

An Analogy Tool to Visualize Bending-Moment Diagrams

Gustavo O. Maldonado¹, Gustavo J. Molina²

Abstract – This work presents an old, but not widely known, educational tool that allows students to use a simple analogy to quickly visualize the shape of bending-moment diagrams for simple beams. The analogy possesses a mathematical base. Two similar ordinary differential equations govern two different equilibrium-based problems in Statics. One involves bending moments in beams, and may present some learning difficulties to students. The other involves the displaced equilibrium configuration of a loaded chain, and may be intuitively visualized. This article compares the two governing equations and indicates the conditions under which both equations become numerically equivalent. It presents the proper use of the analogy to visualize solutions in chains that can be used to generate the corresponding bending moments in beams. For this purpose, five examples, involving beams with different types of supports and load conditions, are presented and analyzed.

Keywords: Beam, Moment, Inextensible, Chain, Analogy.

INTRODUCTION

This work focuses on the use of an analogy that facilitates the visualization of bending-moment diagrams for the analysis of simple beams. Undergraduate students, attending engineering programs in structural, construction, mechanical, and architectural disciplines, learn to analyze the effect of loads on structural elements. For this purpose, they take one or more courses on structural analysis, starting with Statics or Introduction to Structures. These fundamental courses present the analysis of simply-supported beams and include the generation of shear-force and bending-moment diagrams. The determination of shear diagrams is, usually, a straightforward task. On the other hand, the generation of moment diagrams requires more effort and may present some challenges for the students. Not all students may have the same mathematical background. Some construction and/or architectural programs are non-calculus based and students may not have the calculus tools that facilitate the generation of moment diagrams. The objective of this article is to present an old, but not widely known or used, educational tool that allows students to use a simple analogy to visualize the shape of moment diagrams. This a-priori visualization helps to minimize errors during a more rigorous determination of the diagrams, or it can be used to check the general shape of already determined diagrams.

The analogy is not new. However, it is not widely used in American educational institutions. The first author, as a student, learned and used it in class while taking his course on Statics in 1977. Today, many excellent and well-known textbooks on Statics [Beer, 1] [Hibbeler, 3] [Meriam, 6] do not include it. They do use the analogous differential equations in separate sections to cover topics on bending moments and on the analysis of cables, but do not present the analogy. An educational software package, prepared at the University of Washington, does mention the analogy in its worksheets [Miller, 7], but it is used only for simply-supported beams. Also, a website [Farhey, 2] presents several pictures showing the resemblance between the shape of loaded strings and bending-moment

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diagrams for simply-supported beams. The authors have used the analogy in the classroom, for beams with different supports and loadings, and many students have expressly indicated that it is an effective tool. This article presents several examples that illustrate its use.

THE CHAIN ANALOGY

In Statics, we find two equilibrium phenomena that are governed by almost the same ordinary differential equation. One of them is related to the equilibrium of beams and the other to the equilibrium of inextensible chains. The governing equation of the first one relates external, transversally distributed, loads with the internal bending moments resisted by a beam that supports those loads. The equation of the second phenomenon involves the same transversal loads, but relates them to the shape acquired by an inextensible chain, at equilibrium, while subjected to those loads. This second phenomenon can be easily visualized and may be used to understand the first.

Certainly, the differential equations used in the analogy are not new. They are, probably, one of the oldest equations used for engineering purposes. Both are presented in the following paragraphs to analyze its similarities and differences. This analysis leads to the proper use of the analogy.

Differential Equation for Bending Moments

Due to its simplicity, and powerful capabilities, the Euler-Bernoulli beam theory is the first beam theory presented to undergraduate students in courses on Strength of Materials or on Mechanics of Solids. It uses kinematical, constitutive, resultant, and equilibrium relations to derive the fourth order ordinary differential equation that governs the transversal displacements of homogeneous and elastic beams under transversal loadings. In introductory courses, such as Statics, students are presented only with the equilibrium component of beam theory.

Satisfaction of equilibrium conditions produces the load-shear-moment relations. They are differential equations involving the distributed load $q(x)$, the shear force function $V(x)$ and the bending moment function $M(x)$. They are described in most Statics textbooks using the following notation:

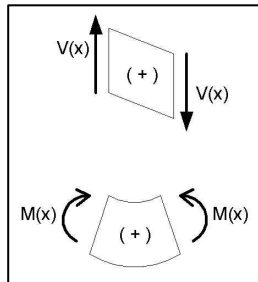


Figure 1: Positive Shear and Positive Moment Effects.

$$\frac{dM(x)}{dx} = V(x) \quad (1)$$

$$\frac{dV(x)}{dx} = -q(x) \quad (2)$$

Where the longitudinal direction of the beam is horizontal, along the x -axis, and $q(x)$ is vertical, but distributed horizontally along the longitudinal x -axis of the beam. In these equations, $q(x)$ is considered positive if it acts downward. The effects of positive shear and positive bending-moment, on a small segment of the beam, are presented in Figure 1.

Substitution of Equation 1 into 2 results in the following differential equation:

$$\frac{d^2 M(x)}{dx^2} = -q(x) \quad (3)$$

It involves the second derivative of the bending-moment function and the distributed loading function.

