AN ANALYTICAL APPROACH TO FRACTIONAL BOUSINESQ-BURGES EQUATIONS

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Original scientific paper
https://doi.org/10.2298/TSCI2004581L

This paper proposes an analytical approach to fractional calculus by the fractional complex transform and the modified variational iteration method. The fractional Bousinesq-Burges equations are used as an example to reveal the main merits of the present technology.

Key words: *fractional Bousinesq-Burges equation, fractional complex transform, modified variational iteration method*

Introduction

In this paper, we consider the following fractional Bousinesq-Burges equations:

$$\mathrm{D}_t^{\alpha} u + 2uu_x - \frac{1}{2} v_x = 0$$
$$\mathrm{D}_t^{\alpha} v + 2(uv)_x - \frac{1}{2} u_x xx = 0 \tag{1}$$

for determining the horizontal velocity $u(x, t)$ and the height $v(x, t)$ of the water surface above a horizontal level at the bottom [1]. Here $0 < \alpha < 1$ is a constant representing the order of fractional derivative, and $\mathrm{D}_t^{\alpha} u$ is He's fractional derivative which is defined by:

$$\mathrm{D}_t^{\alpha} u = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(x,s) - u(x,s)] \mathrm{d}s \tag{2}$$

with a known function $u_0(x, t)$ [2-8]. The $\mathrm{D}_t^{\alpha} v$ is similarly defined by He's fractional derivative of $v(x, t)$. When $\alpha = 1$, the fractional PDE (1) reduce to the classical Bousinesq-Burges equations [9].

The classical Bousinesq-Burges equations can be used to simulate the propagation of shallow water waves, and have been widely applied in the area of fluid dynamics [9, 10]. In the past decades, many different solutions to Bousinesq-Burges equations have been constructed, including travelling wave solution, soliton solution, interaction solution, rational solution, quasi-periodic solution and others, and there are many analytical methods for solving classical Bousinesq-Burges equations, among which the homotopy perturbation method [11-14] and the variational iteration method [15-18] are most effective. The couple of the homotopy perturbation and the Laplace transform is widely used for solving fractional differential equations, the

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technology was called as He-Laplace method [19-21]. To our knowledge, there are few works on the investigation of the numerical or analytical solutions to the fractional Bousinesq-Burges equations [1]. In this paper, we aim at considering the numerical behavior of the fractional Bousinesq-Burges equations by coupling the modified variational iteration method (MVIM) and the fractional complex transform (FCT), the latter was proposed by He and Li to convert the fractional differential equation into ODE [22-25]. Recently Ain and He [25, 26] gave a totally new insight into the transform, it can be considered as the transform of a fractal space or time to its continuous partner on two different scales, so it is also called as the two-scale transform. The VIM was first proposed to solve fractional differential equations in 1998 [15], a modified version of VIM was presented in [27, 28] for solving the linear and non-linear PDE. Motivated by the FCT and the MVIM, we construct an analytical approach named as FCT-MVIM technique. This technique is applied to the initial value problem of the fractional Bousinesq-Burges equations. We can obtain the approximate solutions with high accuracy after few iteration steps. Comparisons with the approximated solutions obtained by FCT-MVIM technique and the exact solutions are given to show its efficiency.

Fractional complex transform

It is difficult to give the analytical or numerical solution of fractional differential equation. To overcome this issue, a special FCT was proposed in [22-25], which can be used to transform the original fractional PDE to ordinary PDE. We consider a fractional PDE:

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_t^{2\alpha}, u_x^{2\beta}, ...) = 0 \tag{3}$$

where $u_t^{\alpha} = [\partial^{\alpha} u(x,t)]/(\partial t^{\alpha})$ denotes He's fractional derivation defined by eq. (2), the function $u(x, t)$ is continuous (but not necessarily differentiable), and $0 < \alpha < 1$, $0 < \beta < 1$.

