Foundations of Mathematics

An Extended Guide and Introductory Text

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To Taddy,
for her understanding, encouragement, and patience during this book's (cetacean) gestation.

Preface

This book was originally conceived as the first of a series to be entitled *What's the Big Idea? Fundamental Concepts in Mathematics and Science for the Dedicated Amateur.* The second book in the series was—and still is—to present the foundations of physics; and a third, if it came to be, would deal with complex systems: physical, biological, etc. It was intended that these books would treat their topics in a way that was accessible to the nonprofessional (the *dedicated amateur* of the subtitle) while providing a depth of understanding usually achieved only by a lengthy course of study. These were to be the books that I wished had been available to me in my youth. In the case of the physics book—which is still incomplete—this has led to a much greater attention to mathematical precision than is usual in that subject, and the result could be described—rather incompletely, albeit not entirely inaccurately—as *Foundations of Physics for Mathematicians.* The present book, on the foundations of mathematics, could be described in the same vein as *Foundations of Mathematics for (All) Mathematicians.*

The emphasis here is on *all.* Mathematicians generally take an interest in the foundations of their subject and have done so since antiquity: witness the attention paid to Euclid's axiomatization of geometry and the logical status of the parallel postulate, for example. Descartes's identification of geometrical points with real numbers is familiar to everyone, as is the work of Bolzano, Weierstrass, Cauchy, Dedekind and others formalizing the notions of limit and continuity, and defining real numbers in terms of sequences or sets of rational numbers. The work of Cantor defining set theory is also well known by reputation, if not in historical detail, and an understanding of basic set theory is universally recognized as an essential part of any mathematician's toolbox today. Less well known is the early work of Frege, Peano, and others on the logical underpinnings of this enterprise, but the conclusion of that development, viz., that the mathematical method is the derivation of theorems from axioms in a formal language using formal logic, is thoroughly understood.

By the year 1900, the centrality of foundational issues in mathematics was sufficiently clear that David Hilbert, the foremost mathematician of his day, placed them at the head of his famous list of problems for mathematicians of the coming century: the first problem asked whether the continuum hypothesis is true, and whether the set of real numbers can be wellordered; the second asked for a proof of the consistency of (Peano) arithmetic. (The tenth problem asked for a decision procedure for the solvability of diophantine equations, which we now understand is also a foundational question.) Most mathematicians are aware that each of these problems has been solved in some sense, but relatively few can state precisely in what sense, and even fewer know how this has been done.

The questions of the axiom of choice and the continuum hypothesis are merely two of many deep issues that arise when one inquires closely into the nature of infinitarity. As Hilbert was later to write, after praising Weierstrass' work on the foundations of analysis:

'. . . in spite of the foundation Weierstrass has provided for the infinitesimal calculus, disputes about the foundations of analysis still go on.

'These disputes have not terminated because the meaning of the *infinite*, as that concept is used in mathematics, has never been completely clarified. Weierstrass's analysis did indeed eliminate the infinitely large and the infinitely small by reducing statements about them to [statements about] relations between finite magnitudes. Nevertheless the infinite still appears in the infinite numerical series which defines the real numbers and in the concept of the real number system which is thought of as a completed totality existing all at once.

'In his foundation for analysis, Weierstrass accepted unreservedly and used repeatedly those forms of logical deduction in which the concept of the infinite comes into play, as when one treats of *all* real numbers with a certain property or when one argues that *there exist* real numbers with a certain property.

'Hence the infinite can reappear in another guise in Weierstrass's theory and thus escape the precision imposed by his critique. It is, therefore, the *problem of the infinite* in the sense just indicated which we need to resolve once and for all.

'. . . The foregoing remarks are intended only to establish the fact that the definitive clarification of *the nature of the infinite*, instead of pertaining just to the sphere of specialized scientific interests, is needed for *the dignity of the human intellect* itself.'[11]

One might think that a course of study in the foundations of mathematics would be part of every mathematician's education (even every thoughtful person's, *per* Hilbert), but this is not the case; indeed, the foundations of mathematics is often regarded as something apart from mathematics proper. Upon reflection, this is not so surprising: although mathematicians have learned that mathematics is not defined by the objects of its attention but rather by the method of its attention, it is a big step to make that method itself the object of one's mathematical attention. This takes some getting used to, and existing texts do not always make it easy for the outsider looking in.

For example, introductory texts in mathematical logic typically attend closely to the necessary distinction between the *use* of an expression (in, say, a mathematical proof one is presenting) and the *mention* of an expression (as, say, an element of a mathematical proof one is talking about). (Gödel's incompleteness theorem, which supplies the solution to Hilbert's second problem, depends critically on attention to this distinction.) Beyond the introductory level, however, the use-mention distinction is typically not scrupulously observed in conventional texts; the reader is presumed to be aware of it and able to edit the text appropriately. I decided at the outset that this was an undue imposition on a reader who is not intent on specializing in the field, and I have introduced some notational innovations that relieve the reader of this interpretive task without being unduly intrusive.

A more substantive issue relates to the natural tendency to refer to an attribute in terms of the class of objects with that attribute. Perfectly harmless, you might say; what's the difference? The difference is that the assertion of the existence of a class as an object, which is itself a member of (some) classes, is a dangerous thing, as shown by *Russell's paradox*, which points out that the class of all classes that are not members of themselves is a member of itself if and only if it is not a member of itself. The Zermelo-Fraenkel theory ZF, in which all classes are sets, avoids paradox by adopting rather limited axioms of class existence. The Gödel-Bernays class theory has a more generous axiom of class existence, but some classes are not sets, which is to say, they are disallowed as members of classes. While inconsistency is thereby avoided, these so-called *proper* classes nevertheless present difficulties in the definition of the most basic relation in the foundations of mathematics, viz., that of satisfaction of a formula in a structure. Such difficulties incline one to work in a pure set theory, but the complete avoidance of proper classes is itself difficult and artificial. Most writers therefore nominally work in a pure set theory such as ZF, but refer to proper classes as though they exist, using various *ad hoc* arrangments to deal with the satisfaction problem and generally relying on the reader to supply correct (ZF) arguments as needed.

Again, I felt this was an undue burden on the nonspecialist, but it was not clear initially how to alleviate it without introducing additional machinery so cumbersome as to defeat its purpose. Fortunately, a solution lay at hand, viz., a simple and robust definition of satisfaction for proper classes[27] which I have found so efficacious in the presentation of this subject that I feel it merits mention in this Preface.

Designed as it is with the nonspecialist in mind, this book has not been constructed as a text in the usual sense. All its roads lead to Big Ideas, without the numerous byways that would be appropriate if its purpose were to prepare the reader directly for research in the field. Likewise, there are very few formal exercises. Some proofs are left to the reader, but these are typically quite straightforward. Proofs are otherwise given in considerable detail. Definitions are likewise given in meticulous detail, so much so that the book may appear at a glance to be more technically demanding than a standard text. It is not: the detail that is here presented on the page must in any event be present in the reader's mind.

That said, the aspiring student of the subject has never been far from my thoughts, and I have been gratified to find that the Big Ideas approach to the organization of the subject has actually resulted in a serviceable introductory text. The serious student will of course consult additional sources that extend what is found here and carry one closer to the research frontier, but this book provides a solid foundation in mathematical logic and set theory and can be a *vade mecum* for the early years of graduate study.

The interests of the tourist and the student coincide in what is perhaps the book's greatest strength, viz., that it is selfcontained. If we need a theorem, we prove it; we do not refer the reader to the literature. This sometimes leads to rather long proofs in the service of rather small Big Ideas, "medium-sized ideas" if you will.[1] Feel free to skip these. In general, there is nothing wrong with omitting any bit of the book until one feels the need for it later. This mode of reading is facilitated by a high density of cross-references. And by all means, if you reach a point where you feel you have learned all you wish to know about the foundations of mathematics then set the book aside. For example, a course of reading consisting of Sections 1.1–5, 2.1–2, 3.1–7, 4.1–11, 7.1–4, and 8.1–10, about 300 pages, may more than satisfy many readers.[2]

We conclude this Preface with an overview of the contents of the book for the purpose of orientation. We begin Chapter 1 with an informal discussion of formal language, then proceed to a formal definition of structure, language, satisfaction, and entailment. By the end of Chapter 1 we are able to prove the celebrated

[1] As a way of providing the requisite level of detail without unduly impeding the flow of the narrative or obscuring the big picture, we have made extensive use of *notes*—footnotes for shorter and endnotes for longer insertions.

[2] The serious student will of course leave no crumb on the plate.

theorem of Gödel and Tarski on the undefinability of truth.

In the second chapter we prove Gödel's completeness theorem, which yields the laws of logic by equating entailment with derivability. We have not at this point assumed the axiom of Infinity, so we do not suppose the existence of infinite sets, although we do suppose the existence of infinite classes (automatically, since proper classes are necessarily infinite), as we must in order to state the completeness theorem. As we continue in this investigation, we begin to glimpse the role that Infinity may play. In Section 2.5 we illustrate some of the ideas we have developed in a discussion of the axiomatic method in the historically important case of geometry. We conclude by proving that the predicative theory of classes without Infinity is essentially finitary, inasmuch as it is a conservative extension of pure set theory without Infinity.

The third chapter develops the theory of membership both as a tool—which we have, of course, already used in the first two chapters—and as an object of metamathematical interest. By the end of the chapter we are able to show that any theory that is capable of talking about itself and its models is incomplete in the sense that there is a sentence true in its intended model that is not provable.

In the fourth chapter we define and develop the science of computation. This allows us to formulate and answer the question as to whether logic is *decidable*, i.e., is there an algorithm that decides whether a sentence is logically valid.[3] We find that there is not. We then prove Gödel's celebrated incompleteness theorems, which provide the solution of Hilbert's second problem by showing that no consistent theory can prove its own consistency. As a bonus, we discover that there is a fascinating world of structure of sets of natural numbers based on relative computability, of which we only scratch the surface.

We have noted above the concern that Hilbert expressed as to the need for clarification of the nature of the infinite. We briefly touch on this problem in our remarks about the complexity of individual sets of natural numbers in Chapter 4, but in Chapter 5 we take up this program in earnest. As Hilbert noted, for foundational purposes real numbers are the simplest infinitary objects, equivalent to sets of natural numbers, and the need for clarification of infinitarity is already evident in this context, as both questions stated in Hilbert's first problem have to do with sets of real numbers. Much progress was made in the early twentieth century regarding definable sets of real numbers in the context of Lebesgue measurability and other regularity properties of sets of real numbers, as well as structural principles such as separation, reduction, and uniformization. We present the most important insights from this period, and then, in the same context, we introduce the extraordinary notion of determinacy, which has come to be regarded as a central insight into the nature of infinitarity. We conclude with the statement of Suslin's hypothesis concerning the structure of the real line, which figures importantly in the sequel.

As the *lingua franca* of mathematics, set theory is naturally the preferred theory to investigate the metatheory of set theory. It is now apparent that the converse is also true: the metatheory of set theory plays a vital role in set theory itself and is essentially inseparable from it. Chapter 6 presents some of the basic metatheory of membership, which we use to show that the axiom of Choice is consistent with the Zermelo-Fraenkel theory ZF, and that the negation of Choice is also consistent with the theory ZFA, which allows *urelements*, or *atoms*, i.e., elements other than sets. This is a nearly satisfactory answer to Hilbert's question about wellordering, which is equivalent to Choice, but we can do better.

[3] It also allows us to formalize Hilbert's tenth problem and reveal its foundational character.

In Chapter 7 we present the constructible universe L, which is a sort of minimum model of ZF, of great importance in foundational studies. This was defined by Gödel for the purpose of showing the consistency ZF with the axiom of choice and the continuum hypothesis, *inter alia*, and we show how these proofs go. The notion of constructibility is far more potent than this, but further development is largely beyond the scope of this book, and we content ourselves with Jensen's proof that Suslin's hypothesis fails in L and Friedman's proof that the Power axiom is needed to prove the determinacy of Borel sets.

In Chapter 8 we introduce another seminal concept in set theory, that of genericity. This completes the solution of Hilbert's first problem by showing that the negations of the axiom of choice and continuum hypothesis are consistent with ZF, but it does much much more than this—indeed, it is an expansion of our understanding of the concept of set that is genuinely revolutionary. We spend a bit more time and space on this topic than is perhaps strictly necessary to fulfill the mandate of this book, but it is well worth it. For example, we are able to present Solovay's construction of a model of ZF in which all sets of real numbers are Lebesgue measurable, an issue left over from Chapter 5.

In Chapter 9 we explore the possibility of extending ZF in such a way as to answer some of the questions it does not settle, which, as we have by now seen, are legion. There are two major currents. Historically, the first to be investigated was the general category of large cardinal hypotheses, by which we mean both large cardinals *per se* and also properties of "small cardinals" with "large cardinal" consequences. Not surprisingly, "small" and "large" jump around in meaning as we apply constructibility and genericity methods. The other major current is determinacy, which we encountered in Chapter 5, and which here comes into its own. The big news is that these concepts, which at first glance appear to have nothing in common except that they answer a lot of questions, are very closely related. Although a detailed explication is beyond the scope of this book, we are able to develop enough of each theory individually and of the implications of each for the other to give a sense of how they are aspects of a single vision of reality.

Bon appétit!

Contents

Preface **iii**

1 Language and Structure **1**
- 1.1 Introduction . . . 2
- 1.2 Structure . . . 26
- 1.3 The structure of language . . . 34
- 1.4 Interpretation . . . 47
- 1.5 Undefinability of satisfaction . . . 57
- 1.6 Summary . . . 59

2 Logic **63**
- 2.1 Deduction, soundness and completeness . . . 64
- 2.2 The completeness theorem . . . 66
- 2.3 Logic . . . 80
- 2.4 Theories . . . 102
- 2.5 Example: theories and models of geometry . . . 113
- 2.6 Other deductive systems . . . 138
- 2.7 Model theory . . . 145
- 2.8 Satisfactoriness and logic . . . 155
- 2.9 A comparison of metatheories . . . 157
- 2.10 Summary . . . 163

3 The Theory of Membership **167**
- 3.1 Introduction . . . 169
- 3.2 Basic theories of membership . . . 172
- 3.3 Basic membership-theoretic constructs . . . 183
- 3.4 Wellfoundedness . . . 200
- 3.5 Relativization . . . 205
- 3.6 Size . . . 216
- 3.7 Extended theories of membership . . . 222
- 3.8 Principles of choice and wellordering . . . 223
- 3.9 Cardinal arithmetic . . . 228
- 3.10 More basic constructs . . . 233
- 3.11 Topology . . . 245
- 3.12 Finitary set theory . . . 253
- 3.13 Inner models . . . 255
- 3.14 Summary . . . 259

4 Definability, Computability, Provability 261
4.1 Paradox . . . 261
4.2 Decidability . . . 262
4.3 The science of computation . . . 263
4.4 Set-theoretic complexity . . . 264
4.5 Evaluation and satisfaction . . . 272
4.6 Satisfaction in the finitary interpretation . . . 275
4.7 Computability . . . 283
4.8 Recursive functions . . . 292
4.9 The unsolvability of the halting problem . . . 297
4.10 The recursion theorem . . . 298
4.11 Consistency and incompleteness . . . 299
4.12 Relative definability over HF . . . 306
4.13 Summary . . . 311

5 Infinitarity 313
5.1 Introduction . . . 314
5.2 Definability of pointsets . . . 320
5.3 Classical descriptive set theory . . . 345
5.4 Structural properties of pointclasses . . . 366
5.5 Perfect sets . . . 382
5.6 Category . . . 385
5.7 Measure . . . 391
5.8 Determinacy . . . 397
5.9 Suslin's operation and analytic sets . . . 407
5.10 Suslin's hypothesis . . . 410
5.11 Limitations of ZF . . . 411
5.12 Summary . . . 412

6 The Metatheory of Membership 415
6.1 Introduction . . . 415
6.2 Satisfaction predicates . . . 418
6.3 Absoluteness . . . 418
6.4 Reflection . . . 421
6.5 Ordinal-definability . . . 425
6.6 Consistency of the axiom of choice . . . 426
6.7 Consistency of ZFA + ¬AC . . . 428
6.8 Summary . . . 432

7 Constructibility 433
7.1 The constructible universe . . . 433
7.2 A definable wellordering of L . . . 438
7.3 The condensation lemma . . . 440
7.4 The continuum hypothesis in L . . . 443
7.5 Relative constructibility . . . 444
7.6 $\mathbb{R}$ in L . . . 445
7.7 Summary . . . 464

8 Possible Worlds **467**
8.1 Introduction . . . 468
8.2 Genericity and forcing . . . 475
8.3 Boolean-valued structures . . . 492
8.4 Logical considerations . . . 503
8.5 The theory of a generic extension . . . 505
8.6 Relative consistency proofs . . . 521
8.7 Chain conditions and saturation . . . 524
8.8 Closure and distributivity . . . 527
8.9 Independence of the continuum hypothesis . . . 529
8.10 Independence of the axiom of choice . . . 532
8.11 Product forcing . . . 534
8.12 Class forcing . . . 540
8.13 Iterated forcing . . . 549
8.14 Some forcing constructions . . . 565
8.15 Summary . . . 575

9 On Beyond ZF **577**
9.1 Introduction . . . 579
9.2 Large cardinals . . . 580
9.3 Large cardinals and constructibility . . . 615
9.4 Ideals over cardinals . . . 629
9.5 Larger cardinals . . . 644
9.6 The continuum problem, continued . . . 653
9.7 Determinacy . . . 654
9.8 Summary . . . 697

10 Notes **701**

Notation **817**

Bibliography **821**

Index **823**

Chapter 1

Language and Structure

Contents

1.1 Introduction **2**
1.1.1 Formal language 3
1.1.2 Formal logic 3
1.1.3 Metalanguage and metatheory 4
1.1.4 Identity, equivalence, and definition 5
1.1.5 Typographical languages 6
1.1.6 The elements of language 9
1.1.7 Syntax and semantics 12
1.1.8 Variables in formulas 13
1.1.9 Identity 15
1.1.10 Domains of discourse 15
1.1.11 A minimal framework 16
1.1.12 Typographical conventions 17
1.1.13 Metalanguage notation for expression-building operations 19
1.1.14 Structure 20
1.1.15 Our metatheory 23
1.1.16 A tantalizing example 24
1.2 Structure **26**
1.2.1 Introduction 26
1.2.2 Eine kleine Mengentheorie 26
1.2.3 Signatures 28
1.2.4 Structures 30
1.2.5 A simpler notation 31
1.2.6 Isomorphism and homology 32
1.2.7 Standard signatures 33
1.3 The structure of language **34**
1.3.1 Language defined 34
1.3.2 Metalanguage conventions 40
1.3.3 Diagrams, occurrences, and binding 41
1.3.4 Substitution 44
1.3.5 A useful quotation convention 45
1.3.6 Change of variables 46

1.4 Interpretation . . . 47
1.4.1 Assignments to variables . . . 48
1.4.2 Interpretation in structures . . . 48
1.4.3 Semantics of substitution and change of variables . . . 55
1.4.4 Theories, satisfiability, and entailment . . . 55
1.5 Undefinability of satisfaction . . . 57
1.6 Summary . . . 59

In the beginning was the Word, ...

The Gospel according to John 1:1

1.1 Introduction

In the beginning was λόγος, says John, by which he meant not just *word* in the English sense, but also *logic* or *reason*. John goes on to say, '. . . and the Word was with God, and the Word was God. The same was in the beginning with God. All things were made by it; and without it was not anything made that was made', expressing the concept of λόγος as the fundamental order of the universe and means of its creation, a concept expressed in the teachings of Heraclitus six hundred years earlier and elaborated in Sophist philosophy and Hebrew wisdom literature. The notion that word is primary, even to the extent of being the source of physical existence, may initially strike you as curious. We need not go along with John on that point, which is probably best regarded as belonging to the domain of *metaphysics*, rather than physics; however, do not be surprised if by the time you finish this series of books you find the idea quite plausible. For the time being, let's just accept that since our understanding of the world exceeds that of the nonverbal animals largely as a result of our command of language; and since the purpose of this book is to communicate by means of language some of the fundamental principles of that understanding; then an examination of language itself properly belongs at the beginning.

Of the several uses of language the one that concerns us here is that of *representation*. A representation in this sense is a linguistic structure that mirrors some other structure of interest. In the early evolution of language the latter structures were aspects of the physical world, and language could be viewed in that context as an *abstract* structure mirroring a *concrete* structure, but the distinction between the abstract and the concrete in this context is of limited utility. On the one hand, language is a concrete structure if its irreducible units, or individuals, are taken to be individual physical utterances or, more recently, writings.[1] On the other hand, the great utility of language in the present day derives in no small measure from its ability to represent structure in the abstract, independent of any specific embodiment, concrete or otherwise.

(1.1) Structure *in a general sense is therefore intrinsic to the representational function of language.*

[1]The (physical) context of an individual utterance contributes to its meaning, so we must properly regard distinct individual utterances as distinct expressions. Morphologic equivalence must be one of the relations of the structure of language in this sense. That said, we ordinarily take the units of language to be the abstract types of which individual utterances are *instances*—to be morphologic equivalence classes, if you will. This works well enough for natural languages, and it works perfectly well for the formal languages that are our only concern in this book, in which the context is the same for all instances of a given expression (in a given *interpretation*).

The founders of philosophy in the Hellenistic world realized the importance of language. It was their great achievement to have invented a process by which such things as beauty or human behavior or physical reality could be analyzed and *talked about*. Most significantly for our present purpose, they invented—perhaps we should say 'discovered'—*logic*. As noted above, the word 'logic' derives from the Greek word 'λόγος', which also means 'word'. Logic, in other words, is an intrinsic part of the way one talks about things; it is the methodology of manipulating linguistic representations so as to illuminate the structures they represent.[2]

This chapter is devoted to a careful analysis of formal language and structure. In the next chapter we will see how logic arises from this analysis. The remainder of the present introductory section is devoted to (relatively) informal remarks that provide a context for these investigations and introduce some of the conventions we will use.

1.1.1 Formal language

Despite the lofty tenor of its title, this chapter has a reasonably circumscribed goal. We will not attempt to deal with natural language as it is commonly spoken and written, which would be a much greater task than that which we have actually undertaken. Instead, we will present and examine a category of relatively simple languages—the *first-order predicate languages*—which differ from natural languages in that they are entirely *formal*.

Like natural languages, formal languages consist of *expressions*, which are related to one another in a structure that constitutes the *grammar* or *syntax* of the language. In general there exists a collection of *simple* or *primitive* expressions from which *complex* expressions may be assembled according to a list of rules. The meaning of a complex expression is determined by its structure from the meaning of its primitive constituents according to rules that constitute the *semantics* of the language. The meaning of the primitive expressions is more or less arbitrary.

Formal languages differ from natural languages in several ways. Firstly, their syntax and semantics are exact. Any expression formed according to the rules is meaningful, and its meaning is unambiguously determined by its form (i.e., structure) and the meanings of its primitive constituents. This is the origin of the designation 'formal'. Secondly, the meanings of the primitive expressions of a formal language really are completely arbitrary, unlike those of a natural language, which usually carry a lot of historical baggage (etymology)—neologisms, like 'quark', being exceptional. Thirdly, formal language expressions are used only to convey or—in the case of computer programming languages, for example—to manipulate information, unlike natural language expressions, which may be used to inquire, exhort, reprimand, soothe, etc. Lastly, formal languages have a much narrower range of expressivity than natural languages, even when the latter are confined to the transmission of information.

1.1.2 Formal logic

Logic is the methodology by which we draw conclusions from premises—conclusions and premises being linguistic expressions. A *formal logic* is such a methodology in the context of formal languages. In a formal logic, the relation of conclusion to premise is based entirely on the structure (i.e., form) of those expressions.

[2] The etymology may not be sound, but the etymythology is.

The purpose of this chapter and the next is to present a concept of language and logic that is rich enough to include mathematical and scientific discourse, and formal enough to be itself amenable to mathematical analysis. Among the most satisfying intellectual achievements of the last century are the surprising and beautiful conclusions to which this analysis has led.

Clearly, formal languages and logics are not suitable for many of the uses of natural languages. The domain of inquiry in which they have been found most useful is mathematics.

(1.2) *Indeed, a reasonable modern definition of 'mathematics' is 'all discussions carried out in a formal language using formal logic'.*

1.1.3 Metalanguage and metatheory

We may, if we wish, consider linguistic expressions to be a specified class of abstract entities, e.g., strings of symbols, but it is not necessary or desirable to do so. To discuss language in general we need not say what a word is, or what a sentence is; we need only say how words and sentences and the like relate to one another. Nevertheless, this discussion has to be carried out within a language, because language is the only tool we have to deal with abstractions—or at least the only tool sufficiently adaptable to serve our present purpose.

In a discussion of this sort, i.e., when the objects under study are themselves linguistic, the language employed for the discussion is called the *metalanguage*, and the languages under study are called *object languages*. The use here of the prefix 'meta' parallels to a certain extent its use in the term 'metaphysics'.[3] We note that the metalanguage might itself be an object language, i.e., it might talk about itself, just as you can talk about yourself.

A *theory* is a collection of premises, along with the conclusions that follow logically from them. The main impetus for the study of formal language and logic is the desire to understand the properties of theories. This study must itself be carried out in the framework of a theory that reflects our understanding of the objects of study, viz., structures, languages, logics, theories and related entities.

[3] The articles on *metaphysics* and *meta-* in the *Oxford English Dictionary* (*OED*) are must reading. This from the second edition: "[Metaphysics is t]hat branch of speculative inquiry which treats of the first principles of things, including such concepts as being, substance, essence, time, space, cause, identity, etc.; theoretical philosophy as the ultimate science of Being and Knowing. [The word derives from the Greek phrase] τὰ μετὰ τὰ φυσικά, meaning 'the (works) after the Physics'. . . , the title applied, at least from the 1st century A.D., to the thirteen books of Aristotle dealing with questions of 'first philosophy' or ontology.

"This title doubtless originally referred (as some of the early commentators state) to the position which the books so designated occupied in the received arrangement of Aristotle's writings (τὰ φυσικά being used to signify, not the particular treatise so called, but the whole collection of treatises relating to matters of natural science). It was, however, from an early period used as a name for the branch of study treated in these books, and hence came to be misinterpreted as meaning 'the science of things transcending what is physical or natural'. This misinterpretation is found, though rarely, in Greek writers, notwithstanding the fact that μετὰ does not admit of any such sense as 'beyond' or 'transcending'. In scholastic Latin writers the error was general (being helped, perhaps, by the known equivalence of the prefixes *meta-* and *trans-* in various compounds); and in English its influence is seen in the custom, frequent down to the 17th c., of explaining *metaphysical* by words like 'supernatural', 'transnatural', etc."

As for *meta-*, again according to the second edition of the *OED*: "The supposed analogy of METAPHYSICS (misapprehended as 'the science of that which transcends the physical') has been followed in the practice of prefixing *meta-* to the name of a science (actual or hypothetical) of the same nature but dealing with ulterior and more fundamental problems."

Such a theory is referred to as a *metatheory* to distinguish it from (or among) the *object theories* of interest.

For the present we will take for our metalanguage the language you see represented before you: the language in which this book is written. This language is not entirely formal, and it will not be an object of our formal consideration. It would be premature at this time to attempt to define a formal metalanguage adequate to the task at hand, not least because we have not yet properly defined 'formal language'.

Similarly, we will not at this time define the metatheory within which this discussion takes place. Rest assured, however, that we will attend to both of these tasks in due course, as we must if we are to reap the full rewards of our labor.

1.1.4 Identity, equivalence, and definition

We use '$=$' to mean 'is' in the sense of 'is the same thing as' or, equivalently, 'is identical to'.

For example, '$1+2=3$' means that $1+2$ is 3, i.e., it asserts that the thing named by the expression '$1+2$' *is* the thing named by the expression '3'. We apply this to things of any sort, not just to numbers or other "mathematical entities". Thus we may write 'the third planet from the sun $=$ the planet on which we live'. Note that 'is' is also used in other ways, as in this sentence, i.e., the sentence you are now reading.

We use '$\leftrightarrow$', '$\iff$', and 'iff' to mean 'if and only if'.

For example, '$x^3 = 8 \leftrightarrow x = 2$' means that $x^3 = 8$ if and only if $x = 2$. One may say that—in the given context, which in this case might be that of real numbers—the statements '$x^3 = 8$' and '$x = 2$' are *equivalent* statements about an unspecified real number represented by 'x'.

We use '$\overset{\text{def}}{=}$' and '$\overset{\text{def}}{\iff}$' to indicate that the expression on the left is being defined *by the expression on the right.*

'$\overset{\text{def}}{=}$' is used to define an expression that names something and may be read 'is, by definition,' or 'is defined to be'. For example,

$$\sqrt[3]{x} \overset{\text{def}}{=} \text{that } y \text{ such that } y^3 = x.$$

'$\overset{\text{def}}{\iff}$' is used to define an expression that asserts something and may be read 'if and only if, by definition,' or 'is defined to mean that'. For example,

(1.3) *n is* prime $\overset{\text{def}}{\iff}$ *n has no divisors other than* 1 *and* n*.*

'$\overset{\text{def}}{=}$' and '$\overset{\text{def}}{\iff}$' are notational conveniences, and not every definition we make will contain one of these symbols.

Note that we often use a distinctive font, as for 'prime' in (1.3), to indicate the portion of the expression on the left that is being defined, although in principle that should be clear from the context, since any expression must be defined exactly once, and its first occurrence must be in its definition. Clearly, though, in a discussion as primordial as this one, one cannot adhere strictly to this rule. Indeed, we diverge extremely from strict adherence: only a tiny proportion of the expressions we use in this discussion do we define. The rest are supposed to be generally understood.

1.1.5 Typographical languages

Like most languages meant to be read by people, our global metalanguage—the language in which this book is written—is a *typographical* language, by which we mean, literally,[4] a language that may be set in type. The expressions of a typographical language are therefore arrangements of typographical characters.

We permit considerable latitude in the choice of characters of a language, but ease of understanding dictates that we adhere to certain conventions that we will specify in due course and which we will refer to as *standard*. The corresponding languages are *standard typographical languages*. Our global metalanguage—the language in which this book is written—has as its core a standard typographical language, one which is however considerably augmented—both to increase readability and to allow us greater freedom of expression than a strictly formal language permits.

1.1.5.1 Concatenation

In the narrowest sense, a typographical language is one whose expressions are finite strings of typographical characters.

Definition *The fundamental operation on strings is* concatenation*, denoted by* '$\frown$'*, that joins strings end to end. Thus, for example, if* $\phi =$ 'abc' *and* $\psi =$ 'de'*, then*

$$\phi \frown \psi = \text{`abc'} \frown \text{`de'} = \text{`abcde'}.$$

Note that concatenation is associative, i.e., $(\alpha \frown \beta) \frown \gamma = \alpha \frown (\beta \frown \gamma)$, so expressions like '$\alpha \frown \beta \frown \gamma$' are unambiguous.[5] When the meaning is clear, we may omit '$\frown$' and form a name for the concatenation of two strings by simply concatenating names of the strings.

Formal language is frequently defined in terms of character strings, but this is unnecessarily restrictive and tends to obscure the essential nature of language. Nevertheless, most familiar languages are largely linear in this sense, and we will use the example of linear typographical languages heavily in the following discussion.

It is perhaps unnecessary to say so, but for the purpose of this discussion we regard a character as an abstract form, distinct from its physical representations as—for example—patterns of ink on paper or illuminated pixels on a screen. Thus the single character 'e' occurs many times in this book. Slight variations in the appearance of a physical representation of a character do not alter the identity of the character. We in fact tolerate considerable variation in appearance when we write by hand (more, usually, than our penmanship teachers allow). Certain variations in appearance, however, do indicate distinct abstract characters. For example, we might use 'e', 'e', and '$\boldsymbol{e}$' as three distinct characters. All of this is familiar from ordinary mathematical writing.

1.1.5.2 Names for typographical expressions

(1.4) *As we have done routinely in the discussion thus far, we use single quotes to create a typographical name for a typographical expression.*

The following sentences illustrate this convention. They are all true.

[4] Literally literally, actually.

[5] We hold this truth to be self-evident.

(1.5)

1. Zero is the least natural number.
2. ‘zero’ is a four-letter word that denotes zero.
3. 0 is the least natural number.
4. ‘0’ is a numeral that denotes the number 0.
5. 0 is zero.
6. ‘0’ is not ‘zero’.
7. ‘0’ denotes zero.
8. ‘ ‘zero’ ’ is an expression—specifically, a four-letter word contained between single quotes—that denotes the four-letter word ‘zero’.
9. ‘0 is zero’ is true, but ‘ ‘0’ is ‘zero’ ’ is false.

Note that the last two sentences involve repeated single quotes, not double quotes.

(1.6) *We will use double quotes informally as an equivalent of the interjection ‘so to speak’ or ‘as it is said’, often to mark an expression that is not quite right or merits scrutiny.*

The use of quotation marks to create names for typographical expressions has limited utility. Consider, for example,

(1.7) ‘a’ and ‘b’.

If we regard the first two quotation marks as forming a pair and the last two as forming another pair, then this phrase refers to two strings, each consisting of one letter. If, however, we regard the first and last quotation marks as forming a pair, then this is a name for a single string consisting of five letters, two spaces and two quotation marks.

(1.8) *To avoid this sort of ambiguity we may use the convention of drawing a line under a typographical expression to create a name for it.*

This convention has the additional virtue of being relatively unobtrusive. With this convention (1.5) may be written

(1.9)

1. Zero is the least natural number.
2. $\underline{\text{zero}}$ is a four-letter word that denotes zero.
3. 0 is the least natural number.
4. $\underline{0}$ is a numeral that denotes the number 0.
5. 0 is zero.
6. $\underline{0}$ is not $\underline{\text{zero}}$.
7. $\underline{0}$ denotes zero.
8. $\underline{\underline{\text{zero}}}$ is an expression—specifically, an underlined four-letter word—that denotes the four-letter word $\underline{\text{zero}}$.
9. $\underline{\text{0 is zero}}$ is true, but $\underline{\underline{0}\text{ is }\underline{\text{zero}}}$ is false.

We should also point out that we employ other means of naming a typographical expression. We may put it in a distinctive font, as in the clause 'we call this operation *conjunction*', where we use an italic font to create a name for the word 'conjunction'; or we may display it in the manner of equations, as we have in (1.7) and (1.9). We rely somewhat on the reader's interpretive participation and good will in thus allowing us to avoid excessively cumbersome notation. For example, we have presumed that you understood that in (1.7) the period (full stop) is not part of the displayed expression but rather part of the sentence in which the displayed expression is the direct object.

Another way to make a name for an expression is by definition. For example, we may say 'let $\phi = \underline{\text{roses are red}}$'. We may also use symbols as variables over expressions. We might, for example, say 'let ϕ be an arbitrary declarative expression'.

By way of illustration, suppose $\phi = \underline{\text{roses are red}}$ and $\psi = \underline{\text{violets are blue}}$. Then

$$\begin{aligned} \phi\,\underline{\text{and}}\,\psi &= \phi\,{}^\frown\,\underline{\text{and}}\,{}^\frown\,\psi = \underline{\text{roses are red}}\,{}^\frown\,\underline{\text{and}}\,{}^\frown\,\underline{\text{violets are blue}} \\ &= \underline{\text{roses are red and violets are blue}}. \end{aligned} \tag{1.10}$$ [6]

The Knight's Song An illustration of the distinction between a thing and its name (and the name of its name) is the following exchange between Alice and the White Knight in *Through the Looking Glass*, by Samuel Dodgson, *aka* Lewis Carroll (two names for the same thing).

> "It's long [his song]," said the Knight, "but it's very, *very* beautiful. Everybody that hears me sing it—either it brings the *tears* into their eyes, or else—"
>
> "Or else what?" said Alice, for the Knight had made a sudden pause.
>
> "Or else it doesn't, you know.[7] The name of the song is called '*Haddocks' Eyes.*' "
>
> "Oh, that's the name of the song, is it?" Alice said, trying to feel interested.
>
> "No, you don't understand," the Knight said, looking a little vexed. "That's what the name is *called.* The name really *is* '*The Aged Aged Man.*' "

[6] Two points:
1) It would be improper to write $\underline{\phi \text{ and } \psi}$ (equivalently, 'ϕ and ψ') where we have written $\underline{\phi\,\underline{\text{and}}\,\psi}$ (equivalently, '$\phi\,\underline{\text{and}}\,\psi$') in (1.10). To put it another way: The expression $\underline{\phi\,\text{and}\,\psi}$ does not denote the expression $\phi\,\underline{\text{and}}\,\psi$. ϕ and ψ are (by definition) declarative expressions. $\underline{\phi}$ and $\underline{\psi}$ are nominative expressions (each consisting of one symbol, a Greek letter) that denote these declarative expressions. ('$\underline{\phi}$' and '$\underline{\psi}$' are in their turn nominative expressions denoting $\underline{\phi}$ and $\underline{\psi}$, respectively.) Thus $\underline{\phi\,\text{and}\,\psi}$ is a nominative expression homologous to $\underline{\text{Mary and John}}$ or $\underline{\text{roses and violets}}$, and refers to a pair of Greek letters. Note that in the expressions named in the preceding sentence, in contrast to (1.10), 'and' does not serve as a logical connective, but rather as an operation symbol that forms a name for a pair of things from names for those things. The former is the 'and' in 'I am cold and I am hungry'; the latter is the 'and' in 'cold and hunger'. These two uses of 'and' should not be confused.
2) We are aware that we have inserted spaces between ϕ, $\underline{\text{and}}$, and ψ instead of just concatenating them. Only the pathologically punctilious would cavil at this concession to convenience.

[7] This is an illustration of the law of the excluded middle—in other words, the tautology (ϕ `or` $\neg\,\phi$), with '$\neg$' indicating negation. This is rejected by the school of intuitionism, which insists that either ϕ or $\neg\,\phi$ must be known to be true before (ϕ `or` $\neg\,\phi$) may be concluded. In this book we present only the standard, or *classical*, logic.

> "Then I ought to have said 'That's what the *song* is called'?" Alice corrected herself.
>
> "No, you oughtn't: That's quite another thing! The *song* is called '*Ways And Means*': but that's only what it's *called*, you know!"
>
> "Well, what *is* the song, then?" said Alice, who was by this time completely bewildered.
>
> "I was coming to that," the Knight said. "The song really is '*A-sitting On A Gate*': and the tune's my own invention."

Apparently the name of the song is '*The Aged Aged Man*'. '*Haddocks' Eyes*' is a name for this name. '*Ways and Means*' is another name for the song (perhaps erroneous, as the Knight emphasizes that this is only what the song is *called*). As written, the Knight's last statement can only be interpreted to mean that the entire content of the song is the phrase '*A-sitting On A Gate*', as in "The first verse of the Gospel according to John really is '*In the beginning was the Word.*' ". Did Lewis Carroll make a mistake? It would make sense to omit the single quotes and write, "The song really is *A-sitting On A Gate*: and the tune's my own invention." This would be a grammatical construction parallel to "My mother really is Lucile", and would imply that '*A-sitting On A Gate*' is yet a third name for the song. I suspect that Carroll did not make a mistake, and that he really did mean the Knight to say that the song consisted of the single phrase '*A-sitting On A Gate*'. That this statement is contradicted by the Knight's subsequent rendition of the song—which is a droll ballad including that phrase, but containing also a great deal more—can hardly be taken as a refutation of this position, given the tenor of Carroll's Alice books.

1.1.6 The elements of language

1.1.6.1 Nominative and declarative expressions

The purpose of the formal languages we discuss here is to make statements about things. In ordinary grammatical terms, things are indicated by nominative expressions (so called because they *name* things). These are inserted into incomplete declarative expressions to make complete declarative expressions. For example, Mary and John are nominative expressions, and · loves · is an incomplete declarative expression with two "slots"—each indicated by ·—into which nominative expressions may be inserted. From these elements we can form four sentences: Mary loves John, John loves Mary, Mary loves Mary, and John loves John.

Mary and John are complete nominative expressions. We may have incomplete nominative expressions as well. For example, the expression the mother of · is an incomplete nominative expression, which may be completed to a nominative expression by inserting any nominative expression in the indicated slot. (We'll ignore for the moment the fact that such an expression may not actually name anything—e.g., the mother of Tokyo or the mother of Eve.) Incomplete nominative expressions may have any finite number of slots—e.g., the eldest child of · and · or the sum of ·, ·, and ·. Likewise, incomplete declarative expressions may have any finite number of slots—e.g., · gave · to ·.

1.1.6.2 Variables

The expression Mary loves Mary illustrates a defect in the rudimentary notion of incomplete expression just described. As presented above, this expression is on a par with Mary loves John, both being derived from · loves · by the insertion of nominative expressions in its two slots. But it is desirable that we distinguish these two sorts of substitution; indeed, in a treatise on narcissism, for example, we might quite naturally wish to define an incomplete expression with just one slot, into which Mary could be inserted to generate Mary loves Mary. Natural languages have devices to accomplish this in simple cases. For example, we might use · loves herself as a template. Inserting Mary gives Mary loves herself, which means the same as Mary loves Mary. But such *ad hoc* devices cannot handle the elaborate substitution patterns that arise in even simple mathematics.

The inadequacy of our notation stems from the use of '·' as a generic label for the slots of an incomplete expression, and a solution is to allow multiple labels. For example, suppose we have labels 'x' and 'y' at our disposal. Then we can form two essentially different types of "incomplete expression" from '· loves ·'. One type is represented by 'x loves y', the other by 'x loves x'. 'x' and 'y' are placeholders, like '·', and have no fixed meaning; we call them *variables*.

Expressions containing variables may be used as templates in much the same way as expressions containing '·', with the following proviso: if we substitute a nominative expression for some occurrence of a variable in an expression, we must substitute the same expression for all occurrences of that variable in that expression. For example, from 'x loves x' we may obtain 'Mary loves Mary' and 'John loves John', but not 'Mary loves John'. Note that there is no rule against substituting the same expression for different variables. From 'x loves y', for example, we may obtain 'Mary loves Mary' as well as 'Mary loves John'.

Note that—*as templates*—'x loves x' is equivalent to 'y loves y', 'z loves z', etc.; and 'x loves y' is equivalent to 'x loves z', 'y loves x', etc.

1.1.6.3 Terms

In the interest of terminological brevity we refer to complete nominative expressions as *terms*, and we include variables and nominative expressions containing variables under this rubric. Examples of terms are 'x' (assuming we have declared 'x' to be a variable), 'Mary', 'the mother of x', 'the mother of John', etc. Note that whenever we have a term that contains a variable we may *substitute* a term for the variable to obtain another term, as we have substituted 'John' for 'x' in the term 'the mother of x' above. We may also form 'the mother of the mother of x', etc. Terms derived in this way are called *complex* or *compound*. All other terms are called *simple* or *primitive*.

A simple term is either a variable or the result of inserting variables into the slots of an incomplete nominative expression that is not the result of combining other expressions of the language. In a typographical language these basic expressions are often indicated by unique symbols. For example, in a language suitable for arithmetic, we might have symbols '$\mathbf{0}$' for zero, 'S' for the successor operation, '$+$' for addition, and '$\times$' for multiplication. We call these *operation symbols* (or, interchangeably, *function symbols*). In the general case of languages that are not necessarily typographical, we have *operation* or *function indices* that play a similar role. In the typographical case an operation symbol may also serve as an operation index, and operation indices are often referred to generically as symbols in discussions of formal language. We will, however, generally preserve the distinction between indices and the symbols that may represent them, at least until we have gotten through these introductory chapters. In this connection it should be noted that familiar typographical languages often do not have a symbol for each basic operation; for example, in a language for arithmetic, we may represent multiplication by juxtaposition, as in '$(1 + 1)(3 + 4) = 14$'. A proper treatment of such a language requires that we have an index for multiplication even if we do not have a symbol for it.

The operation indices of a language correspond to the primitive incomplete nominative expressions we have spoken of previously. Each index has an *arity*, which is the number of its "slots". The arity is a natural number. An operation symbol of arity, say, n, along with an n-sequence of terms, determines a new term by the process of *specification*, which is the insertion of the terms into the slots, in our previous terminology. We call the specified terms the *arguments* of the resulting expression. Note that this is slightly different from the process of substitution, which is the replacement of a variable by a term.

Operation indices of arity 0 are an important special case. They do not take any arguments and are called *constants.* In the above example of arithmetic '**0**' is a constant.

Substitution in terms As discussed above, the essential purpose of a variable is to indicate a place where a term may be substituted in an expression, with the understanding that any substitution of a term τ' for a variable v in a term τ must substitute τ' for all occurrences of v in τ. (The situation is slightly different for formulas, as we will see presently.)

(1.11) **Definition** *Suppose v is a variable and τ and τ' are terms. Then*

$$\tau\binom{v}{\tau'} \stackrel{\text{def}}{=} \textit{ the result of substituting } \tau' \textit{ for every occurrence of } v \textit{ in } \tau.$$

1.1.6.4 Formulas

We refer to complete declarative expressions as *formulas.* Like terms, formulas may be simple or complex, but the processes by which complex formulas are generated differ from those for terms, and we use a slightly different terminology. We call the simplest formulas *atomic.* These result from the insertion of terms into the slots of an incomplete declarative expression that is not analyzable as a compound of simpler expressions. These are similar in several ways to primitive nominative expressions. In a typographical language a basic declarative expression, like a basic nominative expression, is often indicated by symbol. For example, referring again to arithmetic, the symbols '$=$' and '$<$' are often used to indicate the basic relations of identity and order. In general, each primitive declarative expression is associated with a unique *predicate index* or, interchangeably, *relation index.* Like operation indices, each predicate index has an arity. The process of specification for predicate indices creates an atomic formula from an index of arity n and an n-sequence of terms. '$(x + y) < (x + S(y))$' is an example of an atomic formula, in which the terms '$(x + y)$' and '$(x + S(y))$' have been inserted into '$\cdot < \cdot$'.

We defer for the moment the definition of substitution of terms for variables in formulas.

1.1.7 Syntax and semantics

The meaning of an expression depends on its intrinsic structure and on the *interpretation* of its primitive constituents,[8] and the great power of language resides in its recognition of the separate roles of grammatical structure and interpretation in the generation of meaning. *Syntax* is synonymous with *grammar* and refers to the structure of a language and its expressions; *semantics* refers to the way that syntax assigns meaning to expressions given an interpretation. We will not delve too deeply here into the meaning of 'meaning'; we are concerned chiefly with the perhaps narrower—at any rate, precisely defined—concept of the *value* of an expression under an interpretation.

The value of a term is a thing, 'thing' being understood in the most general sense to include both concrete and abstract entities. A formula, on the other hand, becomes either true or false under an interpretation, and we may say that its value is either trueness or falseness—or true or false, construing these adjectives as representing the qualities that they predicate. We often refer to this sort of value as *truth value*.

1.1.7.1 Propositional connectives

Formulas may be combined by means of *propositional connectives*. For example, from the expressions 'roses are red' and 'violets are blue' we may form the expressions 'roses are red and violets are blue', 'roses are red or violets are blue', 'roses are red if violets are blue', etc. In these expressions, 'and', 'or', and 'if' are *propositional connectives*. Semantically, the essential feature of propositional connectives is that the truth value of formulas formed with them depends in a defined way on the truth values of the constituents.

It is common in mathematical writing to represent propositional connectives by individual symbols rather than by the words we use for this purpose in ordinary discourse. For example, we frequently use '$\wedge$' for 'and'. Thus, we might write '$x < y \ \wedge \ y < z$' for '$x < y$ and $y < z$'. We likewise introduce symbols for several other propositional connectives: for any formulas ϕ and ψ

$$\begin{aligned}
\underline{\neg}\,\phi &\text{ means } \underline{\text{not}}\ \phi;\\
\phi\,\underline{\wedge}\,\psi &\text{ means } \phi\ \underline{\text{and}}\ \psi;\\
\phi\,\underline{\vee}\,\psi &\text{ means } \phi\ \underline{\text{or}}\ \psi;\\
\phi\,\underline{\rightarrow}\,\psi &\text{ means } \phi\ \underline{\text{only if}}\ \psi,\ \text{ i.e., } \phi\ \underline{\text{implies}}\ \psi,\ \text{ or } \underline{\text{if}}\ \phi\ \underline{\text{then}}\ \psi; \textit{and}\\
\phi\,\underline{\leftrightarrow}\,\psi &\text{ means } \phi\ \underline{\text{if and only if}}\ \psi.
\end{aligned}$$

1.1.7.2 Quantification

Consider the expression 'roses are red', which we have used above as a example of a formula. This may be analyzed as 'all roses are red', or, equivalently, 'everything that is a rose is red', or, equivalently, 'for every thing, if that thing is a rose then that thing is red'. In the same vein, consider the expression

[8]The primitive constituents of a language are its operation, predicate, and domain[9] indices, and its variables.[10]

[9] We will define 'domain' and 'domain index' presently; at this time we are concerned chiefly with introducing the notion of value.

[10]We will make a special case of the interpretation of a variable, calling it by the name 'assignment', and restricting the formal application of 'interpretation' to the indices of a language.

(1.12) *every cloud has a silver lining.*

This may be analyzed as

(1.13) *for every thing, if that thing is a cloud then there is a thing such that that thing is a silver lining and the first thing has the second thing.*

The phrases 'for every thing' and 'there exists a thing' are called *quantifier phrases* because they make quantitative assertions about the collection of things satisfying the formulas they introduce.[11] The phrase 'for every thing' is called *universal* and the phrase 'there exists a thing' is called *existential*.

(1.13) illustrates the difficulty that arises when one has more than one quantifier phrase in a given expression: one must keep track of the "things" quantified and how they relate to one another. We have used 'that thing', 'the first thing', and 'the second thing' above, but clearly a more systematic approach is wanted, and *variables*, as introduced above, are ideally suited to the task. (1.13), for example, may be written:

for all x, if x is a cloud then there exists y such that y is a silver lining and x has y.

In typographical languages we often use the characters '$\forall$' (an inverted 'A' for *all*) for the phrase 'for all' (which has the same meaning as 'for every'), and '$\exists$' (an inverted 'E' for *exists*) for 'there exists'. Using these symbols and the propositional connective symbols introduced above, we may write

$$\forall x \Big((x \text{ is a cloud}) \to \exists y \left((y \text{ is a silver lining}) \wedge (x \text{ has } y)\right)\Big). \tag{1.14}$$

'$\forall x$' and '$\exists y$' are examples of *quantifier phrases*.

1.1.8 Variables in formulas

1.1.8.1 Occurrences, free and bound

The *raison d'être* of variables is their use repetitively.[§ 1.1.6.2] Our definition[1.11] of substitution of a term τ_0 for a variable v in a term τ_1 applies to terms τ_1 with any number of occurrences of v. The notion of substitution of a term for a variable in a formula is a little more involved.

Let us review the way formulas are constructed. Along the way we will define the notions of *occurrence* of a variable and of a quantifier phrase in an expression, and *free* and *bound occurrences* of variables.

We begin with atomic formulas, which are created by specification of a predicate index P and are of the form $\tilde{P}\langle\tau_0,\ldots,\tau_{n-1}\rangle$, where n is the arity of P, and $\tilde{P}$ is the operation that takes an n-sequence $\langle\tau_0,\ldots,\tau_{n-1}\rangle$ to the formula that results from the specification of P to $\langle\tau_0,\ldots,\tau_{n-1}\rangle$.

[11] These "quantitative assertions" are admittedly of a rather qualitative nature (which does not contradict their being quantitative: the difference in meaning between 'quantitative' and 'qualitative' is quantitative, not qualitative—or, to put it more accurately, if less epigrammatically, they indicate opposite directions along a continuum, not distinct states). They don't tell exactly how many things satisfy the associated formula. We can design more specific quantifier phrases—for example, 'there exists exactly one thing' and 'there exist infinitely many things'—but we will not incorporate these into our basic concept of a formal language.

In the case of a typographical language or any other linear language, in which expressions are finite sequences of symbols, we may define an *occurrence* of a variable v in an atomic formula α to be an ordered pair (m, v), where v occurs in the mth place in α. In the case of an atomic formula we define all the occurrences of variables to be *free*, no occurrences are *bound*, and of course there are no quantifier phrases.

We define *occurrence* of a quantifier phrase in an arbitrary formula in the same way we have defined *occurrence* of variables for atomic formulas: it is simply the quantifier phrase together with an indicator of its position within the formula. The notions of *free* and *bound* do not apply to quantifier phrases.

We now extend these definitions to all formulas by recursion on the complexity of formulas.[12] Atomic formulas all have minimal complexity—i.e., no formula is less complex than an atomic formula. Suppose ϕ is a complex formula. Then it is composed from simpler—i.e., less complex—formulas by the formation of a basic propositional connection (negation, conjunction, disjunction, etc.) or by quantification on a variable. Exactly one of these possibilities is realized.

If ϕ is formed by propositional connection then the occurrences of a variable v in ϕ are in one-one correspondence with the occurrences of v in the constituent formulas in the obvious way. The free (respectively, bound) occurrences of v in ϕ are those that correspond to free (respectively, bound) occurrences of v in the constituent formulas.

Suppose now that ϕ is of the form $\underline{\forall} v\, \psi$ or $\underline{\exists} v\, \psi$. We define the occurrences of v in ϕ to be those that correspond to occurrences of v in ψ. Variables in quantifier phrases do not *occur* there according to this definition—rather, it is the entire quantifier phrase that occurs. We define all free occurrences of v in ψ to be *bound by* the indicated quantifier phrase occurrence, and to be *bound in* ϕ. Any occurrence of v already bound in ψ remains bound in ϕ by the same quantifier phrase occurrence as binds it in ψ. Occurrences of all variables in ψ other than v are bound in ϕ iff they are bound in ψ.

All formulas are obtained from atomic formulas by a succession of operations of this sort, and the above description uniquely defines *occurrence*, *free*, and *bound*.

1.1.8.2 Substitution in formulas

(1.15) **Definition** *Suppose v is a variable, τ is a term, and ϕ is a formula. Then*

$$\phi\binom{v}{\tau} \overset{\text{def}}{=} \textit{the result of substituting } \tau \textit{ for every free occurrence of } v \textit{ in } \phi.$$

A formula ϕ may be thought of as making a statement about the objects denoted by its free variables. We would like $\phi\binom{v}{\tau}$ to make the same statement about the object denoted by the term τ as it makes about the object denoted by v. This is guaranteed by the requirement that no variable in τ is bound in $\phi\binom{v}{\tau}$:

(1.16) **Definition** $[\mathsf{C}^0]$ *A term τ is* free to be substituted for *a variable v in a formula ϕ $\overset{\text{def}}{\iff}$ no variable occurrence in τ is bound in $\phi\binom{v}{\tau}$. We also say briefly that τ is* free for *v in this event.*

[12] We will not define 'complexity' here, as its meaning may be inferred from its use in the definition that follows, and our presentation of language thus far is not so formal as to benefit significantly from a definition of 'complexity' at this point. We assume that you have derived from the preceding discussion a mental image of linguistic expressions, and the following definition is based on that image. A formal definition will follow.

For example, let $\phi = \underline{\forall y\,(x \times y = x)}$. In the standard interpretation in terms of real numbers, ϕ is equivalent to $\underline{x = 0}$. Let $\tau = \underline{y + z}$. To say τ has the property that ϕ asserts for x, we cannot substitute τ for $\underline{x}$ in ϕ, because this yields $\underline{\forall y\,((y+z) \times y = (y+z))}$, in which the occurrence of $\underline{y}$ in τ is inappropriately bound, and which is not a statement about $y + z$, but rather about z (and is not true of any real number). The problem is that τ is not free for $\underline{x}$ in ϕ.

1.1.8.3 Change of variables

Note that when a term is substituted for a variable v in a formula,[1.15] it is substituted only for the free occurrences of v. Substitution of a term for a bound occurrence of v is generally not a useful transformation. If the term is itself a variable, however, say v', and we substitute v' for all occurrences of v bound by the same quantifier phrase, and we substitute v' for v in the quantifier phrase itself, then as long as none of the introduced occurrences of v' are bound by a pre-existing quantifier phrase, the resulting formula is structurally identical to the original. This is called a *change of variables*. This sort of transformation is often used in preparation for a substitution that would otherwise lead to unwanted binding of variables.

In the preceding example, we could effect a change of variables to replace ϕ by the equivalent $\phi' = \underline{\forall w\,(x \times w = x)}$. τ is free for $\underline{x}$ in ϕ', and if we substitute τ for $\underline{x}$ in ϕ', we obtain $\underline{\forall w\,((y+z) \times w = (y+z))}$, which does say that $y + z = 0$.

1.1.9 Identity

Identity is a binary relation that holds for two things iff they are the same thing. Note that by this we do not mean that they are the same type of thing, as in 'Everywhere I look I find the same thing: a pile of dust!', but rather that they are the same individual thing, that "they" are in fact "it", one thing, as in 'I am I.'.[13] By universal convention, '=' is used in typographical languages to denote the identity relation. Unlike other primitive symbols, it is not open to interpretation: it always denotes the identity relation. A language need not have such a dedicated notation and is said to be *with* or *without identity* according as it does or does not.

Since '=' is a typographical character, the foregoing convention applies only to typographical languages. It is convenient to have a similar convention for languages in general. In the general case, predicates are represented by *indices*, which may be elements of any type. We will adopt the convention that, as a predicate index, 0 will be used exclusively for identity, and we will say that a language is with or without identity according as 0 is or is not among its predicate indices.

1.1.10 Domains of discourse

Consider the statement (1.13). Here 'thing' refers *a priori* to anything whatsoever. But note that we immediately restrict the first instance of 'thing' to refer specifically

[13]It must be admitted that this fails to define 'identity'. We can never distinguish between the apparently obvious meaning that you have no doubt gathered from the explication to which this footnote is attached and another meaning in which distinct things may stand in the relation of identity to one another as long as the substitution of an individual for an identical individual does not change the truth value of any relation specification, and such a substitution in an operator specification leads to an expression with an identical (not necessarily the same) value. Fortunately, by its very nature such an interpretation would not differ in any "observable" way from the simplest interpretation. Indeed, we could regard the equivalence classes of such an "identity" relation as the true individuals.

to clouds and the second instance of 'thing' to refer to silver linings. A cloud is a *sort* (or *type* or *species*) of thing. Likewise, a silver lining is a sort of thing. If we were to discuss clouds and silver linings at length we would find it tiresome to use the generic term 'thing' with the qualifying declarations 'thing is a cloud' and 'thing is a silver lining', preferring instead to refer directly to clouds and silver linings, as in (1.12).

The collection of all entities to which 'thing' may refer is the *universe of discourse* or simply the *universe* of the discussion, and the collections of clouds and silver linings are *domains of discourse* or simply *domains*, or *sorts*. To use a more "mathematical" example, consider the theory of vector spaces. Here there are two domains of interest: the collection of scalars (which may, for example, be the collection of real numbers) and the collection of vectors.

Note that the universe may be regarded as a domain also, which may be called the *universal* domain or *the* domain of discourse, but we will not require that it be explicitly a domain.

The introduction of multiple sorts of entities allows us to streamline our notation even further. For example, let 'C' and 'SL' refer respectively to the domains of clouds and silver linings. We may qualify our quantifiers by affixing a subscript for the domain over which the quantification takes place. Then (1.14) may be written

$$\forall_C x\, \exists_{SL} y\, (x \text{ has } y).$$

In a typographical language we often represent domains of discourse by symbols, such as 'C' and 'SL' above, which are naturally called *domain symbols*. In the general case, as for operations and predicates, we use *domain indices* to indicate domains.

