AN EXPLORATION OF THE RIEMANN-HURWITZ FORMULA

CODY DELOS SANTOS

1. Abstract

ABSTRACT. The Riemann-Hurwitz formula is an elegant equation in its own right. The formula illustrates a relation between the integer invariants associated with holomorphic maps between Riemann surfaces. Further, the equation manages to answer the remarkably broad following question: Given all possible holomorphic maps between two compact Riemann surfaces, what may we say about the constraints on the invariants, if any such exist? Indeed, we find that the Riemann-Hurwitz formula tells us precisely what these constraints are.

2. Introduction

This paper serves as an introduction to the Riemann-Hurwitz formula. The goal of this paper is to provide a reader, who has little knowledge of complex analysis, topology, or geometry, with the basic necessities in order to understand the Riemann-Hurwitz formula. Thus, we will need to patiently and deliberately build up adequate understanding of key concepts in these disciplines in order to understand some of the foundational definitions and theorems from Hurwitz theory. However, after a thorough reading of this paper, a reader should be familiar with the role that the Riemann-Hurwitz formula plays and have some understanding of how it can be applied.

Riemann surfaces are a kind of manifold. Manifolds are all around us. Naturally, we would like to consider all of the possible maps between Riemann surfaces. Unfortunately, we find that we are unable to do so. However, if we put a few constraints on the manifolds and the maps between them then we are able to describe all possible maps between manifolds. Namely, we require that our manifolds be Riemann surfaces and we require that our maps between Riemann surfaces be holomorphic. It is through the Riemann-Hurwitz formula that we are able to desribe these maps.

The Riemann-Hurwitz formula is worthy of an entire paper for two reasons. First, the formula allows us to make very strong claims about the relationships that our discrete invariants have when looking at holomorphic maps between two compact Riemann surfaces. Second, the Riemann-Hurwitz formula allows us to deal with just integer values despite the continuous nature of our maps.

Hurwitz Theory, the discipline that we find our formula of interest nested in, seeks to utilize connections between complex analysis, geometry, and algebra. Namely, Hurwitz theory is the enumerative study of analytic functions between Riemann Surfaces. The Riemann-Hurwitz formula relates integer invariants associated to all holomorphic maps between compact Riemann surfaces via an equation. Riemann Surfaces, one-dimensional complex manifolds, are locally identified with $\mathbb{C}$. Thus, it is natural to require maps between Riemann surfaces locally to be identified with holomorphic maps from $\mathbb{C}$ to $\mathbb{C}$. The study of such objects is important for many reasons, just one of which is because when studying maximal domains of complex analytic functions we find that

Date: May 23, 2020.

looking at geometric spaces which are locally indistinguishable from $\mathbb{C}$, but globally are different from $\mathbb{C}$ is optimal.

Riemann surfaces are found in different areas of mathematics. Bernhard Riemann (1826-1866) was the first to study such objects. As such, they are named after him. Riemann first introduced Riemann surfaces when dealing with complex functions in his dissertation in 1851. Riemann, at age 25, would go on to lay the building blocks of a general theory of complex functions, which would launch a systematic study of topology, revolutionize algebraic geometry and allow him to create his famous approach to differential geometry.

In 1851 and in 1857, Riemann discussed Riemann surfaces and the holomorphic maps between them. One of the tools that would emerge from his discussion would be a formula. This formula would become the Riemann-Hurwitz formula. Riemann claimed that if we had a *surjective* map φ *Riemann surfaces* X and Y, then we could relate the *genus* of X,Y, the *degree* of φ and the amount of *ramification*. This formula will be explored in section 4.

The structure of our paper will be as follows: we will start by unpacking some of the necessary preliminary facts, definitions, and examples. Then we will discuss *Riemann surfaces* in detail. After devoting some time to Riemann surfaces, we will talk about the analytic maps between Riemann surfaces. Finally, we will be able to discuss the Riemann-Hurwitz formula.

2.1. **The Preliminaries.** We can not properly understand the Riemann-Hurwitz formula without first building the necessary foundation. Here, that means taking some time to become acquainted with some terminology and definitions. Let us first define what it means for a function to be *analytic*, since these will be our functions of interest. We should note that *holomorphic* functions are equivalent to analytic functions on an open set $\Omega \subset \mathbb{C}$, thus we will take a moment to define both.

Definition. A function $f : \Omega \to \mathbb{C}$ defined on an open set $\Omega \subset \mathbb{C}$ is said to be **analytic** at a point $z_0 \in \Omega$ if there exists a power series $\sum a_n(z-z_0)^n$ centered at z_0 with positive radius of convergence such that, for all z in a neighborhood of z_0, $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$

Definition. Let ω be an open set in $\mathbb{C}$ and f be a complex-valued function on ω. The function is **holomorphic** at a point $z_0 \in \Omega$ if $\frac{f(z_0+h)-f(z_0)}{h}$ converges to a limit when $h \to 0$

With this in mind, we will have some idea of the kinds of functions we will be examining. In our introduction, we stated that Hurwitz theory was the study of analytic functions between Riemann Surfaces. So, we have some notion of the kind of mappings we will be looking at, but what of the mathematical structures? Thus, a natural course of action would be to define a *Riemann Surface*. However, we must not get too quickly ahead of ourselves.