Consider the following fractional complex transform [22-25]:

$$T = \frac{pt^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{qx^{\beta}}{\Gamma(1+\beta)} \tag{4}$$

with non-zero constants p and q. The physical explanation of eq. (4) was given by in 2019 [25], and it is also called as the two-scale transform, on the small scale of (x, t), spatiotemporal variables are discontinuous, while on the larger scale of (X, T), they become approximately continuous, so the traditional calculus can be applied. In view of eq. (4), we have:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = p\frac{\partial u}{\partial T}, \quad \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = q\frac{\partial u}{\partial X}$$

Therefore, we can rewrite the fractional differential eq. (3) as an ordinary PDE. It is noted that this FCT can also be used to the fractional PDE with various definitions of derivative [29-32].

Modified variational iteration method

The original variational iteration method was first proposed by He [15, 16], and has been widely discussed for solving the linear and non-linear differential equations. For speeding up the convergence and reducing the computation cost of VIM, a MVIM was proposed in [27, 28]. We consider the following non-linear PDE to illustrate the basic idea of MVIM:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{5}$$

where $L = \partial/\partial t$ and R are two linear operators with the partial derivative, N is a non-linear operator, and $g(x, t)$ is an inhomogeneous term. Then we can construct the following iteration formula:

$$u_{(n+1)}(x,t)=u_n(x,t)-\int_0^t \{R[u_n(x,\xi)-u_{(n-1)}(x,\xi)]+[G_n(x,\xi)-G_{(n-1)}(x,\xi)]\mathrm{d}\xi \tag{6}$$

where $u_{-1}=0,\ u_0=f(x),\ u_1=u_0-\int_0^t [R(u_0-u_{-1})+(G_0-G_{-1})]\mathrm{d}\xi,$ and $G_n(x,t)$ is given by $Nu_n(x, t) = G_n(x, t) + 0(t^{n+1})$.

The FCT-MVIM technique

The FCT-MVIM technique is a combination of the FCT [22-25] and the MVIM [27, 28]. We first apply He's FCT to a fractional PDE, and obtain an ordinary PDE. Then, the analytical solution to the ODE can be given by the MVIM. The detailed procedure of FCT-MVIM technique is given below.

Numerical example

To show the efficiency of FCT-MVIM, we consider the initial value problem of the fractional Bousinesq-Burges eq. (1) with the following initial conditions:

$$u(x,0)=-\frac{\omega}{2k}+\frac{1}{2}k\tanh(kx),\quad v(x,0)=-\frac{1}{2}k^2\operatorname{sech}^2(kx) \tag{7}$$

where k and ω are given constants. The single soliton solutions to the initial value problem associated with eqs. (1) are given by [33]:

$$u(x,t)=-\frac{\omega}{2k}+\frac{1}{2}k\tanh(kx-\omega t),\quad v(x,t)=-\frac{1}{2}k^2\operatorname{sech}^2(kx-\omega t)$$

By FCT-MVIM technique with $T = t^{\alpha}/[\Gamma(1 + \alpha)]$, the previous initial value problem can be equivalently transformed to the ordinary PDE:

$$\frac{\partial u}{\partial T}+2uu_x-\frac{1}{2}v_x=0$$

$$\frac{\partial v}{\partial T}+2(uv)_x-\frac{1}{2}u_{xxx}=0$$

with the initial conditions (7).

By using MVIM, it is easy to obtain the iteration formulae:

$$u_{(n+1)}(x,T)=u_n(x,T)+\int_0^T\left\{\frac{1}{2}[v_nx(x,\xi)-v_{(n-1x)}(x,\xi)]-[G_n(x,\xi)-G_{(n-1)}(x,\xi)]\right\}\mathrm{d}\xi$$

$$v_{(n+1)}(x,T)=v_n(x,T)+\int_0^T\left\{\frac{1}{2}[u_nxxx(x,\xi)-u_{(n-1xxx)}(x,\xi)]-[H_n(x,\xi)-H_{(n-1)}(x,\xi)]\right\}\mathrm{d}\xi$$

where $G_n(x, t)$ and $H_n(x, t)$ are defined by $2u_nu_{nx} = G_n(x, t) + \mathrm{O}(t^{n+1})$ and $2(u_nv_n)_x = H_n(x, t) + \mathrm{O}(t^{n+1})$, respectively.

