Algorithms are great:

What about the mathematics that underpins them?

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A description of some of the essential mathematics that underpins the use of algorithms in multiplication and division, examined through a series of learning pathways.

Our previous article, *Algorithms are great: Understanding them is even better*, presented evidence from students and suggested that some were attempting to use written algorithms without having an adequate level of understanding of what they were doing and why. This article will describe some of the essential mathematics that underpins the use of algorithms through a series of learning pathways. The graphic below depicts the mathematical ideas and concepts that underpin the learning of algorithms for multiplication and division. The understanding and use of algorithms is informed by two important ideas—grid multiplication and extended multiplication facts. The graphic combines a number of learning pathways that lead to those two ideas. The discussion that follows shows how each element builds part of the underpinning structure needed to understand algorithms and to use them efficiently.

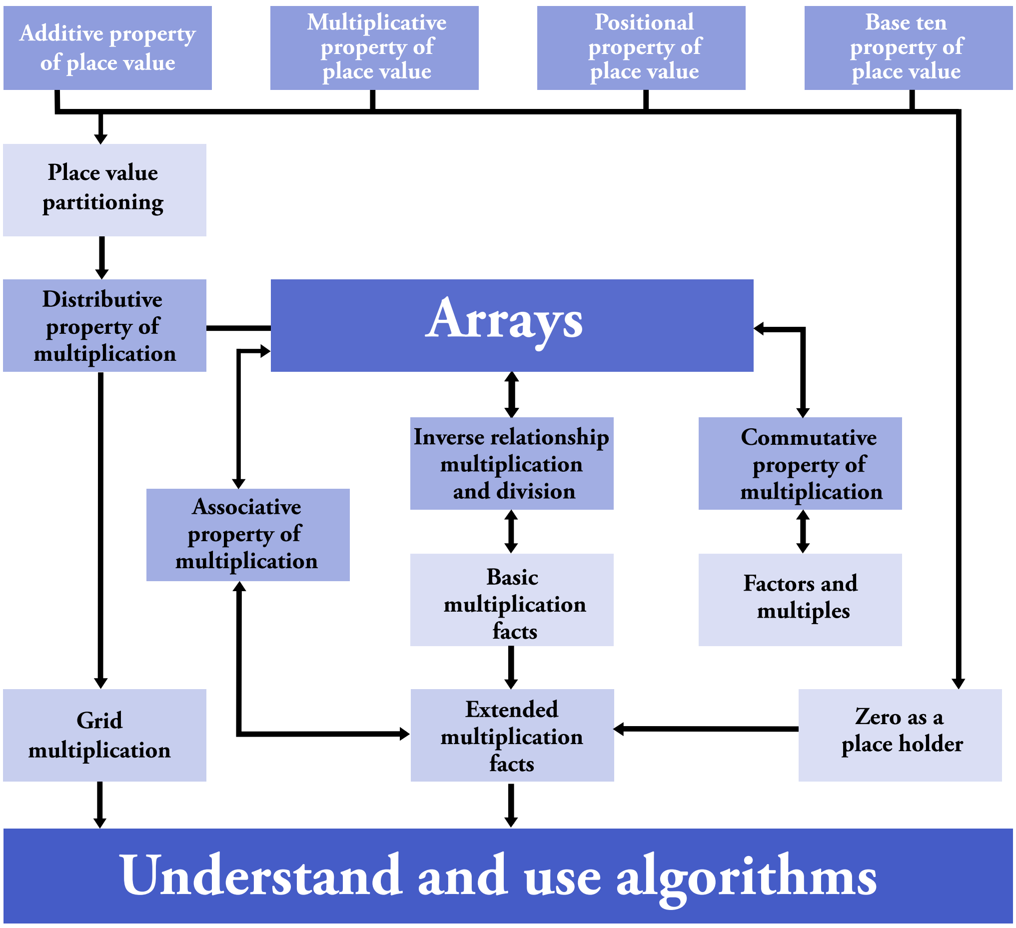


Figure 1: The mathematical ideas and concepts underpinning the learning of algorithms for multiplication and division.

