All human knowledge thus begins with intuitions, proceeds thence to concepts and ends with ideas.

Kant, *Critique of Pure Reason,* "Elements of Transcendentalism," *Second Part*, II.

INTRODUCTION

Geometry, like arithmetic, requires only a few and simple principles for its logical development. These principles are called the **axioms** of geometry. The establishment of the axioms of geometry and the investigation of their relationships is a problem which has been treated in many excellent works of the mathematical literature since the time of Euclid. This problem is equivalent to the logical analysis of our perception of space.

This present investigation is a new attempt to establish for geometry a **complete**, and **as simple as possible**, set of axioms and to deduce from them the most important geometric theorems in such a way that the meaning of the various groups of axioms, as well as the significance of the conclusions that can be drawn from the individual axioms, come to light.

CHAPTER I

THE FIVE GROUPS OF AXIOMS

§ 1. The Elements of Geometry and the Five Groups of Axioms

DEFINITION. Consider three distinct sets of objects. Let the objects of the **first** set be called *points* and be denoted by *A, B, C,* . . . ; let the objects of the **second** set be called *lines* and be denoted by *a, b, c,* . . .; let the objects of the **third** set be called *planes* and be denoted by α, β, γ. . . . The points are also called the *elements of line geometry*; the points and the lines are called the *elements of plane geometry*; and the points, the lines and the planes are called the *elements of space geometry* or the *elements of space*.

The points, lines and planes are considered to have certain mutual relations and these relations are denoted by words like **"lie," "between," "congruent."** The precise and mathematically complete description of these relations follows from the **axioms of geometry**.

The axioms of geometry can be divided into five groups. Each of these groups expresses certain related facts basic to our intuition. These groups of axioms will be named as follows:

I,	1 - 8	Axioms of *Incidence,*
II,	1 - 4	Axioms of *Order,*
III,	1 - 5	Axioms of *Congruence,*
IV,		Axiom of *Parallels,*
V,	1 - 2	Axioms of *Continuity.*

§ 2. Axiom Group I: Axioms of Incidence

The axioms of this group establish an *incidence* relation among the above-introduced objects–points, lines and planes, and read as follows:

I, 1. *For every two points A, B there exists a line a that contains each of the points A, B.*

I, 2. *For every two points A, B there exits no more than one line that contains each of the points A, B.*

Here as well as in what follows, two, three, . . . points or lines, planes are always to be understood as distinct points or lines, planes.

Instead of **"contains"** other expressions will also be used, e.g., *a* **passes through** *A* **and through** *B, a* **joins** *A* **and** *B* **or joins** *A* **with** *B, A*

lies on a, A **is a point of** a, **there exists a point** A **on** a, etc. If A lies on the line a as well as on another line b the expressions used will be '**The lines** a **and** b **intersect at** A, **have the point** A **in common**', etc.

I, 3. *There exist at least two points on a line. There exist at least three points that do not lie on a line.*

I, 4. *For any three points A, B, C that do not lie on the same line there exits a plane* α *that contains each of the points A, B, C. For every plane there exists a point which it contains.*

The expressions 'A **lies in** α; A **is a point of** α,' etc. will also be used.

I, 5. *For any three points A, B, C that do not lie on one and the same line there exists no more than one plane that contains each of the three points A, B, C.*

I, 6. *If two points A, B of a line a lie in a plane* α *then every point of a lies in the plane* α.

In this case it is said that **the line** a **lies in the plane** α, etc.

I, 7. *If two planes* α, β *have a point A in common then they have at least one more point B in common.*

I, 8. *There exist at least four points which do not lie in a plane.*

Axiom I, 7 expresses the fact that space has no more than three dimensions, whereas Axiom I, 8 expresses the fact that space has no less than three dimensions.

Axioms I, 1-3 can be called the *plane axioms of group* I in distinction to Axioms I, 4-8 which will be called the *space axioms of group* I.

Of the theorems that ensue from Axioms I, 1-8 only the following two are mentioned:

THEOREM 1. Two lines in a plane either have one point in common or none at all. Two planes have no point in common, or have one line and otherwise no other point in common. A plane and a line that does not lie in it either have one point in common or none at all.

THEOREM 2. Through a line and a point that does not lie on it, as well as through two distinct lines with one point in common, there always exists one and only one plane.

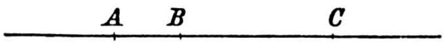
§ 3. Axiom Group II: Axioms of Order[1]

[1] These axioms were first studied in detail by M. Pasch in his *Vorlesungen über neuere Geometrie* (Leipzig, 1882). In particular, Axiom II, 4 is essentially due to him.

The axioms of this group define the concept of "**between**" and by means of this concept the *ordering* of points on a line, in a plane, and in space is made possible.

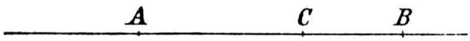
DEFINITION. The points of a line stand in a certain relation to each other and for its description the word *"between"* will be specifically used.

II, 1. *If a point B lies between a point A and a point C then the points A, B, C*



are three distinct points of a line, and B then also lies between C and A.

II, 2. *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*

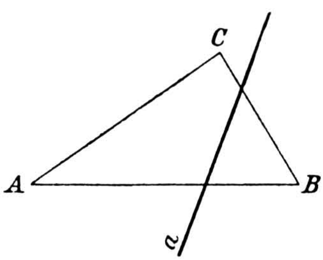


II, 3. *Of any three points on a line there exists no more than one that lies between the other two.*

Besides these *line axioms of order* a *plane axiom of order* is still needed.

DEFINITION. Consider two points, A and B, on a line a. The set of the two points A and B is called a *segment*, and will be denoted by AB or by BA. The points between A and B are called the points of the segment AB, or are also said to lie *inside* the segment AB. The points A, B are called the *end points* of the segment AB. All other points of the line a are said to lie *outside* the segment AB.

II, 4. Let A, B, C be three points that do not lie on a line and let a be a line in the plane ABC which does not meet any of the points A, B, C. If the line a passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.



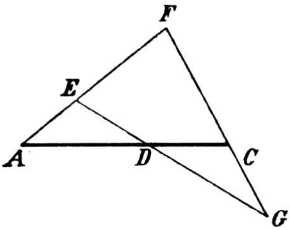
Expressed intuitively, if a line enters the interior of a triangle, it also leaves it. The fact that both segments AC and BC are not intersected by the line a can be proved. (See Supplement I, 1.)

§ 4. Consequences of the Axioms of Incidence and Order

The subsequent theorems follow from Axioms I and II:

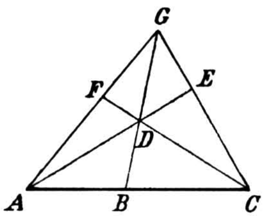
THEOREM 3. For two points A and C there always exists at least one point D on the line AC that lies between A and C.

PROOF. By Axiom I, 3 there exists a point E outside the line AC, and by Axiom II, 2 there exists on AE a point F such that E is a point of the segment AF. By the same axiom and by Axiom II, 3 there exists on FC a point G, that does not lie on the segment FC. By Axiom II, 4 the line EG must then intersect the segment AC at a point D.



THEOREM 4. Of any three points A, B, C on a line there always is one that lies between the other two.