1.1.11 A minimal framework

In a language with identity, operations may be eliminated.

We could, for example, dispense with the binary operation '$\cdot + \cdot$' in favor of a ternary predicate, say '$S(\cdot,\cdot,\cdot)$' defined by

$$S(z,x,y) \overset{\text{def}}{\iff} x + y = z.$$

S has the special property that for any x and y there is a unique z such that $S(z,x,y)$. For any natural number n, any n-ary operation may be replaced by an $(n+1)$-ary predicate in this way. Operations are therefore a convenience, not a necessity, but "operational" predicates are so common and useful that we prefer to incorporate them explicitly in our definition of formal language.

Similarly, any reference to a domain may be replaced by a reference to a unary predicate.

For example, we could formulate the theory of vector spaces with unary predicate symbols 'S' and 'V' and replace, for example, 'for every scalar α' by 'for every thing α, if $S(\alpha)$ then' and 'there exists a vector a such that' by 'there exists a thing a such that $V(a)$ and'. As in this example, we often find it convenient to use a different family of variable symbols for each sort of entity. We say that 'α' and 'a' *range over* the domains of scalars and vectors, respectively.

(1.17) **Definition** *We call a language* relational $\overset{\text{def}}{\iff}$ *it has only relation (i.e., predicate) indices. We call it* operational $\overset{\text{def}}{\iff}$ *it has only operation indices with the possible exception of an index for the identity relation.*

If we wish, we can make do with only two propositional connectives, say negation and conjunction. For example, we may replace $\phi \underline{\vee} \psi$ by

$$\underline{\neg((\neg}\phi\underline{) \wedge (\neg}\psi\underline{))}.$$

In fact, we can get by with just one propositional connective: popular choices for such a connective in the information processing world are *nor* and *nand*, where $\phi \underline{\text{ nor }} \psi$ is equivalent to $\underline{\neg(}\phi\underline{\vee}\psi\underline{)}$ and $\phi \underline{\text{ nand }} \psi$ is equivalent to $\underline{\neg(}\phi\underline{\wedge}\psi\underline{)}$.

We can also make do with one quantifier, as $\underline{\forall} v\, \phi$ is equivalent to

$$\underline{\neg\exists} v\, \underline{\neg}\phi,$$

and *vice versa*, for any variable v and formula ϕ.

Definitions and proofs concerning languages in general can often be effectively indicated by reference to one or another of the above simplified types.

(1.18) *We will frequently take $\neg$, $\rightarrow$, and $\exists$ as a minimal set of logical operations.*[14]

1.1.12 Typographical conventions

Formal languages may be represented typographically in a number of ways. In standard typographical languages the expressions are strings of characters. Certain characters are *reserved*. Among these are the characters used for propositional connectives and quantifiers, as well as characters used to make variables. A potentially infinite supply of variables is required, so these must, from a practical standpoint, be composite. We could, for example, say that a variable is a string of the form $\underline{v}^\frown N$, where N is an arabic numeral, so that, for example, $\underline{v0}$, $\underline{v1}$, and $\underline{v65537}$ are variables. In this case $\underline{v}$, $\underline{0}$, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, $\underline{5}$, $\underline{6}$, $\underline{7}$, $\underline{8}$, $\underline{9}$ are reserved characters. More neatly, we may form variables by using subscripts, so that a variable is of the form

$$\underline{v}_N,$$

where N is an arabic numeral, and our variables are

$$\underline{v_0}, \underline{v_1}, \ldots.$$

We regard each of these as a reserved character, leaving $\underline{v}$, $\underline{0}$, $\underline{1}$, etc., free for general use.

One might have other reserved characters as well, such as $\underline{(}$, $\underline{)}$, and $\underline{,}$. In addition, one typically has a character for each relation and operation index, which are referred to respectively as *relation* and *operation symbols*. These are distinct from the reserved characters and from one another.

To construct expressions from these characters, perhaps the simplest convention to describe is so-called *Polish notation*, in which each expression begins with a quantifier, a propositional connective, a relation symbol, or an operation symbol, which is followed by expressions of the types and number appropriate to it. For example, in a language appropriate to discussing rational numbers, we might use

[14] $\phi \underline{\vee} \psi$ is equivalent to $\underline{(\neg}\phi\underline{)} \underline{\rightarrow} \psi$, etc.

'+' for addition, '×' for multiplication, '0' for zero and '1' for one, and '=' (as always) for identity. The expression

$$\forall\, v_0 \rightarrow \neg = v_0\, 0\, \exists\, v_1 = \times\, v_0\, v_1\, 1$$

means that every nonzero element has a multiplicative inverse, as may be seen by introducing brackets and the familiar order for binary operations and relations:

$$\begin{aligned}
&\forall\, v_0 \rightarrow \neg = v_0\, 0\, \exists\, v_1 = \times\, v_0\, v_1\, 1\\
&\forall\, v_0 \rightarrow \neg(v_0 = 0)\, \exists\, v_1 = (v_0 \times v_1)\, 1\\
&\forall\, v_0 \rightarrow (\neg(v_0 = 0))\, \exists\, v_1((v_0 \times v_1) = 1)\\
&\forall\, v_0 \rightarrow (\neg(v_0 = 0))\, (\exists v_1\, ((v_0 \times v_1) = 1))\\
&\forall\, v_0((\neg(v_0 = 0)) \rightarrow (\exists v_1\, ((v_0 \times v_1) = 1)))
\end{aligned}$$

One virtue of Polish notation is that no bracketing characters are necessary to indicate which substrings constitute subexpressions, and Polish notation is efficient for communicating with computers, but humans have trouble with it. One common modification is to place the arguments of a binary symbol on either side of it, as above. Once we do this, however, we must indicate which substrings are subexpressions, usually by the use of paired brackets, such as $\underline{(}$ and $\underline{)}$.

To economize on the use of brackets for grouping in formulas, we follow precedence rules similar to those used in arithmetic for negation, multiplication, and addition. Negation and quantification take precedence over disjunction and conjunction, which take precedence over implication and bi-implication.

For example,

$$\neg\, \phi_0 \wedge \forall u\, \phi_1 \rightarrow \phi_2 = \big((\neg\, \phi_0) \wedge (\forall u\, \phi_1)\big) \rightarrow \phi_2$$

Another common modification is to separate the arguments of a relation or operation symbol by commas and to demarcate the entire list by brackets, as in

$$H(A, B, C, D).$$

We may also place arguments as subscripts or superscripts, as in '$\delta_{\mu\nu}$', or in more elaborate arrangements, as in '$\sum_{i=1}^{\infty} a_i$'. We even have terms in which the "operation symbol" is lacking altogether, as in 'ab' or 'a^b', denoting respectively the product of a and b and a raised to the power b. We also have, quite commonly, "symbols" that consist of more than one character. For example, in the expression '$\sin\theta$', 'sin' is the symbol for the sine function.

A cursory perusal of ordinary mathematical writing reveals that the types of construction illustrated above, while they characterize "math" in the popular imagination and while they admittedly allow a page of "mathematical" writing to be instantly recognized as such, do not include the most common type of relation and operation specifications. For example, in the expression 'V is a vector space', 'is a vector space' functions as a unary predicate symbol. Similarly, the expression 'V has dimension n over W' is regarded as derived from some such ternary predicate symbol as '$\cdot$ has dimension $\cdot$ over $\cdot$'.

It might be conceptually convenient to limit typographical languages to some standard construction, and many expositions of mathematical logic do so—but, at least in our approach, nothing of substance is gained thereby, and there are moreover good reasons not to do this. First, our metalanguage has nonstandard constructions, and we certainly want it to fall within the purview of our general analysis. Second, to properly understand language we must consider it in the abstract, and it will be conceptually easier to make that transition if our notion of typographical language is already relatively fluid.

1.1.13 Metalanguage notation for expression-building operations

(1.19) *Languages of the sort we have described are called* first-order predicate languages.[15] *They are*

1. *predicate languages in that they involve predicates applied to terms; and*
2. *first-order in that they involve quantification over the arguments of predicates but not over predicates themselves, i.e., they do not employ "predicate variables'.*

Allowing quantification over predicate variables yields a second-order language, and if we allowed quantification over variables that range over predicates applicable to predicates, we would have a third-order language, etc. Disallowing these higher-order entities is not a significant limitation, as the full expressivity of higher-order languages can be achieved in the context of a first-order theory of the membership relation (the notion of something being a member of a collection). On the other hand, excluding predicates altogether, along with the machinery of variables and quantification that goes with them, leaves so-called propositional languages, which are significantly less expressive than predicate languages and are not adequate for general purposes.

It will be convenient to have a uniform way to refer (in the metalanguage) to the expression-forming operations of first-order predicate languages. These are of three types:

1. argument specification;
2. logical connection; and
3. quantification.

1.1.13.1 Argument specification

We indicate the operation of specification as follows.

(1.20) **Definition** *Suppose X is a relation or operation index with arity n. Then $\tilde{X} \overset{\text{def}}{=}$ the operation that takes an n-sequence $\tau = \langle \tau_0, \ldots, \tau_{n-1} \rangle$ of terms to the expression that is the result of specifying the arguments of X to be $\tau_0, \ldots, \tau_{n-1}$.*

1.1.13.2 Propositional connection

For each propositional connective we have a corresponding operation on expressions. For example, for a standard typographical language,

if ϕ and ψ are formulas,

$$\phi \wedge \psi \overset{\text{def}}{=} \underline{(}\phi\underline{)\wedge(}\psi\underline{)}.$$

[15]Languages with operations and/or multiple domains fall under this rubric if, as indicated above,[§ 1.1.11] an operation or a domain is regarded as a particular sort of predicate.

Note the difference between '$\boldsymbol{\land}$' and '$\land$'. The typographical difference is that the former is boldface and the latter is lightface. The difference in significance is that whereas '$\boldsymbol{\land}$' is a(n object-language) propositional connective, which joins two formulas to make a formula, '$\land$' is a (metalanguage) operation symbol that denotes the operation of conjunction (as applied to object-language formulas). This is a type of use/mention relationship: '$\boldsymbol{\land}$' is part of an object-language formula, whereas '$\land$' is part of a metalanguage term that names that formula. We define metalanguage symbols for all the object-language connectives in the same way. For a standard typographical language,

$$
\begin{aligned}
\neg\,\phi &\overset{\text{def}}{=} \underline{\neg(}\phi\underline{)} \\
\phi \land \psi &\overset{\text{def}}{=} \underline{(}\phi\underline{)\ \boldsymbol{\land}\ (}\psi\underline{)} \\
\phi \lor \psi &\overset{\text{def}}{=} \underline{(}\phi\underline{)\ \boldsymbol{\lor}\ (}\psi\underline{)} \\
\phi \rightarrow \psi &\overset{\text{def}}{=} \underline{(}\phi\underline{)\ \boldsymbol{\rightarrow}\ (}\psi\underline{)} \\
\phi \leftrightarrow \psi &\overset{\text{def}}{=} \underline{(}\phi\underline{)\ \boldsymbol{\leftrightarrow}\ (}\psi\underline{)}.
\end{aligned}
\tag{1.21}
$$

Note that there is nothing to prevent us from using the lightface characters '$\neg$', etc., as metalanguage connectives. For example, we might write '(ϵ is a term) $\rightarrow$ (ϵ is not a formula)'.

1.1.13.3 Quantification

Similarly,

$$
\begin{aligned}
\forall v\,\phi &\overset{\text{def}}{=} \underline{\forall} v\,\underline{(}\phi\underline{)} \\
\exists v\,\phi &\overset{\text{def}}{=} \underline{\exists} v\,\underline{(}\phi\underline{)},
\end{aligned}
\tag{1.22}
$$

for any variable v and formula ϕ. This prescription presumes that variables are domain-specific, so that if we wish to quantify over individuals of a given domain D, we use a variable of that sort.

Alternatively, we may apply the domain specification to the quantifier and allow the variable to be generic, thusly:

$$
\begin{aligned}
\forall_D v\,\phi &\overset{\text{def}}{=} \underline{\forall}_D v\,\underline{(}\phi\underline{)} \\
\exists_D v\,\phi &\overset{\text{def}}{=} \underline{\exists}_D v\,\underline{(}\phi\underline{)},
\end{aligned}
$$

for any domain index D, variable v, and formula ϕ. The former convention allows us to assign domains to terms that contain variables, so that we can tell at once whether they and formulas that contain them are well formed. We adopt the position that an expression of the latter form is understood to stand for the expression of the former type obtained by replacing v in the quantifier phrase and every free occurrence of v in ϕ by the "D-specific" version of v.

1.1.14 Structure

As noted above,[1.1] the notions of *structure*, *language*, and *logic* are inextricably intertwined. We have also indicated above[1.2] that formal language, with formal logic, is essentially coextensive with mathematics. Mathematicians have had millennia to figure out the range of topics to which their methods apply, and the existing body of mathematics is a representative sample of the sort of structure that is the subject of formal language and logic. Let's examine some familiar mathematical theories with this in mind.

Take number theory for example. In the most restricted sense this seems to be talking about natural numbers $\{0, 1, \ldots\}$. What does it say about them? First note that it doesn't say what they are. This is not to say that people have not talked about what numbers are, but this is not the concern of number theory. Instead, number theory talks about how numbers are related. It talks, for example, about prime versus composite numbers. (Recall that n is prime if and only if for all k and l, if $k \cdot l = n$ then either $k = n$ or $l = n$.) A typical theorem of number theory is the statement:

(1.23) *There are infinitely many prime numbers.*

One way to interpret this statement is to say:

(1.24) *The collection of prime numbers is infinite.*

If we do this, then we are talking not only about numbers, but also about collections of numbers, and distinguishing finite and infinite collections.

There is, however, another way to interpret (1.23) that does not talk about collections:

(1.25) *For every number m there is a subsequent number n such that n is prime.*

Note that, inasmuch as they talk about primality, which is defined in terms of multiplication, both (1.23) and (1.25) (as well as (1.24)) concern not just numbers but also the *multiplication* operation. Additionally (1.25) talks about the *order* relation—the notion of one number following another.

Let's look at another theorem of number theory, Fermat's theorem:[16]

If p is prime and p does not divide a, then $a^{p-1} - 1$ is divisible by p.

Now we're talking about subtraction and exponentiation as well as multiplication (implicit in the notions of primality and divisibility). Or Wilson's theorem:

For all primes p, $(p-1)! \equiv -1 \mod p$.

We've now introduced the factorial operation and modular equivalence; and so it goes.

Evidently the question of what number theory is about is one of some delicacy. Let's look at another mathematical theory, projective geometry. Plane projective geometry talks about points and lines, while solid projective geometry talks about points, lines, and planes. Let's restrict ourselves for the time being to plane geometry. The fundamental relation is that of *incidence*. Intuitively, a point P and a line l are *incident* iff P lies on l, or, equivalently l runs through P. A typical statement of projective geometry is:

If P and Q are distinct points, then there is a unique line l such that l is incident with both P and Q.

Another is:

[16] This is a theorem of Fermat, but it is not *Fermat's last theorem*.

If l and m are distinct lines, then there is a unique point P such that P is incident with both l and m.

(Actually, both of these are true in projective geometry. Projective geometry differs from euclidean geometry in this regard: In euclidean geometry there exist pairs of distinct lines that are not both incident with any point: parallel lines.)

Note that incidence is distinct from membership, i.e., we do not regard a line as the set of points incident with it, any more than we regard a point as the set of lines incident with it. Indeed, the concepts of a set of points and a set of lines are not a part of projective geometry in its elementary form.

Our next example is set theory, which talks about sets. The fundamental relation of set theory is *membership*, denoted by '$\in$', i.e., '$x \in y$' means 'x is a member of y'. A typical statement of set theory is:

If for all z, $z \in x$ iff $z \in y$, then x is y.

Another is:

There is a set x such that for all y, it is not the case that $y \in x$.

In other words, x has no members; it is *empty*. Along the same lines, we have the statement:

If for some x, $x \in z$, then for some x, $x \in z$ and for all y, if $y \in z$ then it is not the case that $y \in x$.

A common feature of the preceding examples is the existence of a single structure whose properties the relevant theories are meant to elucidate. In the case of number theory, that structure is the collection $\{0, 1, \ldots\}$ with the usual notion of order.[17] In the case of plane projective geometry, that structure is the "ordinary" plane with the adjunction of points at infinity and a line at infinity. In the case of set theory, the structure in question is the class of all pure sets. Following Euclid, we systematize our understanding of these structures by positing certain statements about them as axioms. Traditionally these are chosen because they are "self-evident", and are sufficient to prove any other self-evident statement. The theory then consists of all statements derivable from the axioms (which must include some statements that are not self-evident if the enterprise is to have any merit).

As mathematics developed, mathematicians began to observe certain structural elements occurring in a variety of settings, and they came to see the utility of talking about structures in the abstract. The concept of a group is an early example of this. The integers (positive, negative, and zero) form a group under the operation of addition. The rational numbers do as well. The nonzero rational numbers form a group under the operation of multiplication. So do the positive rational numbers. The bijections (one-one onto mappings) of a set to itself form a group under composition. These are just a few of the many and varied examples of groups in mathematics.

Other types of structures—like ring, field, topological space, etc.—have also been found to arise in diverse settings. A large part of mathematics is the study of defined classes of structures. What can we say about all groups? Or all finite groups? Or commutative groups? All topological vector spaces? Our point of

[17] We are not at present concerned with the various operations and relations that we mentioned above, like multiplication and exponentiation, which have definitions for this structure that uniquely characterize them.

view in this endeavor is different than when we are trying to find out all that is true about a single structure, like the natural numbers, the projective plane, or the universe of all pure sets. Instead of asking what are all the statements true of a given structure, we ask what are all the structures of which given statements are true and what additional statements are true of all these structures.

1.1.15 Our metatheory

Up to this point our discussion has been largely descriptive, but we will soon begin to draw conclusions about the objects we are describing. We could go about this informally, using the sort of "common sense" that mathematicians develop over time, but this approach is unsatisfactory on two counts. First, we want our study of the foundations of mathematics to be itself mathematical, so we want to use formal language and formal logic—which, as discussed above, characterize mathematics. One of the purposes of our study of the foundations of mathematics is to examine the issue of the "reliability" of mathematics: How do we know that formal methods are "correct"; what does that even mean? If we arrive at answers to these questions by informal methods, then the reliability of our conclusions will be suspect. On the other hand, if we clearly state the premises of our analysis, and we apply formal logic, then at the end of the process we will have a clear idea of where we stand.

As it happens, the "amount" of mathematics required for this task is rather small, so that only the most violent skeptic would seriously question the validity of our analysis. There is, however, another reason to be specific as to the mathematical framework for this analysis, which we will illustrate presently.[§ 1.1.16]

The desiderata for our metatheory are as follows. We want to be able to talk about linguistic expressions and how they may be combined. We want to be able to talk about finite collections and finite sequences of expressions, as well as finite collections and sequences of these collections and sequences, finite collections and sequences of these, etc., up to some level. We want to be able to define 'finite'. We want to be able to use natural numbers. We want to be able to talk about functions and relations. We want to be able to carry out common-sense deductions regarding these objects. In particular, we want to be able to use definition by recursion and proof by induction.

There are several ways to formulate a suitable metatheory. All of these are equivalent in a sense to be made precise—as long as we restrict ourselves to "natural" theories that are simply described and are not excessively strong. The theory PA of natural numbers known as Peano arithmetic is a popular choice of metatheory, but it suffers from the drawback of artificiality. A much more natural theory is the basic theory of membership developed in Section 3.2. Specifically, we will use the theory C^0,[3.17] which is the theory of membership that allows proper classes and allows, but does not mandate, the existence of infinite sets. Note that C^0 omits the $\mathsf{Foundation}$ axiom, which is not required for the utilitarian purposes of this chapter and the next, and we prefer it for this reason over $\mathsf{C} = \mathsf{C}^0 + \mathsf{Foundation}$, whose use would convey the erroneous impression that $\mathsf{Foundation}$ is essential.[18] In the interest of simplicity, we exclude consideration of proper elements (urelements, atoms). Thus, our analysis does not apply directly to typographical languages, for example, whose expressions are arrangements of graphical symbols; but via appropriate coding conventions, these are obviously equivalent to pure sets. We treat C^0 as a

[18] It is perhaps another shortcoming of PA as a metatheory that it cannot conveniently omit its equivalent of $\mathsf{Foundation}$, which is the schema of induction.

unisorted theory with a predicate that specifies that an object is an element (i.e., a set).

We regard C^0 as the ideal theory for these introductory chapters for three reasons:

1. The ability of C^0 to deal with infinite classes is sufficient to treat the essential elements of the semantics of countable languages; most importantly, it suffices to prove the fundamental theorem of semantics: the completeness theorem.

2. C^0 is a conservative extension of pure set theory S^0 without an axiom of infinity, i.e., anything that C^0 can prove that refers only to sets can be proved in S^0. Thus, any theorem of C^0 concerning logic that is purely syntactical is a theorem of S^0. In this regard C^0 is only a convenience, but it is a great convenience.

3. Confronting the salient shortcoming of C^0 *vis-à-vis* a theory with an axiom of infinity in the present context, which is its inability to prove that satisfaction relations exist for all structures of interest (countable structures, for example), provides an early insight into the expressive and deductive limitations that characterize the foundations of mathematics.

It should be noted that at no point do we deny the existence of the infinite, we simply do not always assume it. We examine some of the implications of infinitarity in the later part of Chapter 3, and in Chapter 5 we commence our investigation of infinitarity in earnest, which will occupy us for the remainder of the book.

1.1.16 A tantalizing example

We conclude this introduction with an example of the sort of revelation we may expect from a close examination of language and logic. This is not a rigorous demonstration—we don't yet have the tools for one—so don't be discouraged if you are not quite able to follow the argument, much less "fill in the details". Its purpose is to entice rather than convince. Consider it an hors d'oeuvre to the coming feast.

In our examination of the foundations of mathematics we will present an axiomatic theory of natural numbers (the "counting numbers": 0, 1, 2, ...) known as Peano arithmetic (PA). The question will arise whether these axioms are sufficient to prove all true sentences in the language of arithmetic. The answer is 'no'. How do we show this? We will describe a sentence σ in the language of arithmetic whose "meaning" is 'σ is not provable from Peano's axioms'. We enclose 'meaning' in double quotes in the preceding sentence according to the convention (1.6) to draw attention to the fact that the ordinary meaning of a sentence in the language of arithmetic is something about numbers, whereas we are now saying that the meaning of σ is that a particular sentence in the language of arithmetic is not provable.

In order that we be able to interpret σ—which is a statement in the language of arithmetic—as meaning 'σ is not provable' we must be able to regard numbers as linguistic expressions. Since we have left the identity of linguistic entities unspecified, this is not a problem: all we have to do is define relations among numbers that have the structure of a language. (This language is of course not a typographical language, as typographical symbols are not numbers in any ordinary sense. Even numerals are not numbers.) We now show that σ is indeed not provable and is therefore true (since it says it's not provable). The proof is by *reductio ad absurdum*. Suppose toward a contradiction that σ is provable, i.e., there exists a PA-proof π of σ. Then in PA we can prove

1. that π is a proof of σ (simply by checking that all premises of π are axioms of PA, each step in π is justified by one of the rules of our standard deductive system, and the conclusion of π is σ);

2. that therefore there exists a proof of σ; so that therefore

3. it is not the case that that there is no proof of σ.

In other words, if PA proves σ, then PA proves $\neg\,\sigma$, the negation of σ, so PA is inconsistent.[19] As this is absurd, the hypothesis that PA proves σ is false; in other words, σ is not provable and is by that token true.

Thus we have the following theorem of Gödel:

(1.26) **First incompleteness theorem** *There exists a true arithmetical sentence that is not provable in* PA*.*

The argument just sketched has a delightful corollary. By way of preparation, let us acknowledge that, while the example of σ is interesting as far as it goes, in that it is an arithmetical truth that cannot be proved (not in PA, that is; we have, of course, just proved σ, but evidently in a stronger theory), we otherwise have no reason to be interested in σ. But what about the "stronger theory" just alluded to? What did we just use over and above PA to prove σ? An analysis of the above argument shows that it can be formulated in the language of arithmetic and every step can be justified (i.e., proved) in PA except possibly the one where we say that the notion that PA is inconsistent is absurd—where we say, in other words, that PA is consistent. Leaving aside for an instant the question whether that step can also be justified in PA, we are content to note that in any event

(1.27) **First incompleteness theorem**

$$\mathsf{PA} + \mathrm{Con}(\mathsf{PA}) \vdash \sigma,$$

where Con(PA) *is the the sentence in the language of arithmetic that "says" that* PA *is consistent, and '$\vdash$' means 'proves', i.e., what's on the right follows logically from what's on the left.*

(1.27) is a sharp form of (1.26) in that it lays out the hypotheses we have used to derive σ, viz., $\mathsf{PA} + \mathrm{Con}(\mathsf{PA})$. It is also entirely syntactic, inasmuch as it does not employ the semantic concept of meaning, as (1.26) does in saying that σ is true.

Now suppose for a moment that $\mathsf{PA} \vdash \mathrm{Con}(\mathsf{PA})$. Then, by virtue of (1.27), $\mathsf{PA} \vdash \sigma$. As we have just shown, if PA is consistent, this is false, so the supposition that PA proves that Con(PA) is false. We therefore have Gödel's

Second incompleteness theorem *If* PA *is consistent then* $\mathsf{PA} \nvdash \mathrm{Con}(\mathsf{PA})$.

[19] A theory is *consistent* just in case it cannot derive any *contradiction*, i.e., for any sentence ϕ, it cannot prove both ϕ and $\neg\,\phi$. Note that any sentence follows by simple logic from any pair of the form $\{\phi, \neg\,\phi\}$, so an inconsistent theory can prove any sentence that can be formulated in its language.

Unlike σ, Con(PA) is an arithmetical assertion that is intrinsically interesting, at least to logicians.[20] It is, however, not at all "number-theoretical" in nature and not of interest to number theorists or to mathematicians in general, except insofar as they are interested in the logical foundations of their subject. There are statements about numbers that are not overtly "logical" that have been shown to be true but not provable in PA, although even these are not mainstream number theory. When we turn our attention to more powerful theories, however, such as set theory, we find a profusion of mathematically interesting statements—such as the axiom of choice and the continuum hypothesis—that are neither provable nor disprovable (assuming in this case, of course, that set theory is consistent). In fact, most of set theory as it exists today consists of things not provable in the standard axiomatization, and much of set theory deals with various extensions of the axioms. What naïvely seems extraordinary, viz., being neither provable nor disprovable, turns out to be ordinary.

1.2 Structure

1.2.1 Introduction

In Section 1.1 we have introduced all the elements of formal languages, and we have described how we are going to write about them in our metalanguage. We have also given an indication of the sort of structure to which formal language applies. We may now give adequate definitions of both *language* and *structure*. We begin with structure.

As discussed in Section 1.1.3 these definitions and analysis are carried out in the basic theory of membership, which is our metatheory. This theory is defined and developed in Chapter 3. That treatment depends in part on the material in the present chapter, so there is a certain circularity to the overall discussion. Some such circularity or *self-dependence* is unavoidable in an explication of the foundations of formal discourse (i.e., mathematics), and it does not undermine the logical cogency of the discussion.

1.2.2 Eine kleine Mengentheorie

We will make use of some membership-theoretic ideas, all of which are covered in Chapter 3. For convenience, we will present most of the relevant concepts briefly here. The reader is nevertheless advised at least to peruse Chapter 3, paying particular attention to the concepts of proper classes and proper elements.

(x, y) is the ordered pair $\{\{x\}, \{x, y\}\}$.[21] We also call this a 2-tuple. n-tuples are also defined[3.58] for any $n > 0$, but in this chapter—indeed, throughout this book—we will make limited use of this notion for $n \neq 2$. A *prefunction* is a class of ordered

[20] Although we have shown that (assuming Con(PA)) PA is incomplete in the semantic sense that there are true statements that it does not prove (e.g., Con(PA) and the *Gödel sentence* σ), we have not shown that PA is incomplete in the purely syntactic sense that there is a sentence ϕ such that PA proves neither ϕ nor $\neg\,\phi$. We can show this for $\phi = \sigma$ if we also assume the ω-consistency PA, of which need just the instance 'if PA proves $\neg\,\sigma$—i.e., PA proves that there is a proof of σ—then it is not the case that, for every proof π, PA proves that π is not a proof of σ'. If we replace σ by the *Rosser sentence* ρ, which says that for every proof of ρ there is a shorter proof of $\neg\,\rho$, then the unprovability of both ρ and $\neg\,\rho$ is provable (in PA) from Con(PA).

[21] This is, for us, an unusual use of round brackets with a specific meaning; ordinarily they are generic grouping symbols. Angle, square, and curly brackets, on the other hand, usually have specific defined meanings.

pairs. As discussed after (3.25), what we are defining here as a prefunction is often called a *relation*, but we define *relation* to mean a class of finite sequences.[3.62]

For R a prefunction, $\operatorname{dom} R \stackrel{\text{def}}{=} \{x \mid \exists y\,(x,y) \in R\}$. $\operatorname{im} R \stackrel{\text{def}}{=} \{y \mid \exists x\,(x,y) \in R\}$. For R a prefunction and X a class, $R^{\rightarrow}X \stackrel{\text{def}}{=} \{y \mid \exists x \in X\,(x,y) \in R\}$, and $R^{\leftarrow}X \stackrel{\text{def}}{=} \{y \mid \exists x \in X\,(y,x) \in R\}$. A *function* F is a prefunction such that for all $x \in \operatorname{dom} F$ there is a unique y such that $(x,y) \in F$, and $Fx \stackrel{\text{def}}{=} F(x) \stackrel{\text{def}}{=} y$.

Given a class X and a set Y, ${}^{Y}X$ is the class of functions f such that $\operatorname{dom} f = Y$ and $\operatorname{im} f \subseteq X$, i.e., $f : Y \to X$.

Definition [C^0] *A is a* family[3.37] *(of classes)* $\stackrel{\text{def}}{\Longleftrightarrow}$ *A is a prefunction and*

$$\forall i \in \operatorname{dom} A\,\big((i,0) \in A \vee \forall c\,\big((i,c) \in A \to \exists d\, c = \{d\}\big)\big).$$

If A is a family and $i \in \operatorname{dom} A$, then the class indexed by i in A $\stackrel{\text{def}}{=}$

$$A_{[i]} \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } (i,0) \in A \\ \{d \mid (i,\{d\}) \in A\} & \textit{otherwise.} \end{cases}$$

We use von Neumann ordinal notation. In particular, $0 \stackrel{\text{def}}{=}$ the empty set, $1 \stackrel{\text{def}}{=} \{0\}$, $2 = \{0,1\}$, etc. In general $n = \{0,1,\ldots,n-1\}$, and $\omega \stackrel{\text{def}}{=} \{0,1,\ldots\}$.[22] We frequently have occasion to indicate the enumeration of a finite ordinal, and the following notation is useful in this context.

(1.28) **Definition** [C^0] *Suppose α is a successor ordinal.* $\alpha^{-} \stackrel{\text{def}}{=} \alpha - 1 \stackrel{\text{def}}{=}$ *the predecessor of α.*

A *finite sequence* is a function with domain n for some $n \in \omega$. We call n the *length* of the sequence and refer to the sequence as an n-sequence. We often use subscript notation for finite sequences, so that if σ is an n-sequence, $\sigma_m \stackrel{\text{def}}{=} \sigma(m)$ for $m \in n$ (i.e., for $m < n$). We also use the notation '$\langle \sigma_0, \sigma_1, \ldots, \sigma_{n^-} \rangle$' for an n-sequence σ.[1.28] There is only one 0-sequence, viz., 0, the empty set.

For $n \in \omega$, an *n-ary relation* is a class of n-sequences. 'Nulary' means 0-ary, 'unary' means '1-ary', 'binary' means '2-ary', etc. Note that if an n-ary relation is nonempty then it is not an m-ary relation for any $m \neq n$. The *arity of a* nonempty multivariate relation is therefore well defined. The empty class $\{\,\} = 0$ is an n-ary relation for all $n \in \omega$, and its arity is undefined. There is only one 0-sequence, viz., 0, so a nulary relation is a subclass of $\{0\}$, of which there are exactly two, viz., the empty subclass, $\{\,\} = 0$, and the full subclass, $\{0\} = 1$.[23]

We define two versions of cartesian products:

$$\begin{aligned} A_0 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} A_{n^-} &\stackrel{\text{def}}{=} \dot{\bigtimes}_{m \in n} A_m \stackrel{\text{def}}{=} \dot{\bigtimes}_{m=0}^{n^-} A_m \\ &\stackrel{\text{def}}{=} \begin{cases} \{(a_0,\ldots,a_{n^-}) \mid a_i \in A_i,\ i = 0,\ldots,n-1\} & \text{if } n > 0 \\ 0 & \text{if } n = 0. \end{cases} \end{aligned}$$

[22] The existence of ω cannot be proved in the basic theory of membership, which does not have an axiom of infinity. In general, therefore, we regard a formula of the form '$n \in \omega$' as standing for 'n is a finite ordinal', i.e., a natural number.

[23] This will serve as the rationale for defining *trueness* as 1 and *falseness* as 0.

$$A_0 \times \cdots \times A_{n^{-}} \overset{\text{def}}{=} \bigtimes_{m \in n} A_m$$
$$\overset{\text{def}}{=} \begin{cases} \{\langle a_0, \ldots, a_{n^{-}} \rangle \mid a_i \in A_i,\ i = 0, \ldots, n-1\} & \text{if } n > 0 \\ \{0\} & \text{if } n = 0. \end{cases}$$

Note that
$$^{n}A = \bigtimes_{m \in n} A_m,$$
where $A_m = A$ for all $m \in n$.

For $n \in \omega$, an *n-ary function* is a function F such that $\operatorname{dom} F$ consists entirely of n-sequences. If $F \neq 0$ then F is an n-ary function for at most one n, and we call n the *arity* of F. The nulary functions are the empty function, which is the empty set 0, and the functions $\{(0, e)\}$ for elements e.

The *composition* of prefunctions R and $S \overset{\text{def}}{=}$

$$R \circ S \overset{\text{def}}{=} \{(x, y) \mid \exists z\, ((x, z) \in S \wedge (z, y) \in R)\}.$$

In the event that R and S are functions, $R \circ S$ is the composition in the familiar sense that
$$(R \circ S)x = R(Sx).$$
Suppose F is a function and R is a family. Then $R \circ F$ is a family, and for any $x \in \operatorname{dom} F$
$$(R \circ F)_{[x]} = R_{[Fx]}$$
We use '$\{F\circ\}$' to name the map that sends any function G with $\operatorname{im} G \subseteq \operatorname{dom} F$ to the function $\langle F(Gx) \mid x \in \operatorname{dom} G \rangle$, i.e., $x \mapsto F(Gx)$. We use this most often in the combination '$\{F\circ\}^{\rightarrow}$', as in

$$\{F\circ\}^{\rightarrow}\mathcal{F} = \{F \circ f \mid f \in \mathcal{F}\},$$

where $\mathcal{F}$ is a class of functions such that for all $f \in \mathcal{F}$, $\operatorname{im} f \subseteq \operatorname{dom} F$.

1.2.3 Signatures

A language is determined for all practical purposes by its domain, relation, and operation indices. This information is summarized in its *signature*, and this signature is likewise an attribute of the structures to which the language applies. As discussed above,[§ 1.1.11] domains and operations are a convenience, not a necessity, and the essential principles are adequately illustrated by consideration of languages that refer only to relations. We will use 'unsorted' to refer to the absence of explicit domains (or *sorts*), and 'relational' to indicate the omission of both domains and operations.[1.17] We will give the following definitions first in a restricted context and then in full generality.

For the purpose of orientation, Π and Φ are classes of predicate and function indices, respectively; and T is the arity function. Note that a signature may be a proper class. This allows us to have infinite signatures even when there are no infinite sets.

As we are now operating in the framework of precise mathematical definition and proof, we initiate the practice which we will follow henceforth of indicating the theory within the context of which a definition is made or a proof is constructed.

(1.29) **Definition** $[\mathsf{C}^0]$

1. *A* relational signature $\stackrel{\text{def}}{=}$ *a* 2*-indexed family* $[\Pi, T]$ *such that*
   1. Π *is nonempty;*
   2. $V \backslash \Pi$ *is infinite, where* V *is the class of all elements;*
   3. $T : \Pi \to \omega$*; and*
   4. $T(0) = 2$ *if* $0 \in \Pi$.[24]
2. *A* unisorted signature $\stackrel{\text{def}}{=}$ *a* 3*-indexed family* $[\Pi, \Phi, T]$ *such that*
   1. $\Pi \cup \Phi$ *is nonempty;*
   2. $V \backslash (\Pi \cup \Phi)$ *is infinite;*
   3. $\Pi \cap \Phi = 0$*;*
   4. $0 \notin \Phi$*;*
   5. $T : \Pi \cup \Phi \to \omega$*; and*
   6. $T(0) = 2$ *if* $0 \in \Pi$.
3. *Suppose* $\rho = [\Pi, \Phi, T]$ *and* $\rho' = [\Pi', \Phi', T']$ *are signatures.*
   1. ρ' expands ρ $\stackrel{\text{def}}{\iff}$ $\Pi \subseteq \Pi'$*,* $\Phi \subseteq \Phi'$*, and* $T \subseteq T'$.[25] *We also say that* ρ' *is an* expansion *of* ρ *and that* ρ *is a* contraction *of* ρ'.
   2. ρ *and* ρ' *are* compatible $\stackrel{\text{def}}{\iff}$ *they have a common expansion (iff* $[\Pi \cup \Pi', \Phi \cup \Phi', T \cup T']$ *is a signature, in which case it is the minimum common expansion of* ρ *and* ρ'*).*

The requirement (1.29.1.2) (likewise (1.29.2.2)) is to allow expansion of signatures. This sort of expandability is used routinely in mathematical discourse, as new predicates and operations are introduced by definition, and as new constants are introduced in the process of manipulating formulas involving quantification. We could get along without this requirement, but we would have to take special pains when it did not apply. This will be first become relevant in Chapter 2 when we define a system of deduction that depends on the introduction of new constant indices. Note that if we restricted Π and Φ to be sets (not proper classes), then this requirement would be fulfilled automatically (as V is not a set), but this is unduly restrictive for our purposes.

Definition $[\mathsf{C}^0]$ *For* $\rho = [\Pi, \Phi, T]$ *as in (1.29),*

1. $\Pi^\rho \stackrel{\text{def}}{=} \rho_{[0]} = \Pi$*,* $\Phi^\rho \stackrel{\text{def}}{=} \rho_{[1]} = \Phi$*, and* $T^\rho \stackrel{\text{def}}{=} \rho_{[2]} = T$*; and*
2. *for each* $X \in \Pi \cup \Phi$*, we define the* arity of X in ρ *to be* $T^\rho(X)$.

Corresponding definitions are made mutatis mutandis *for the case of relational signatures.*

We refer to the members of Π as *predicate* or *relation indices*, and to the members of Φ as *function* or *operation indices*. Recall that we follow the convention that if 0 is an index then it indexes the identity relation, which is binary.

The definition of a general signature is given in Note 10.1. Briefly, a general signature is a 4-indexed family $[\Delta, \Pi, \Phi, T]$, where Δ, Π, and Φ are the classes of domain, relation, and operation indices, respectively. T indicates not only the arity of each relation or operation index X but also the domains to which its arguments—and, in the case of an operation index, its value—may belong.

[24] This reflects the fact that 0 is reserved as an index for the identity predicate, which is binary.

[25] We leave unstated the analogous definitions for purely relational and for multisorted signatures. Note that if f and f' are functions then $f \subseteq f'$ iff $\operatorname{dom} f \subseteq \operatorname{dom} f'$ and $\forall x \in \operatorname{dom} f\ f'x = fx$.

1.2.4 Structures

(1.30) **Definition** $[\mathsf{C}^0]$ *Given a relational signature* $\rho = [\Pi, T]$*, a* ρ-structure $\mathfrak{S} \stackrel{\text{def}}{=}$ *a* 3*-indexed family* $[\rho, U, \pi]$ *with the following properties:*

1. U *is a nonempty class.*
2. π *is* Π*-indexed family.*
3. *For each* $P \in \Pi$, $\pi_{[P]} \subseteq {}^{T(P)}U$*. If* $0 \in \Pi$ *then* $\pi_{[0]} = \{\langle x, x\rangle \mid x \in U\}$.

Given a unisorted signature $\rho = [\Pi, \Phi, T]$*, a* ρ*-structure* $\stackrel{\text{def}}{=}$ 4*-indexed family* $[\rho, U, \pi, \phi]$ *with properties 1–3 above, together with*

4. *For each* $F \in \Phi$, $\phi_{[F]}$ *is a function from* ${}^{T(F)}U$ *into* U.

Note that a structure incorporates its signature.

If $P \in \Pi$ *and* $T(P) = 0$ *then*[1.30.3] $\pi_{[P]} \subseteq {}^0U = \{0\}$*. Hence, if* P *is nulary then* $\pi_{[P]}$ *is* $\{0\} = 1$ *or* $\{\} = 0$*. In the former case '*$s \in \pi_{[P]}$*' is true for all* 0*-sequences* s*, i.e., for* $s = 0$*; while in the latter case '*$s \in \pi_{[P]}$*' is false for all* s*. We accordingly define* trueness *(*truth, verity*, …) to be (the nulary relation)* 1*, and* falseness *(*falsehood, fallacy*, …) to be (the nulary relation)* 0.

If $F \in \Phi$ and $T(F) = 0$ then[1.30.4] $\operatorname{dom} \phi_{[F]} = \{0\}$. With only one element in its domain, $\phi_{[F]}$ is necessarily constant. Accordingly, we refer to nulary operation indices as *constant operation indices*, or as *constant indices*, or often simply as *constants*. Nulary predicate indices may also be referred to as *constant predicate indices*.

Note that we have required U to be nonempty. If U were empty then it would not be possible to assign a value to $\phi_{[F]}$ for a nulary operation index F, i.e., a constant. There is also good reason to exclude even purely relational structures with empty domain, as will become clear when we discuss the semantic and logical aspects of structure.

Definition $[\mathsf{C}^0]$ *For* $\mathfrak{S}$ *as in (1.30) we define*

1. $\rho^{\mathfrak{S}} \stackrel{\text{def}}{=} \rho$, $|\mathfrak{S}| \stackrel{\text{def}}{=} U$, $\pi^{\mathfrak{S}} \stackrel{\text{def}}{=} \pi$*, and* $\phi^{\mathfrak{S}} \stackrel{\text{def}}{=} \phi$*;*
2. $\Pi^{\mathfrak{S}} \stackrel{\text{def}}{=} \Pi^{\rho^{\mathfrak{S}}}$, $\Phi^{\mathfrak{S}} \stackrel{\text{def}}{=} \Phi^{\rho^{\mathfrak{S}}}$*, and* $T^{\mathfrak{S}} \stackrel{\text{def}}{=} T^{\rho^{\mathfrak{S}}}$*; and*
3. *for all* $P \in \Pi$, $P^{\mathfrak{S}} \stackrel{\text{def}}{=} \pi_{[P]}$*, and for all* $F \in \Phi$, $F^{\mathfrak{S}} \stackrel{\text{def}}{=} \phi_{[F]}$.

The corresponding definitions in the general case are similar.[10.3] Briefly, a structure for a signature $\rho = [\Delta, \Pi, \Phi, T]$ is a 4-indexed family $[\rho, \delta, \pi, \phi]$, where δ, π, ϕ interpret the respective classes of indices. For example, for each $F \in \Phi$, $\phi_{[F]}$ is a function with the appropriate domain and range (defined by $T_{[F]}$).

We will usually use 'D' and related symbols to denote domain indices; 'P', 'R', and related symbols to denote predicate (relation) indices; 'F', 'O', and related symbols to denote function(operation) indices; and 'X', 'Y', and related symbols to denote relation or operation indices without prejudice.

1.2.5 A simpler notation

Suppose $\rho = [\Pi, \Phi, T]$ is a unisorted signature with identity and $\mathfrak{A} = [\rho, U, \pi, \phi]$ is a ρ-structure. All or nearly all of the information in $\mathfrak{A}$ is contained in the relation $\mathcal{A}$ defined by the conditions

1. $\operatorname{dom}\mathcal{A} = \Pi \cup \Phi$ (so $0 \in \operatorname{dom}\mathcal{A}$, since ρ is with identity);

2. $\mathcal{A}_{[0]} = U$;[26]

3. $\mathcal{A}_{[i]} = \pi_{[i]}$ if $i \in \Pi\backslash\{0\}$; and

4. $\mathcal{A}_{[i]} = \phi_{[i]}$ if $i \in \Phi$.

Suppose $\rho' = [\Pi', \Phi', T']$ is another unisorted signature and $\mathfrak{A}' = [\rho', U', \pi', \phi']$ is a ρ'-structure, and suppose the above construction from $\mathfrak{A}'$ yields the same class $\mathcal{A}$. It is straightforward to show that $\Pi' = \Pi$, $\Phi' = \Phi$, $U' = U$, $\pi' = \pi$, $\phi' = \phi$, $\forall i \in \Phi\, T'(i) = T(i)$, and $\forall i \in \Pi\, \big(\pi_{[i]} \neq 0 \to T'(i) = T(i)\big)$. In other words, the only way that ρ and ρ' may differ is in the arity they assign to a predicate index that is interpreted as the empty relation (i.e., the empty set, which is an n-ary relation for every $n \in \omega$.)

As a concession to simplicity of notation, at a tolerable cost in lost precision, we will use indexed families like $\mathcal{A}$ as surrogates for the corresponding unisorted structures. If the index class happens to be an ordinal then $\mathcal{A} = [U, R, S, \ldots]$.

1. *To emphasize the informality of this representation of a structure, instead of square brackets indicating an indexed family, we will typically use round brackets in their generic capacity as grouping symbols, and—to emphasize its distinctive role—we separate the universe U from the predicates and operations by a semicolon: '$(U; R, S, \ldots)$'.*

2. *In this notation, we do not presume that the index class is an ordinal; we simply omit to mention indices, showing only the relations and operations they index. If we want to indicate indices, we may represent a structure $\mathfrak{A}$ as '$(|\mathfrak{A}|; i^{\mathfrak{A}}, j^{\mathfrak{A}}, \ldots)$', where $i, j, \ldots$ are indices.*

3. *We may also indicate classes of indices in a subscript. For example, if $\mathfrak{A}$ is a structure in a signature with a constant c_n for every $n \in \omega$, we may represent $\mathfrak{A}$ by '$(|\mathfrak{A}|; c_n^{\mathfrak{A}}, \ldots)_{n\in\omega}$', where '$\ldots$' stands for other relations and operations.*

4. *We use a similar informal notation for multisorted structures, with the domains listed first, followed by a semicolon, and then the predicates and operations.*

For example, a vector space might be indicated by '$(V, F; +_V, \cdot_V, +_F, \cdot_F)$', where V and F are respectively the domains of vectors and scalars, and $\cdot_V$, $\cdot_V$, $+_F$ and $\cdot_F$ are respectively vector addition, scalar multiplication, and addition and multiplication of scalars. Note that the universal domain has been omitted here. Many variations may be played on this theme, in the expectation of the reader's cooperation.

[26]Note the dual use of 0 as the index of the identity predicate in the structure $\mathfrak{A}$ and as the index of U in the family $\mathcal{A}$. This is just a happy accident.

1.2.6 Isomorphism and homology

As discussed in Section 1.1.14, we are usually not interested in the identity of the individuals of a structure but rather in the "structure" of the structure, i.e., that aspect of it that is not altered when we replace its individuals by other elements without changing their relationships. Since the individuals of a structure are intrinsic to its relations and operations, any discussion of these relationships *per se* is indirect to a degree. The basic notion in this regard is that of *isomorphism*.

(1.31) **Definition** $[\mathsf{C}^0]$ *ι is an* isomorphism of $\mathfrak{A}$ to $\mathfrak{B}$ $\overset{\text{def}}{\iff}$

1. *$\mathfrak{A}$ and $\mathfrak{B}$ are structures with the same signature, say* $\rho = [\Delta, \Pi, \Phi, T]$*;*
2. $\iota : |\mathfrak{A}| \overset{\text{bij}}{\to} |\mathfrak{B}|$*, and for all* $D \in \Delta$, $\iota^{\to} D^{\mathfrak{A}} = D^{\mathfrak{B}}$*;*
3. *for every* $P \in \Pi$, $P^{\mathfrak{B}} = \{\iota\circ\}^{\to} P^{\mathfrak{A}} \ \left(\overset{\text{def}}{=} \left\{ \iota \circ \sigma \ \middle|\ \sigma \in P^{\mathfrak{A}} \right\} \right)$*;*[27] *and*
4. *for every* $O \in \Phi$, $O^{\mathfrak{B}} \circ \{\iota\circ\} = \iota \circ O^{\mathfrak{A}}$*, i.e.* $(\forall \sigma \in \operatorname{dom} O^{\mathfrak{A}})\, O^{\mathfrak{B}}(\iota \circ \sigma) = \iota(O^{\mathfrak{A}}(\sigma))$.

If an isomorphism exists we say that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic *and that each is an* isomorph *of the other.*

Clearly,

1. the identity map is an isomorphism of a structure with itself;
2. if $\iota : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, then $\iota^{-1} : \mathfrak{B} \to \mathfrak{A}$ is an isomorphism, and
3. if $\iota : \mathfrak{A} \to \mathfrak{B}$ and $\iota' : \mathfrak{B} \to \mathfrak{C}$ are isomorphisms, then $\iota' \circ \iota : \mathfrak{A} \to \mathfrak{C}$ is an isomorphism.

Hence, isomorphic is an equivalence relation.

Note that any structure has many isomorphs. All we have to do to create an isomorph of $\mathfrak{A}$ is to take an injective function ι with domain $|\mathfrak{A}|$ and define a structure $\mathfrak{B}$ with the same signature so as to make (1.31) true.

Isomorphic structures clearly have the same "structure", just different individuals. The simplest embodiment of "pure structure" would be an *isomorphism type*, i.e., a collection consisting of all the structures isomorphic to a given structure. We say 'would' because such a "collection" is too large to be usefully regarded as an individual. If $\mathfrak{A}$ is a set, its isomorphs constitute a proper class; if $\mathfrak{A}$ is a proper class, its isomorphs constitute a collection of proper classes too large to be represented as an indexed family. We prefer to regard 'isomorphism type'—insofar as we use the expression at all—as *une façon de parler*, which may be systematically eliminated from the discussion.

Note that by definition[1.31.1] isomorphic structures have the same signature, so that corresponding domains, relations, and operations in isomorphic structures are called by the same name, as it were. It is natural to extend the notion of isomorphism to structures that have signatures that are not identical but are homologous in an appropriate sense. For example, it should not matter whether we index the multiplication operation of a ring, say, by '×' or '·', or for that matter by 1 or 2 or 'Australia', or even by Australia itself. To relate structures that differ in this

[27] Remember that an n-sequence $\sigma = \langle a_0, \ldots, a_{n^-} \rangle$ is the function with domain n such that $(\forall m \in n)\, \sigma m = a_m$, so $\iota \circ \sigma$ is a function with domain n, and $(\forall m \in n)\, (\iota \circ \sigma) m = \iota(a_m)$, i.e., $\iota \circ \sigma = \langle \iota a_0, \ldots, \iota a_{n^-} \rangle$.

way requires that we have a correspondence between the indices of their respective similarity types. In effect, we require that their signatures be "isomorphic", but since a signature is not technically a structure[28] we use 'homologous' in place of 'isomorphic'. The definition is the obvious one. We give it here for relational structures.

Definition $[\mathsf{C}^0]$ *Suppose* $\rho = [\Pi, T]$ *and* $\bar{\rho} = [\bar{\Pi}, \bar{T}]$ *are relational signatures.* h *is a* homology of ρ to $\bar{\rho}$ $\overset{\text{def}}{\iff} h : \Pi \overset{\text{bij}}{\to} \bar{\Pi}$ *and* $(\forall P \in \Pi)\, \bar{T}(h(P)) = T(P)$.

Homologous signatures are equivalent in every relevant way. It is obvious that if a signature did not satisfy (1.29.1.2) (or (1.29.2.2)), we could replace it by an homologous type that does, so this condition is merely a convenience.

The following definition is also given specifically for relational structures; its generalization to arbitrary structures is obvious.

(1.32) **Definition** $[\mathsf{C}^0]$ *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are respectively a* ρ*- and a* $\bar{\rho}$*-structure, with* $\rho = [\Pi, T]$ *and* $\bar{\rho} = [\bar{\Pi}, \bar{T}]$. $\langle h, \iota \rangle$ *is a* homology of $\mathfrak{A}$ to $\mathfrak{B}$ *iff*

1. h *is a homology of* ρ *to* $\bar{\rho}$*;*
2. $\iota : |\mathfrak{A}| \overset{\text{bij}}{\to} |\mathfrak{B}|$*;*
3. *for every* $P \in \Pi$, $h(P)^{\mathfrak{B}} = \{\iota\circ\}^{\to} P^{\mathfrak{A}}$.

In this case we say that $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* homologous *and that each is a* homolog *of the other.*

We use *homology type* to refer to the collection of all structures homologous to a given structure with the same provisos as for 'isomorphism type' above.

1.2.7 Standard signatures

We have defined signature broadly so as to permit arbitrary objects as indices. For some purposes, however, it is convenient to be able to refer to a fixed standard signature and its subsignatures, all of which will be referred to as *standard*. In the interest of simplicity, we will define a standard unisorted signature. An extension to a multisorted signature is straightforward.

(1.33) **Definition** $[\mathsf{C}^0]$ *The* standard *unisorted signature has the binary predicate index* 0 *(for identity), and for each* $m, n \in \omega$ *it has an* m*-ary predicate index* $\langle 0, m, n \rangle$ *and an* m*-ary operation index* $\langle 1, m, n \rangle$.

Thus there are infinitely many predicate and operation indices of each arity. Note that the standard signature is included in HF and any finite standard signature is in HF.

[28]We could define a signature in such a way as to be a structure, but this would generate more confusion than clarity.

1.3 The structure of language

A structure as defined in Section 1.2 is explicitly designed to be the subject of a language—to be "what the language is talking about". There are a number of ways to talk about a structure or structures, several of which we will discuss, but there is one way that it is tailor-made to the "structure" of a structure—neither too weak nor too strong—viz., its first-order predicate language.[1.19] It is languages of this type that we have been considering in our informal remarks above. The qualifier 'predicate' in 'first-order predicate language' refers to the whole apparatus of predicates, variables, and quantifiers, (also operations and domains, if these are present) and serves to distinguish these from *propositional* languages, in which propositions, i.e., sentences, are the primitive elements, from which expressions are formed by the action of propositional connectives. Propositional logic is essentially trivial. The qualifier 'first-order' indicates that predicates (and operations and domains, if present) are fixed. If instead we allow these to be variable, and we quantify over them and apply *second-order* predicates to them, then we have a *second-order* predicate language, and we can move up this hierarchy as far as we wish. A theory of membership (regarding predicates as sets, predicates applicable to predicates as sets of sets, and so on) can largely substitute for higher-order predication.

Thus, 'formal language' and 'logic' most naturally refer to first-order predicate language and logic, and for the duration of this section we will use 'language' to mean 'first-order predicate language'. A language $\mathcal{L}$ appropriate to a structure $\mathfrak{S}$ consists of expressions interpretable in $\mathfrak{S}$, along with relationships among these expressions that determine relationships among their interpretations. The signature ρ of a structure determines the languages appropriate to it, and we say that these languages *implement* ρ and that they are *ρ-languages*.

The reader is warned that there are necessarily quite a few definitions in this section, and we will be rather compulsive in our attention to detail. Since this is the foundation of the rest of the work, it is best that it be perfectly sound, but it is not necessary that the reader attend compulsively to the details.

1.3.1 Language defined

As noted above, the interpretation of an expression ϵ of a language $\mathcal{L}$ for a signature ρ is determined by the interpretation of its domain, relation, and operation indices in conjunction with the structure of ϵ, which is in turn a component of the structure of $\mathcal{L}$. A language is therefore naturally itself a structure, and we will treat it as such.

For a given ρ, the ρ-languages constitute an homology type, and the simplest way to define this type is by paradigm. We therefore define the standard ρ-language $\mathcal{L}^\rho$ *below.*

The important principles are adequately illustrated by the example of unisorted languages.

(1.34) *Suppose $\rho = [\Pi, \Phi, T]$ is a unisorted signature.*[29] *We define the standard ρ-language $\mathcal{L}^\rho$ as follows.*

[29] Recall that the elements of Π are the *predicate* or *relation indices*, and those of Φ are the *operation* or *function indices*. $\Pi \cap \Phi = 0$. For each index $X \in \Pi \cup \Phi$, $T(X)$ is its arity, i.e., the number of arguments it takes.

1. $\mathrm{i_v} \overset{\text{def}}{=} 0$. $\mathrm{v}_n \overset{\text{def}}{=} \langle \mathrm{i_v}, n\rangle$ *if* $n \in \omega$*; otherwise* 0. $\mathcal{V} = \{\mathrm{v}_n \mid n \in \omega\}$. *These are the variables, which we regard as ordered according to their ordinal components:* $\mathrm{v}_0 \prec \mathrm{v}_1 \prec \cdots$.
2. $\mathrm{i_{pred}} \overset{\text{def}}{=} 1$. *For each* $P \in \Pi$, $\check{P} \overset{\text{def}}{=} \langle \mathrm{i_{pred}}, P\rangle$. $\mathcal{P}^\rho \overset{\text{def}}{=} \{\check{P} \mid P \in \Pi\}$. *These are the predicate symbols.*
3. $\mathrm{i_{op}} \overset{\text{def}}{=} 2$. *For each* $O \in \Phi$, $\check{O} \overset{\text{def}}{=} \langle \mathrm{i_{op}}, O\rangle$. $\mathcal{O}^\rho \overset{\text{def}}{=} \{\check{O} \mid O \in \Phi\}$. *These are the operation symbols.*
4. $\mathrm{i}_\neg \overset{\text{def}}{=} 3$, $\mathrm{i}_\wedge \overset{\text{def}}{=} 4$, $\mathrm{i}_\vee \overset{\text{def}}{=} 5$, $\mathrm{i}_\rightarrow \overset{\text{def}}{=} 6$, $\mathrm{i}_\leftrightarrow \overset{\text{def}}{=} 7$. $\mathcal{C} \overset{\text{def}}{=} \{\mathrm{i}_\neg, \mathrm{i}_\vee, \mathrm{i}_\wedge, \mathrm{i}_\rightarrow, \mathrm{i}_\leftrightarrow\}$. *These are the propositional connectives.*
5. $\mathrm{i}_\exists \overset{\text{def}}{=} 8$, $\mathrm{i}_\forall \overset{\text{def}}{=} 9$. $\mathcal{Q} \overset{\text{def}}{=} \{\langle \mathrm{q}, v\rangle \mid \mathrm{q} \in \{\mathrm{i}_\exists, \mathrm{i}_\forall\} \wedge v \in \mathcal{V}\}$. *These are the quantifier phrases.*
6. $\mathcal{S}^\rho \overset{\text{def}}{=} \mathcal{V} \cup \mathcal{P}^\rho \cup \mathcal{O}^\rho \cup \mathcal{C} \cup \mathcal{Q}$. *These are the* syntactical elements *or* signs.