In this example, we let $k = 0.2$ and $\omega = 0.04$. By previous iteration formulae, we obtain the following approximated solutions:

$$u_1 = -0.1 + 0.1\tanh(0.2x) + 0.004T\text{sech}^2(0.2x)$$

$$v_1 = -0.02\text{sech}^2(0.2x) + 0.0016T\text{sech}^2(0.2x)\text{tanhh}^2(0.2x)$$

$$u_2 = -0.1 + 0.1\tanh(0.2x) + 0.004T\text{sech}^2(0.2x) + 0.00016T^2\text{sech}^2(0.2x)\tanh(0.2x)$$

$$v_2 = -0.02\text{sech}^2(0.2x) + 0.0016T\text{sech}^2(0.2x)\tanh(0.2x) + 3.2\cdot 10^{-5}T^2$$
$$\text{sech}^2(0.2x)[\text{sech}^2(0.2x) + 2\tanh^2(0.2x)]$$

Higher-order approximate series solution [34] can be obtained if the iteration continues.

Recalling $T = t^\alpha/[\Gamma(1 + \alpha)]$, we can obtain the following second order approximations:

$$u_2 = -0.1 + 0.1\tanh(0.2x) + 0.004\text{sech}^2(0.2x)\frac{t^\alpha}{\Gamma(1+\alpha)} +$$
$$+ 0.00016\text{sech}^2(0.2x)\tanh(0.2x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2$$

$$v_2 = 0.02\text{sech}^2(0.2x) + 0.001\text{sech}^2(0.2x)\tanh(0.2x)\frac{t^\alpha}{\Gamma(1+\alpha)} +$$
$$+ 3.2\cdot 10^{-5}T^2\text{sech}^2(0.2x)[\text{sech}^2(0.2x) + 2\tanh^2(0.2x)]\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2$$

By setting $\alpha = 1$, we have the approximated solutions to the classical Bousinesq-Burges eq. (1):

$$\hat{u}_2 = -0.1 + 0.1\tanh(0.2x) + 0.004t\text{sech}^2(0.2x) + 0.00016t^2\text{sech}^2(0.2x)\tanh(0.2x)$$

$$\hat{v}_2 = -0.02\text{sech}^2(0.2x) + 0.0016t\text{sech}^2(0.2x) + \tanh(0.2x) +$$
$$+ 3.2\cdot 10^{-5}t^2\text{sech}^2(0.2x)[\text{sech}^2(0.2x) + 2\tanh^2(0.2x)]$$

Figure 1 plots the compared results of the second order approximated solution $\hat{u}_2$ and the exact solution $u(x, t)$ when $\alpha = 1$. The comparisons of the approximate solution $\hat{v}_2$ and the exact solution $v(x, t)$ are presented in fig. 2. It is easy to see that the approximated solutions obtained by FCT-MVIM agree well with the exact solutions to the classical Bousinesq-Burges equations. We then consider the behavior of the solutions to eq. (1) when the time, t, is set as a constant. Figures 3 and 4 show the paired curves of $[\hat{u}_2, u(x,t)]$, $[\hat{v}_2, v(x,t)]$, respectively. The absolute errors of $\hat{u}_2$ and $\hat{v}_2$ are given in tab. 1. We remark that the accuracy of the approximated solutions can be further improved by considering more iteration steps of FCT-MVIM.

In order to further illustrate the effectiveness of FCT-MVIM for the fractional Bousinesq-Burges eq. (1), we provide the numerical results of the approximations with different α and time, t. The numerical solutions can be obtained without linearization, perturbation or complicated iterations. Figures 5 and 6 show the numerical behavior of the approximated solutions obtained by FCT-MVIM for the fractional Bousinesq-Burges equations with $\alpha = 0.4$ and $\alpha = 0.8$.

