- computer science: computational complexity, quantum information theory
- optimization: self-concordance, higher-order optimality conditions, polynomial optimization
- applied physics: elasticity, piezoelectricity, X-ray crystallography
- theoretical physics: quantum mechanics (state space of multiple quantum systems), statistical mechanics (Yang-Baxter equations), particle physics (quark states), relativity (Einstein equation)
- signal processing: antenna array processing, blind source separation, CDMA communication
- statistics: multivariate moments and cumulants, sparse recovery and matrix completion

venue: Room B3-01, Instituto para a Investigação Interdisciplinar da Universidade de Lisboa

dates: July 23–24, 2013

Affine Varieties

for further information

easy

- S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, 1990
- B. Hassett, *Introduction to Algebraic Geometry*, 2007
- K. Hulek, *Elementary Algebraic Geometry*, 2003
- M. Reid, *Undergraduate Algebraic Geometry*, 1989
- K. Smith et al., *An Invitation to Algebraic Geometry*, 2004 (our main text)

medium

- J. Harris, *Algebraic Geometry: A First Course*, 1992
- I. Shafarevich, *Basic Algebraic Geometry*, Vols. I & II, 2nd Ed., 1994

standard

- P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, 1978
- R. Hartshorne, *Algebraic Geometry*, 1979

recent

- D. Arapura, *Algebraic Geometry over the Complex Numbers*, 2012
- S. Bosch, *Algebraic Geometry and Commutative Algebra*, 2013
- T. Garrity et al., *Algebraic Geometry: A Problem Solving Approach* , 2013
- A. Holme, *A Royal Road to Algebraic Geometry*, 2012

basic and not-so-basic objects

affine varieties: subsets of $\mathbb{C}^n$ cut out by polynomials

projective varieties: subsets of $\mathbb{P}^n$ cut out by homogeneous polynomials

quasi-projective varieties: open subsets of projective varieties

algebraic varieties: affine varieties glued together

affine schemes: affine varieties with ‘non-closed points’ added

schemes: affine schemes glued together

furthermore: schemes $\subseteq$ algebraic spaces $\subseteq$ Deligne–Mumford stacks $\subseteq$ algebraic stacks $\subseteq$ stacks

but to a first-order approximation,

algebraic geometry is the study of algebraic varieties

just like differential geometry is, to a first-order approximation, the study of differential manifolds

what is an algebraic variety

manifold: objects locally resembling Euclidean spaces

algebraic variety: objects locally resembling affine varieties

or:

manifold: open subsets glued together

algebraic variety: affine varieties glued together

differences:

1. machinery for gluing things

   manifold: usually charts/atlases/transition maps

   algebraic varieties: usually sheaves

2. dimension

   manifold: glue together subsets of same dimension

   algebraic varieties: can have different dimensions

sheaf: neatest tool for gluing things — works for Riemann surfaces, manifolds, algebraic varieties, schemes, etc

what is an affine variety

- zero loci of polynomials, i.e., common zeros of a collection of complex polynomials in n variables $\{F_j\}_{j\in J}$,

$$\{(x_1,\dots,x_n)\in\mathbb{C}^n : F_j(x_1,\dots,x_n)=0 \text{ for all } j\in J\}$$

- J arbitrary index set, can be uncountable
- notation: $\mathbb{V}(\{F_j\}_{j\in J})$ or $\mathbb{V}(F_1,\dots,F_n)$ if finite
- caution: actually these are just Zariski closed subsets of $\mathbb{C}^n$, the actual definition of affine variety will come later
- may define manifolds as subsets of $\mathbb{R}^n$ but unwise; want affine varieties to be independent of embedding in $\mathbb{C}^n$ too
- simplest examples

empty set: $\varnothing = \mathbb{V}(1)$

singleton: $\{(a_1,\dots,a_n)\} = \mathbb{V}(x_1-a_1,\dots,x_n-a_n)$

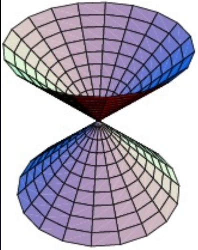
hyperplane: $\mathbb{V}(a_0+a_1x_1+\cdots+a_nx_n)$

hypersurface: $\mathbb{V}(F)$

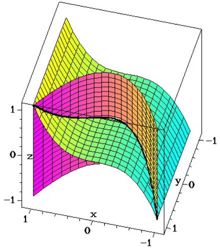
whole space: $\mathbb{C}^n = \mathbb{V}(0)$