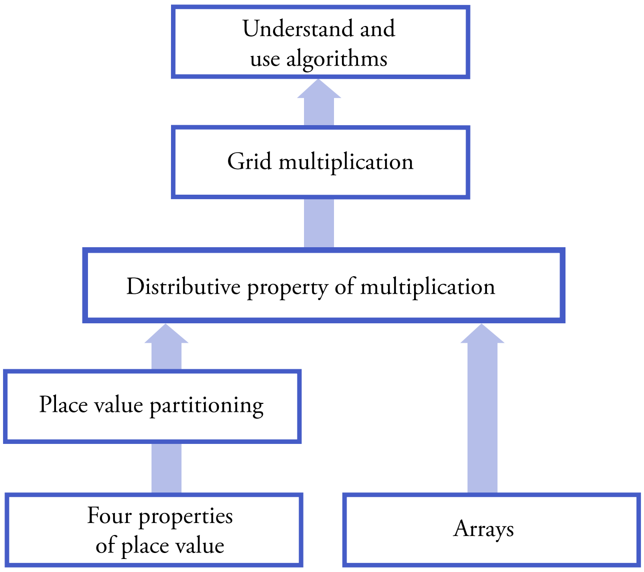


Figure 2: Learning Pathway 1.

Learning Pathway 1

Learning Pathway 1 stems from two elements of the overall representation (Figure 1) of the mathematical ideas and concepts that we believe underpin the learning of algorithms for multiplication and division. They are the four properties of place value, and the use and understanding of arrays.

Element 1: The four properties of place value

The four properties of place value (Figure 1, top row of boxes) are important conceptual building blocks of mathematics and underpin a number of aspects of the effective use of algorithms. Each of the four properties directly informs the notion of partitioning. Partitioning is the recognition that a number can be split into smaller parts—an important concept in effectively using the distributive property of multiplication. The four properties of place value are:

- **Additive property:** The quantity represented by the whole numeral is the sum of the values represented by the individual digits, e.g., 796 = 700 + 90 + 6.
- **Multiplicative property**: The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position, e.g., 796 = (7 × 100) + (9 × 10) + (6 × 1).
- **Positional property:** The quantities represented by the individual digits are determined by the positions that they hold in the whole numeral, e.g., in 796, the 9 is worth 9 tens or 90.
- **Base ten property:** The values of the positions increase in powers of ten from right to left, e.g., in 666, the six in the right hand place is worth 6 ones, the 6 to its left is worth 6 tens (or ten times as much), and the six in the next place is worth ten times as much again, or six times ten times ten (Ross, 2002).

The base ten property also informs the development of extended number facts which are applied when students use a two-line algorithm. For example, knowing that 60 times seven, will give an answer ten times bigger than six times seven, because 60 is ten times bigger than six, is very important. A conceptually deep understanding of this principle is essential and replaces the over simplistic and fundamentally incorrect explanation given around the increasing and decreasing place value in terms of 'adding a zero'.

Element 2: Arrays

Arrays are of critical importance and should be used with students in all primary years to develop and consolidate understanding of the properties of multiplication and the multiplicative situation (Hurst, 2014; Siemon et al., 2016). In Figure 1, the properties of multiplication are shown. One property of multiplication that can be powerfully demonstrated with the array, is the distributive property.

Distributive property of multiplication

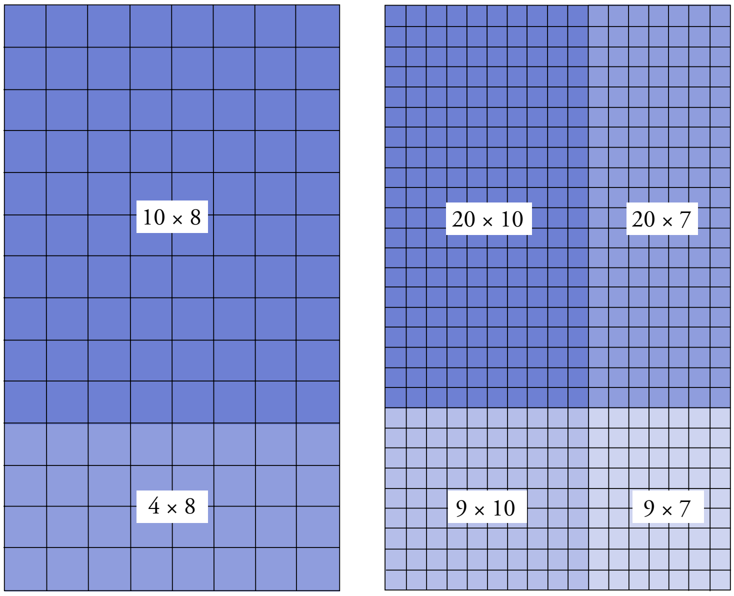
The distributive property directly impacts the setting out of the standard algorithm and it can also be developed through the use of the array. Kinzer and Stafford (2013) refer to the distributive property as the core of multiplication. The examples shown below are for 14 × 8 and 29 × 17.

