# Algorithm Analysis

with

# Big Oh

Data Structures and Design with Java and JUnit
Chapter 12

# *Algorithm Analysis*

- Objectives
  - Analyze the efficiency of algorithms
  - Analyze a few classic algorithms
    - Linear Search, Binary Search, Selection Sort
  - Know the differences between O(1), O(n), O(log n), and $O(n^2)$
  - Visualize runtime differences with experiments

# *Algorithms* continued

- Computer Scientists focus on problems such as
  - How fast do algorithms run
  - How much memory does the process require
- Example Applications
  - Make the Internet run faster
    - Pink-Degemark's routing algorithms
    - Gene Meyers determined the sequences of the Human genome using his whole genome shotgun algorithm

# *Analysis of Algorithms*

- We have ways to compare algorithms
  - Generally, the larger the problem, the longer it takes the algorithm to complete
  - Sorting 100,000 elements can take much more time than sorting 1,000 elements
    - and more than 10 times longer
  - the variable *n* suggests the "*n*umber of things"
  - If an algorithm requires $0.025n^2 + 0.012n + 0.0005$ seconds, just plug in a value for n

# *A Computational Model*

- To summarize algorithm runtimes, we can use a computer independent model
  - instructions are executed sequentially
  - count all assignments, comparisons, and increments there is infinite memory
  - every simple instruction takes one unit of time

# *Simple Instructions*

- Count the simple instructions
  - assignments have cost of 1
  - comparisons have a cost of 1
  - let's count all parts of the loop

```
for (int j = 0; j < n; j++)
```

    - j=0 has a cost of 1, j<n executes n+1 times,and j++ executes n times for a total cost of 2n+2
  - each statement in the repeated part of a loop have have a cost equal to number of iterations

# *Examples*

```
                                          Cost
sum = 0;                          ->      1
sum = sum + next;                 ->      1       Total Cost: 2
```

```
                                          Cost
for (int i = 1; i <= n; i++)      ->      1 + n+1 + n = 2n+2
  sum = sum++;                    ->      n       Total Cost: 3n + 2
```

```
                                          Cost
k = 0                             ->      1
for (int i = 0; i < n; i++)       ->      2n+2
  for (int j = 0; j < n; j++)     ->      n(2n+2) = 2n^2 +2n
    k++;                          ->      n^2     Total Cost: 3n^2 + 4n + 3`
```

# *Total Cost of Sequential Search*

```
                                                        Cost
for (int index = 0; index < n; index++)  ->   2n + 2
  if(searchID.equals(names[index])       ->   n
    return index;                        ->   0 or 1

return -1 // if not found                ->   0 or 1
```

*Total cost = 3n+3*

# *Different Cases*

- The total cost of sequential search is 3n + 3
  - But is it always exactly 3n + 3 instructions?
  - The last assignment does not always execute
    - But does one assignment really matter?
  - How many times will the loop actually execute?
    - that depends
  - If searchID is found at index 0: _____ iterations
    - best case
  - If searchID is found at index n-1:_____ iterations
    - worst case

# *Typical Case of sequential (linear)*

- The average describes the more typical case
- First, let the the entire cost be simplified to n
  - Assume the target has a 50/50 chance of being in the array
    - n comparisons are made: worst-case occurs 1/2 the time
  - Assume if it's in **a**, it's as likely to be in one index as another