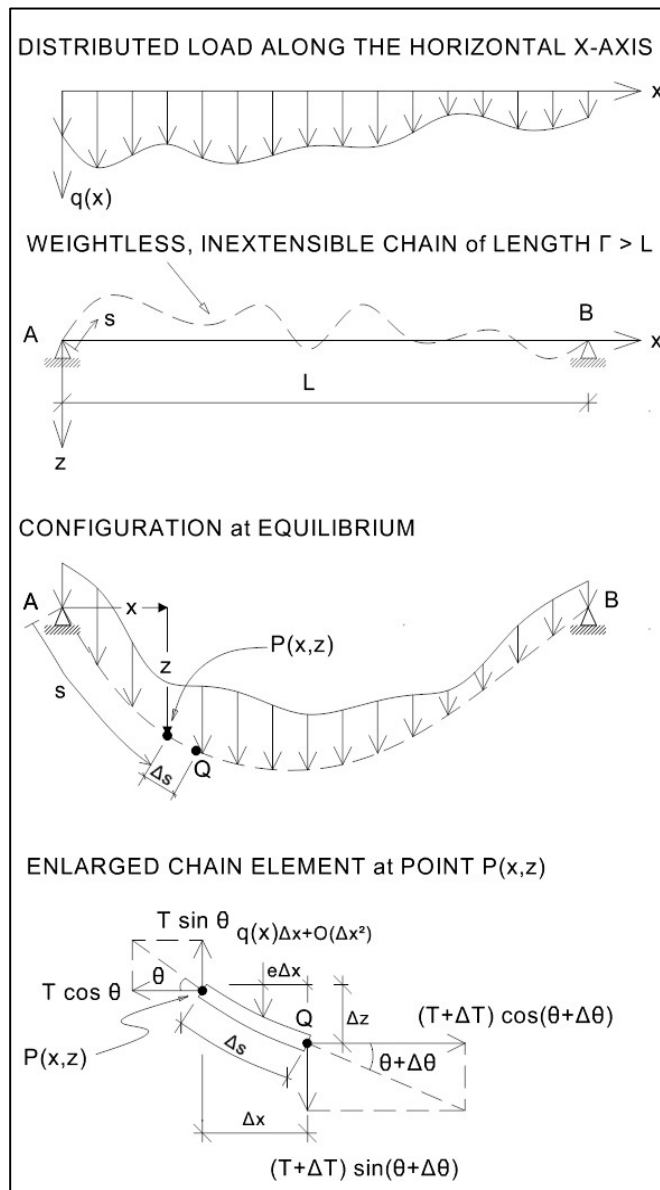
Differential Equation for the Profile of a Weightless, Inextensible Chain

Since the differential equation presented in this section is the essence of the analogy used in this work, its full derivation is presented in the following paragraphs. This equation is old and has been used extensively, especially in the design of cable supported structures. It has been indicated [Truesdell, 8] [Irvine, 5] that a similar equation, considering a uniform load distributed along the x -axis, was probably used by Beekman circa 1615, but the solution did not become known until 1794, when it was used by Fuss, Euler's son in law. Today, derivation of this equation, or similar ones, is found elsewhere [Hibbeler, 3] [Irvine, 4, 5]. However, they may not use the same coordinate

system, or notation, or may start by including the weight of the chain. Hence, its derivation is presented in the following paragraphs.

Consider a weightless, inextensible chain, of length $\Gamma > L$. This chain has total flexural flexibility, but it is axially rigid. Oftentimes, it is referred to as a flexible, inextensible chain (or cable). Also, consider that its first and last links are attached to fixed supports A and B, which are at the same level, as shown in Figure 2. The end links can freely rotate about supports A and B. The horizontal distance between supports is L . The chain is subjected to a vertical load, $q(x)$, distributed along the horizontal x -axis (not along the length of the chain). The attained equilibrium configuration is sketched in Figure 2. An arbitrary point P , of the chain, is shown in the equilibrium configuration. It is located at coordinates (x, z) .

Figure 2 shows the free-body diagram of an incremental element of chain, from P to Q . It has tangential tensional forces, T and $T+\Delta T$, acting at P and at Q , respectively. Only the horizontal and vertical components of those



tangential forces are indicated. The external force acting on the element is represented by a single vertical force at a distance $(e \Delta x)$ from point Q , with $0 < e < 1$. The magnitude of this force is $q(x)\Delta x + O(\Delta x^2)$. Where $O(\Delta x^2)$ represents a term of order Δx^2 .

In this element, the equilibrium of forces along the horizontal direction yields:

$$(T + \Delta T) \cos(\theta + \Delta\theta) - T \cos \theta = 0 \quad (4)$$

After dividing this expression by Δx , and taking the limit as $\Delta x \rightarrow 0$, we obtain: $\frac{d T \cos \theta}{dx} = 0$. That is,

the horizontal component of the tensional tangential force T is constant along the chain. This constant value is here designated as:

$$H = T \cos \theta \quad (5)$$

The equilibrium of forces along the vertical direction, z , results in

$$(T + \Delta T) \sin(\theta + \Delta\theta) - T \sin \theta + q(x)\Delta x + O(\Delta x^2) = 0 \quad (6)$$

Dividing Equation 6 by Δx , and taking the limit as $\Delta x \rightarrow 0$, yields:

$$\frac{d T \sin \theta}{dx} = -q(x) \quad (7)$$

Similarly, as $\Delta x \rightarrow 0$, the limit of the equation of equilibrium of moments about point Q , produces: $dz/dx = \tan \theta$. Solving Equation 5 for T , and substituting it into 7 results in:

$$\frac{d}{dx} (H \tan \theta) = -q(x) \quad (8)$$

In this last expression, substitution of $\tan \theta$ by

Figure 2: Weightless and Inextensible Chain

dz/dx gives:

$$H \frac{d^2 z}{dx^2} = -q(x) \quad (9)$$

Equation 9 is the differential equation governing the vertical profile, $z(x)$, of a weightless, inextensible chain at equilibrium under a vertical load $q(x)$ distributed along the horizontal x -axis.