Learning Pathway 2

Learning Pathway 2 is the suite of further understandings that are inherent in the use of the array, and are instrumental in developing extended number facts, which are, in turn, integral to developing an understanding of algorithms and the capacity to effectively use them. Whilst these understandings are inherent in the representation of the array, the array alone will not 'teach' these understandings. A developmental sequence of activities and sound mathematical pedagogies are required.

Basic multiplication facts

Knowing basic multiplication facts is important, not only from a perspective of the learning and teaching of 'school' mathematics, but as a handy tool in the wider world. However, knowing basic multiplication facts, that is being able to recall multiplication facts from 0 × 0 to 10 × 10, in itself does not show multiplicative thinking. Basic multiplication facts are an integral part of multiplicative thinking and the question should never be: "Should they be taught and learned?" They should. The question should be, "How are they to be taught and learned?" There are far more conceptual ways of developing multiplication facts than purely the "drill and kill" approach and a thorough undertanding of the commutative property is one way we can help children learn these facts.



The array and distributive property can be developed into a grid method of multiplication which directly informs the standard algorithm.

$10 \times 8 = 80$
$4 \times 8 = 32$
$(10 \times 8) + (4 \times 8) = 112$

$20 \times 10 = 200$
$20 \times 7 = 140$
$9 \times 10 = 90$
$9 \times 7 = 63$
$(20 \times 10) + (20 \times 7) + (9 \times 10) + (9 \times 7) = 493$
$29 \times 17 = 493$

Figures 3 & 4. Multiplicative arrays.

	2 9	
×	1 7	
	6 3	7 × 9
1	4 0	7 × 20
	9 0	10 × 9
2	0 0	10 × 20
4	9 3	

	2 9	
×	1 7	
2	0 3	7 × 29
2	9 0	10 × 29
4	9 3	

A grid method can then be readily developed into the standard written algorithm for multiplication, in either its extended or shorter format.

Figures 5 & 6. Grid and standard written algorithms for multiplication.

What may not be obvious when teaching basic multiplication facts is that children need to be systematically guided to an understanding that what they are dealing with is actually not a single basic multiplication fact, but a starting point to consider the multiplicative situation (Hurst, 2014). The multiplicative situation is seeing the inverse relationships and the commutative properties which are inherent in dealing with multiplication, and therefore division. The array is proven to be important in the development of a conceptual understanding of the multiplicative situation (Hurst, 2014). Instead of considering multiplication and division separately and as two different concepts, it is suggested that the two must be taught simultaneously, as they are two ways of representing the same situation, that is, they involve the same components. For multiplication this looks like:

Number of groups × Number in each group
= Total or product

For division this looks like:

Total ÷Number of groups = Number in each group;
or
Total ÷ Number in each group = Number of groups

It is also important to embed correct mathematical language in the everyday teaching of mathematics and for children to understand terms such as factor and multiple.

The multiplicative situation can be conceptualised in the following ways:

- If the number of groups, and the number in each group, are known, we multiply to find the total or product.
- If either the number of groups or the number in each group, and the total/product are known, we divide to find the unknown.
- If both factors are known, we multiply to find the multiple.