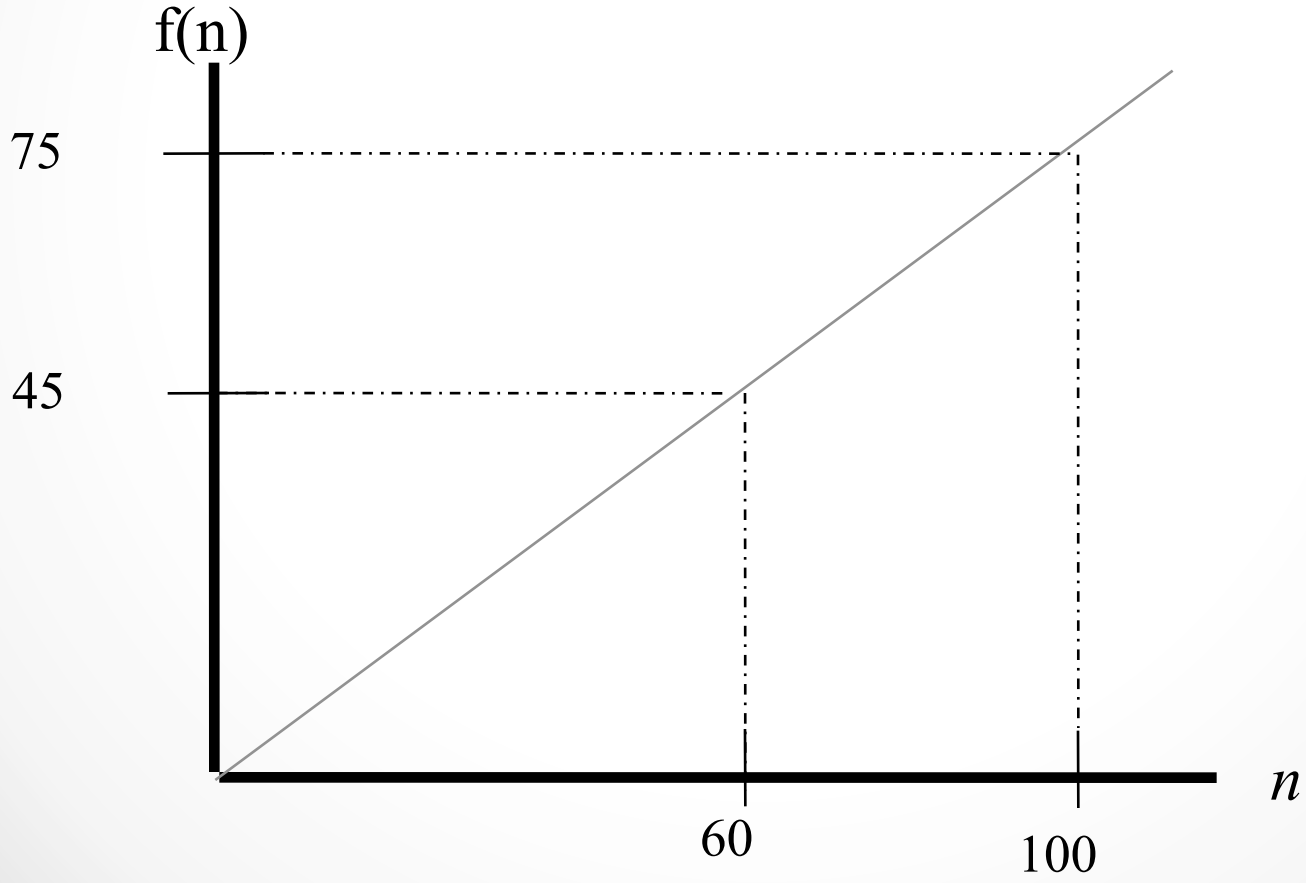
$$\frac{1}{2}n + \frac{1}{2} \times \frac{n}{2} = \frac{n}{2} + \frac{n}{4} = \frac{3}{4}n$$

Half the time it is n comparisons, the other half it is n/2 comparisons

  - So the typical case is 3/4 n comparisons

# *The Essence of Linear Search*

Plot the function *this is why sequential search is also called linear search. As n increases, runtime forms a line*



# *Linear Search Continued*

- This equation is a polynomial: $3n + 3$
- The fastest growing term is the high-order term
- The high order term (which could be $n^2$ or $n^3$), represents the most important part of the analysis function
- We refer to the rate of growth as the *order of magnitude*, which measure the rate of growth

# *Rate of Growth*

- Imagine two functions:

  **`f(n) = 100n`** **`g(n) =`** $\mathbf{n^2 + n}$

  – When n is small, which is the bigger function?