Constant H is the horizontal component of the internal tensional force existing at any point of the chain at equilibrium. The value of T may change from point to point, but H remains constant. The value of H depends on the load function $q(x)$, on the length between supports L , and on the total length of the chain Γ (with $\Gamma > L$) [Irvine, 4, 5].

Analysis of the Analogy

Equations 3 and 9 are analogous. They can be rewritten as:

$$\frac{d^2 [M(x)]}{dx^2} = -q(x) \quad (10)$$

$$\frac{d^2 [H z(x)]}{dx^2} = -q(x) \quad (11)$$

Both expressions are linear, non-homogeneous, second order, ordinary differential equations. They both contain one second derivative term and the same forcing term. Both were derived by satisfying equilibrium conditions. They involve different quantities within the square brackets, but those quantities, $M(x)$ and $H z(x)$, are dimensionally equal.

By using the same dimensional units in both equations, and by selecting an appropriate length $\Gamma > L$ for the chain, the horizontal component H of the tensional force may become equal to a unit force. In that case, the numerical values of the vertical ordinates $z(x)$ of the chain will coincide with the numerical values of the bending moments $M(x)$ of the beam. Even if the constant H were not equal to 1, the vertical profile of the chain will still be similar to the corresponding bending-moment diagram, but numerically different.

USE OF THE ANALOGY

Since the approximate profile shape of loaded chains is prone to visualization, it is used in this article to estimate the shape of the corresponding bending moment diagram. This section presents several examples showing different aspects of the proper use of the analogy. They include beams with different support conditions and with different types of loadings.

Example 1 (Figure 3)

In order to introduce the use of the Chain Analogy, we first consider a simply-supported beam with end supports at points A and B. Its total length is L and its uniformly distributed load is $q(x) = q$. The left-hand side of Figure 3 shows the reactions for this beam, its diagrams for shear force and bending moment. Assume that the bending-moment diagram is not known yet. We will use the analogy to visualize and qualitatively estimate its shape.

The weightless, inextensible chain to be used in the analogy is shown on the right-hand side of Figure 3. Supports A and B are at the same level and must be separated by a distance equal to the total length L of the beam. The loads on the chain must be the same as those on the beam. The length of the chain may be any length $\Gamma > L$.

After visualizing the chain attached to its supports, students may also visualize the acquired shape of the loaded chain at equilibrium. The symmetric image of that shape, with respect to the x -axis, should be similar to the shape of the bending-moment diagram. The need to obtain that symmetric image arises from the fact that positive values of the profile are plotted below the x -axis, and positive bending moments are plotted above the x -axis.

As seen in Figure 3, an excessively long chain results in a shape similar to that of the actual moment diagram, but with larger vertical dimensions. In spite of this difference, the shape of the profile and, therefore, the shape of the moment diagram may still be *qualitatively* visualized.

By using an appropriate length of chain, its profile at equilibrium should be the same as the bending-moment diagram. However, that length is not known a priori and requires some calculation effort to determine it. Since the main purpose of using the chain analogy is to visualize the moment diagram, preferably without performing calculations, the need to determine the proper length of chain will be circumvented by calculating just one value of the bending moment at a selected point of the beam. For this purpose, in this example, we choose the point at the mid-span of the beam. The bending moment at this point can be obtained from the available shear-force function (diagram). It may be integrated, from its origin to the selected point, or we may just calculate the area under its curve between those two points. That calculation produces $qL^2/8$. Hence, the lowest point of the profile must have an ordinate equal to $qL^2/8$ to be the same as the bending-moment diagram. After this calculation, we know the approximate shape of the profile and the position of three of its points: A, B and the selected mid-span point. Adjustments can now be made to produce the second profile shown in Figure 3. This one has the *appropriate* length of chain and its symmetric image (with respect to the x-axis) should be similar to the actual shape of the bending-moment diagram.