For one to understand a Riemann surface, one must have some notion of a topological space, a manifold, and a surface.

Definition. A **topological space** is an ordered pair (X,τ) in which X is set and τ is a collection of subsets (namely, open sets) of X, satisfying three axioms: [2]
(i) The empty set and X itself belong to τ.
(ii) Any union of elements in τ is still in τ.
(iii) The intersection of any finite number of elements in τ is in τ.

Here we have an example that should adequately demonstrate what we mean by a topological space.

Example 2.1.1 Consider $x = \mathbb{R}^2$ or $X = \mathbb{C}$. We then define a subset $U \subset X$ to be Euclidean-open if for all $x \in U$ there exists $\varepsilon \in \mathbb{R}$ such that $B_\varepsilon(x) \subset U$. Here $B_\varepsilon(x)$ denotes the ball of radius ε centered at x i.e. $B_\varepsilon(x) = \{y \in \mathbb{R}^2 \mid d(x,y) < \varepsilon\}$. The Euclidean topology on $\mathbb{R}^2$ is $T = \{$Euclidean-open sets , $U \subset \mathbb{R}^2\}$.

Having defined a topological space, we are now able to define a *homeomorphism*.

Definition. A **homeomorphism** is a function $f \to Y$ such that f is continuous and the inverse of f is also continuous.

Definition. If X, Y are topological spaces, a function $f : X \to Y$ is **continuous** if $f^{-1}(U) \subset X$ is open whenever $U \subset Y$ is open.

With some understanding of topological spaces and homeomorphisms, we may now turn our attention to a *manifold*, which we will not define formally, but will make use of a visual aid.

Here we have an image to visualize precisely what we are talking about. We have some topological space X, with a neighborhood $U_x \subset X$ of x and a homeomorphism φ_x. We call the pair (U_x, φ_x) *local charts*. We call $T_{y,x}$ and $T_{x,y}$ *transition functions.*. We require these transition functions be *smooth* in order for X to be a manifold.

In Figure 1, below, X is covered by open sets U_x, each of which has a homeomorphism $\varphi_x : U_x \to V_x$, where $V_x \subset \mathbb{R}^n$ is open in the Euclidean topology. Our transition functions $T_{y,x}, T_{x,y}$ compares different local coordinates for the same points in X. Here $T_{x,y} = \varphi_x \circ \varphi_y$

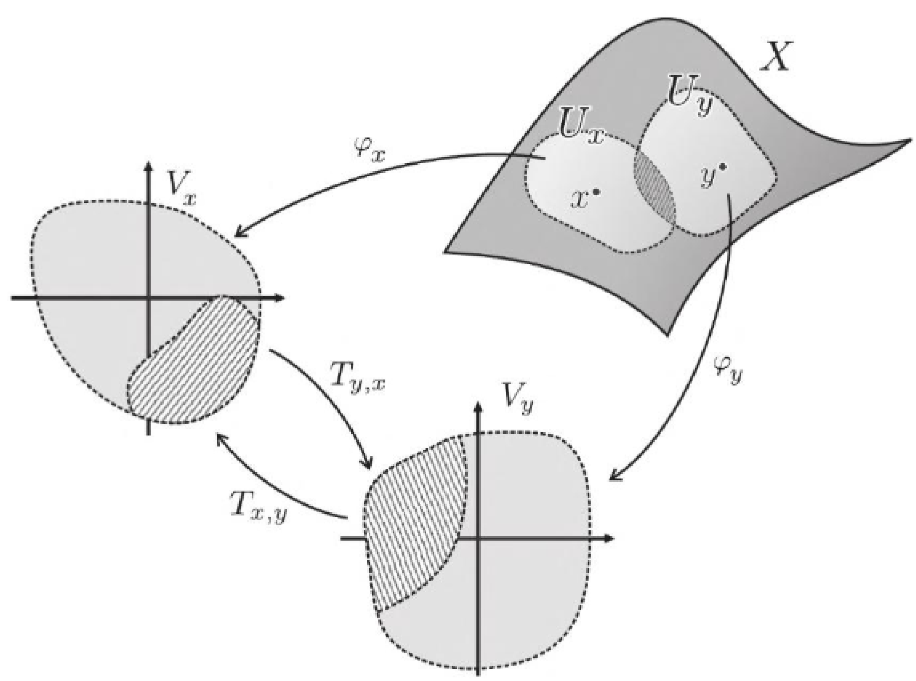

FIGURE 1. A visual of a manifold [3]

With this notion of a manifold in mind, we will define a surface. We note that the *dimension* of a manifold is the n in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition. A **surface** is a manifold of dimension 2.

