An Essay on the Quadratic Formula:
Origins, Derivation, and Applications

Foreword:

The quadratic equation is a formula that is used to solve equations in the form of quadratics. A quadratic is an equation in which the degree, or highest exponent, is a square. The degree also describes the number of possible solutions to the equation (therefore, the number of possible solutions for a quadratic is two). The quadratic formula is as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are other methods used for solving quadratics, such as graphing, factoring, and completing the square. Depending on the quadratic in question, there is an appropriate time for each method. However, the quadratic formula is advantageous in the fact that it is applicable to all quadratics and will always yield the correct solution. This essay will discuss the origins of the quadratic formula, its applications, and derivation. To do so, however, the methods stated above must be understood.

Completing the Square and Factoring:

To solve a quadratic equation through completing the square, it is important that the equation is in the standard form of a quadratic, as follows:

$$ax^2 + bx + c = 0$$

Note that both *a* and *b* in the equation are coefficients of the variable x. C is the constant, which, in this context, is a real number lacking a variable. If the equation is not in this format, simply subtract terms so the equation is in standard form. The next step of the process is to attempt to factor the equation. To factor the equation, multiply the coefficient of the x^2 term by the constant, a times c. Attempt to find two numbers that multiply to a times c and add to the coefficient of x, or *b*. It is important to include the appropriate signs of the terms when multiplying and adding. If the *a* value is 1, then the equation can be written as two sets of binomials where each of the multiplying and addition factors are added to *x*. This is an example of simple factorization. The next possibility for factoring involves the *a* value being a real number other than one. The difference occurs when the two binomials must be written. The two numbers that multiply to a times c and add to b should replace the b value in the equation. Then, group the first two terms and the last two terms. Remove the greatest common factor in each group. The addition of the greatest common factors is the first binomial solution, and the grouped terms (which are equal) is the second binomial. The following examples may be helpful in describing both methods of factoring.

Example 1:

1. $x^2+7x+12=0$	1. Find two numbers that add to 12 (The coefficient of x^2 is one, so multiply the constant, or *c* term, by one) Thus, *ac*=12, or 12(1)=12.
2. 3*4=12 and 3+4=7	2. The numbers 3 and 4 multiply to 12 and add to the *b* value of 7.
3. (x+3)(x+4)=0	3. Since the *a* value is one, the factors (binomial solutions) can immediately be written as the factors added to *x*.

4. $x+3=0 \quad x+4=0$

4. Since the original equation is equal to 0, the binomial factors can be set equal to 0.

5. $x=-3 \quad x=-4$

5. Solve for *x*.

Example 2: In this example, the equation is not in standard form.

1. $-2x^2=-3x-5$

1. Add *3x* to each side of the equation and add *10* to each side so the equation is in standard form.

2. $-2x^2+3x+5=0$

2. Multiply *a* by *c.* Find two numbers that will multiply to *ac* and add to the *b* value.

3. $(-2x^2-2x)(5x+5)=0$

3. The two numbers are -2 and 5. Since the *a* value is not one, the terms must be grouped accordingly.

4. $-2x(x+1)+5(x+1)=0$

4. Remove the greatest common factor from each set of parenthesis. Notice that the binomials in the parenthesis are equal.

5. $(-2x+5)=0 \quad (x+1)=0$

5. Form the two binomial expressions and set each equal to zero.

6. $x=\frac{5}{2} \qquad x=-1$

6. Solve for x.

If the equation cannot be factored, then completing the square can be implemented. The equation should first be in standard form. Group each term that contains the variable *x*. If the value of *a* is not *1*, then divide each of the grouped terms by the coefficient of *a*. Next, add $(\frac{b}{2})^2$ to the equation as well as the opposite of $(\frac{b}{2})^2$. Group the *x* terms and the $(\frac{b}{2})^2$ value to form a trinomial. This is now a perfect square trinomial and can be factored. Add the $-(\frac{b}{2})^2$ term and the constant value. Make sure to distribute the *a* value to the $-(\frac{b}{2})^2$ term if the *a* value was removed when grouping the *x* terms. Factor out the trinomial and subtract the constant, thus setting each equal to one another. Solve for *x*. See the example below:

1. $2x^2+8x+9=0$

1. Make sure that the equation is in standard form and group the x terms.

2. $2(x^2+4x)+9=0$

2. Remove the a value from the grouped terms.

3. $2(x^2+4x+4)-4+9=0$

3. Add the value of $(b/2)^2$ and subtract $(b/2)^2$ and group accordingly.

4. $(x+2)^2-8+9=0$

4. Factor the trinomial and make sure to distribute the a value to the $-(b/2)^2$ value.

5. $(x+2)^2=1$

5. Move the constant to the other side of the equation.

6. $x+2=1$

6. Solve for x.

7. $x=-1$

In step four of the problem, the quadratic is in vertex form, or $f(x) = a(x-h)^2 + K$. This equation is very important when graphing. If the *a* value is greater than 1, then the graph stretches vertically. If *a* is less than 1 and greater than 0, then the graph shrinks vertically. The axis of symmetry is x= h. The axis of symmetry contains the vertex of the parabola, the point at which the y value is at the greatest or least. This can be expressed as the relative minimum or maximum of the equation. The vertex is represented as (h,k). The equation can also be written in expanded form. To do so, simply multiply the binomial that is squared and combine like terms.

Completing the square is not always simple, especially when fractions are involved. It can also be difficult and time consuming to determine which method (graphing, factoring, and completing the square) is best for a specific quadratic equation. Thus, a general method needed to be developed that is applicable to every quadratic equation. And so the quadratic formula was born.

Origins:

The development of the quadratic formula has spanned millennia. The original problem of unifying quadratics arose around 2000 B.C.E., when Egyptian, Chinese, and Babylonian engineers found need of a way to measure the scale of a figure to its area. This information would be used to determine walls in architectural floor plans. The Egyptians first embarked on the search for such a formula around approximately 1500 B.C.E. Egyptian mathematicians did not follow the logical flow of information from specific to general, but rather created a broad table of measurements for certain figures. Thus, no formula was needed.   Errors occurred, however, when translations of the tablet were necessary and some of the information was unreliable.   This method of calculation is closely connected to the social order of Egypt at the time. The Egyptians were not allowed to question any form of authority. Thus, the Egyptians had no proof of any of the calculations on the tablets, nor were able to extirpate errors.

The next advancements were made by the Babylonians and Chinese in 400 B.C.E., when a general formula was desired. The Babylonian's system of mathematics was more advanced than the Egyptian's in the sense that it was based on a hexagesimal system (base 60). This base system was advantageous in the simplicity of addition and multiplication. At this point in history, completing the square had been developed to solve problems pertaining to a shape's area. It seems that the procedure was still slightly flawed due to the fact that guesswork was involved. This leads to the conclusion that a specific procedure was not used. The Chinese arrived at this same method at around the same time, despite the fact that they, like the Egyptians, did not have a numerical system. However, the use of the abacus confirmed calculations.

The next major advancements in the creation of the quadratic formula are credited to Pythagoras and Euclid. Pythagoras of Italy discovered in 500 B.C.E. that the ratio of the area of a square to its side lengths, essentially the square root of a number, is not necessarily an integer. However, he dismissed this fact because the notion of a non-real number was far too radical and ridiculous in the Mediterranean region. In fact, it was a laughing matter. Pythagoras swore his close associates to secrecy in order to avoid having his reputation impaired. One of the associates, however, released the concept of non-integers to the public, infuriating Pythagoras. Pythagoras had the associate executed. Paul Erdös referred to this event as the "Pythagorean Scandal." In 300 B.C.E. Egypt, Euclid also found the proportion of a rectangles area and side lengths may not result in an integer. Euclid accepted this discovery and called his solutions irrational numbers. In Euclid's famous collection of mathematics, the Elements, regarded as one of the greatest and most influential works in mathematics of all times, Euclid expanded the mathematics necessary for applications involving square roots. The notation and formulas, however, were very different than today, making it impossible for precise evaluations of a square root. This was, therefore, disregarded by engineers at the time.