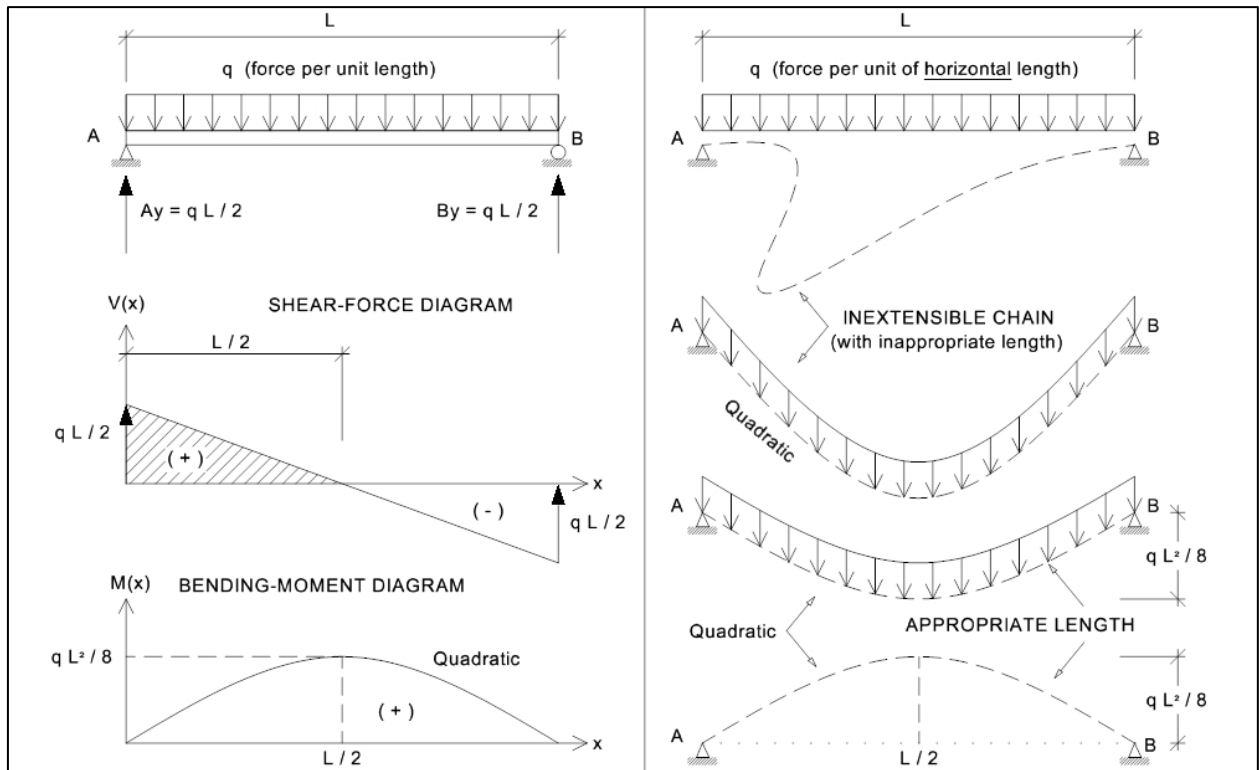


Figure 3: Simply-Supported Beam with Uniformly Distributed Load

Consider the chain with the proper length. By solving the equilibrium equations corresponding to one full half of this chain, it can be shown that $H=1$. Also, those solutions show that the vertical components of the reactions for the chain coincide with the vertical reactions for the beam and, as expected, the horizontal components of the reactions for the chain are $H=1$. The beam has no horizontal reactions.

If a mathematical expression for $M(x)$ is needed, it can be obtained by satisfying the equilibrium of moments at an arbitrary point P located at coordinate x on the beam. This is the approach presented in most textbooks. The resulting expression is: $M(x) = (qL^2/2)[(x/L)-(x/L)^2]$. Alternatively, the same expression is obtained by integrating Equation 10 twice and using $M(0)=0$ and $M(L)=0$ as boundary conditions. Similarly, two successive integrations of Equation of 11, with $z(0)=0$ and $z(L)=0$, produces the analogous expression: $H_z(x) = (qL^2/2)[(x/L)-(x/L)^2]$. For

$H=1$, the chain position at mid-span is $z(L/2) = qL^2/8$, which coincides with the value of the bending moment at the mid-span of the beam.

Example 2 (Figure 4)

This example considers a simply-supported beam with a concentrated load \mathbf{P} at a distance a from the left support. Even though the governing differential equations, 10 and 11, have been derived for a continuous forcing function $q(x)$, the chain analogy is still applicable for concentrated loads. The left-hand side of Figure 4 shows the beam with its support reactions, shear, and moment diagrams. Assume that the moment diagram is not known yet.

The right-hand side of Figure 4 shows a weightless, inextensible chain supported by hinges A and C at a distance L from each other. Consider that this chain has an arbitrary length $\Gamma > L$. The same load \mathbf{P} acting on the beam is applied on the chain. It acts along a vertical line at a distance a from the left support.

Since most students are knowledgeable on chain behavior, they will have no difficulties in visualizing the chain profile at equilibrium. The shape of the symmetric image of this profile should be similar to the shape of the bending moment diagram. However, since the length of the chain was arbitrarily selected, the vertical distances measured along the profile, may not coincide with the values of the bending moment. This coincidence is attained only when an appropriate length of chain is used. Such length makes the horizontal component of the internal tensional force to be equal to 1 ($H=1$). The appropriate length can be determined by using equilibrium equations. However, for quick visualization purposes, the approach described in the following paragraph is preferred.

We will determine the value of the bending moment at a selected point (location) of the beam. In this example, the chosen point is B. The area under the shear diagram, calculated from its origin to the location of point B, gives the following value for the moment at B: $P a b / L$. This value is used to adjust the initial chain profile. The adjusted profile corresponds to the appropriate length of chain. Then, the symmetric image of the adjusted profile should be similar, in shape and magnitude, to the corresponding bending moment.