(1.35) **Definition** $[\mathsf{C}^0]$

1. *Let* $S^\rho : \mathcal{S}^\rho \to \omega$ *be given by*
$$S^\rho\varsigma \overset{\text{def}}{=} \begin{cases} 0 & \text{if } \varsigma \in \mathcal{V} \\ 1 & \text{if } \varsigma = \mathrm{i}_\neg \text{ or } \varsigma \in \mathcal{Q} \\ 2 & \text{if } \varsigma = \mathrm{i}_\wedge,\ \mathrm{i}_\vee,\ \mathrm{i}_\rightarrow, \text{ or } \mathrm{i}_\leftrightarrow \\ T(X) & \text{if } \varsigma = \check{X} \text{ for some } X \in \Pi \cup \Phi. \end{cases}$$
S^ρ extends the arity function T to the full class of syntactical elements. We will also refer to $S^\rho\varsigma$ as the arity *of a sign ς.*
2. *For $\varsigma \in \mathcal{S}^\rho$, we define the n-ary operation $\hat{\varsigma}$ by*
$$\hat{\varsigma}\langle x_0, \ldots, x_{n^-}\rangle \overset{\text{def}}{=} \langle \varsigma, x_0, \ldots, x_{n^-}\rangle,$$
where $n = S^\rho\varsigma$.

In the above definitions we have distinguished those classes $\mathcal{P}^\rho$, $\mathcal{O}^\rho$, $\mathcal{S}^\rho$, and S^ρ that are ρ-specific by the superscript '$^\rho$'. Many of the definitions that follow are also ρ-specific, but we will generally omit a distinguishing superscript when this will occasion no confusion. Keep in mind that everything we define is germane to the standard language $\mathcal{L}^\rho$ but has a homolog *vis-à-vis* any other ρ-language, as these are all homologous to $\mathcal{L}^\rho$.

If $S^\rho(\varsigma) = 0$, then $\operatorname{dom}\hat{\varsigma} = \{0\}$, i.e., the domain of $\hat{\varsigma}$ has just one member—the empty sequence, which is the empty set, 0—and $\hat{\varsigma}0 = \langle\varsigma\rangle$.[1.35.2] In particular, $S^\rho(v) = 0$ for any variable v, and $\hat{v}0 = \langle v\rangle$. The only other nulary grammatical elements are those corresponding to nulary predicate and operation indices, which are called *constants*. We do not use constant predicates very much, but we use constant operations frequently.

Definition $[\mathsf{C}^0]$ *The following definitions simplify the notation.*

1. *For* $X \in \Pi \cup \Phi$, $\tilde{X}s \overset{\text{def}}{=} \hat{\check{X}}s$, *for any $T^\rho(X)$-sequence s of ρ-terms.*
2. *For* $v \in \mathcal{V}$, $\bar{v} \overset{\text{def}}{=} \hat{v}0$.
3. *For* $X \in \Pi \cup \Phi$, *if* $T^\rho X = 0$, $\bar{X} \overset{\text{def}}{=} \tilde{X}0\ (= \hat{\check{X}}0)$.

We also use these for the homologous notions in an arbitrary ρ-language.

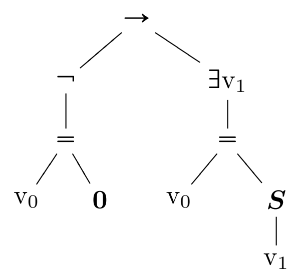

Figure 1.1: The expression (1.36) visualized as a tree growing downward.

1.3.1.1 Example

By way of illustration, suppose ρ has, in addition to the binary predicate index 0 for identity, a nulary and a unary operation index. Let $\mathrm{i}_= = \langle \mathrm{i}_{\mathrm{pred}}, 0\rangle$ be the standard sign for the index 0, and let i_0 and i_S be the standard signs for the additional operation indices.[30] Let $\mathbf{0}$ and $\mathbf{S}$ be respectively the constant term and the unary term-building function defined by

$$\mathbf{0} \stackrel{\text{def}}{=} \bar{\mathrm{i}}_0 = \hat{\mathrm{i}}_0 0 = \langle \mathrm{i}_0 \rangle$$
$$\mathbf{S}\langle \tau \rangle \stackrel{\text{def}}{=} \hat{\mathrm{i}}_S \langle \tau \rangle = \langle \mathrm{i}_S, \tau \rangle.$$

For example, suppose[31]

$$\epsilon = \bar{\mathrm{v}}_0 \neq \mathbf{0} \mathbin{\boldsymbol{\rightarrow}} \exists \mathrm{v}_1\, \bar{\mathrm{v}}_0 \mathbin{\boldsymbol{=}} \mathbf{S}(\bar{\mathrm{v}}_1),$$

where we have used boldface symbols as in (1.21) and (1.22) to name expression-building operations. Then

$$\epsilon = \langle \mathrm{i}_\rightarrow, \langle \mathrm{i}_\neg, \langle \mathrm{i}_=, \langle \mathrm{v}_0 \rangle, \langle \mathrm{i}_0 \rangle \rangle\rangle, \langle\langle \mathrm{i}_\exists, \mathrm{v}_1 \rangle, \langle \mathrm{i}_=, \langle \mathrm{v}_0 \rangle, \langle \mathrm{i}_S, \langle \mathrm{v}_1 \rangle\rangle\rangle\rangle\rangle. \tag{1.36}$$

The data comprised by (1.36) may be visualized as a tree growing downward, as in Figure 1.1. To define how expressions are built from signs (syntactical elements), we must define, for each category of sign, the type of expressions that the sign acts on (if any), and the type of expression it produces.

Definition [C^0]

1. TermType $\stackrel{\text{def}}{=}$ 1.
2. FormulaType $\stackrel{\text{def}}{=}$ 2.
3. *The* in-type *of a sign* $\varsigma \stackrel{\text{def}}{=}$

$$\mathrm{InType}\,\varsigma \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } \varsigma \in \mathcal{V} \\ \mathrm{TermType} & \textit{if } \varsigma \in \mathcal{P}^\rho \cup \mathcal{O}^\rho \\ \mathrm{FormulaType} & \textit{if } \varsigma \in \mathcal{C} \cup \mathcal{Q}. \end{cases}$$

[30] Note that in the case of the operation indices and signs, we have simply introduced names for the signs. We have not specified the indices and therefore have not specified the signs; and there is no reason to do so.

[31] If we interpret i_0 as 0 and i_S as the successor operation in Peano arithmetic, then ϵ says that if the number assigned to v_0 is not 0 then it is the successor of a number.

4. *The* out-type *of a sign* $\varsigma \overset{\text{def}}{=}$

$$\operatorname{OutType}\varsigma \overset{\text{def}}{=} \begin{cases} \text{TermType} & \textit{if } \varsigma \in \mathcal{V} \cup \mathcal{O}^\rho \\ \text{FormulaType} & \textit{if } \varsigma \in \mathcal{P}^\rho \cup \mathcal{C} \cup \mathcal{Q}. \end{cases}$$

The class $\mathcal{E}^\rho$ of *expressions* of $\mathcal{L}^\rho$ is the closure of 0—the empty set—under the operations $\hat{\varsigma}$, for $\varsigma \in \mathcal{S}^\rho$. Note that $\mathcal{E}^\rho$ contains $\bar{v}$ $(= \hat{v}0 = \langle v \rangle)$ for every variable v. Likewise, $\mathcal{E}^\rho$ contains $\bar{c}$ $(= \hat{c}0 = \langle c \rangle)$ for any constant predicate or operation symbol c. All other expressions are built from these by operations $\hat{\varsigma}$, for $\varsigma \in \mathcal{S}^\rho$ with $S^\rho(\varsigma) > 0$, operating on sequences of expressions. In the following discussion, the superscript ρ may be omitted to reduce notational clutter; it is to be understood to be present where appropriate.

The following is a formal definition of $\mathcal{E}$. In the interest of efficiency we will simultaneously define the notions of subexpression and height of expressions.

(1.37) **Definition** [C^0] *Suppose ρ is a signature.*

1. *Define classes $\mathcal{E}^\rho_n$ and binary relations $\preccurlyeq^\rho_n$ by recursion on $n \in \omega$ so that*
   1. $\mathcal{E}^\rho_0 = \preccurlyeq^\rho_0 = 0$*; and*
   2. *for each $n \in \omega$,*
      1. $\mathcal{E}^\rho_{n+1}$ *consists of the sets $\hat{\varsigma}t$, where $\varsigma \in \mathcal{S}^\rho$ and t is an $S^\rho(\varsigma)$-sequence from $\mathcal{E}^\rho_n$; and*
      2. $\preccurlyeq^\rho_{n+1}$ *consists of the 2-sequences $\langle \epsilon, \hat{\varsigma}t \rangle$ where $\varsigma \in \mathcal{S}^\rho$, and t is an $S^\rho(\varsigma)$-sequence from $\mathcal{E}^\rho_n$ and*
         1. $\epsilon = \hat{\varsigma}t$*; or*
         2. $\epsilon \preccurlyeq^\rho_n t_m$ *for some $m \in \operatorname{dom} t$.*
2. $\mathcal{E}^\rho \overset{\text{def}}{=} \bigcup_{n\in\omega} \mathcal{E}^\rho_n$. *$x$ is a ρ-*expression $\overset{\text{def}}{\iff} x \in \mathcal{E}^\rho$. *The* height *of a ρ-expression ϵ is the least $n \in \omega$ such that $\epsilon \in \mathcal{E}^\rho_n$.*
3. $\preccurlyeq^\rho \overset{\text{def}}{=} \bigcup_{n\in\omega} \preccurlyeq^\rho_n$. *$x$ is a* subexpression of *y* $\overset{\text{def}}{\iff} x \preccurlyeq^\rho y$.

(1.38) **Theorem** [C^0]

1. *$\mathcal{E}$ is closed under the operations $\hat{\varsigma}$ for $\varsigma \in \mathcal{S}$.*
2. *If $\mathcal{E}'$ is closed under the operations $\hat{\varsigma}$ for $\varsigma \in \mathcal{S}$, then $\mathcal{E} \subseteq \mathcal{E}'$.*

Remark In other words, $\mathcal{E}$ is the $\subseteq$-least class closed under the operations $\hat{\varsigma}$ for $\varsigma \in \mathcal{S}$, i.e., it is the closure (of the empty class 0) under the expression-forming operations $\hat{\varsigma}$.

Proof 1 Suppose $\varsigma \in \mathcal{S}$ and t is an $S^\rho(\varsigma)$-sequence from $\mathcal{E}^\rho$. Since $S^\rho(\varsigma)$ is finite, for some $n \in \omega$, $\operatorname{im} t \subseteq \mathcal{E}^\rho_n$, so $\hat{\varsigma}t \in \mathcal{E}^\rho_{n+1} \subseteq \mathcal{E}^\rho$.

2 Suppose $\mathcal{E}'$ is closed under the operations $\hat{\varsigma}$ for $\varsigma \in \mathcal{S}$. It is straightforward to show by induction on $n \in \omega$ that $\mathcal{E}^\rho_n \subseteq \mathcal{E}'$. $\square$1.38

(1.39) **Theorem: Unique readability** [C^0] *For any $\epsilon \in \mathcal{E}^\rho$ there exist a unique $\varsigma \in \mathcal{S}^\rho$ and t such that $\epsilon = \hat{\varsigma}t$.*

Remark This is the *unique readability property*, which is required of any language.

Proof The existence of ς and t follow directly from (1.37), and the uniqueness follows immediately from the definition of $\hat{\varsigma}$. $\square$[1.39]

(1.40)

1. *Theorem 1.38 is the basis for the very useful method of proof by* induction on complexity *of expressions: Suppose Θ is a property of sets that is definable by a formula (of our metalanguage C^0) in which all quantified variables are set variables, so that we may infer that there is a class T that consists of exactly the sets satisfying Θ. Note that Θ may involve proper classes as constants or unquantified variables. If we show that $\hat{\varsigma}t \in T$ for any $\varsigma \in \mathcal{S}^\rho$ and any $t : S^\rho(\varsigma) \to T$, then we may conclude that $\mathcal{E} \subseteq T$, i.e., Θ holds for all ρ-expressions.*
2. *Similarly, a function F may be defined for all expressions by* recursion on complexity *by giving $F(\hat{\varsigma}t)$ in terms of $\langle F\eta \mid \eta \in \operatorname{im} t\rangle$.*

(1.41) **Definition** $[\mathsf{C}^0]$

1. *Suppose $\epsilon = \hat{\varsigma}t$ is an expression.* $\operatorname{ImSubexpr}\epsilon \stackrel{\text{def}}{=} \operatorname{im} t$. *The members of* $\operatorname{ImSubexpr}\epsilon$ *are the* immediate subexpressions of ϵ.
2. *We define the function* $\operatorname{Subexpr} : \mathcal{E} \to \mathcal{P}\mathcal{E}$ *by recursion on complexity of expressions so that*

$$\operatorname{Subexpr}\epsilon = \{\epsilon\} \cup \bigcup_{\eta \in \operatorname{ImSubexpr}\epsilon} \operatorname{Subexpr}\eta.$$

The members of $\operatorname{Subexpr}\epsilon$ *are the* subexpressions of ϵ.

3. *Note that ϵ is a subexpression of ϵ. The* proper subexpressions of ϵ *are the subexpressions of ϵ other than ϵ.*

Note that if $\epsilon = \hat{\varsigma}t$ then[1.41.2]

$$\operatorname{Subexpr}\epsilon = \{\epsilon\} \cup \bigcup_{\eta \in \operatorname{im} t} \operatorname{Subexpr}\eta,$$

from which we see that (1.41.2) is indeed a definition by recursion on complexity as described in (1.40.2).

(1.40) reflects the wellfoundedness[3.76] of $\preccurlyeq$. (1.41) takes advantage of this to give another definition of *subexpression*, already defined in (1.37). It has the advantage of allowing an easy inductive proof that the subexpressions of given expression constitute a finite set. Note that (1.41) does not render (1.37) superfluous, as the latter is the primary definition of $\mathcal{E}^\rho$ and also provides a direct definition of the height of an expression. The following theorem summarizes these observations.

(1.42) **Theorem** $[\mathsf{C}^0]$ *For all $\epsilon \in \mathcal{E}^\rho$*

1. $\operatorname{ImSubexpr}\epsilon$ *and* $\operatorname{Subexpr}\epsilon$ *are finite sets.*
2. $\operatorname{Subexpr}\epsilon = \{\epsilon' \in \mathcal{E}^\rho \mid \epsilon' \preccurlyeq^\rho \epsilon\}$.[1.37.3]
3. *$\preccurlyeq^\rho$ is a wellfounded partial order on $\mathcal{E}^\rho$.*

Proof 1, 2 Induction on complexity.

3 As noted above, this is inherent in (1.40). To see it directly, note that any class of expressions contains a member with minimum height (using the fact that $(\omega; <)$ is wellordered), which must be $\preccurlyeq^\rho$-minimal. $\square^{1.42}$

Definition $[\mathsf{C}^0]$

1. *A ρ-expression ϵ is a ρ-*term $\overset{\text{def}}{\iff} \epsilon = \hat{\varsigma}t$, *where* $\mathrm{OutType}\,\varsigma = \mathrm{TermType}$.
    1. $\mathcal{T}^\rho \overset{\text{def}}{=}$ *the class of ρ-terms.*
    2. *A ρ-term τ is a* variable-term $\overset{\text{def}}{\iff} \tau = \bar{v}\ (= \hat{v}0)$, *where* $v \in \mathcal{V}$.[32]
2. *A ρ-expression ϵ is a ρ-*formula $\overset{\text{def}}{\iff} \epsilon = \hat{\varsigma}t$, *where* $\mathrm{OutType}\,\varsigma = \mathrm{FormulaType}$.
    1. $\mathcal{F}^\rho \overset{\text{def}}{=}$ *the class of ρ-formulas.*
    2. *A ρ-formula ϕ is* atomic *or* simple $\overset{\text{def}}{\iff} \phi = \hat{\varsigma}t$, *where $\varsigma \in \mathcal{P}^\rho$, i.e., ς is a predicate sign.*
    3. *A formula ϕ is* complex *or* compound $\overset{\text{def}}{\iff}$ *it is not simple.*
3. *An expression is* quantifier-free $\overset{\text{def}}{\iff}$ *it has no subexpression $\hat{\varsigma}s$, where ς is a quantifier phrase.*

(1.43) **Theorem** $[\mathsf{C}^0]$

1. *Every subexpression of a term is a term.*
2. *Every proper subexpression of an atomic formula is a term.*
3. *Every term and every atomic formula is quantifier-free.*

Proof A straightforward induction on complexity of expressions.[1.40] $\square^{1.43}$

(1.44) **Definition** C^0 *We now formally define $\mathcal{L}^\rho$ to be a multisorted structure with the following features:*

1. *The domains are $\mathcal{V}$, $\mathcal{T}^\rho$, $\mathcal{F}^\rho$, and $\mathcal{E}^\rho$; the last being the universal domain for this structure, i.e., everything is an expression.*
2. *The operations are as follows.*
    1. $\hat{\varsigma}$ *for each sign* $\varsigma \in V \cup \mathcal{P}^\rho \cup \mathcal{O}^\rho \cup \mathcal{C}$.
    2. $\tilde{\mathrm{q}}$ *for each* $\mathrm{q} \in \{\mathrm{i}_\exists, \mathrm{i}_\forall\}$, *where*

    $$\tilde{\mathrm{q}}\langle v, \phi\rangle \overset{\text{def}}{=} \langle\langle \mathrm{q}, v\rangle, \phi\rangle,$$

    for any $v \in \mathcal{V}$ and $\phi \in \mathcal{F}^\rho$, i.e., $\tilde{\mathrm{q}}\langle v, \phi\rangle = \hat{Q}\phi$, where $Q = \langle \mathrm{q}, v\rangle$.
3. *There is just one predicate, viz., identity.*

Note that (1.44) specifies $\mathcal{L}^\rho$ only up to homologic equivalence, inasmuch as it does not specify a signature, and indeed there is no real need to be any more specific than this. Nevertheless, for some purposes it is convenient to presume the specification of a signature to facilitate reference to expressions in the corresponding (meta)language. We will return to this after we present some convenient typographical conventions that also serve this purpose.

[32]Note that all standard languages have the same variable-terms, independent of the signature ρ.

1.3.2 Metalanguage conventions

Definition [C^0] *A* ρ-translation *(or simply* translation *if ρ is understood) is a homology of $\mathcal{L}^\rho$.*[1.32]

A ρ-translation establishes a connection between $\mathcal{L}^\rho$ and a structure $\mathcal{S}$ by virtue of which $\mathcal{S}$ may be considered a language appropriate to ρ.

Definition [C^0] *We define a* ρ-language *to be any structure $\mathcal{S}$ related to $\mathcal{L}^\rho$ by a translation, together with such a translation.*

There is often no need to specify a ρ-language in much detail. It is much more useful to have a natural system of metalanguage names for the domains and expression-building operations, and we have already indicated the elements of such a system. These names may be regarded as applying to any ρ-language.

(1.45)

1. *We adopt '$\mathcal{V}$', '$\mathcal{T}$', '$\mathcal{F}$', and '$\mathcal{E}$' as metalanguage names for the various domains.*
2. *A systematic nomenclature for argument-specification operations has been presented above.*[1.20] *In this system, if 'X', for example, is used in the metalanguage to refer to a predicate or operation index, then '$\tilde{X}$' may be used to refer to the corresponding argument-specification operation.*
3. *The operations that form complex (i.e., nonatomic) formulas we often represent by the bold versions—'$\boldsymbol{\neg}$', '$\boldsymbol{\wedge}$', '$\boldsymbol{\vee}$', '$\boldsymbol{\rightarrow}$', '$\boldsymbol{\leftrightarrow}$', '$\boldsymbol{\forall}$', and '$\boldsymbol{\exists}$'—of the usual symbols for these logical notions.*[1.21,1.22]
4. *We may extend this practice to (some) predicate and operation symbols in a typographical object language. To give a nearly universal example, '$\boldsymbol{=}$' represents the operation of forming the formula that relates two terms by the identity predicate. When we discuss the language of membership we often use '$\boldsymbol{\in}$' in this way.*
5. *Binary propositional connectives and binary object-language predicates and operations lead to binary expression-building operations, and for these we routinely use the convention of placing the corresponding metalanguage symbol between the arguments, as we do with binary operation symbols in typographical languages generally. The object-language quantifiers also correspond to binary expression-building operations—taking a variable and a formula as arguments—but for these we retain in the metalanguage the same order that is standard for quantifier expressions themselves in typographical languages.*
6. *To reduce notational clutter, we will often conflate a variable v with the corresponding term $\bar{v}$ $(= \hat{v}0)$.*[1.35.3] *Similarly, we may conflate a constant (nulary) predicate or operation index X with the corresponding formula or term $\tilde{X}$. For the duration of this chapter, however, for the sake of consistency we will adhere to precise usage.*

Definition [C^0]

1. *A language $\mathcal{L}$ is* a standard language $\overset{\text{def}}{\iff}$ *$\mathcal{L}$ is the standard ρ-language for a standard*[1.33] *signature ρ.*

2. $\mathcal{L}$ *is* the standard language $\overset{\text{def}}{\iff}$ *$\mathcal{L}$ is the standard ρ-language for the (full) standard signature ρ.*

3. *A* standard expression $\overset{\text{def}}{=}$ *an expression of the standard language.*

Note that any countable language is homologous to a sublanguage of the standard language, and we could restrict our attention to the standard language if we wished.

1.3.3 Diagrams, occurrences, and binding

As we have previously observed, the definition we have given of *expression* is obviously only one of many that would serve our purpose. For some purposes it is convenient to use an alternative construct that renders more explicit the relationships among subexpressions and signs within an expression. We call these objects *diagrams*. By way of illustration, consider the expression ϵ of (1.36) written in an "exploded" format, with each sign on a separate line and its place within ϵ represented numerically in the obvious way.

$$\begin{array}{l}
\langle \mathrm{i}_{\to}, \langle \mathrm{i}_{\neg}, \langle \mathrm{i}_{=}, \langle \mathrm{v}_0 \rangle, \langle \mathrm{i}_0 \rangle\rangle\rangle, \langle\langle \mathrm{i}_{\exists}, \mathrm{v}_1\rangle, \langle \mathrm{i}_{=}, \langle \mathrm{v}_0\rangle, \langle \mathrm{i}_S, \langle \mathrm{v}_1\rangle\rangle\rangle\rangle\rangle \\
\langle \mathrm{i}_{\to}, \\
0 \quad \langle \mathrm{i}_{\neg}, \\
00 \quad \langle \mathrm{i}_{=}, \\
000 \quad \langle \mathrm{v}_0 \rangle, \\
001 \quad \langle \mathrm{i}_0 \rangle\rangle\rangle, \\
1 \quad \langle\langle \mathrm{i}_{\exists}, \mathrm{v}_1 \rangle, \\
10 \quad \langle \mathrm{i}_{=}, \\
100 \quad \langle \mathrm{v}_0 \rangle, \\
101 \quad \langle \mathrm{i}_S, \\
1010 \quad \langle \mathrm{v}_1 \rangle\rangle\rangle\rangle\rangle
\end{array}$$

Note that all the relevant information is contained in the function

$$\begin{aligned}
\{&(\langle\,\rangle, \mathrm{i}_{\to}), \\
&(\langle 0\rangle, \mathrm{i}_{\neg}), \\
&(\langle 0,0\rangle, \mathrm{i}_{=}), \\
&(\langle 0,0,0\rangle, \mathrm{v}_0), \\
&(\langle 0,0,1\rangle, \mathrm{i}_0), \\
&(\langle 1\rangle, \langle \mathrm{i}_{\exists}, \mathrm{v}_1\rangle), \\
&(\langle 1,0\rangle, \mathrm{i}_{=}), \\
&(\langle 1,0,0\rangle, \mathrm{v}_0), \\
&(\langle 1,0,1\rangle, \mathrm{i}_S), \\
&(\langle 1,0,1,0\rangle, \mathrm{v}_1)\},
\end{aligned} \tag{1.46}$$

where we have used '$\langle\,\rangle$' to denote the empty sequence—which is also the empty set—to emphasize the regularity of the construction. This is the *diagram* of ϵ.

Figure 1.2 illustrates the domain of the diagram (1.46) as a tree growing downward, as is conventional for sequence trees. This figure may be overlaid on Figure 1.1 to represent the diagram *per se*.

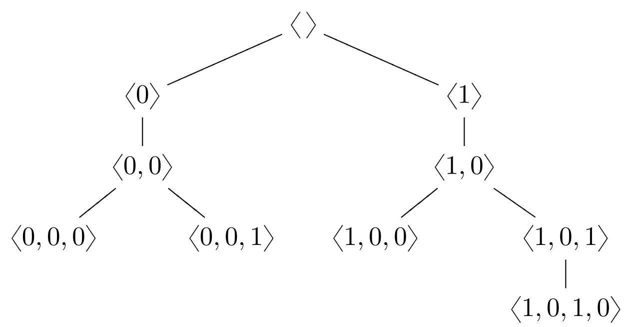

Figure 1.2: The domain of the diagram (1.46) visualized as a tree growing downward.

Definition $[\mathsf{C}^0]$ *We define the* diagram *of an expression by recursion on complexity as follows.*

$$\Delta(\hat{\varsigma}s) \stackrel{\text{def}}{=} \{(0,\varsigma)\} \cup \big\{ \big(\langle m\rangle ^\frown p, (\Delta\, s_m)(p)\big) \,\big|\, m \in |s| \wedge p \in \operatorname{dom}(\Delta\, s_m)\big\}.$$

It is clear that the diagram of a ρ-expression is a *ρ-diagram* as defined below. It is also clear that for each ρ-diagram δ in this sense, there is a unique ρ-expression ϵ such that $\Delta\,\epsilon = \delta$.

Definition $[\mathsf{C}^0]$

1. *A ρ-diagram is a finite function δ such that*
   1. $\operatorname{dom}\delta \subseteq {}^{<\omega}\omega$ *is a nonempty sequence tree;*[33]
   2. $\operatorname{im}\delta \subseteq \mathcal{S}^\rho$*;*
   3. *for every* $p \in \operatorname{dom}\delta$
      1. $\{m \mid p^\frown\langle m\rangle \in \operatorname{dom}\delta\} = S^\rho(\delta p)$*;*[34]
      2. $\forall m \in S^\rho(\delta p)\ \operatorname{OutType}\delta(p^\frown\langle m\rangle) = \operatorname{InType}\delta(p)$.
2. *Given a ρ-diagram δ, $E\delta \stackrel{\text{def}}{=}$ the (unique) ρ-expression ϵ such that $\Delta\,\epsilon = \delta$.*

Definition $[\mathsf{C}^0]$

1. *If δ is a diagram, we call the elements of* $\operatorname{dom}\delta$ places in δ.
2. *A* place *in an expression ϵ is a place in $\Delta\,\epsilon$.*
3. *Suppose p is a place in a diagram δ.*
   1. *The* subdiagram of δ at p $\stackrel{\text{def}}{=} \Delta^{\delta,p} \stackrel{\text{def}}{=}$
   $$\{(p',\varsigma) \mid (p^\frown p',\varsigma) \in \delta\}.$$
   2. $\Delta^{\delta,p}$ occurs at p in δ.
4. *Suppose p is a place in an expression ϵ.*

[33] $T \subseteq {}^{<\omega}\omega$ is a sequence tree iff for every $s \in T$ and $m \in |s|$, $s \restriction m \in T$. Sequence trees are conventionally visualized as "growing downward" as in Figure 1.2.

[34] Note that $\{m \mid p^\frown\langle m\rangle \in \operatorname{dom}\delta\}$ is required to be the ordinal $S^\rho(\delta(p))$, i.e., the set $\{0, 1, \ldots, S^\rho(\delta(p)) - 1\rangle$.

   1. *The* subexpression of ϵ at $p \stackrel{\text{def}}{=} E^{\epsilon,p} \stackrel{\text{def}}{=} E\big(\Delta^{\Delta\,\epsilon,p}\big)$.
   2. $E^{\epsilon,p}$ occurs at p in ϵ.

5. *An expression* ϵ' occurs in $\epsilon \stackrel{\text{def}}{\iff}$ *for some place* p *in* ϵ, ϵ' *occurs at* p *in* ϵ. *If* $\epsilon' = \hat{Q}\phi$, *where* $Q \in \mathcal{Q}$ *is a quantifier phrase, we will also refer to* Q *as occurring at* p *in* ϵ.

6. *An* occurrence in ϵ *is a 2-sequence* $\langle p, x\rangle$ *such that* p *is a place in* ϵ *and* x *is the expression—or quantifier phrase—that occurs at* p *in* ϵ. *We will refer to an occurrence of a quantifier phrase as a 'quantifier phrase occurrence' or simply as a 'quantifier occurrence'.*

(1.47) *In practice, we use 'diagram' and 'expression' interchangeably, so that we may refer to a diagram occurring in an expression, for example.*

Using the example (1.46), the diagram

$$\begin{aligned}&\{(\langle\,\rangle, \mathrm{i}_=),\\ &\ (\langle 0\rangle, \mathrm{v}_0),\\ &\ (\langle 1\rangle, \mathrm{i}_S),\\ &\ (\langle 1,0\rangle, \mathrm{v}_1)\}\end{aligned}$$

occurs at $\langle 1, 0\rangle$ in ϵ and the corresponding subexpression is

$$\bar{\mathrm{v}}_0 = \mathbf{S}(\bar{\mathrm{v}}_1)\ \ (= \langle \mathrm{i}_=, \langle \mathrm{v}_0\rangle, \langle \mathrm{i}_S, \langle \mathrm{v}_1\rangle\rangle\rangle).$$

A simple proof by induction on complexity of expressions,[1.40] shows that the subexpressions of ϵ are exactly those expressions that occur in ϵ; a given expression may occur at more than one place in ϵ.

Employing the convention (1.45.6) we may refer to an occurrence of a variable term $\bar{v}$ *as an occurrence of the corresponding variable* v.

Definition $[\mathsf{C}^0]$ *Suppose* ϵ *is an expression.*

1. *Suppose* $\varpi = \langle p, \bar{v}\rangle$, *is an occurrence of a variable* v *at* p *in* ϵ.
   1. ϖ *is* bound by *a quantifier phrase occurrence* $\varpi' = \langle p', Q\rangle \stackrel{\text{def}}{\iff}$
      1. Q *is* $\langle \mathrm{i}_\exists, v\rangle$ *or* $\langle \mathrm{i}_\forall, v\rangle$*;*
      2. $p' \subseteq p$*; and*
      3. *there does not exist a quantifier occurrence* $\varpi'' = \langle p'', Q'\rangle$, *such that* Q' *is* $\langle \mathrm{i}_\exists, v\rangle$ *or* $\langle \mathrm{i}_\forall, v\rangle$, *and* $p' \subsetneqq p'' \subseteq p$.

      In other words, a variable occurrence is bound by the first corresponding quantifier phrase occurrence encountered moving up the diagram tree (as in Figure 1.1), if any.
   2. ϖ *is* bound in $\epsilon \stackrel{\text{def}}{\iff}$ *it is bound by some quantifier occurrence in* ϵ.
   3. ϖ *is* free in $\epsilon \stackrel{\text{def}}{\iff}$ *it is not bound in* ϵ.
2. *Suppose* v *is a variable.* v *is* free *in* $\epsilon \stackrel{\text{def}}{\iff}$ *there is an free occurrence of* v *in* ϵ.
3. $\mathrm{Free}\,\epsilon \stackrel{\text{def}}{=}$ *the set of variables free in* ϵ.

If ϵ'' is a subexpression of ϵ', which is a subexpression of ϵ, then ϵ'' is also a subexpression of ϵ, and we informally conflate any occurrence of ϵ'' in ϵ' with the corresponding occurrence in ϵ—i.e., if $\langle p', \epsilon' \rangle$ is an occurrence in ϵ, and $\langle p'', \epsilon'' \rangle$ is an occurrence in ϵ', then, letting $p = p' {}^\frown p''$, we refer to $\langle p, \epsilon'' \rangle$ as the "same occurrence" of ϵ'' (in ϵ) as $\langle p'', \epsilon'' \rangle$ (in ϵ').

With this notion of an occurrence "maintaining its identity" as a complex expression is "built" from simpler expressions, we may state the critical observation that if a variable occurrence is bound by a quantifier occurrence in an expression ϵ, it remains bound by that occurrence in any expression ϵ' of which ϵ is a subexpression.

1.3.4 Substitution

(1.48) **Definition** [C^0] *The result of* substituting *a diagram η at a place p in a diagram δ* $\stackrel{\text{def}}{=}$

$$\delta\left\{{p \atop \eta}\right\} \stackrel{\text{def}}{=} \{(p', \varsigma) \in \delta \mid p \nsubseteq p'\} \cup \{(p {}^\frown q, \varsigma) \mid (q, \varsigma) \in \eta\}.$$

We use the same terminology and notation with reference to expressions.[1.47]

In effect, $\delta\left\{{p \atop \eta}\right\}$ is the result of excising $\Delta^{\delta,p}$ from δ and replacing it with η. Letting $\eta' = \Delta^{\delta,p}$, we therefore also refer to the substitution of η at p as substitution of η for the occurrence of η' at p in δ. Note that $\delta\left\{{p \atop \eta}\right\}$ is a diagram iff $\mathrm{OutType}(\eta 0) = \mathrm{OutType}(\delta p)$. In other words, we may substitute any term for any occurrence of a term and any formula for any occurrence of a formula, but we may not substitute a term for an occurrence of a formula or *vice versa*.[35]

1.3.4.1 Simultaneous substitutions

Multiple substitutions may be performed sequentially; if the respective places of substitution are incomparable, i.e., none is an initial segment (i.e., subset) of any other, then the order of substitution is irrelevant, and the substitutions may be regarded as simultaneous.

(1.49) **Definition** [C^0] *We indicate the result of the simultaneous substitution of expressions $\epsilon_0, \ldots, \epsilon_{n^-}$ at incomparable places $p_0, \ldots, p_{n^-}$ in an expression ϵ as*[36]

$$\epsilon\left\{{p_0 \cdots p_{n^-} \atop \epsilon_0 \cdots \epsilon_{n^-}}\right\}.$$

The commonest substitutions are of terms for variables.[1.11, 1.15] It usually only makes sense to substitute a given term for all free occurrences of a variable in a given formula or term—i.e., we typically do not want to substitute for bound occurrences of variables, and we typically do not want to substitute for some, but not all, free occurrences. We frequently want to describe multiple simultaneous such substitutions. Since variables are nulary signs, the respective places of distinct occurrences of variables are automatically incomparable.

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[35] For a multisorted language, terms must also be of the correct sorts.

[36] Note that this is a variation on the notation (3.56) for finite functions. We can also indicate this substitution as $\epsilon\{S\}$, where $S = \left\langle{p_0 \cdots p_{n^-} \atop \epsilon_0 \cdots \epsilon_{n^-}}\right\rangle$. The use of curly brackets simply indicates that this is a substitution of expressions at places.

(1.50) **Definition** $[\mathsf{C}^0]$ *Suppose ϵ is an expression, $v_0, \ldots, v_{n^-}$ are distinct variables, and $\tau_0, \ldots, \tau_{n^-}$ are terms. Then $\epsilon\binom{v_0 \cdots v_{n^-}}{\tau_0 \cdots \tau_{n^-}} \stackrel{\text{def}}{=}$ the result of substituting τ_m for all the free occurrences of v_m, for all $m \in n$.*[37]

Note that this notation is specific to substitution for variables, and as a way of emphasizing this specificity we show variables v rather than their terms $\bar{v}$ as the destinations of the substitutions. To compare this notation to (1.49), note that if $p_m^0, \ldots, p_m^{k_m{}^-}$ are the places where v_m occurs free in ϵ, then

$$\epsilon\binom{v_0 \cdots v_{n^-}}{\tau_0 \cdots \tau_{n^-}} = \epsilon\left\{\begin{matrix} p_0^0 \cdots p_0^{k_0{}^-} \cdots p_{n^-}^0 \cdots p_{n^-}^{k_{n^-}{}^-} \\ \tau_0 \cdots \tau_0 \cdots \tau_{n^-} \cdots \tau_{n^-} \end{matrix}\right\}.$$

It is not always necessary to be specific as to the identity of the variables involved in a substitution.

(1.51) **Definition** $[\mathsf{C}^0]$ *Suppose ϵ is an expression with exactly one free variable u, and τ is a term. Then*

$$\epsilon(\tau) \stackrel{\text{def}}{=} \epsilon\binom{u}{\tau}.$$

If ϵ has more than one free variable we may use a similar notation, with the understanding that the same variables occur in the same order in all substitutions so indicated in a given expression in a given discussion. Ordinarily in such a case the substitutions involve all the free variables of ϵ, so it is only the order of variables that is left unstated. As a convenience, if ϵ has no free variables, $\epsilon(\tau) \stackrel{\text{def}}{=} \epsilon$.

1.3.5 A useful quotation convention

(1.52)

1. *When an informal expression is flanked by corner quotes, the result is a metalanguage term that refers to a formal expression of $\mathcal{L}^{\mathsf{s}}$ that is equivalent to the quoted expression over whatever theory is in force at the time. For example,*

   $$\ulcorner \textit{no set is a member of itself} \urcorner$$

   refers to an expression such as

   $$\forall v\ \neg\, v \in v,$$

   where v is a variable, which need not be specified.[38]

2. *We extend this notation to construct metalanguage terms for object-language expressions, with (often implicit) variables for which metalanguage terms denoting object-language expressions may be substituted.*[39] *For example,*

   (1.53) $$\ulcorner \textit{if } (\phi \leftrightarrow \psi) \textit{ then } (\theta) \urcorner$$

[37] This is again a variation on the notation (3.56) for finite functions. We can also indicate this substitution as $\epsilon(S)$, where $S = \left\langle\begin{matrix} v_0 \cdots v_{n^-} \\ \tau_0 \cdots \tau_{n^-}\end{matrix}\right\rangle$. The use of round brackets simply indicates that this is a substitution of terms for free variables.

[38] The critical difference between this sort of quotation and previous quotation conventions, viz., (1.4) and (1.8), is that a name formed by quotation in this way does not denote a specific formal expression: any instance of it may be replaced by any equivalent expression. Its purpose is to promote readability by extending the informal style of our metalanguage to descriptions of object language expressions.

[39] Such a metalanguage expression is therefore a "pattern" for constructing object-language expressions, often referred to as a 'schema'.

may be understood to refer to

$$(\phi \leftrightarrow \psi) \rightarrow \theta.$$

(1.53) would look more conventional if we used metalanguage variables, say 'u' and 'v', and let τ be the metalanguage term <u>if</u> u <u>then</u> v.[1.8] Then (1.53) is

$$\tau\binom{u \quad v}{\phi \leftrightarrow \psi \;\; \theta},$$

with the customary use of round brackets to indicate substitution.[1.50] The use of round brackets as in (1.53) to indicate substitution "in stream" is therefore appropriate; we use a light typeface for the brackets in this situation both to distinguish them from round brackets that may otherwise be part of the quoted text, and to be relatively unobtrusive. They are usually omitted altogether in discussion of this sort, so that (1.53) is rendered as

$$\ulcorner\text{if } \phi \leftrightarrow \psi \text{ then } \theta\urcorner.$$

The quotation marks are also frequently omitted, leaving

$$\text{if } \phi \leftrightarrow \psi \text{ then } \theta.$$

These conventional notations—while ultimately harmless—are technically meaningless, and misleading to the extent that they are meaningful, and require more interpretive effort from the reader than a precise notation. When clarity is not at risk, however, we may indulge in notational abuses of this sort.

1.3.6 Change of variables

(1.54) **Definition** [C^0] *Suppose ϵ is an expression and ι is a function that assigns a variable to each quantifier occurrence in ϵ. Let $\delta = \Delta\epsilon$ and let δ' be the diagram such that* $\operatorname{dom}\delta' = \operatorname{dom}\delta$ *and for every* $p \in \operatorname{dom}\delta$, *letting* $\varsigma = \delta p$, $\varpi = \langle p, \varsigma\rangle$, *and* $\varsigma' = \delta' p$,

1. *if* $\varsigma \in \mathcal{Q}$ *then, letting* $\varsigma = \langle \mathrm{q}, v\rangle$, *where* $\mathrm{q} \in \{\mathrm{i}_\exists, \mathrm{i}_\forall\}$ *and* $v \in \mathcal{V}$, $\varsigma' = \langle \mathrm{q}, \iota\varpi\rangle$;
2. *if* $\varsigma \in \mathcal{V}$ *and* ϖ *is bound by a quantifier occurrence* ϖ_0 *in* ϵ *then* $\varsigma' = \iota\varpi_0$;
3. *otherwise* $\varsigma' = \varsigma$.

Note that δ' is indeed a diagram. A change of variables[§ 1.1.8.3] *for ϵ is a function ι as above such that the binding structure of δ' is the same as that of δ, i.e., for any* $p, p_0 \in \operatorname{dom}\delta$, $\langle p_0, \delta p_0\rangle$ *binds* $\langle p, \delta p\rangle$ *in δ iff* $\langle p_0, \delta' p_0\rangle$ *binds* $\langle p, \delta' p\rangle$ *in δ'. Letting* $\epsilon' = E\delta'$, *we say that ϵ' is the result of applying the change of variables ι to ϵ. Note that free variable occurrences are not affected by a change of variables.*

For example, let

$$\epsilon = \forall u\, \forall v\, (\tilde{R}\langle u, w\rangle \rightarrow \forall u\, \tilde{R}\langle v, u\rangle).$$

Let $\varpi_0, \varpi_1, \varpi_2$ be the quantifier occurrences in ϵ in the order shown above, i.e.,

$$\begin{aligned}
\varpi_0 &= \langle 0, \langle \mathrm{i}_\forall, u\rangle\rangle \\
\varpi_1 &= \langle\langle 0\rangle, \langle \mathrm{i}_\forall, v\rangle\rangle \\
\varpi_2 &= \langle\langle 0, 0, 1\rangle, \langle \mathrm{i}_\forall, u\rangle\rangle.
\end{aligned}$$

Let ι be given by

$$\begin{aligned}\varpi_0 &\mapsto v\\ \varpi_1 &\mapsto u\\ \varpi_2 &\mapsto w.\end{aligned}$$

Then

$$\epsilon' = \forall v\,\forall u\,(\tilde{R}\langle v,w\rangle \to \forall w\,\tilde{R}\langle u,w\rangle).$$

Since ϵ' has the same binding structure as ϵ, ι is a change of variables for ϵ.

Suppose instead that ι is given by

$$\begin{aligned}\varpi_0 &\mapsto v\\ \varpi_1 &\mapsto u\\ \varpi_2 &\mapsto u.\end{aligned}$$

Then

$$\epsilon' = \forall v\,\forall u\,(\tilde{R}\langle v,w\rangle \to \forall u\,\tilde{R}\langle u,u\rangle),$$

which does not have the same binding structure as ϵ (the first argument of the second occurrence of R is bound too soon), so ι is not a change of variables for ϵ.

Now suppose ι is given by

$$\begin{aligned}\varpi_0 &\mapsto u\\ \varpi_1 &\mapsto w\\ \varpi_2 &\mapsto u.\end{aligned}$$

Then

$$\epsilon' = \forall u\,\forall w\,(\tilde{R}\langle u,w\rangle \to \forall u\,\tilde{R}\langle w,u\rangle),$$

which also does not have the same binding structure as ϵ (the second argument of the first occurrence of R is bound), so ι is not a change of variables for ϵ.

(1.55) *Suppose ϵ is an expression, B and F are the sets of variables that occur respectively bound and free in ϵ,*[40] *$i : B \overset{\text{inj}}{\to} \mathcal{V}$, and $F \cap \operatorname{im} i = 0$. Let ι be a function that assigns to each quantifier occurrence $\varpi = \langle p, \langle \mathrm{q}, v\rangle\rangle$ in ϵ the variable iv. Then (it is easy to see that) ι is a change of variables for ϵ.*

This is useful if we want to substitute a term τ (for a free variable) in ϵ, and we don't want a variable in τ to be accidentally bound by a quantifier in ϵ. Of course, in this case we would impose the additional condition that $\operatorname{im} i \cap \operatorname{Free} \tau = 0$.

1.4 Interpretation

Suppose $\mathcal{L}$ and $\mathfrak{S}$ are respectively a language and a structure with the same signature ρ. Then it makes sense to interpret the expressions of $\mathcal{L}$ in the context of $\mathfrak{S}$—indeed, that is the whole point of $\mathcal{L}$. Here, as in most of what follows, we restrict our comments to unsorted signatures. It is not difficult to work out the general theory.

[40] Note that B and F need not be disjoint.

1.4.1 Assignments to variables

As we have already noted, an interpretation does not give a specific denotation to variables. Nevertheless, in order to give a formal definition of when a formula containing quantifiers is true in a given interpretation, we must have a notion of assigning specific values to variables.

It is clear that in a formula such as $\forall v\, \phi$ or $\exists v\, \phi$ it would not make sense to assign a specific value to v. On the other hand, if v occurs free in ϕ then it does make sense to assign v a specific denotation, and the meaning of ϕ depends on the assignment.

Definition [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure and ϵ is a ρ-expression. An* $\mathfrak{S}$-assignment for ϵ *is a finite function from a subset of $\mathcal{V}$ into $|\mathfrak{S}|$ whose domain includes* $\mathrm{Free}\,\epsilon$*.*

Note that—being a finite class—an assignment is necessarily a set.

It will be convenient to be able to form chains of assignments, with later assignments taking precedence over earlier assignments in case of disagreement, i.e., if A and B are assignments, then AB is the assignment with domain $\operatorname{dom} A \cup \operatorname{dom} B$ and

$$(AB)v = \begin{cases} Bv & \text{if } v \in \operatorname{dom} B \\ Av & \text{otherwise.} \end{cases}$$

For assignments, as for substitutions, we frequently use the notation (3.56) for finite functions. Note that if

$$A = \left\langle \begin{smallmatrix} u_0 & \cdots & u_{m^-} \\ x_0 & \cdots & x_{m^-} \end{smallmatrix} \right\rangle$$

$$B = \left\langle \begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ y_0 & \cdots & y_{n^-} \end{smallmatrix} \right\rangle,$$

then

$$AB = \left\langle \begin{smallmatrix} u_0 & \cdots & u_{m^-} & v_0 & \cdots & v_{n^-} \\ x_0 & \cdots & x_{m^-} & y_0 & \cdots & y_{n^-} \end{smallmatrix} \right\rangle.$$

(See the comment following Definition 3.56.) To indicate the role of such a finite function as an assignment, we use square brackets in a manner analogous to the use of round brackets for substitutions, e.g., '$\left[\begin{smallmatrix} v \\ a \end{smallmatrix}\right]$' for '$\left\langle\begin{smallmatrix} v \\ a \end{smallmatrix}\right\rangle$'.

1.4.2 Interpretation in structures

1.4.2.1 Evaluation of terms

We define the value of a ρ-term τ in the ρ-structure $\mathfrak{S}$ at an $\mathfrak{S}$-assignment A for τ by recursion on $\langle \tau, A \rangle$. In the terminology of our basic theorem on recursive definition,[3.80]

1. let X be the class of 2-sequences $\langle \tau, A \rangle$, where τ is a ρ-term and A is an $\mathfrak{S}$-assignment A for τ;

2. let R be the binary relation X defined by the condition that

   (1.56) $\langle \tau', A' \rangle \, R \, \langle \tau, A \rangle$ *iff* $\tau' \in \mathrm{ImSubexpr}\, \tau$ *and* $A' \subseteq A$*;*

3. let G be the function whose domain consists of all 2-sequences $\langle x, f \rangle$ such that $x \in X$ and f is a function with $\operatorname{dom} f = R^{\leftarrow}\{x\}$, such that for all $\langle x, f \rangle \in \operatorname{dom} G$,

1. if $x = \langle \bar{v}, A \rangle$, where $v \in \mathcal{V}$ (so $R^{\leftarrow}\{x\} = 0$), then $G\langle x, 0 \rangle = Av$; and
2. if $x = \left\langle \tilde{F}\langle \tau_0, \ldots, \tau_{n^-} \rangle, A \right\rangle$, where F is an n-ary operation index (so $\forall m \in n \; \langle \tau_m, A \rangle \in \operatorname{dom} f$), then

$$G\langle x, f \rangle = F^{\mathfrak{S}} \left\langle f\langle \tau_0, A \rangle, \ldots, f\langle \tau_{n^-}, A \rangle \right\rangle.$$

R is obviously wellfounded. By virtue of (1.56), since assignments are finite by definition, for any $\langle \tau, A \rangle \in X$, $R^{\leftarrow}\{\langle \tau, A \rangle\}$ is finite and is therefore a set; hence, R is setlike. Theorem 3.80 therefore applies, and the following definition is justified.

(1.57) **Definition** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure. We define* $\operatorname{Val}^{\mathfrak{S}} \tau[A]$ *by R-recursion on $\langle \tau, A \rangle$ so that for any ρ-term τ and $\mathfrak{S}$-assignment A for τ,*

1. *if $\tau = \bar{v}$ then* $\operatorname{Val}^{\mathfrak{S}} \tau[A] = Av$*; and*
2. *if $\tau = \tilde{F}\langle \tau_0, \ldots, \tau_{n^-} \rangle$ then*

$$\operatorname{Val}^{\mathfrak{S}} \tau[A] = F^{\mathfrak{S}} \langle \operatorname{Val}^{\mathfrak{S}} \tau_0[A], \ldots, \operatorname{Val}^{\mathfrak{S}} \tau_{n^-}[A] \rangle.$$

1.4.2.2 Evaluation of formulas

We may describe the evaluation of a formula ϕ under a given interpretation as the assignment of a truth-value, and we have previously argued that the sets 1 and 0 naturally correspond to *true* and *false*, respectively.

A *valuation function* for formulas for a structure $\mathfrak{S}$ is naturally regarded as an extension of the valuation function $\operatorname{Val}^{\mathfrak{S}}$ for terms. As will become clear, when $\mathfrak{S}$ is a proper class it is not possible to formulate this definition as a recursion as in (1.57), and we must deal with partial valuation functions. As always, we require that the domain of an assignment contain all the free variables of the formula under consideration, but it may be larger than that.

(1.58) **Definition** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure.*

1. *Suppose $E \subseteq \mathcal{E}^\rho$, i.e., E is a class of ρ-expressions. Let $\overline{E}$ be the class of subexpressions of members of E. An E*-valuation function for $\mathfrak{S}$ $\stackrel{\text{def}}{=}$ *a function F such that $\operatorname{dom} F$ consists of all 2-sequences $\langle \epsilon, A \rangle$ such that $\epsilon \in \overline{E}$ and A is an $\mathfrak{S}$-assignment for ϵ, and for any $\langle \epsilon, A \rangle \in \operatorname{dom} F$*
    1. *if $\epsilon = \bar{v}$ then $F\langle \epsilon, A \rangle = Av$;*
    2. *if $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-} \rangle$, where X is an n-ary operation index, then*

    $$F\langle \epsilon, A \rangle = X^{\mathfrak{S}} \langle F\langle \tau_0, A \rangle, \ldots, F\langle \tau_{n^-}, A \rangle \rangle;$$

    3. *if $\epsilon \in \mathcal{F}^\rho$ then $F\langle \epsilon, A \rangle \in 2$ $(= \{0, 1\})$;*
    4. *if $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-} \rangle$, where X is an n-ary predicate index, then*

    $$F\langle \epsilon, A \rangle = 1 \leftrightarrow \left\langle F\langle \tau_0, A \rangle, \ldots, F\langle \tau_{n^-}, A \rangle \right\rangle \in X^{\mathfrak{S}};$$

    5. *if $\epsilon = \neg \psi$ then $F\langle \epsilon, A \rangle = 1 \leftrightarrow F\langle \psi, A \rangle = 0$;*
    6. *if $\epsilon = \psi_0 \vee \psi_1$ then $F\langle \epsilon, A \rangle = 1 \leftrightarrow F\langle \psi_0, A \rangle = 1 \vee F\langle \psi_1, A \rangle = 1$;*
    7. *if $\epsilon = \psi_0 \wedge \psi_1$ then $F\langle \epsilon, A \rangle = 1 \leftrightarrow F\langle \psi_0, A \rangle = 1 \wedge F\langle \psi_1, A \rangle = 1$;*
    8. *if $\epsilon = \psi_0 \rightarrow \psi_1$ then $F\langle \epsilon, A \rangle = 1 \leftrightarrow (F\langle \psi_0, A \rangle = 1 \rightarrow F\langle \psi_1, A \rangle = 1)$;*

9. *if* $\epsilon = \psi_0 \leftrightarrow \psi_1$ *then* $F\langle \epsilon, A\rangle = 1 \leftrightarrow (F\langle \psi_0, A\rangle = 1 \leftrightarrow F\langle \psi_1, A\rangle = 1)$;
10. *if* $\epsilon = \exists v\, \psi$ *then* $F\langle \epsilon, A\rangle = 1 \leftrightarrow \exists a \in |\mathfrak{S}|\ F\big\langle \psi, A\big\langle {v \atop a} \big\rangle\big\rangle = 1$; *and*
11. *if* $\epsilon = \forall v\, \psi$ *then* $F\langle \epsilon, A\rangle = 1 \leftrightarrow \forall a \in |\mathfrak{S}|\ F\big\langle \psi, A\big\langle {v \atop a} \big\rangle\big\rangle = 1$.

2. *A* partial valuation function for $\mathfrak{S}$ $\stackrel{\text{def}}{=}$ *a E-valuation function for $\mathfrak{S}$ for some* $E \subseteq \mathcal{E}^\rho$.
3. *A partial valuation function F for $\mathfrak{S}$* covers E $\stackrel{\text{def}}{\Longleftrightarrow}$ *F includes a E-valuation function for $\mathfrak{S}$.*
4. *A* valuation function for $\mathfrak{S}$ $\stackrel{\text{def}}{=}$ *an $\mathcal{E}^\rho$-valuation function for $\mathfrak{S}$.*

(1.59) **Theorem** $[\mathsf{C}^0]$ *Suppose ρ is a signature, $E \subseteq \mathcal{E}^\rho$, and $\mathfrak{S}$ is a ρ-structure. If there exists an E-valuation function for $\mathfrak{S}$ then it is unique.*

Proof This is a straightforward induction on the complexity of expressions occurring in the 2-sequences $\langle \epsilon, A\rangle$, using the wellfoundedness of $\preccurlyeq^\rho$.[1.42.3] $\square$[1.59]

It is somewhat more natural and parsimonious to refer to ϕ as true or false under an interpretation $(\mathfrak{S}, A)$ rather than having the value trueness or falseness. We may also say that a structure $\mathfrak{S}$ *satisfies* a formula ϕ *at* an assignment A. A *satisfaction relation* for $\mathfrak{S}$ is the class of 2-sequences $\langle \phi, A\rangle$ such that A is an $\mathfrak{S}$-assignment for ϕ and $\mathfrak{S}$ *satisfies* ϕ at A. We define satisfaction in terms of valuation for formulas as follows.

Definition $[\mathsf{C}^0]$ *Suppose ρ is a signature and $E \subseteq \mathcal{E}^\rho$.*

1. *If F is a E-valuation function for $\mathfrak{S}$ then* the corresponding E-satisfaction relation for $\mathfrak{S}$ $\stackrel{\text{def}}{=}$ *the class of $\langle \phi, A\rangle$ such that $\phi \in \mathcal{F}^\rho$, $\langle \phi, A\rangle \in \operatorname{dom} F$, and $F\langle \phi, A\rangle = 1$.*
2. *S is an* E-satisfaction relation for $\mathfrak{S}$ $\stackrel{\text{def}}{\Longleftrightarrow}$ *S corresponds to an E-valuation function for $\mathfrak{S}$.*
3. *We define* partial satisfaction relation, *etc., correspondingly.*
4. *A partial satisfaction relation S* covers E $\stackrel{\text{def}}{\Longleftrightarrow}$ *S is an E'-satisfaction relation for some $E' \subseteq \mathcal{E}^\rho$ such that $E \subseteq E'$.*

Obviously, a valuation function and a satisfaction relation for formulas are interchangeable entities, and we will use one or the other as suits our convenience.

In view of Theorem 1.59 we often refer to *the* satisfaction relation in a given context, even if we have not shown (or cannot show) that one exists. This is reasonable in that only in exceptional and rather artificial circumstances can it be shown that a satisfaction relation does not exist.

(1.60) **Definition** $[\mathsf{C}^0]$ *Suppose $\mathfrak{S}$ is a ρ-structure.*

1. $\mathfrak{S}$ *is* weakly satisfactory $\stackrel{\text{def}}{\Longleftrightarrow}$ *for every finite set Φ of ρ-formulas there is a Φ-satisfaction relation for $\mathfrak{S}$.*
2. $\mathfrak{S}$ *is* strongly satisfactory *or simply* satisfactory $\stackrel{\text{def}}{\Longleftrightarrow}$ *there is a satisfaction relation for $\mathfrak{S}$.*

(1.61) **Definition** $[\mathsf{C}^0]$ *Suppose $\mathfrak{S}$ is a ρ-structure.*

1. *Suppose ϕ is a ρ-formula and A is an $\mathfrak{S}$-assignment for ϕ. Then $\mathfrak{S}$* satisfies ϕ at A $\overset{\text{def}}{\iff}$ $\mathfrak{S} \models \phi[A]$ $\overset{\text{def}}{\iff}$ *for every $\{\phi\}$-satisfaction relation S for $\mathfrak{S}$, $\langle \phi, A\rangle \in S$.*
2. *If ϕ is a sentence, i.e.,* $\operatorname{Free}\phi = 0$*, then* $\mathfrak{S} \models \phi \overset{\text{def}}{\iff} \mathfrak{S} \models \phi[0]$.
3. *If Θ is a class of ρ-sentences then $\mathfrak{S} \models \Theta \overset{\text{def}}{\iff} \forall\theta \in \Theta\ \mathfrak{S} \models \theta$. We also say that $\mathfrak{S}$* models *or* is a model of Θ.
4. *As a convenience, given a ρ-formula ϕ, a partial satisfaction relation S for some ρ-structure $\mathfrak{S}$, and an $\mathfrak{S}$-assignment A for ϕ, $\models^S \phi[A] \overset{\text{def}}{\iff} S$ is a partial satisfaction relation that covers $\{\phi\}$, and $\langle\phi, A\rangle \in S$ (in terms of valuation functions, $\langle\phi, A\rangle \in \operatorname{dom} S$ and $S\langle\phi, A\rangle = 1$).*

Following a convention similar to that for substitutions[1.51] *we may omit explicit mention of variables in the indication of assignments. Thus, '$\epsilon[a]$' may be used for '$\epsilon\left[{u \atop a}\right]$' if* $\operatorname{Free}\epsilon = \{u\}$*. Expressions like '$\epsilon[a_1, \ldots, a_n]$' may also be used.*

(1.62) *Assignment to variables may be indicated "in stream" in a manner analogous to substitution.*[1.52.2]

1. *Thus, for example,*
$$\operatorname{Val}^{\mathfrak{S}} \ulcorner \ldots [x] \ldots [y] \ldots \urcorner$$
and
$$\mathfrak{S} \models \ulcorner \ldots [x] \ldots [y] \ldots \urcorner$$
indicate the assignment of individuals $x, y \in |\mathfrak{S}|$ to variables implicitly present in the expressions—nominative and declarative, respectively—flanked by corner quotes.
2. *We may also indicate the structure as a superscript, so*
$$\ulcorner \ldots [x] \ldots [y] \ldots \urcorner^{\mathfrak{S}} \overset{\text{def}}{=} \operatorname{Val}^{\mathfrak{S}} \ulcorner \ldots [x] \ldots [y] \ldots \urcorner$$
for nominative expressions, and
$$\ulcorner \ldots [x] \ldots [y] \ldots \urcorner^{\mathfrak{S}} \overset{\text{def}}{\iff} \mathfrak{S} \models \ulcorner \ldots [x] \ldots [y] \ldots \urcorner$$
for declarative expressions.[41]

Definition 1.61 is a little peculiar in that it does not require the existence of a $\{\phi\}$-satisfaction relation in order that $\mathfrak{S} \models \phi[A]$—indeed, if there is no $\{\phi\}$-satisfaction relation then $\mathfrak{S} \models \phi[A]$ by default! It might seem more reasonable to substitute existential quantification over satisfaction relations for the universal quantification in (1.61.1). Note that by virtue of the uniqueness of satisfaction relations,[1.59] satisfaction in the latter sense would imply satisfaction in the sense we have defined. The two notions are equivalent to the extent that the existence of satisfaction relations is demonstrable. As we will see, this existence question, for proper class structures, is just at the boundary of provability, and the definition we have given is precisely calibrated to serve in this setting.