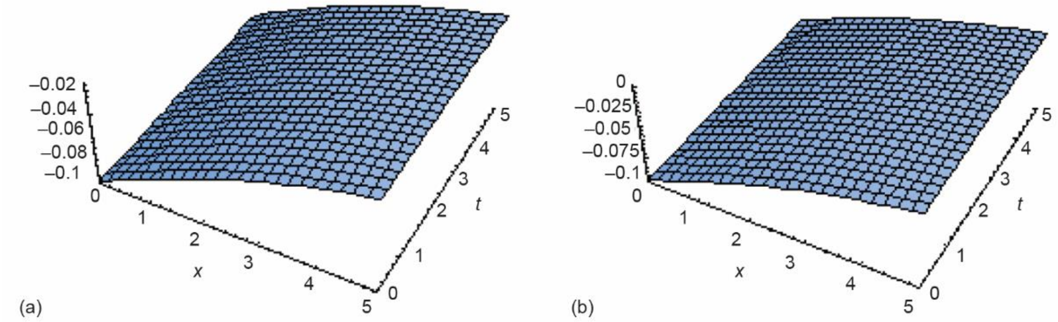

Figure 1. The approximated solution $\hat{u}_2$ (a) and the exact solution *u*(*x*, *t*) (b) for classical Boussinesq-Burger equation [9]

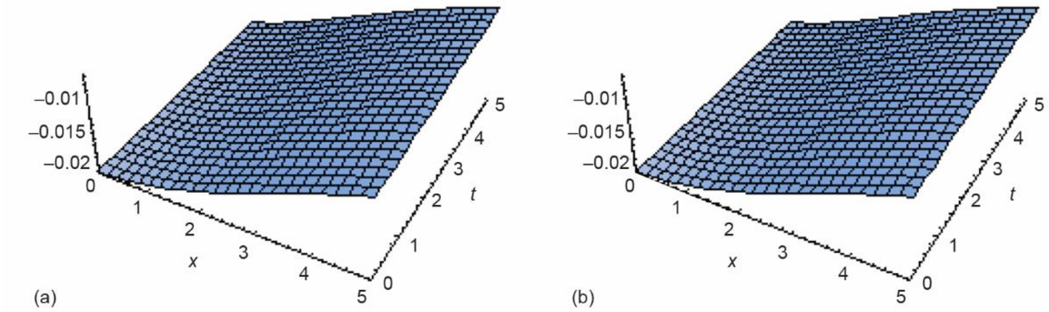

Figure 2. The approximated solution $\hat{v}_2$ (a) and the exact solution *v*(*x*, *t*) (b) for classical Boussinesq-Burger equation [9]

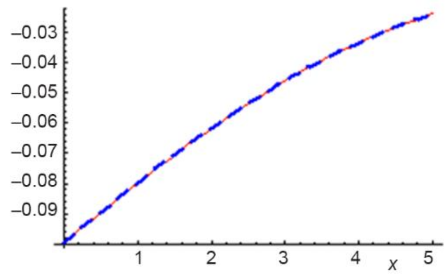

Figure 3. Compared results of $\hat{u}_2$ (red) and *u*(*x*, *t*) (blue dashed) for Boussinesq-Burger equation when *t* = 0.1 (*for color image see journal web site*)

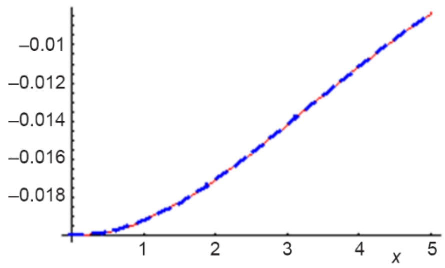

Figure 4. Compared results of $\hat{v}_2$ (red) and *v*(*x*, *t*) (blue dashed) for Boussinesq-Burger equation when *t* = 0.1 (*for color image see journal web site*)