  – When n is big, which is the bigger function?

  – We can say: *g(n) grows faster than f(n)*

# *Rate of Growth, another view*

Function growth and weight of terms as a percentage of all terms as n increases for

$$f(n) = n^2 + 80n + 500$$

| n | f(n) | $n^2$ | 80n | 500 |
|---|---|---|---|---|
| **10** | 1,400 | 100 ( 7%) | 800 (57%) | 500 (36%) |
| **100** | 18,500 | 10,000 (54%) | 8,000 (43%) | 500 ( 3%) |
| **1000** | 1,0805,000 | 1,000,000 (93%) | 80,000 ( 7%) | 500 ( 0%) |
| **10000** | 100,800,500 | 100,000,000 (99%) | 800,000 ( 1%) | 500 ( 0%) |

Conclusion: consider highest order term with the coefficient dropped, also drop all lower order terms

# *Definition*

- The *asymptotic growth* of an algorithm
  - describes the relative growth of an algorithm as n gets very large
  - With speed and memory increases doubling every two years, the asymptotic efficiency *where n is very large* is the thing to consider
  - There are many sorting algorithm that are "on the order of" $n^2$ (there are roughly n×n instructions executed)
  - Other algorithms are "on the order of" $n \times \log_2 n$
    - and this is a huge difference when n is very large

# *Constant Function*

- Some functions don't grow with n
  - If the sorting program initializes a few variables first, the time required does not change when n increases
  - These statements run in *constant time*
    - e.g. construct an empty List with capacity 20
  - The amount of time can be described as a constant function f(n) = k, where k is a constant
  - it takes ~0.0003 seconds no matter the size of n

# *Big O*

- Linear search is "on the order of n", which can be written as O(n) to describe the upper bound on the number of operations
- This is called *big O* notation
- Orders of magnitude:

  O(1) *constant (the size of n has no effect)*

  O(n) *linear*

  O(log n) *logarithmic*

  O(n log n) *no other way to say it, John K's License plate*

  $O(n^2)$ *quadratic*

  $O(n^3)$ *cubic*

  $O(2^n)$ *exponential*

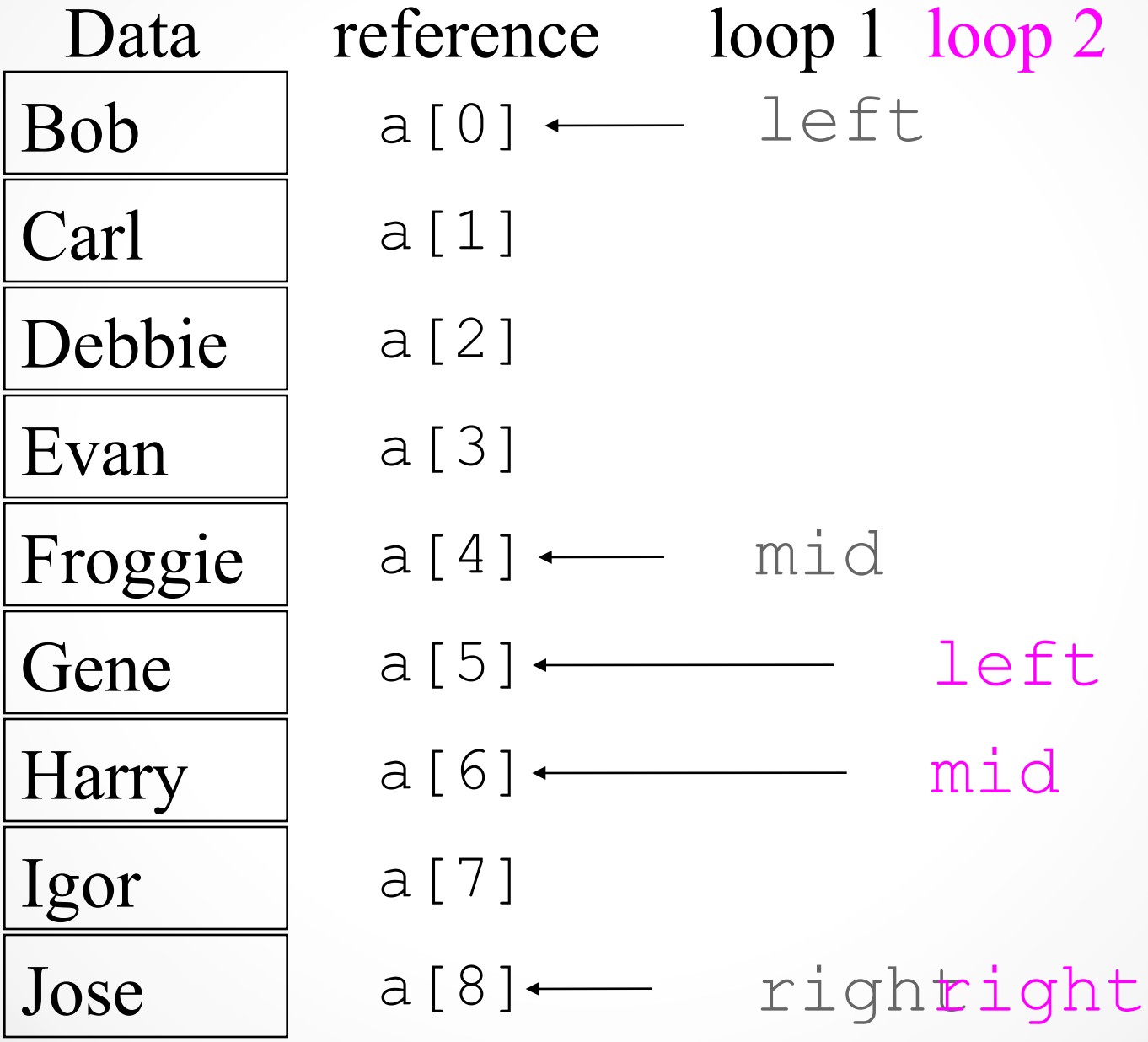
# *Binary Search*

- We'll see that binary search can be a more efficient algorithm for searching

```
If the element in the middle is the target
        report target was found and the search is done
if the key is smaller
        search the array to the left
Otherwise
        search the array to the right
```

- This process repeats until the target is found or there is nothing left to search
- Each comparison narrows search by half

*Binary Search Harry*



# *How fast is Binary Search?*

- Best case: 1
- Worst case: when target is not in the array
- At each pass, the "live" portion of the array is narrowed to half the previous size.
- The series proceeds like this:
  - n , n/2, n/4, n/8, ...
- Each term in the series represents one comparison How long does it take to get to 1?
  - This will be the number of comparisons

# *Binary Search (con.)*

- Could start at 1 and double until we get to n

  $1, 2, 4, 8, 16, \ldots, k >= n$ or

  $2^0, 2^1, 2^2, 2^3, 2^4, \ldots, 2^c >= n$

- The length of this series is c+1
- The question is

  2 to what power *c* is greater than or equal to n?

  - if n is 8, *c* is 3
  - if n is 1024, *c* is 10
  - if n is 16,777,216, *c* is 24