PROOF.[1] Let A not lie between B and C and let also C not lie between A and B. Join a point D that does not lie on the line AC with B and choose by Axiom II, 2 a point G on the connecting line such that D lies between B and G. By an application of Axiom II, 4 to the triangle BCG and to the line AD it follows that the lines AD and CG intersect at a point E that lies between C and G. In the same way, it follows that the lines CD and AG meet at a point F that lies between A and G.



If Axiom II, 4 is applied now to the triangle AEG and to the line CF it becomes evident that D lies between A and E, and by an application of the same axiom to the triangle AEC and to the line BG one realizes that B lies between A and C.

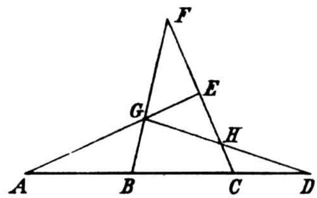
THEOREM 5. Given any four points on a line, it is always possible to label them A, B, C, D in such a way that the point labeled B lies between A and C and also between A and D, and furthermore, that the point labeled C lies between A and D and also between B and D.[2]

[1] This proof is due to A. Wald.

[2] This theorem, which had been given in the First Edition as an axiom, was recognized by E. H. Moore, *Trans. Am. Math. Soc.*, 1902, to be a consequence of the plane axioms of incidence and order formulated above. Compare also the works subsequent to this by Veblen, *Trans. Am. Math. Soc.*, 1904, and Schweitzer, *American Journal*, 1909. A thorough investigation of independent sets of line axioms of order that postulate ordering on straight lines is found in E. v. Huntington, "A New Set of Postulates for Betweenness with Proof of Complete Independence," *Trans. Am. Math. Soc.*, 1924. Compare also *Trans. Am. Math. Soc.*, 1917.

PROOF. Let A, B, C, D be four points on a line g. The following will now be shown:

1. If B lies on the segment AC and C lies on the segment BD then the points B and C also lie on the segment AD. By Axioms I, 3 and II, 2 choose a point E that does not lie on g, on a point F such that E lies between C and F. By repeated applications of Axioms II, 3 and II, 4 it follows that the segments AE and BF meet at a point G, and moreover, that the line CF meets the segment GD at a point H. Since H thus lies on the segment GD and since, however, by Axiom II, 3, E does not lie on the segment AG, the line EH, by Axiom II, 4, meets the segment AD, i.e., C lies on the segment AD. In exactly the same way one shows analogously that B also lies on this segment.



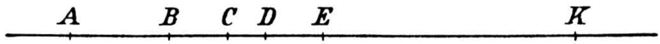
2. If B lies on the segment AC and C lies on the segment AD then C also lies on the segment BD and B also lies on the segment AD. Choose one point G that does not lie on g, and another point F such that G lies on the segment BF. By Axioms I, 2 and II, 3 the line CF meets neither the segment AB nor the segment BG and hence, by Axiom II, 4 again, does not meet the segment AG. But since C lies on the segment AD, the straight line CF meets then the segment GD at a point H. Now by Axiom II, 3 and II, 4 again the line FH meets the segment BD. Hence C lies on the segment BD. The rest of Assertion 2 thus follows from 1.

Now let any four points on a line be given. Take three of the points and label Q the one which by Theorem 4 and Axiom II, 3 lies between the other two and label the other two P and R. Finally label S the last of the four points. By Axiom II, 3 and Theorem 4 again it follows then that the following five distinct possibilities for the position of S exist:

R lies between P and S,
or P lies between R and S,
or S lies between P and R simultaneously when Q lies between P and S,
or S lies between P and Q,
or P lies between Q and S.

The first four possibilities satisfy the hypotheses of 2 and the last one satisfies those of 1. Theorem 5 is thus proved.

THEOREM 6 (generalization of Theorem 5). Given any finite number of points on a line it is always possible to label them A, B, C, *D, E, . . ., K* in such a way that the point labeled *B* lies between *A* and *C, D, E, . . ., K*, the point labeled *C* lies between *A, B*

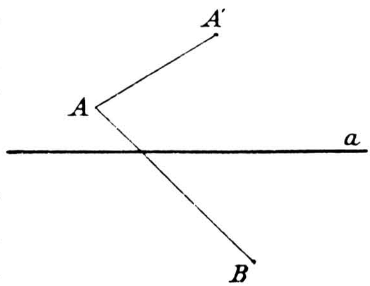


and *D, E, . . ., K, D* lies between *A, B, C* and *E, . . ., K*, etc. Besides this order of labeling there is only the reverse one that has the same property.

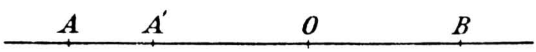
THEOREM 7. Between any two points on a line there exists an infinite number of points.

THEOREM 8. Every line *a* that lies in a plane α separates the points which are not on the plane α into two regions with the following property: Every point *A* of one region determines with every point *B* of the other region a segment *AB* on which there lies a point of the line *a*. However any two points *A* and *A′* of one and the same region determine a segment *AA′* that contains no point of *a*.

DEFINITION. The points *A,A′* are said to lie *in the plane* α *on one and the same side of the line a* and the points *A, B* are said to lie *in the plane* α *on different sides of the line a*.



DEFINITION. Let *A, A′, O, B* be four points of the line *a* such that *O* lies between *A* and *B* but not between *A* and *A′*. The points *A, A′* are then said to lie *on the line a on one and the same side of the point O* and the points *A, B* are said to lie *on the line a on different sides of the point O*. The totality of points of the line *a* that



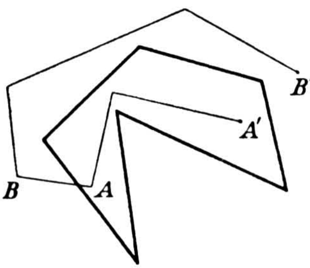
lie on one and the same side of *O* is called a *ray* emanating from *O*. Thus every point of a line partitions it into two rays.

DEFINITION. A set of segments *AB, BC, CD, . . . , KL* is called a *polygonal segment* that connects the points *A* and *L*. Such a polygonal segment will also be briefly denoted by *ABCD . . . KL*. The points inside the segments *AB, BC, CD, . . . , KL* as well as the points *A, B, C, D, . . . , K, L* are collectively called the *points of the polygonal segment*. If the points *A, B, C, D, . . . , K, L* all lie in a plane and the point *A* coincides with the point *L* then the polygonal segment is called a *polygon* and is denoted as the polygon *ABCD . . . K*. The segments *AB, BC, CD, . . . , KA* are also called the *sides of the polygon*. The points $A, B, C, D, \ldots, K$ are called the *vertices of the polygon*. Polygons of 3, 4, . . . , *n vertices* are called *triangles, quadrilaterals, . . . , n-gons*.

DEFINITION. If the vertices of a polygon are all distinct, none of them falls on a side and no two of its nonadjacent sides have a point in common, the polygon is called *simple*.

With the aid of Theorem 8 the following theorems can now be obtained:* (See the bibliography at the end of Supplement I, 1.)

THEOREM 9. Every single polygon lying in a plane α separates the points of the plane α that are not on the polygonal segment of the polygon into two regions, the *interior* and the *exterior*, with the following property: If A is a point of the interior **(an inner point)** and B is a point of the exterior **(an exterior point)** then every polygonal segment that lies in α and joins A with B has at least one point in common with the polygon. On the other hand if A, A' are two points of the interior and B, B' are two points of the exterior then there exist polygonal segments in α which join A with A' and others which join B with B', none of which have any point in common with the polygon. By suitable labeling of the two regions there exist lines in α that always lie entirely in the exterior of the polygon. However, there are no lines that lie entirely in the interior of the polygon.