Table 1. The absolute errors of $\hat{u}_2$ and $\hat{v}_2$

X	$\lvert \hat{u}_2 - u(x,t) \rvert$	$\lvert \hat{v}_2 - u(x,t) \rvert$	x	$\lvert \hat{u}_2 - u(x,t) \rvert$	$\lvert \hat{v}_2 - u(x,t) \rvert$
0.1	$2.1296 \cdot 10^{-9}$	$7.1590 \cdot 10^{-10}$	0.6	$2.0109 \cdot 10^{-9}$	$3.9624 \cdot 10^{-10}$
0.2	$2.1190 \cdot 10^{-9}$	$1.3928 \cdot 10^{-10}$	0.7	$19684 \cdot 10^{-9}$	$4.5494 \cdot 10^{-10}$
0.3	$2.1017 \cdot 10^{-9}$	$2.0603 \cdot 10^{-10}$	0.8	$1.9201 \cdot 10^{-9}$	$5.1067 \cdot 10^{-10}$
0.4	$2.0779 \cdot 10^{-9}$	$2.7139 \cdot 10^{-10}$	0.9	$1.8663 \cdot 10^{-9}$	$5.6311 \cdot 10^{-10}$
0.5	$2.0475 \cdot 10^{-9}$	$3.3493 \cdot 10^{-10}$	1.0	$1.8076 \cdot 10^{-9}$	$6.1198 \cdot 10^{-10}$

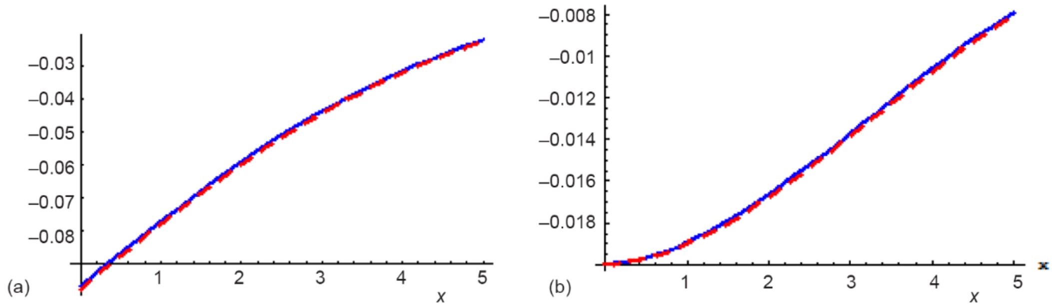

Figure 5. Results of u_2 (a) and v_2 (b) with α = 0.4 (blue) and α = 0.8 (red dashed) when t = 0.5
(*for color image see journal web site*)

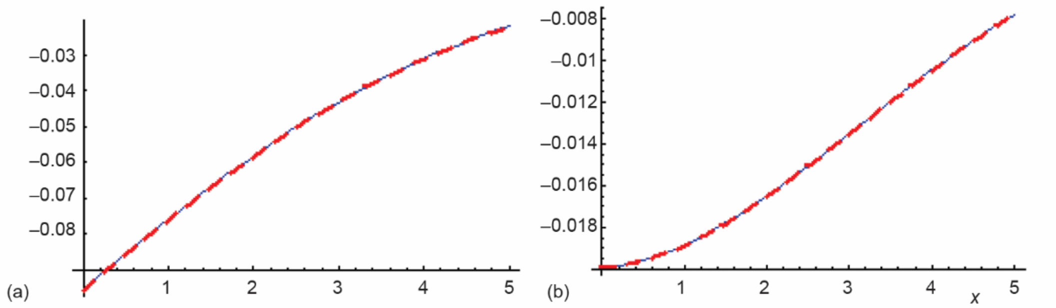

Figure 6. Results of u_2 (a) and v_2 (b) with α = 0.4 (blue) and α = 0.8 (red dashed) when t = 1
(*for color image see journal web site*)

Conclusion

This paper focused on the numerical behavior of the fractional Bousinesq-Burges equations. The approximated solutions to the initial value problem of the fractional Bousinesq-Burges equations were constructed by an analytical approach (FCT-MVIM), which is based on the FCT and the MVIM. Numerical results shown that the FCT-MVIM technique provides the approximated solutions with high accuracy without linearization and perturbations. Therefore, we can conclude that FCT-MVIM is efficient for solving the fractional Bousinesq-Burges equations. In our future work, we will further consider the convergence analysis of this approach, and extend it to other non-linear fractional differential equations.

Acknowledgment

The work is supported by the Natural Science Foundation of Zhejiang Province (LY17A010001).

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Paper submitted: April 27, 2019
Paper revised: November 1, 2019
Paper accepted: November 1, 2019