An equilibrium analysis of the last chain (the chain with the appropriate length) shows that $H=1$. Also, for this chain, the vertical components of the support reactions coincide with the reactive forces of the beam. As expected, the horizontal components of the reactions for the chain are equal to $H=1$. The beam has no horizontal reactions.

Example 3 (Figure 5)

This example presents a simply-supported beam with a concentrated moment \mathbf{M} . A concentrated moment causes a step in the bending moment diagram. The vertical length of the step is equal to the magnitude of the moment. This example shows how the chain analogy can be used to properly visualize the effect of concentrated moments. Figure 5, shows the beam on its left-hand side. Its span is L and the concentrated moment is acting at a distance $(0.6)L$ from the left support.

The corresponding weightless, inextensible chain is shown on the right-hand side of Figure 5. Supports A and C are at the same level and separated by a distance L . Moment \mathbf{M} is applied at the same location that it acts on the beam. Since a concentrated moment causes a step in the bending-moment diagram, the equilibrium profile of the loaded chain should be able to show this effect. For this purpose, the chain has to be slightly modified. It must contain a long rigid link at the location of moment \mathbf{M} . The length of this link should be equal to the magnitude of moment \mathbf{M} . Initially, the long link should be accommodated horizontally in the unloaded, loose chain. Once the chain is attached to its supports, moment \mathbf{M} is applied to the long link. If the chain were of the proper length, moment \mathbf{M} should cause the long rigid link to rotate (in the direction indicated by the applied moment) and attain a vertical position at equilibrium.

As it was the case in the previous examples, the proper length of the chain is not known a priori. However, the proper magnitude of the profile can still be obtained by calculating the bending moment at a single selected point of the beam. To perform this calculation, we may integrate the shear diagram (i.e., consider the area under its curve). However, concentrated moments are not accounted for by integration of shear-force functions. If concentrated moments are located on the left of the selected point, they should be appropriately added, or subtracted, from the calculation obtained via the shear diagram.

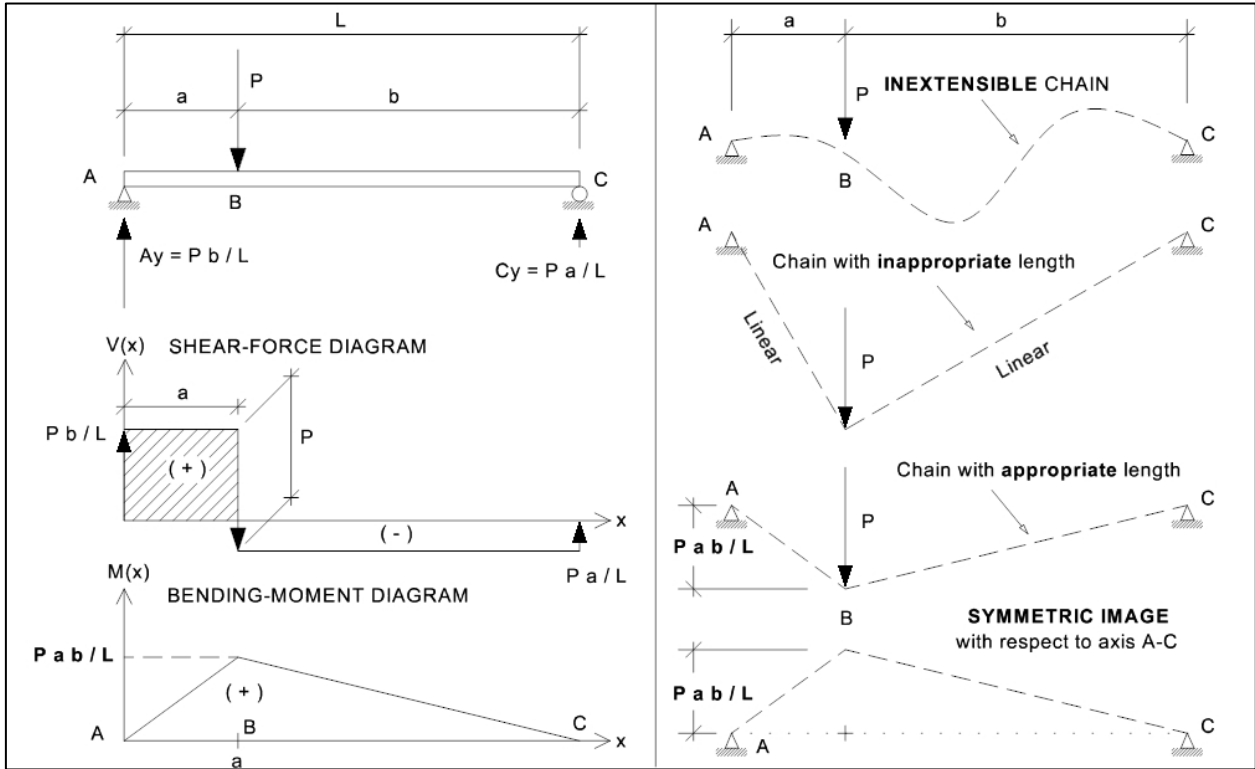


Figure 4: Simply-Supported Beam with a Concentrated Load.