THEOREM 10. Every plane α separates the other points of space into two regions with the following property: Every point A of one region determines with every point B of the other region a segment AB on which there lies a point of α; whereas, two points A and A' of one and the same region always determine a segment AA' that contains no point of α.

DEFINITION. In the notation of Theorem 10 it is said that the points A, A' lie in space *on one and the same side of the plane* α and the points A, B lie in space *on different sides of the plane* α.

Theorem 10 expresses the most important facts about the ordering of the elements of **space**. These facts are thus merely consequences of

*This sentence is one of the corrections alluded to in the preface to the Ninth Edition of this book. Hilbert's original statement, which was retained in previous editions, was "With the aid of Theorem 8 one obtains without much difficulty the following theorem." (Translator's note) the axioms considered so far and thus no new **space** axiom is required in Group II.

§ 5. Axiom Group III: Axioms of Congruence

The axioms of this group define the concept of congruence and with it also that of displacement.

DEFINITION. Segments stand in a certain relation to each other and for its description the words "*congruent*" or "*equal*" will be used.

III, 1. *If A, B are two points on a line a, and A' is a point on the same or on another line a' then it is always possible to find a point B' on a given side of the line a' through A' such that the segment AB is congruent or equal to the segment $A'B'$. In symbols*

$$AB \equiv A'B'.$$

This axiom requires the **possibility of** *constructing* **segments. Its uniqueness** will be proved later.

A segment was simply defined as a set of two points A, B and was denoted by AB or BA. In the definition the order of the two points was not specified. Therefore, the formulas

$$AB \equiv A'B', \qquad AB \equiv B'A',$$
$$BA \equiv A'B', \qquad BA \equiv B'A'$$

have equal meanings.

III, 2. *If a segment $A'B'$ and a segment $A''B''$, are congruent to the same segment AB, then the segment $A'B'$ is also congruent to the segment $A''B''$, or briefly, if two segments are congruent to a third one they are congruent to each other.*

Since congruence or equality is introduced in geometry only through these axioms, it is by no means obvious that **every segment is congruent to itself**. However, this fact follows from the first two axioms on congruence if the segment AB is constructed on a ray so that it is congruent, say, to $A'B'$ and Axiom III, 2 is applied to the congruences $AB \equiv A'B'$, $AB \equiv A'B'$.

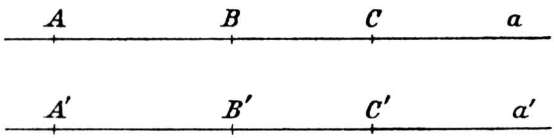
On the basis of this the *symmetry* and the *transitivity* of segment congruence can be established by an application of Axiom III, 2; i.e., the validity of the following theorems:

If $$AB \equiv A'B',$$
then $$A'B' \equiv AB;$$
if $$AB \equiv A'B'$$

and $$A'B' \equiv A''B'',$$
then $$AB \equiv A''B''.$$

Due to the symmetry of segment congruence one may use the expression "Two segments are *congruent to each other.*"

III, 3. *On the line a let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line a′ let*



$A'B'$ and $B'C'$ be two segments which except for B' also have no point in common. In that case, if

$$AB \equiv A'B' \quad \textit{and} \quad BC \equiv B'C'$$

then $$AC \equiv A'C'.$$

This axiom expresses the requirement of **additivity** of segments.

The construction of angles is dealt with precisely as the construction of segments. Besides the **possibility** of constructing angles it is by all means also necessary to require **uniqueness** axiomatically. However, transitivity and additivity can be proved.

DEFINITION. Let α be a plane and h, k any two distinct rays emanating from O in α and lying on **distinct lines.** The pair of rays h, k is called an *angle* and is denoted by $\measuredangle(h, k)$ or by $\measuredangle(k, h)$.

The rays h, k are called the *sides* of the angle and the point O is called the *vertex* of the angle.

Degenerate and obtuse angles are excluded by this definition.

Let the ray h lie on the line $\bar{h}$ and the ray k on the line $\bar{k}$. The rays h and k together with the point O partition the points of the plane into two regions. All points that lie on the same side of $\bar{k}$ as those on h, and also those that lie on the same side of $\bar{h}$ as those on k, are said to lie in the **interior** of the angle $\measuredangle(h, k)$. All other points are said to lie in the **exterior** of, or outside, this angle.

It is easy to see by Axioms I and II that both regions contain points and that a segment that connects two points **inside** the angle lies entirely in the interior. The following facts are just as easy to prove: If a point H lies on h and a point K lies on k then the segment HK lies entirely in the interior. A ray emanating from O lies either entirely inside or entirely outside the angle. A ray that lies in the interior meets the segment HK. If A is a point of one region and B is a point of the other region, then every polygonal segment that connects A and B either passes through O or has at least one point in common with h or with k. However, if A, A' are points of the same region then there always exists a polygonal segment that connects A with A' and passes neither through O nor through any point of the rays h, k.

DEFINITION. Angles stand in a certain relation to each other, and for the description of which the word "*congruent*" or "*equal*" will be used.

III, 4. *Let $\sphericalangle(h, k)$ be an angle in a plane α and a' a line in a plane α' and let a definite side of a' in α' be given. Let h' be a ray on the line a' that emanates from the point O'. Then there exists in the plane α' one and only one ray k' such that the angle $\sphericalangle(h, k)$ is congruent or equal to the angle $\sphericalangle(h', k')$ and at the same time all interior points of the angle $\sphericalangle(h', k')$ lie on the given side of a'.*
Symbolically

$$\sphericalangle (h, k) \equiv \sphericalangle (h', k').$$

Every angle is congruent to itself, i.e.,

$$\sphericalangle (h, k) \equiv \sphericalangle (h, k).$$

is always true.

One also says briefly that every angle in a given plane can be *constructed* on a given side of a given ray in a uniquely determined way.

In the definition of an angle just as little consideration will be given to its orientation as has been given to the sense of a segment. Consequently the designations $\sphericalangle (h, k)$, $\sphericalangle (k, h)$ will have the same meaning.

DEFINITION. An angle with a vertex B on one of whose sides lies a point A and on whose other side lies a point C will also be denoted by $\sphericalangle ABC$ or briefly by $\sphericalangle B$. Angles will also be denoted by small Greek letters.

III, 5. *If for two triangles*[1] *ABC and $A'B'C'$ the congruences*

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \sphericalangle BAC \equiv \sphericalangle B'A'C'$$

hold, then the congruence

$$\sphericalangle ABC \equiv \sphericalangle A'B'C'$$

is also satisfied.

[1] Here, and in what follows, the vertices of a triangle shall always be supposed not to lie on the same line.

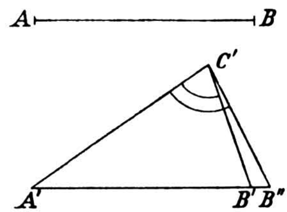
The concept of a triangle is defined on p. 9. Under the hypotheses of the axiom it follows, by a change of notation, that **both congruences**

$$\measuredangle\, ABC \equiv \measuredangle\, A'B'C' \quad \text{and} \quad \measuredangle\, ACB \equiv \measuredangle\, A'C'B'$$

are satisfied.