[41] The last form should only be used when a suitable partial satisfaction relation exists.

The following theorem shows that the fussiness of (1.61) is unnecessary for structures that are sets. The reader should be aware that the extension of the theory of satisfaction to proper class structures as defined above and used throughout this book is not standard in the literature. Its status and purpose are described at some length in the summary of this chapter and in the article [27].

(1.63) **Theorem** [C^0] *Every set structure is satisfactory.*

Proof This follows from our general theorem on definition by recursion.[3.80] It is convenient to frame the argument in terms of valuation functions. Suppose $\mathfrak{S}$ is a ρ-structure that is a set. We must exercise a little caution in view of the fact that without the axiom of infinity, the class of variables may be proper. Thus, we initially let X be the class of 2-sequences $\langle \phi, A\rangle$ where $\phi \in \mathcal{F}^\rho$ and $\operatorname{dom} A : \operatorname{Free}\phi \to |\mathfrak{S}|$. Define the relation R on X so that $\langle \psi, B\rangle R \langle \phi, A\rangle$ iff $\psi \in \operatorname{ImSubexpr}\phi$ and $\forall u \in \operatorname{Free}\psi \cap \operatorname{Free}\phi\ Bu = Au$. Note that $\operatorname{Free}\psi \subseteq \operatorname{Free}\phi$ unless $\phi = \hat{\varsigma} t$ where $\varsigma = \langle \mathrm{q}, u\rangle$ is a quantifier phrase,[1.34.5] in which case $\operatorname{Free}\psi \subseteq \operatorname{Free}\phi \cup \{u\}$. In the former case, if $\langle \psi, B\rangle R\langle \phi, A\rangle$ then $B = A \restriction \operatorname{Free}\psi$, whereas in the latter case, $B = A \cup \{(u, a)\}$ for some $a \in |\mathfrak{S}|$ if $u \in \operatorname{Free}\psi$, otherwise $B = A$. In any event, since ϕ has only finitely many (0, 1 or 2) immediate subexpressions, $R^{\leftarrow}\langle \phi, A\rangle$ is a set.

R is wellfounded because $\preccurlyeq$ is wellfounded,[1.42.3] and it is clearly irreflexive, so Theorem 3.80 applies, and it is straightforward to rework the definition[1.58] of the satisfaction relation on X to refer instead to a valuation function $F : X \to 2$.

Now we simply eliminate the restriction that $\operatorname{dom} A = \operatorname{Free}\phi$ for $\langle \phi, A\rangle \in X$ by letting $F\langle \phi, A\rangle = F\langle \phi, A \restriction \operatorname{Free}\phi\rangle$, where A is any $\mathfrak{S}$-assignment for ϕ. □[1.63]

Since languages are infinite, satisfaction relations are infinite, even for finite structures. If all sets are finite, therefore, satisfaction relations are proper classes. For this reason, (1.63) is not a theorem of S^0. Note, however, that if $\mathfrak{S}$ is an infinite set structure (and infinite sets therefore exist) then its satisfaction relation is also a set. We therefore have the following version of (1.63).

(1.64) **Theorem** [ZF] *Every structure is satisfactory.*

Given (1.64) and the well established adequacy of ZF as a metatheory, one may well ask whether our emphasis on proper class structures and the attendant issues of satisfactoriness is warranted. Rest assured, the small additional effort that is required to treat this topic now will be amply compensated in the chapters that follow.

(1.65) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure.*

1. *Suppose ϕ is an atomic ρ-formula. Then the $\{\phi\}$-satisfaction relation for $\mathfrak{S}$ exists.*
2. *Suppose ψ_0, ψ_1, ϕ are ρ-formulas, v is a variable, and the $\{\psi_0\}$- and $\{\psi_1\}$-satisfaction relations for $\mathfrak{S}$ exist. Then the $\{\phi\}$-satisfaction relation for $\mathfrak{S}$ exists if*
   1. *ϕ is a subformula of ψ_0;*
   2. *$\phi = \psi_0(T)$, where T is a substitution for (some or all) free variables of ψ_0;*
   3. *$\phi = \neg\,\psi_0$, $\psi_0 \vee \psi_1$, $\psi_0 \wedge \psi_1$, $\psi_0 \to \psi_1$, $\psi_0 \leftrightarrow \psi_1$, $\exists v\, \psi_0$, or $\forall v\, \phi_0$.*

Proof Straightforward. □[1.65]

We use (1.65) to show that the satisfaction predicate as defined in (1.61.1) has the expected dependence on syntax even for proper class structures, with the exception of the negation operation.

(1.66) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure and ϕ is a ρ-formula.*

1. *Suppose $\phi = \tilde{P}\langle\tau_0, \ldots, \tau_{n^-}\rangle$, where P is an n-ary ρ-predicate index and $\tau_0, \ldots, \tau_{n^-}$ are ρ-terms. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \Big\langle \operatorname{Val}^{\mathfrak{S}} \tau_0[A], \ldots, \operatorname{Val}^{\mathfrak{S}} \tau_{n^-}[A]\Big\rangle \in P^{\mathfrak{S}}.$$

2. *Suppose $\phi = \neg\,\psi$. If there exists a $\{\phi\}$-satisfaction relation for $\mathfrak{S}$ then there exists a $\{\psi\}$-satisfaction relation for $\mathfrak{S}$, and*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \mathfrak{S} \not\models \psi[A];$$

but if there does not exist a $\{\phi\}$-satisfaction relation for $\mathfrak{S}$ then there does not exist a $\{\psi\}$-satisfaction relation, and

$$\mathfrak{S} \models \phi[A] \wedge \mathfrak{S} \models \psi[A].$$

3. *Suppose $\phi = \psi_0 \wedge \psi_1$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \big(\mathfrak{S} \models \psi_0[A] \wedge \mathfrak{S} \models \psi_1[A]\big).$$

4. *Suppose $\phi = \psi_0 \vee \psi_1$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \big(\mathfrak{S} \models \psi_0[A] \vee \mathfrak{S} \models \psi_1[A]\big).$$

5. *Suppose $\phi = \psi_0 \rightarrow \psi_1$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \big(\mathfrak{S} \models \psi_0[A] \rightarrow \mathfrak{S} \models \psi_1[A]\big).$$

6. *Suppose $\phi = \psi_0 \leftrightarrow \psi_1$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \big(\mathfrak{S} \models \psi_0[A] \leftrightarrow \mathfrak{S} \models \psi_1[A]\big).$$

7. *Suppose $\phi = \exists v\,\psi$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \exists a \in |\mathfrak{S}|\ \mathfrak{S} \models \psi\big[A\big\langle {v \atop a} \big\rangle\big].$$

8. *Suppose $\phi = \forall v\,\psi$. Then*

$$\mathfrak{S} \models \phi[A] \leftrightarrow \forall a \in |\mathfrak{S}|\ \mathfrak{S} \models \psi\big[A\big\langle {v \atop a} \big\rangle\big].$$

Proof 1 The $\{\phi\}$-satisfaction relation for $\mathfrak{S}$ exists[1.65.1] and is defined[1.58.1.1] so as to make the conclusion true.

2–8 Using (1.65.2) to move up or down in the diagram of ϕ, we can show that partial satisfaction relations for $\mathfrak{S}$ either exist for all or exist for none of the relevant formulas. In the former case, these relations are defined[1.58.1] so as to make the conclusion true. In the latter case, all the satisfaction statements are true by default, which also makes the conclusion true. Note that only in the case of negation does the conclusion differ when satisfaction relations do not exist. □[1.66]

For any particular explicitly given formula, (1.65) can be used repeatedly to show in C^0 that the corresponding partial satisfaction relation exists for any appropriate structure. The same can be done for the entire class of formulas with some particular bound on their complexity. To formulate such a theorem it is useful to have a numerical definition of complexity, for which we will use the classification (1.37), where $\mathcal{E}^\rho_n$ is the class of expressions of height $< n$. Let $\mathcal{F}^\rho_n = \mathcal{F}^\rho \cap \mathcal{E}^\rho_n$.

(1.67) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure.*

1. *There exists an $\mathcal{E}^\rho_0$-valuation function for $\mathfrak{S}$.*
2. *For any $n \in \omega$, if there exists an $\mathcal{E}^\rho_n$-valuation function for $\mathfrak{S}$ then there exists an $\mathcal{E}^\rho_{n+1}$-valuation function for $\mathfrak{S}$.*

Proof 1 Trivial, as $\mathcal{E}^\rho_0$ is empty.

2 Suppose $n \in \omega$ and F is an $\mathcal{E}^\rho_n$-valuation function for $\mathfrak{S}$. Let F' be the function such that dom F' consists of all $\langle \epsilon, A \rangle$ such that $\epsilon \in \mathcal{E}^\rho_{n+1}$ and A is an $\mathfrak{S}$-assignment for ϵ, and for any $\langle \epsilon, A \rangle \in \operatorname{dom} F'$,[1.58]

1. if $\epsilon = \bar{v}$ then $F'\langle \epsilon, A \rangle = Av$;
2. if $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-} \rangle$, where X is an n-ary operation index, then
$$F'\langle \epsilon, A \rangle = X^{\mathfrak{S}}\langle F\langle \tau_0, A \rangle, \ldots, F\langle \tau_{n^-}, A \rangle \rangle;$$
3. if $\epsilon \in \mathcal{F}^\rho$ then $F\langle \epsilon, A \rangle \in 2$;
4. if $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-} \rangle$, where X is an n-ary predicate index, then
$$F'\langle \phi, A \rangle = 1 \leftrightarrow \big\langle F\langle \tau_0, A \rangle, \ldots, F\langle \tau_{n^-}, A \rangle \big\rangle \in X^{\mathfrak{S}};$$
5. if $\epsilon = \neg\, \psi$ then $F'\langle \epsilon, A \rangle = 1 \leftrightarrow F\langle \psi, A \rangle = 0$;
6. if $\epsilon = \psi_0 \wedge \psi_1$ then $F'\langle \epsilon, A \rangle = 1 \leftrightarrow F\langle \psi_0, A \rangle = 1 \wedge F\langle \psi_1, A \rangle = 1$;
7. if $\epsilon = \psi_0 \vee \psi_1$ then $\langle \epsilon, A \rangle \in F' \leftrightarrow \langle \psi_0, A \rangle \in F \vee \langle \psi_1, A \rangle \in F$;
8. if $\epsilon = \psi_0 \rightarrow \psi_1$ then $\langle \epsilon, A \rangle \in F' \leftrightarrow (\langle \psi_0, A \rangle \in F \rightarrow \langle \psi_1, A \rangle \in F)$;
9. if $\epsilon = \psi_0 \leftrightarrow \psi_1$ then $\langle \epsilon, A \rangle \in F' \leftrightarrow (\langle \psi_0, A \rangle \in F \leftrightarrow \langle \psi_1, A \rangle \in F)$;
10. if $\epsilon = \exists v\, \psi$ then $\langle \epsilon, A \rangle \in F' \leftrightarrow \exists a \in |\mathfrak{S}|\ \big\langle \psi, A\big\langle {v \atop a} \big\rangle \big\rangle \in F$; and
11. if $\epsilon = \forall v\, \psi$ then $\langle \epsilon, A \rangle \in F' \leftrightarrow \forall a \in |\mathfrak{S}|\ \big\langle \psi, A\big\langle {v \atop a} \big\rangle \big\rangle \in F$.

Clearly, F' is an $\mathcal{E}^{\rho}_{n+1}$-valuation function for $\mathfrak{S}$. $\square^{1.67}$

One might think to use (1.67) to prove ⌜for any signature ρ, ρ-structure $\mathfrak{S}$, and $n \in \omega$, there exists an $\mathcal{E}^{\rho}_{n}$-valuation function for $\mathfrak{S}$⌝, using induction. For this we would have to show that there exists a class $C \subseteq \omega$ such that $n \in C$ iff there exists an $\mathcal{E}^{\rho}_{n}$-valuation function for $\mathfrak{S}$. If $\mathfrak{S}$ is a proper class, then partial valuation functions for $\mathfrak{S}$ are proper classes, so the comprehension axiom schema of C^0—which is restricted to formulas involving quantification over sets—does not permit us to conclude the existence of C.

1.4.3 Semantics of substitution and change of variables

The proofs of the following two theorems are entirely straightforward.

(1.68) **Theorem** $[\mathsf{C}^0]$ *Suppose ϵ is a ρ-expression and τ is a ρ-term. Suppose* $\mathrm{Free}\,\tau \subseteq \mathrm{Free}\,\epsilon\binom{v}{\tau}$.[42] *Suppose $\mathfrak{S}$ is a ρ-structure and A is an $\mathfrak{S}$-assignment for* $\epsilon\binom{v}{\tau}$. *Let* $t = \mathrm{Val}^{\mathfrak{S}}\,\tau[A]$. *Then*

1. *if ϵ is a term,*
$$\mathrm{Val}^{\mathfrak{S}}\,\epsilon\tbinom{v}{\tau}[A] = \mathrm{Val}^{\mathfrak{S}}\,\epsilon[A]\left[\begin{smallmatrix}v\\t\end{smallmatrix}\right];$$
2. *if ϵ is a formula,*
$$\mathfrak{S} \models \epsilon\tbinom{v}{\tau}[A] \rightarrow \mathfrak{S} \models \epsilon[A]\left[\begin{smallmatrix}v\\t\end{smallmatrix}\right].$$[43]

Theorem $[\mathsf{C}^0]$ *Suppose ϕ is a ρ-formula and ϕ' results from ϕ by a change of variables. Suppose $\mathfrak{S}$ is a ρ-structure and A is an $\mathfrak{S}$-assignment for ϕ. Then*[44]

$$\mathfrak{S} \models \phi'[A] \leftrightarrow \mathfrak{S} \models \phi[A].$$

1.4.4 Theories, satisfiability, and entailment

Recall the definition[1.29.3] of *expansion* and *compatibility* of signatures.

Definition $[\mathsf{C}^0]$ *Suppose $\mathfrak{S}$ is a ρ-structure. An* expansion of $\mathfrak{S}$ *is a ρ'-structure $\mathfrak{S}'$ such that*

1. *ρ' is an expansion of ρ;*
2. $|\mathfrak{S}'| = |\mathfrak{S}|$*; and*
3. *for every ρ-index X,* $X^{\mathfrak{S}'} = X^{\mathfrak{S}}$.

Definition $[\mathsf{C}^0]$ *Suppose ρ is a signature.*

1. *A ρ-*theory *is a class of ρ-sentences.*

---

[42] I.e., no variable occurrence in τ is bound in $\epsilon\binom{v}{\tau}$.[1.55]

[43] We do not assert a bi-implication in (1.68.2) because the existence of a $\{\epsilon\binom{v}{\tau}\}$-satisfaction relation does not directly imply the existence of a $\{\epsilon\}$-satisfaction relation. On the assumption that $\mathfrak{S}$ is weakly satisfactory,[1.60.1] of course, the bi-implication is demonstrable.

[44] Recall[1.54] that ϕ and ϕ' have the same free variables.

2. *Suppose* $\mathfrak{S}$ *is a satisfactory* ρ*-structure.* $\mathrm{Th}\,\mathfrak{S} \overset{\mathrm{def}}{=}$ *the class of* ρ*-sentences* σ *such that* $\mathfrak{S} \models \sigma$*.*

Note that if $\mathfrak{S}$ is only weakly satisfactory, one would have to quantify over partial satisfaction relations to define $\mathrm{Th}\,\mathfrak{S}$, which is not permitted in C^0 (since $\mathfrak{S}$ is necessarily a proper class in this case).

Suppose ρ is a signature and ϵ is a ρ-expression. Then ϵ is also a ρ'-expression for any compatible signature ρ' that contains the indices that occur in ϵ. Similarly, a theory Θ is a ρ-theory for any signature ρ that has the indices that occur in Θ. We will say that a signature that has the indices that occur in an expression or class of expressions is *appropriate* to it. The question naturally arises whether a theory Θ that has a weakly satisfactory or satisfactory ρ-model for some appropriate ρ has a respectively weakly satisfactory or satisfactory ρ-model for every appropriate ρ.[45] The following rather prosaic theorem answers this question in the affirmative and generally clarifies semantic issues having to do with multiple or unspecified signatures.

(1.69) **Theorem** $[\mathsf{C}^0]$ *Suppose* $\mathfrak{S}$ *is a (weakly) satisfactory* ρ*-structure and* ρ' *is an expansion of* ρ*. Then there is a (weakly) satisfactory* ρ'*-structure* $\mathfrak{S}'$ *that is an expansion of* $\mathfrak{S}$*.*

Remark We will call a ρ'-index *new* iff it is not a ρ-index. To expand $\mathfrak{S}$ to a ρ'-structure one simply assigns relations and operations on $|\mathfrak{S}|$ to the new indices X of ρ'. We have to show that the structure $\mathfrak{S}'$ so defined is (weakly) satisfactory. If $|\mathfrak{S}|$ is a proper class this does not follow for an arbitrary choice of denotations, but by choosing carefully we can arrange that the Φ'-satisfaction relation for $\mathfrak{S}'$ for any $\Phi' \subseteq \mathcal{F}^{\rho'}$ is definable from the Φ-satisfaction relation for $\mathfrak{S}$ for an appropriate $\Phi \subseteq \mathcal{F}^{\rho}$.

Proof See Note 10.2. $\square^{1.69}$

Given a theory Θ in some signature, let ρ be the subtype that has just the indices occurring in Θ. It follows from Theorem 1.69 that a theory Θ has a (weakly) satisfactory ρ-model iff it has a (weakly) satisfactory ρ'-model, where ρ' is any expansion of ρ. Hence

(1.70) *if* ρ' *and* ρ'' *are signatures appropriate to* Θ *then there is a (weakly) satisfactory* ρ'*-model of* Θ *iff there is a (weakly) satisfactory* ρ*-model of* Θ *iff there is a (weakly) satisfactory* ρ''*-model of* Θ*.*

Keeping this in mind we make the following definition.

(1.71) **Definition** $[\mathsf{C}^0]$

1. *A theory* Θ *is* satisfiable $\overset{\mathrm{def}}{\iff}$ *some satisfactory structure satisfies* Θ*.*
2. *A theory* Θ entails *a sentence* σ $\overset{\mathrm{def}}{\iff}$ *every for every satisfactory structure* $\mathfrak{S}$*, if* $\mathfrak{S} \models \Theta$ *then* $\mathfrak{S} \models \sigma$*.*

Note that Θ entails σ iff $\Theta \cup \{\neg\,\sigma\}$ is not satisfiable. Note also that Θ is not satisfiable iff Θ entails every sentence.

[45] The corresponding question without the qualification of (weak) satisfactoriness is trivially answered in the affirmative.

(1.72) *By virtue of (1.70) the notions of satisfiability and entailment may be defined—as we have done—without reference to a specific signature.*

Note that we have not defined a notion of "weak satisfiability" by substituting weak satisfactoriness for satisfactoriness in (1.71). We will see[2.37] that this notion is equivalent to satisfiability.

1.4.4.1 Empty models

Our decision[1.30.1] to exclude empty structures from consideration now becomes pertinent. It is instructive to consider how the inclusion of empty structures changes the meaning of satisfiability and entailment. Let us therefore, temporarily, permit structures to be empty. The signature ρ of an empty structure $\mathfrak{S}$ can have no nulary operation indices, and for each n-ary ρ-predicate index R, $R^{\mathfrak{S}} = 0$ if $n > 0$, since ${}^n 0 = 0$ in this case. If R is a nulary predicate symbol then $R^{\mathfrak{S}}$ may be 0 or 1, since ${}^0 x = \{0\}$ for any class x. It is easy to show in C^0 that a satisfaction relation S exists for any empty ρ-structure $\mathfrak{S}$ (since $|\mathfrak{S}|$ is a set; that ρ may be a proper class is irrelevant).

Suppose Θ is a ρ-theory and Θ has exactly one model, which is empty. For example, $\Theta = \{\forall v\, (\neg(\phi \rightarrow \phi))\}$, where ϕ is a ρ-formula with $\mathrm{Free}\, \phi \subseteq \{v\}$. Then if we allow structures to be empty, Θ is satisfiable; otherwise it is not. There are corresponding effects on the relation of entailment. We will address the issue of empty structures again when we develop the syntactical equivalent of entailment, i.e., deducibility, as we prove the completeness theorem.

1.4.4.2 Decidability

The most fundamental problem of logic is to decide when a given sentence is entailed by a given theory. In Chapter 2 we will show that a theory Θ entails a sentence σ iff there exists a proof of σ from Θ. We have yet to define 'proof', but once we have, we will see that the class of proofs is effectively enumerable. As long as Θ is also effectively enumerable, the class of sentences entailed by Θ is effectively enumerable—given an unbounded memory capacity, we could program a computer to generate all proofs from Θ, and we would thereby generate a list of exactly the sentences entailed by Θ.

If we could similarly generate a list of exactly the sentences *not* entailed by Θ, then then the decision problem for Θ would be solved. To determine whether a given sentence σ is entailed by Θ, we could just generate both lists. σ would have to appear on exactly one of them, and as soon as it appeared we would know whether it was entailed by Θ. Alas!—or, if you prefer, Whew!—there is in general no algorithm, i.e., effective procedure, for generating the list of sentences *not* entailed by a theory Θ, in particular the empty theory 0. Hence there is no algorithm for deciding in general whether a sentence is entailed by a theory. This is the *undecidability* of predicate logic, which is proved in Chapter 4.

1.5 Undefinability of satisfaction

What is truth?

Pontius Pilate, in *The Gospel* according to John 18:38

NB: In this section, as in other discussions of undefinability, unprovability, etc., what is meant is un· in particular systems, not absolute un·. An exception is uncomputability, as there is only one system of computation (up to equivalence).

As mentioned following the proof of Theorem 1.67, if $\mathfrak{S}$ is a proper class the proof of Theorem 1.63 breaks down because the dependency relation R for formula-assignment pairs as defined there is not setlike, inasmuch as one needs to know $F\big\langle\psi, A\big\langle{u \atop a}\big\rangle\big\rangle$ for all $a \in |\mathfrak{S}|$ to determine $F\langle\exists u\,\psi, A\rangle$ (or $F\langle\forall u\,\psi, A\rangle$), where F is the valuation function for $\mathfrak{S}$.

In the absence of an axiom of infinity this is a significant limitation, as infinite structures are an essential part of the semantics of predicate logic, and these are proper classes if there are no infinite sets. In the presence of an axiom of infinity the problem is much less severe, as set structures are sufficient for most purposes.[46]

One may wonder whether by some other means we might demonstrate the existence of satisfaction relations for proper class structures, but we will show that there are limitations on the demonstrability of the existence of satisfaction relations, which are related to limitations on the definability of satisfaction—on the definability of truth, if you will. In general, in the case of a ρ-structure $\mathfrak{S}$ within which language can be modeled, the satisfaction relation for $\mathfrak{S}$ is not definable by a ρ-formula interpreted in $\mathfrak{S}$. The following theorem states this for the paradigmatic case of the structure within which we have modeled language in this chapter, viz., $\mathfrak{V}_\omega = (V_\omega; \in)$, the hereditarily finite sets with the membership relation. The signature of $\mathfrak{V}_\omega$ is s, with two predicate indices, for identity and membership. The following theorem is due to Kurt Gödel and to Alfred Tarski independently. The first published statement and proof were given by Tarski in 1936[25].

(1.73) **Theorem** [C^0] *There does not exist a satisfaction relation for $\mathfrak{V}_\omega$ that is definable over $\mathfrak{V}_\omega$. That is to say, it is not the case that there exists a satisfaction relation S for $\mathfrak{V}_\omega$ and an s-formula φ with two free variables, u and v, such that for every s-formula ψ with one free variable w and every $a \in V_\omega$, letting $A = \big\langle{w \atop a}\big\rangle$,*

(1.74) $$\mathfrak{V}_\omega \models \varphi\big[{u \atop \psi}\ {v \atop A}\big] \leftrightarrow \mathfrak{V}_\omega \models \psi\big[{w \atop a}\big].$$ [47]

Proof Suppose toward a contradiction that there is a satisfaction relation for $\mathfrak{V}_\omega$ and an s-formula φ such that (1.74) holds. By a simple modification of φ we obtain an s-formula φ' with free variables u and v such that for every s-formula ψ with one free variable w and every $a \in V_\omega$,

(1.75) $$\mathfrak{V}_\omega \models \varphi'\big[{u \atop \psi}\ {v \atop a}\big] \leftrightarrow \mathfrak{V}_\omega \models \psi\big[{w \atop a}\big].$$ [48]

[46] Inner models of set theory, which are by definition proper classes, are an exception, and it is mainly in this context that the comment following (1.62) is significant.

[47] Recall that by definition,[1.61.1]

for every s-formula θ and $\mathfrak{V}_\omega$-assignment B for θ, $\mathfrak{V}_\omega \models \theta[B]$ iff for every *$\{\theta\}$-satisfaction relation S for $\mathfrak{V}_\omega$, $\langle\theta, B\rangle \in S$.*

Since we are assuming that a full satisfaction relation S exists for $\mathfrak{V}_\omega$,

for every s-formula θ and $\mathfrak{V}_\omega$-assignment B for θ, $\mathfrak{V}_\omega \models \theta[B]$ iff $\langle\theta, B\rangle \in S$.

so all occurrences of '$\models$' in this context are interpretable with reference to the single relation S. In particular,

$$\mathfrak{V}_\omega \models \neg\,\theta[B] \leftrightarrow \neg\,\mathfrak{V}_\omega \models \theta[B],$$

which we use to establish (1.76).

[48] Where φ refers to an assignment $\big\langle{w \atop a}\big\rangle$, φ' refers to a directly.

Let $\psi = \neg\,\varphi'\binom{u\ v}{\bar{w}\ \bar{w}}$. Then

$$\mathfrak{V}_\omega \models \varphi'\begin{bmatrix} u & v \\ \psi & \psi \end{bmatrix} \leftrightarrow \mathfrak{V}_\omega \models \psi\begin{bmatrix} w \\ \psi \end{bmatrix} \leftrightarrow \mathfrak{V}_\omega \models \neg\,\varphi'\begin{bmatrix} u & v \\ \psi & \psi \end{bmatrix} \leftrightarrow \neg\,\mathfrak{V}_\omega \models \varphi'\begin{bmatrix} u & v \\ \psi & \psi \end{bmatrix}, \tag{1.76}$$

a contradiction. $\square$[1.73]

This proof is our first use of a so-called *diagonal argument*. If we think of u and v as "coordinates" in a 2-dimensional space, then ϕ' is a ($\{0,1\}$-valued) function on this space, and ψ is, in effect, the complement of this function along the "diagonal" of this space, since $\psi(w) = \neg\,\phi'(w,w)$ (using informal notation).

1.6 Summary

Formal language and structure are the obverse and reverse of the coin of the realm of mathematics. We began this chapter with a relatively informal description of the sort of formal language that has been found by experience to be both necessary and sufficient for the discussion of all things mathematical.[1.2] The formality of language required to achieve the rigor and certainty that characterize mathematics also defines formal language as a suitable object of mathematical analysis. Most concretely conceived, a formal language consists of *expressions* that have a precise *form*, constructed according to rules that constitute its grammar or syntax. The purpose of these expressions is to convey meaning via *interpretations*. The dependence of the meaning of expressions on their form is the *semantics* of the language. *First-order predicate language* is the simplest syntactical/semantical system that is sufficiently expressive for general use, and it is the system we have described in this chapter.

The first-order predicate system is characterized by fixed rules of *propositional connection*, *quantification* over *variables*, and *specification of arguments*. A *language* in this system is characterized by its *signature*, which consists of one or more *predicate* or *operation indices*, together with information as to the (finite) number of arguments each takes. An inessential generalization allows for *domain* indices as well. In a typographical realization of a language these indices are represented by symbols, and 'symbol' is used synonymously with 'index' for this reason.

The use of 'predicate' to refer to this system is a convenience that does not indicate any prejudice against operations. The use of 'first-order' is an historical convention, and it would be quite reasonable to replace 'first-order predicate' by 'predicate', as we often will.[49]

Although 'interpretation' may be understood in an informal way, for all practical purposes it is synonymous with 'structure', which has a specific membership-theoretical definition. A structure consists of a class of things—its *universe* of *individuals*—together with relations and functions on individuals corresponding respectively to the predicate and operation indices of its signature.[50] Note that a structure interprets a language just in case they have the same signature.

[49] If we omit quantification we have the *propositional* system, which may be regarded as having order 0, but it also has no predicates, operations, or domains. The predicates of a first-order language have a fixed meaning under a given interpretation. Higher-order predicate systems may be obtained by allowing variable predicates that may serve as arguments of higher-order predicates and are themselves subject to quantification; but these are only secondarily of interest. Indeed, the discussion of such systems, like all mathematical discussions, is done in the usual way: using first-order predicate language.

[50] 'predicate' and 'operation' are often used synonymously with 'relation' and 'function', respectively, but for clarity we preferentially restrict the former to their use in reference to language.

An expression is either a *term* or a *formula*. Given a signature ρ and a ρ-structure $\mathfrak{S}$, an $\mathfrak{S}$-*assignment* for ϵ assigns an individual in $|\mathfrak{S}|$ to each of its free variables, whereupon the expression acquires a value. This map is the *valuation function* for $\mathfrak{S}$. In the case of a term τ with an assignment A, the value $\mathrm{Val}^{\mathfrak{S}}\,\tau[A]$ is a member of $|\mathfrak{S}|$. A formula ϕ, on the other hand, is either satisfied or not at an assignment A, i.e., it is either true or false, and we regard $\mathrm{Val}^{\mathfrak{S}}\,\phi[A]$ as either 1 or 0, respectively. A valuation operation Val for formulas is equivalent to a *satisfaction predicate* $\models$, and we define $\mathfrak{S} \models \phi[A] \overset{\text{def}}{\iff} \mathrm{Val}^{\mathfrak{S}}\,\phi[A] = 1$.

The definition[1.61] of $\models$ used in this book necessarily deviates from standard practice because of our decision to permit the universe of a structure to be an arbitrary class, rather than requiring it to be a set. At this point this decision appears to do nothing more than introduce a gratuitous complication in the interest of a trivial generality, but it is actually motivated by the desire to render the rest of the story significantly more accessible to the general reader, who is not necessarily expected to become a specialist in this field.

By way of explanation, we note that, as discussed in Chapter 3, there are various theories of membership, which differ in their treatment of elements, classes, and sets. By definition, an element is anything that is a member of a class, a class is anything that is a collection of elements (possibly empty), and set is something that is both an element and a class, i.e., a class that is a member of a class. A proper element is an element that is not a class, and a proper class is a class that is not an element. Although it is quite reasonable to allow the individuals of a structure to be proper elements—indeed, it would appear to be quite unnatural to disallow this—in practice, nothing essential is lost by excluding proper elements from the discussion, and we typically do so. In fact, all of mathematics may be formalized in the theory of membership with proper elements excluded.

It is also common practice to exclude proper classes, but this exclusion comes at a cost. Certain essential ideas that are easy to state in the terminology of a class theory require *ad hoc* arrangements for their formal statement in a pure set theory. As a result, proper classes are often referred to informally with the understanding that they can be eliminated from the discussion. This is acceptable to the specialist, who will either perform the requisite elimination or rest easy in the knowledge gained from experience that it could be done if desired. In our opinion, this is too great a burden to place on someone who does not wish to gain this level of mastery.

The advantage of our approach can be seen already in this chapter, for which our metatheory is C^0, which asserts the existence of any class whose membership is defined by a formula in which all quantified variables range over elements. The corresponding pure set theory is S^0. Neither theory posits the existence of infinite sets, but C^0 allows proper classes, which are necessarily infinite. Theorem 2.183 asserts that any statement about sets that is provable in C^0 is provable in S^0: essentially, any C^0-proof of a sentence of pure set theory may be replaced by a S^0-proof in which all references to proper classes have been replaced by purely set-theoretical statements involving their defining formulas. Thus, C^0 provides a convenient framework in which to refer to infinite objects such as languages and other definable structures in an essentially finitary way.

This is not a sterile exercise in minimality: for example, it renders Gödel's celebrated second incompleteness theorem an easy corollary of (the existence of the proof of) the first incompleteness theorem in Chapter 4. As will become apparent in later chapters, the use of proper class structures with the definition (1.61) of the satisfaction predicate has additional benefits that significantly outweigh its initial cost. Specialists in the field should also find it advantageous.

Apart from this, the concept of infinitarity, by which we mean the attitude that infinite objects may be supposed to exist in the fullest sense, as constituents of other objects—as elements, in short—is the great ontological threshold of mathematics; and it is essential to an understanding of the foundations of mathematics to know when this threshold must be crossed.

The definition of $\models$ for structures that may not be sets is done in terms of *partial satisfaction relations*. We prove an S^0-theorem asserting the existence of a C^0-proof of the existence of partial satisfaction relations for all structures for any explicitly given formula. We also define *satisfactory* and *weakly satisfactory* structures in terms of the existence of partial satisfaction relations for arbitrary formulas. The reader who takes the time to fully understand these distinctions will be well on the way to an appreciation of the subtleties that distinguish the mathematics of mathematics (metamathematics) from the mathematics of everything else.

We define entailment: a theory Θ *entails* a sentence σ iff every satisfactory structure that satisfies Θ satisfies σ. This is the heart of the mathematical enterprise: does Θ entail σ? The standard way to a positive answer is of course to prove σ from Θ, and it is easy to see that this is a legitimate method. But entailment is a semantic notion, having to do with the meaning of sentences in structures, and it refers to all (satisfactory) structures, whereas proof is a syntactical notion, having to do with manipulation of expressions according to formal rules of deduction. Is it possible that all sentences entailed by a theory are provable from it? This fundamental question is answered in Chapter 2, along with many others.

We conclude with the first of the celebrated "negative" results that make the formal theory of logic so intriguing to the general public:[1.73] Suppose $\mathfrak{S}$ is a structure that is capable of defining the basic notions of syntax and operations on expressions. Then $\mathfrak{S}$ is not capable of defining its own satisfaction relation. Put simply, we demonstrate the undefinability of truth. Again, it is critical that the reader fully appreciate the meaning and significance of this result; there will be many more such to follow.

Chapter 2

Logic

Contents

2.1 Deduction, soundness and completeness **64**
2.2 The completeness theorem **66**
2.2.1 The Henkin procedure 67
2.2.2 The model 69
2.2.3 Validation of the model 69
2.2.4 A system of deduction 72
2.2.5 Conclusion 74
2.3 Logic **80**
2.3.1 Inference, implication, validity 81
2.3.2 Proof trees 84
2.3.3 Propositional logic 86
2.3.4 Truth tables 88
2.3.5 Propositional logic in S^0 89
2.3.6 Another deductive system 90
2.3.7 Universal quantification 91
2.3.8 Universal closure 94
2.3.9 Logical equivalence 94
2.3.10 Disjunction and conjunction 95
2.3.11 Identity 97
2.3.12 Substitution of equivalents 100
2.3.13 Normal forms 101
2.4 Theories **102**
2.4.1 Extensions of theories 103
2.4.2 Herbrand's theorem 104
2.4.3 Skolemization 106
2.4.4 Logic with identity conservatively extends logic without 109
2.4.5 Extension by definition 110
2.4.6 Relativization 112
2.4.7 Substructure 112
2.4.8 Interpretations in languages and theories 113
2.5 Example: theories and models of geometry **113**
2.5.1 Analytic projective geometry: modules over division rings 115

2.5.2 Synthetic projective geometry: axiomatic systems . . . 117
2.5.3 Duality . . . 119
2.5.4 Solid projective geometry . . . 120
2.5.5 Desargues's principle . . . 122
2.5.6 Models of the axioms . . . 125
2.5.7 Pappus's principle . . . 130
2.5.8 Duality: synthetic and analytic . . . 132
2.5.9 Projectivity and the fundamental theorem . . . 133
2.6 Other deductive systems . . . 138
2.6.1 Hilbert systems . . . 139
2.6.2 Gentzen systems . . . 139
2.6.3 Cut-elimination and the subformula property . . . 142
2.6.4 Completeness of **LK** and $\mathbf{LK}^-$ . . . 142
2.7 Model theory . . . 145
2.7.1 Completeness . . . 145
2.7.2 Elementary substructures and embeddings . . . 146
2.7.3 Elementary directed families . . . 148
2.7.4 Löwenheim-Skolem theorems . . . 149
2.7.5 Ultraproducts . . . 152
2.7.6 Ultrapowers of proper classes . . . 154
2.8 Satisfactoriness and logic . . . 155
2.8.1 Satisfaction of logical validities . . . 156
2.9 A comparison of metatheories . . . 157
2.9.1 Predicative class theory conservatively extends pure set theory . . . 158
2.9.2 Peano arithmetic is equi-interpretable with S^0 . . . 161
2.10 Summary . . . 163

I was at the mathematical school, where the master taught his pupils after a method scarce imaginable to us in Europe. The proposition, and demonstration, were fairly written on a thin wafer, with ink composed of a cephalic tincture. This, the student was to swallow upon a fasting stomach, and for three days following, eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success has not hitherto been answerable, partly by some error in the quantum or composition, and partly by the perverseness of lads, to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards, before it can operate; neither have they been yet persuaded to use so long an abstinence, as the prescription requires.

Gulliver's Travels by Jonathan Swift

2.1 Deduction, soundness and completeness

Our primary goal in this chapter is to define a system of deduction so that any sentence that is entailed by a theory Θ is provable from Θ. Remember[1.71] that a sentence σ is said to be *entailed* by Θ iff for every satisfactory structure $\mathfrak{S}$, if $\mathfrak{S} \models \Theta$ then $\mathfrak{S} \models \sigma$. s On the other hand, σ is said to be *provable* from Θ just in case there exists a *proof* of σ all of whose premises are in Θ. We represent this by '$\Theta \vdash \sigma$', keeping in mind that the notion of proof, and with it the relation $\vdash$, is yet to be defined.

Definition [C^0] *A system of deduction is* complete $\overset{\text{def}}{\Longleftrightarrow}$ *any sentence entailed by a theory Θ is provable from Θ.*

A system of deduction is just a set of rules for generating conclusions from premises. Thus, the notions of proof and provability depend on the particular system of deduction we choose. An easy way to achieve completeness of a deductive system would be to design it so that anything could be proven. Such a system would be of no value and of no interest, because it lacks the property of *soundness*, i.e., the property that any sentence provable from a theory Θ is entailed by Θ. We state this formally as the first of several conditions that a suitable deductive system must satisfy.

(2.1) **Condition 0: Soundness** *For any theory Θ and sentence σ, if $\Theta \vdash \sigma$ then Θ entails σ.*[1]

Now suppose we have two systems of deduction that are both sound and complete. Then for any theory Θ, the sentences provable from Θ are exactly those entailed by Θ, whichever notion of provability (i.e., whichever system of deduction) we use. Thus, once we come up with one system of deduction that is both sound and complete, we know that any other such system is equivalent to it. From that point on, we are justified in using the notion of provability without reference to any specific system of deduction.

Another way to view the completeness theorem is from the standpoint of *consistency*.

Definition [C^0] *A theory Θ is* consistent *(*vis-à-vis *a given system of deduction)* $\overset{\text{def}}{\Longleftrightarrow}$ *for every sentence σ, one does not have both $\Theta \vdash \sigma$ and $\Theta \vdash \neg\sigma$.*

(2.2) **Theorem** [C^0] *If a given system of deduction is complete then*

(2.3) *every consistent theory is satisfiable.*

Proof Suppose a theory Θ is not satisfiable. Then vacuously every satisfactory model of Θ satisfies every sentence. By completeness, every sentence is provable from Θ, so Θ is inconsistent. $\square^{2.2}$

Property (2.3) of a deductive system does not, however, imply that the system is complete. For example, let $\vdash'$ be defined as follows.

$$\Theta \vdash' \sigma \overset{\text{def}}{\Longleftrightarrow} \Theta \text{ is not satisfiable.}$$

$\vdash'$ is sound and satisfies (2.3), but it is not complete. The proof is left as an exercise.

What's lacking is the following.

(2.4) **Condition 1: Reductio ad absurdum** *If $\Theta \cup \{\neg\sigma\}$ is inconsistent then $\Theta \vdash \sigma$.*

[1] Recall[1.72] that the notion of entailment is independent of signature, so we already have here an indication that the notion of provability is also absolute in this sense.

(2.5) *A deductive system satisfying Condition 1*[2.4] *that also satisfies (2.3) is complete.*

The proof is left as an exercise.

Since we are ultimately going to show that 'proves' and 'entails' are equivalent, each of the conditions we impose on our deductive system must be satisfied by the entailment relation. In fact the final proof of soundness of the deductive system amounts to demonstrating that this is so. It is a useful exercise to check this for each condition we impose. For Condition 0,[2.1] of course, replacing '$\vdash$' by 'entails' yields a tautology. We note that replacing '$\vdash$' by 'entails' in the definition of 'inconsistent' yields 'unsatisfiable', so Condition 1[2.4] becomes

(2.6) *'if $\Theta \cup \{\neg\,\sigma\}$ is unsatisfiable then every satisfactory model of Θ satisfies σ'.*

A satisfactory model $\mathfrak{S}$ of Θ, by definition, has a satisfaction relation S, and necessarily either $\langle \sigma, 0\rangle \in S$ or $\langle \neg\,\sigma, 0\rangle \in S$. If $\Theta \cup \{\neg\,\sigma\}$ is unsatisfiable, then $\langle \neg\,\sigma, 0\rangle \notin S$, so $\langle \sigma, 0\rangle \in S$. Hence (2.6) is true.

We will now suppose that we have a deductive system satisfying Conditions 0 and 1 and a theory Θ that is consistent according to this system, and attempt to construct a model of Θ. In the process we will discover several additional conditions our system must satisfy in order to guarantee the success of this endeavor. Once we have a sufficient set of conditions, we will define a deductive system based on rules that guarantee these conditions. As discussed above, the conditions we impose on deducibility will be true for entailment, and the rules we posit will not permit any inferences beyond those required by completeness, so that the system they define has Condition 0 also, i.e., it is sound. This system will, in fact, be the natural system that we use in daily mathematical practice. The completeness theorem will then be the assertion that this system of deduction is complete, and we will already have given the proof.

2.2 The completeness theorem

(2.7) *In the interest of simplicity we will prove the completeness theorem first for languages that have only countably many indices, with no domain indices, that do not contain identity, that have only the logical connectives '$\neg$' and '$\rightarrow$', and that have only the existential quantifier.*

All the important ideas are illustrated by this case. As noted above,[§ 1.1.11] it is straightforward to formulate any theory in such a reduced language, with the exception that the elimination of operations can only be accomplished in languages with identity. We will extend our results to languages with identity in due course.[§ 2.3.11]

As described above, we assume our deductive system satisfies Conditions 0 (soundness) and 1 (*reductio ad absurdum*), and given a consistent theory Θ_0, we wish to construct a countable satisfactory structure $\mathfrak{M}$ such that $\mathfrak{M} \models \Theta_0$. To see where we're heading, suppose that $\mathfrak{M} \models \Theta_0$, and let $\Theta' = \mathrm{Th}\,\mathfrak{M}$, the class of sentences satisfied by $\mathfrak{M}$. We first observe that Θ' is consistent. This is because we have assumed that our system of deduction is sound, so any sentence it can prove from Θ' is true in $\mathfrak{M}$; hence, we cannot prove both σ and $\neg\,\sigma$ from Θ' for any σ.

Θ' is also *maximal*:

Definition [C^0] *A consistent ρ-theory Θ is* maximal *iff for any ρ-sentence σ, either $\sigma \in \Theta$ or $\neg\,\sigma \in \Theta$.*

Maximal consistent theories are called 'maximal' because they are not included in any larger consistent theory. Any larger theory would contain both σ and $\neg\sigma$ for some σ and would therefore be inconsistent. Actually, in making this assertion we're assuming something about our system of deduction, viz., that every sentence of a theory Θ is provable from Θ. This seems so obvious that it might be overlooked, but it has to be stated explicitly as a condition on our deductive system, and we will state it in due course as Condition 6.[2.18]

So in constructing a model of a consistent theory Θ_0, we are necessarily also constructing a maximal consistent extension of Θ_0. This suggests that—as a step toward the construction of a model of Θ_0—we first generate a maximal consistent extension.

2.2.1 The Henkin procedure

The following procedure is a general outline for obtaining a maximal consistent extension of a theory Θ_0. We will modify it in due course.

(2.8) *Suppose ρ is a countable signature, Θ_0 is a ρ-theory, and ρ has infinitely many constants, i.e., nulary operation indices, that do not occur in Θ_0. Let $\langle \sigma_0, \sigma_1, \dots \rangle$ be an enumeration of the ρ-sentences. Let $\langle \theta_n \mid n \in \omega \rangle$ be the sequence of ρ-sentences defined by the condition that for each $n \in \omega$, letting $\Theta_n = \Theta_0 \cup \{\theta_0, \dots, \theta_{n^-}\}$,*

$$\theta_n = \begin{cases} \sigma_n & \text{if } \Theta_n \cup \{\sigma_n\} \text{ is consistent} \\ \neg\sigma_n & \text{otherwise.} \end{cases}$$

Let $\Theta_\omega = \bigcup_{n\in\omega} \Theta_n$.

It will follow from the definition of our system of deduction that $\Theta_n \cup \{\sigma_n\} = \Theta_0 \cup \{\theta_0, \dots, \theta_{n^-}, \sigma_n\}$ is consistent iff there does not exist a proof of inconsistency from it, and that a proof is a set. Thus, the map $\langle \theta_n \mid n \in N \rangle \mapsto \theta_N$ is definable from Θ_0, $\langle \sigma_0, \sigma_1, \dots \rangle$, and $N \in \omega$ by a formula with quantification restricted to sets, so in C^0 we may infer that this function exists. Theorem 3.80 (definition by recursion) then implies that $\langle \theta_n \mid n \in \omega \rangle$ exists as described.

If we assume our system satisfies the following condition then we can derive the consistency of Θ_{n+1} from the consistency of Θ_n and show by induction on $n \in \omega$ that Θ_n is consistent for all $n \in \omega$.

(2.9) **Condition 2** *If $\Theta \vdash \sigma$ and $\Theta \cup \{\sigma\} \vdash \theta$, then $\Theta \vdash \theta$.*

For suppose Θ_n is consistent. If $\Theta_n \cup \{\sigma_n\}$ is inconsistent then for some ρ-sentence θ, $\Theta_n \cup \{\sigma_n\} \vdash \theta$ and $\Theta_n \cup \{\sigma_n\} \vdash \neg\theta$. If $\Theta_n \cup \{\neg\sigma_n\}$ is also inconsistent then Condition 1[2.4] implies that $\Theta_n \vdash \sigma_n$, and Condition 2[2.9] then implies that $\Theta_n \vdash \theta$ and $\Theta_n \vdash \neg\theta$, contradicting the consistency of Θ_n.

Clearly Condition 2 holds for entailment in place of deducibility.

To conclude that Θ_ω is consistent we need to know that the union of an increasing sequence of consistent theories is consistent. This is guaranteed by the following condition.

(2.10) **Condition 3** *If every finite subset of a theory Θ is consistent then Θ is consistent.*

Θ_ω is therefore a maximal consistent extension of Θ.

As discussed above, each condition we impose on our deductive system must *mutatis mutandis* apply to the entailment relation, and for the most part the proof of the latter is straightforward. Condition 3[2.10] is an exception to this general rule. As a statement about structures, it asserts the *compactness* of a certain topology in which structures play the role of points, and it is not particularly easy to prove.[2] Given the completeness theorem, of course, it follows from the fact that the provability relation satisfies Condition 3; there does not appear to be any proof much more direct than this, so the compactness property may be regarded as one of the important consequences of the completeness theorem.

How to define a model of Θ_ω? In general, there is no straightforward way to do this. But if Θ_ω has a full class of *witnesses* in the following sense, it is easy.

Definition [C^0] *Suppose* Θ *is a theory and* $\phi \in \Theta$ *is* existential*, i.e.,* $\phi = \exists v\, \psi$ *for some formula* ψ*. If* c *is a constant and* $\psi\binom{v}{\bar{c}}$[3] *is in* Θ*, then we say that* c witnesses ϕ *in* Θ*.* Θ has witnesses $\overset{\text{def}}{\iff}$ *every existential sentence in* Θ *is witnessed in* Θ *by some constant.*

Thus we are led to

(2.11) **Condition 4** *If* Θ *is a consistent theory,* $\exists v\, \psi \in \Theta$*, and* c *is a constant that does not occur in any sentence in* Θ*, then* $\Theta \cup \left\{\psi\binom{v}{\bar{c}}\right\}$ *is consistent.*

Clearly this is true for entailment in place of deducibility (satisfiability in place of consistency).

With the addition of this condition we can extend any consistent theory Θ_0 to a maximal consistent theory with witnesses, provided that there is an ω-sequence of distinct constants that do not occur in Θ_0.

(2.12) *Thus, we suppose there is an* ω*-sequence of distinct constants that do not occur in* Θ_0*, and we carry out the construction*[2.8] *used above to generate a maximal extension, but at each step, if* σ_n *happens to be an existential sentence and* $\sigma_n \in \Theta_{n+1}$*, we also add a sentence that designates the first as yet unused constant as a witness of* σ_n*.*[4] *This is the* Henkin construction.[5]

We have assumed that ρ has infinitely many constant indices not occurring in Θ_0 and that ρ is countable, so there is bijection of ω with the constants of ρ, and this construction may be carried out. In general, for a given theory Θ_0 in a given signature ρ, we will expand ρ if necessary to satisfy this condition.[1.29.2.2] After carrying out the preceding construction and using the resulting theory to define a model, we discard the added constant indices and we are left with a model of Θ_0 of the original type. To ensure that the addition of constants does not affect the consistency of Θ_0 we require the following condition.

[2] Our proof of the soundness of our deductive system will not require us to have previously proved this result.

[3] Recall[1.35.3.2] that $\bar{c}$ is the term corresponding to the nulary operation index c, and[1.11] $\psi\binom{v}{\bar{c}}$ is the result of substituting $\bar{c}$ for every free occurrence of v in ψ.

[4] Repetitions are permissible in the sequence $\sigma_0, \ldots$. If σ_n has already occurred and been witnessed, it simply receives another witness at this stage.

[5] The completeness theorem was first proved by Gödel in his doctoral dissertation[6] of 1929, and was subsequently published in a more succinct form in 1930[7]. Henkin published his more transparent proof in 1949[8].

(2.13) **Condition 5** *Suppose Θ is a ρ-theory, σ is a ρ-sentence, and ρ' is an expansion of ρ by the addition of a countable class of constants. Then $\Theta \vdash \sigma$ in the context of ρ iff $\Theta \vdash \sigma$ in the context of ρ'.*

We have already shown[1.72] that the entailment relation satisfies this condition.

2.2.2 The model

In the preceding section we have shown how to extend a consistent theory Θ_0 to a maximal consistent theory Θ with witnesses, with consistency defined in terms of a deductive system satisfying Conditions 0–5. We will now define a structure $\mathfrak{H} = \mathfrak{H}^{\Theta}$ such that $\mathfrak{H} \models \Theta$.

2.2.2.1 The universe

The universe $|\mathfrak{H}|$ of our model is the class $\mathcal{K}$ of variable-free terms. The operation and predicate indices are interpreted as follows.

2.2.2.2 Operation indices

If F is an n-ary operation index, we let $F^{\mathfrak{H}}$ be given by

$$F^{\mathfrak{H}}\langle \tau_0, \ldots, \tau_{n^-} \rangle = \tilde{F}\langle \tau_0, \ldots, \tau_{n^-} \rangle. \tag{2.14}$$

2.2.2.3 Predicate indices

If P is an n-ary predicate index, we let

$$P^{\mathfrak{H}} = \{\langle \tau_0, \ldots, \tau_{n^-} \rangle \mid \tilde{P}\langle \tau_0, \ldots, \tau_{n^-} \rangle \in \Theta\}. \tag{2.15}$$

2.2.3 Validation of the model

Having defined $\mathfrak{H} = \mathfrak{H}^{\Theta}$, we must now show that $\mathfrak{H}$ is satisfactory and $\mathfrak{H} \models \Theta$. Recall that we are working in the theory C^0, which does not have an axiom of infinity, so we cannot show in general that a structure has a satisfaction relation, although we can show that there is at most one satisfaction relation.[1.4.2.2] We may, however, define a satisfaction relation for $\mathfrak{H}$ from Θ if we impose certain additional conditions on our system of deduction. These conditions will be stated as they arise in the validation of the model.

(2.16) **Theorem** [C^0] *Suppose Θ is a maximal consistent ρ-theory with witnesses. Let $\mathfrak{H} = \mathfrak{H}^{\Theta}$ be defined as above.*[§ 2.2.2] *Let H be the class of $\langle \phi, A \rangle$ such that*

1. *ϕ is a ρ-formula;*
2. *A is an $\mathfrak{H}$-assignment for ϕ; and*
3. *$\phi(A) \in \Theta$.*[6]

Assuming the system of deduction, in terms of which 'consistent' is defined, satisfies—in addition to Conditions 0–5 already stated—also Conditions 6–10, to be stated below, H is a (the) satisfaction relation for $\mathfrak{H}$.

[6] Since $|\mathfrak{H}|$ consists of ρ-terms, an $\mathfrak{H}$-assignment, i.e., a function from variables to elements of $|\mathfrak{H}|$, is also a ρ-substitution, i.e., a function from variables to ρ-terms.

Proof We proceed by induction on the complexity of formulas, i.e., we let Γ be the class of formulas ϕ such that H satisfies the definition of a satisfaction relation for every subformula ϕ' of ϕ with every $\mathfrak{H}$-assignment for ϕ'. We will show that Γ contains all atomic formulas and is closed under the formula-building operations; hence, Γ contains all formulas.[1.38] We give the essence of the proof, leaving it to the reader to formulate it precisely in these terms.

Atomic sentences Suppose $\phi = \tilde{P}\langle v_0, \ldots, v_{n^-}\rangle$ (where there may be repetitions in the sequence $\langle v_0, \ldots, v_{n^-}\rangle$), and suppose A is an $\mathfrak{H}$-assignment for ϕ. Then

$$\begin{aligned}\langle \phi, A\rangle \in H \leftrightarrow \phi(A) \in \Theta &\leftrightarrow \tilde{P}\langle Av_0, \ldots, Av_{n^-}\rangle \in \Theta \\ &\leftrightarrow \langle Av_0, \ldots, Av_{n^-}\rangle \in P^{\mathfrak{H}},\end{aligned}$$

so $\phi \in \Gamma$.

Negation Suppose $\phi = \neg\,\psi$ and $\psi \in \Gamma$. Then $\operatorname{Free}\psi = \operatorname{Free}\phi$. Let A be an $\mathfrak{H}$-assignment for ϕ. Then A is also an $\mathfrak{H}$-assignment for ψ, and

$$\langle \psi, A\rangle \in H \leftrightarrow \psi(A) \in \Theta.$$

We now wish to assert that since Θ is maximal and consistent,

$$\neg\,\psi(A) \in \Theta \leftrightarrow \psi(A) \notin \Theta, \tag{2.17}$$

from which it will follow that

$$\langle \phi, A\rangle \in H \leftrightarrow \phi(A) \in \Theta \leftrightarrow \psi(A) \notin \Theta \leftrightarrow \langle \psi, A\rangle \notin H.$$

To obtain (2.17) we impose the following condition on our system of deduction.

(2.18) **Condition 6** *If Θ is a theory and $\sigma \in \Theta$, then $\Theta \vdash \sigma$.*

Thus, if Θ is consistent, $\psi(A)$ and $\neg\,\psi(A)$ are not both in Θ, and since Θ is maximal, one of them is.

2.2.3.1 Implication

Next suppose that $\phi = \psi \to \eta$, with $\psi, \eta \in \Gamma$. Then $\operatorname{Free}\psi, \operatorname{Free}\eta \subseteq \operatorname{Free}\phi$. Let A be an $\mathfrak{H}$-assignment for ϕ. Then A is also an $\mathfrak{H}$-assignment for ψ and η, and

$$\begin{aligned}\langle \psi, A\rangle \in H &\leftrightarrow \psi(A) \in \Theta \\ \langle \eta, A\rangle \in H &\leftrightarrow \eta(A) \in \Theta.\end{aligned}$$

We now wish to assert that since Θ is maximal and consistent,

$$\psi(A) \to \eta(A) \in \Theta \leftrightarrow \big(\psi(A) \in \Theta \to \eta(A) \in \Theta\big), \tag{2.19}$$

from which it will follow that

$$\begin{aligned}\langle \phi, A\rangle \in H &\leftrightarrow \phi(A) \in \Theta \\ &\leftrightarrow (\psi(A) \to \eta(A)) \in \Theta \\ &\leftrightarrow (\psi(A) \in \Theta \to \eta(A) \in \Theta) \\ &\leftrightarrow (\langle \psi, A\rangle \in H \to \langle \eta, A\rangle \in H).\end{aligned}$$

Since

$$\big(\psi(A) \in \Theta \to \eta(A) \in \Theta\big) \leftrightarrow \big(\psi(A) \notin \Theta \vee \eta(A) \in \Theta\big),$$

to obtain (2.19) it suffices that our system of deduction allow us to infer ζ from $\theta \to \zeta$ and θ and to infer $\theta \to \zeta$ from either $\neg\theta$ or ζ (since $\theta \notin \Theta \to \neg\theta \in \Theta$ by maximality).