The Indian system of commerce in 700 C.E. is responsible for the next advancement in the quadratic equation. The Indian commerce was developed on the same decimal system that is in current use. Indian merchants also used zero, and negative numbers to explain business. The Hindu mathematician Brahmagupta found a general quadratic equation using numbers and irrational numbers. He was also the first to realize that two solutions are possible when solving a quadratic. The complete solution developed by 1100 C.E. when the Hindu mathematician Baskhara realized that all positive numbers have two square roots.

In 820 C.E. near Baghdad, the Islamic mathematician Mohammad bin Al-Khwarismi also derived the quadratic formula, but he rejected negative solutions. This may seem insignificant, but it, in fact, had quite a large impact on the development of the formula we use today. Mohammad's formula was brought to Europe, first in Barcelona, around 1100 C.E. by the astronomer Abraham bar Hiyya.

The conflict presented by Hiyya was resolved in 1500 C.E. in the Renaissance period. This time period was a rebirth for mathematics in the sense that attention reverted to original math problems. In 1545, the algebraist Girolamo Cardano combined Al-Khwarismi's solutions with those of Euclid. He also took into account complex, or imaginary numbers. Mathematician François Viète of France introduced a more modern notation and symbolism by the 1500s. Finally, in 1637, René Descartes published a modern volume of geometry, which contained the quadratic formula in the form used today.

It is clear that the development of the quadratic formula was influenced by those who participated in its creation, as well as the events in Europe and Asia and their effect on the development of mathematics.

Derivation:

The Quadratic Formula is derived from the completing the square method. Recall that in this process a quadratic is made into a perfect square trinomial, regardless of the values of the coefficient and constant value in the equation. Begin with the standard form of a quadratic equation:

$$ax^2+bx+c=0$$

Group the monomials which contain *x* using parenthesis.

$$(ax^2+bx)+c=0$$

Factor out the value of *a*, to account for the fact that *a* may not be *1*.

$$a(x^2+\frac{b}{a}x)+c=0$$

To transform the quadratic into a perfect square trinomial, follow the completing the square method by adding and subtracting the value of $(\frac{b}{2})^2$. This value must be added and subtracted so that zero is being added to the equation. Thus, the equation is not being changed and remains balanced. Note that dividing by two is no different than multiplying by its reciprocal, or $\frac{1}{2}$.

$$a(x^2+\frac{b}{a}x+(\frac{b}{2a})^2-(\frac{b}{2a})^2)+c=0$$

Continue by grouping the equation into a trinomial and binomial as follows:

a($x^2+\frac{b}{a}$x+$(\frac{b}{2a})^2$)+(- $\frac{b}{2a})^2$+c=0

Evaluate the equation by applying the powers to a quotient property.

a($x^2+\frac{b}{a}$x+$\frac{b^2}{4a^2}$)+(- $\frac{b^2}{4a^2}$)+c=0

Distribute the *a* to the (- $\frac{b^2}{4a^2}$), since this term was removed by division from the original equation. This keeps the equation balanced. Doing so yields:

a($x^2+\frac{b}{a}$x+$\frac{b^2}{4a^2}$)+($\frac{-ab^2}{4a^2}$)+c=0

Proceed by factoring the perfect square trinomial as follows:

a(x+$\frac{b}{2a})^2$ - $\frac{b^2}{4a}$ + $\frac{c}{1}$=0

Notice that the term $\frac{-ab^2}{4a^2}$was simplified by dividing the *a* from the numerator. Next, multiply $\frac{c}{1}$ by $\frac{4a}{4a}$ so a common denominator is produced. Thus, $\frac{-b^2}{4a}$ and $\frac{c}{1}$ can be added:

a(x+$\frac{b}{2a})^2$ +(-b^2+$\frac{4ac}{4a}$)=0

To begin isolating the *x* value, remove the second term in the equation.

a(x+$\frac{b}{2a})^2$ = b^2+$\frac{4ac}{4a}$

Divide *a* from both sides of the equation. On the right side of the equation, multiply by the reciprocal, $\frac{1}{a}$.