Axioms III, 1-3 contain statements about the congruence of segments. They may therefore be called the *line* axiom of group III. Axiom III, 4 contains statements about the congruence of angles. Axiom III, 5 relates the concepts of congruence of segments to that of angles. Axioms III, 4 and III, 5 contain statements about the elements of plane geometry and may therefore be called the *plane* axioms of group III.

The **uniqueness of segment construction** follows from the uniqueness of angle construction with the aid of Axiom III, 5.



Suppose that the segment AB is constructed in two ways on a ray emanating from A' to B' and to B''. Choosing a point C' not on the line $A'B'$ the congruences

$$A'B' \equiv A'B'', \quad A'C' \equiv A'C', \quad \measuredangle\, B'A'C' \equiv \measuredangle\, B''A'C',$$

are obtained and so by Axiom III, 5

$$\measuredangle\, A'C'B' \equiv \measuredangle\, A'C'B'',$$

in contradiction to the uniqueness of angle construction required by Axiom III, 4.

§ 6. Consequences of the Axioms of Congruence

DEFINITION. Two angles that have a vertex and one side in common and whose separate sides form a line are called *supplementary angles*. Two angles with a common vertex whose sides form two lines are called *vertical angles*. An angle that is congruent to one of its supplementary angles is called a *right angle*.

The following theorems will now be proved:

THEOREM 11. In a triangle the angles opposite two congruent sides are congruent, or briefly, the base angles of an isosceles triangle are equal.

This theorem follows from Axiom III, 5 and the last part of Axiom III, 4.

DEFINITION. A triangle ABC is said to be congruent to a triangle $A'B'C'$ if all congruences

$$AB \equiv A'B', \quad AC \equiv A'C', \quad BC \equiv B'C'$$

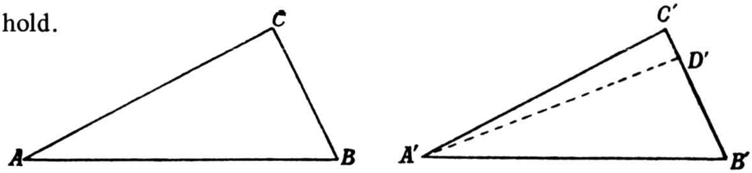
$$\sphericalangle A \equiv \sphericalangle A', \quad \sphericalangle B \equiv \sphericalangle B', \quad \sphericalangle C \equiv \sphericalangle C'$$

are satisfied.

THEOREM 12. (**first congruence theorem for triangles**). A triangle ABC is congruent to a triangle $A'B'C'$ whenever the congruences

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \sphericalangle A \equiv \sphericalangle A'$$

hold.



PROOF. By Axiom III, 5 the congruences

$$\sphericalangle B \equiv \sphericalangle B' \text{ and } \sphericalangle C \equiv \sphericalangle C'$$

are satisfied and thus it is only necessary to prove the validity of the congruence $BC \equiv B'C'$. If it is assumed to the contrary that BC is not congruent to $B'C'$ and a point D' is determined on $B'C'$ so that $BC \equiv B'D'$ then Axiom III, 5, applied to both triangles ABC and $A'B'D'$, will indicate that $\sphericalangle BAC \equiv \sphericalangle B'A'D'$. Then $\sphericalangle BAC$ would be congruent to $\sphericalangle B'A'D'$ as well as to $\sphericalangle B'A'C'$. This is impossible, as by Axiom III, 4 every angle can be constructed on a given side of a given ray in a plane in only **one** way. It has thus been proved that the triangle ABC is congruent to the triangle $A'B'C'$.

The following is just as easy to prove:

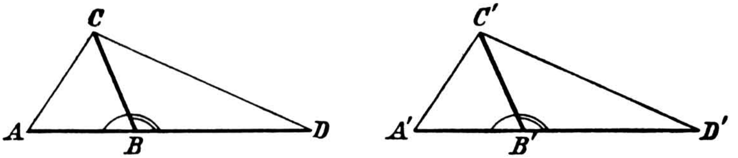
THEOREM 13 (**second congruent theorem for triangles**). A triangle ABC is congruent to another triangle $A'B'C'$ whenever the congruences

$$AB \equiv A'B', \quad \sphericalangle A \equiv \sphericalangle A', \quad \sphericalangle B \equiv \sphericalangle B'$$

hold.

THEOREM 14. If an angle $\sphericalangle ABC$ is congruent to another angle

$\sphericalangle A'B'C'$ then its supplementary angle $\sphericalangle CBD$ is congruent to the supplementary angle $\sphericalangle C'B'D'$ of the other angle.



PROOF. Choose the points A', C', D' on the sides that pass through B' in such a way that

$$AB \equiv A'B', \qquad CB \equiv C'B', \qquad DB \equiv D'B'.$$

It follows then from Theorem 12 that the triangle ABC is congruent to the triangle $A'B'C'$, i.e., the congruences

$$AC \equiv A'C' \quad \text{and} \quad \sphericalangle BAC \equiv \sphericalangle B'A'C'$$

hold.

Since moreover by Axiom III, 3 the segment AD is congruent to the segment $A'D'$, it follows again by Theorem 12 that the triangle CAD is congruent to the triangle $C'A'D'$, i.e., the congruences

$$CD \equiv C'D' \quad \text{and} \quad \sphericalangle ADC \equiv \sphericalangle A'D'C'$$

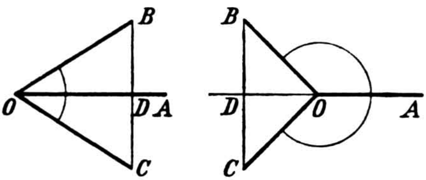
hold and hence, by considering the triangles BCD and $B'C'D'$, it follows by Axiom III, 5 that

$$\sphericalangle CBD \equiv \sphericalangle C'B'D'.$$

An immediate corollary of Theorem 14 is the **congruence theorem for vertical angles.**

The **existence of right angles** also follows from this theorem (see p. 15).

If angles constructed on both sides of a ray OA emanating from O and if the two noncommon sides of the angle are made equal, $OB \equiv OC$, then the segment BC intersects the line OA at a point D. If D



coincides with O then $\sphericalangle COA$ and $\sphericalangle BOA$ are equal supplementary angles and hence are right angles. If D lies on the ray OA then by the construction $\sphericalangle DOB \equiv \sphericalangle DOC$. If D lies on the other ray then the congruence follows from Theorem 14. By Axiom III, 2 every segment is congruent to itself: $OD \equiv OD$. Hence, by Axiom III, 5, it follows that $\sphericalangle ODB \equiv \sphericalangle ODC$.