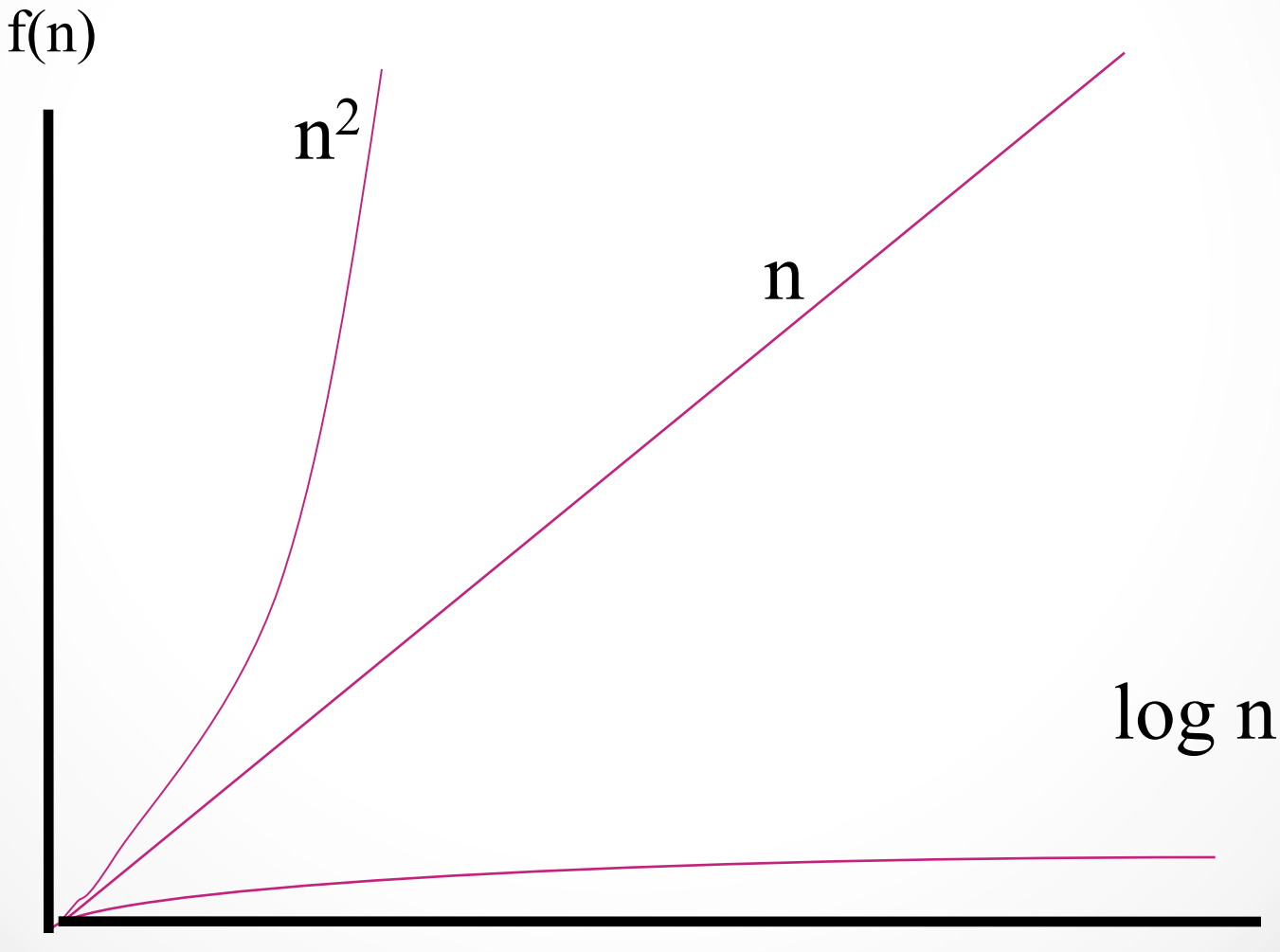
- Binary search runs O(log n) *logarithmic*

# *Comparing O(n) to O(log n)*

Rates of growth and logarithmic functions

| Power of 2 | n | $\log_2 n$ |
|---|---|---|
| $2^4$ | 16 | 4 |
| $2^8$ | 128 | 8 |
| $2^{12}$ | 4,096 | 12 |
| $2^{24}$ | 16,777,216 | 24 |