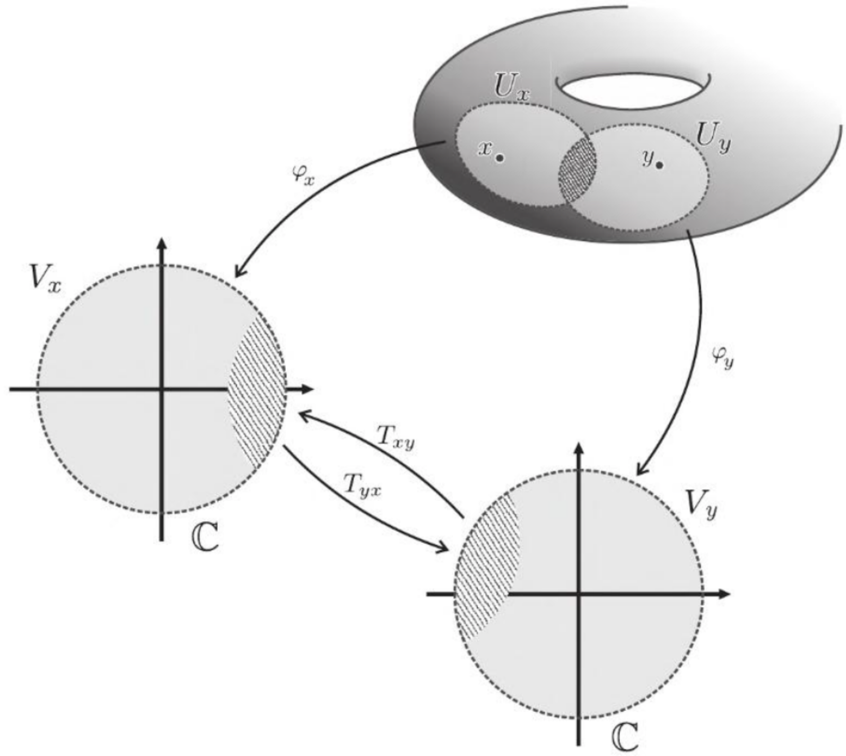

FIGURE 2. This image from Renzo Cavalieri's *Riemann Surfaces and Algebraic Curves* gives us a visual intuition of the components of a Riemann surface.[3]

With these definitions, examples, and aids at our disposal, we can now discuss *Riemann surfaces*.

3. Riemann Surfaces and Holomorphic Maps

Since Riemann surfaces are the most important objects of this narrative, we will define them formally and carefully below.

Definition. A **Riemann Surface** is a complex analytic manifold of dimension 1. To expand, we have two conditions in order for X to be a Riemann Surface:
(i) X is a real 2 dimensional manifold.
(ii) For any U_x, U_y, such that $U_x \cap U_y = \varnothing$ the transition function
$T_{y,x} := \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \to \varphi(U_x \cap U_y)$ is analytic.

To further understand the structure of a Riemann surface, let us consider figure 2. We can see that we have a Riemann surface, in which local charts (U_x, φ_x) and (U_y, φ_y) allow us to see that that φ_x, φ_y take points from our Riemann surface and map them to $\mathbb{C}$. Further, we can see that our transition functions T_{yx}, T_{xy} allows us to transition from V_x to V_y and from V_y to V_x respectively.

3.1. Examples Of Riemann Surfaces.

Next, we will look at some basic examples of Riemann surfaces in order to gain some insight into the mathematical structures of primary interest. Firstly, we will look at how we might naturally come to consider Riemann surfaces, by starting with a function. Second, we will consider the most basic example of a Riemann surface. Then we will look at the prototypical example of a Riemann sphere.

Example 3.1.1

We first observe that natural algebraic expressions have some nature of ambiguity in their solutions. In other words they define multi-valued rather than single valued functions. When we are working in $\mathbb{R}$, it becomes fairly obvious how to go about fixing said ambiguity. We simply select one branch of the function. To illustrate this, consider $f(x)$=$\sqrt{x}$.

We know that the domain of this function is $[0,\infty)$. Then we have a continuous function $f:[0,\infty)\to\mathbb{R}$, which is analytic everywhere but 0. If we approach the same function from a complex perspective, we take w=$\sqrt{z}$ to mean w^2=z, but to have a single-valued function of z we need to make a choice. If we want to preserve continuity, we require a "cut" in the domain. We may do this by defining $\sqrt{z}$: $\mathbb{C}$-$\mathbb{R}^-\to\mathbb{C}$ to be the square root with positive real part. Then our function is holomorphic away from the negative real axis.

So there is no continuous extension of f over this missing half-line. If we take the limits of the chosen values of $\sqrt{z}$ our results will be opposite signs. Then clearly our problem is at 0. We may fix this by considering Riemann surfaces. Our idea will be to have the graph of the function replace the complex plane as domain of the multi-valued function. In the case of w=$\sqrt{z}$, the graph of the function is a closed subset of $\mathbb{C}^2$. $S=\{(z,w)\in\mathbb{C}^2\mid w^2=z\}$. Then, in fact, the graph of S is a simple example of a Riemann surface.