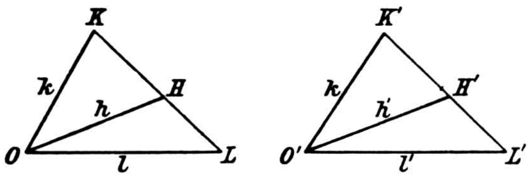
THEOREM 15. Let h, k, l and h', k', l' be rays emanating from O and O' in the planes α and α', respectively. Let h, k and h', k' lie simultaneously on the same or on different sides of l and l', respectively. If the congruences

$$\sphericalangle (h, l) \equiv \sphericalangle (h', l') \quad \text{and} \quad \sphericalangle (k, l) \equiv \sphericalangle (k', l')$$

are satisfied then so is the congruence

$$\sphericalangle (h, k) \equiv \sphericalangle (h', k').$$

The **proof** will be given for the case when h and k lie on the same side of l and hence by the hypothesis when h' and k' lie on the same side of l'. The second case will be reduced to the first case by an application of Theorem 14. From the definition on p. 11 it follows that either h lies in the angle $\sphericalangle (k, l)$ or that k lies in the angle $\sphericalangle (h, l)$. Now label so that h lies in the angle $\sphericalangle (k, l)$. Choose the points K, K', L, L' on the sides k, k', l, l' so that $OK \equiv O'K'$ and $OL \equiv O'L'$. By a theorem stated on p. 11 h intersects the segment KL at a point H.



Determine H' on h' so that $OH \equiv O'H'$. By Theorem 12 the congruences $\sphericalangle OLH \equiv \sphericalangle O'L'H'$, $\sphericalangle OLK \equiv \sphericalangle O'L'K'$,

$$LH \equiv L'H', \qquad LK \equiv L'K'$$

and also

$$\sphericalangle OKL \equiv \sphericalangle O'K'L'$$

are obtained in the triangles OLH and $O'L'H'$ or OLK and $O'L'K'$.

Since by Axiom III, 4 every angle can be constructed on a given side of a given ray in a plane in only one way and since by hypothesis

H' and K' lie on the same side of l' the first two mentioned angle congruences show that H' lies on $L'K'$. Hence the two segment congruences show easily by Axiom III, 3 and the uniqueness of segment construction that $HK \equiv H'K'$. The assertion is now deduced by Axiom III, 5 from the congruence $OK \equiv O'K'$, $HK \equiv H'K'$ and $\measuredangle OKL \equiv \measuredangle O'K'L'$.

The following result is obtained in a similar way:

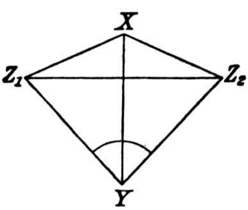
THEOREM 16. Let the angle $\measuredangle(h, k)$ in the plane α be congruent to the angle $\measuredangle(h', k')$ in the plane α', and let l be a ray in the plane α that emanates from the vertex of the angle $\measuredangle(h, k)$ and which lies in the interior of this angle. Then there always exists one and only one ray l' in the plane α' that emanates from the vertex of the $\measuredangle(h', k')$ and which lies in the interior of this angle in such a way that

$$\measuredangle(h, l) \equiv \measuredangle(h', l') \quad \text{and} \quad \measuredangle(k, l) \equiv \measuredangle(k', l').$$

In order to obtain the third congruence theorem and the symmetry property of angle congruence the following theorem is now deduced from Theorem 15:

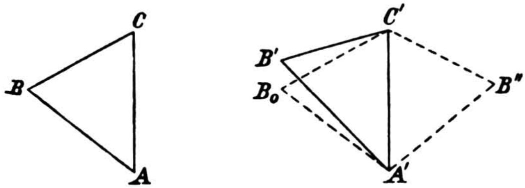
THEOREM 17. If two points Z_1 and Z_2 are placed on different sides of a line XY and if the congruences $XZ_1 \equiv XZ_2$ and $YZ_1 \equiv YZ_2$ hold, then the angle $\measuredangle XYZ_1$ is congruent to the angle $\measuredangle XYZ_2$.

PROOF. By Theorem 11 $\measuredangle XZ_1Z_2 \equiv \measuredangle XZ_2Z_1$ and $\measuredangle YZ_1Z_2 \equiv \measuredangle YZ_2Z_1$. Hence the congruence $\measuredangle XZ_1Y \equiv \measuredangle XZ_2Y$ follows from Theorem 15. The special cases when X or Y lies on Z_1Z_2 can be disposed of in an even simpler manner. From the last congruence and the assumed congruences $XZ_1 \equiv XZ_2$ and $YZ_1 \equiv YZ_2$ the assertion $\measuredangle XYZ_1 \equiv \measuredangle XYZ_2$ follows by Axiom III, 5.



THEOREM 18 (**third congruence theorem for triangles**). If in two triangles ABC and $A'B'C'$ each pair of corresponding sides is congruent then so are the triangles.

PROOF. By virtue of the symmetry of segment congruence proved on p. 11 it is sufficient to prove that the triangle ABC is congruent to the triangle $A'B'C'$. Construct the angle $\measuredangle BAC$ at A' on both sides of



the ray $A'C'$. Choose the point B_0 on the side that lies on the same side of $A'C'$ as B' so that $A'B_0 \equiv AB$. On the other side let B'' be chosen in such a way that $A'B'' \equiv AB$. By Theorem 12 $BC \equiv B_0C'$ and $BC \equiv B''C'$. These congruences together with those in the hypothesis yield by Axiom III, 2 the congruences

$$A'B'' \equiv A'B_0, \quad B''C' \equiv B_0C'$$

and correspondingly

$$A'B'' \equiv A'B', \quad B''C' \equiv B'C'.$$

Both the triangles $A'B''C'$ and $A'B_0C'$ as well as the triangles $A'B''C'$ and $A'B'C'$ satisfy the hypotheses of Theorem 17, i.e., the angle $\measuredangle B''A'C'$ is congruent to the angle $\measuredangle B_0A'C'$ as well as the angle $\measuredangle B'A'C'$. But since by Axiom III, 4 every angle can be constructed on a given side of a given ray in a plane in only **one** way, the ray $A'B_0$ coincides with the ray $A'B'$, i.e., the angle that is congruent to $\measuredangle BAC$, constructed on the given side of $A'C'$, is the angle $\measuredangle B'A'C'$. The assertion follows then by Theorem 12 from the congruence $\measuredangle BAC \equiv \measuredangle B'A'C'$ and the assumed segment congruences.

THEOREM 19. If two angles $\measuredangle (h', k')$ and $\measuredangle (h'', k'')$ are congruent to a third angle $\measuredangle (h, k)$ then the angle $\measuredangle (h', k')$ is also congruent to angle $\measuredangle (h'', k'')$.[1]

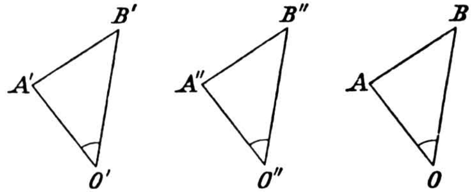
This theorem which corresponds to Axiom III, 2 can also be formulated in this way. If two angles are congruent to a third one then they are congruent to each other.

PROOF. Let the vertices of the three given angles be O', O'' and O. On one side of each angle choose the points A', A'' and A so that $O'A' \equiv OA$ and $O''A'' \equiv OA$. Similarly, on the third sides choose the

[1] The proof given here for Theorem 19, which in the First Edition was taken as an axiom, is due to A. Rosenthal. Cf. *Math. Ann.*, Vol. 71.