(x+$\frac{b}{2a})^2$ = b^2 + $\frac{4ac}{4a^2}$

Take the square root of each side of the equation to isolate *x*. The following equation remains:

x+ $\frac{b}{2a}$ = $\frac{\sqrt{b^2-4ac}}{2a}$

Subtract $\frac{b}{2a}$ from each side. *X* is now isolated and the equation is the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be applied to any quadratic. Though it is fairly simple to understand and apply, concentration is required as numerous small errors can occur. Sign errors and errors in multiplication or division may occur without focus. Keep in mind that the roots of the equation may be imaginary. This will be identified if the discriminant, $b^2 - 4ac$, yields a negative number. The square root of a negative number is imaginary. Remove an i, the square root of negative one, from the discriminant if necessary. The following example of a quadratic may be helpful:
Given: $4x^2+7x+7=0$

Determine the *a*,*b*, and *c* values of the equation.

a=4 b=7 c=7

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Insert the values into the appropriate positions in the quadratic formula.

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(4)(7)}}{2(4)}$$

Simplify the discriminant and the denominator.

$$\frac{-7 \pm \sqrt{49 - 112}}{8}$$

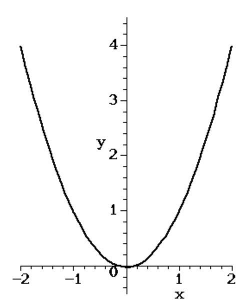
$$\frac{-7 \pm \sqrt{-63}}{8}$$

Remove i, and continue to simplify the discriminant (simplify the square root).

$$\frac{-7 \pm 9i\sqrt{7}}{8}$$

Above is the final solution. Note that there are two roots, one for the positive and one for the negative case.

Another application of the quadratic formula lies in the discriminant. If the value of the discriminant is positive (i.e. greater than zero), then the result of the equation is two real roots. If the discriminant is a negative number (i.e. less than zero), then the result is two imaginary roots. If the value of the discriminant is 0, then the result is a double root. A double root occurs when there is only one value for x (there are really two values, but they are equal, and, thus, it is superfluous to list the same value twice). The information obtained through utilizing the discriminant's properties can be used to graph quadratics, which are in the shapes of parabolas (see the figure below for the parent graph, $y=x^2$, of a parabola).



Conclusion:

The quadratic formula is a useful alternative to obtaining information about quadratics. It has disadvantages and advantages in relation to such methods as completing the square, factoring, and graphing. Thus, it should be used appropriately. The derivation of the formula is both elegant and an example of the fluctuations of mathematics in response to historical events and ancient and recent societies. The quadratic formula can be applied in numerous ways. It can be applied to the changes in motion of object, as well as nexus the Cartesian plane to physics. The quadratic formula is essential to calculus, the study of objects in motion (more generally, it is the study of change). The quadratic equation and its applications must be fully understood in order to appreciate the beauty and complexity of mathematics.

Works Cited:

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Abstract

The purpose of this essay is to act as a supplement for students who wish to reinforce their knowledge of the quadratic formula. It is not intended to be used as a primary source of education. If so, it should be used in collaboration with an expert of mathematics and quadratics. This essay focuses primarily on the ways a quadratic equation can be solved. This includes an explanation of factoring, completing the square, the quadratic formula, and references to graphical approaches and analysis. This essay also highlights the history and derivations of the formula, which allows students insight into the profession of a mathematician and the response of mathematics to the world around it. The intent of this essay is to present the quadratic formula as a device that students can utilize according to their understanding of quadratics and mathematical ingenuity.