This example illustrates the basic idea behind the theory of Riemann surfaces, to replace the domain of a multi-valued function, i.e. a polynomial equation $P(z,w)$ by its graph $S=\{(z,w)\in\mathbb{C}^2\mid P(z,w)$=$0\}$

Example 3.1.2

The most straightforward example of a Riemann surface is the complex plane $\mathbb{C}$ itself. We define our map as the identity map, with $U_x=X$ and $\varphi_x=id$, which defines a chart for $\mathbb{C}$ and the set containing f acts as an *atlas* for $\mathbb{C}$. By *atlas*, we just mean a collection of local charts that cover all of X, or in this case, $\mathbb{C}$. Note, this is just one possible chart and atlas for $\mathbb{C}$.

Example 3.1.3

The Riemann sphere, denoted $\mathbb{P}^1(\mathbb{C})$,viewed as a one-dimensional complex manifold, may be described by two charts with their domains equal to $\mathbb{C}$. To construct our sphere we do the following:

(i) We let ζ be some complex number in one copy of the complex plane
(ii) We then let ξ be a complex number in another copy of the complex plane.
(iii) Identify every nonzero complex number ζ of the first $\mathbb{C}$ with the nonzero complex number $\frac{1}{\zeta}$ of the second $\mathbb{C}$. Thus, for example, $\zeta=2$ is the same as $\xi=1/2$

Further, we can view $\mathbb{P}^1(\mathbb{C})$ as the following: $\mathbb{P}^1(\mathbb{C}):=\frac{\mathbb{C}^2-\{(0,0)\}}{\sim}$, where $(z,w)\sim(\lambda z,\lambda w)$ for all $\lambda\in\mathbb{C}^*$. This allows us to have useful notation. The equivalence class of (z,w) is denoted at $[z:w]$. For example, $(1,2)\in\mathbb{C}^2-\{(0,0)\}$ and $(3,6)\in\mathbb{C}^2-\{(0,0)\}$ with $(1,2)\sim(3,6)$ so $[1:2]\in\mathbb{P}^1$ and $[3:6]\in\mathbb{P}^1$. Thus, we write $[1:2]=[3:6]$. Next, we must check that that there are local charts that form an atlas for $\mathbb{P}^1(\mathbb{C})$. Further, we must verify that the transition function is holomorphic. This will confirm that $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

Indeed we have two local charts: $U_1=\{[z:w]\mid w\neq 0\}$ paired with $\varphi_1:U_1\to\mathbb{C}$, where $\varphi_1([z:w])=\frac{z}{w}$. $U_2=\{[z:w]\mid z\neq 0\}$ paired with $\varphi_2:U_2\to\mathbb{C}$, where $\varphi_2([z:w])=\frac{w}{z}$. We also do have

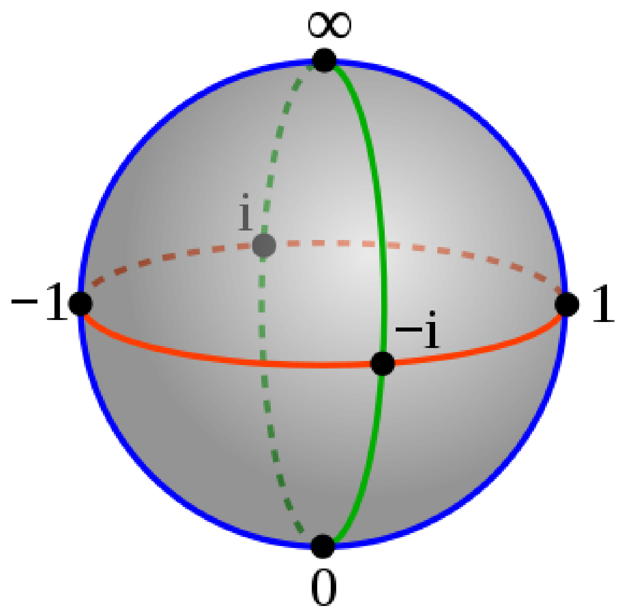

FIGURE 3. Here, we may see another example of a Riemann surface, the Riemann sphere.

a holomorphic transition function: $\mathbb{P}^1(\mathbb{C})$:$(\varphi_2 \circ {\varphi_1}^{-1})(u) = \varphi_2([u:1]) = \frac{1}{u}$, which is holomorphic. Thus we have verified that $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

Intuitively, we may view the transition maps as providing instruction on how to go about gluing our two copies of $\mathbb{C}$ together. The planes are glued in an inside out manner, so the overlap is almost ubiquitous. However, each copy of $\mathbb{C}$ contributes just one point (its origin) missing from the other plane. To elaborate, almost every point in the Riemann sphere has two values, both a ζ and a ξ value, related by $\zeta = \frac{1}{\xi}$. Here the points where ξ=0 takes on the value $\frac{1}{0}$. In this sense, the origin of the ξ charts plays the role of ∞ in the ζ chart.

To gain some further intuition of example 3.1.3, let us consider an example of something a little more earthly, our globe. Indeed we see, from figure 4, that our process is analogous to working with the globe. We may think about how Earth may be represented on our computer screens (i.e. an open set in $\mathbb{C}$). Then we are really considering how would we would about including every point on the surface of our Earth by flattening our globe.

Now armed with two examples, a formal definition, and an intuitive aid, we will look at holormorphic maps between Riemann surfaces.