# *Graph Illustrating Relative Growth n, log n, $n^2$*



# *Other logarithm examples*

- The guessing game:
  - Guess a number from 1 to 100
    - try the middle, you could be right
    - if it is too high
      - check near middle of 1..49
    - if it is too low
      - check near middle of 51..100
  - Should find the answer in a maximum of 7 tries
    - If 1..250, a maximum of $2^c >= 250$, c == 8
    - If 1..500, a maximum of $2^c >= 500$, c == 9
    - If 1..1000, a maximum of $2^c >= 1000$, c == 10

# *Logarithmic Explosion*

- Assuming an infinitely large piece of paper that can be cut in half, layered, and cut in half again as often as you wish.
  - How many times do you need to cut and layer until paper thickness reaches the moon?
  - Assumptions
    - paper is 0.002 inches thick
    - distance to moon is 240,000 miles
      - 240,000 * 5,280 feet per mile * 12 inches per foot = 152,060,000,000 inches to the moon

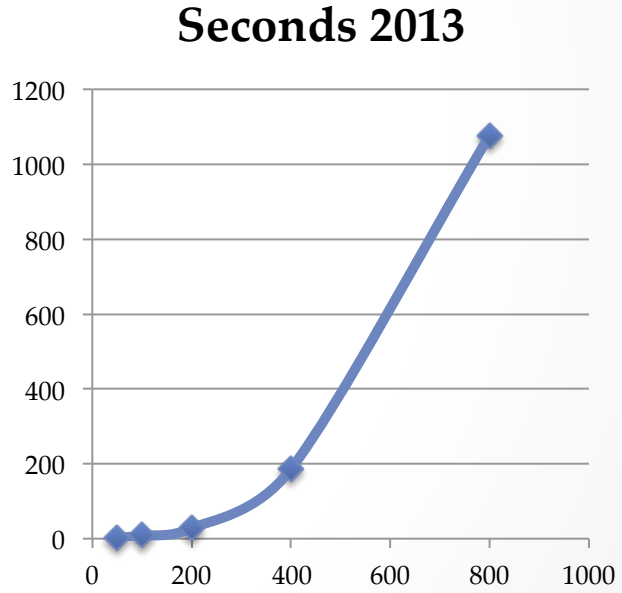
# *Examples of Logarithmic Explosion*

- The number of bits required to store a binary number is logarithmic *add 1 bit to get much larger ints*
  - 8 bits stored 256 values $\log_2 256 = 8$
  - log 2,147,483,648 = 31
- The inventor of chess asked the Emperor to be paid like this:
  - 1 grain of rice on the first square, 2 on the next, double grains on each successive square $2^{63}$

# *Compare Sequential and Binary Search*

- Output from CompareSearches.java (1995)