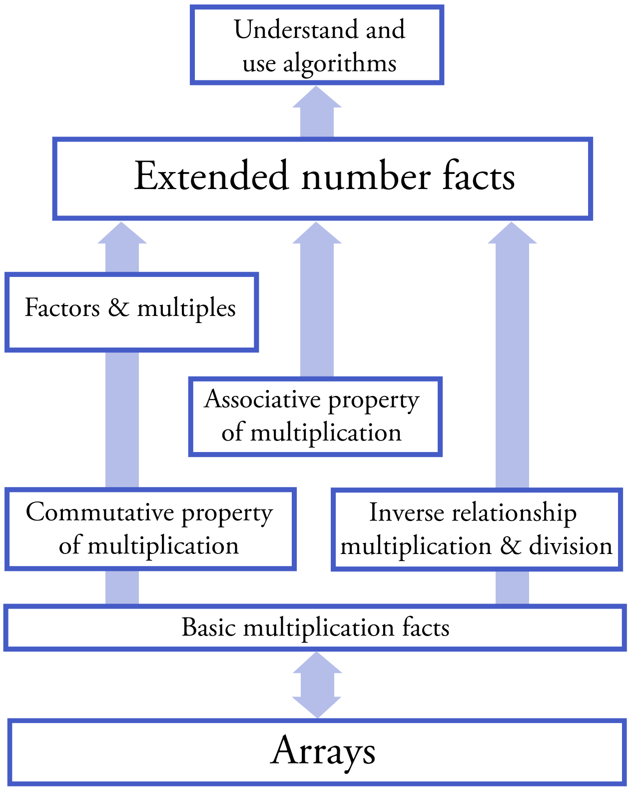
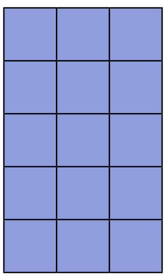


Figure 7. Learning pathway 2.

- If the multiple and one factor are known, we divide to find the unknown.

If the multiplicative situation is considered in these ways, and with the array as a representation, it is relatively easy for students to understand the inverse relationship between multiplication and division in terms of factors and multiples. Consideration of the multiplicative situation also facilitates the learning of number facts, as for each multiplication fact, there are actually six facts that can be generated. Jacob and Mulligan (2014, p. 39) noted that "Arrays provide a vehicle for teachers to focus students' attention on the nature of the quantities involved, the associated language, the relationship between multiplication and division, and commutativity". For example:

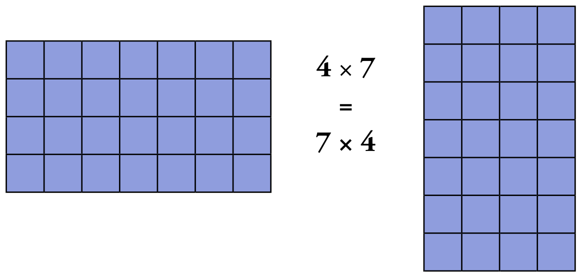


- This array depicts 5 × 3 = 15. It can be rotated to show 3 × 5 = 15 (commutative)
- This array can be divided into 5 rows of 3 so 15 ÷ 5 = 3
- This rotated array can be divided into 3 rows of 5 so 15 ÷ 3 = 5
- Each row of 3 is a fifth of the array so $\frac{1}{5} \times 15 = 3$
- In this rotated array, each row of 5 is a third of the array so $\frac{1}{3} \times 15 = 5$

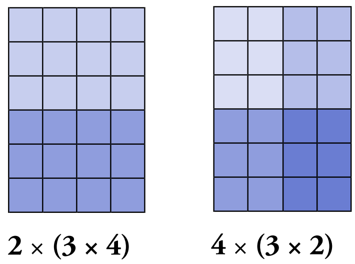
Commutative property of multiplication

Students often explain the commutative property of multiplication in terms of 'switching the numbers around' rather than through the language of factors. That is, the product will remain the same even if the order of the factors is changed. Students need to understand that four groups of seven (4 × 7) is conceptually different to seven groups of four (7 × 4) but that the order of factors can be changed and the product will remain the same. This is easily shown by rotating an array or by overlaying four strips of seven and seven strips of four. Although none of the parts (factors) have 'changed', the picture is different.

The commutative property is important as it reduces the number of multiplication facts to be memorised. Students can also use it to understand that multiplying numbers can be done in any order. For example, 8 × 78 can also be done as 78 × 8. The factors are the same so the product will be the same. Being able to manipulate numbers with this level of flexibility is both important and useful.