3.2. **Holomorphic Maps Between Riemann Surfaces.** We shall start by formally defining a *holomorphic* map between Riemann surfaces:

Definition. Let X, Y be Riemann surfaces and $f : X \to Y$ be a set function.
(i) We say that f is **holomorphic** at $x \in X$ if for every choice of charts $\varphi_x, \varphi_{f(x)}$ the function $\varphi_{f(x)} \circ f \circ {\varphi_x}^{-1}$ is holomorphic at x.
(ii)If $U_x \subset X$ is open, we say that f is holomorphic on U if f is holomorphic at each $x \in U$
(iii) If f is holomorphic on $U_x = X$ then we say that f is a holomorphic map.

3.3. **Some Important Terminology.** As mentioned in our introduction, it is natural to require maps of Riemann surfaces locally to be identified with holomorphic maps from $\mathbb{C}$ to $\mathbb{C}$. This

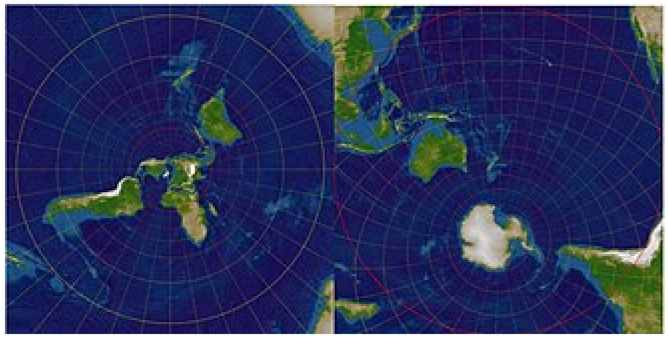

FIGURE 4. In this image, we first consider our roughly-spherical globe and we suppose that we want to somehow include every point on the surface by flattening out the globe.

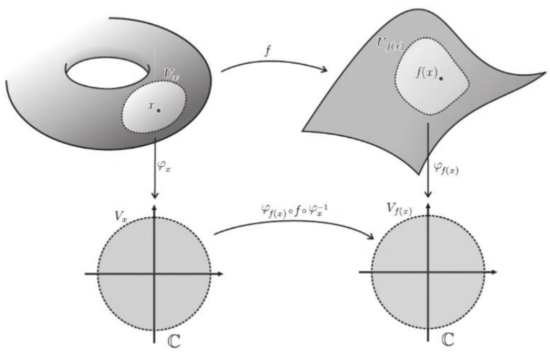

FIGURE 5. Here X,Y are Riemann surfaces and $f : X \to Y$ is a holomorphic map. We see our *local expression* is given by $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$. A chart (U_x, φ_x) is centered at if $\varphi_x(x) = 0$. This image shows how a Riemann surface is locally indistinguishable from $\mathbb{C}$. In fact, this image will be valuable because it allows us to see Riemann surfaces of *genus*, the number of holes our Riemann surface has, greater than 1. [3]

identification is referred to as a *local expression* and requires that both our source and target have associated chosen charts. This is a feature, and, as a consequence we have two useful theorems we will state.

Theorem 3.1. *Let $f : X \to Y$ be a non-constant holomorhpic map of Riemann surfaces. For any $x \in X$ there are charts centered at $x, f(x)$, such that the local expression of f using these charts is $z \to z^k$ for some integer $k \geq 1$.*

It's Theorem 3.1 that allows us to develop the notion of ramification index. In this way, if our map f, between Riemann surfaces, is a holomorphic map, we are able to ensure the existence of charts such that our local expression will be a power function.

As we mentioned above, we are indeed always able to find a local expression such that our function will be a power function. Figure 6 shows us we are able to alter charts to get local expression of z^k. Essentially, this explains theorem 3.1, visually. We note that we start with any charts ϕ, ψ centered at $x, f(x)$, giving us a local expression $F := \psi \circ f \circ \phi^{-1}$. We consider the Taylor expansion of F at 0 and let k be the smallest positive integer such that our coefficient z^k will not vanish. Then we find that $F(z) = z^k(\sum_{n=0}^{\infty} a_{k+n} z^n)$ because $F(0) = 0, k \geq 1$. We denote the second

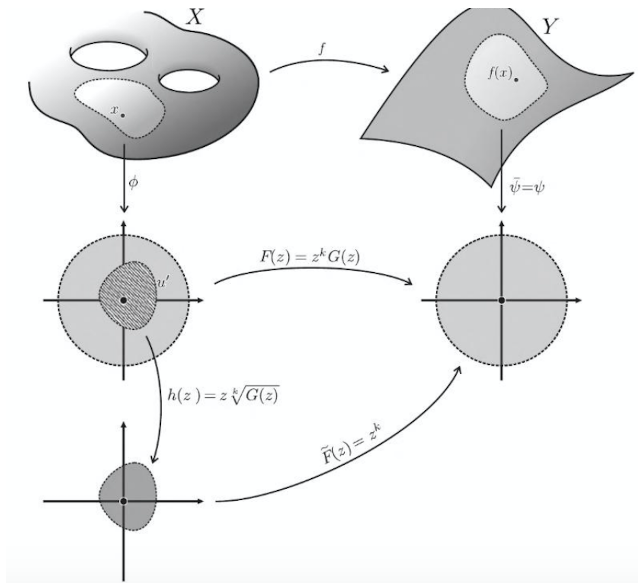

FIGURE 6. Via an image from Cavalieri's text, we are able to see how we are able to alter charts to have our local expression be a power function.[3]

factor in $F(z)$ by $G(z)$=$a_k + a_{k+1}z + ...$ Then $G(z)$ is holomorphic at 0 and $G(0) \neq 0$. We choose the branch so that our map $\sqrt[k]{G(z)}$ is well-defined and holomorphic around $\phi(x)$=0.