Thus, for any sentences ζ and θ, we impose

(2.20) **Condition 7** $\{\theta, \theta \to \zeta\} \vdash \zeta$,

(2.21) **Condition 8** $\{\neg\theta\} \vdash \theta \to \zeta$,

and

(2.22) **Condition 9** $\{\zeta\} \vdash \theta \to \zeta$

2.2.3.2 Quantification

Lastly, suppose $\phi = \exists v\,\psi$, with $\psi \in \Gamma$. Then $\mathrm{Free}\,\psi \subseteq \mathrm{Free}\,\phi \cup \{v\}$. Let A be an $\mathfrak{H}$-assignment for ϕ. Then for any variable-free term τ

$$\left\langle \psi, A\left\langle {v \atop \tau} \right\rangle \right\rangle \in H \leftrightarrow \psi\big(A\left\langle {v \atop \tau} \right\rangle\big) \in \Theta,$$

where $A\left\langle {v \atop \tau} \right\rangle = \{(u, Au) \mid u \in \mathrm{Free}\,\psi \backslash \{v\}\} \cup \{(v, \tau)\}$, i.e., the assignment (also substitution) A followed by the assignment $\left\langle {v \atop \tau} \right\rangle$ (a reassignment, if $\mathrm{dom}\,A$ superfluously contains v). We now wish to assert that since Θ is maximal and consistent,

$$(\exists v\,\psi)(A) \in \Theta \leftrightarrow \exists \tau \in |\mathfrak{H}|\ \psi\big(A\left\langle {v \atop \tau} \right\rangle\big) \in \Theta, \tag{2.23}$$

from which it will follow that

$$\begin{aligned} \langle \phi, A \rangle \in H \leftrightarrow \phi(A) \in \Theta \leftrightarrow & (\exists v\,\psi)(A) \in \Theta \leftrightarrow \exists \tau \in |\mathfrak{H}|\ \psi\big(A\left\langle {v \atop \tau} \right\rangle\big) \in \Theta \\ & \leftrightarrow \exists \tau \in |\mathfrak{H}|\ \left\langle \psi, A\left\langle {v \atop \tau} \right\rangle \right\rangle \in H. \end{aligned}$$

The $\to$ direction of (2.23) follows from the fact that Θ has witnesses. The $\leftarrow$ direction requires that another condition be imposed, viz.,

(2.24) **Condition 10** *If* $\mathrm{Free}\,\psi \subseteq \{v\}$ *and* τ *is a variable-free term, then* $\left\{\psi\binom{v}{\tau}\right\} \vdash \exists v\,\psi$.[7]

$\square^{2.16}$

This completes the outline of the proof. Now it only remains to define a system of deduction that satisfies the conditions set forth above. 'proof' and 'consistency' will then be defined in terms of that system, and we shall have proved the completeness theorem.

[7] The limitation on $\mathrm{Free}\,\psi$ is required to ensure that $\exists v\,\psi$ is a sentence, as our deductive system will be based on sentences.

2.2.4 A system of deduction

In the previous section we have shown that if a suitable system of deduction can be devised, and we define 'proof' in terms of that system, then we can prove the completeness theorem, which states that any sentence that is entailed by a theory can be proved from that theory. We imposed certain conditions on the deductive system that allowed the argument to go through. These conditions are recapitulated below. The conditions lead us directly to a system of deduction, one that corresponds precisely to the natural system with which you are no doubt already familiar, the one we have been using since the beginning of this book and will use through to the end.

2.2.4.1 The conditions summarized

0. For any theory Θ and sentence σ, if $\Theta \vdash \sigma$ then Θ entails σ.[2.1]

1. If $\Theta \cup \{\neg\,\sigma\}$ is inconsistent then $\Theta \vdash \sigma$.[2.4]

2. If $\Theta \vdash \sigma$ and $\Theta \cup \{\sigma\} \vdash \theta$, then $\Theta \vdash \theta$.[2.9]

3. If every finite subset of a theory Θ is consistent then Θ is consistent.[2.10]

4. If Θ is a consistent theory, $\exists v\, \psi \in \Theta$, and c is a constant that does not occur in any sentence in Θ, then $\Theta \cup \left\{\psi\binom{v}{\bar{c}}\right\}$ is consistent.[2.11]

5. Suppose Θ is a ρ-theory and σ a ρ-sentence, and suppose ρ' is an expansion of ρ by the addition of new constant indices. Then $\Theta \vdash \sigma$ in the context of ρ iff $\Theta \vdash \sigma$ in the context of ρ'.[2.13]

6. If Θ is a theory and $\sigma \in \Theta$, then $\Theta \vdash \sigma$.[2.18]

7. For all sentences θ and ζ, $\{\theta, \theta \to \zeta\} \vdash \zeta$.[2.20]

8. For all sentences θ and ζ, $\{\neg\,\theta\} \vdash \theta \to \zeta$.[2.21]

9. For all sentences θ and ζ, $\{\zeta\} \vdash \theta \to \zeta$.[2.22]

10. If ψ has no free variables other than v, and τ is a variable-free term, then $\left\{\psi\binom{v}{\tau}\right\} \vdash \exists v\, \psi$.[2.24]

2.2.4.2 The system

The system of deduction is based on the concepts of *sequent* and *inference rule.*

(2.25) **Definition** $[\mathsf{C}^0]$ *Suppose ρ is a signature. A* c-expansion of ρ *is a signature that expands ρ with all the added indices being constants, i.e., nulary operation indices.*

1. *A* ρ-sequent *is a 2-sequence $\langle \Sigma, \sigma \rangle$, where Σ is a finite set of ρ'-sentences and σ is a single ρ'-sentence for some c-expansion ρ' of ρ. For its mnemonic value, we use the special notation:*

$$\Sigma \Rightarrow \sigma \stackrel{\text{def}}{=} \langle \Sigma, \sigma \rangle$$

for sequents.

2. Σ *is the* antecedent *and* σ *the* succedent *of* $\Sigma \Rightarrow \sigma$.
3. *We may also refer to the elements of* Σ *as the* premises *and to* σ *as the* conclusion *of the sequent.*
4. *A sequent is* valid $\overset{\text{def}}{\iff}$ Σ *entails* σ.[8]
5. *A* ρ-proof *is a finite sequence of* ρ*-sequents, each item of which is* justified *(within the sequence) by one of the* rules *to be listed presently.*
6. *A sequent is* ρ*-provable* $\overset{\text{def}}{\iff}$ *it is a* ρ*-sequent and it occurs in some* ρ*-proof.*

(2.26) *Note that if* ρ' *is a c-expansion of* ρ *then any* ρ'*-proof is also a* ρ*-proof.*

Sequent derives from the German *Sequenz* used by Gerhard Gentzen to describe his *logischer klassischer Kalkül* **LK** (*sequent* being an English neologism to replace the direct translation *sequence*, already in widespread use with the modern definition). A sequent in the original sense is an ordered pair of finite sequences of formulas, rather than a finite set of formulas and a single formula, as defined above. Systems of deduction based on sequents in the original sense are elegant and useful, and we will present Gentzen's system in Section 2.6.2. For the present, we restrict our attention to so-called *natural* systems of deduction, for which the present notion of sequent is appropriate.

In the following rules, Σ and Σ' refer to arbitrary finite sets of sentences. σ, θ, and ζ are arbitrary sentences. ψ is an arbitrary formula with at most one free variable, v. c is an arbitrary constant, and τ is an arbitrary variable-free term. Some of the rules justify the appearance of a sequent by virtue of the antecedent appearance of one or two related sequents. Others state that sequents of a particular form may appear anywhere in a proof. In the representation of the rules, the sequent below the line is inferred from the sequent(s) above the line. Note that there may be 0, 1, or 2 sequents above the line.

(2.27) **Inference rules for the natural deduction system ND**

0. $\dfrac{\Sigma \Rightarrow \sigma}{\Sigma' \Rightarrow \sigma}$ *if* $\Sigma' \supseteq \Sigma$.
1. $\dfrac{}{\Sigma \Rightarrow \sigma}$ *if* $\sigma \in \Sigma$.
2. $\dfrac{\Sigma \cup \{\neg\,\sigma\} \Rightarrow \theta \qquad \Sigma \cup \{\neg\,\sigma\} \Rightarrow \neg\,\theta}{\Sigma \Rightarrow \sigma}$
3. $\dfrac{\Sigma \Rightarrow \sigma \qquad \Sigma \cup \{\sigma\} \Rightarrow \theta}{\Sigma \Rightarrow \theta}$
4. $\dfrac{\Sigma \cup \left\{\psi\binom{v}{c}\right\} \Rightarrow \sigma}{\Sigma \cup \{\exists v\, \psi\} \Rightarrow \sigma}$ *if* c *does not occur in the lower sequent.*
5. $\dfrac{}{\{\theta, \theta \to \zeta\} \Rightarrow \zeta}$
6. $\dfrac{\Sigma \cup \{\theta\} \Rightarrow \zeta}{\Sigma \Rightarrow \theta \to \zeta}$
7. $\dfrac{}{\left\{\psi\binom{v}{\tau}\right\} \Rightarrow \exists v\, \psi}$

[8] Recall[1.72] that the notion of entailment is independent of signature.

2.2.5 Conclusion

2.2.5.1 Statement and proof of the theorem

We can now provide a formal definition of the provability relation, with which we can state the completeness theorem, the proof of which is now straightforward. It is no more difficult to prove the following useful sharp version, which makes reference to a signature ρ. It will be easy to show that the ρ-dependence may be dropped.[2.34]

(2.28) **Definition** $[\mathsf{C}^0]$ *Suppose* Θ *is a* ρ*-theory and* σ *is a* ρ*-sentence.*

1. Θ ρ-proves σ $\overset{\text{def}}{\Longleftrightarrow}$ $\Theta \vdash^{\rho} \sigma$ $\overset{\text{def}}{\Longleftrightarrow}$ *there is a* ρ*-provable sequent* $\Sigma \Rightarrow \sigma$ *with* $\Sigma \subseteq \Theta$.
2. *A* ρ-proof of σ from Θ *is a* ρ*-proof of any such sequent.*
3. *We also say that* σ *is* deducible *or* inferable *from* Θ.
4. Θ *is* ρ*-consistent* $\overset{\text{def}}{\Longleftrightarrow}$ *for all* ρ*-sentences* θ *it is not the case that* $\Theta \vdash^{\rho} \theta$ *and* $\Theta \vdash^{\rho} \neg\, \theta$.

(2.29) **Theorem** $[\mathsf{C}^0]$ *Suppose* Θ *is a* ρ*-theory.* Θ *is* ρ*-consistent iff* Θ *is satisfiable.*

Remark Recall[1.70] that, in general, a theory Θ has a satisfactory ρ-model iff it has a satisfactory ρ'-model, where ρ, ρ' are any signatures appropriate to Θ, so we may say that Θ is satisfiable without troubling to specify a signature.[1.71]

Proof We first note that given a theory Θ_0 in a countable signature ρ with infinitely many constant operation indices that do not occur in Θ_0, the theory Θ_ω generated by the Henkin construction[2.8] is definable from Θ_0 and an enumeration of ρ without the use of class quantification. This follows from the fact that proofs are finite and are therefore sets, and consistency is defined in terms of the existence of proofs.

We have already shown—on the assumption that $\vdash^{\rho}$ satisfies CONDITIONS 0–10[2.2.4.1]—that the structure $\mathfrak{H}$ defined from Θ_ω is satisfactory and that $\mathfrak{H} \models \Theta$.

The only thing left is to check that $\vdash^{\rho}$ meets all the conditions we have imposed. All expressions are presumed to be ρ-expressions or, as appropriate, ρ'-expressions, where ρ' is a c-expansion of ρ.

Condition 0 We need to check that the rules generate only valid sequents. Since we are working in C^0, a straightforward induction on the length of proofs cannot be carried out, as the validity property involves quantification over structures, which may be proper classes. We proceed instead as follows.

(2.30) *Suppose toward a contradiction that* Θ *is a* ρ*-theory,* σ *is a* ρ*-sentence,* $\Theta \vdash^{\rho} \sigma$*, and* Θ *does not entail* σ.

Let π be a ρ-proof of a sequent $\Sigma \Rightarrow \sigma$ with $\Sigma \subseteq \Theta$. Let ρ' be a c-expansion of ρ such that all the sentences in π are ρ'-sentences. Since Θ does not entail σ, neither does Σ, so there is a satisfactory model of $\Sigma \cup \{\neg\, \sigma\}$, and this is independent of signature, so

(2.31) *let* $\mathfrak{S}$ *be a satisfactory* ρ'*-model of* Σ *and* $\neg\, \sigma$.

Let $\pi = \langle \pi_0 \ldots, \pi_{n^-} \rangle$, and let $\pi_m = \Sigma_m \Rightarrow \sigma_m$ for $m \in n$. Let Γ be the set of constant indices that occur in π, and for each $c \in \Gamma$ let v_c be a distinct variable that does not occur in π. Let $\hat{\Gamma} = \{v_c \mid c \in \Gamma\}$. For every expression ϵ that occurs in π, let $\hat{\epsilon}$ be the result of substituting $\bar{v}_c$ for every occurrence of $\bar{c}$, for every constant index c in ϵ. If S is a set of expressions, let $\hat{S} = \{\hat{\epsilon} \mid \epsilon \in S\}$. Let A range over $\mathfrak{S}$-assignments to $\hat{\Gamma}$.

(2.32) **Claim** *For every $m \in n$*

$$\forall A\big(\mathfrak{S} \models \hat{\Sigma}_m[A] \rightarrow \mathfrak{S} \models \hat{\sigma}_m[A]\big).$$

Proof By induction on m, which is a legitimate C^0-proof because only set-quantification is involved, with the satisfaction relation for $\mathfrak{S}$ as a parameter. Given $m \in n$, $\Sigma_m \Rightarrow \sigma_m$ is justified by one of the rules (2.27) with reference to 0, 1, or 2 preceding sequents. Our induction hypothesis is that

$$\forall m' \in m\, \forall A\big(\mathfrak{S} \models \hat{\Sigma}_{m'}[A] \rightarrow \mathfrak{S} \models \hat{\sigma}_{m'}[A]\big).$$

For convenience of reference we will generally use the notation of the rules. We give the argument for Rule 0 in detail. For the rest we give the heart of the argument with the understanding that the expressions that arise are assumed to occur in π as appropriate.

Rule 0 Suppose $\pi_m = \Sigma' \Rightarrow \sigma$ and $\Sigma \subseteq \Sigma'$ and for some $m' < m$, $\pi_{m'} = \Sigma \Rightarrow \sigma$. Then by induction hypothesis, $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$. It follows that $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}'[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$.

Rule 1 If $\sigma \in \Sigma$ then $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$.

Rule 2 If $\forall A\,\big(\mathfrak{S} \models (\hat{\Sigma} \cup \{\widehat{\neg\sigma}\})[A] \rightarrow \mathfrak{S} \models \hat{\theta}[A]\big)$ and $\forall A\,\big(\mathfrak{S} \models (\hat{\Sigma} \cup \{\widehat{\neg\sigma}\})[A] \rightarrow \mathfrak{S} \models \widehat{\neg\theta}[A]\big)$ then $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$, using the fact that $\widehat{\neg\sigma} = \neg\hat{\sigma}$ and $\widehat{\neg\theta} = \neg\hat{\theta}$.

Rule 3 If $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$ and $\forall A\,\big(\mathfrak{S} \models (\hat{\Sigma} \cup \{\hat{\sigma}\})[A] \rightarrow \mathfrak{S} \models \hat{\theta}[A]\big)$ then $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \rightarrow \mathfrak{S} \models \hat{\theta}[A]\big)$.

Rule 4 Suppose $c \in \Gamma$ and c does not occur in ψ, in σ, or in Σ. Suppose $\forall A\left(\mathfrak{S} \models \left(\hat{\Sigma} \cup \left\{\psi\binom{v}{\bar{c}}^{\wedge}\right\}\right)[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\right)$, i.e.

$$\forall A\,\big(\mathfrak{S} \models \big(\hat{\Sigma} \cup \{\hat{\psi}\tbinom{v}{\bar{v}_c}\}\big)[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big), \tag{2.33}$$

since $\psi\binom{v}{\bar{c}}^{\wedge} = \hat{\psi}\binom{v}{\bar{v}_c}$. We want to show that $\forall A\,\big(\mathfrak{S} \models \big(\hat{\Sigma} \cup \{\widehat{\exists v\,\psi}\}\big)[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big)$, i.e.,

$$\forall A\,\big(\mathfrak{S} \models \big(\hat{\Sigma} \cup \{\exists v\,\hat{\psi}\}\big)[A] \rightarrow \mathfrak{S} \models \hat{\sigma}[A]\big),$$

since $\widehat{\exists v\,\psi} = \exists v\,\hat{\psi}$. To this end, suppose A is an $\mathfrak{S}$-assignment to Γ, and suppose $\mathfrak{S} \models \big(\hat{\Sigma} \cup \{\exists v\,\hat{\psi}\}\big)[A]$. Then for some $a \in |\mathfrak{S}|$,

$$\mathfrak{S} \models \hat{\psi}[A]\big[{}^{v}_{a}\big],$$

which is to say

$$\mathfrak{S} \models \hat{\psi}\binom{v}{\bar{v}_c}[A]\big[{}^{v_c}_{a}\big],$$

since v_c does not occur in $\hat{\psi}$; whence

$$\mathfrak{S} \models \big(\hat{\Sigma} \cup \{\hat{\psi}\binom{v}{\bar{v}_c}\}\big)[A]\big[{}^{v_c}_{a}\big],$$

since v_c does not occur in $\hat{\Sigma}$. Hence[2.33]

$$\mathfrak{S} \models \hat{\sigma}[A]\big[{}^{v_c}_{a}\big].$$

Since v_c does not occur in $\hat{\sigma}$,

$$\mathfrak{S} \models \hat{\sigma}[A],$$

as claimed.

Rule 5 $\forall A\,\big(\mathfrak{S} \models \{\hat{\theta}[A], \widehat{\theta \to \zeta}[A]\} \to \mathfrak{S} \models \hat{\zeta}[A]\big)$, using the fact that $\widehat{\theta \to \zeta} = \hat{\theta} \to \hat{\zeta}$.

Rule 6 If $\forall A\,\big(\mathfrak{S} \models (\hat{\Sigma} \cup \{\hat{\theta}\})[A] \to \mathfrak{S} \models \hat{\zeta}[A]\big)$ then $\forall A\,\big(\mathfrak{S} \models \hat{\Sigma}[A] \to \mathfrak{S} \models \widehat{\theta \to \zeta}[A]\big)$, using again the fact that $\widehat{\theta \to \zeta} = \hat{\theta} \to \hat{\zeta}$.

Rule 7 $\forall A\,\Big(\mathfrak{S} \models \widehat{\psi\binom{v}{\tau}}[A] \to \mathfrak{S} \models \widehat{\exists v\,\psi}[A]\Big)$, using the fact that $\widehat{\psi\binom{v}{\tau}} = \hat{\psi}\binom{v}{\hat{\tau}}$ and, as above, $\widehat{\exists v\,\psi} = \exists v\,\hat{\psi}$.

This completes the proof of the claim. □[2.32]

By hypothesis, the sequent of interest, $\Sigma \Rightarrow \sigma$, is π_m for some $m < n$, so[2.32] for any $A : \hat{\Gamma} \to |\mathfrak{S}|$

$$\mathfrak{S} \models \hat{\Sigma}[A] \to \mathfrak{S} \models \hat{\sigma}[A],$$

in particular for $A = \{(v_c, c^{\mathfrak{S}}) \mid c \in \Gamma\}$, where we indulge in the usual mild abuse of notation to let $c^{\mathfrak{S}} = c^{\mathfrak{S}}0$ for a constant (i.e., nulary operation index) c. For this assignment, for any sentence θ that occurs in π,

$$\mathfrak{S} \models \hat{\theta}[A] \leftrightarrow \mathfrak{S} \models \theta,$$

so

$$\mathfrak{S} \models \Sigma \to \mathfrak{S} \models \sigma,$$

which contradicts (2.31) and therefore invalidates the supposition (2.30), so Θ does entail σ, and this completes the proof that **ND** meets Condition 0.□Condition 0

Condition 1 is met by virtue of Rule 2, as the following argument shows. Suppose $\Theta \cup \{\neg\,\sigma\}$ is ρ-inconsistent, i.e., for some sentence θ, $\Theta \cup \{\neg\,\sigma\}$ ρ-proves both θ and $\neg\,\theta$. Specifically, suppose there are ρ-proofs that contain the sequents $\Sigma \Rightarrow \theta$ and $\Sigma' \Rightarrow \neg\,\theta$, where Σ and Σ' are finite subsets of $\Theta \cup \{\neg\,\sigma\}$. Let $\Sigma'' = \Sigma \cup \Sigma'$. Using Rule 0, we can construct a ρ-proof contains both $\Sigma'' \cup \{\neg\,\sigma\} \Rightarrow \theta$ and $\Sigma'' \cup \{\neg\,\sigma\} \Rightarrow \neg\,\theta$. By virtue of Rule 2, this ρ-proves $\Sigma'' \Rightarrow \sigma$. Since $\Sigma'' \subseteq \Theta$, $\Theta \vdash^{\rho} \sigma$, as claimed. □Condition 1

Condition 2 follows from RULE 3.

Condition 3 follows from the definition[2.28] of $\vdash^{\rho}$ in terms of ρ-provable sequents, since the set of premises of a sequent is finite.

Condition 4 is met by virtue of RULE 4, as the following argument shows. Suppose Θ is a ρ-consistent theory, $\exists v\, \psi \in \Theta$, and c is a constant index that does not occur in any sentence in Θ; and suppose toward a contradiction that $\Theta \cup \left\{\psi\binom{v}{\bar{c}}\right\}$ is ρ-inconsistent—say $\Theta \cup \left\{\psi\binom{v}{\bar{c}}\right\}$ ρ-proves both θ and $\neg\,\theta$ for some θ. RULE 2 implies that $\Theta \cup \left\{\psi\binom{v}{\bar{c}}\right\} \vdash^{\rho} \sigma$ for any ρ-sentence σ. Let σ be some sentence that does not involve the index c, and—as in the proof of CONDITION 1—let $\Sigma \subseteq \Theta$ be finite such that $\Sigma \cup \left\{\psi\binom{v}{\bar{c}}\right\} \Rightarrow \sigma$ and $\Sigma \cup \left\{\psi\binom{v}{\bar{c}}\right\} \Rightarrow \neg\,\sigma$ are both ρ-provable sequents. Since c does not occur in any sentence in Σ or in ψ (since $\Sigma \cup \{\exists v\, \psi\} \subseteq \Theta$), RULE 4 applies, and $\Sigma \cup \{\exists v\, \psi\} \Rightarrow \sigma$ and $\Sigma \cup \{\exists v\, \psi\} \Rightarrow \neg\,\sigma$ are ρ-provable. Since $\Sigma \cup \{\exists v\, \psi\} \subseteq \Theta$, Θ is ρ-inconsistent, contrary to hypothesis. $\square$CONDITION 4

Condition 5 follows from the observation (2.26).

Condition 6 follows from RULE 1.

Condition 7 follows from RULE 5.

Condition 8 follows from RULE 6 with $\Sigma = \{\neg\,\theta\}$ (using RULE 2 with $\Sigma = \{\theta, \neg\,\theta\}$ and $\sigma = \zeta$, and RULE 1).

Condition 9 follows from RULE 6 with $\Sigma = \{\zeta\}$ (and RULE 1).

Condition 10 follows from RULE 7.

Together with the discussion in Section 2.2 this concludes the proof. $\square$[2.29]

Now suppose Θ is a theory and ρ, ρ' are signatures appropriate to Θ, i.e., Θ is both a ρ-theory and a ρ'-theory. Then by Theorem 2.29

$$\begin{aligned}\Theta \text{ is } \rho\text{-consistent} &\leftrightarrow \Theta \text{ is satisfiable}\\ &\leftrightarrow \Theta \text{ is } \rho'\text{-consistent},\end{aligned}$$

so the following definition is reasonable. Note that there is no dependence on signature (cf., (2.28)).

(2.34) **Definition** [C^0] *Suppose Θ is a theory and σ is a sentence.*

1. Θ proves σ $\overset{\text{def}}{\iff}$ *there is a provable sequent $\langle \Sigma, \sigma \rangle$ with $\Sigma \subseteq \Theta$.*
2. *A* proof of σ from Θ *is a proof of any such sequent.*
3. *We also say that σ is* deducible *or* inferable *from Θ.*
4. Θ *is consistent* $\overset{\text{def}}{\iff}$ *for all sentences θ it is not the case that $\Theta \vdash \theta$ and $\Theta \vdash \neg\,\theta$.*

We can now state Gödel's completeness theorem in its final form.

Theorem: Completeness [C^0] *Suppose* Θ *is a theory.* Θ *is consistent iff* Θ *is satisfiable.*

Using the previous form[2.29] of the completeness theorem we have

Theorem [C^0] *If* $\Theta \vdash \sigma$ *then there is a* ρ*-proof of* σ *from* Θ*, where* ρ *is any signature appropriate to* $\Theta \cup \{\sigma\}$*. In particular, there is a proof that involves no indices that do not occur in* Θ *or* σ *other than constant indices.*

As a simple corollary we have the important *compactness* property of predicate logic, mentioned above following the statement of CONDITION 3.[2.10]

(2.35) **Theorem: Compactness** [C^0] *Suppose* Θ *is a theory.* Θ *is satisfiable iff every finite subset of* Θ *is satisfiable.*

Proof Θ is unsatisfiable iff Θ is inconsistent iff there is a proof of inconsistency from Θ iff there is a proof of inconsistency from a finite subset of Θ iff a finite subset of Θ is inconsistent iff a finite subset of Θ is unsatisfiable. □[2.35]

Note that according to Definition 2.34, by virtue of Rule 0, if Σ *is a finite set of sentences and* σ *is a sentence, then* $\Sigma \vdash \sigma$ *iff* $\Sigma \Rightarrow \sigma$ *is provable.*

Rules 0–7[2.27] defining a justified sequence of sequents therefore have the following immediate consequences for the provability relation.

(2.36) **Theorem** [C^0] *Suppose* Θ *is a class of sentences.*

0. *If* $\Theta \vdash \sigma$ *then* $\Theta' \vdash \sigma$ *for any* $\Theta' \supseteq \Theta$.
1. *If* $\sigma \in \Theta$*, then* $\Theta \vdash \sigma$.
2. *If* $\Theta \cup \{\neg\sigma\} \vdash \theta$ *and* $\Theta \cup \{\neg\sigma\} \vdash \neg\theta$ *then* $\Theta \vdash \sigma$.
3. *If* $\Theta \vdash \sigma$ *and* $\Theta \cup \{\sigma\} \vdash \theta$ *then* $\Theta \vdash \theta$.
4. *If* c *does not occur in* ψ*, in* σ*, or in any of the sentences of* Θ*, and* $\Theta \cup \left\{\psi\binom{v}{c}\right\} \vdash \sigma$*, then* $\Theta \cup \{\exists v\, \psi\} \vdash \sigma$.
5. $\{\theta, \theta \to \zeta\} \vdash \zeta$.
6. *If* $\Theta \cup \{\theta\} \vdash \zeta$ *then* $\Theta \vdash \theta \to \zeta$.
7. $\left\{\psi\binom{v}{\tau}\right\} \vdash \exists v\, \psi$.

Proof Immediate. □[2.36]

2.2.5.2 Significance of the theorem

The notion of interpretation we have considered so far in this chapter is that of a satisfactory structure, but we may consider generalizations of this. Note that enlargement of the category of allowed interpretations potentially weakens the notion of satisfiability and strengthens the corresponding notion of entailment. With the completeness theorem, however, it is easy to show that any *reasonable* notion of interpretation leads to the same notion of satisfiability (and entailment).

For the purpose of this discussion, we will say that a notion of interpretation has the soundness property iff any theory that is true in some interpretation is consistent. Consider a notion of interpretation that is *reasonable* in the sense that it has the soundness property and includes all satisfactory structures. Suppose Θ is a theory. If Θ is true in some interpretation then by soundness it is consistent. On the other hand, if Θ is consistent then by the completeness theorem it is true in some interpretation (in some satisfactory structure, in fact).

Thus, satisfiability is a more stable concept than might at first appear, as it is equivalent to consistency for any reasonable notion of interpretation; consequently, a theory is satisfiable in any reasonable sense iff it is satisfiable in any other. In particular, we have the following theorem equating weak satisfiability with satisfiability.

(2.37) **Theorem** [C^0] *Suppose Θ is a ρ-theory.*

1. *Suppose $\mathfrak{S}$ is a weakly satisfactory ρ-structure and $\mathfrak{S} \models \Theta$. Then Θ is consistent.*
2. *Hence, Θ has a weakly satisfactory model iff it has a satisfactory model.*

Proof 1 Suppose toward a contradiction that Θ is inconsistent. Let θ be a ρ-sentence such that $\Theta \vdash \theta$ and $\Theta \vdash \neg\theta$. Let π^+ and π^- be proofs of θ and $\neg\theta$, respectively, from Θ. Let Σ^+ and Σ^- be finite subsets of Θ such that $\Sigma^+ \Rightarrow \theta$ occurs in π^+ and $\Sigma^- \Rightarrow \neg\theta$ occurs in π^-. Let Φ be the set of sentences that occur in either π^+ or π^-, and let S be a Φ-satisfaction relation for $\mathfrak{S}$. It is easily shown by induction on the position of sequents in any proof π that for every sequent $\Sigma \Rightarrow \sigma$ in π, if $\models^S \Sigma$[1.61.4] then $\models^S \sigma$. Since $\models^S \Sigma^+$ and $\models^S \Sigma^-$, it follows that $\models^S \theta$ and $\models^S \neg\theta$, which is impossible.

2 Immediate. □[2.37]

One of the fascinating aspects of the completeness theorem is that it equates provability—which, as we have seen, is a very concrete concept—with entailment—which, in the general sense of interpretation, is perhaps the most abstract concept in mathematics. We have restricted our attention to countable languages, so there are only countably many expressions in a given language, countably many sequents, and countably many proofs. By enumerating proofs we can enumerate all valid sequents. If a theory Θ is effectively enumerable, to determine whether a sentence σ follows from Θ, we can simply enumerate valid sequents looking for one of the form $\Sigma \Rightarrow \sigma$ for some finite $\Sigma \subseteq \Theta$. If $\Theta \vdash \sigma$ we will find a proof in a finite time.[9]

Contrast this with the notion of entailment. To verify directly that Θ entails σ would require a search over all interpretations; for each interpretation we would have to ascertain whether it satisfied Θ, and—if it did—whether it satisfied σ. This is obviously a highly infinitary undertaking, the more so as the notion of interpretation, as discussed above, is quite open-ended.[10]

[9] If $\Theta \nvdash \sigma$, we may continue enumerating valid sequents forever without a conclusion. More to come on this.

[10]

> There are more things in heaven and earth, Horatio,
> Than are dreamt of in your philosophy.

The Tragedy of Hamlet, Prince of Denmark
Act I, Scene V
William Shakespeare

Given a consistent theory Θ in a countable language such as we have been considering,[2.7] our proof of the completeness theorem constructs a countable satisfactory model of Θ, so we succeed in proving the ostensibly stronger statement:

If Θ is consistent, then Θ has a countable satisfactory model.

This is the foundation for the *Skolem paradox* in the metatheory of membership. In ZF we can prove the existence of uncountably many sets. If ZF is consistent then there exists a countable model $\mathfrak{M}$ of ZF. How can this countable model satisfy the sentence 'there exist uncountably many sets'? The resolution of the paradox lies in realizing that while we in the "outside world" realize that $\mathfrak{M}$ is countable—i.e., we have a function that maps ω onto $|\mathfrak{M}|$—there is no such function in $\mathfrak{M}$.

2.3 Logic

Up to now our working theory has been primarily C^0. For the definition of language and the development of basic syntactical ideas, the use of the class theory C^0 instead of the pure set theory S^0 has been a mere convenience, allowing us to refer to definable but infinite collections of finitary objects (such as variables, expressions, etc.) as objects in their own right, and in some cases providing simpler definitions or proofs than S^0 could do. Semantical concepts, on the other hand, cannot reasonably be discussed without reference to infinite structures, and S^0 is not adequate for this purpose.

The completeness theorem, in particular, requires the admission of infinite classes for its statement and proof; and we have used the completeness theorem to define a deductive system **ND** that precisely captures the semantic notion of entailment. With this accomplished, much of the subsequent development and investigation of systems of deduction may be undertaken without regard to their semantic derivation. This sort of discussion is often referred to as *proof-theoretic*, in contrast to the *model-theoretic* approach, in which—as the name implies—notions of structure and satisfaction are employed. Some model-theoretic arguments can be carried out in C^0, but often the model-theoretic approach relies not just on the existence of satisfactory models of syntactically consistent theories (i.e., the completeness theorem, provable in C^0) but also on the existence of satisfaction relations for arbitrary structures, for which the $\mathsf{Infinity}$ axiom is required. The natural theory for a typical model-theoretic argument is therefore $\mathsf{S}^0 + \mathsf{Infinity}$. From the foundational point of view, the admission of $\mathsf{Infinity}$ is a giant step, and one of our main concerns in these early chapters is to understand its import. We will therefore provide proof-theoretic proofs for syntactical theorems where possible.

By virtue of Theorem 2.183, C^0 is a conservative extension of S^0, i.e., any statement in the language of pure set theory that is provable in C^0 is provable in S^0, and we have used this to justify the position that C^0 has not crossed the ontologic threshold of infinitarity. From a practical standpoint, we have used (2.183) to justify the practice of accepting a presentation of a C^0 proof of a purely set-theoretical statement as proof that an S^0-proof exists. We note that the proof of (2.183) provides a method for constructing an S^0-proof from a C^0-proof, so this is really just a sophisticated variation on the usual practice, which is to provide a sketch of a proof in a given theory T as "proof" that a T-proof exists. Only in the simplest cases, such as (2.41.1), do we actually present a proof *per se*.

Any individual application of (2.183) for this purpose could of course be obviated by simply sketching the S^0-proof whose existence it asserts. If the proof of (2.183) required a powerful theory such as $\mathsf{S}^0 + \mathsf{Infinity}$ we might be reluctant to use it for this purpose and might prefer to do the tedious work of presenting all S^0-proofs directly. In other words, even though we might believe (2.183) to be true, we might be reluctant to rely on it.

In fact, Infinity is not required, and we will present a finitary proof of (2.183) later in this chapter. To avoid any appearance of impropriety we will take the trouble to present an S^0-proof of (2.183) directly, in preference to presenting a C^0-proof of the theorem and then invoking the theorem to infer the existence of an S^0-proof of itself.

(2.38) *To do this, it is of course necessary not just to present the proof of (2.183) in S^0; we must also have presented the proofs of any results referenced in that proof in S^0. In the remainder of this chapter, therefore, we will take care to show how to treat syntactical concepts in the context of S^0. It is easy to check that any relevant syntactical theorems presented prior to this point can be formulated and proved in S^0, and we will presume that this has been done.*

We should point out that S^0 does not have a unique status as a theory of the finitary. It is, in fact, much more powerful than is necessary for most proof-theoretical purposes. The term *finitistic* has historically been used to characterize a minimal set of proof-theoretic methods. In the modern view, this methodology is identified with *primitive recursive arithmetic* PRA. The proof-theoretic results of this chapter are actually all provable in PRA.

2.3.1 Inference, implication, validity

We now derive some important properties of the provability relation. These are for the most part familiar proof techniques, and—once derived—they will subsequently often be used without explicit recognition.

(2.39) **Theorem** $[\mathsf{S}^0]$ $\Theta \vdash \theta \to \zeta$ *iff* $\Theta \cup \{\theta\} \vdash \zeta$.

Remark In other words ζ may be inferred from θ and Θ iff the implication $\theta \to \zeta$ may be inferred from Θ.

Proof Suppose $\Theta \vdash \theta \to \zeta$. Then $\Theta \cup \{\theta\} \vdash \theta \to \zeta$. (This follows from the fact that by definition $\Theta \vdash \sigma$ iff $\Sigma \vdash \sigma$ for some finite $\Sigma \subseteq \Theta$. We typically only need to invoke Rule 0 explicitly when describing an actual proof sequence.) Also, $\Theta \cup \{\theta, \theta \to \zeta\} \vdash \zeta$ (from Rule 5). So from Rule 3 we conclude that $\Theta \cup \{\theta\} \vdash \zeta$. The converse direction follows from Rule 6. $\square$2.39

(2.40) **Definition**

1. $[\mathsf{S}^0]$ *A* validity *is a sentence that is derivable from the empty set (of premises). It is a theorem of pure logic.*
2. $[\mathsf{C}^0]$ *Equivalently, (by virtue of the completeness theorem) a validity is a sentence that is true under any interpretation.*

Thus, θ is a validity iff $0 \vdash \theta$, or, as we usually write it, $\vdash \theta$.

(2.41) **Theorem** $[\mathsf{S}^0]$ *Suppose σ, θ, and ζ are sentences.*

1. $\{\neg\neg\sigma\} \vdash \sigma$.

2. $\{\sigma\} \vdash \neg\neg\sigma$.
3. $\{\sigma, \neg\sigma\} \vdash \theta$.
4. $\{\sigma \to \neg\sigma)\} \vdash \neg\sigma$.
5. $\{\zeta\} \vdash \theta \to \zeta$.
6. $\{\neg\theta\} \vdash \theta \to \zeta$.

Proof 1 The sequence

$$\begin{aligned} \{\neg\neg\sigma, \neg\sigma\} &\Rightarrow \neg\sigma \\ \{\neg\neg\sigma, \neg\sigma\} &\Rightarrow \neg\neg\sigma \\ \{\neg\neg\sigma\} &\Rightarrow \sigma \end{aligned}$$

of sequents is justified by virtue of Rules 1, 1, and 2 (with $\{\neg\neg\sigma\}$ for Σ and $\neg\sigma$ for θ). Hence

$$\{\neg\neg\sigma\} \vdash \sigma. \tag{2.42}$$

2 Replacing σ by $\neg\sigma$ in (2.42), we have $\{\neg\neg\neg\sigma\} \vdash \neg\sigma$. Hence

$$\{\sigma, \neg\neg\neg\sigma\} \vdash \neg\sigma,$$

and since $\{\sigma, \neg\neg\neg\sigma\} \vdash \sigma$ as well, Rule 2 yields $\{\sigma\} \vdash \neg\neg\sigma$.

3 In (2.27.2) let Σ be $\{\sigma, \neg\sigma\}$, let σ be θ, and let θ be σ.

4 It suffices to show that $\{\sigma \to \neg\sigma\} \vdash \neg\sigma$. By Rule 5, $\{\sigma \to \neg\sigma, \sigma\} \vdash \neg\sigma$. By (2.42), $\{\sigma \to \neg\sigma, \neg\neg\sigma\} \vdash \sigma$, so by Rule 3, $\{\sigma \to \neg\sigma, \neg\neg\sigma\} \vdash \neg\sigma$. Now Rule 2 gives the desired result.

5 The sequence

$$\begin{aligned} \{\theta, \zeta\} &\Rightarrow \zeta \\ \{\zeta\} &\Rightarrow \theta \to \zeta \end{aligned}$$

is justified as a proof by Rule 1 and Rule 6.

6 By (2.41.3) $\{\theta, \neg\theta\} \vdash \zeta$, so by Rule 6 $\{\neg\theta\} \vdash \theta \to \zeta$. $\square$[2.41]

Theorem 2.41 essentially provides the following set of inference rules, which may be added to (2.27) without altering the set of derivable sequents.

(2.43) **Some derived inference rules for ND**

1. $\dfrac{}{\{\neg\neg\sigma\} \Rightarrow \sigma}$

2. $\dfrac{}{\{\sigma\} \Rightarrow \neg\neg\sigma}$

3. $\dfrac{}{\{\sigma, \neg\sigma\} \Rightarrow \theta}$

4. $\dfrac{}{\{\sigma \to \neg\sigma)\} \Rightarrow \neg\sigma}$

5. $$\frac{}{\{\zeta\} \Rightarrow \theta \to \zeta}$$

6. $$\frac{}{\{\neg\,\theta\} \Rightarrow \theta \to \zeta}$$

The use of one of these derived inference rules in a proof π should be understood as the insertion into π of a proof of the final sequent.

The following theorem gives provides additional derived inference rules, which differ from the preceding in having sequents above the line.

(2.44) **Theorem** [S^0]

1. *Suppose Θ is a set of sentences, and σ and θ are sentences.*
   1. *Suppose*
   $$\Theta \cup \{\neg\,\sigma\} \vdash \theta.$$
   Then
   $$\Theta \cup \{\neg\,\theta\} \vdash \sigma.$$
   2. *Suppose*
   $$\Theta \cup \{\sigma\} \vdash \theta.$$
   Then
   $$\Theta \cup \{\neg\,\theta\} \vdash \neg\,\sigma.$$
   3. *Suppose*
   $$\begin{aligned} &\Theta \cup \{\sigma\} \vdash \theta \\ \textit{and} \quad &\Theta \cup \{\neg\,\sigma\} \vdash \theta. \end{aligned}$$
   Then
   $$\Theta \vdash \theta.$$
2. *Equivalently, the following inference rules are valid for any finite set Σ of sentences.*
   1. $$\frac{\Sigma \cup \{\neg\,\sigma\} \Rightarrow \theta}{\Sigma \cup \{\neg\,\theta\} \Rightarrow \sigma}$$
   2. $$\frac{\Sigma \cup \{\sigma\} \Rightarrow \theta}{\Sigma \cup \{\neg\,\theta\} \Rightarrow \neg\,\sigma}$$
   3. $$\frac{\Sigma \cup \{\sigma\} \Rightarrow \theta \qquad \Sigma \cup \{\neg\,\sigma\} \Rightarrow \theta}{\Sigma \Rightarrow \theta}$$

Proof 1.1 Let π be a proof of $\Sigma \cup \{\neg\,\sigma\} \Rightarrow \theta$, where Σ is a finite subset of Θ. Let π' be π extended by the following sequence:

$$\begin{gathered} \Sigma \cup \{\neg\,\sigma, \neg\,\theta\} \Rightarrow \theta \\ \Sigma \cup \{\neg\,\sigma, \neg\,\theta\} \Rightarrow \neg\,\theta \\ \Sigma \cup \{\neg\,\theta\} \Rightarrow \sigma \end{gathered}$$

Then π' is a proof by virtue of Rules 0, 1, and 2.

1.2 Let π be a proof of $\Sigma \cup \{\sigma\} \Rightarrow \theta$, where Σ is a finite subset of Θ. Use RULE 0 to extend π to a proof of $\Sigma \cup \{\neg\neg\sigma, \sigma\} \Rightarrow \theta$; append a proof of $\{\neg\neg\sigma\} \Rightarrow \sigma$;[2.43.1] and then append

$$\begin{gathered}\Sigma \cup \{\neg\neg\sigma\} \Rightarrow \sigma \\ \Sigma \cup \{\neg\neg\sigma\} \Rightarrow \theta\end{gathered}$$

The result is a proof π' of $\Sigma \cup \{\neg\neg\sigma\} \Rightarrow \theta$. Now apply (2.44.1.1) with $\neg\sigma$ for σ to conclude that there is a proof of $\Sigma \cup \{\neg\theta\} \Rightarrow \neg\sigma$.

1.3 By virtue of (2.44.1, 2), there exist finite $\Sigma_0, \Sigma_1 \subseteq \Theta$ and proofs π_0 and π_1 of

$$\begin{aligned} &\Sigma_0 \cup \{\neg\theta\} \Rightarrow \sigma \\ \text{and}\quad &\Sigma_0 \cup \{\neg\theta\} \Rightarrow \neg\sigma. \end{aligned}$$

Let $\Sigma = \Sigma_0 \cup \Sigma_1$. Let π be $\pi_0 {}^\frown \pi_1$ extended by the sequence

$$\begin{gathered}\Sigma \cup \{\neg\theta\} \Rightarrow \sigma \\ \Sigma \cup \{\neg\theta\} \Rightarrow \neg\sigma \\ \Sigma \Rightarrow \theta\end{gathered}$$

Then π is justified as a proof by RULES 0, 0, and 2. $\square$[2.44]

2.3.2 Proof trees

Our definition[2.255] of a proof in the system **ND** of natural deduction as a linear sequence of sequents corresponds to the way proofs are traditionally presented, but it should be recognized that this mode of presentation is natural only in the context of the ubiquitous linearity of printed text. Just as we have chosen to represent linguistic expressions as tree structures, rather than as linear strings of symbols, so it is often useful to represent proofs as trees that reveal their logical structure.

The purpose of a proof tree is to keep track of the justification history of sequents within a proof according to the inference rules (2.27), and a proof tree may be represented graphically as a concatenation of inference rules, for example:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\sigma} \quad \dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\neg\sigma}}{\{\neg\neg\sigma\}\Rightarrow\sigma}}{\{\sigma\to\neg\sigma,\neg\neg\sigma\}\Rightarrow\sigma} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\sigma} \quad \dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\neg\sigma}}{\{\neg\neg\sigma\}\Rightarrow\sigma}}{\{\sigma\to\neg\sigma,\neg\neg\sigma\}\Rightarrow\sigma} \quad \dfrac{\dfrac{}{\{\sigma\to\neg\sigma,\sigma\}\Rightarrow\neg\sigma}}{\{\sigma\to\neg\sigma,\neg\neg\sigma,\sigma\}\Rightarrow\neg\sigma}}{\{\sigma\to\neg\sigma,\neg\neg\sigma\}\Rightarrow\neg\sigma}}{\{\sigma\to\neg\sigma\}\Rightarrow\neg\sigma}$$

Note that every time a given sequent is used in a proof tree its proof must be included as well. Thus, the subtree

$$\dfrac{\dfrac{\dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\sigma} \quad \dfrac{}{\{\neg\neg\sigma,\neg\sigma\}\Rightarrow\neg\neg\sigma}}{\{\neg\neg\sigma\}\Rightarrow\sigma}}{\{\sigma\to\neg\sigma,\neg\neg\sigma\}\Rightarrow\sigma}$$

occurs twice in the above tree. In a linear proof we avoid this duplication, but proof-theoretic manipulations are often more difficult as a result.

In order to distinguish multiple occurrences of sequents we regard the domain of a proof tree not as the set of its sequents but rather as the set of sequences leading from the base of the tree up to each occurrence of each sequent. A proof tree is therefore a finite tree of sequences in the sense of (3.180.2). For convenience, we stipulate that there be only one sequent "at the bottom", which we call the *root* of the tree, and which may be regarded as the sequent that the tree proves, although the tree actually proves every sequent in it, just as a linear proof does. It is convenient to replace RULES 5 and 7 by the following rules, which are equivalent in light of RULE 0.

5′. $\dfrac{}{\Sigma \cup \{\theta, \theta \to \zeta\} \Rightarrow \zeta}$

7′. $\dfrac{}{\Sigma \cup \left\{\psi\binom{v}{\tau}\right\} \Rightarrow \exists v\, \psi}$

We therefore have the following definition:

(2.45) **Definition** [S^0] *π is an* **ND**-proof tree $\overset{\text{def}}{\iff}$ *π is a finite tree of sequences of sequents such that there is a unique $S \in \pi$ of length 1, and for every nonzero $S \in \pi$, letting s be the last item in S, either*

1. *S has no proper extension in π, and either*
   1. $s = \Sigma \Rightarrow \sigma$ and $\sigma \in \Sigma$,
   2. $s = \Sigma \cup \{\theta, \theta \to \zeta\} \Rightarrow \zeta$, or
   3. $s = \Sigma \cup \left\{\psi\binom{v}{\tau}\right\} \Rightarrow \exists v\, \psi$; or
2. *S has exactly one immediate extension $S^\frown\langle s'\rangle$ in π, and either*
   1. $\dfrac{s'}{s} = \dfrac{\Sigma \Rightarrow \sigma}{\Sigma' \Rightarrow \sigma}$ *and* $\Sigma' \supseteq \Sigma$,
   2. $\dfrac{s'}{s} = \dfrac{\Sigma \cup \left\{\psi\binom{v}{c}\right\} \Rightarrow \sigma}{\Sigma \cup \{\exists v\, \psi\} \Rightarrow \sigma}$ *and c does not occur in the lower sequent, or*
   3. $\dfrac{s'}{s} = \dfrac{\Sigma \cup \{\theta\} \Rightarrow \zeta}{\Sigma \Rightarrow \theta \to \zeta}$ *; or*
3. *S has exactly two immediate extensions $S^\frown\langle s'\rangle$ and $S^\frown\langle s''\rangle$ in π, and either*
   1. $\dfrac{s' \qquad s''}{s} = \dfrac{\Sigma \cup \{\neg\,\sigma\} \Rightarrow \theta \qquad \Sigma \cup \{\neg\,\sigma\} \Rightarrow \neg\,\theta}{\Sigma \Rightarrow \sigma}$ *, or*
   2. $\dfrac{s' \qquad s''}{s} = \dfrac{\Sigma \Rightarrow \sigma \qquad \Sigma \cup \{\sigma\} \Rightarrow \theta}{\Sigma \Rightarrow \theta}$.

Given a linear proof π, of which s is the final sequent, it is easy to construct a proof tree π' with root s, consisting of sequences in π in reverse order. In general, there is more than one way to do this. Conversely, given a proof tree π with root s, we may construct a linear proof of s as a single sequence of which each member of π is a subsequence in reverse order. We therefore have the following theorem.

(2.46) **Theorem** [S^0] *A sequent is* **ND**-*provable iff it occurs in a proof tree iff it is the root sequent of a proof tree.*

2.3.3 Propositional logic

For certain purposes it is convenient to delineate that portion of logic that deals with the propositional connectives only, omitting quantification. This is *propositional logic*, and it operates in the context of *propositional languages*. A propositional language is a unisorted operational structure. Its individuals are referred to as *expressions* or more particularly as *propositions* to emphasize the context. Its operations correspond to propositional connectives, which may be taken to be any sufficient subset of negation, disjunction, conjunction, implication and bi-implication (or whatever else one wishes to use). The operations satisfy the *unique readability condition*[1.39] whereby any proposition p is the value of an operation in at most one way, i.e., if $p = Ft$ for some operation F and argument sequence t, then F and t are uniquely determined by p. The propositions that are not the value of any operation are *prime*.

The class Π of prime propositions of a propositional language is analogous to the signature of a predicate language, but it contains much less information, as prime propositions have no distinguishing characteristics from the point of view of propositional logic. A propositional language is determined up to homologic equivalence (suitably defined) by the cardinality of its class of prime propositions. We use '$\mathcal{L}^{\Pi}$' loosely to refer to a propositional language of which Π is the class of prime propositions, acknowledging that the precise identity of $\mathcal{L}^{\Pi}$ depends on the expression-building operations. We could, of course, define a standard propositional language built from Π, but we have no use for such a notion, as the propositional languages we will be concerned with will be derived from predicate languages,[2.47.1] and we will interpret expressions like '$\mathcal{L}^{\Pi}$' in this context.

(2.47) **Definition** [C^0] *Suppose $\mathcal{L}$ is predicate language.*

1. *The* propositional part of $\mathcal{L}$ $\overset{\text{def}}{=}$ $\mathcal{L}^{\mathsf{P}}$ *is that structure whose domain is the class of sentences of $\mathcal{L}$ and whose operations are the expression-building operations of $\mathcal{L}$ corresponding to propositional connectives.*

2. *A formula of $\mathcal{L}$ is* prime $\overset{\text{def}}{\iff}$ *it is the result of an argument specification or quantification operation.*

The prime propositions of $\mathcal{L}^{\mathsf{P}}$ are therefore the prime sentences of $\mathcal{L}$.

Suppose Π is a class of prime propositions. A Π-*interpretation* is a function $\mathfrak{I} : \Pi \to 2$, where 2 is the set $\{0, 1\}$ of truth values (1 for true and 0 for false). We let $p^{\mathfrak{I}} \overset{\text{def}}{=} \mathfrak{I}p$ so that our notation for the interpretation of propositions corresponds to our notation for interpretations of predicate and operation indices. Given such a function $\mathfrak{I}$, we define the truth value $\operatorname{Val}^{\mathfrak{I}} p \in 2$ of any proposition $p \in \mathcal{L}^{\Pi}$ by recursion on complexity in the expected way:

1. if $p \in \Pi$ then $\operatorname{Val}^{\mathfrak{I}} p = p^{\mathfrak{I}}$;

2. if $p = \neg\, q$ then
$$\operatorname{Val}^{\mathfrak{I}} p = 1 \leftrightarrow \operatorname{Val}^{\mathfrak{I}} q = 0;$$

3. etc.

We define *satisfaction* as the relational equivalent of the valuation operation, as for predicate logic:
$$\mathfrak{I} \textit{ satisfies } p \overset{\text{def}}{\iff} \mathfrak{I} \models p \overset{\text{def}}{\iff} \operatorname{Val}^{\mathfrak{I}} p = 1.$$

Note that—just as for valuation of terms of a predicate language—the existence of a (unique) valuation function extending any interpretation of the class Π of prime propositions of a propositional language can be demonstrated in C^0 even if Π is a proper class—it is only the quantification step in the corresponding definition for predicate formulas that requires that the domain of the relevant structure be a set. Hence, we will not always differentiate between an interpretation of the prime propositions of a propositional language and the entire valuation operation it induces.

The propositional analog of Theorem 1.69 is easily seen to be true.

(2.48) **Theorem** $[\mathsf{C}^0]$ *Suppose $\Pi \subseteq \Pi'$ are classes of propositions and $\mathfrak{I}$ is a Π-interpretation. Then there is a Π'-interpretation that is an expansion of $\mathfrak{I}$.*

We will say that a class Π of prime propositions (in whatever context) is *appropriate to* a class Θ of propositions $\overset{\text{def}}{\Longleftrightarrow}$ Π contains every prime proposition that occurs in Θ. The propositional analog of (1.70) is easily seen to follow from Theorem 2.48:

(2.49) *If Π and Π' are classes of prime propositions appropriate to a class Θ of propositions, then there is a Π-interpretation that satisfies Θ iff there is a Π'-interpretation that satisfies Θ.*

(2.50) **Definition** $[\mathsf{C}^0]$

1. *A propositional expression is* tautological $\overset{\text{def}}{\Longleftrightarrow}$ *it is true in every interpretation.*
2. *A* tautology *is a tautological propositional expression.*
3. *A class Θ of sentences is* propositionally satisfiable $\overset{\text{def}}{\Longleftrightarrow}$ *there is an interpretation that makes all the propositions in Θ true.*

Clearly, if ϕ is a sentence of a predicate language $\mathcal{L}$, and ϕ is tautological, then ϕ is valid. The converse is not true. For example, $\forall u\,(\phi \rightarrow \phi)$ is valid, but it is not a tautology, because as an expression of the propositional part $\mathcal{L}^{\mathsf{P}_{2.47.1}}$ of $\mathcal{L}$, it is a prime proposition, which may be false in an interpretation of $\mathcal{L}^{\mathsf{P}}$.

The goal of a propositional deductive system is to be able to derive exactly the tautologies of any propositional language, and we may obtain such a system by the method we used for predicate languages. Given the observation in the previous paragraph, this system must be (equivalent to) a fragment of the deductive system for predicate logic.

To carry out the the Henkin procedure for propositional systems, there are of course no constants and no steps having to do with quantification. Otherwise the construction is essentially unaltered, and we obtain the following system.

(2.51) **Natural deductive system for propositional logic NDP** *The notions of sequent, justification, and proof for propositional logic are just as for predicate logic, except that the inference rules are limited to* Rules 0, 1, 2, 3, 5, 6.

Definition $[\mathsf{C}^0]$ *Suppose Θ is a class of propositions.*

1. *Suppose σ is a proposition.* Θ propositionally proves σ $\overset{\text{def}}{\Longleftrightarrow}$ $\Theta \vdash^{\mathsf{P}} \sigma$ $\overset{\text{def}}{\Longleftrightarrow}$ *there exists a finite $\Sigma \subseteq \Theta$ and a proof π using the deductive system* **NDP**[2.51] *such that $\Sigma \Rightarrow \sigma$ occurs in π.*

2. Θ *is* propositionally consistent $\overset{\text{def}}{\Longleftrightarrow}$ *it is not the case that* $\Theta \vdash^{\mathsf{P}} \sigma$ *and* $\Theta \vdash^{\mathsf{P}} \neg\sigma$ *for any proposition* σ*.*

It follows easily that

(2.52) *Θ is propositionally inconsistent iff there exists a finite $\Sigma \subseteq \Theta$, a proposition σ, and a propositional proof π such that $\Sigma \Rightarrow \sigma$ and $\Sigma \Rightarrow \neg\sigma$ both occur in π.*

With this definition we have the completeness theorem for propositional logic:

(2.53) **Theorem** $[\mathsf{C}^0]$ *A countable theory Θ is propositionally consistent iff it is propositionally satisfiable.*

Proof The proof is just a simplified version of the proof of the completeness theorem for predicate logic. □[2.53]

As for predicate logic,[2.35] the compactness property of propositional logic follows directly:

(2.54) **Theorem** $[\mathsf{C}^0]$ *A countable theory Θ is propositionally consistent iff every finite subset of Θ is propositionally consistent; hence,*[2.53] *Θ is propositionally satisfiable iff every finite subset of Θ is propositionally satisfiable.*

2.3.4 Truth tables

A critical difference between propositional and predicate logic is that in the case of a proposition there is an algorithm for ascertaining whether it is valid (which is to say, tautological), whereas, in general, in the case of a predicate formula there is no such algorithm, as we will see.[4.65] A simple method for propositional logic is that of *truth tables.* Given a propositional expression E, built from a (necessarily finite) set Π of prime expressions, a truth table for E is basically a list of all possible interpretations for Π, of which there are $2^{|\Pi|}$, together with the value of E under each interpretation. The terminology derives from the natural presentation of such a list in tabular form, which we illustrate by example. Suppose

$$E = (P \wedge Q) \leftrightarrow (P \rightarrow Q).$$

We can make a truth table for E as follows:

P	Q	$P \wedge Q$	$P \rightarrow Q$	$(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
1	1	1	1	1
1	0	0	0	1
0	1	0	1	0
0	0	0	1	0

Clearly E is not tautological.

On the other hand, if

$$E = (P \vee R) \rightarrow (\neg P \rightarrow (\neg Q \rightarrow R))$$

then the truth table

P	Q	R	$P \vee R$	$\neg Q$	$\neg Q \to R$	$\neg P$	$\neg P \to (\neg Q \to R)$	E
1	1	1	1	0	1	0	1	1
1	1	0	1	0	1	0	1	1
1	0	1	1	1	1	0	1	1
1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	1
0	1	0	0	0	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	0	1	0	1	0	1

demonstrates that E is a tautology.

2.3.5 Propositional logic in S^0

Unlike predicate logic—in which the syntax can be treated in the framework of S^0, but the semantics requires the admission of infinite classes, which are available in C^0, but not necessarily in S^0—both the syntax and the semantics of propositional logic may be treated within the context of S^0. We are primarily interested proof-theoretical issues, and proofs are finitary objects, so we will focus on finite sets of propositions, but much of what we say is true for infinite sets as well.

Given a proposition p, if p is not prime then p is derived from one or two propositions by the action of a propositional connective: $\neg$, $\to$, ... This proposition or propositions are the *immediate subpropositions* of p. By iteration of this process, we obtain all the subpropositions of p. Note that if p is an expression of a predicate language, p may have subexpressions that are not subpropositions in this sense. Thus, for example, the subpropositions of

$$\forall u\,(\phi \wedge \psi) \vee (\exists u\,\theta \wedge \exists u\,\sigma)$$

are

1. $\forall u\,(\phi \wedge \psi) \vee (\exists u\,\theta \wedge \exists u\,\sigma)$,
2. $\forall u\,(\phi \wedge \psi)$,
3. $\exists u\,\theta \wedge \exists u\,\sigma$,
4. $\exists u\,\theta$, and
5. $\exists u\,\sigma$,

regardless of the syntactical structure of ϕ, ψ, θ, and σ. Of these, the prime propositions are $\forall u\,(\phi \wedge \psi)$, $\exists u\,\theta$, and $\exists u\,\sigma$.