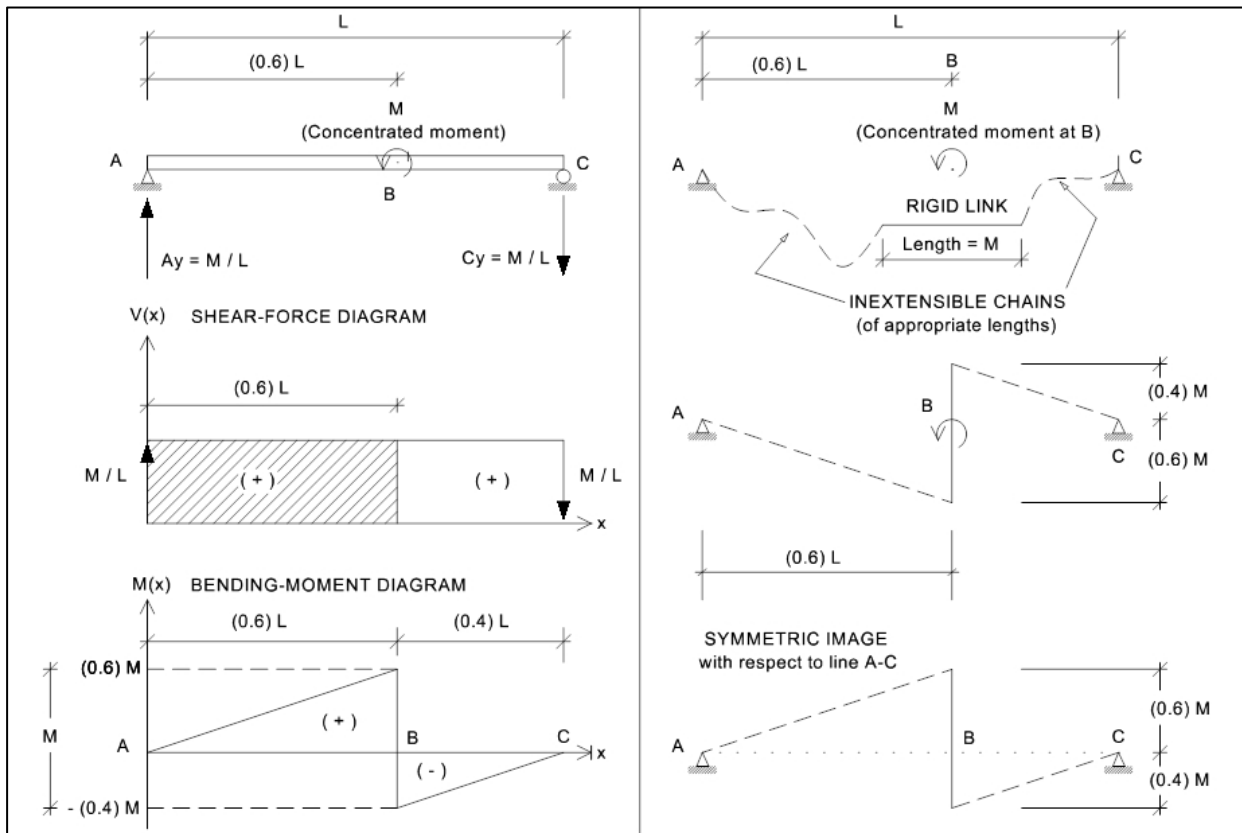


Figure 5: Simply-Supported Beam with a Concentrated Moment

We choose to calculate the bending moment at point B. It is $(M/L)(0.6)L = (0.6)M$. This information is used to adjust the position of the chain profile. The adjusted chain should now have the proper length required by the analogy (and H should be equal to 1). The last step is to obtain the symmetric image of the adjusted profile. It should adequately represent the bending-moment diagram of the beam.

An equilibrium analysis of the long, vertical link, in the chain with the proper length, shows that the horizontal component of the internal tension in the chain is equal to $N=I$. The vertical components of the support reactions of the chain are equal to the support reactions of the beam. As expected, the horizontal component of the reactions are equal to $H=I$. The beam has no horizontal reactions.

Example 4 (Figure 6)

This example shows a simply-supported beam with overhang. The total length of the beam is L . Even though the distance between the beam supports, points A and C, is $(2/3)L$, the supports for the chain, points A and D, must still be separated by the total length L of the beam.

The loads applied on the chain are the same as those acting on the beam. However, in this case, an additional force has to be applied on the chain. That force is the beam reaction at support C. Since the chain has no support at C, the beam reaction at C must be considered as a load on the chain at point C.

A qualitative visualization of the bending-moment diagram is attained in the same fashion as described in the previous examples. The visualization of the shape with the proper dimensions can be assisted by calculating the bending moments at points B and C of the beam.