We define $h(z)$=$z\sqrt[k]{G(z)}$, so that gives us that F=h^k. The function h is holomorphic in a neighborhood U of $\phi(x)$=0, $h(0)$=0, and $h'(0)$=0. We claim that that composition $\tilde{\phi}$=$h \circ \phi$ gives us a local chart for X centered at x. We relate the local coordinate $\tilde{z}$ coming from $\tilde{\phi}$ to z via $\tilde{z}$=$h(z)$. Finally, the local expression for f around x is obtained by changing coordinates from z to $\tilde{z}$ in F: $\tilde{F}(\tilde{z})$=$F(z(\tilde{z})$=$h(z)^k$

We will see that this theorem allows us to meaningfully define new terms. As a result of the above theorem, the complex analytic notion of order of vanishing of a function at a point may be extended to Riemann surfaces. We call this vanishing the *ramification index*. More formally, the integer k_x such that there exists a local expression centered at p of the form F(z)=z^{k_x} is the ramification index of f at x. Then we may simply define the *differential length* of f at x as $k_x - 1$.

Our second theorem will be equally important. In the way that our previous allows us to consider the concept of ramification index, theorem 3.2 allows us speak about further useful concepts.

Theorem 3.2. *Let $f : X \to Y$ be a non-constant holomorhpic map of compact Riemann surfaces. If $y_0, y_1 \in Y$ are not in the branch locus of f, then $\mid f^{-1}(y_0) \mid$ = $\mid f^{-1}(y_1) \mid$*

The number of inverse images is referred to as the *degree* of the map. Lastly, we should note that a chart (U_x, ϕ_x) for a Riemann surface X is *centered at* x if $\varphi_x(x) = 0$

A point x such that $k_x \geq 2$ is called a *ramification point*. The *ramification locus R* is the subset of X consisting of all ramification points. Lastly, If x is a ramification point, then *f(x)* $\in Y$ is called a *branch point*. The *branch locus B* is a subset of Y consisting of all branch points. In other words the branch locus is the image via f of the ramification locus, as seen in figure 7.

3.4. **Some Examples of Holomorphic Maps Between Riemann Surfaces.** We will get a feel for the kinds of maps that we are talking about by explicitly looking at two examples. In these examples, we will be using the notation as we introduced in Example 3.1.3.

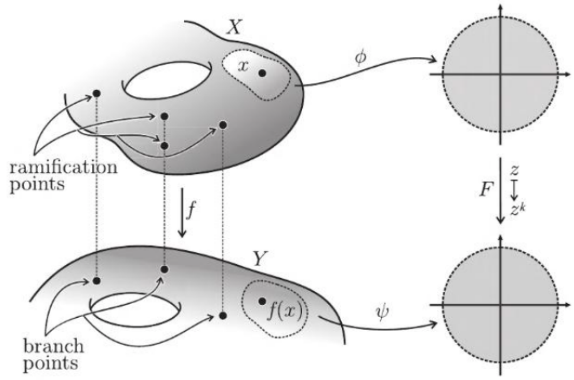

FIGURE 7. This image visualizes *ramification points* and *branch points* [3]

Example 3.4.1

Consider two Riemann surfaces, X=Y=$\mathbb{P}^1(\mathbb{C})$. So, both our target and source are Riemann spheres. Consider the point $x=[0:1]$. Then suppose that our map is defined by $f([z:w])$=$[z^2:w^2]$. So, $f(x)=f([0:1])=[0,1]$. Consider the chart ϕ_x given by $\phi_x=\frac{z}{w}$ and ${\phi_x}^{-1}(u)$=$[u:1]$. We also define $\phi_{f(x)}$ by $\phi_{f(x)}([z,w])=\frac{z}{w}$.

Then to confirm that f is holomorphic requires that we check our composition.

$$\begin{aligned}\phi_{f(x)}\circ f\circ {\phi_x}^{-1}(u) &= (\phi_{f(x)}\circ f)([u:1])\\ &= (\phi_{f(x)})([u^2:1^2)\\ &= \frac{u^2}{1^2}\\ &= u^2\end{aligned}$$

Thus, we have found a chart centered at $[0:1]$ in which f looks like $u\to u^2$. We conclude that our ramification index of f at $[0:1]$ is 2. This tells us that $[0:1]$ is the only point that maps to $[0:1]$, but for other points in the chart, two points in our source map to any one point in our target. Further then, our differential length of f at $[0:1]$ is $v_x=2-1$=1. Lastly, we say that the *degree* of *f* is 2 because the fiber of *f* over any y other than $[1:0]$ and $[0:1]$ has cardinality 2.