It is easy to show in S^0 by induction on complexity that for every proposition p there is a finite set of propositions that consists of exactly the subpropositions of p. It follows that for every finite set Π of propositions there is a finite set $\overline{\Pi}$ of propositions consisting of exactly the subpropositions of members of Π.

For the purpose of the program (2.38) we note that

(2.55) *it is straightforward to show in S^0 that any interpretation of the prime propositions in a set Π of propositions may be extended uniquely to an interpretation of Π. (2.48) and (2.49) also hold in the context of S^0. Definition 2.50 is also applicable; and (2.53) is also a theorem of S^0.*

2.3.6 Another deductive system

We can use the notion of the propositional part[2.47] of a predicate language to isolate what are in effect the propositional and quantificational parts of our deductive system. Suppose ρ is a countable signature, Θ_0 is a ρ-theory, and ρ has infinitely many constant operation indices that do not occur in Θ_0. Recall that the Henkin procedure uses a fixed list $\langle \sigma_n \mid n \in \omega \rangle$ of the ρ-sentences to construct a sequence of consistent ρ-theories $\Theta_0 \subseteq \cdots \subseteq \Theta_n \subseteq \cdots$ $(n \in \omega)$. At the nth stage, after Θ_n has been defined, we consider σ_n for inclusion. If $\Theta_n \cup \{\sigma_n\}$ is inconsistent we let $\Theta_{n+1} = \Theta_n \cup \{\neg\, \sigma_n\}$; whereas if $\Theta_n \cup \{\sigma_n\}$ is consistent we let $\Theta_{n+1} = \Theta_n \cup \{\sigma_n\}$ unless $\sigma = \exists v\, \psi$ for some v and ψ, in which case we let $\Theta_{n+1} = \Theta_n \cup \{\sigma_n, \psi\binom{v}{\bar{c}}\}$, where c is the first constant operation index of ρ (with respect to some fixed enumeration) that does not occur in $\Theta_n \cup \{\sigma\}$.

Note that in forming Θ_{n+1} when $\sigma_n = \exists v\, \psi$ for some v and ψ, instead of adding $\psi\binom{v}{\bar{c}}$ just in case σ_n is added, we can achieve the same effect by adding

$$\sigma_n \to \psi\tbinom{v}{\bar{c}} \tag{2.56}$$

regardless of whether $\neg\, \sigma_n$ or σ_n is added. Let Ξ_0 be the class of sentences of the form (2.56) that would be added in this way. These sentences are a propositional embodiment of Rule 4 of our deduction system, and we may reasonably expect that any sentence deducible from Θ_0 is deducible from $\Theta_0 \cup \Xi_0$ without using Rule 4. If we let Ξ_1 be the class of all sentences

$$\psi\tbinom{v}{\tau} \to \exists v\, \psi,$$

where $\exists v\, \psi$ is an existential ρ-sentence and τ is a variable-free ρ-term, then any sentence deducible from Θ_0 should be deducible from $\Theta_0 \cup \Xi_0 \cup \Xi_1$ without using Rule 4 or Rule 7, i.e., using only the propositional rules.

The following definition and theorem state this result proof-theoretically. A sufficiently inclusive signature is to be understood.

(2.57) **Definition** $[\mathsf{S}^0]$

1. *A* witness sequence *for a set Σ of sentences is a sequence $W = \langle \xi_n \mid n \in N \rangle$ such that $N \leqslant \omega$ and for each $n < N$,*

$$\xi_n = \exists v_n\, \psi_n \to \psi_n\tbinom{v_n}{\bar{c}_n},$$

where c_n is a constant that does not occur in Σ or in $\{\xi_m \mid m \in n\}$.

2. *An* instance class *is a class of sentences of the form*

$$\psi\tbinom{v}{\tau} \to \exists v\, \psi.$$

(2.58) *To state the notion of inconsistency proof-theoretically we make use of some fixed sentence F that is tautologically false, e.g., $\mathsf{F} = \neg(\sigma \to \sigma)$ for some sentence σ in the relevant signature. Then a theory Θ is inconsistent iff $\Theta \vdash \mathsf{F}$, and Θ is propositionally inconsistent iff $\Theta \vdash^{\mathsf{P}} \mathsf{F}$.*

(2.59) **Theorem** $[\mathsf{S}^0]$ *A theory Θ is inconsistent iff there exist a finite $\Sigma \subseteq \Theta$, finite witness sequence W for Σ, and finite instance set I such that $\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent.*

The preceding description of a modified Henkin procedure constitutes a proof of this theorem in C^0. An S^0-proof is given in Note 10.3.

In view of Theorem 2.59 we have the following definition.

(2.60) **Definition** [S^0] *Suppose Θ is a theory.*

1. *A* proof of inconsistency of Θ in the sense of Definition 2.60 $\overset{\text{def}}{=}$ *a 3-sequence $\langle \Sigma, W, I\rangle$ such that*
   1. *Σ is a finite subset of Θ;*
   2. *W is a finite witness sequence for Σ;*
   3. *I is a finite instance set; and*
   4. *$\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent.*
2. *A* proof of σ from Θ $\overset{\text{def}}{=}$ *a proof of inconsistency of $\Theta \cup \{\neg\,\sigma\}$.*

2.3.7 Universal quantification

We now extend our languages and deductive system to incorporate the rest of the standard connectives and quantifiers, beginning with the universal quantifier.

(2.61) **Inference rules for universal quantification**

1. $\dfrac{}{\{\forall v\,\psi\} \Rightarrow \neg\,\exists v\,\neg\,\psi}$,
2. $\dfrac{}{\{\neg\,\exists v\,\neg\,\psi\} \Rightarrow \forall v\,\psi}$,

where ψ is an arbitrary formula with at most one free variable v. We extend $\vdash$ accordingly.

(2.62) **Theorem** [C^0] *The completeness theorem extends to universal quantification with the rules (2.61).*

Proof In keeping with the program outlined above[2.38] we will present the essence of the argument proof-theoretically in S^0, although the theorem itself—since it refers to the existence of models—is necessarily stated and proved in C^0.

(2.63) *By recursion on complexity, define $\phi \mapsto \phi^*$ for formulas ϕ so that*

1. *if ϕ is atomic then $\phi^* = \phi$;*
2. *if $\phi = \neg\,\psi$ then $\phi^* = \neg\,\psi^*$;*
3. *if $\phi = \psi_0 \rightarrow \psi_1$ then $\phi^* = \psi_0^* \rightarrow \psi_1^*$;*
4. *if $\phi = \exists v\,\psi$ then $\phi^* = \exists v\,\psi^*$; and*
5. *if $\phi = \forall v\,\psi$ then $\phi^* = \neg\,\exists v\ \neg\,\psi^*$.*

Thus, ϕ^* is the result of eliminating universal quantification from ϕ.

(2.64) **Claim** *For any formula ϕ,*

1. *$\{\phi\} \vdash \phi^*$; and*
2. *$\{\phi^*\} \vdash \phi$.*

Proof By induction on the complexity of formulas, where we use a notion of complexity that does not distinguish among formulas related by substitution of terms; e.g., we could use the number of quantifier phrases and propositional connectives as the measure of complexity. The induction steps for $\neg$ and $\rightarrow$ are entirely straightforward. It is also easily shown by induction on complexity that for any formula ψ, variable v and term τ,

$$\psi\tbinom{v}{\tau}^{*} = \psi^{*}\tbinom{v}{\tau}. \tag{2.65}$$

Suppose $\phi = \exists v\, \psi$ and the claim holds for all formulas of lower complexity, in particular for $\psi\binom{v}{\bar{c}}$ for any constant c. Thus,

$$\left\{\psi\tbinom{v}{\bar{c}}\right\} \Rightarrow \psi\tbinom{v}{\bar{c}}^{*}$$

is a provable sequent. Hence,[2.65]

$$\left\{\psi\tbinom{v}{\bar{c}}\right\} \Rightarrow \psi^{*}\tbinom{v}{\bar{c}}$$

is a provable sequent.

Let π be a proof of it. Let c be a constant that does not occur in ψ. If we append the sequence

$$\begin{aligned} \left\{\psi^{*}\tbinom{v}{\bar{c}}\right\} &\Rightarrow \exists v\, \psi^{*} \\ \left\{\psi\tbinom{v}{\bar{c}}, \psi^{*}\tbinom{v}{\bar{c}}\right\} &\Rightarrow \exists v\, \psi^{*} \\ \left\{\psi\tbinom{v}{\bar{c}}\right\} &\Rightarrow \exists v\, \psi^{*} \\ \exists v\, \psi &\Rightarrow \exists v\, \psi^{*} \end{aligned}$$

to π we have a proof of

$$\exists v\, \psi \Rightarrow \exists v\, \psi^{*},$$

which proves (2.64.1) for this case. To prove (2.64.2) we reverse the roles of ψ and ψ^{*}.

Suppose $\phi = \forall v\, \psi$ and the claim holds for ψ and its substituents. Let $\theta = \neg \exists v \neg \psi$. We use the inductive steps for $\neg$ and $\exists$ to show that $\{\theta\} \vdash \theta^{*}$ and $\{\theta^{*}\} \vdash \theta$. By definition[2.63.5] $\theta^{*} = \phi^{*}$, so $\{\theta\} \vdash \phi^{*}$ and $\{\phi^{*}\} \vdash \theta$. By virtue of the inference rules (2.61), $\{\phi\} \vdash \theta$ and $\{\theta\} \vdash \phi$, so $\{\phi\} \vdash \phi^{*}$ and $\{\phi^{*}\} \vdash \phi$. □[2.64]

Suppose Θ is a consistent class of sentences. Let $\Theta^{*} = \{\theta^{*} \mid \theta \in \Theta\}$. It is easy to use the claim[2.64] to show that Θ^{*} is consistent. Let $\mathfrak{S}$ be a satisfactory structure such that $\mathfrak{S} \models \Theta^{*}$. It is straightforward to show by induction on complexity that for any θ, $\mathfrak{S} \models \theta$ iff $\mathfrak{S} \models \theta^{*}$, from which it follows that $\mathfrak{S} \models \Theta$. □[2.62]

The following theorem states what amount to inference rules for universal quantification that are dual to Rules 7 and 4. Their derivation is left to the reader.

(2.66) **Theorem** $[\mathsf{S}^0]$ *Suppose ψ is a formula with* $\mathrm{Free}\,\psi \subseteq \{v\}$.

1. *For any term τ, $\{\forall v\, \psi\} \vdash \psi\binom{v}{\tau}$.*
2. *Suppose Θ is a set of sentences, c is a constant that does not occur in ψ or in any of the sentences of Θ, and $\Theta \vdash \psi\binom{v}{\bar{c}}$, then $\Theta \vdash \forall v\, \psi$.*

Clearly, we could have developed our deductive system with universal instead of existential quantification using the rules embodied in Theorem 2.66. In this formulation, in the proof of the completeness theorem using the Henkin construction, in forming Θ_{n+1} when $\sigma_n = \forall v\, \psi$, if σ_n is *not* added to Θ_{n+1} and $\neg\, \sigma_n$ therefore *is* added, then we also add $\neg\, \psi\binom{v}{\bar{c}}$ for some new constant c.

Equivalently, we could add

$$\psi\tbinom{v}{\bar{c}} \rightarrow \forall v\, \psi$$

to Θ_{n+1} regardless of the status of $\forall v\, \psi$, this being propositionally equivalent to

$$\neg\, \forall v\, \psi \rightarrow \neg\, \psi\tbinom{v}{\bar{c}}.$$

This observation leads to variations on the notion of proof described in Definition 2.60, suitable for languages in that use only the universal quantifier or both quantifiers:

(2.67) **Theorem** $[\mathsf{S}^0]$ *We obtain an equivalent notion of provability for languages with only the universal quantifier if in Definition 2.60 we use* witness sentences $\psi\binom{v_n}{\bar{c}_n} \rightarrow \forall v\, \psi$ *and instance sentences* $\forall v\, \psi \rightarrow \psi\binom{v}{\tau}$.
For languages with both quantifiers, we allow both forms for witnesses and instances.

Proof Straightforward. $\square^{2.67}$

We will make use of the convention that a symbol with the overarrow accent represents a finite sequence whose elements we represent by the corresponding unaccented symbol with subscripts, e.g., $\vec{v} = \langle v_0, \ldots, v_{n^-}\rangle$. An expression such as '$\forall\vec{v}$' or '$\exists\vec{v}$' is understood to abbreviate the corresponding sequence of quantifier phrases: '$\forall v_0 \ldots \forall v_{n^-}$' or '$\exists v_0 \cdots \forall v_{n^-}$', respectively; while $\forall\vec{v}$ and $\exists\vec{v}$ are the corresponding formula-generating operations. If $\vec{v}$ is the empty sequence, then $\forall\vec{v}$ and $\exists\vec{v}$ are the identity operation on formulas.

(2.68) **Theorem** $[\mathsf{S}^0]$ *Suppose* ψ *is a formula and* $\mathrm{Free}\,\psi \subseteq \{v_0, \ldots, v_{n^-}\}$. *Let* $\pi : n \overset{\text{bij}}{\rightarrow} n$ *be a permutation. (We often use 'permutation' to refer to a bijection of a finite ordinal or, more generally, any set, with itself.) Let* $\vec{v} = \langle v_0, \ldots, v_{n^-}\rangle$ *and* $\vec{v}\,' = \langle v_{\pi(0)}, \ldots, v_{\pi(n^-)}\rangle$. *Then*

1. $\forall\vec{v}\,\psi \vdash \forall\vec{v}\,'\,\psi$.
2. $\exists\vec{v}\,\psi \vdash \exists\vec{v}\,'\,\psi$.

Proof 1 Let $\vec{c} = \langle c_0, \ldots, c_{n^-}\rangle$ be an n-tuple of distinct constants, and let $\vec{\bar{c}} = \langle \bar{c}_0, \ldots, \bar{c}_{n^-}\rangle$. By induction on $m \in \{1, \ldots, n\}$ we show that $\forall\vec{v}\,\psi \vdash \forall v_m \cdots \forall v_{n^-}\psi\binom{v_0 \cdots v_{m^-}}{\bar{c}_0 \cdots \bar{c}_{m^-}}$. Letting $m = n$ we have, in particular

$$\forall\vec{v}\,\psi \vdash \psi\tbinom{v_0 \cdots v_{n^-}}{\bar{c}_0 \cdots \bar{c}_{n^-}}.$$

Now suppose c_m does not occur in ψ for any $m \in n$. Let $\vec{\bar{c}}\,' = \langle \bar{c}_{\pi(0)}, \ldots, \bar{c}_{\pi(n^-)}\rangle$. Note that

$$\psi\tbinom{v_0 \cdots v_{n^-}}{\bar{c}_0 \cdots \bar{c}_{n^-}} = \psi\tbinom{v'_0 \cdots v'_{n^-}}{\bar{c}'_0 \cdots \bar{c}'_{n^-}}.$$

By induction on $m \in \{1, \ldots, n\}$, using (2.66), we now show that

$$\forall\vec{v}\,\psi \vdash \forall v'_{m'} \cdots \forall v'_{n^-}\psi\tbinom{v'_0 \cdots v'_{m'^-}}{\bar{c}'_0 \cdots \bar{c}'_{m'^-}},$$

where $m' = n - m$. In particular, for $m = n$ (so $m' = 0$),

$$\forall \vec{v}\, \psi \vdash \forall v'_0 \cdots \forall v'_{n^-} \psi,$$

i.e.,

$$\forall \vec{v}\, \psi \vdash \forall \vec{v}'\, \psi,$$

as claimed.

2 Homologous. □2.68

2.3.8 Universal closure

Definition [S^0]

1. *A* universal closure *of a formula ϕ is a formula $\forall \vec{v}\, \phi$, where $\vec{v}$ is any enumeration of the free variables of ϕ. By Theorem 2.68 if ϕ' and ϕ'' are universal closures of ϕ, then $\phi' \vdash \phi''$, so all universal closures of a given formula are equivalent, and we may speak loosely of* the *universal closure of a formula ϕ. For definiteness, we may specify that the free variables be quantified in their natural order, with v_m before v_n if $m < n$, and we define $\bar{\forall}\phi$ to be this universal closure.*
2. *We extend the definition of validity by defining any formula ϕ as a validity $\overset{\text{def}}{\Longleftrightarrow}$ its universal closures are validities, and we accordingly define $\vdash \phi$ $\overset{\text{def}}{\Longleftrightarrow} \vdash \phi'$ for any (equivalently, every) universal closure ϕ' of ϕ.*

Theorem [S^0] *For any formula ϕ, variable v, and term τ,*

$$\vdash \forall v\, \phi \rightarrow \phi\binom{v}{\tau}.$$

2.3.9 Logical equivalence

We now extend our notion of language to include bi-implication with the following inference rules.

(2.69) **Inference rules for bi-implication** *Suppose σ and θ are sentences.*

1. $\dfrac{}{\{\sigma \rightarrow \theta, \theta \rightarrow \sigma\} \Rightarrow \sigma \leftrightarrow \theta}$
2. $\dfrac{}{\{\sigma \leftrightarrow \theta\} \Rightarrow \sigma \rightarrow \theta}$
3. $\dfrac{}{\{\sigma \leftrightarrow \theta\} \Rightarrow \theta \rightarrow \sigma}$

It follows that a complete consistent theory Θ contains $\sigma \leftrightarrow \theta$ iff it contains $\sigma \rightarrow \theta$ and $\theta \rightarrow \sigma$, whence it follows that the completeness theorem holds.

(2.70) **Theorem** [S^0] *Suppose ϕ and ψ are formulas. Then*

1. $\vdash (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi))$;

2. $\vdash(\phi\leftrightarrow\psi)\rightarrow(\phi\rightarrow\psi)$; *and*
3. $\vdash(\phi\leftrightarrow\psi)\rightarrow(\psi\rightarrow\phi)$.

Remark Remember that these statements assert that the universal closure of the formula following '$\vdash$' is provable.

Proof In general, to derive the universal closure of a formula χ, with distinct free variables $v_0,\dots,v_{n^-}$, it suffices to derive

$$\chi\binom{v_0\ \cdots\ v_{n^-}}{\bar c_0\ \cdots\ \bar c_{n^-}}, \tag{2.71}$$

where $c_1,\dots,c_{n^-}$ are distinct constants that do not occur in χ, and then use (2.66.2). In the present case, the corresponding sentences (2.71) follow fairly directly from the rules 2.69. $\square^{2.70}$

Definition [S^0] *Formulas ϕ and ψ are* equivalent $\overset{\text{def}}{\iff}$ *they have the same free variables and*

$$\vdash\phi\leftrightarrow\psi.$$

We relativize this notion in two ways:

1. *Suppose $\mathfrak{A}$ is a structure that interprets ϕ and ψ. Then ϕ and ψ are* equivalent for *or over* $\mathfrak{A}$ $\overset{\text{def}}{\iff}$ $\phi\overset{\mathfrak{A}}{\equiv}\psi$ $\overset{\text{def}}{\iff}$ *they have the same free variables and*

$$\mathfrak{A}\models\bar\forall\,(\phi\leftrightarrow\psi).$$

2. *ϕ and ψ are* equivalent modulo *or* over *a theory* Θ $\overset{\text{def}}{\iff}$ $\phi\overset{\Theta}{\equiv}\psi$ $\overset{\text{def}}{\iff}$ *they have the same free variables and*

$$\Theta\vdash\bar\forall\,(\phi\leftrightarrow\psi).$$

Clearly, ϕ and ϕ' are equivalent iff they are equivalent over every structure that interprets them (by the completeness theorem) iff they are equivalent modulo 0; however, formulas may be equivalent over some structures or theories and not others.

(2.72) **Theorem** [S^0] *Suppose ϕ is a formula and ϕ' is the result of applying a change of variables to ϕ.*[11] *Then*

$$\vdash\phi\leftrightarrow\phi'.$$

Proof Straightforward. $\square^{2.72}$

2.3.10 Disjunction and conjunction

Incorporation of the final two standard logical connectives into our deductive system may now be effected by adding the following two inference rules.

Inference rules for disjunction and conjunction *Suppose σ and θ are sentences.*

[11] Remember that a change of variables changes only bound variables.

1. $\dfrac{}{0 \Rightarrow (\sigma \vee \theta) \leftrightarrow ((\neg\, \sigma) \rightarrow \theta)}$

2. $\dfrac{}{0 \Rightarrow (\sigma \wedge \theta) \leftrightarrow \neg(\sigma \rightarrow \neg\, \theta)}$

It follows that a complete consistent theory Θ contains $\sigma \vee \theta$ iff it contains σ or θ; and Θ contains $\sigma \wedge \theta$ iff it contains σ and θ. From this it follows that the completeness theorem holds.

(2.73) **Theorem** $[\mathsf{S}^0]$

1. $\vdash (\phi \vee \psi) \leftrightarrow ((\neg\, \phi) \rightarrow \psi)$.
2. $\vdash (\phi \wedge \psi) \leftrightarrow \neg(\phi \rightarrow \neg\, \psi)$.

Proofs of the following useful propositions are left to the reader.

Theorem $[\mathsf{S}^0]$ *Suppose ϕ and ψ are formulas, σ and θ are sentences.*

1. $\vdash \phi \rightarrow (\phi \vee \psi)$.
2. $\vdash \psi \rightarrow (\phi \vee \psi)$.
3. $\vdash (\phi \wedge \psi) \rightarrow \phi$.
4. $\vdash (\phi \wedge \psi) \rightarrow \psi$.
5. $\{\sigma, \theta\} \vdash \sigma \wedge \theta$.

(2.74) **Theorem** $[\mathsf{S}^0]$ *For any formulas ϕ, ψ, and η,*

1. $\vdash (\phi \vee \psi) \leftrightarrow (\psi \vee \phi)$.
2. $\vdash (\phi \wedge \psi) \leftrightarrow (\psi \wedge \phi)$.
3. $\vdash ((\phi \vee \psi) \vee \eta) \leftrightarrow (\phi \vee (\psi \vee \eta))$.
4. $\vdash ((\phi \wedge \psi) \wedge \eta) \leftrightarrow (\phi \wedge (\psi \wedge \eta))$.

The proof is straightforward.

(2.75) **Definition** $[\mathsf{S}^0]$ *We define $\bigvee_{n\in N} \phi_n$ and $\bigwedge_{n\in N} \phi_n$ by recursion on $N = 1, 2, \ldots$:*

1. $\bigvee_{n\in 1} \phi_n = \bigwedge_{n\in 1} \phi_n = \phi_0$.
2. $\bigvee_{n\in N+1} \phi_n = (\bigvee_{n\in N} \phi_n) \vee \phi_N$.
3. $\bigwedge_{n\in N+1} \phi_n = (\bigwedge_{n\in N} \phi_n) \wedge \phi_N$.

We define

1. $\phi_0 \vee \cdots \vee \phi_{N^-} \stackrel{\text{def}}{=} \bigvee_{n\in N} \phi_n$ *and*
2. $\phi_0 \wedge \cdots \wedge \phi_{N^-} \stackrel{\text{def}}{=} \bigwedge_{n\in N} \phi_n$.

By virtue of Theorem 2.74, if $\pi : N \stackrel{\text{bij}}{\rightarrow} N$ is a permutation, then

Theorem $[\mathsf{S}^0]$

1. $\vdash \bigvee_{n\in N} \phi_n \leftrightarrow \bigvee_{n\in N} \phi_{\pi(n)}$.
2. $\vdash \bigwedge_{n\in N} \phi_n \leftrightarrow \bigwedge_{n\in N} \phi_{\pi(n)}$.

The proof is a straightforward induction.

If Φ is a finite set of formulas then '$\bigvee \Phi$' and '$\bigwedge \Phi$', while they do not have a definite *syntactical* meaning, nevertheless have a definite *semantical* meaning, as all of the syntactical interpretations arising from the various orderings of Φ are logically equivalent. We will freely make use of these equivalences without explicit recognition. We will also regard $\bigvee 0$ and $\bigwedge 0$ (where 0 is the empty set) as meaningful. The former is always false, while the latter is always true. Note that with these values we could have begun the recursion in Definition 2.75 at $N = 0$.

With these conventions, we have the following validities.

Theorem $[\mathsf{S}^0]$ *For any finite set Φ of formulas and any $\phi \in \Phi$,*

1. $\vdash \phi \rightarrow \bigvee \Phi$.
2. $\vdash \bigwedge \Phi \rightarrow \phi$.

Theorem $[\mathsf{S}^0]$ *If σ is a sentence and Σ is a finite set of sentences then $\Sigma \vdash \sigma$ iff $\vdash \bigwedge \Sigma \rightarrow \sigma$.*

2.3.11 Identity

To incorporate identity we again look ahead to the proof of the completeness theorem. Given a maximal consistent theory Θ with witnesses, we must now construct a model $\mathfrak{A}$ so that for any terms τ_0, τ_1, if $\tau_0 = \tau_1 \in \Theta$, then $\tau_0^{\mathfrak{A}} = \tau_1^{\mathfrak{A}}$. The simplest way to effect this is to let the elements of $|\mathfrak{A}|$ be equivalence classes of terms modulo the relation

$$\tau_0 \equiv^{\Theta} \tau_1 \overset{\text{def}}{\iff} (\tau_0 = \tau_1) \in \Theta. \tag{2.76}$$

The first issue is that if we do not have an axiom of infinity, these equivalence classes may be proper classes, and they are technically unsuitable as elements of a structure. Since we are dealing with countable languages, there exists an enumeration $\langle \epsilon_0, \epsilon_1, \dots \rangle$ of the expressions, and a simple solution is to use the first term in each equivalence class of terms as the representative of the class. In the following discussion we will use the nomenclature of equivalence classes for convenience.

The more substantive issue is that we must ensure that (2.76) defines an equivalence relation, and that (2.14) and (2.15) may be modified so as to apply to equivalence classes. The following rules suffice.

(2.77) **Inference rules for identity** *Suppose $c_0, \dots, c_{n^-}, c'_0, \dots, c'_{n^-}$ are constant indices (not necessarily distinct), and F and P are n-ary operation and predicate indices, respectively.*

1. $$\frac{}{\{\bar{c}_m = \bar{c}'_m \mid m \in n\} \Rightarrow \tilde{F}\langle \bar{c}_0, \dots, \bar{c}_{n^-}\rangle = \tilde{F}\langle \bar{c}'_0, \dots, \bar{c}'_{n^-}\rangle}$$
2. $$\frac{}{\{\bar{c}_m = \bar{c}'_m \mid m \in n\} \Rightarrow \tilde{P}\langle \bar{c}_0, \dots, \bar{c}_{n^-}\rangle \leftrightarrow \tilde{P}\langle \bar{c}'_0, \dots, \bar{c}'_{n^-}\rangle}$$

(2.78) **Definition** $[\mathsf{S}^0]$ *When we wish to draw attention to the distinction, we use '$\vdash^=$' to denote the provability relation for languages with identity using the inference rules 2.77 in addition to the rules previously set forth.*

We will soon see[§ 2.4.4] that it is not necessary to be specific on this point, since if $\Theta \vdash^= \theta$, where Θ and θ do not involve the identity predicate, then $\Theta \vdash \theta$.

Another way to treat identity is with axioms instead of inference rules.

(2.79) **Axioms of identity**

1.
$$\bigwedge_{m\in n} \bar{v}_m = \bar{v}'_m \to \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle = \tilde{F}\langle \bar{v}'_0, \ldots, \bar{v}'_{n^-}\rangle,$$
for each n-ary operation index F, and

2.
$$\bigwedge_{m\in n} \bar{v}_m = \bar{v}'_m \to \tilde{P}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle \leftrightarrow \tilde{P}\langle \bar{v}'_0, \ldots, \bar{v}'_{n^-}\rangle,$$
for each n-ary predicate index P,

where $v_0, \ldots, v_{n^-}, v'_0, \ldots, v'_{n^-}$ are any distinct variables.

It is straightforward to show that the inference rules (2.77) may be eliminated if the axioms (2.79) are available.

(2.80) **Theorem** [S^0] *Suppose u, v, w are variables.*

1. $\vdash u = u$.
2. $\vdash u = v \to v = u$.
3. $\vdash u = v \wedge v = w \to u = w$.

Proof 1 Let c be an arbitrary constant. Then by (2.77.1) with $n = 0$ and $F = c$, since $\bar{c} = \tilde{c}0$,
$$\vdash \bar{c} = \bar{c}.$$
By (2.66.2)
$$\vdash \forall u\, u = u,$$
i.e.,
$$\vdash u = u.$$

2 Let c, c' be distinct constants. By (2.80.1)
$$\vdash \bar{c} = \bar{c}.$$
By (2.77.2) with $n = 2$, $P = 0$ (the standard index for identity), $\langle c_0, c_1\rangle = \langle c, c\rangle$ and $\langle c'_0, c'_1\rangle = \langle c', c\rangle$,
$$\{\bar{c} = \bar{c}', \bar{c} = \bar{c}\} \vdash \bar{c} = \bar{c} \leftrightarrow \bar{c}' = \bar{c},$$
so (omitting a few steps)
$$\vdash \bar{c} = \bar{c}' \to \bar{c}' = \bar{c},$$
so
$$\vdash u = v \to v = u.$$

3 The proof is left to the reader. $\square$[2.80]

(2.81) **Theorem** [S^0]

1. *Suppose τ is a term, and $v_0, \ldots, v_{n^-}$ are distinct variables in* $\operatorname{Free}\tau$. *Suppose $\tau_0, \ldots \tau_{n^-}, \tau'_0 \ldots, \tau'_{n^-}$ are terms. Then*

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \tau\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) = \tau\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right).$$

2. *Suppose ϕ is a formula, and $v_0, \ldots, v_{n^-}$ are distinct variables in* $\operatorname{Free}\phi$. *Suppose $\tau_0, \ldots \tau_{n^-}, \tau'_0 \ldots, \tau'_{n^-}$ are terms, and for each $m \in n$, τ_m and τ'_m are free for v_m in ϕ.*[1.16] *Then*

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \left(\phi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \phi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right).$$

Proof 1 A straightforward induction on the complexity of terms.

2 A straightforward induction on the complexity of formulas. The condition on variable binding enters into the quantifier steps of the induction. Suppose $\phi = \exists v\, \psi$, for example. Then $v \notin \{v_0, \ldots, v_{n^-}\}$, so

$$\phi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) = (\exists v\, \psi)\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) = \exists v\, \left(\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right)\right).$$

If τ_m and τ'_m are free for v_m in ϕ then $v \notin \bigcup_{m\in n} \operatorname{Free}\tau_m \cup \bigcup_{m\in n}\operatorname{Free}\tau'_m$. τ_m and τ'_m are also free for v_m in ψ, so by induction hypothesis,

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \left(\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right).$$

Since $v \notin \bigcup_{m\in n} \operatorname{Free}\tau_m \cup \bigcup_{m\in n}\operatorname{Free}\tau'_m$, v does not occur in the antecedent, so

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \forall v\, \left(\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right). \tag{2.82}$$[12]

Hence

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \left(\exists v\, \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \exists v\, \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right),$$

i.e.,

$$\vdash \left(\bigwedge\nolimits_{m\in n} \tau_m = \tau'_m\right) \to \left(\phi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \phi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right).$$

$\square$[2.81]

The following theorem summarizes our deductive system.

(2.83) **Theorem** [S^0]

1. *The following properties of the provability relation define a complete system of deduction for first-order predicate logic without identity, with operations, and with the full set of logical connectives and quantifiers: $\neg, \to, \leftrightarrow, \wedge, \vee, \exists, \forall$. σ, θ, and ζ are arbitrary sentences. Σ and Σ' are arbitrary finite sets of sentences; ψ is an arbitrary formula with at most one free variable, v; c is an arbitrary constant; and τ is an arbitrary variable-free term.*

---

[12] Recall that $\vdash \theta \overset{\text{def}}{\iff} \vdash \bar\forall\theta$, where $\bar\forall\theta$ is the universal closure of θ. Thus, (2.82) is of the form $\vdash \forall u, v\, (A(u) \to B(u,v))$, which implies $\vdash \forall u\, (A(u) \to \forall v\, B(u,v))$.

0. *If* $\Sigma \vdash \sigma$ *then* $\Sigma' \vdash \sigma$ *for any* $\Sigma' \supseteq \Sigma$.
1. *If* $\sigma \in \Sigma$, *then* $\Sigma \vdash \sigma$.
2. *If* $\Sigma \cup \{\neg\sigma\} \vdash \theta$ *and* $\Sigma \cup \{\neg\sigma\} \vdash \neg\theta$ *then* $\Sigma \vdash \sigma$.
3. *If* $\Sigma \vdash \sigma$ *and* $\Sigma \cup \{\sigma\} \vdash \theta$ *then* $\Sigma \vdash \theta$.
4. *If* c *does not occur in* ψ, *in* σ, *or in any of the sentences of* Σ, *and* $\Sigma \cup \left\{\psi\binom{v}{\bar{c}}\right\} \vdash \sigma$, *then* $\Sigma \cup \{\exists v\,\psi\} \vdash \sigma$.
5. $\{\theta, \theta \to \zeta\} \vdash \zeta$.
6. *If* $\Sigma \cup \{\theta\} \vdash \zeta$ *then* $\Sigma \vdash \theta \to \zeta$.
7. $\left\{\psi\binom{v}{\tau}\right\} \vdash \exists v\,\psi$.
8. $\{\forall v\,\psi\} \vdash \psi\binom{v}{\tau}$.
9. *If* c *is a constant that does not occur in* ψ *or in any of the sentences of* Σ, *and* $\Sigma \vdash \psi\binom{v}{\bar{c}}$, *then* $\Sigma \vdash \forall v\,\psi$.
10. $\vdash (\sigma \to \theta) \to ((\theta \to \sigma) \to (\sigma \leftrightarrow \theta))$.
11. $\vdash (\sigma \leftrightarrow \theta) \to (\sigma \to \theta)$.
12. $\vdash (\sigma \leftrightarrow \theta) \to (\theta \to \sigma)$.
13. $\vdash (\sigma \vee \theta) \leftrightarrow ((\neg\sigma) \to \theta)$.
14. $\vdash (\sigma \wedge \theta) \leftrightarrow \neg(\sigma \to \neg\theta)$.

2. *For logic with identity, the following additional properties suffice.* $v_0, \ldots, v_{n^-}$ *are distinct variables,* τ *is a term, and* ϕ *is a formula, with* $\operatorname{Free}\tau = \operatorname{Free}\phi = \{v_0, \ldots, v_{n^-}\}$. $\tau_0, \ldots, \tau_{n^-}, \tau'_0, \ldots, \tau'_{n^-}$ *are terms, and for each* $m \in n$, τ_m *and* τ'_m *are free for* v_m *in* ϕ.[13]
   1. $\{\tau_m = \tau'_m \mid m \in n\} \vdash \tau\binom{v_0 \cdots v_{n^-}}{\tau_0 \cdots \tau_{n^-}} = \tau\binom{v_0 \cdots v_{n^-}}{\tau'_0 \cdots \tau'_{n^-}}$.
   2. $\{\tau_m = \tau'_m \mid m \in n\} \vdash \phi\binom{v_0 \cdots v_{n^-}}{\tau_0 \cdots \tau_{n^-}} \leftrightarrow \phi\binom{v_0 \cdots v_{n^-}}{\tau'_0 \cdots \tau'_{n^-}}$.

Remark This list comprises the original set of seven rules[2.27] and the additional rules to handle universal quantification,[2.61] bi-implication,[2.69] conjunction and disjunction,[2.73] and identity.[2.77]

Proof In proving the original completeness theorem we have shown that (2.83.1.0–7) suffice to handle $\neg$, $\to$, and $\exists$; and we have essentially proved the rest in our discussion of the various extensions. We leave it to the interested reader to supply any omitted details. $\square$[2.83]

2.3.12 Substitution of equivalents

(2.84) **Theorem** [S^0] *Suppose* ϕ *is a formula and* η *is the expression that occurs at a place* p *in* ϕ. *Let* $\vec{v} = \langle v_0, \ldots, v_{n^-}\rangle$ *be an enumeration of those variables* v *for which there is an occurrence of* v *in* η *that is free in* η *but bound in* ϕ.[14] *Suppose* η' *is an expression of the same type (term or formula) as* η *with the same free variables. Let* $\phi' = \phi\left\{{p \atop \eta'}\right\}$.[1.48]

[13] As it is no harder to state these properties in a general form than it is in the original specific form,[2.77] we have chosen to do so here. Note that (2.77) was already more general than it had to be: it would have sufficed to state it for a single substitution.

[14] Note that if any occurrence of v that is free in η is bound in ϕ, then all occurrences of v that are free in η are bound (by the same quantifier occurrence) in ϕ.

1. *Suppose η is a term. Then*
$$\vdash \forall \vec{v}\ \eta = \eta' \rightarrow (\phi \leftrightarrow \phi').$$
2. *Suppose η is a formula. Then*
$$\vdash \forall \vec{v}\,(\eta \leftrightarrow \eta') \rightarrow (\phi \leftrightarrow \phi').$$

Remark η and η' may have free variables not in $\{v_0, \ldots, v_{n\text{-}}\}$, which may also occur free in ϕ. These are universally quantified in the process of forming the universal closures of the expressions shown, which are the implicit arguments of '$\vdash$' above.

If we let ϕ be a term then η and η' are necessarily terms, and all variable occurrences are free, so $n = 0$, and the analog of (2.84.1) is
$$\vdash \eta = \eta' \rightarrow \phi = \phi',$$
which is an instance of (2.81).

Proof This is easily proved by induction on the complexity of expressions in which η occurs. □2.84

Note that this theorem does not require that η and η' be equivalent expressions, i.e., that $\vdash \eta = \eta'$, if η, η' are terms, or $\vdash \eta \leftrightarrow \eta'$, if η, η' are formulas. Of course, if η and η' *are* logically equivalent, then ϕ and ϕ' are logically equivalent.

2.3.13 Normal forms

We began this chapter by limiting the set of propositional connectives for technical reasons to negation and implication. Another useful set of propositional connectives consists of negation, disjunction, and conjunction. Unlike the previous set, this set is not minimal, as we could obviously omit either disjunction or conjunction; however, the symmetry of including both members of this dual pair of operations more than compensates for the lack of parsimony.

2.3.13.1 Disjunctive and conjunctive normal forms

Recall that a prime formula is one that does not occur as the value of an expression-building operation corresponding to a propositional connective, i.e., it is either atomic, existential, or universal. In the following definition and discussion, disjunctions and conjunctions are assumed to be of non-empty finite sets of formulas.

Definition $[\mathsf{S}^0]$

1. *A formula is in* conjunctive normal form $\overset{\text{def}}{\Longleftrightarrow}$ *it is a conjunction of disjunctions of prime formulas and their negations.*
2. *A formula is in* disjunctive normal form $\overset{\text{def}}{\Longleftrightarrow}$ *it is a disjunction of conjunctions of prime formulas and their negations.*

(2.85) **Theorem** $[\mathsf{S}^0]$ *Every formula is propositionally equivalent to a formula in disjunctive normal form and to a formula in conjunctive normal form.*

Proof See Note 10.4. □

2.3.13.2 Prenex form

(2.86) **Definition** [S^0] *A formula ϕ is* prenex $\overset{\text{def}}{\iff}$ $\phi = \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n^-} v_{n^-}\, \psi$*, where $v_0, \ldots, v_{n^-}$ are distinct variables, $\mathsf{Q}_m = \exists$ or $\forall$ for each $m \in n$, and ψ is quantifier-free. ψ is the* matrix *of ϕ.*

(2.87) **Theorem** [S^0] *Any formula ϕ is logically equivalent to a prenex formula.*

Proof By induction on complexity making use of the following validities:

1. $\neg \exists v\, \phi \leftrightarrow \forall v\, \neg \phi$;
2. $\neg \forall v\, \phi \leftrightarrow \exists v\, \neg \phi$;
3. $(\exists v\, \psi \vee \psi') \leftrightarrow \exists u \left(\psi\binom{v}{u} \vee \psi'\right)$, if u does not occur free in ψ or ψ';
4. $(\exists v\, \psi \wedge \psi') \leftrightarrow \exists u \left(\psi\binom{v}{u} \wedge \psi'\right)$, if u does not occur free in ψ or ψ';
5. $(\forall v\, \psi \vee \psi') \leftrightarrow \forall u \left(\psi\binom{v}{u} \vee \psi'\right)$, if u does not occur free in ψ or ψ';
6. $(\forall v\, \psi \wedge \psi') \leftrightarrow \forall u \left(\psi\binom{v}{u} \wedge \psi'\right)$, if u does not occur free in ψ or ψ'.

The point in (2.87.3–6) is that if u does not occur free in ψ', then $\psi' \leftrightarrow \exists u\, \psi'$ is valid.[15]

If we wish, we may extend this list to cover implication and bi-implication, or we may suppose we have first converted to a form involving only negation, disjunction, and conjunction. $\square^{2.87}$

(2.88) **Definition** [S^0] *A formula is* purely existential *(*purely universal*)* $\overset{\text{def}}{\iff}$ *it is prenex with only existential (universal) quantifiers.*

2.4 Theories

Suppose ρ is a signature.[16] Recall that a ρ-theory is a class of ρ-sentences. A theory Θ is consistent iff $\Theta \nvdash (\sigma \wedge \neg \sigma)$ for some (equivalently, for all) ρ-sentences σ. By the completeness theorem, Θ is consistent iff Θ has a satisfactory model.

Definition [C^0] *Given a theory Θ in a signature ρ, $\rho^\Theta \overset{\text{def}}{=}$ the smallest contraction ρ' of ρ such that Θ is a ρ'-theory, i.e., ρ' has just the ρ-indices that occur in Θ.*

Definition [C^0]

1. *A theory Θ is* complete $\overset{\text{def}}{\iff}$ *for every ρ^Θ-sentence σ, $\Theta \vdash \sigma$ or $\Theta \vdash \neg \sigma$.*
2. *A* complete ρ-theory *is a theory Θ such that $\rho^\Theta = \rho$ and Θ is complete.*

Clearly, a complete consistent ρ-theory is maximal among consistent theories Θ such that $\rho^\Theta = \rho$. It is also true that if Θ is maximal among theories Θ such that $\rho^\Theta = \rho$, Θ is complete. For suppose σ is a ρ-sentence. If $\Theta \nvdash \sigma$ then $\Theta \cup \{\neg \sigma\}$ is consistent, and since Θ is maximal, $\neg \sigma \in \Theta$, so $\Theta \vdash \neg \sigma$.

[15] It is worth noting that the validity of this equivalence depends in general on our having excluded empty structures. For example, if ψ' is, say, $\sigma \vee \neg \sigma$ for some sentence σ, which is true in any structure, then in the empty structure, since all existential formulas are false, $\exists u\, \psi'$ is false, while ψ' is true.

[16] We continue to restrict our attention to countable signatures.

2.4.1 Extensions of theories

Definition [C^0]

1. *The* deductive closure *or simply the* closure *of a theory* $\Theta \overset{\text{def}}{=} \overline{\Theta} \overset{\text{def}}{=}$ *the class of* ρ^{Θ}*-sentences* σ *such that* $\Theta \vdash \sigma$. *The* ρ*-closure of* Θ *is the class of* ρ*-sentences* σ *such that* $\Theta \vdash \sigma$.

2. Θ *is* deductively closed *or simply* closed $\overset{\text{def}}{\iff} \overline{\Theta} = \Theta$.[17]

3. *Suppose* ρ *is a signature, and* ρ' *expands* ρ.

    1. *Suppose* Θ' *is a* ρ'*-theory. The* restriction of Θ' to $\rho \overset{\text{def}}{=} \Theta'|\rho \overset{\text{def}}{=} \overline{\Theta}' \cap \mathcal{F}^{\rho}$.[18]

    2. *Suppose* Θ *is a* ρ*-theory,* ρ' *expands* ρ*, and* Θ' *is a* ρ'*-theory.*

        1. Θ' extends *or* is an extension of $\Theta \overset{\text{def}}{\iff} \Theta \subseteq \Theta'|\rho$ $(\leftrightarrow \Theta \subseteq \overline{\Theta'} \leftrightarrow \overline{\Theta} \subseteq \overline{\Theta'})$.

        2. Θ' *is a* conservative extension *of* $\Theta \overset{\text{def}}{\iff} \Theta'$ *extends* Θ *and* $\overline{\Theta'}|\rho = \overline{\Theta}$.

(2.89) **Theorem** [C^0]

1. *If* Θ' *is a conservative extension of* Θ *then, in particular, if* Θ *is consistent then* Θ' *is consistent.*

2. *Suppose* Θ *is a theory and* ρ *is an expansion of* ρ^{Θ}. *Then the* ρ*-closure of* Θ *is a conservative extension of* Θ.

Proof 1 Suppose Θ' is inconsistent. Let σ be a ρ^{Θ}-sentence. Then $\Theta' \vdash (\sigma \wedge \neg\, \sigma)$. If Θ' is a conservative extension of Θ then $\Theta \vdash (\sigma \wedge \neg\, \sigma)$, so Θ is inconsistent.

2 See (2.13). $\square^{2.89}$

(2.90) **Theorem** [S^0] *Suppose* Θ *is a set of sentences,* ψ *is a formula,* $v_0, \ldots, v_{n\text{-}}$ *are distinct variables,* $\operatorname{Free}\psi \subseteq \{v_0, \ldots, v_{n\text{-}}\}$*, and* $c_0, \ldots, c_{n\text{-}}$ *are distinct constants that do not occur in* Θ *or in* ψ.

1. *If* Θ *is consistent then*

    1. $\Theta \cup \left\{\exists v_0, \ldots, v_{n\text{-}}\, \psi\right\} \rightarrow \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} \\ \bar{c}_0 & \cdots & \bar{c}_{n\text{-}} \end{smallmatrix}\right)\}$ *is consistent; and*

    2. $\Theta \cup \left\{\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} \\ \bar{c}_0 & \cdots & \bar{c}_{n\text{-}} \end{smallmatrix}\right) \rightarrow \forall v_0, \ldots, v_{n\text{-}}\, \psi\right\}$ *is consistent.*

2. *If* $\Theta \cup \{\exists v_0, \ldots, v_{n\text{-}}\, \psi\}$ *is consistent then* $\Theta \cup \left\{\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} \\ \bar{c}_0 & \cdots & \bar{c}_{n\text{-}} \end{smallmatrix}\right)\right\}$ *is consistent.*

3. *If* $\Theta \cup \left\{\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} \\ \bar{c}_0 & \cdots & \bar{c}_{n\text{-}} \end{smallmatrix}\right)\right\}$ *is consistent then* $\Theta \cup \{\forall v_0, \ldots, v_{n\text{-}}\, \psi\}$ *is consistent.*

[17] Theories with the same deductive closure are equivalent for all practical purposes, and in the following definitions we avoid awkward and superfluous circumlocutions by dealing essentially with the closures of the relevant theories.

[18] Note that $\Theta'|\rho$ is deductively closed (as a ρ-theory), since any ρ-deduction from Θ' is also a ρ'-deduction.

Proof We will prove (2.90.1.2); (2.90.1.1) is dual to (2.90.1.2) and therefore has a dual proof, and it also follows directly from it. Let

$$\sigma = \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \bar{c}_0 & \cdots & \bar{c}_{n^-} \end{smallmatrix}\right) \to \forall v_0, \ldots, v_{n^-}\, \psi.$$

We will prove the contrapositive. Thus, suppose $\Theta \cup \{\sigma\}$ is inconsistent. Then for some finite $\Sigma \subseteq \Theta$, some finite witness sequence W for $\Sigma \cup \{\sigma\}$, and some finite instance set I as in (2.67),

$$\Sigma \cup \{\sigma\} \cup \operatorname{im} W \cup I \tag{2.91}$$

is propositionally inconsistent.

Let

$$W' = \langle \xi'_0, \ldots, \xi'_{n^-} \rangle ^\frown W,$$

where for each $m \in n$

$$\xi'_m = \forall v_{m+1} \cdots \forall v_{n^-}\, \psi\left(\begin{smallmatrix} v_0 & \cdots & v_m \\ \bar{c}_0 & \cdots & \bar{c}_m \end{smallmatrix}\right) \to \forall v_m\, \forall v_{m+1} \cdots \forall v_{n^-}\, \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{m^-} \\ \bar{c}_0 & \cdots & \bar{c}_{m^-} \end{smallmatrix}\right).$$

Since the new constants in W are necessarily distinct from $c_0, \ldots, c_{n^-}$, and $c_0, \ldots, c_{n^-}$ do not occur in Σ, W' is a witness sequence for Σ. Note that $\{\xi'_m \mid m \in n\} \vdash^{\mathsf{P}} \sigma$, so $\operatorname{im} W' \vdash^{\mathsf{P}} \sigma$. Hence[2.91]

$$\Sigma \cup \operatorname{im} W' \cup I$$

is propositionally inconsistent. Thus, Θ is inconsistent.

(2.90.2) and (2.90.3) are direct consequences of (2.90.1). $\square$[2.90]

We have presented (2.90) as a theorem of S^0 in line with the program (2.38), but it extends directly to the setting of C^0, as in the following theorem, which formulates the result in terms of conservative extension.

(2.92) **Theorem** [C^0] *Suppose Θ is a theory, ψ is a formula, $v_0, \ldots, v_{n^-}$ are distinct variables,* $\operatorname{Free}\psi \subseteq \{v_0, \ldots, v_{n^-}\}$*, and $c_0, \ldots, c_{n^-}$ are distinct constants that do not occur in Θ or in ψ. Then*

$$\Theta \cup \left\{\exists v_0, \ldots, v_{n^-}\, \psi \to \psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \bar{c}_0 & \cdots & \bar{c}_{n^-} \end{smallmatrix}\right)\right\}$$

and

$$\Theta \cup \left\{\psi\left(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ \bar{c}_0 & \cdots & \bar{c}_{n^-} \end{smallmatrix}\right) \to \forall v_0, \ldots, v_{n^-}\, \psi\right\}$$

are conservative extensions of Θ.

2.4.2 Herbrand's theorem

(2.93) **Theorem** [S^0] [9] *Suppose ρ is a signature with at least one constant (nulary operation index), and σ is a purely universal ρ-sentence.*[2.88] *Let $\sigma = \forall v_0 \cdots \forall v_{N^-} \mu$, where μ is quantifier-free. Then $\{\sigma\}$ is inconsistent iff there exists $M \in \omega$ and variable-free ρ-terms τ^m_n, $m \in M$, $n \in N$, such that*

$$\left\{\mu\left(\begin{smallmatrix} v_0 & \cdots & v_{N^-} \\ \tau^m_0 & \cdots & \tau^m_{N^-} \end{smallmatrix}\right) \,\middle|\, m \in M\right\}$$

is propositionally inconsistent.

Remark The 'if' direction is trivial. The 'only if' direction is a good example of a theorem concerning finitary objects which has an easy infinitary proof, but whose proof by finitary methods is substantially more involved. To prove it infinitarily we proceed as follows.

Suppose the theorem is false for σ. Let Θ be the set of all ρ-sentences of the form $\mu\binom{v_0 \cdots v_{N^-}}{\tau_0 \cdots \tau_{N^-}}$. Then (by compactness), Θ is propositionally consistent. Let $\mathfrak{I}$ be a propositional interpretation such that $\mathfrak{I} \models \Theta$. Extend $\mathfrak{I}$ to an interpretation $\mathfrak{I}'$ such that $\operatorname{dom}\mathfrak{I}'$ contains every atomic ρ-sentence. This can be done by assigning some definite value, say 1 (*true*), to each new sentence of the form $\tilde{P}\langle\tau_0, \ldots, \tau_{n^-}\rangle$, and then defining $\mathfrak{I}'$ to extend $\mathfrak{I}$ by recursion on complexity. Define a structure $\mathfrak{S}$ as follows.

1. $|\mathfrak{S}|$ is the class of variable-free ρ-terms.
2. If F is an n-ary ρ-operation index then $F^{\mathfrak{S}}$ is the function that assigns to each n-sequence $\langle\tau_0, \ldots, \tau_{n^-}\rangle$ of variable-free ρ-terms the term $\tilde{F}\langle\tau_0, \ldots, \tau_{n^-}\rangle$.
3. If P is an n-ary ρ-predicate index then $P^{\mathfrak{S}}$ is the class of n-sequences $\langle\tau_0, \ldots, \tau_{n^-}\rangle$ of variable-free ρ-terms such that
$$\mathfrak{I}' \models \tilde{P}\langle\tau_0, \ldots, \tau_{n^-}\rangle.$$

Assuming the axiom of infinity, $\mathfrak{S}$ is a set, so it has a satisfaction relation. By construction $\mathfrak{S} \models \Theta$, so Θ is consistent.

Proof For a finitary proof see Note 10.5. $\square$

The following generalization of (2.93) is a useful corollary.

(2.94) **Theorem** $[\mathsf{S}^0]$ *Suppose ρ is a signature with at least one constant and Σ is an inconsistent finite set of universal ρ-sentences. Then there is a finite set of constant instances of members of Σ in the sense of (2.93) that is propositionally inconsistent.*

Proof We may assume without loss of generality that variables occurring in distinct members of Σ are distinct from one another. Let $\Sigma = \{\sigma_k \mid k \in K\}$, and let $\sigma_k = \forall v_{k,0} \cdots \forall v_{k,N_k^-}\ \mu_k$. Let
$$\mu = \mu_0 \wedge \cdots \wedge \mu_{K^-},$$
and let
$$\sigma = \forall v_{0,0} \cdots \forall v_{0,N_0^-} \cdots \forall v_{K^-,0} \cdots \forall v_{K^-,N_{K^-}^-}\ \mu.$$
Then $\{\sigma\} \vdash \{\sigma_k \mid k \in K\}$, so $\{\sigma\}$ is inconsistent, so[2.93] there exist $M \in \omega$ and ρ-terms $\tau^m_{k,n}$, with $m \in M$, $k \in K$, and $n \in N_k$ such that
$$\Big\{\mu\binom{v_{0,0} \cdots v_{0,N_0^-} \cdots v_{K^-,0} \cdots v_{K^-,N_{K^-}^-}}{\tau^m_{0,0} \cdots \tau^m_{0,N_0^-} \cdots \tau^m_{K^-,0} \cdots \tau^m_{K^-,N_{K^-}^-}} \Bigm| m \in M\Big\}$$
is propositionally inconsistent. It follows that
$$\Big\{\mu_k\binom{v_{k,0} \cdots v_{k,N_k^-}}{\tau^m_{k,0} \cdots \tau^m_{k,N_k^-}} \Bigm| m \in M, k \in K\Big\}$$
is propositionally inconsistent. $\square$[2.94]

2.4.3 Skolemization

Recall that a *witness* for an existential sentence $\exists v\,\psi$ is a nulary operation index F that does not occur in ψ, whose intended use is to form the sentence $\exists v\,\psi \to \psi\binom{v}{F}$, i.e.

$$\psi\binom{v}{\tilde{F}0}. \tag{2.95}$$

The analogous concept for an existential *formula* $\exists v\,\psi$, that is not necessarily a sentence, is a *Skolem operation*. Suppose $\langle v_0, \dots, v_{n^-}\rangle$ enumerates $\mathrm{Free}\,\exists v\,\psi$ and F is an n-ary operation index that does not occur in ψ. Then

$$\psi\binom{v}{\tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle} \tag{2.96}$$

is the general formula of which (2.95) is the instance corresponding to $n = 0$.

(2.96) is an example of *skolemization*, and this process may be iterated. In general, to *skolemize* a formula we first put it in prenex form[2.86, 2.87] and then proceed as follows.

(2.97) **Definition** $[\mathsf{S}^0]$

1. *Suppose ϕ is prenex and $\phi = \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n^-} v_{n^-}\,\exists v\,\psi$, where the Q_m, $m \in n$, are individually existential or universal quantifiers.*[19] *A* simple skolemization of ϕ at ψ *is a formula*

   $$\phi' = \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n^-} v_{n^-}\,\psi\binom{v}{\tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle}, \tag{2.98}$$

   where F is an n-ary operation index that does not occur in ϕ. Note that ϕ' is prenex.

2. *A* partial skolemization *of a prenex formula ϕ is a formula resulting from successive simple skolemizations.*

3. *A* complete skolemization *of a prenex formula ϕ is a partial skolemization of ϕ with no existential quantifiers (which therefore cannot be further skolemized).*

4. *A* standard skolemization *of a prenex formula ϕ is a complete skolemization in which the existential subformulas are skolemized in decreasing order of logical complexity (i.e., "from the outside in").*

5. *A* skolemization *of an arbitrary formula ϕ is a complete skolemization of any prenex formula logically equivalent to ϕ.*

6. *Suppose Θ is a set of formulas.*

   1. *A skolemization is* over Θ $\overset{\text{def}}{\iff}$ *the introduced operation indices do not occur in Θ.*

   2. *A* skolemization of Θ *is a set Θ' such that there exists $f : \Theta \overset{\text{bij}}{\to} \Theta'$ such that for all $\phi \in \Theta$, $f\phi$ is a complete skolemization of ϕ over $\Theta' \setminus \{f\phi\}$.*[20]

---

[19] Note that ψ is necessarily prenex, but it need not be quantifier-free. Note also that since by definition[2.86] each variable in the quantifier prefix of a prenex formula occurs there just once, v_m is free for v in ψ for each $m \in n$.

[20] Prenexification and skolemization can be done in a canonical way, so no choice axiom is necessary to infer the existence of skolemizations for infinite classes.

Note that a simple skolemization of ϕ has one fewer existential quantifier than ϕ, so repeated (simple) skolemization of a prenex formula eventually yields a complete skolemization.

Recall[2.90.1] that for any theory Θ and sentence $\exists v\,\psi$, if F is a nulary operation index that does not occur in Θ or ψ, then if Θ is consistent, so is $\Theta \cup \left\{\exists v\,\psi \to \psi\binom{v}{\tilde{F}0}\right\}$.

The general theorem is as follows.

(2.99) **Theorem** [S^0] *Suppose Θ is a set of sentences, ψ is a formula, $\langle v, v_0, \ldots, v_{n^-}\rangle$ is an enumeration of* Free ψ, *and $\bar{v}_m$ is free for v in ψ for all $m \in n$. Let F be an n-ary operation index that does not appear in Θ or ψ. Suppose Θ is consistent. Then*

$$\Theta \cup \left\{\forall v_0 \cdots \forall v_{n^-}\left(\exists v\,\psi \to \psi\binom{v}{\tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle}\right)\right\}$$

is consistent.