The modified form of Axioms I, 3 and I, 4 is also due to A. Rosenthal. Cf. *Math. Ann.*, Vol. 69.

points B', B'' and B so that $O'B' \equiv OB$ and $O''B'' \equiv OB$. These congruences together with the assumption $\sphericalangle(h', k') \equiv \sphericalangle(h, k)$ and $\sphericalangle(h'', k'') \equiv \sphericalangle(h, k)$ yield by Theorem 12 the congruences



$$A'B' \equiv AB \quad \text{and} \quad A''B'' \equiv AB.$$

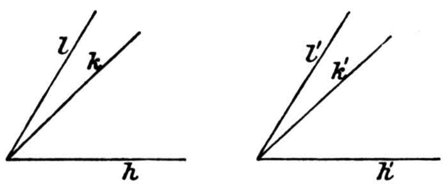
By Axiom III, 2 the triangles $A'B'O'$ and $A''B''O''$ coincide in their three sides and hence by Theorem 18

$$\sphericalangle(h', k') \equiv \sphericalangle(h'', k'').$$

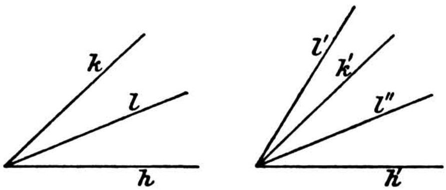
The symmetry property of angle congruence follows from Theorem 19 just as it does for segments from Axiom III, 2; i.e., if $\sphericalangle \alpha \equiv \sphericalangle \beta$ then $\sphericalangle \alpha$ and $\sphericalangle \beta$ are **congruent to each other**. In particular Theorems 12-14 can be expressed now in symmetric form.

The **quantitative comparison of angles** can now be established.

THEOREM 20. Let any two angles $\sphericalangle(h, k)$ and $\sphericalangle(h', l')$ be given. If the construction of $\sphericalangle(h, k)$ on h' on the side of l' yields an **interior** ray k' then the construction of $\sphericalangle(h', l')$ on h on the side of k yields an **exterior** ray l, and conversely.



PROOF. By hypothesis l lies in the interior of $\sphericalangle(h, k)$. Since $\sphericalangle(h, k) \equiv \sphericalangle(h', k')$ by Theorem 16 there exists for the interior ray l a ray l'' in the **interior** of $\sphericalangle(h', k')$ for which the congruence $\sphericalangle(h, l) \equiv \sphericalangle(h', l'')$ holds. By hypothesis and by virtue of the symmetry of angle congruence $\sphericalangle(h, l) \equiv \sphericalangle(h', l')$ where l' and l'' are necessarily



distinct, contrary to the uniqueness of angle construction III, 4. The converse is proved similarly.

If the construction of $\sphericalangle (h, k)$ described in Theorem 20 yields an interior ray k' in $\sphericalangle (h', l')$ it is said that $\sphericalangle (h, k)$ is *smaller than* $\sphericalangle (h', l')$; symbolically $\sphericalangle (h, k) < \sphericalangle (h', l')$. If it yields an exterior ray it is said that $\sphericalangle (h, k)$ *is greater than* $\sphericalangle (h', l')$; symbolically $\sphericalangle (h, k) > \sphericalangle (h', l')$.

It should be realized that for two angles α and β **one and only one of the three cases**

$$\alpha < \beta \text{ and } \beta > \alpha, \quad \alpha \equiv \beta, \quad \alpha > \beta \text{ and } \beta < \alpha$$

can exist. The quantitative comparison of angles is *transitive*, i.e., from each of the three assumptions

$$1.\, \alpha > \beta,\ \beta > \gamma; \quad 2.\, \alpha > \beta,\ \beta \equiv \gamma; \quad 3.\, \alpha \equiv \beta,\ \beta > \gamma$$

follows

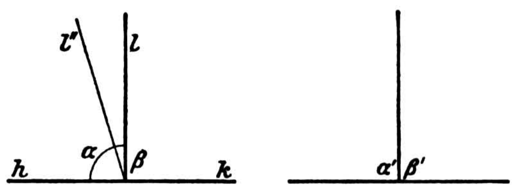
$$\alpha > \gamma.$$

The **quantitative comparison of segments** with corresponding properties follows immediately from Axioms II and III, 1-3 as well as from the uniqueness of segment construction (see p. 13).

On the basis of the quantitative comparison of angles it is possible to obtain a proof for the following simple theorem which, subjectively speaking, *Euclid* listed unjustifiedly among the axioms:

THEOREM 21. *All right angles are congruent to each other.*

PROOF.[1] By definition a right angle is one that is congruent to its complementary angle. Let the angles α or $\sphericalangle (h, l)$ and β or $\sphericalangle (k, l)$ be supplementary angles and let α' and β' also be such angles. Let $\alpha \equiv \beta$ and $\alpha' \equiv \beta'$. Suppose that α' is not congruent to α, contrary to the hypothesis of Theorem



[1] The idea of this proof can be found as early as the Euclid commentator **Proclus**, who indeed, instead of Theorem 14 used the hypothesis that the construction of one right angle always yields another right angle, i.e., yields an

21. Then the construction of the angle α' on h on the side on which l lies yields a ray l'' that is distinct from l. l'' lies then either in the interior of α or in the interior of β. If l'' lies in the interior of α then

$$\sphericalangle(h, l'') < \alpha,\ \alpha \equiv \beta,\ \ \beta < \sphericalangle(k, l'').$$

By the transitivity of the quantitative comparison of angles it follows from this that $\sphericalangle(h, l'') < \sphericalangle(k, l'')$. On the other hand, by the hypothesis and by Theorem 14

$$\sphericalangle(h, l'') \equiv \alpha',\ \alpha' \equiv \beta',\ \ \beta' \equiv \sphericalangle(k, l''),$$

and hence follows

$$\sphericalangle(h, l'') \equiv \sphericalangle(k, l''),$$

contrary to the relation $\sphericalangle(h, l'') < \sphericalangle(k, l'')$. If l'' lies in the interior of β a completely analogous contradiction is obtained and Theorem 21 is thus proved.

DEFINITION. An angle that is greater than its supplementary angle is called an *obtuse* angle. An angle that is smaller than its supplementary angle is called an *acute* angle.

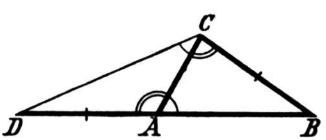
A fundamental theorem that already played an important role for Euclid and from which follows a series of important results is the theorem of the exterior angle.

DEFINITION. The angles $\sphericalangle ABC$, $\sphericalangle BCA$ and $\sphericalangle CAB$ of the triangle ABC are called the *interior angles* of the triangle. Their supplementary angles are called the *exterior angles* of the triangle.

THEOREM 22 (**theorem of the exterior angle**). The exterior angle of a triangle is greater than any interior angle that is not adjacent to it.

PROOF. Let $\sphericalangle CAD$ be an exterior angle of the triangle ABC. D may be chosen so that $AD \equiv CB$.

It will be shown next that $\sphericalangle CAD \not\equiv \sphericalangle ACB$. If $\sphericalangle CAD \equiv \sphericalangle ACB$ held then so would $\sphericalangle ACD \equiv \sphericalangle CAB$ by virtue of the congruence $AC \equiv CA$ and by Axiom III, 5. It would follow from Theorems 14 and 19 that $\sphericalangle ACD$ would be congruent to the supplementary angle of $\sphericalangle ACB$. By Axiom III, 4 D would thus lie on the



angle that is equal to its supplementary angle.