Example 3.4.2

We suppose that X and Y are as in Example 3.4.1, but we change our map f to $f([z:w])=[w:z]$ Then suppose $x=[0:1]$, then $f(x)=[1:0]$.

Let $\phi_x([z:w])=\frac{z}{w}$. Then ${\phi_x}^{-1}(u)=[u:1]$. Suppose $\phi_{f(x)}([z:w])=\frac{w}{z}$

Then, as in our previous example, we check that our map f is holomorhpic by checking that the composition is holomorphic.

$$\begin{aligned}(\phi_{f(x)} \circ f \circ \phi_x{}^{-}1)(u) &= (\phi_{f(x)} \circ f)([u:1]) \\ &= \phi_{f(x)}([1:u]) \\ &= \frac{u}{1} \\ &= u\end{aligned}$$

Thus, we found a chart centered at $[0:1]$ in which f looks like $u \to u$. We then say that the ramification index of f at $[0:1]$ is 1. We say that f is *unramified* at $[0,1]$. Note, this implies that our *differential length* is 1-1=0. Lastly, we observe that our map is injective, which is another way of saying that the fiber of any point has cardinality one, meaning the degree of *f* is 1.

4. The Riemann-Hurwitz Formula and Applications

4.1. **An Explanation.** Finally, we are ready to start discussing the Riemann-Hurwitz formula. Before we state the formula, there still remains some unpacking of terminology to do. Namely, we will give some examples of various Riemann surfaces of differing *genus*.

Roughly speaking, the genus of a surface is the number of holes that the surface has. We see in figure 8 and figure 9 that we may have Riemann surfaces of genus greater than 1.

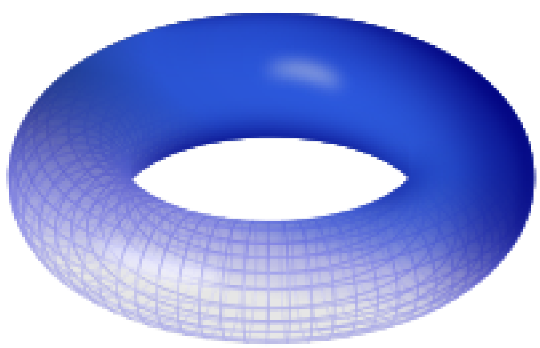

FIGURE 8. A torus is a Riemann surface of genus 1. We may see that it has one hole, analogus to a donut.

Indeed, we may consider Riemann surfaces of various genus. We will see below that the discrete number that is the genus of a Riemann surface is found on both sides of our equation and is thus of significance.

An important detail in the Riemann-Hurwitz formula is the condition that two Riemann surfaces X and Y are both compact. In other words, X and Y are compact as topological spaces. Thus, we would be well served to look at an important fact:

Theorem 4.1. *If* X *is a compact Riemann surface and $f: X \to Y$ is a nonconstant holomorphic map of Riemann surfaces, then the ramification locus R is a finite set. Since the branch locus is the image of R via* f, *it follows that the branch locus is also a finite set.*

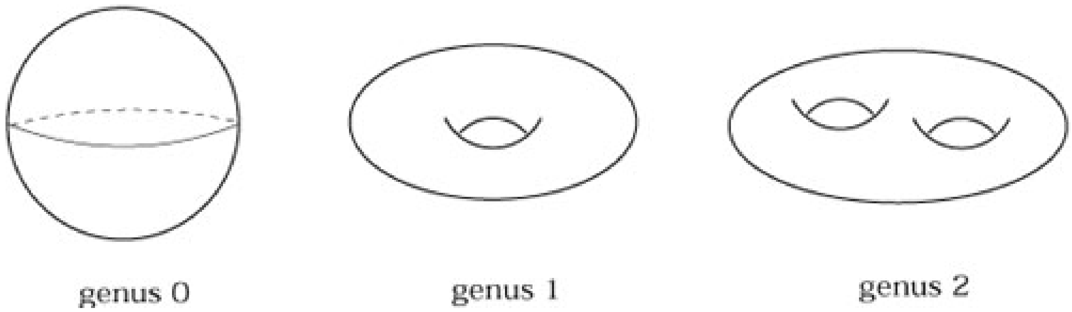

FIGURE 9. From left to right, we have Riemann surfaces of genus 0, genus 1, and genus 2, respectively. [1]

Theorem 4.1 will be important because it tells us that our summation in the Riemann-Hurwitz formula is indeed a summation and is not an infinite series, which might have the potential to diverge.

4.2. **The Formula Itself.** The Riemann-Hurwitz formula gives the following relation among all discrete invariants associated with maps of compact Riemann surfaces:

The Riemann-Hurwitz Formula

Theorem 4.2. *Suppose $f : X \to Y$ is a non-constant, degree d, holomorphic map of compact Riemann surfaces. We denote g_X and g_Y as the genus of X and Y respectively. Then we have:*
$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X}(v_x)$ *where v_x is the differential length of f at x.*

With the Riemann-Hurwitz formula explicitly stated via Theorem 4.2, let us now consider two examples that demonstrate how the formula can be used. For example 4.3.0, we will see that the Riemann-Hurwitz formula allows us to make some strong claims about the possibility, or lack thereof of ramification points. Example 4.3.1 will show us that we are able to solve a previous example, in which we found the degree of f, the ramification index of f at x, and knew the genus of X and Y.