Remark As we have done in (2.90), we emphasize that this is a theorem of S^0. As in the case of Herbrand's theorem,[2.93] there is an infinitary proof that is considerably simpler: If Θ is consistent it has a countable satisfactory model $\mathfrak{S}$. Supposing a fixed enumeration of $|\mathfrak{S}|$, and letting a' be some fixed member of $|\mathfrak{S}|$, we define a function $f : {}^n|\mathfrak{S}| \to |\mathfrak{S}|$ by letting $f\langle a_0, \ldots, a_{n^-}\rangle = a$, where a is the first member of $|\mathfrak{S}|$ such that

$$\mathfrak{S} \models \psi\begin{bmatrix} v & v_0 & \cdots & v_{n^-} \\ a & a_0 & \cdots & a_{n^-}\end{bmatrix},$$

if there is any; otherwise $f\langle a_0, \ldots, a_{n^-}\rangle = a'$. Let $\mathfrak{S}'$ be the expansion of $\mathfrak{S}$ for which $F^{\mathfrak{S}'} = f$. Given the axiom of infinity, we know there is a satisfaction relation for $\mathfrak{S}'$, and clearly

$$\mathfrak{S}' \models \Theta \cup \left\{\forall v_0 \cdots \forall v_{n^-}\left(\exists v\,\psi \to \psi\binom{v}{\tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle}\right)\right\},$$

so the latter theory is consistent.

Proof An S^0-proof of Theorem 2.99 is given in Note 10.6. $\square$

The full skolemization theorem is as follows.

(2.100) **Theorem** [S^0] *Suppose Θ is a theory, ϕ is a prenex sentence, and ϕ' is a partial skolemization of ϕ over Θ. Then $\vdash \phi' \to \phi$, and $\Theta \cup \{\phi\}$ is consistent iff $\Theta \cup \{\phi'\}$ is consistent.*

Proof We have the general result by induction once we have proved it for simple skolemization. We operate in the setting of (2.98). To show that $\vdash \phi' \to \phi$ in (2.97.1) we note that

$$\psi\binom{v}{\tau} \to \exists v\,\psi$$

is valid for any term τ that is free for v in ψ, in particular for $\tau = \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle$.[21] It follows that if $\Theta \cup \{\phi'\}$ is consistent then $\Theta \cup \{\phi\}$ is consistent.

To prove the converse, suppose $\Theta \cup \{\phi\}$ is consistent. By Theorem 2.99

$$\Theta \cup \left\{\phi, \forall v_0 \cdots \forall v_{n^-}\left(\exists v\,\psi \to \psi\binom{v}{\tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle}\right)\right\} \tag{2.101}$$

[21] See the footnote to (2.98).

is consistent. The (universal closures of the) following are easily shown to be valid:

$$\begin{gathered}\forall u\,(\eta \to \theta) \to (\exists u\,\eta \to \exists u\,\theta)\\ \forall u\,(\eta \to \theta) \to (\forall u\,\eta \to \forall u\,\theta).\end{gathered}$$

Using these repeatedly, we see that

$$\forall v_0 \cdots \forall v_{n\text{-}}\,(\eta \to \theta) \to \big(\mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}\,\eta \to \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}\,\theta\big)$$

is valid for any quantifier string $\mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}$. Hence[2.101]

$$\Theta \cup \left\{\phi, \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}\,\exists v\,\psi \to \mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}\,\psi\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n\text{-}}\rangle\end{smallmatrix}\right)\right\}$$

is consistent, so

$$\Theta \cup \left\{\mathsf{Q}_0 v_0 \cdots \mathsf{Q}_{n\text{-}} v_{n\text{-}}\,\psi\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n\text{-}}\rangle\end{smallmatrix}\right)\right\}$$

is consistent. □[2.100]

Example By way of illustration, suppose Θ is a theory, $\phi = \exists v_0\,\forall v_1\,\exists v_2\,\forall v_3\,\exists v_4\,\psi$ is a prenex sentence, and $\Theta \cup \{\phi\}$ is consistent. By definition[2.86] v_0, v_1, v_2, v_3, v_4 are distinct variables. We can skolemize the existential quantifications in any order. We'll begin with v_2. Let F_1 be a binary operation index not mentioned in Θ or ϕ, and let

$$\begin{aligned}\phi_1 &= \exists v_0\,\forall v_1\,(\forall v_3\,\exists v_4\,\psi)\left(\begin{smallmatrix} v_2 \\ \tilde{F}_1\langle \bar{v}_0, \bar{v}_1\rangle\end{smallmatrix}\right)\\ &= \exists v_0\,\forall v_1\,\forall v_3\,\exists v_4\,\psi\left(\begin{smallmatrix} v_2 \\ \tilde{F}_1\langle \bar{v}_0, \bar{v}_1\rangle\end{smallmatrix}\right).\end{aligned}$$

Then[2.100] $\Theta \cup \{\phi_1\}$ is consistent. We now skolemize, say, v_0, letting F_0 be a nulary operation index not mentioned in Θ or ϕ_1, and letting

$$\begin{aligned}\phi_2 &= \left(\forall v_1\,\forall v_3\,\exists v_4\,\psi\left(\begin{smallmatrix} v_2 \\ \tilde{F}_1\langle \bar{v}_0, \bar{v}_1\rangle\end{smallmatrix}\right)\right)\left(\begin{smallmatrix} v_0 \\ \tilde{F}_0 0\end{smallmatrix}\right)\\ &= \forall v_1\,\forall v_3\,\exists v_4\,\psi\left(\begin{smallmatrix} v_0 & v_2 \\ \tilde{F}_0 0 & \tilde{F}_1\langle \tilde{F}_0 0, \bar{v}_1\rangle\end{smallmatrix}\right).\end{aligned}$$

Then $\Theta \cup \{\phi_2\}$ is consistent. Finally we skolemize v_4, letting F_2 be a binary operation index not mentioned in Θ or ϕ_2, and letting

$$\begin{aligned}\phi_3 &= \forall v_1\,\forall v_3\,\left(\psi\left(\begin{smallmatrix} v_0 & v_2 \\ \tilde{F}_0 0 & \tilde{F}_1\langle \tilde{F}_0 0, \bar{v}_1\rangle\end{smallmatrix}\right)\right)\left(\begin{smallmatrix} v_4 \\ \tilde{F}_2\langle \bar{v}_1, \bar{v}_3\rangle\end{smallmatrix}\right)\\ &= \forall v_1\,\forall v_3\,\psi\left(\begin{smallmatrix} v_0 & v_2 & v_4 \\ \tilde{F}_0 0 & \tilde{F}_1\langle \tilde{F}_0 0, \bar{v}_1\rangle & \tilde{F}_2\langle \bar{v}_1, \bar{v}_3\rangle\end{smallmatrix}\right).\end{aligned} \tag{2.102}$$

Then $\Theta \cup \{\phi_3\}$ is consistent.

Skolemizing in a different order gives slightly different final forms. For example, the order v_0, v_2, v_4 yields

$$\forall v_1\,\forall v_3\,\psi\left(\begin{smallmatrix} v_0 & v_2 & v_4 \\ \tilde{F}'_0 0 & \tilde{F}'_1\langle \bar{v}_1\rangle & \tilde{F}'_2\langle \bar{v}_1, \bar{v}_3\rangle\end{smallmatrix}\right), \tag{2.103}$$

which is most efficient, while the order v_4, v_2, v_0 yields

$$\forall v_1\,\forall v_3\,\psi\left(\begin{smallmatrix} v_0 & v_2 & v_4 \\ \tilde{F}''_0 0 & \tilde{F}''_1\langle \tilde{F}''_0 0, \bar{v}_1\rangle & \tilde{F}''_2\langle \tilde{F}''_0 0, \bar{v}_1, \tilde{F}''_1\langle \tilde{F}''_0 0, \bar{v}_1\rangle, \bar{v}_3\rangle\end{smallmatrix}\right), \tag{2.104}$$

which is least efficient. These are all equiconsistent over any theory that does not mention any of the Fs.

2.4.4 Logic with identity conservatively extends logic without

To interpret the statement that forms the title of this section as it stands, we must treat logic with identity as a theory in logic without identity. This is entirely straightforward. Recall that 0 is reserved as the predicate index for identity. Suppose ρ is a signature and $0 \notin \Pi^\rho$. Let $\rho^=$ be the expansion of ρ by the addition of 0 as a binary predicate index.

(2.105) *Let $\Theta^{\rho,=}$ be the $\rho^=$-theory that consists of the following sentences:*[2.77]

1. *For each $n \in \omega$, n-ary ρ-operation index F, and distinct variables $v_0, \ldots, v_{n^-}$, $v'_0, \ldots, v'_{n^-}$,*

$$\forall v_0, \ldots, v_{n^-}, v'_0, \ldots, v'_{n^-} \Big(\bigwedge\nolimits_{m\in n} \bar{v}_m = \bar{v}'_m \to \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle = \tilde{F}\langle \bar{v}'_0, \ldots, \bar{v}'_{n^-}\rangle\Big).$$

2. *For each $n \in \omega$, n-ary ρ-predicate index P, and distinct variables $v_0, \ldots, v_{n^-}$, $v'_0, \ldots, v'_{n^-}$,*

$$\forall v_0, \ldots, v_{n^-}, v'_0, \ldots, v'_{n^-} \Big(\bigwedge\nolimits_{m\in n} \bar{v}_m = \bar{v}'_m \to (\tilde{P}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle \leftrightarrow \tilde{P}\langle \bar{v}'_0, \ldots, \bar{v}'_{n^-}\rangle)\Big).$$

Recall[2.78] that $\vdash^=$ is provability in logic with identity. It is easy to see that for any $\rho^=$-theory Θ and $\rho^=$-sentence σ,

$$\Theta \vdash^= \sigma \leftrightarrow \Theta \cup \Theta^{\rho,=} \vdash \sigma.$$

We therefore wish to show that if Θ is a ρ-theory and σ is a ρ-sentence then

$$\Theta \cup \Theta^{\rho,=} \vdash \sigma \to \Theta \vdash \sigma.$$

Since $\Theta \vdash \sigma$ iff $\Theta \cup \{\neg\sigma\}$ is inconsistent, and the same is true for $\vdash^=$, the result may be stated as follows.

(2.106) **Theorem** $[\mathsf{S}^0]$ *If Θ is a ρ-theory and Θ is consistent then $\Theta \cup \Theta^{\rho,=}$ is consistent.*[22]

Remark Once again, the infinitary proof is quite easy. Suppose $\mathfrak{S}$ is a countable satisfactory ρ-structure and $\mathfrak{S} \models \Theta$. Let $\mathfrak{S}^=$ be the expansion of $\mathfrak{S}$ to a $\rho^=$-structure with the identity predicate interpreted as the identity relation on $|\mathfrak{S}|$, i.e., $0^{\mathfrak{S}^=} = \{\langle a, a\rangle \mid a \in |\mathfrak{S}|\}$. Given the axiom of infinity, we know that $\mathfrak{S}^=$ has a satisfaction relation, and by construction $\mathfrak{S}^= \models \Theta^{\rho,=}$. By the definition of *expansion*, all operation and predicate indices other than identity have the same interpretation in $\mathfrak{S}^=$ as in $\mathfrak{S}$, so $\mathfrak{S}^= \models \Theta$. Hence $\Theta \cup \Theta^{\rho,=}$ is consistent.

Proof For a finitary proof see Note 10.7. □

Note that Theorem 2.106 confirms the remark following Definition 2.78 that there is no need to distinguish between the provability relations $\vdash$ and $\vdash^=$ for sentences that do not involve the identity predicate. In the interest of notational simplicity, in the context of logic with identity, we will use '$\vdash$' to mean '$\vdash^=$'.

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[22] Clearly, 'consistent' in the statement and proof of Theorem 2.106 refers to the provability relation $\vdash$, as opposed to $\vdash^=$.

2.4.5 Extension by definition

In our proof of the completeness theorem and in the deductive system derived from it, the introduction of new constants is an intrinsic element. This is not necessary, and if we had been willing to consider sequents involving formulas other than sentences, we could have replaced additional constants by new variables. Indeed, in practice, when one says, for example, 'let a be an arbitrary thing', one typically does not trouble to state whether 'a' is a constant or a variable, nor can this always be inferred from the argument.

The advantage of using variables instead of constants for this purpose is that deductions from a theory Θ are then conducted entirely within the original signature of Θ.[23] One reason we have not taken any trouble to gain this advantage is that in practice deductions are almost never carried out in the original signature of a theory. Instead, as a way of organizing the work, the signature is routinely extended by the addition of predicate and operation symbols to the language (with 'symbols' broadly understood to include words or phrases, etc.), while their definitions are simultaneously added to the theory.

(2.107) **Definition** $[\mathsf{S}^0]$ *Suppose ρ is a signature, and ρ' is an extension of ρ. We will refer to a ρ'-index that is not a ρ-index as a* new *index.*

1. *Suppose P is new predicate index. A* definition of P over ρ *is a sentence*

$$\forall v_1, \ldots, v_n \left(\tilde{P}\langle v_1, \ldots, v_n\rangle \leftrightarrow \phi\right),$$

   where ϕ is a ρ-formula and $\operatorname{Free}\phi \subseteq \{v_1, \ldots, v_n\}$.

2. *Suppose F is a new operation index and Θ is a ρ-theory. A* definition of F over Θ *is a sentence*

$$\forall v_0, \ldots, v_n \left(v_0 = \tilde{F}\langle v_1, \ldots, v_n\rangle \leftrightarrow \phi\right),$$

   where ϕ is a ρ-formula, $\operatorname{Free}\phi \subseteq \{v_0, \ldots, v_n\}$*, and* $\Theta \vdash \forall v_1, \ldots, v_n\, \exists! v_0\, \phi$.

The fundamental theorem of definition is

(2.108) **Theorem** $[\mathsf{S}^0]$ *Suppose ρ is a signature and Θ is a ρ-theory.*

1. *Suppose ϕ is a ρ-formula with* $\operatorname{Free}\phi \subseteq \{v_1, \ldots, v_n\}$*, and suppose P is an n-ary predicate index not in ρ. Let*

$$\Theta' = \Theta \cup \left\{\forall v_1, \ldots, v_n \left(\tilde{P}\langle v_1, \ldots, v_n\rangle \leftrightarrow \phi\right)\right\}.$$

   Then Θ' is a conservative extension of Θ.

2. *Suppose ϕ is a ρ-formula with* $\operatorname{Free}\phi \subseteq \{v_0, \ldots, v_n\}$ *and*

$$\Theta \vdash \forall v_1, \ldots, v_n\, \exists! v_0\, \phi,$$

   and suppose F is an n-ary operation index not in ρ. Let

$$\Theta' = \Theta \cup \left\{\forall v_0, \ldots, v_n \left(v_0 = \tilde{F}\langle v_1, \ldots, v_n\rangle \leftrightarrow \phi\right)\right\}.$$

   Then Θ' is a conservative extension of Θ.

[23] The disadvantage is that when substituting a term τ for a variable v in a formula ϕ one generally has to specify that τ is free for v in ϕ.

Proof Again, it is sufficient to show (for every Θ) that if Θ is consistent then Θ' is consistent, and the infinitary proof is trivial: Given a satisfactory ρ-model $\mathfrak{S}$ of Θ, expand it to a ρ'-model $\mathfrak{S}'$ (with the same domain), where ρ' is the expansion of ρ by the new index, interpreting the new index according to its definition. To use this method to prove the theorem in C^0, we must show that $\mathfrak{S}'$ has a satisfaction relation, which we can accomplish by defining the satisfaction relation S' for $\mathfrak{S}'$ in terms of the satisfaction relation S for $\mathfrak{S}$ using only set quantification. The idea is that, given a ρ'-formula η and an $\mathfrak{S}'$-assignment for η, we put $\langle \eta, A\rangle \in S'$ iff $\langle \zeta, A\rangle \in S$, where ζ is obtained from η by elimination of the new index in favor of its definition. This is quite easy for (2.108.1), somewhat more difficult for (2.108.2).

Note, however, that we have set ourselves the task of proving the theorem in S^0, so model-theoretic methods are not available. We have done this to conform with the program outlined above,[2.38] the goal of which is to provide a S^0-proof of Theorem 2.183, which states that C^0 is a conservative extension of S^0. Actually, it is only (2.108.1) that is used in the proof of (2.183), but it turns out that a proof-theoretic proof of (2.108.2) is—in the context of our previous work on skolemization—easier than a model-theoretic proof in C^0.

1 Let ρ' be the expansion of ρ by the addition of P as a predicate index. Suppose $\Theta' \vdash \sigma$, where σ is a ρ-sentence. Then there exists a ρ'-proof π of a sequent $\Sigma \cup \big\{\forall v_1, \dots, v_n\,\big(\tilde{P}\langle v_1, \dots, v_n\rangle \leftrightarrow \phi\big)\big\} \Rightarrow \sigma$, where $\Sigma \subseteq \Theta$. Let $\tilde{\phi}$ be obtained from ϕ be a change of (bound) variables so no variable that occurs in π occurs bound in $\tilde{\phi}$. For each formula η that occurs in π, let $T\eta$ be the formula obtained by substituting for each occurrence in η of a subformula $\tilde{P}\langle \tau_1, \dots, \tau_n\rangle$ the formula $\tilde{\phi}\big(\begin{smallmatrix} v_1 & \cdots & v_n \\ \tau_1 & \cdots & \tau_n\end{smallmatrix}\big)$. Let $T\pi$ be the result of replacing each sentence θ in each sequent in π by $T\theta$. It is easy to see that $T\pi$ is a ρ-proof. Obviously, if θ is a ρ-sentence then $T\theta = \theta$, and

$$T\big(\forall v_1, \dots, v_n\,(\tilde{P}\langle v_1, \dots, v_n\rangle \leftrightarrow \phi)\big) = \forall v_1, \dots, v_n\,(\tilde{\phi} \leftrightarrow \phi).$$

Hence, $T\pi$ is a ρ-proof of $\Sigma \cup \{\forall v_1, \dots, v_n\,(\tilde{\phi} \leftrightarrow \phi)\} \Rightarrow \sigma$. But $\forall v_1, \dots, v_n\,(\tilde{\phi} \leftrightarrow \phi)$ is a logical validity, which may therefore be eliminated as a premise (by inserting a proof of it), yielding a ρ-proof of $\Sigma \Rightarrow \sigma$.

2 Let ρ' be the expansion of ρ by the addition of F as an operation index. It suffices to show (for all Θ) that if Θ is consistent and

$$\Theta \vdash \forall v_1, \dots, v_n\, \exists! v_0\, \phi, \tag{2.109}$$

then

$$\Theta \cup \big\{\forall v_0, \dots, v_n\,\big(v_0 = \tilde{F}\langle v_1, \dots, v_n\rangle \leftrightarrow \phi\big)\big\} \tag{2.110}$$

is consistent. Suppose, therefore, that Θ is consistent and (2.109). By (2.99)

$$\Theta \cup \big\{\forall v_1, \dots, v_n\,\big(\exists v_0\,\phi \rightarrow \phi\big(\begin{smallmatrix} v_0 \\ \tilde{F}\langle \bar{v}_1, \dots, \bar{v}_n\rangle\end{smallmatrix}\big)\big)\big\}$$

is consistent. Given (2.109),

$$\Theta \cup \big\{\forall v_1, \dots, v_n\,\big(\exists v_0\,\phi \rightarrow \phi\big(\begin{smallmatrix} v_0 \\ \tilde{F}\langle \bar{v}_1, \dots, \bar{v}_n\rangle\end{smallmatrix}\big)\big)\big\} \\ \vdash \forall v_0, \dots, v_n\,\big(v_0 = \tilde{F}\langle v_1, \dots, v_n\rangle \leftrightarrow \phi\big),$$

so (2.110) is consistent. $\square$[2.108]

(2.111) *We will often indicate extension by definition by means of the superscript '$^+$'. Thus, if Θ is a ρ-theory, then Θ^+ is a ρ^+-theory that extends Θ by the inclusion of definitions of new predicate and/or operation indices that when added to ρ yield ρ^+. Note that such extensions are often sequential, with subsequent definitions formulated in terms of previously defined indices.*

2.4.6 Relativization

(2.112) **Definition** [S^0] *Suppose ρ is a signature.*

1. *Suppose P is a unary ρ-predicate index. For ρ-formulas ψ we define the* relativization of ψ to $P \stackrel{\text{def}}{=} \psi^P$ *by recursion on the complexity of ψ as follows.*
   1. *If ψ is atomic then $\psi^P = \psi$.*
   2. $(\neg\,\psi)^P = \neg\,\psi^P$, $(\psi_0 \vee \psi_1)^P = \psi_0^P \vee \psi_1^P$, *etc.*
   3. $(\exists v\,\psi)^P = \exists v\,(\tilde{P}\langle \bar{v}\rangle \wedge \psi^P)$ *and* $(\forall v\,\psi)^P = \forall v\,(\tilde{P}\langle \bar{v}\rangle \rightarrow \psi^P)$.
2. *We may define relativization to an arbitrary ρ-formula ϕ with one free variable u by positing a definition*
$$\forall u\,\tilde{P}\langle \bar{u}\rangle \leftrightarrow \phi,$$
where P is a new unary predicate index, and letting ψ^ϕ be ψ^P.
3. *We may also define ψ^ϕ directly by substituting ϕ for P in (2.112.1.1–3) and replacing $\tilde{P}\langle \bar{v}\rangle$ by $\phi\binom{u}{\bar{v}}$ in (2.112.1.3), but we must take care that $\bar{v}$ is free for u in ϕ in this case. This can be done by replacing ϕ by an equivalent formula ϕ' obtained by a suitable change of variables. Note that (2.112.1) may be regarded as a special case of this construction with $\phi = \tilde{P}\langle \bar{u}\rangle$.*

2.4.7 Substructure

(2.113) **Definition** [C^0] *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are structures.* $\mathfrak{B}$ is a substructure of $\mathfrak{A} \stackrel{\text{def}}{\iff}$

1. *$\mathfrak{A}$ and $\mathfrak{B}$ have the same signature, say ρ;*
2. $|\mathfrak{B}| \subseteq |\mathfrak{A}|$;
3. *for every ρ-predicate index P with arity n and any $x_0, \ldots, x_{n^-} \in |\mathfrak{B}|$*
$$\langle x_0, \ldots, x_{n^-}\rangle \in P^{\mathfrak{B}} \leftrightarrow \langle x_0, \ldots, x_{n^-}\rangle \in P^{\mathfrak{A}};$$
4. *and for every ρ-operation index F with arity n and any $x_0, \ldots, x_{n^-} \in |\mathfrak{B}|$*
$$F^{\mathfrak{B}}\langle x_0, \ldots, x_{n^-}\rangle = F^{\mathfrak{A}}\langle x_0, \ldots, x_{n^-}\rangle.$$

Substructure and relativization are related in the following obvious way.

(2.114) **Theorem** [C^0] *Suppose ρ is a signature and ρ' is the extension of ρ with one additional unary predicate index P. Suppose $\mathfrak{A}$ is a ρ'-structure with the property that for every operation index F of ρ, $P^{\mathfrak{A}}$ is closed under the function $F^{\mathfrak{A}}$, i.e., for all $x_0, \ldots, x_{n^-} \in P^{\mathfrak{A}}$, $F^{\mathfrak{A}}\langle x_0, \ldots, x_{n^-}\rangle \in P^{\mathfrak{A}}$, assuming F is n-ary.*

1. *There is a unique substructure $\mathfrak{B}'$ of $\mathfrak{A}$ with $|\mathfrak{B}'| = P^{\mathfrak{A}}$. (Note that $P^{\mathfrak{B}'} = |\mathfrak{B}'|$.)*

2. *Let* $\mathfrak{B}$ *be the restriction of* $\mathfrak{B}'$ *to* ρ. *Then for any* ρ*-formula* ψ *and* $\mathfrak{B}$*-assignment* A *for* ψ,
$$\mathfrak{B} \models \psi[A] \leftrightarrow \mathfrak{A} \models \psi^P[A].$$

Proof Straightforward. $\square$[2.114]

Note that any substructure $\mathfrak{B}$ of a given ρ-structure $\mathfrak{A}$ may be obtained in this way from the expansion $\mathfrak{A}'$ of $\mathfrak{A}$ by the addition of a predicate index P to ρ such that $P^{\mathfrak{A}'} = |\mathfrak{B}|$. Thus the syntactical operation of relativization corresponds precisely to the semantical relationship of substructure.

2.4.8 Interpretations in languages and theories

(2.115) **Definition** [C^0] *An* interpretation ι *of a language* $\mathcal{L}^\rho$ *in a language* $\mathcal{L}^{\rho'}$ *is determined by an isomorphism* σ *of* ρ *with a subsignature of* ρ',[24] *which we regard as extending in the obvious way to an isomorphism of* $\mathcal{L}^\rho$ *with a sublanguage of* $\mathcal{L}^{\rho'}$, *and a* ρ'*-formula* ϕ *with one free variable, such that for any* ρ*-formula* ψ, $\iota\psi = (\sigma\psi)^\phi$.

Definition [C^0] *Suppose* Θ *and* Θ' *are respectively a* ρ*- and a* ρ'*-theory.*

1. *An* interpretation *of* Θ *in* Θ' *is an interpretation* ι *of* $\mathcal{L}^\rho$ *in* $\mathcal{L}^{\rho'}$ *such that for each sentence* $\theta \in \Theta$, $\Theta' \vdash \iota\theta$.

2. Θ *is* interpretable in Θ' $\overset{\text{def}}{\iff}$ *there is an extension-by-definition* Θ'' *of* Θ' *and an interpretation of* Θ *in* Θ''.

3. Θ *and* Θ' *are* equi-interpretable $\overset{\text{def}}{\iff}$ *each is interpretable in the other.*

2.5 Example: theories and models of geometry

[This is a rather long digression that is not required for anything that follows. Its purpose is to illustrate metamathematical principles in the context of a body of mathematics that is of interest in its own right, viz., projective geometry. The main thread is picked up again in Section 2.6. W. T. Fishback's *Projective and Euclidean Geometry*[4] and Alfred North Whitehead's *The Axioms of Projective Geometry*[28] are recommended sources for conventional treatments of projective geometry.]

The axiomatic method has a long and illustrious history in mathematics; for millennia Euclid's *Elements* served as the paradigm of deductive inference. From the standpoint of formal logic as presented in this chapter, however, the *Elements* is not entirely satisfactory. In the first place, it relies on unstated assumptions that often make their way into proofs by way of diagrams, which—by their physical nature—incorporate such notions as betweenness and continuity that Euclid does not address explicitly. In the second place, the *Elements* posits the existence of higher-order objects, such as sets of points, without stating axioms governing these entities. Defects of the first sort were identified by a number of authors in the nineteenth century, and David Hilbert famously presented an adequate system of axioms in his *Grundlagen der Geometrie*[10]. Hilbert's system, however, failed to

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[24] In other words, σ is injective, the domain of σ is the class of ρ-predicate and -operation indices, and for each ρ-index X, σX is a ρ'-index of the same type (predicate or operation) and arity.

address the second sort of defect: specifically, Hilbert expressed the notion of continuity by saying—in effect—that any set of points on a line with an upper bound has a least upper bound, without explicitly acknowledging such sets as individuals governed by axioms. Alfred Tarski gave the first satisfactory first-order axiomatization of euclidean geometry[26], expressing the continuity principle as a schema of axioms, one for each formula that—interpreted in a model of the theory—would define a set of points on a line, in the same manner that the separation principle, for example, is expressed by an axiom schema in the Zermelo-Fraenkel axiomatization of set theory.

Hilbert's second-order formulation of the continuity principle has the advantage of uniquely specifying the structure of interest, viz., $\mathbb{R}^2$ in the case of plane geometry. As we will see, this is not possible with a first-order theory: by the downward Löwenheim-Skolem theorem[2.159] there are countable models of Tarski's axioms, and by the upward Löwenheim-Skolem theorem there are models of every infinite cardinality. On the other hand, since the first-order predicate logic we have developed does not apply to second-order axioms, if we wish to use them for the purpose of deduction we must either extend our logic to higher order, or we must expand the universe of discourse so that higher-order objects, such as sets of points, may be regarded as individuals in a larger structure, i.e., we must employ a theory of membership.

The precise delineation of first-order vs. higher-order methods, and the recognition of the necessity of a formal theory of membership to deal with higher-order objects, only came in the late nineteenth and early twentieth centuries. Along with these developments came an interest in the scope of first-order methods in various branches of mathematics. In this connection we recall that in our initial remarks concerning the notion of structure[§ 1.1.14] we noted that axioms are not always used—as in Euclid's *Elements*—to formalize our understanding of a specific structure, but are often used in the converse way, to define a class of structures.

(2.116) *An* elementary class *is a class $\mathcal{C}$ of structures of a given signature ρ, such that for some ρ-theory Θ (a set of axioms), $\mathcal{C}$ is the class of ρ-structures $\mathfrak{S}$ such that $\mathfrak{S} \models \Theta$. In this context* elementary *is essentially synonymous with* first-order*: membership of a structure $\mathfrak{S}$ in an elementary class is determined entirely by its first-order theory.*

The classes of groups, rings, fields, and many other classes of structures are elementary, but many classes of mathematical interest are not, e.g., the class of finite groups. Even in the case of an elementary class, its first-order theory is usually of greater metamathematical than primarily mathematical interest. For example, while we may define a group as a structure with a binary (multiplication) operation having certain first-order properties—viz., associativity and the existence of an identity and of inverses—the first-order theory of groups is only a tiny part of group theory as we know it: we are mostly interested in subgroups, homomorphisms of groups, etc. The same is true of many other branches of mathematics, as the notions of *sequence*, *set*, *function* and the like are almost always pertinent to the subject.

A major development in the foundations of mathematics in the late nineteenth and early twentieth centuries was the definition of the role of these ubiquitous "higher-order" notions and the recognition that they too must be dealt with axiomatically. As noted above, the standard approach today is to regard these as individuals in the context of a theory of membership, in which the notion of the *order* of an object (as in *first-* and *second-order*, etc.) has no relevance: everything is a set.[25]

Given its relatively rich elementary theory—which nevertheless falls short of its full higher-order theory—together with its historical familiarity, geometry is well suited to the purpose of illustrating some of the concepts developed in this chapter in the context of mathematics as it is usually practiced; and for this purpose projective geometry is particularly well suited.

Projective geometry originated as the study of geometric properties invariant under projective transformations. It is apparent that the euclidean plane is itself not invariant under such transformations. For example, suppose we project the horizontal plane $\sigma = \{\langle x, y, -1\rangle \mid x, y \in \mathbb{R}\}$ onto the vertical plane $\pi = \{\langle 1, y, z\rangle \mid y, z \in \mathbb{R}\}$, using the point $O = \langle 0, 0, 0\rangle$ as the center of perspectivity—as we would in painting a landscape arrayed in σ on a canvas at π with the observer's eye at O. Let T be this transformation. Then $T\langle x, y, -1\rangle = \langle 1, y/x, -1/x\rangle$ for $x \neq 0$, because these points, in σ and π, respectively, are collinear with O. It follows that $\operatorname{dom} T$ omits the line $\{\langle 0, y, -1\rangle \mid y \in \mathbb{R}\}$ in σ, and $\operatorname{im} T$ omits the line $h = \{\langle 1, y, 0\rangle \mid y \in \mathbb{R}\}$ in π, which represents the "horizon" in the parlance of perspective drawing. Parallel lines in σ project to lines in π that meet at a point on h. It is customary in projective geometry to define the projective plane as an extension of the euclidean (or *affine*) plane by the addition of *points at infinity* or *ideal points*, together with a *line at infinity*, or *ideal line*, containing them. All lines parallel to a given affine line are regarded as meeting at a single ideal point.

The incidence properties of the plane so extended are particularly simple.

1. Any two distinct lines are incident with a unique point—"parallel lines meet at infinity".

2. Any two distinct points are incident with a unique line.

2.5.1 Analytic projective geometry: modules over division rings

An elegant model of the projective plane is obtained by taking the points to be 1-dimensional subspaces of $\mathbb{R}^3$, regarded as a vector space of dimension 3 over the field $\mathbb{R}$, and taking the lines to be the 2-dimensional subspaces of $\mathbb{R}^3$. A point and line are incident $\overset{\text{def}}{\iff}$ the former is a subspace of the latter. This model is $\mathbb{RP}^2$. Thus, a point of $\mathbb{RP}^2$ is a line through the origin in $\mathbb{R}^3$, and a line in $\mathbb{RP}^2$ is a plane through the origin in $\mathbb{R}^3$.

The same construction yields the real projective space $\mathbb{RP}^n$ for each $n > 0$: the points are 1-dimensional subspaces of $\mathbb{R}^{n+1}$, and the lines, planes, etc., are respectively subspaces of dimension 2, 3, etc. We will confine our remarks to $n = 1, 2, 3$. Importantly, the same construction can be performed with any division ring $\mathbb{K}$ in place of $\mathbb{R}$. As a reminder, a division ring is a ring with (multiplicative) identity 1 such that $\forall a \neq 0\, \exists b\, (ab = ba = 1)$. Multiplication need not be commutative; if it is, the division ring is a field.

A right- or left-module over a division ring is analogous to a vector space over a field. A right-module M over a ring $\mathbb{K}$ is a structure with an addition operation, with respect to which it is an abelian group, and a scalar multiplication operation

[25] One might say that in the theory of membership, sets are of first order and classes of second order, but one may just as well regard all classes as first-order, with a set just being a special sort of class.

$a, u \mapsto ua$, where $a \in |\mathbb{K}|$ and $u \in |\mathsf{M}|$, with the properties familiar from linear algebra. In particular, $(ua)b = u(ab)$, writing the scalar to the right of the vector so that the rule has the appearance of an associative law. Alternatively, we may define a left-module over $\mathbb{K}$ by writing the scalar multiplication operation as $a, u \mapsto au$, where $a \in |\mathbb{K}|$ and $u \in |\mathsf{M}|$, and specifying that $a(bu) = (ab)u$.

Clearly, any general statement about right-modules over division rings applies *mutatis mutandis* to left-modules, and *vice versa*.

The notions of *linear combination*, *linearly independent*, *subspace*, *span*, and *basis* (linearly independent spanning set) have the familiar meaning. M is *finite-dimensional* $\overset{\text{def}}{\iff}$ it has a finite subset whose span is $|\mathsf{M}|$. In this case, it can be shown that any linearly independent set is extendible to a basis, any spanning set has a subset that is a basis, and all bases have the same number of elements, which is the *dimension* of M.

As in the case of vector spaces over fields, for each $n \in \omega$, ${}^n|\mathbb{K}|$ has a natural structure as a right-module over a ring $\mathbb{K}$. Writing the elements of ${}^n|\mathbb{K}|$ as column vectors (i.e., $n \times 1$ matrices), we have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} a = \begin{bmatrix} u_1 a \\ \vdots \\ u_n a \end{bmatrix}.$$

We call this module $\mathbb{K}^n$, or $\mathbb{K}^n_r$ to emphasize that it is a right-module. The corresponding left-module is $\mathbb{K}^n_l$, consisting of row vectors with

$$a \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} au_1 & \cdots au_n \end{bmatrix}.$$

Via a basis, any n-dimensional right-module over a division ring $\mathbb{K}$ is isomorphic to $\mathbb{K}^n_r$; and any n-dimensional left-module over a division ring $\mathbb{K}$ is isomorphic to $\mathbb{K}^n_l$

Linear transformation has the usual meaning, and a linear transformation $T : \mathbb{K}^n_r \to \mathbb{K}^m_r$ is represented by an $m \times n$ matrix so that

$$T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} t_{1,1} & \cdots & t_{1,n} \\ \vdots & \ddots & \vdots \\ t_{m,1} & \cdots & t_{m,n} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

Note the order of matrix factors. Be aware that scalar multiplication, i.e., the map

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} a,$$

is not generally linear; whereas

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto a \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

is.

Similarly, a linear transformation $T : \mathbb{K}^n_l \to \mathbb{K}^m_l$ is represented by an $n \times m$ matrix operating on row matrices by right multiplication.

Given a right-module M over $\mathbb{K}$ of dimension $n+1$, we define PM to be the set of subspaces A of M such that $0 < \dim A \leqslant n$. We confer on PM an *incidence structure* as follows. Let pg^n be an n-sorted signature with a single binary relation of *incidence* in addition to identity. The individuals of sort k are the elements of PM of dimension k (as subspaces of M) for $k = 1, \ldots, n$. A and B are incident $\overset{\text{def}}{\iff} A$ is *on* $B \overset{\text{def}}{\iff} A \mathrel{\square} B \overset{\text{def}}{\iff}$ either $A \subseteq B$ or $B \subseteq A$ (as subspaces of M). We refer to this pg^n-structure as the *incidence structure of* PM, or of M itself, and we call it a *projective space*. Colloquially, the 1-, 2-, and 3-dimensional subspaces of M are the *points*, *lines*, and *planes* of PM. If M is a right-module then, given a nonzero $u \in |\mathsf{M}|$, $[u] \overset{\text{def}}{=} \{ua \mid a \in \mathbb{K}\}$, which is the "point" of PM "represented by" u. If M is a left-module then $[u] \overset{\text{def}}{=} \{au \mid a \in \mathbb{K}\}$.

We define $\mathbb{KP}^n_r$ to be PM, where $\mathsf{M} = \mathbb{K}^{n+1}_r$. As in the original case of $\mathbb{RP}^n$, the points of the projective space $\mathbb{KP}^n_r$ are the 1-dimensional subspaces of $\mathbb{K}^{n+1}_r$, the lines of $\mathbb{KP}^n_r$ are the 2-dimensional subspaces of $\mathbb{K}^{n+1}_r$, etc. To demonstrate the correspondence between, say, the projective $\mathbb{K}$-plane $\mathbb{KP}^2_r$ and the affine $\mathbb{K}$-plane $\mathbb{K}^2_r$ extended by the addition of ideal points and an ideal line, it is convenient to let $\mathbb{K}^2_r$ be represented by the set $A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \mid x, y \in \mathbb{K} \right\}$, which is the plane through the point $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ parallel to the "x, y-plane". For each point $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ of A there is a unique 1-dimensional subspace $\left\{ \begin{bmatrix} xa \\ ya \\ a \end{bmatrix} \mid a \in \mathbb{K} \right\}$ of $\mathbb{K}^3_r$ that contains it, which is by definition a point of $\mathbb{KP}^2_r$. The remaining points of $\mathbb{KP}^2_r$ are the 1-dimensional subspaces $\left\{ \begin{bmatrix} xa \\ ya \\ 0 \end{bmatrix} \mid a \in \mathbb{K} \right\}$, where $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{K}^2_r$. These do not intersect A, and they correspond to points at infinity for A. Similarly, for each line of A, there is a unique 2-dimensional subspace of $\mathbb{K}^3_r$ that includes it, which is by definition a line of $\mathbb{KP}^2_r$. The 2-dimensional space $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{K} \right\}$ does not intersect A, and corresponds to the line at infinity for A.

The preceding discussion naturally applies *mutatis mutandis* to left-modules.

2.5.2 Synthetic projective geometry: axiomatic systems

The treatment of euclidean geometry via the representation of the plane as $\mathbb{R}^2$ is the *analytic* approach, pioneered by René Descartes, in contrast to the *synthetic* or *axiomatic* approach exemplified by Euclid's *Elements*. The study of the projective spaces $\mathbb{KP}^n$ is likewise termed the *analytic* approach to projective geometry. For the axiomatic approach, we will focus on two theories, PG^2 and PG^3, of projective geometry in 2 and 3 dimensions, respectively, also known as *plane* and *solid* projective geometry. These are theories in the respective signatures pg^2 and pg^3. In PG^2 we refer to the domains as *points* and *lines*; in PG^3, we have the additional domain of *planes*. In the following formal presentation of the axioms we will use subscripts '0', '1', and '2' on quantifiers to indicate these respective domains. We generally follow the convention of using upper case Roman, lower case Roman, and lower case Greek letters for points, lines, and planes, respectively. We use the bold '=' and '$\square$' as usual to denote the formula-building operations corresponding to these predicate symbols (identity and incidence).

As noted above for the special case of structures $\mathbb{KP}^n$, the incidence relation may hold between individuals of any respective domains, and it is symmetric; thus, for example, a point A is incident with a line a iff a is incident with A. Intuitively, A is incident with a iff A is *in* a, if the latter is regarded as a set of points, but we wish to de-emphasize this point of view for reasons that will become apparent. Instead, we will say that A is *on* a, and we will also say that a is *on* A. Symbolically, $A \square a \leftrightarrow a \square A$. (Incidentally, objects of the same sort are incident iff they are identical, although we will not emphasize this equivalence.)

In the following axioms, all variables are presumed to be distinct. As we present the axioms, we will verify that they hold for $\mathbb{KP}^2$ for any division ring $\mathbb{K}$. Note that this should not be taken to suggest that these are the only models of the axioms. By convention, in a ring with identity, $0 \neq 1$, so $\mathbb{K}$ has at least two elements.

Axioms of PG^2

$\mathsf{PG}^2.0$

$$\forall_0 A \, \forall_1 a \, (A \square a \leftrightarrow a \square A)$$

This establishes the symmetry of the incidence relation. We could do without this axiom if we restricted the order of arguments of $\square$ to point-line or line-point.

$\mathsf{PG}^2.1\mathsf{a}$

$$\exists_1 a \;\; a = a$$

i.e., ⌜there exists a line⌝. This obviously holds for $\mathbb{KP}^2$.

$\mathsf{PG}^2.1\mathsf{b}$

$$\forall_1 a \, \exists_0 A \;\; A \boxslash a$$

i.e., ⌜given any line there is a point not on it⌝. Suppose l is a line of $\mathbb{KP}^2$, i.e., a 2-dimensional subspace of $\mathbb{K}^3$. Let u be an element of $\mathbb{K}^3$ not in l (necessarily nonzero), and let $A = \{ua \mid a \in \mathbb{K}\}$ be the 1-dimensional space containing u. Then A is a point of $\mathbb{KP}^2$ not on l.

$\mathsf{PG}^2.1\mathsf{c}$

$$\forall_1 a \, \exists_0 A_1, A_2, A_3 \, (A_1 \square a \wedge A_2 \square a \wedge A_3 \square a \wedge A_1 \neq A_2 \wedge A_1 \neq A_3 \wedge A_2 \neq A_3)$$

i.e., ⌜on any line are at least three distinct points⌝. As a subspace of $\mathbb{K}^3$, a line l of $\mathbb{KP}^2$ is of the form $\{ua + vb \mid a, b \in \mathbb{K}\}$ for linearly independent $u, v \in \mathbb{K}^3$. u, v, and $u + v$ are distinct points on l (because $0 \neq 1$ in $\mathbb{K}$).

$\mathsf{PG}^2.2\mathsf{a}$

$$\forall_1 a, b \left(a \neq b \rightarrow \exists_0 ! A \, (A \square a \wedge A \square b) \right)$$

i.e., ⌜given any two distinct lines there is a unique point that is on both of them⌝. The intersection of distinct 2-dimensional subspaces of $\mathbb{K}^3$ is a 1-dimensional subspace.

$\mathsf{PG}^2.2\mathsf{b}$

$$\forall_0 A, B \left(A \neq B \rightarrow \exists_1 ! a \, (a \square A \wedge a \square B) \right)$$

i.e., ⌜given any two distinct points there is a unique line that is on both of them⌝. The span of two distinct 1-dimensional subspaces is a 2-dimensional subspace.

We now extend the basic signature by defining a binary operation that takes as arguments either two distinct points or two distinct lines, and returns the unique line or point respectively that is on both of them. Axioms $\mathsf{PG}^2.2$ allow the following definition.

(2.117) **Definition** [PG^2]

1. *Suppose A, B are distinct points.* $(A, B) \stackrel{\text{def}}{=}$ *the unique line a such that $A \,\square\, a$ and $B \,\square\, a$.*
2. *Suppose a, b are distinct lines.* $(a, b) \stackrel{\text{def}}{=}$ *the unique point A such that $a \,\square\, A$ and $b \,\square\, A$.*[26]

The following theorems establish the *duality principle*.

(2.118) **Theorem** [PG^2] $\exists_0 A\ A = A$, *i.e., there exists a point.*

Proof This follows trivially from the fact that there exists a line and there is a point on it. $\square^{2.118}$

(2.119) **Theorem** [PG^2] *Given any point there is a line not on it.*

Proof Let A be a point. By $\mathsf{PG}^2.1\mathsf{a}$ there exists a line a. If $A \,\boxslash\, a$ we are finished. If $A \,\square\, a$ then let B be a point on a other than A (by $\mathsf{PG}^2.1\mathsf{c}$) and let C be a point not on a (by $\mathsf{PG}^2.1\mathsf{b}$). Then $B \neq C$. Let $b = (B, C)$. Then $b \neq a$, since $C \,\square\, b$ and $C \,\boxslash\, a$. Hence, $A \,\boxslash\, b$; otherwise both a and b would be on both A and B, contrary to $\mathsf{PG}^2.2\mathsf{b}$. $\square^{2.119}$

(2.120) **Theorem** [PG^2] *On any point are at least three distinct lines.*

Proof Suppose A is a point. Let a be a line not incident with A, and let B_1, B_2, B_3 be distinct points on a. Then $(A, B_1), (A, B_2), (A, B_3)$ are distinct lines on A. $\square^{2.120}$

2.5.3 Duality

Note that (2.118), (2.119), and (2.120) become $\mathsf{PG}^2.1\mathsf{a}$, $\mathsf{PG}^2.1\mathsf{b}$, and $\mathsf{PG}^2.1\mathsf{c}$, respectively, if we substitute 'point' for 'line' and 'line' for 'point' throughout. We say that the former and the latter are respectively *dual* to each other. Clearly, $\mathsf{PG}^2.0$ is selfdual in this sense, and $\mathsf{PG}^2.2\mathsf{a}$ and $\mathsf{PG}^2.2\mathsf{b}$ are each dual to the other. We refer to the transformation of a formula by swapping the point- and line-domains as *dualization*, and we call the result the *dual* of the original. Given a formula ϕ in the signature of PG^2 we let $\phi^* \stackrel{\text{def}}{=}$ its dual. Note that $\phi^{**} = \phi$.

Dualization is therefore an interpretation of $\mathcal{L}^{\mathsf{pg}^2}$ in itself. It has the very useful consequence that the set of theorems of PG^2 is closed under dualization, i.e.,

$$\text{if } \mathsf{PG}^2 \vdash \sigma \text{ then } \mathsf{PG}^2 \vdash \sigma^*. \tag{2.121}$$

We call a pg^2-theory *selfdual* $\stackrel{\text{def}}{\iff}$ it is closed under dualization. We therefore have the theorem:

[26]For completeness, since operations must be defined for all arguments (of specified types), we stipulate that $(A, A) = A$ and $(a, a) = a$ for any point A and line a. This never arises in practice.

(2.122) **Theorem** $[\mathsf{S}^0]$ PG^2 *is selfdual.*

Note that this is a theorem *about* PG^2, not *of* PG^2; and we may refer to it as a metatheorem. We have shown it as a theorem of S^0, which we have adopted as our standard theory of finitary objects, and which is suitable for a discussion of languages, axiom systems, and proofs.

Note that in saying that a pg^2-theory Θ is selfdual we only mean that for any pg^2-sentence σ,

$$\Theta \vdash \sigma \rightarrow \Theta \vdash \sigma^*,^{2.121}$$

not that

$$\Theta \vdash \sigma \rightarrow \sigma^*.$$

2.5.4 Solid projective geometry

We now enlarge the system to three dimensions. There are three domains: points, lines, and planes. The incidence relation takes as arguments any two distinct sorts, and it is symmetric. This theory exhibits duality under the interchange of 'point' and 'plane'. At the cost of some redundancy, we will present a selfdual axiomatization.

As we present the axioms, you may wish to verify that they hold for $\mathbb{KP}^3$ for any division ring $\mathbb{K}$. As we will see, in this case, these are the only models of the axioms.

Axioms of PG^3

$\mathsf{PG}^3.0\mathsf{a}$

$$\forall_0 A\, \forall_1 a\, (A \,\square\, a \leftrightarrow a \,\square\, A)$$
$$\forall_0 A\, \forall_2 \alpha\, (A \,\square\, \alpha \leftrightarrow \alpha \,\square\, A)$$
$$\forall_1 a\, \forall_2 \alpha\, (a \,\square\, \alpha \leftrightarrow \alpha \,\square\, a)$$

$\mathsf{PG}^3.0\mathsf{b}$ ⌜Given a point A, a line a, and a plane α, if $A \,\square\, a$ and $a \,\square\, \alpha$ then $A \,\square\, \alpha$.⌝

$\mathsf{PG}^3.1\mathsf{a}$ ⌜There exists a line.⌝

$\mathsf{PG}^3.1\mathsf{b}$ ⌜Given a line a on a $\left\{\begin{matrix}\text{point } A\\ \text{plane } \alpha\end{matrix}\right\}$ there is a $\left\{\begin{matrix}\text{plane on } A\\ \text{point on } \alpha\end{matrix}\right\}$ not on a.⌝

$\mathsf{PG}^3.1\mathsf{c}$ ⌜On any line are at least three distinct $\left\{\begin{matrix}\text{points}\\ \text{planes}\end{matrix}\right\}$⌝.

$\mathsf{PG}^3.2\mathsf{a}$ ⌜Any two distinct $\left\{\begin{matrix}\text{points}\\ \text{planes}\end{matrix}\right\}$ are on a unique line.⌝

$\mathsf{PG}^3.2\mathsf{b}$ ⌜Any nonincident $\left\{\begin{matrix}\text{point}\\ \text{plane}\end{matrix}\right\}$ and line are incident with a unique $\left\{\begin{matrix}\text{plane}\\ \text{point}\end{matrix}\right\}$.⌝

$\mathsf{PG}^3.2\mathsf{c}$ ⌜If two distinct $\left\{\begin{matrix}\text{points}\\ \text{planes}\end{matrix}\right\}$ on a line a are on a $\left\{\begin{matrix}\text{plane } \alpha\\ \text{point } A\end{matrix}\right\}$ then a is on $\left\{\begin{matrix}\alpha\\ A\end{matrix}\right\}$.⌝

Axioms $\mathsf{PG}^3.0$ correspond to the "housekeeping" axiom $\mathsf{PG}^2.0$; axioms $\mathsf{PG}^3.1$ to the "existence" axioms $\mathsf{PG}^2.1$; and axioms $\mathsf{PG}^3.2$ to the "incidence" axioms $\mathsf{PG}^2.2$.

Definition $[\mathsf{PG}^3]$ *We adapt the operation* $(\cdot,\cdot)^{2.117}$ *for the* 3*-dimensional setting.*

1. *Suppose A, B are distinct points.* $(A, B) \stackrel{\text{def}}{=}$ *the unique line on both A and B.*
2. *Suppose α, β are distinct planes.* $(\alpha, \beta) \stackrel{\text{def}}{=}$ *the unique line on both α and β.*
3. *Suppose A and a are a nonincident point and line.* $(A, a) \stackrel{\text{def}}{=} (a, A) \stackrel{\text{def}}{=}$ *the unique plane on both A and a.*
4. *Suppose α and a are a nonincident plane and line.* $(\alpha, a) \stackrel{\text{def}}{=} (a, \alpha) \stackrel{\text{def}}{=}$ *the unique point on both α and a.*

The following existence and incidence principles could naturally have been listed as axioms, but they are easily derived from the axioms given.

(2.123) **Theorem** [PG^3]

1. *Given any* $\left\{\begin{matrix}\textit{plane}\\ \textit{point}\end{matrix}\right\}$ *there is a* $\left\{\begin{matrix}\textit{point}\\ \textit{plane}\end{matrix}\right\}$ *not on it.*
2. *Any three distinct* $\left\{\begin{matrix}\textit{points}\\ \textit{planes}\end{matrix}\right\}$ *not all on a line are on a unique* $\left\{\begin{matrix}\textit{plane}\\ \textit{point}\end{matrix}\right\}$.
3. *Distinct lines on a* $\left\{\begin{matrix}\textit{plane } \alpha\\ \textit{point } A\end{matrix}\right\}$ *are on a unique* $\left\{\begin{matrix}\textit{point}\\ \textit{plane}\end{matrix}\right\}$, *which is also on* $\left\{\begin{matrix}\alpha\\ A\end{matrix}\right\}$.

Proof 1 We will prove the top version. The dual of this proof is a proof of the bottom version. Suppose α is a plane. Let a be a line. If there is a point on a not on α we are finished; if not, a is on α. Let β be another plane on a, and let A be a point on β not on a. Then $\beta = (A, a)$, i.e., β is the only plane on both A and a. Since $a \,\square\, \alpha$ and $\alpha \neq \beta$, it follows that $A \boxslash \alpha$.

2 Again, we will just prove the top version. Suppose A, B, C are distinct points. Let $a = (B, C)$ and $\alpha = (A, a)$. Then $A \,\square\, \alpha$, and $a \,\square\, \alpha$, so $B, C \,\square\, \alpha$. Suppose α' is another plane on A, B, C. Then $A \,\square\, \alpha'$ and $a \,\square\, \alpha'$, so $\alpha' = \alpha$.

3 Again, we will just prove the top version. Suppose b and c are distinct lines on a plane α. Let A be a point not on α. Let $\beta = (A, b)$ and $\gamma = (A, c)$. Since A is on β and γ, $\alpha \neq \beta$ and $\alpha \neq \gamma$. Hence, $b = (\alpha, \beta)$ and $c = (\alpha, \gamma)$. It follows that $\beta \neq \gamma$. Thus, α, β, and γ are distinct planes. They are not all on a line, because that line would have to be both b and c, which are distinct. Let[2.123.2] D be the unique point on α, β, and γ. Then $D \,\square\, b$; otherwise, since D and b are both on α and β, $\alpha = (D, b) = \beta$. Likewise, $D \,\square\, c$. $\square$[2.123]

Clearly, if π is a plane, then $\mathsf{PG}^3.\mathsf{0a}$ (top line), $\mathsf{PG}^3.\mathsf{1a}$, $\mathsf{PG}^3.\mathsf{1b}$ (lower), $\mathsf{PG}^3.\mathsf{1c}$ (upper), $\mathsf{PG}^3.\mathsf{2a}$ (upper), and Theorem 2.123.3 (upper) yield axioms $\mathsf{PG}^2.\mathsf{0}$–$\mathsf{2}$ for the structure consisting of the points and lines on π. In other words, PG^2 is interpretable in the theory obtained from PG^3 by adding a constant to its signature with the "axiom" that this constant denotes a plane, so PG^2 is—in this slightly extended sense—interpretable in PG^3.

We have remarked above that the historically original models of PG^2 and PG^3 are the projective spaces $\mathbb{RP}^2$ and $\mathbb{RP}^3$, and we can replace $\mathbb{R}$ by any division ring. We have also mentioned—but not yet proved—that every model of PG^3 is isomorphic to $\mathbb{KP}^3$ for some division ring $\mathbb{K}$. This is shown by providing geometrical definitions of the algebraic operations of addition and multiplication on the points of a line with one point removed, which—when interpreted in a model $\mathfrak{S}$ of PG^3—constitute a division ring $\mathbb{K}$, and using this to establish an isomorphism of $\mathfrak{S}$ with $\mathbb{KP}^3$ by assigning *homogeneous coordinates* to the points, lines, and planes of $\mathfrak{S}$. We defer this for the moment.

For each $n > 1$, let $\mathcal{D}^n$ be the class of pg^n-structures of the form $\mathbb{KP}^n$, where $\mathbb{K}$ is a division ring. It follows from the preceding remarks that $\mathcal{D}^3$ is elementary in the sense introduced above.[2.116] Let Θ^n be the theory of $\mathcal{D}^n$, i.e., the set of sentences true in all members of $\mathcal{D}^n$. As we have stated above (but still not proved), $\Theta^3 = \overline{\mathsf{PG}^3}$. As we will see shortly, the class of models of PG^2 properly includes $\mathcal{D}^2$. Note that this does not imply that Θ^2 properly includes $\overline{\mathsf{PG}^2}$: it could conceivably happen that every sentence true in all structures in $\mathcal{D}^2$ is derivable from PG^2. If this were so then $\mathcal{D}^2$ would not be elementary. We will see, however, that $\mathcal{D}^2$ is elementary, so Θ^2 properly includes $\overline{\mathsf{PG}^2}$.

2.5.5 Desargues's principle

In this section we begin to justify the assertions made above concerning the theories PG^2 and PG^3. We have already noted that for any $\sigma \in \mathsf{PG}^2$, $\mathsf{PG}^3 \vdash \sigma^\Pi$, where Π is a constant denoting a plane, and the relativization $\sigma \mapsto \sigma^\Pi$ is accomplished by restricting quantified variables to points and lines on Π. The key question is whether the converse holds, i.e., if σ is a pg^2-sentence and $\mathsf{PG}^3 \vdash \sigma^\Pi$, does it follow that $\mathsf{PG}^2 \vdash \sigma$; and, if not, can we simply axiomatize the theory $\{\sigma \mid \mathsf{PG}^3 \vdash \sigma^\Pi\}$. The answers are respectively 'no' and 'yes', and one fundamental theorem of PG^3 is central to the discussion. We begin with a definition.

Definition [PG^3]

1. *Lines or planes are* concurrent $\overset{\text{def}}{\iff}$ *they are all on a single point.*
2. *Points or planes are* collinear $\overset{\text{def}}{\iff}$ *they are all on a single line.*
3. *Lines or points are* coplanar $\overset{\text{def}}{\iff}$ *they are all on a single plane.*
4. *A* triangle *is a sequence A, B, C, a, b, c of points and lines such that*
    1. *A, B, C are noncollinear;*
    2. *a, b, c are nonconcurrent;*
    3. *$a = (B, C)$, $b = (C, A)$, and $c = (A, B)$; equivalently, $A = (b, c)$, $B = (c, a)$, and $C = (a, b)$.*

    We regard different orderings of given set of points and lines as different triangles. We may refer to the triangle A, B, C, a, b, c as 'ABC' or as 'abc'.
5. *Triangles A, B, C, a, b, c and A', B', C', a', b', c' are*
    1. perspective from a point D $\overset{\text{def}}{\iff}$ *D is collinear with A and A', with B and B', and with C and C';*
    2. perspective from a line d $\overset{\text{def}}{\iff}$ *d is concurrent with a and a', with b b', and with c and c'.*

Note that since the points of a triangle are noncollinear they are on a unique plane, and the lines of the triangle are on the same plane.

Figure 2.1 shows perspectivity of a triangle from a point D and from a line d. It is no accident that a single triangle serves to illustrate both forms of perspectivity, as they are equivalent for $\mathbb{RP}^2$. In fact, they are equivalent for $\mathbb{KP}^n$ for any division ring $\mathbb{K}$, and their equivalence is a theorem of PG^3: *Desargues's theorem.* It is not, however, a theorem of PG^2.