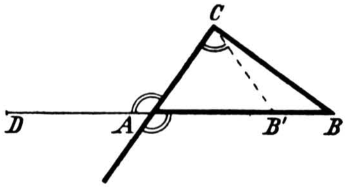
A French translation of the commentary of **Proclus** with an introduction and notes by P. Ver Eecke, "Proclus de Lycie"–Les Commentaires sur le premier livre des éléments d'Euclide, was published in *Collection de travaux de l'Acad. internat. d'histoire des sciences*, No. 1 (Brügge, 1948).

line CB, contrary to Axiom I, 2. It must then be that

$$\measuredangle CAD \not\equiv \measuredangle ACB.$$

It is also impossible that $\measuredangle CAD < \measuredangle ACB$ since then the construction of the exterior angle $\measuredangle CAD$ on CA at C on the side on which B lies would yield a side that lies in the interior of the angle $\measuredangle ACB$, and thus would meet the segment AB at point B'. The exterior angle $\measuredangle CAD$ would then be congruent to the angle $\measuredangle ACB'$ in the triangle $AB'C$. This however, as shown above, is impossible. It remains then only the possibility

$$\measuredangle CAD > \measuredangle ACB.$$



In exactly the same way one obtains the fact that the vertical angle of the angle $\measuredangle CAD$ is greater than the angle $\measuredangle ABC$, and from the congruence of vertical angles and the transitivity of the quantitative comparison of angle sizes it follows that

$$\measuredangle CAD > \measuredangle ABC.$$

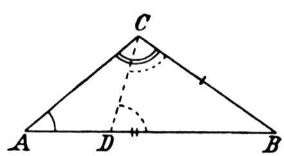
The assertion is thus completely proved.

Important corollaries from this theorem are the following theorems:

THEOREM 23. In every triangle the greater angle lies opposite the greater side.

PROOF. In the given triangle construct the smaller of two sides with common end points on the greater one. The assertion follows then from Theorems 11 and 12 by virtue of the transitivity of the quantitative comparison of angle sizes.

THEOREM 24. A triangle with two equal angles is isosceles.



This converse of Theorem 11 is an immediate consequence of Theorem 23.

Moreover, from Theorem 22 follows in a simple way an extension to the second congruence theorem for triangles.

THEOREM 25. Two triangles ABC and $A'B'C'$ are congruent to each other if the congruences.

$$AB \equiv A'B' \quad \measuredangle A \equiv \measuredangle A' \quad \text{and} \quad \measuredangle C \equiv \measuredangle C'$$

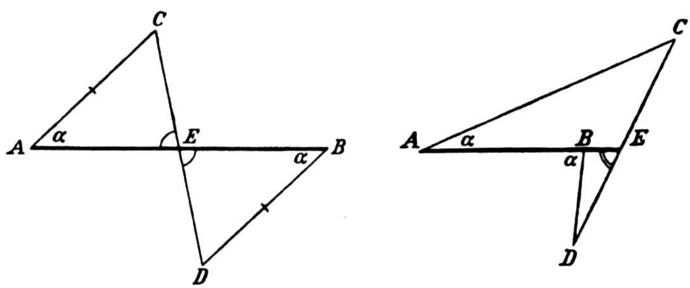
are satisfied.

THEOREM 26. Every segment can be bisected.

PROOF. On different sides of the given segment AB construct the same angle α at its end points and lay off equal segments on the third sides of the angles so that $AC \equiv BD$. Since C and D lie on different sides of AB the segment CD meets the line AB at a point E.

The assumption that E coincides with A or with B is an immediate contradiction of Theorem 22. Let it then be assumed that B lies between A and E. By Theorem 22 it would follow that

$$\measuredangle ABD > \measuredangle BED > \measuredangle BAC,$$



contrary to the construction. The assumption that A lies between B and E yields the same contradiction.

By Theorem 4 E lies then on the segment AB. Therefore $\measuredangle AEC$ and $\measuredangle BED$, as vertical angles, are congruent. Hence Theorem 25 is applicable to the triangles AEC and BED and yields

$$AE \equiv EB.$$

An immediate consequence of Theorems 11 and 26 is the fact that every angle can be bisected.

The concept of congruence can be extended to any figure.

DEFINITION. If $A, B, C, D, \ldots, K, L$ and $A', B', C', D', \ldots, K', L'$ are two sequences of points on a and a', respectively, such that the segments AB and $A'B'$, AC and $A'C'$, BC and $B'C'$, $\ldots$, KL and $K'L'$ are congruent to each other in pairs the two sequences of points are said to be congruent to each other. A and A', B and B', $\ldots$, L and L' are called the corresponding points of the *congruent sequences of points.*

THEOREM 27. If the first of the two sequences of congruent points $A, B, \ldots, K, L$ and $A', B', \ldots, K', L'$ is so ordered that B lies between A and $C, D, \ldots, K, L$; C lies between A, B, and $D, \ldots, K, L$, etc.; then the points $A', B', \ldots, K', L'$ are ordered in the same way, i.e., B' lies between A' and $C', D', \ldots, K', L'$; C' lies between A', B', and $D', \ldots, K', L'$, etc.

DEFINITION. A finite number of points is called a *figure*. If all points of a figure lie in a plane it is called a *plane figure*.

Two figures are said to be *congruent* if their points can be ordered in pairs so that the segments and the angles that become ordered in this way are all congruent to each other.

As becomes evident from Theorems 14 and 27 congruent figures have the following properties: If three points of a figure are collinear then the corresponding points in any congruent figure are also collinear. The ordering of points in corresponding planes with respect to corresponding lines is the same in congruent figures. The same holds for sequences of corresponding points on corresponding lines.

The most general congruence theorem for the plane and for space is expressed as follows:

THEOREM 28. If $(A, B, C, \ldots, L)$ and $(A', B', C', \ldots, L')$ are congruent plane figures and P denotes a point in the plane of the first figure then it is possible to find a point P' in the plane of the second figure so that $(A, B, C, \ldots, L, P)$ and $(A', B', C', \ldots, L', P')$ are again congruent figures. If the figure $(A, B, C, \ldots, L)$ contains at least three noncollinear points then the construction of P' is possible in only **one** way.

THEOREM 29. If $(A, B, C, \ldots, L)$ and $(A', B', C', \ldots, L')$ are congruent figures and P is any point, then it is always possible to find a point P' so that the figures $(A, B, C, \ldots, L, P)$ and $(A', B', C', \ldots, L', P')$ are congruent. If the figure $(A, B, C, \ldots, L)$ contains at least four noncoplanar points then the construction of P' is possible in only **one** way.

By invoking axiom groups I and II, Theorem 29 expresses the important result that all **space** properties of congruence and thus the properties of displacement in **space** are consequences of the five **line** and **plane** axioms of congruence formulated above.

§ 7. Axiom Group IV: Axiom of Parallels

Let α be any plane, a any line in α and A a point in α not lying on a. If a line c is drawn in α so that it passes through A and intersects a, and a line b is drawn in α through A so that the line c intersects the lines a, b at the same angles then it follows easily from the exterior angle theorem, Theorem 22, that the lines a, b have no point in common, i.e., in a plane α it is always possible to draw a line through a point A not on a line a so that it does not intersect a.

DEFINITION. Two lines are said to be parallel if they lie in the same plane and do not intersect.