Example 4.3.0
Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a degree d holomorphic map. Suppose that two points x_1 and $x_2 \in \mathbb{P}^1(\mathbb{C})$ have full ramification (i.e. $k_{x_1} = k_{x_2} = d$). Then, using our formula, we will show that there do not exists any more ramification points for f.
We set $g_X = g_Y = 0$ and we have

$$2(0) - 2 = d(2(0) + \sum_{x \in X} v_x = -2 = d(-2) + (d-1) + (d-1) + \sum_{x_1 \neq x_2} (k_x - 1)$$
$$0 = \sum_{x_1 \neq x_2} (k_x - 1)$$

Since each differential length is a non-negative integer, we have that each v_x is 0.

Example 4.3.1 Now, let us apply the Riemann-Hurwitz formula to Example 3.4.1, in which we found that when both our target and source was $\mathbb{P}^1(\mathbb{C})$. We considered the function $f([z:w]) = [z^2 : w^2]$. Further, we concluded that the degree of f was 2. Lastly, we found that the ramification index of f at $[0:1]$ was 2. Then using the Riemann-Hurwitz formula to verify this, we have:

$$\begin{aligned}
2(0) - 2 &= -2d + \sum_{x \in X} (k_x - 1) \\
2(0) - 2 &= -2d + 2 \\
-2 &= -2d + 2 \\
-4 &= -2d \\
2 &= d
\end{aligned}$$

Indeed in our example we did find that the degree of f was 2.

Example 4.3.2

The Riemann-Hurwitz formula may also be applied to Riemann surfaces that are not the Riemann sphere of course, in this case we will apply it to a projective plane curve. We will find the genus of C.

Consider the Riemann surface $C = \{[x:y:z] \in \mathbb{P}^2 \mid y^2z^2 = x^4 - z^4\}$. We define a map $f : C \to \mathbb{P}^1(\mathbb{C})$ by $f([x:y:z]) = [x:z]$.

Claim: the degree of f is 2 and the branch points of are $[1:1], [-1:1], [i:1], [-i:1]$.

We see this by letting $[x:z] \in \mathbb{P}^1(\mathbb{C})$. If $z \neq 0$, then we may assume $z = 1$, so $f^{-1}([x:1]) = \{[x : y:1] \mid y^2 = x^2 - 1$. Then given x, there are two choices for y, unless $x \in \{1, -1, i, -i\}$.

By using the Riemann-Hurwitz formula, we will find the genus of C. We have that the degree of f is 2, that the genus of the Riemann sphere is 0, and that over our summation $1, -1, i, -i$ will contribute an addition of 1 each. So we have:

$$\begin{aligned}
2(g_C) - 2 &= -2(0-2) + \sum_{x \in X} (k_x - 1) \\
2(g_C) - 2 &= -4 + (1+1+1+1) \\
2(g_C) - 2 &= 0 \\
2(g_C) &= 2 \\
g_C &= 1
\end{aligned}$$

We may draw some interesting conclusions from the Riemann-Hurwitz Formula as well:

- $\sum_{x \in X} v_x$ is even.
- $g_X \geq g_Y$. A Riemann Surface X can never map (nontrivially) to a Riemann Surface Y of a higher genus.
- If $\sum_{x \in X} v_x = 0$ then $g_X = dg_Y - d + 1$

5. Conclusion

The Riemann-Hurwitz formula, as we discovered, allows us to draw conclusions about certain invariants if we consider values for the other integer invariants. For example, we were able to conclude in example 4.3.0, by using the formula, that if two points in our source x_1, x_2 were fully ramified in our source, then we were able to conclude that the formula allowed us conclude that no other ramification points existed.

If our goal is to describe maps between manifolds, we find that the Riemann-Hurwitz formula only requires that we limit our consideration to non-constant holomorphic maps between compact Riemann surfaces. If we abide by these constraints, we find that we are able to relate the genus of compact Riemann surfaces, the degree of our holomorphic map, and the ramification index of ramification points via an equation.

REFERENCES

1. Tech. report, Also available as https://ned.ipac.caltech.edu/level5/March01/Carroll3/Carroll2.html.
2. Mark Anthony Armstrong, *Basic topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1983, Corrected reprint of the 1979 original. MR 705632
3. Renzo Cavalieri and Eric Miles, *Riemann surfaces and algebraic curves*, London Mathematical Society Student Texts, vol. 87, Cambridge University Press, Cambridge, 2016, A first course in Hurwitz theory. MR 3585685

DEPARTMENT OF MATHEMATICS, SAN FRANCISCO STATE UNIVERSITY, SAN FRANCISCO, CA 94132, U.S.A.
Email address: cdelossantos@mail.sfsu.edu