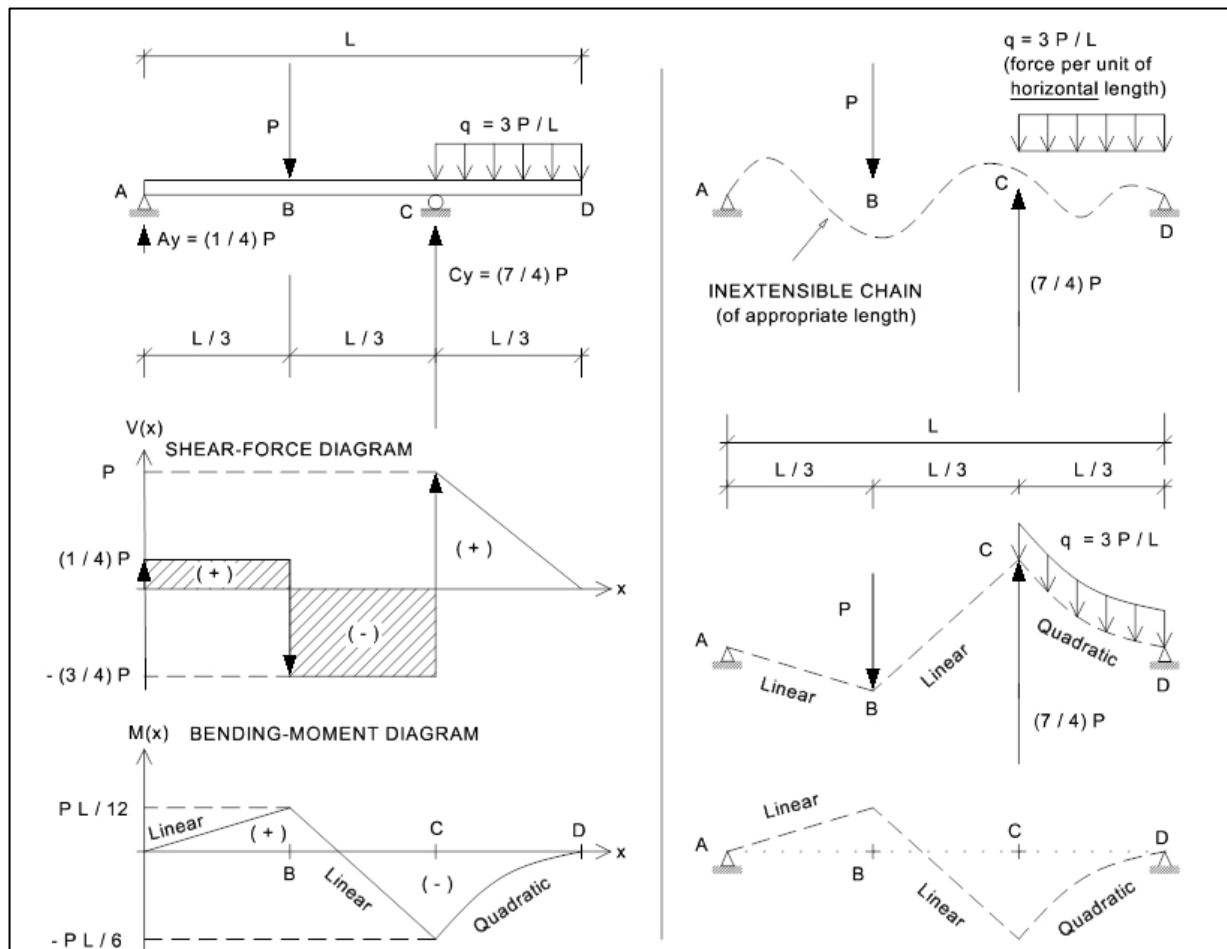


Figure 6: Simply-Supported Beam with Overhang

An equilibrium analysis of point B in the loaded chain, with the proper length, shows that the vertical reaction of the chain at point A is the same as the reaction of the beam at A. Also, the horizontal component of the reaction at A is $H=I$. The beam has no horizontal reactions.

Example 5 (Figure 7)

A Cantilever beam with a uniformly distributed load is presented in this example. The total length of the beam is L . Even though this beam is supported only at point B, the chain, as required by the analogy, will be supported at both end points, A and B. They are separated by a distance equal to the total length of the beam L .

The beam has a reactive moment at point B. Its value is $M = qL^2/2$. This moment should be considered in the chain as a concentrated moment applied at point B. As it was the case in Example 3, a long rigid link must be incorporated in the chain to properly model this concentrated moment. In this case, the long link will be the end link of the chain and should be attached to support B. The proper length of this link should be equal to the magnitude of the moment $M = qL^2/2$.