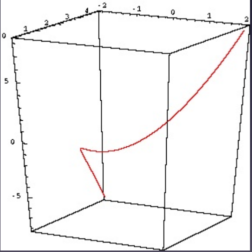
more affine varieties

quadratic cone: $\mathbb{V}(x^2 + y^2 - z^2) = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 = z^2\}$



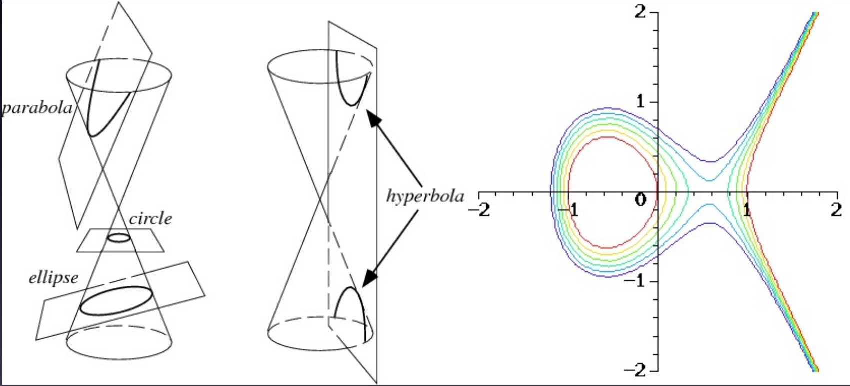
twisted cubic: $\mathbb{V}(x^2 - y, x^3 - z) = \{(t, t^2, t^3) \in \mathbb{C}^3 : t \in \mathbb{C}\}$





more affine varieties

conic sections: $\mathbb{V}(x^2 + y^2 - z^2, ax + by + cz)$



elliptic curve: $\mathbb{V}(y^2 - x^3 + x - a)$ for $a = 0, 0.1, 0.2, 0.3, 0.4, 0.5$

earlier examples revisited

linear affine variety: solutions to linear equation

$$\{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{b}\} = \mathbb{V}(\{a_{i1}x_1 + \cdots + a_{in}x_n - b_i\}_{i=1,\dots,m})$$

determinantal variety: rank-r matrices

$$\{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) \le r\} = \mathbb{V}(\{\text{all } (r+1)\times(r+1) \text{ minors}\})$$

special linear group: determinant-1 matrices

$$\mathrm{SL}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} : \det(X) = 1\} = \mathbb{V}(\det - 1)$$

- in algebraic geometry, we identify $\mathbb{C}^{m\times n} \equiv \mathbb{C}^{mn}$
- why did we say $\mathrm{GL}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} : \det(X) \neq 0\}$ is an affine variety?

non-examples

assume Euclidean/norm topology, following *not* affine varieties:

open ball: $B_\varepsilon(\mathbf{x}) = \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| < \varepsilon\}$

closed ball: $B_\varepsilon[\mathbf{x}] = \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| \leq \varepsilon\}$

unitary group: $\mathrm{U}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} : X^*X = I\}$

general linear group: $\mathrm{GL}_n(\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} : \det(X) \neq 0\}$

punctured line/plane: $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, $\mathbb{C}^2\backslash\{(0,0)\}$

set with interior points: $S \supseteq B_\varepsilon(\mathbf{x})$ for some $\varepsilon > 0$

graphs of transcendental functions: $\{(x,y) \in \mathbb{C}^2 : y = e^x\}$

- $\mathrm{GL}_n(\mathbb{C})$ and $\mathbb{C}^\times$ affine varieties via actual definition
- complex conjugation is not an algebraic operation
- inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$ not polynomial
- every affine variety is closed in Euclidean topology
- converse almost never true
- need another topology: Zariski