Associative property of multiplication



The associative property of multiplication underpins an important aspect of algorithm use, and can also be developed through the use of the array. It becomes important in helping students to understand what is happening when they multiply two-digit numbers by two-digit numbers, and why the procedure works. When multiplying 28 by 37 using the standard algorithm, the second line of multiplication involves multiplying 20 by 30. Instead of teaching children to 'multiply two by three and add two zeros', it is better to take a conceptual approach and use the associative property, in combination with the commutative property and base ten property of place value, to help them understand why 20 × 30 = 600.

20 × 30

$$(2 \times 10) \times (3 \times 10) = 2 \times 10 \times 3 \times 10$$
$$2 \times 3 \times 10 \times 10$$
$$6 \times 10 \times 10$$
$$60 \times 10 = 600$$

Learning Pathway 3

Extended number facts

When students understand the base ten property of place value and the role of zero as a place holder, they are in a position to build their knowledge of basic multiplication facts into extended facts. This is an understanding which is essential if they are to properly know how algorithms work. Once they know that 4 × 7 = 28, and understand that if one of the factors is increased by a power of ten (e.g. 40 × 7), application of an understanding of the base ten property will help them know that the product must be ten times bigger, hence 40 × 7 = 280. Students are better placed to understand algorithms if they learn this conceptually rather than think of it in terms of 'adding a zero'. That is, they will know why a zero is placed in the second line of the algorithm shown earlier, not just place it there because 'that's what you do!'

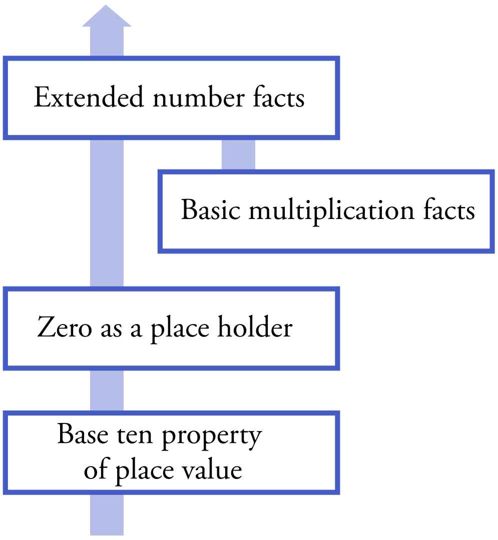


Figure 8. Learning pathway 3.

Connected teaching

As noted earlier, multiplication and division are better taught simultaneously to reinforce the notion of the inverse relationship that exists between them. It follows that the algorithm for both should be developed simultaneously as well, so that students can see how the two processes in fact represent the same situation in different ways. Figure 9 shows how links, even at this abstract stage, can be made between the two algorithms.

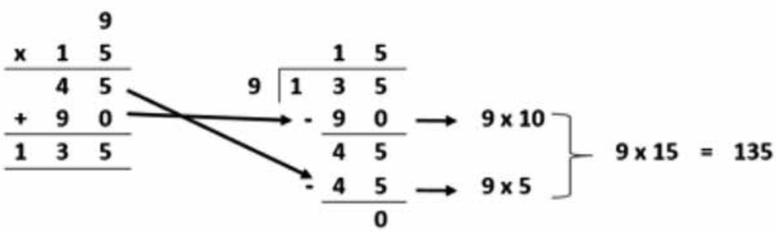


Figure 9. Links between multiplication and division.

Although this connection is not necessarily a connection one would want to record in a written format (as in Figure 9), in working with students, it is certainly one that should feature in discussions. Many people are of the opinion that the thinking in a division algorithm is somehow difficult. However, once the conceptual understanding of what the act of division does is understood, the mechanics are actually quite achievable. In completing the steps of a division algorithm we actually do not "do" division, we engage in multiplication and subtraction. These discussions should highlight this use of the inverse operation, which may to some degree alleviate the concerns of some students attempting division.

Conclusion

It may appear to some that we are sceptical of algorithms. This is certainly not the case. The correct algorithm, used at the appropriate time, that has understanding at its core, is truly wonderful. It is a tool that organises thinking in a succinct and often elegant manner. We just believe that the benefits of teaching the algorithm properly, by developing the many understandings that sit inside of it, bears many fruits. To develop this understanding with our students, we as teachers sometimes need to consider the many parts that come together to be able to develop a conceptual understanding of what is sometimes 'hidden' in an algorithm. We hope that this article has reminded you of these parts.

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