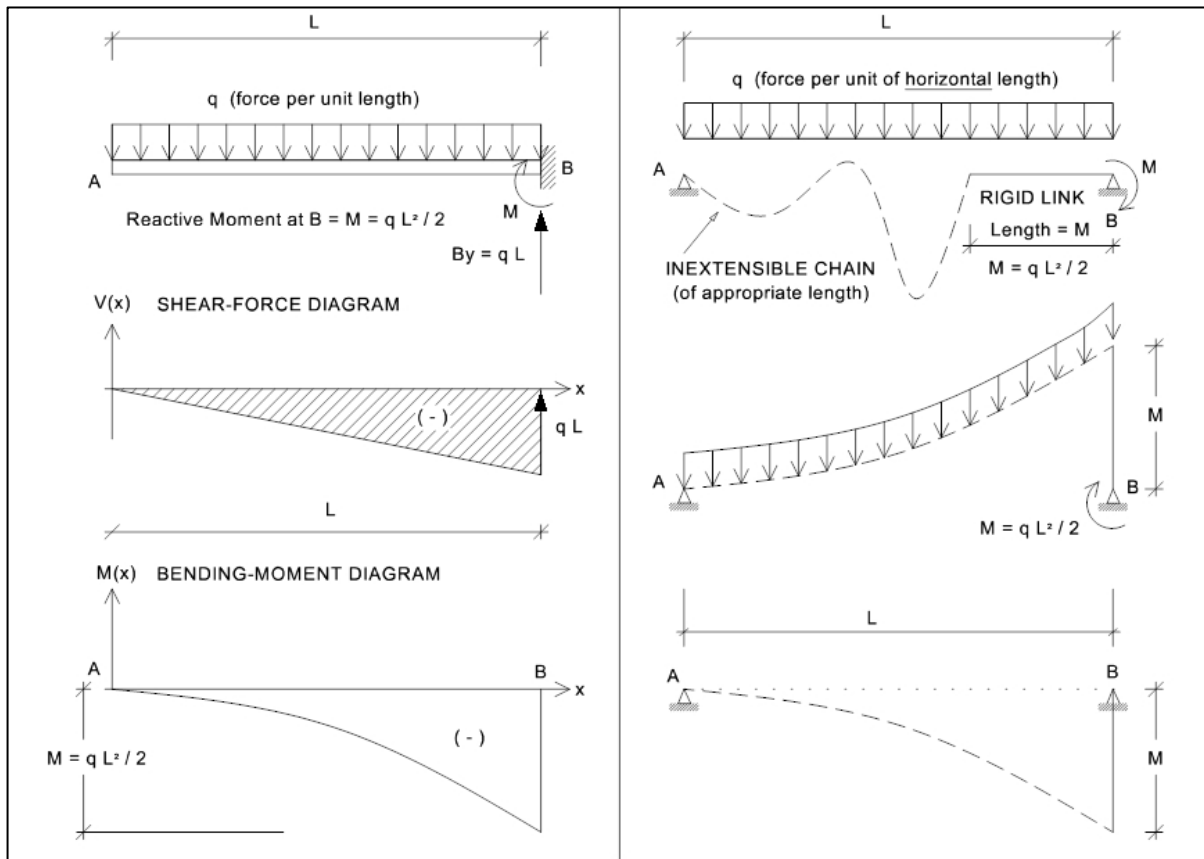


Figure 7: Cantilever Beam with Uniformly Distributed Load.

For visualization purposes, initially, the loose chain should contain the long link in the horizontal position. After attaching the ends to supports A and B, the visualized chain should look as indicated on top of Figure 7. Before loading it with the distributed load, the concentrated moment M should be considered. The long link should be rotated in the direction indicated by moment M and should attain a vertical position. Then, the distributed load should be applied to the chain. If the proper length of chain is used, the equilibrium configuration will correspond to the vertical position of the long link. The symmetric image of that chain profile should coincide with the bending moment of the beam.

If the chain has the proper length, an equilibrium analysis of the vertical link of length $M=qL^2/2$ will show that the horizontal component of the tensional force in the chain is equal to $H=I$.

The analogy may also be used in more complex, statically determinate beams, such as compound beams with internal hinges. In such cases, the chain should pass through fixed hoops at the location of the hinges. Also, it is possible to use the analogy to qualitatively visualize the moment diagrams of statically indeterminate beams.

CONCLUSIONS

This work presented the use of an analogy that assists in the generation of bending-moment diagrams in beams. This Chain Analogy is based on two similar differential equations describing two different equilibrium problems in Statics. One of them can be easily visualized and is used to assist in the solution of the other. The shape of bending-moment diagrams, in transversally loaded beams, is analogous to the shape of the vertical profile, at equilibrium, of a weightless, inextensible chain with the same loadings that affect the beam. Several examples were presented to illustrate the proper use of details that need to be considered in the chain so its visualized profile is similar to the moment diagram of the corresponding beam. Also, it is shown that the profiles of chains with proper lengths are exactly the same as the moment diagrams of the corresponding beams. Chains with that appropriate length have the horizontal component of their tensional forces equal to a unit force. The presented cases indicate that the analogy is not restricted to certain support or loading conditions. Five examples show how to properly use it in simply-supported beams, cantilevered beams, and in beams with overhangs. They included distributed loads and concentrated moments and loads. Students who used the analogy indicated that it is a useful tool.

REFERENCES

- [1] Beer, Ferdinand, Russell Johnston and Elliot Eisenberg, *Vector Mechanics for Engineers, Statics*, McGraw-Hill, New York, 2004.
- [2] Farhey, Daniel, *Conceptual Visualization of Moment Diagrams for a Simply Supported Beam*, Webpage: <http://academic.udayton.edu/DanielFarhey/mconcept.htm>, 2002.
- [3] Hibbeler, R.C., *Engineering Mechanics, Statics*, Prentice Hall, New York, 2001.
- [4] Irvine, H. Max, *Studies in the Statics and Dynamics of Simple Cable Systems*, Thesis, California Institute of Technology, Pasadena, California, 1974.
- [5] Irvine, Max, *Cable Structures*, Dover Publications, New York, 1992.
- [6] Meriam, James and Glenn Kraige, *Engineering Mechanics, Statics*, John Wiley & Sons, 2002.
- [7] Miller, Gregory and Stephen Cooper, *Visual Mechanics: Beams and Stress States*, Software Package, Wadsworth/ITP, 1998.
- [8] Truesdell, Clifford, III, *The Rational Mechanics of Flexible or Elastic Bodies 1638-1788*, L. Euleri Opera Omnia, series II, vol. II, part 2. Zurich: Fussli, 1960.

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