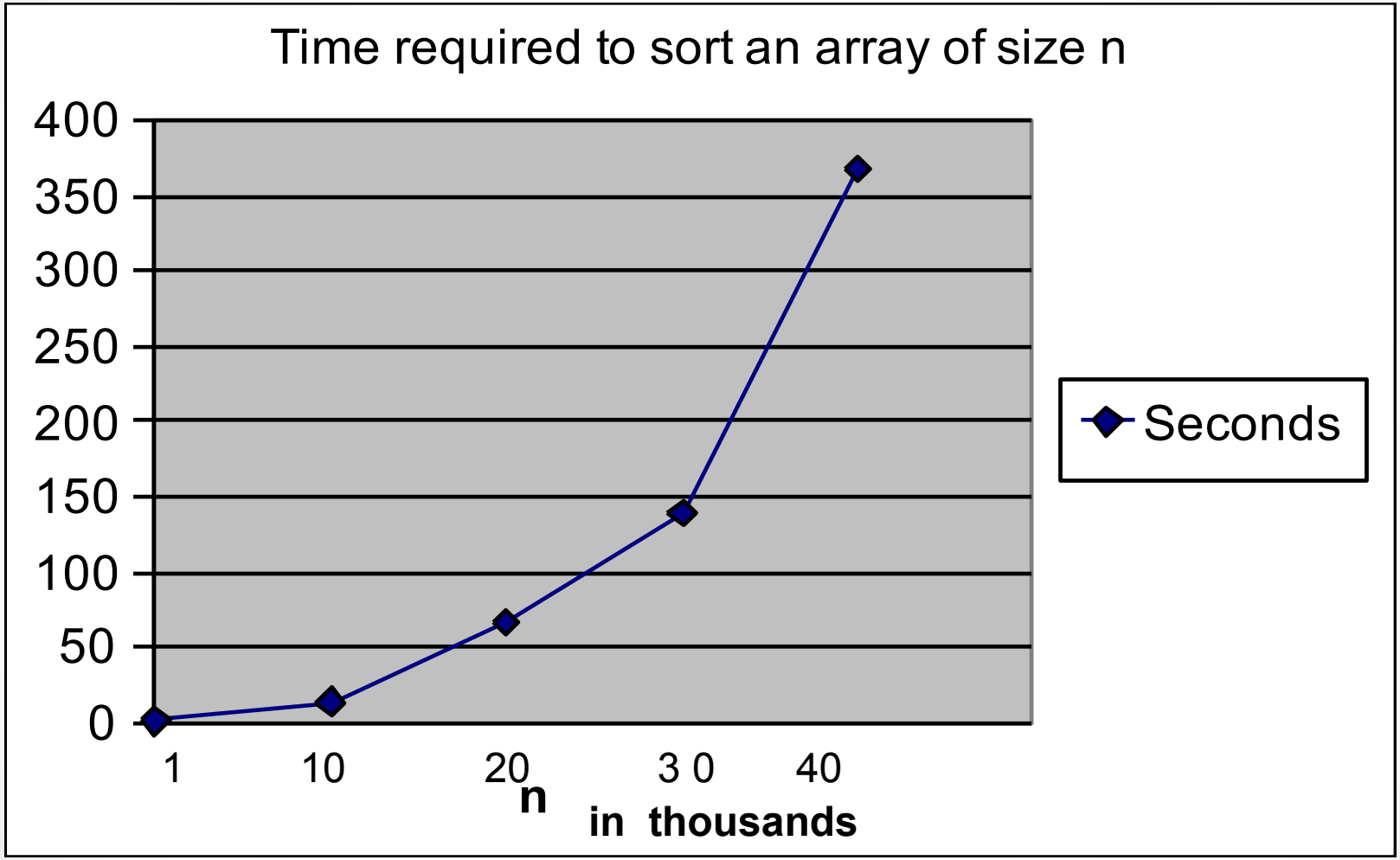
```
Search for 20000 objects

Binary Search
#Comparisons: 267248
     Average: 13
    Run time: 20ms

Sequential Search
#Comparisons: 200010000
     Average: 10000
    Run time: 9930ms

Difference in comparisons : 199742752
Difference in milliseconds: 9910
```



# *O($n^2$) quadratic*

- O($n^2$) reads on the order of n squared or quadratic
- When n is small, rates of growth don't matter
- Quadratic algorithms are greatly affected by increases in n
  - Consider the selection sort algorithm
    - Find the largest, n-1 times

# Actual observed data for $O(n^2)$ sort



# *Two $O(n^2)$ algorithms*

- Many known sorting algorithms are $O(n^2)$
- Given n points, find the pair that are closest

```
Compare p1 with p2, p3, p4, p5   (4 comparisons)
Compare p2 with p3, p4, p5       (3 comparisons)
Compare p3 with p4, p5           (2 comparisons)
Compare p4 with p5               (1 comparisons)
```

  - When n is 5, make 10 comparisons
  - In general, #comparisons is

    $n(n-1) / 2 == n^2/2 - n/2$

  - highest order term is $n^2$, drop ½ and runtime is $O(n^2)$

# $O(n^3)$ algorithms

- Matrix Multiplication (naïve):

```
for(int i = 0; i < m.length; i++) {
  for(int j = 0; j < m2.length - 1; j++) {
    for(int k = 0; k < m2.length; k++){
      m[i][j] += m[i][k] * m2[k][j];
    }
  }
}
```