The axiom of parallels can be stated now as follows:

IV **(Euclid's Axiom).** *Let a be any line and A a point not on it. Then there is at most one line in the plane, determined by a and A, that passes through A and does not intersect a.*

From the foregoing and on the basis of the axiom of parallels it can be seen that there is exactly one parallel to a line through a point not on it.

The axiom of parallels IV is equivalent to the following requirement:

If two lines a, b in a plane do not meet a third line c in the same plane then they also do not meet each other.

In fact if a, b had a point A in common these two lines would pass through A in the same plane without meeting c. This situation would contradict the axiom of parallels IV. Conversely, the axiom of parallels also follows easily from this requirement.

The axiom of parallels is a *plane axiom.*

The introduction of the axiom of parallels **simplifies** the foundation of geometry and **facilitates** its development to a considerable degree.

Adjoining to the axioms of congruence the axiom of parallels the following familiar fact is obtained:

THEOREM 30. If two parallels are intersected by a third line then the corresponding and the alternate angles are congruent, and conversely, the congruence of the corresponding or the alternate angles implies that the lines are parallel.

THEOREM 31. The angles of a triangle add up to two right angles.[1]

[1]Concerning the question of how far this theorem can replace the converse of the axiom of parallels, compare the remarks in Section 12 at the end of Chapter II.

DEFINITION. If M is any point in a plane α then the collection of all points A in α for which the segments MA are congruent to each other is called a *circle*. M is called the *center of the circle.*

On the basis of this definition the familiar theorems about the circle follow easily with the aid of axiom groups III - IV—in particular, the possibility of constructing a circle through any three noncollinear points as well as the theorem about the congruence of inscribed angles over the same chord and the theorem of the angles in an inscribed quadrilateral.

§ 8. Axiom Group V: Axioms of Continuity

V, 1 (**Axiom of measure or Archimedes' Axiom**). *If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.*

V, 2 (**Axiom of line completeness**). *An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III, and from V, 1 is impossible.*

By the fundamental properties is meant the order properties formulated in Axioms II, 1-3 and in Theorem 5 as well as the congruence properties formulated in Axioms III, 1-3 along with the uniqueness of segment construction.[1] It is further meant that on extending the set of points the order and congruence relations carry over to the extended point region.

It should be noted that Axiom I, 3 is preserved at every extension *eo ipso* and that the validity of Theorem 3 at such extensions is a consequence of the persistence of Archimedes' Axiom V, 1.

The satisfaction of the axiom of completeness depends essentially on the fact that it contains Archimedes' Axiom among the axioms whose validity is required. In fact, it can be shown that to a set of points on a line that satisfies the previously enumerated axioms and theorems of order and congruence it is always possible to adjoin other points such that these axioms are also valid in the resulting extended

[1] A precise classification of the conditions to be required here for line ordering and congruence was carried out by F. Bachmann and was incorporated in the formulation of Axiom V, 2 in the Seventh Edition.

set; i.e., a completeness axiom that requires only the validity of these axioms but not that of Archimedes, or one that is equivalent to it, would entail a contradiction.

Both continuity axioms are *line* axioms.

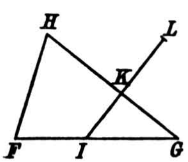
The following fact is essentially obtained **from the line completeness axiom**:

THEOREM 32 (**theorem of completeness**).[1] The elements (i.e., the points, the lines and the planes) of geometry form a system which cannot be extended by points, lines and planes because of the persistence of the axioms of incidence, order, congruence and Archimedes, and so only because of the persistence of all axioms.[2]

PROOF. Let the elements that exist before the extension be designated as the **old** elements; those that arise from the extension be designated as the **new** elements. The assumption of new elements leads immediately to the assumption of a new point *N*.

By Axiom I, 8 there exist four old noncoplanar points *A, B, C, D*. The labels can be so chosen that *A, B, N* are not collinear. The two distinct planes *ABN* and *ACD* by Axiom I, 7 have besides *A*, another point *E* in common. *E* does not lie on the line *AB* for then *B* would lie in the plane *ACD*. If *E* is a new point then a new point *E* lies in the old plane *ACD*. On the other hand, if *E* is an old point then the new point *N* lies in an old plane, namely, in the plane *ABE*. In any case a new point lies in an old plane.

There exists an old triangle *FGH* in an old plane and on the segment *FG* and old point *I*. If a new point *L* is joined with *I* then by Axiom II, 4 the lines *IL* and *FH* or the lines *IL* and *GH* meet at a point *K*. If *K* is new then a new point *K* lies on an old line *FH* or *GH*. If on the other hand *K* is old then a new point *L* lies on an old line *IK*. All these assumptions are thus contrary to the axiom of line completeness. The assumption of a new point in an old plane must therefore be dropped and thereby the assumption of new elements.



[1] The observation that the line completeness axiom is sufficient is due to P. Bernays.

[2] This assertion was stated in the previous editions as an axiom of completeness.

The completeness theorem can even be sharpened. The persistence of some of the mentioned axioms need not be unconditionally required for it. However, essential for its validity is that Axiom I, 7 be contained among the axioms whose persistence it requires. In fact it can be shown that to a set of elements which satisfies Axioms I-V it is always possible to adjoin points, lines and planes so that these same axioms with the exception of Axiom I, 7 hold in the set that arises from the adjunction, i.e., a completeness theorem in which Axiom I, 7 or one that is equivalent to it, is not contained would entail a contradiction.

The **completeness axiom is not a consequence of Archimedes' Axiom.** In fact in order to show with the aid of Axioms I-IV that this geometry is identical to the ordinary analytical "Cartesian" geometry Archimedes' Axiom by itself is insufficient (cf. Sections 9 and 12). However, by invoking the completeness axiom, although it contains no direct assertion about the concept of convergence, it is possible to prove the existence of a limit that corresponds to a Dedekind cut as well as the Bolzano-Weierstrass theorem for the existence of condensation points, whereby this geometry appears to be identical to Cartesian geometry.

By the above treatment the requirement of continuity has been decomposed into two essentially different parts, namely, into Archimedes' Axiom whose role is to prepare the requirement of continuity and into the completeness axiom **which forms the cornerstone of the entire system of axioms.**[1]

The subsequent investigations rest essentially only on Archimedes' Axiom and the completeness axiom is in general not assumed.

[1] Compare also the remarks at the end of Section 17, as well as my lecture on the concept of a number, "Berichte der Deutschen Mathematiker-Vereinigung, 1900." The investigation of the theorem on the equality of the two base angles of an isosceles triangle will lead to two more continuity axioms. Cf. Appendix II of this book, p. 114 and my article "Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck," *Proceedings of the London Mathematical Society*, Vol. 35 (1903).

As additional investigations of the axioms of continuity, the following examples are mentioned here: R. Baldus, "Zur Axiomatik der Geometrie," I-III, I in *Math. Ann.*, (1928), 100, 321-33; II in *Atti d. Conge. int. d. Mat.* (Bologna, 1928), IV (1931); III in *Sitzber. d. Heidelberger Akad. Wiss.*, 1930, Fifth Proceedings. A. Schmidt, "Die Stetigkeit in der absoluten Geometrie." *Ibid.*, 1931, Fifth Proceedings. P. Bernays, "Betrachtungen über das Vollständigkeitsaxiom und verwandte Axiome," *Math. Zeitschr.* 63 (1955), 219-92.