# *Big O and Style Guidelines*

- Big O is similar to saying the runtime is less than or equal to Big O notation.
  - O(f) is an upper bound
- Don't use constants or lower-order terms
  - These are no-nos for now (you will use coefficients in C Sc 345)

  $O(n^2 + n)$ should be written $O(n^2)$

  O(5500n) should be written O(n)

  O(2.5n) should be written O(n)

# *Properties of Big-O*

Summarizing two main properties

If $f(n)$ is a sum of several terms, the one with the largest growth rate is kept, and all others omitted

If $f(n)$ is a product of several factors, any constants (terms in the product that do not depend on $n$) are omitted – which means you can drop coefficients

# *Properties of Big-O*

We can drop coefficient

Example:

$f(n) = 100*n$

then $f(n)$ is $O(n)$

# *Summation of same Orders*

The property is useful when an algorithm contains several loops of the same order

Example:

f(n) is O(n)

f2(n) is O(n)

then f(n) + f2(n) is O(n) + O(n), which is O(n)

# *Summation of different Orders*

This property works because we are only concerned with the term of highest growth rate

Example:

$f_1(n)$ is $O(n^2)$

$f_2(n)$ is $O(n)$

so $f_1(n) + f_2(n) = n^2 + n$ is $O(n^2)$

# *Product*

This property is useful for analyzing segments of an algorithm with nested loops

Example:

$f_1(n)$ is $O(n^2)$

$f_2(n)$ is $O(n)$

then $f_1(n) \times f_2(n)$ is $O(n^2) \times O(n)$, which is $O(n^3)$

# *Limitations of Big-Oh Analysis*

- Constants sometimes make a difference
  - n log n may be faster than 10000n
  - Doesn't differentiate between data cache memory, main memory, and data on a disk--there is a *huge* time difference to access disk data
    - thousands of times slower
  - Worst case doesn't happen often
    - it's an overestimate

# *Quick Analysis*

- Can be less detailed
- Running time of nested loops is
  - the product of each loop's number of iterations
- Several consecutive loops
  - the longest running loop
    - 3n is O(n) after all

# *Runtimes with for loops*

```
int n = 1000;
int[] x = new int[n];
```

- O(n)

```
for(int j = 0; j < n; j++)
  x[j] = 0;
```

- $O(n^2)$

```
int sum = 0;
for (int j = 0; j < n; j++)
  for (int k = 0; k < n; k++)
     sum += j * k;
```

# *Run times with for loops*

- $O(n^3)$

```
for (int j = 0; j < n; j++)
  for (int k = 0; k < n; k++)
    for (int l = 0; l < n; l++)
      sum += j * k * l;
```

- $O(n)$

```
for (int j = 0; j < n; j++)
  sum++;
for (int j = 0; j < n; j++)
  sum--;
```

- $O(\log n)$

```
for (int j = 1; j < n; j = 2 * j)
  sum += j;
```

# *Analyze this*

```
public void swap(int[] a, int left, int right) {
  int temp = array[left];
  array[left] = array[right];
  array [right] = temp;
}
```

# *Analyze that*

```
for (int j = 0; j < n; j++)
 sum += l;

for (int k = 0; k < n; k++)
 sum += l;

for (int l = 0; l < n; l++)
  sum += l;
```

# *Analyze that*

```
for (int j = 0; j < n; j++)
  for (int k = 0; k < n; k++)
    sum += k + l;

for (int l = 0; l < n; l++)
  sum += l;
```

# *Analyze this*

```
for (int top = 0; top < n - 1; top++) {
  int smallestIndex = top;
  for (int index = top; index < n; index++)  {
    if(a[index] < a[smallestIndex])
      smallestIndex = index;
  }
  // Swap smallest to the top index
  swap(a, top, smallestIndex);
}
```