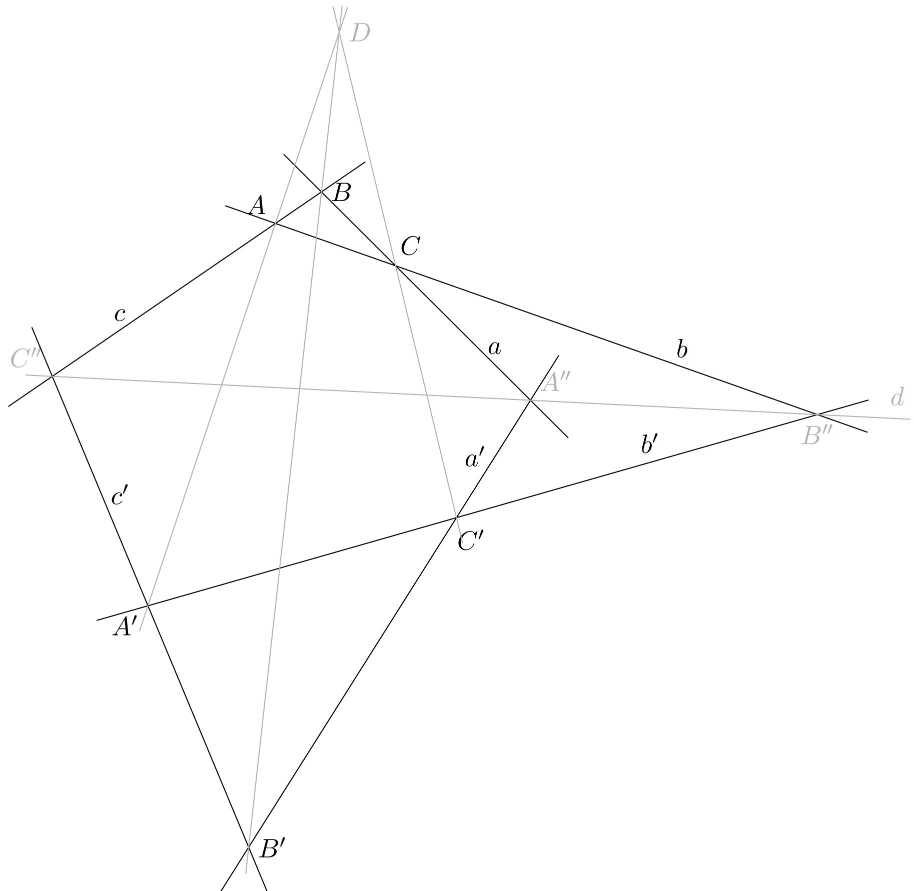

Figure 2.1: The triangles ABC and $A'B'C'$ are perspective from the point D and from the line d.

1. *For the purpose of this discussion we define* Desargues's principle DP *to be the statement that any triangles perspective from a point are perspective from a line.*
2. *Note that the dual* DP^* *of* DP *in plane projective geometry is also the converse of* DP*: Any triangles perspective from a line are perspective from a point.*

(2.124) **Theorem (Desargues)** [PG^3] *Two triangles perspective from a point are perspective from a line.*

Proof Suppose A, B, C, a, b, c and A', B', C', a', b', c' are triangles perspective from a point D. Suppose first that they are on distinct planes π and π', respectively. Let $d = (\pi, \pi')$. By hypothesis there is a line a'' on D, A, A', and a line b'' on D, B, B'. Since a'' and b'' are both on D, there is a plane γ such that a'', b'' are on γ. Thus, A, A', B, B' are on γ, so (A, B) and (A', B') are on γ. There is therefore a point C'' on both $c = (A, B)$ and $c' = (A', B')$. C'' is on both π and π', so it is on d. Hence, c, c', d are concurrent. Similarly, a, a', d and b, b', d are concurrent, so the triangles are perspective from d.

Now suppose $\pi = \pi'$. Let e be a line on D not on π, and let E and E' be distinct points on e distinct from D. By the hypothesis of perspectivity, there exist planes α, β, γ such that A, A' are on α; B, B' are on β; C, C' are on γ; and e and hence also D, E, E' are on all three of them. E and E' are both distinct from A, B, C, A', B', C'. Since E, E', A, A' are on α there is a point A'' on α that is on (E, A) and (E', A'). Similarly, there exist points B'' on (E, B) and (E', B'), and C'' on (E, C) and (E', C').

(E, A) is not (E, B); otherwise, $A = ((E, A), \pi) = ((E, B), \pi) = B$. Thus, if $A'' = B''$ then, since A'' is on the distinct lines (E, A) and (E, B), $A'' = E$. Likewise, if $A'' = B''$ then $A'' = E'$. Since $E \neq E'$, it follows that $A'' \neq B''$.

C'' cannot be identical with both E and E', since the latter are distinct. Suppose without loss of generality that $C'' \neq E$. (A'', B'') is on the plane (E, c). If C'' is on (A'', B'') then C'' is also on (E, c). Since E is on (E, c) and is distinct from C'', (E, C'') is on (E, c). Since C is on (E, C''), C is on (E, c). But the only points on both π and (E, c) (which are distinct, since E is not on π) are on the common line $c = (A, B)$. This is impossible, since by the definition of triangle, C is not collinear with A, B. It follows that A'', B'', C'' are not collinear, so they form a triangle. Let $a'' = (B'', C'')$, $b'' = (C'', A'')$, and $c'' = (A'', b'')$.

Let π'' be the plane on A'', B'', C''. It is easy to see that if A'' is on π then $A = A'' = A'$; likewise for B'' and C''. Thus, if $\pi'' = \pi$, then the triangles are identical. In this case they are perspective from any line in π. Suppose therefore that $\pi'' \neq \pi$. ABC and $A''B''C''$ are perspective from E, so by the theorem for the noncoplanar case, they are perspective from the line $f = (\pi, \pi'')$; likewise for $A'B'C'$ and $A''B''C''$. a'' has only one point in common with f, so this point must be on both a and a'; hence, they are concurrent with f. Similarly, b, b' and c, c' are concurrent with f; hence, ABC and $A'B'C'$ are perspective from f. $\square$[2.124]

The first metamathematical point to make about Desargues's theorem is that it does not follow from PG^2. To show this it suffices to exhibit a model of PG^2 in which it fails. A simple such model is due to Moulton. A finite, or *affine*, *Moulton plane* has the same points as the euclidean plane $\mathbb{R}^2$. The lines of the Moulton plane include the vertical euclidean lines and those of negative slope. The remaining Moulton lines consist of a segment in the left halfplane of positive slope together with a segment in the right halfplane with half the slope, as illustrated in Figure 2.2. This plane has the same incidence properties as the ordinary euclidean plane. Note particularly that is has parallel—i.e., nonintersecting—lines of arbitrary slope. We form the corresponding projective Moulton plane by adjoining a point at infinity for each equivalence class of parallel lines, and a line at infinity containing all the points at infinity. It is easy to see that this is a model of PG^2. Figure 2.3, which is just Figure 2.1 "moultonized", shows that it does not satisfy Desargues's theorem. Likewise, DP^* is not a theorem of PG^2: its failure in the Moulton plane may be demonstrated by a simple modification of Figure 2.3.

Not surprisingly, DP^* is a theorem of PG^3. It is tempting to use the duality principle to infer this from the fact that $\mathsf{PG}^3 \vdash \mathsf{DP}$, and there are multiple instances in the expository literature on this subject of erroneous "proofs" of DP^* based on duality. Recall, however, that DP^* is the *plane* dual of DP. The duality principle for PG^3 only yields the *solid* dual of DP, in which 'point' is replaced by 'plane' throughout, and the conclusion is still a statement about perspectivity from a line. The duality principle for PG^2 is irrelevant to the matter at hand.

The reason this error is so seductive is that the theory $\mathsf{PG}^2 + \mathsf{DP}$ is, in fact, selfdual, but this has to be proved:

(2.125) **Theorem** $[\mathsf{PG}^2]$ $\mathsf{DP} \rightarrow \mathsf{DP}^*$.

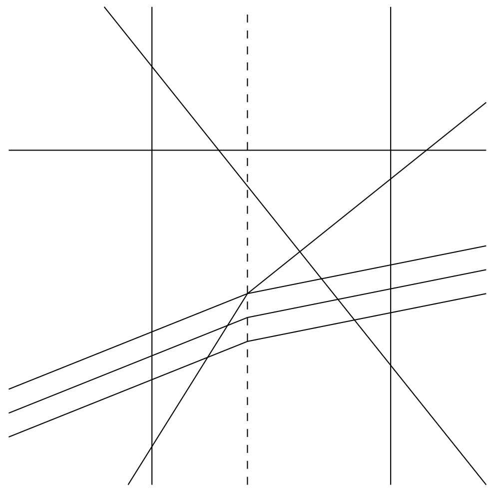

Figure 2.2: A Moulton plane.

Proof See Note 10.8. □

Note that ⌜$\mathsf{DP}^* \to \mathsf{DP}$⌝ is dual to ⌜$\mathsf{DP} \to \mathsf{DP}^*$⌝, so the duality principle for PG^2 allows us to infer the existence of a PG^2-proof of ⌜$\mathsf{DP}^* \to \mathsf{DP}$⌝ from the existence of a PG^2-proof of ⌜$\mathsf{DP} \to \mathsf{DP}^*$⌝:

Theorem [PG^2] $\mathsf{DP}^* \to \mathsf{DP}$.

Putting (2.124) together with (2.125) and the fact that $\mathsf{PG}^3 \vdash$ ⌜for every plane π, PG^{2^π}⌝, we know that PG^3 proves that any coplanar triangles perspective from a line are perspective from a point. If we want the full converse—for noncoplanar triangles as well—an additional argument is required: neither plane nor solid duality is applicable. We omit the proof.

2.5.6 Models of the axioms

It is of interest that, although the 3-dimensional converse of DP is not obtainable via duality, a different metamathematical device exists for showing that it is a theorem of PG^3 without actually sketching a proof, viz., the completeness theorem. This is a corollary of the above-mentioned characterization of the models of PG^3 as the projective spaces $\mathbb{KP}^3$, where $\mathbb{K}$ is a division ring. Desargues's theorem is the key to this, and we have the analogous result that the models of $\mathsf{PG}^2 + \mathsf{DP}$ are the projective spaces $\mathbb{KP}^2$, $\mathbb{K}$ a division ring.

Let us briefly indicate how this is done. Given a model $\mathfrak{M}$ of $\mathsf{PG}^2 + \mathsf{DP}$ or PG^3 we obtain the ring $\mathbb{K}$ by providing geometric definitions of addition and multiplication operations on the points of any line with one point (which may be thought of as the "point at infinity") deleted. Figures 2.4 and 2.5 illustrate these definitions. In Figure 2.4, k is an arbitrary line, and I is an arbitrary point on k, which we regard as the point at infinity on k. Let $\tilde{k}$ be the set of points on k other than I. O is an arbitrary point distinct from I that will be the 0 element of $\mathbb{K}$. Given points A and B on k other than I, to construct $A + B$ let m and m' be arbitrary distinct lines on I distinct from k, and let X be an arbitrary point on m distinct from I. Let X' on m' be collinear with X and A, let Y' on m' be collinear with X and O, and let

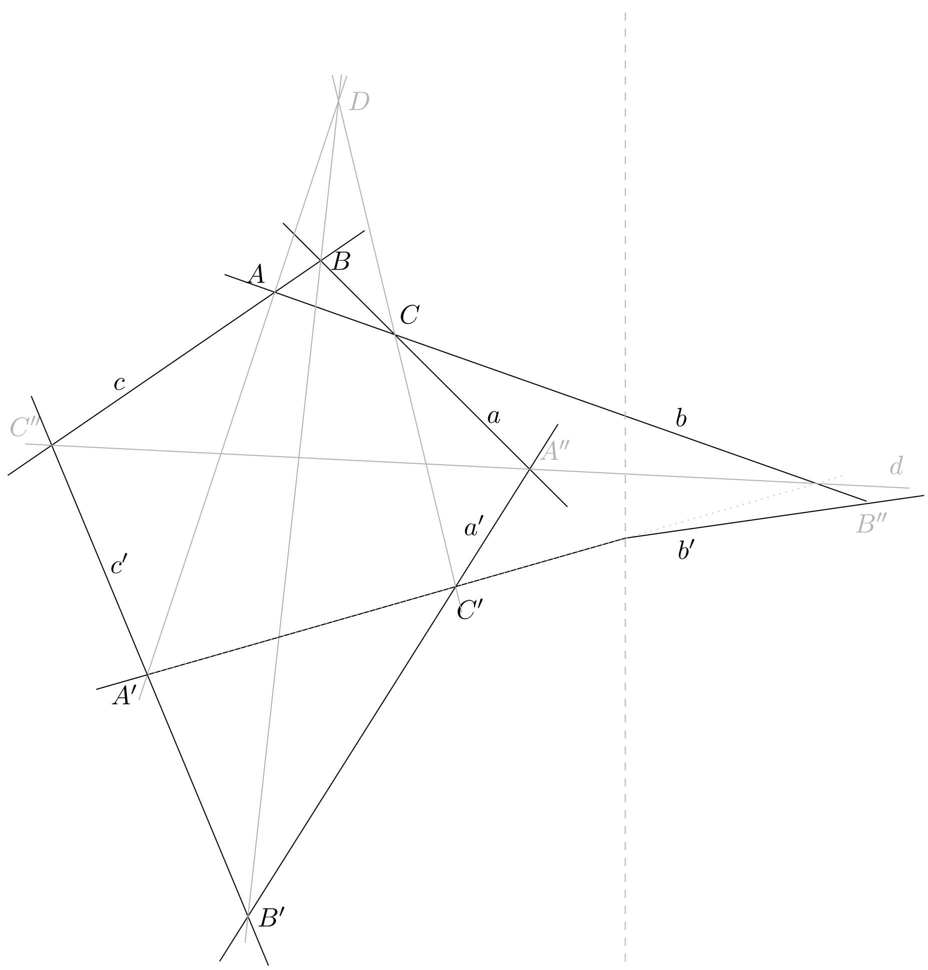

Figure 2.3: The triangles ABC and $A'B'C'$ in this Moulton plane are perspective from the point D, but the points of intersection of corresponding lines are not collinear, as the line d on A'' and C'' is not on B''. (This is Figure 2.1 modified as in Figure 2.2.)

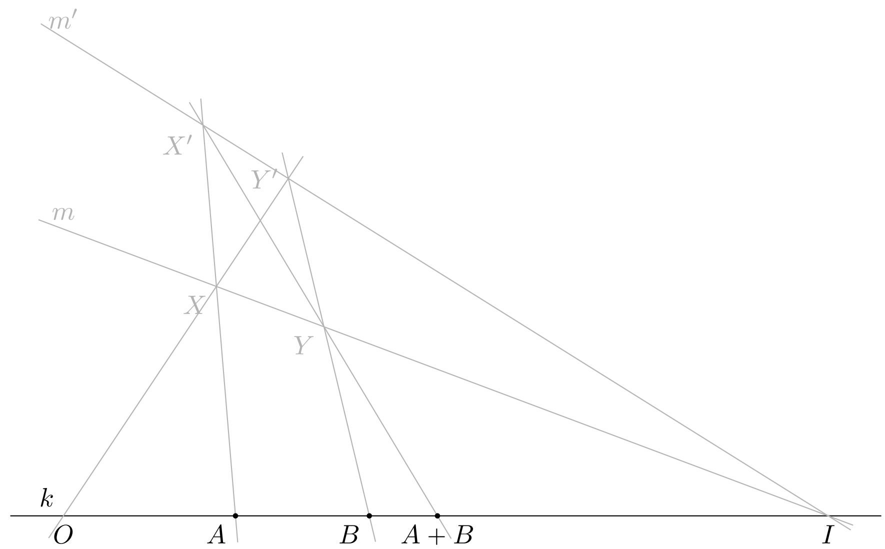

Figure 2.4: Geometrical construction of the sum of points A and B on an arbitrary line k with a point I omitted. O is the 0 element.

Y on m be collinear with Y' and B. We define $A+B$ as the point on k collinear with X' and Y. To validate this as a definition we must first show that if k, I, O, X, A, and B be given then X', Y', Y, and $A+B$ are uniquely determined; then we must show that $A+B$ so constructed is independent of the choice of m, m', and X, for which Desargues's principle is required. We must then show that $(\tilde{k}; O, +)$ is an abelian group. We omit the proofs.

In Figure 2.5 we continue the construction by choosing an arbitrary point U on k distinct from I and O, which will be the multiplicative identity. Given points A and B on k other than I, to construct $A \cdot B$ let m and n be arbitrary lines on I and O, respectively, distinct from k, and let U_m and U_n be arbitrary points on m and n, respectively, collinear with U. Let X on m be collinear with U_n and A, and let Y on n be collinear with U_m and B. We define $A \cdot B$ as the point on k collinear with X and Y. As before, we must show that this construction yields a definite point $A \cdot B$, and that this is independent of the choice of m, n, U_m and U_n. We must then verify that $(\tilde{k}; O, U, +, \cdot)$ is a division ring. We omit the proofs.

It is fairly straightforward to show that the above constructions applied to any l, I, O, U in $\mathbb{KP}^2_r$ do in fact yield an isomorph of $\mathbb{K}$. Perhaps unsurprisingly, applied to $\mathbb{KP}^2_l$, the construction in Figure 2.5 defines $B \cdot A$, rather than $A \cdot B$.

Still working in $\mathsf{PG}^2 + \mathsf{DP}$, we can now define a structure isomorphic to $\mathbb{KP}^2$, where $\mathbb{K}$ is the division ring just defined. This is equivalent to the definition of *homogeneous coordinates*, as follows. Suppose ι is an isomorphism of a projective plane $\mathfrak{P}$ with $\mathbb{KP}^2$, and suppose A is a point of $\mathfrak{P}$. Let $W = \iota A$. W is a 1-dimensional subspace $\mathbb{KP}^3$, and the components of any nonzero element of W serve as coordinates of A. They are *homogeneous* inasmuch as coordinate tuples that are proportional to each other (by right multiplication by a nonzero scalar) represent the same point. We naturally represent these coordinate tuples as column matrices.

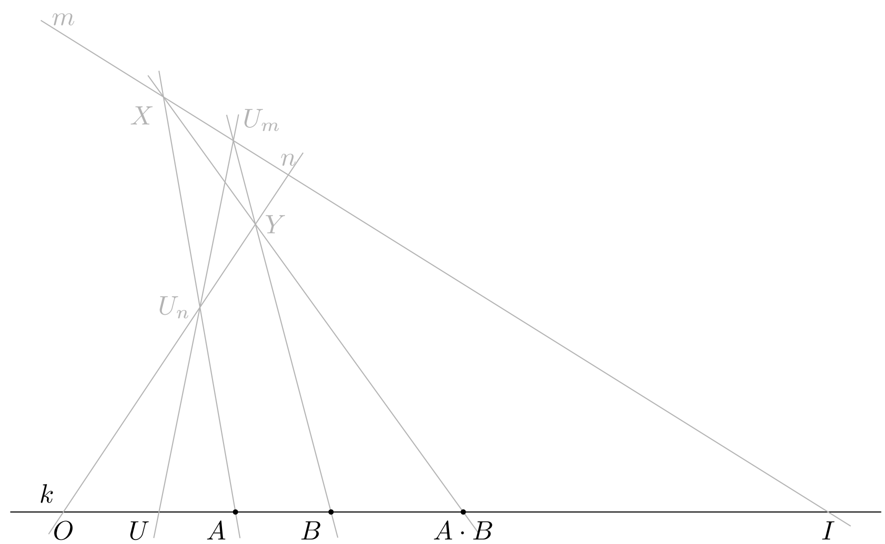

Figure 2.5: Geometrical construction of the product of points A and B on an arbitrary line k with a point I omitted. O is the 0 element, U the 1 element.

Now suppose l is a line of $\mathfrak{P}$. Let $W = \iota l$. W is a 2-dimensional subspace of $\mathbb{K}^3$ and is therefore the nullspace (kernel) of a linear transformation $T : \mathbb{K}_r^3 \to \mathbb{K}$. Note that T is represented by a nonzero row matrix, and any transformation proportional to T (by left multiplication) has the same nullspace. Thus, the components of (the matrix representation of) T are homogeneous coordinates of l. A line and a point are incident iff the product of their coordinate (row and column) matrices is 0.

To define homogeneous coordinates for a projective plane synthetically, we proceed as illustrated in Figures 2.6. We define a division ring $\mathbb{K}$ by choosing a line a and points O, U, and I on a as above. Thus, $|\mathbb{K}|$ consists of the points of a other than I, which is the "point at infinity in the direction of" a. All coordinates will be obtained by reference to a. We establish an isomorphism of $\mathbb{K}$ with the ring structure on another line a' with the same origin O by a perspectivity through a point J not on a or a'. Note that $I' \neq I$. $i = (I, I')$ is the "line at infinity" for this coordinatization of the plane. Given a point P not on (I, I') (a point of the finite or affine plane, if you will), we obtain an "x" coordinate by projecting P through I' to a, and a "y" coordinate by projecting first through I to a' and then through J to a. These are the affine coordinates of P, and the corresponding homogeneous coordinate triple is $\langle X, Y, 1\rangle$, i.e., $\langle X, Y, U\rangle$. Any other homogeneous coordinate triple is $\langle X \cdot A, Y \cdot A, A\rangle$, where A on a is not O or I. (To save space, we are representing column matrices by 3-sequences.)

To obtain coordinates for a point Q on i, let P be any point on (O, Q) other than O and Q, and let $\langle X, Y\rangle$ be the affine coordinates of P. Then $\langle X, Y, 0\rangle$, i.e., $\langle X, Y, O\rangle$ is a homogeneous coordinate triple for Q, as of course is any nonzero (right) multiple thereof; indeed, any such multiple arises from another choice of P on (O, Q).

We omit the somewhat complicated proof that this assignment of coordinates establishes an isomorphism of $\mathfrak{P}$ with $\mathbb{KP}^2$, the essential facts being that the points and lines of $\mathfrak{P}$ do indeed correspond to the 1- and 2-dimensional subspaces of $\mathbb{K}_l^3$, respectively. The same can be done for solid projective geometry. Note that in

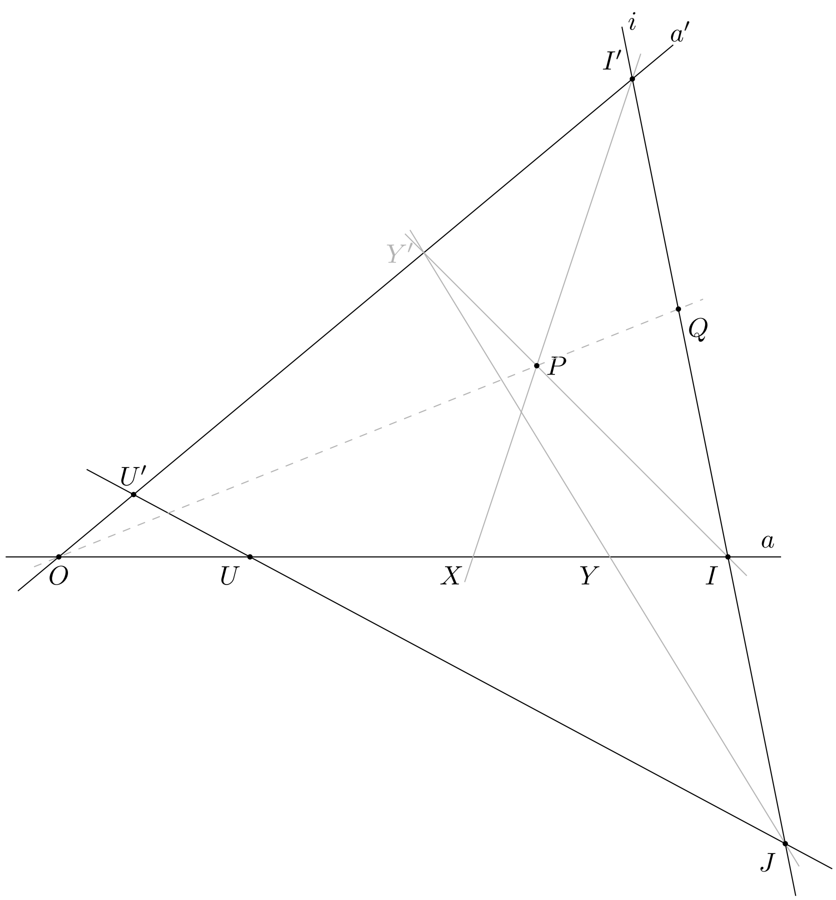

Figure 2.6: The homogeneous coordinates of a point P not on (I, I') are $\langle X, Y, U\rangle$ and all nonzero multiples thereof, i.e., $\langle X \cdot A, Y \cdot A, A\rangle$, where A is a point on (O, I) other than O, I.

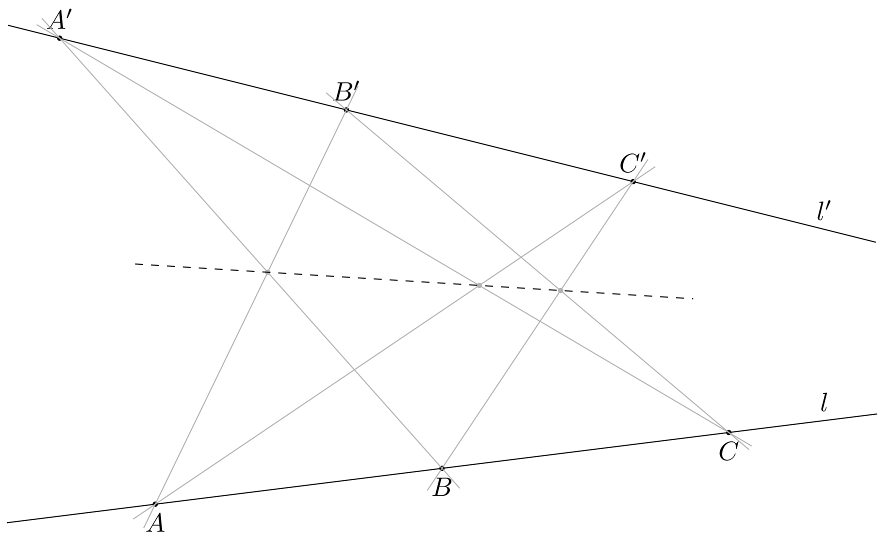

Figure 2.7: Pappus's principle

this case, Desargues's principle does not have to be separately posited, as it is a theorem of PG^3. The following theorem sums it up.

(2.126) **Theorem** [ZF]

1. *The models of* $\mathsf{PG}^2 + \mathsf{DP}$ *are exactly the isomorphs of the structures* $\mathbb{KP}^2$*,* $\mathbb{K}$ *a division ring.*
2. *The models of* PG^3 *are exactly the isomorphs of the structures* $\mathbb{KP}^3$*,* $\mathbb{K}$ *a division ring.*

2.5.7 Pappus's principle

The following principle derives from a theorem of Pappus.

(2.127) **Definition** Pappus $\overset{\text{def}}{\iff}$ *for any two distinct coplanar lines* l, l'*, and six distinct points* A, B, C, A', B', C'*, such that* A, B, C *are on* l *and* A', B', C' *are on* l'*, the points* $((A, B'), (A', B))$*,* $((A, C'), (A', C))$*, and* $((B, C'), (B', C))$ *are collinear. See Figure 2.7.*

It is easy to show that (A, B') and (A', B) are distinct lines, so $((A, B'), (A', B))$ is well defined; and the same is true of the other two pairs of pairs of points.

The following theorem is not required for our purposes. It merely allows us to shorten the description of the theory $\mathsf{PG}^2 + \mathsf{DP} + \mathsf{Pappus}$ to '$\mathsf{PG}^2 + \mathsf{Pappus}$'. We omit its proof.

Theorem [$\mathsf{PG}^2 + \mathsf{Pappus}$] DP.

The following theorem gives the model theoretic equivalent of Pappus.

(2.128) **Theorem** [ZF] *The models of* $\mathsf{PG}^2 + \mathsf{Pappus}$ *are exactly the isomorphs of the structures* $\mathbb{KP}^2$*,* $\mathbb{K}$ *a field.*

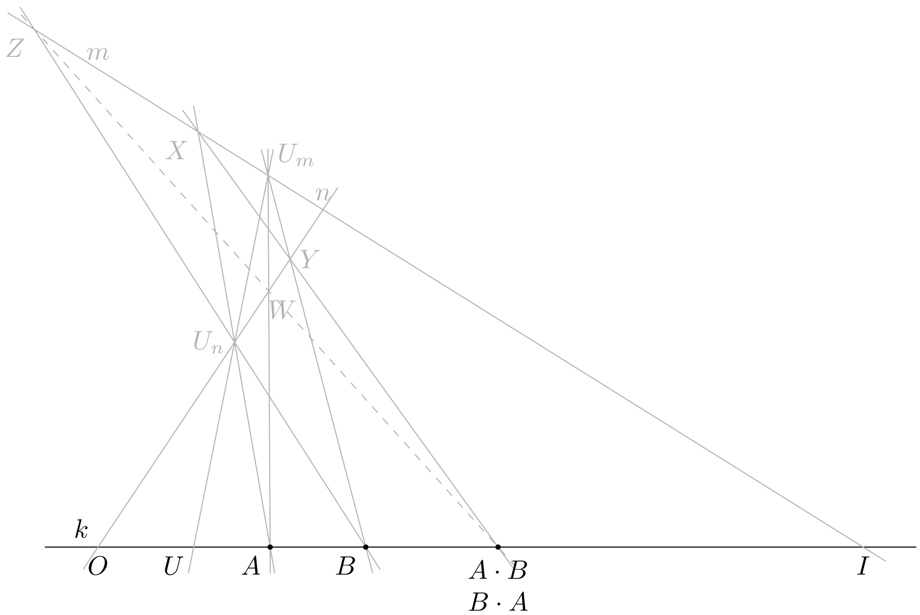

Figure 2.8: Geometrical construction of the products $A \cdot B$ and $B \cdot A$.

Proof In the forward direction the theorem says that Pappus implies that the multiplication operation $A, B \mapsto A \cdot B$ defined by Figure 2.5 is commutative. To see that this is so, consider the diagram in Figure 2.8, where we have additionally shown the construction defining $B \cdot A$, letting $Z = (m, (B, U_n))$, $W = (n, (A, U_m))$, and $B \cdot A = (k, (Z, W))$. Ignoring, in the interest of brevity, possible degeneracies, we show that $B \cdot A = A \cdot B$ by showing that $A \cdot B$ is collinear with Z and W. This follows from Pappus applied to the points A, U_n, X, Y, U_m, B in that order; i.e., we note that $W = ((A, U_m), (U_n, Y))$, $Z = ((U_n, B), (X, U_m))$, and $A \cdot B = ((X, Y), (A, B))$, which are collinear by Pappus.

The reverse direction is the statement that if $\mathbb{K}$ is a field then $\mathbb{KP}^2 \models$ Pappus.[27] Using the notation of Definition 2.127 and Figure 2.7, we first observe that at most one of the points A, B, C, A', B', C' can lie on both l and l', and by renaming, we may suppose that any such point is either C or C'. Thus, no three of the points in $\{A, A', B, B'\}$ are collinear. Recall that a point D of $\mathbb{KP}^2$ is a 1-dimensional subspace of $\mathbb{K}^3$. Any nonzero $\delta \in D$ is a *representative* of D, its components (as a 3-sequence) are *homogeneous coordinates* of D, and $[\delta] = D$. Collinearity of three points in $\mathbb{KP}^2$ is equivalent to linear dependence of their representatives in $\mathbb{K}^3$. Let $\alpha, \alpha', \beta, \beta'$ be representatives of A, A', B, B', respectively. Then any three vectors in $\{\alpha, \alpha', \beta, \beta'\}$ are linearly independent, while $\{\alpha, \alpha', \beta, \beta'\}$ is linearly dependent (being a set of four vectors in a 3-dimensional vector space). Thus there exist a, a', b, b' such that

$$a\alpha + a'\alpha' + b\beta + b'\beta' = 0,$$

and these coefficients are necessarily nonzero. By renaming the representative vectors, we may suppose that

$$\alpha + \alpha' + \beta + \beta' = 0 \tag{2.129}$$

[27]For $\mathbb{K} = \mathbb{R}$, which is of course what is actually illustrated by Figure 2.7, this is Pappus's theorem.

By the same sort of reasoning, there exist representatives γ, γ' of C, C', respectively, and $c, c' \in \mathbb{K}$, such that

$$\gamma = \alpha + c\beta, \qquad \gamma' = \alpha' - c'\beta'.$$

Let

$$\gamma'' = \alpha' + \beta.$$

Then[2.129]

$$-\alpha - \beta' = \gamma'' = \alpha' + \beta,$$

so $[\gamma'']$ is on both (A, B') and (A', B); hence, $[\gamma''] = ((A, B'), (A', B))$. Let

$$\beta'' = c'\alpha + c(1 + c')\alpha' + cc'\beta.$$

Then

$$c'(1 - c)\alpha + c\gamma' = \beta'' = c(1 + c')\alpha' + c'\gamma,$$

so $[\beta''] = ((A, C'), (A', C))$. Let

$$\alpha'' = c'\alpha + (1 + c')\alpha' + (1 - c + c')\beta.$$

Then

$$(1 - c)\beta + \gamma' = \alpha'' = -(1 + c')\beta' - \gamma,$$

so $[\alpha''] = ((B, C'), (B', C))$.

Since

$$\alpha'' - \beta'' - (1 - c)(1 + c')\gamma'' = 0,$$

$\{[\alpha''], [\beta''], [\gamma'']\}$ is linearly dependent. □[2.128]

2.5.8 Duality: synthetic and analytic

We can now use the completeness theorem to relate analytic to synthetic ideas in projective geometry. Consider first the duality principle. Recall that we established the synthetic version of this for the theories PG^2 and PG^3 by proving the dual of each of the axioms from the axioms. The analytic version of duality is formulated as follows. Suppose $\mathbb{K}$ is a division ring and $n \in \omega$. Recall that $\mathbb{K}_r^n$ and $\mathbb{K}_l^n$ are the right- and left-$\mathbb{K}$-modules whose elements are n-sequences from $\mathbb{K}$. We conventionally represent the elements of $\mathbb{K}_r^n$ and $\mathbb{K}_l^n$ respectively as column and row matrices, with either right- or left-multiplication of vectors by scalars. $\mathbb{K}_r^n$ and $\mathbb{K}_l^n$ are dual in the usual linear algebraic sense that there exists an evaluation tensor, which by our conventions is matrix multiplication. Thus, given $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{K}_r^n$ and $\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{K}_l^n$, the *value* of the pair is

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{K}.$$

Given a subspace A of $\mathbb{K}_r^n$, the *annihilator of* $A \overset{\text{def}}{=} A^\perp \overset{\text{def}}{=}$ the set of $v \in \mathbb{K}_l^n$ such that $vu = 0$ for all $u \in A$. $A^\perp$ is a subspace of $\mathbb{K}_l^n$, and $\dim A + \dim A^\perp = n$.

Similarly, the annihilator of a subspace of $\mathbb{K}^n_l$ is a subspace of $\mathbb{K}^n_r$ of complementary dimension. $A^{\perp\perp} = A$.

The elements of the projective spaces $\mathbb{KP}^n_r$ and $\mathbb{KP}^n_l$ are the subspaces of dimension $m = 1, \ldots, n-1$, so the annihilator operation determines a bijection of these spaces. It is easy to see that $A \subseteq B \leftrightarrow B^\perp \subseteq A^\perp$. Since, by definition, elements of $\mathbb{KP}^n$ are *incident* iff one is included in the other, the annihilator operation is an isomorphism of the incidence structures of $\mathbb{KP}^n_r$ and $\mathbb{KP}^n_l$. (Since it takes an element of sort m to an element of the dual sort $n - m$, it is not an isomorphism *per se*.)

The transpose operation, which converts a row to a column matrix and *vice versa*, clearly induces an isomorphism of $\mathbb{KP}^n$ with $\mathbb{K}^{\mathsf{op}}\mathbb{P}^n$, where $\mathbb{K}^{\mathsf{op}}$ is the *opposite* of $\mathbb{K}$, which is the ring with the same domain and addition operation, but with the order of multiplication reversed: $u \cdot^{\mathsf{op}} v = v \cdot u$.

We define the *dual* operation $A \mapsto A^*$ on $\mathbb{KP}^n$ as the composition of the annihilator and transpose operations. It is therefore an automorphism of the incidence structure of $\mathbb{KP}^n$ that maps objects of sort m to objects of sort $n - m$. If $n = 2$, the dual operation swaps points and lines; if $n = 3$, it swaps points and planes, and it takes each line to a (typically different) line.

The duality operation on sentences clearly transforms the theory of $\mathbb{KP}^n$ to the theory of $\mathbb{K}^{\mathsf{op}}\mathbb{P}^n$. To obtain a more useful result, let Θ^n be the theory of the class of structures $\mathbb{KP}^n$, where $\mathbb{K}$ is an arbitrary division ring, i.e., the set of sentences true in all structures $\mathbb{KP}^n$. Then Θ^n is closed under dualization, since the class of division rings is closed under the map $\mathbb{K} \mapsto \mathbb{K}^{\mathsf{op}}$. Thus we have the following corollary of (2.126).

(2.130) **Theorem** [ZF]

1. $\mathsf{PG}^2 + \mathsf{DP}$ *is selfdual.*
2. PG^3 *is selfdual.*

By virtue of (2.130.1), since $\mathsf{PG}^2 + \mathsf{DP} \vdash \mathsf{DP}$, it follows that $\mathsf{PG}^2 + \mathsf{DP} \vdash \mathsf{DP}^*$, i.e., $\mathsf{PG}^2 \vdash \mathsf{DP} \to \mathsf{DP}^*$. If we believe that ZF would not tell us that a PG^2-proof of $\mathsf{DP} \to \mathsf{DP}^*$ exists when a proof does not actually exist, then we need not actually exhibit PG^2-proof of $\mathsf{DP} \to \mathsf{DP}^*$ to know that one exists. In effect, (2.130.1) justifies the authors mentioned in the remarks preceding (2.125)—who have used "duality" to infer DP^* from DP. This is, of course, far more elaborate than the direct argument given in Note 10.8. It is also less than satisfactory in that it uses the infinitary theory ZF[28] to infer the existence of a finitary object (a proof), just the sort of thing we have taken pains elsewhere in this chapter to avoid.

2.5.9 Projectivity and the fundamental theorem

The notion of *projectivity*, which is central to projective geometry, is easily defined analytically, and it is of interest to see how metatheoretical considerations bear on its synthetic definition.

Suppose M and N are right-modules of the same finite dimension over a division ring $\mathbb{K}$. Let PM and PN be their corresponding projective spaces. Suppose T : M $\to$ N is a nonsingular linear transformation. We define $\mathsf{P}T$: PM $\to$ PN to be

[28] We do not need the full power of ZF to prove (2.126), but the avoidance of the Infinity axiom altogether would be awkward at best.

the induced map on the projective spaces: Given $A \in |\mathsf{PM}|$, A is by definition a subspace of M, and we let $\mathsf{P}T(A) = T^{\rightarrow}A$.

Clearly, $\mathsf{P}T$ is an isomorphism of the incidence structures of PM and PN. A *projectivity* $\overset{\text{def}}{\Longleftrightarrow}$ any isomorphism of incidence structures of projective spaces PM and PN obtained from a linear transformation from M to N in this way. Note that in general there are isomorphisms of incidence structures that are not projectivities. For example, any automorphism of the division ring $\mathbb{K}$ induces an automorphism of the incidence structure of any module over $\mathbb{K}$, which is in general not projective. We will limit our remarks to projectivities.

Note that the following theorem is specific to the case of commutative division rings, i.e., fields.

(2.131) **Theorem: Fundamental theorem of projective geometry** [ZF] *Suppose* M *and* M' *are* 2*-dimensional vector spaces over a field* $\mathbb{K}$*,* A, B, C *are distinct points in* PM*, and* A', B', C' *are distinct points in* PM'*. Then there exists a unique projective map* $P : \mathsf{PM} \to \mathsf{PM}'$ *such that* $PA = A'$*,* $PB = B'$*, and* $PC = C'$*.*

Proof There is a simple heuristic dimensional argument. The space of linear maps $T : \mathsf{M} \to \mathsf{M}'$ is 4-dimensional. The conditions that $\mathsf{P}TA = A'$, etc., independently reduce the dimension of the solution space by 1, leaving a space of dimension 1. The nonzero transformations in this space all generate the same projective transformation.[29]

For a rigorous argument, choose bases so that A, B, C are represented respectively by

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}r\\s\end{bmatrix},$$

and A', B', C' are represented respectively by

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}r'\\s'\end{bmatrix}.$$

Note that $r, s, r', s' \neq 0$. A projective map as stipulated in the theorem is represented by a 2×2 matrix T acting by left multiplication on column vectors, such that there exist nonzero $a, b, c \in \mathbb{K}$ such that

$$T\begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix},\ T\begin{bmatrix}0\\1\end{bmatrix} = b\begin{bmatrix}0\\1\end{bmatrix},\ T\begin{bmatrix}r\\s\end{bmatrix} = c\begin{bmatrix}r'\\s'\end{bmatrix},$$

i.e.,

$$T = \begin{bmatrix}a & 0\\0 & b\end{bmatrix},\ ar = cr',\ bs = cs'.$$

The suitable matrices T are therefore the nonzero scalar multiples of

$$\begin{bmatrix}r'/r & 0\\0 & s'/s\end{bmatrix}.$$ [30]

[29] The author thanks Tim Penttila for pointing out that the intuition behind this argument is specific to vector spaces and is not applicable to modules over noncommutative division rings.

[30] In the case of a noncommutative division ring we must be particular about the fact that scalars act by right multiplication, so we must satisfy

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}a,\ T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}b,\ T\begin{bmatrix}r\\s\end{bmatrix} = \begin{bmatrix}r'\\s'\end{bmatrix}c,$$

$\square$[2.131]

The issue to be addressed now is whether and how the notion of projectivity and the fundamental theorem pertaining to it can be formulated synthetically, i.e., in the elementary theory of points, lines, and incidence. In light of Theorem 2.131 and the footnote accompanying its proof, we will restrict our attention to models of the form $\mathbb{KP}^2$ where $\mathbb{K}$ is a field, and in light of Theorem 2.128, we will focus on the theory $\mathsf{PG}^2 + \mathsf{Pappus}$.

We begin with the notion of *perspectivity*. The essential case is that of a perspectivity from a line to a line. Suppose therefore that l and l' lines on a plane, and O is a point on the plane that is not on either l or l'. Given a point A on l, A is not O, so the line (O, A) is well defined and is not l', so $((O, A), l')$ is well defined. The *perspectivity from l to l' through O* $\overset{\text{def}}{\iff}$ the map $A \mapsto ((O, A), l')$. Note that if $A' = ((O, A), l')$ then $A = ((O, A'), l)$, so the inverse map is the perspectivity from l' to l through O, and a perspectivity is a bijection.

For the remainder of this section we will confine our attention to commutative division

(2.132) **Theorem** [ZF] *Suppose $\mathbb{K}$ is a field, l and l' are distinct lines in $\mathbb{KP}^2$, and O is a point in $\mathbb{KP}^2$ not on l or l'. Then the perspectivity from l to l' through O is a projectivity.*

Proof Let $\mathsf{M} = \mathbb{K}^3$. As a point of PM, O is a 1-dimensional subspace of M. Let e be a nonzero element of O, so $O = [e]$. l' is a 2-dimensional subspace of M that does not contain e. Let $e_0, e_1 \in l'$ be such that $\{e_0, e_1, e\}$ is a basis for M. Define $T : \mathsf{M} \to l'$ as the unique linear map such that

$$\begin{aligned} Te_0 &= e_0 \\ Te_1 &= e_1 \\ Te &= 0. \end{aligned}$$

Since $e \notin l$, $T \restriction l$ is injective, and its nullspace is $\{\mathbf{0}\}$. Let P be the map from the points on l to the points on l' induced by $T \restriction l$. Thus, given a point A on l (i.e., a 1-dimensional subspace of l), let u be a nonzero element of A, and let $A' = [Tu]$. Then A' is independent of the choice of u, and $P(A) = A'$. Let $a_0, a_1, a \in \mathbb{K}$ be such that

$$u = e_0a_0 + e_1a_1 + ea.$$

Then

$$Tu = e_0a_0 + e_1a_1.$$

Thus

$$ea - u + Tu = 0.$$

Since $[e] = O$, this shows that O, A, and $P(A)$ are collinear, so P is the perspectivity from l to l' through O. As P is induced by T, it is by definition a projectivity. $\square$[2.132]

---

i.e.,

$$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \; ar = r'c, \; bs = s'c,$$

for which the solutions are

$$T = \begin{bmatrix} r'cr^{-1} & 0 \\ 0 & s'cs^{-1} \end{bmatrix},$$

with c an arbitrary nonzero scalar. In general, different choices of c yield different projective maps, so we have existence, but not uniqueness.

By virtue of (2.132) any perspectivity is a projectivity, so—since any (finite) product (i.e., composition) of projectivities is a projectivity—any product of perspectivities is a projectivity. As we will see, every projectivity of lines in $\mathbb{KP}^2$ is a product of perspectivities. This result suggests a geometric definition of projectivity as a (finite) product of perspectivities. This makes sense in any model of PG^2 (including nonpappian and even nonarguesian planes), and this is indeed often taken as the definition of *projectivity* in the synthetic theory. Note, however, that while the notion of *perspectivity* can be formulated in PG^2—by identifying the operation with the point and lines that determine it (O, l, and l' in the theorem)—as can the notion of the product of two, or of three, or of any specified finite number of perspectivities, the notion of an arbitrary finite product of perspectivities is beyond the expressive capability of the elementary language of projective geometry, which can only talk about points, lines and incidence.

The following theorem provides the key to the elementary definition of projectivity.

(2.133) **Theorem** [ZF] *Suppose $\mathbb{K}$ is a field, and l, l' are lines in $\mathbb{KP}^2$. Any projective transformation from l to l' is a product of 3 perspectivities.*

Proof

(2.134) **Claim** *Suppose A, B, C are distinct points on l, and A', B', C' are distinct points on l'. Then there is a product of three perspectivities that takes A, B, C to A', B', C', respectively. If $l \neq l'$ then two perspectivities suffices.*

Proof Suppose first that $l \neq l'$. It is easy to show (as a theorem of PG^2, so it must hold in $\mathbb{KP}^2$) that there is a product of two perspectivities that takes A, B, C respectively to A', B', C'. (Let the first perspectivity P_0 be to a line l'' distinct from l', such that A' is on l'' and $P_0 A = A'$. There is then a unique perspectivity from l'' to l' that accomplishes required transformation.)

Now suppose $l = l'$. Let l'' be any other line, and let Q be any perspectivity from l to l''. Let P' be a product of two perspectivities that takes QA, QB, QC respectively to A', B', C'. Then $P'Q$ is a product of three perspectivities that takes A, B, C respectively to A', B', C'. □[2.134]

Suppose P is a projectivity from l to l'. Let A, B, C be any distinct points on l, and let $A' = PA$, $B' = PB$ and $C' = PC$. Let[2.134] P' be a product of three perspectivities that takes A, B, C to A', B', C'. By (2.132) P' is a projectivity, so by the uniqueness statement of (2.131) $P = P'$; hence, P is a product of three perspectivities. If $l \neq l'$ then we may take P' to be a product of two perspectivities,[2.134] so P is a product of two perspectivities. □[2.133]

For the purpose of this discussion we formulate the following elementary *perspectivity principle.*

Definition [PG^2] *The* perspectivity principle *holds* $\overset{\text{def}}{\iff}$ PP $\overset{\text{def}}{\iff}$ *any product of four perspectivities is a product of three perspectivities.*

As indicated, this definition is formulated in the language of PG^2, i.e., in terms of the existence of points and lines satisfying certain incidence conditions. The following theorem is key to the synthetic definition of projectivity:

(2.135) **Theorem** [PG^2 + Pappus] PP.

We will not prove (2.135) directly. Instead, we will prove the following metatheorem.

(2.136) **Theorem** [ZF] $\mathsf{PG}^2 + \mathsf{Pappus} \vdash \mathsf{PP}$

Proof By the completeness theorem, it suffices to show that PP holds in every model of $\mathsf{PG}^2 + \mathsf{Pappus}$, i.e.,[2.128] in $\mathbb{KP}^2$ for every field $\mathbb{K}$. This follows from the fact that any perspectivity is a projectivity,[2.132] so any product of four perspectivities is a projectivity, and any projectivity is a product of three perspectivities.[2.133] □[2.136]

Note that (2.136) is not precisely (2.135). When we wrote (2.135) we were asserting that there exists a proof of PP in the theory $\mathsf{PG}^2 + \mathsf{Pappus}$. Ordinarily we justify such a statement by exhibiting a proof—which is to say we provide a sketch of a proof that satisfies the reader that a formal proof exists in the indicated theory. When we wrote (2.136) we were asserting that there exists a proof in ZF that there exists a proof of in $\mathsf{PG}^2 + \mathsf{Pappus}$ of PP. In this case, we did provide (a sketch of) a proof in the usual way. As discussed following (2.130), if we believe that all theorems of ZF of this simple form (i.e., that there exists a finitary object with a finitarily verifiable property) are true then we are confident that there is a proof of PP in $\mathsf{PG}^2 + \mathsf{Pappus}$. In a fully elementary treatment we must actually exhibit a proof, but it is nice to know ahead of time that one exists.

Although we cannot refer to mappings *per se* in the context of a pure incidence theory, we may nevertheless define the perspectivity *operation*. We will use 'Per' to name the composition of any finite number of perspectivity operations, relying on the argument list to indicate how many operations are being composed. As the preceding discussion makes clear, the composition of three simple perspectivities is the key to the synthetic definition of projectivity.

(2.137) **Definition** [PG^2]

1. *Suppose A is a point, O is a point distinct from A, and l is a line not on O. The* image of A via the perspectivity through O to l $\stackrel{\text{def}}{=} \mathrm{Per}(A, O, l) \stackrel{\text{def}}{=} ((A, O), l)$.
2. *Suppose A is a point, O_1, O_2, O_3 are points, l_1, l_2, l_3 are lines, such that $A \neq O_1$, l_1 is not on O_1 or O_2, l_2 is not on O_2 or O_3, and l_3 is not on O_3. The* image of A on l_3 via the perspectivities through O_1, O_2, O_3 to l_1, l_2, l_3 $\stackrel{\text{def}}{=} \mathrm{Per}(A, O_1, l_1, O_2, l_2, O_3, l_3) \stackrel{\text{def}}{=} \mathrm{Per}(\mathrm{Per}(\mathrm{Per}(A, O_1, l_1), O_2, l_2), O_3, l_3)$.

The existence assertion of (2.131) may now be stated as follows:

(2.138) **Theorem** [$\mathsf{PG}^2 + \mathsf{Pappus}$] *Suppose l and l' are lines, A, B, C are distinct points on l, and A', B', C' are distinct points on l'. Then there exist points O_1, O_2, O_3, and lines l_1, l_2, such that O_1 is not on l, and $O_1, O_2, O_3, l_1, l_2, l'$ are as in (2.137.2) with l' for l_3, such that*

$$\begin{aligned}
\mathrm{Per}(A, O_1, l_1, O_2, l_2, O_3, m) &= A', \\
\mathrm{Per}(B, O_1, l_1, O_2, l_2, O_3, m) &= B', \\
\textit{and } \mathrm{Per}(C, O_1, l_1, O_2, l_2, O_3, m) &= C'.
\end{aligned}$$

We will not supply a proof of this theorem, but we will show that a proof exists:

(2.139) **Theorem** [ZF] $\mathsf{PG}^2 + \mathsf{Pappus} \vdash$ (2.138).

Proof Arguing as in the proof of (2.136), we need only show that theorem holds in $\mathbb{KP}^2$ for any field $\mathbb{K}$. This is the existence assertion of (2.131) combined with (2.133). $\square^{2.139}$

Of course, O_1, O_2, O_3, l_1, l_2 are not uniquely determined in (2.138). The uniqueness assertion of (2.131) may, however, be stated as follows:

(2.140) **Theorem** [$\mathsf{PG}^2 + \mathsf{Pappus}$] *Suppose l and l' are lines, and suppose O_1, O_2, O_3, l_1, l_2, and $O_1', O_2', O_3', l_1', l_2'$ are each as in (2.138)* vis-à-vis *l and l'. Suppose A, B, C are distinct points on l, and D is on l. If*

$$\begin{aligned}
\mathrm{Per}(A, O_1, l_1, O_2, l_2, O_3, l') &= \mathrm{Per}(A, O_1', l_1', O_2', l_2', O_3', l'),\\
\mathrm{Per}(B, O_1, l_1, O_2, l_2, O_3, l') &= \mathrm{Per}(B, O_1', l_1', O_2', l_2', O_3', l'),\\
\textit{and}\ \mathrm{Per}(C, O_1, l_1, O_2, l_2, O_3, l') &= \mathrm{Per}(C, O_1', l_1', O_2', l_2', O_3', l'),\\
\textit{then}\ \mathrm{Per}(D, O_1, l_1, O_2, l_2, O_3, l') &= \mathrm{Per}(D, O_1', l_1', O_2', l_2', O_3', l').
\end{aligned}$$

Once again, we will only show that a proof exists:

(2.141) **Theorem** [ZF] $\mathsf{PG}^2 + \mathsf{Pappus} \vdash$ (2.140).

Proof Working in any model $\mathbb{KP}^2$ of $\mathsf{PG}^2 + \mathsf{Pappus}$, the maps

$$D \mapsto \mathrm{Per}(D, O_1, l_1, O_2, l_2, O_3, l')$$

and

$$D \mapsto \mathrm{Per}(D, O_1', l_1', O_2', l_2', O_3', l')$$

are projectivities from l to l' that agree on the three distinct points A, B, C, so they are identical.[2.131] $\square^{2.141}$

With (2.138) and (2.140) in hand we may now formulate an elementary definition of projectivity:

Definition [$\mathsf{PG}^2 + \mathsf{Pappus}$] *Suppose l, l' are lines.*

1. *For each $n \geqslant 4$, we formulate the following definition: Distinct points $A_1, \ldots, A_n$ on l are* projectively related *to points $A_1', \ldots, A_n'$ on l' $\overset{\text{def}}{\iff}$ there is a product of three perspectivities that takes A_1 to A_1', A_2 to A_2', $\ldots$, and A_n to A_n'.*
2. *Suppose A, B, C are distinct points on l, A', B', C' are distinct points on l', and D is a point on l. $P_{A,B,C,A',B',C'}(D) \overset{\text{def}}{=}$ the unique point D' on l' such that A, B, C, D are projectively related to A', B', C', D'.*

In the elementary theory we may informally refer to $P_{A,B,C,A',B',C'}(\cdot)$ as the "projectivity" determined by A, B, C and A', B', C', but (not being either a point or a line) it does not actually exist.

2.6 Other deductive systems

The natural deductive system **ND** is only one of several reasonable systems of deduction. One way to modify **ND** to conform even more closely to standard mathematical practice is to define entailment for formulas with free variables as well as those without (sentences).

Definition [C^0] *A class Θ of formulas* entails *a formula ϕ iff for every structure $\mathfrak{A}$ of the signature of Θ and ϕ, for every assignment A of elements of $\mathfrak{A}$ to the free variables occurring in ϕ or in any member of Θ, if $\mathfrak{A} \models \Theta[A]$ (i.e., $\mathfrak{A} \models \theta[A]$ for all $\theta \in \Theta$) then $\mathfrak{A} \models \phi[A]$.*

Clearly, if S is a bijection of the free variables occurring in $\Theta \cup \{\phi\}$ with a class of constants not occurring in $\Theta \cup \{\phi\}$, then Θ entails ϕ iff $\Theta(S)$ entails $\phi(S)$.

To adapt our system of deduction to this notion of entailment, we permit arbitrary formulas, rather than just sentences, in proofs. The role of constants in the system **ND**[2.27] is now more or less taken over by free variables. In RULE 7[2.27.7] we must allow for substitution terms containing variables, but the variables must behave syntactically like constants—in particular, they must not be bound by the substitution. Thus, RULE 7 becomes

Modified inference rule for ND with free variables

$$7'.\ \frac{}{\left\{\psi\binom{v}{\tau}\right\} \Rightarrow \exists v\, \psi} \text{ if } \tau \text{ is free for } v \text{ in } \psi.$$ [1.16]

We may construct proofs without the introduction of new constants if we also replace RULE 4 by

Modified inference rule for ND with free variables

$$4'.\ \frac{\Sigma \cup \{\psi\} \Rightarrow \sigma}{\Sigma \cup \{\exists v\, \psi\} \Rightarrow \sigma} \text{ if } v \text{ does not occur free in } \Sigma \text{ or } \sigma.$$

2.6.1 Hilbert systems

An alternate system of deduction may be based on an appropriate class T of validities and the rule *modus ponens*, which states that σ may be inferred from ζ and $\zeta \to \sigma$. In such a system, a *proof* from a class Θ of premises is a sequence of formulas such that for each formula σ in the sequence, either $\sigma \in \Theta \cup T$ or there exists a formula ζ such that ζ and $\zeta \to \sigma$ precede σ in the sequence.[31]

It is easy to show if T is the class of all validities then this system is complete, but this is, of course, not a very useful system as it is hard to know whether a formula is a member of T (we will see in the next chapter just how hard). It is not difficult, however, to define a manageable class of validities that is sufficient for completeness. Such a system—with only one or two inference rules—is often called a *Hilbert system*.

2.6.2 Gentzen systems

For fine analysis of the structure of proofs and the strengths of theories, *Gentzen systems* are particularly useful. These are elegant, highly symmetrical systems, introduced by Gerhard Gentzen originally for the purpose of proving the consistency of Peano arithmetic (PA) in PA with a minimal strengthening of the induction schema. We will present this theory in some detail, because of its intrinsic interest and its prominent position in proof theory, and because it is a good way of introducing a deductive system with the *subformula property*, which we introduce below[§ 2.6.3] and later use to give a finitary proof of an important theorem about satisfaction of

[31] Often the term 'inference rule' is restricted to rules other than validities, in which case a system of this type has just one inference rule: modus ponens.

logical validities. As this is the only use we make of this theory, however, it may be omitted without much harm, and reading may be resumed with Section 2.7. In the other direction, Takeuti's *Proof Theory*[24] is a good source for further reading in this subject.

To obtain a Gentzen system we generalize the notion of *sequent* given in Definition 2.25 as follows.

(2.142) **Definition** [C^0] *Suppose ρ is a signature.*

1. *A* ρ-sequent *in the Gentzen sense is a* 2*-sequence* $\langle\Gamma,\Delta\rangle$ *of finite sets of* ρ*-formulas. We let*
$$\Gamma \Rightarrow \Delta \overset{\text{def}}{=} \langle\Gamma,\Delta\rangle,$$
as in (2.25).
2. Γ *is the* antecedent *and* Δ *the* succedent *of* $\Gamma \Rightarrow \Delta$*.*
3. *A sequent is* valid $\overset{\text{def}}{\iff} \bigwedge\Gamma$ *entails* $\bigvee\Delta$*, i.e., under any interpretation of all formulas in* Γ *and* Δ*, if all the members of* Γ *are true then at least one of the members of* Δ *is true.*

Note that we allow arbitrary formulas, not just sentences. As a consequence, there is no need to permit expansion of the original signature by additional constants as in (2.25).

For the purpose of comparison with the natural deductive system **ND**,[2.27] we will define a Gentzen system with propositional connectives $\neg$ and $\rightarrow$, and quantifier $\exists$. This is, for all practical purposes, Gentzen's system **LK**: the ***l**ogischer **k**lassischer Kalkül*. It should be noted that this system is often described with finite sequences instead of finite sets of formulas for the antecedent and succedent of a sequent. This requires inference rules to establish the equivalence of sequences with the same image.

For the time being, we restrict our attention to logic without identity.

(2.143) **The system LK** *The inference rules are as follows, where u, v are variables, ϕ,ψ are formulas, τ is a term, and $\Gamma,\Delta,\Pi,\Lambda$ are finite sets of formulas, subject only to the indicated restrictions.*

1. $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta}$ *, if* $\Gamma \subseteq \Gamma'$*.*
2. $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta'}$ *, if* $\Delta \subseteq \Delta'$*.*
3. $\dfrac{\Gamma \Rightarrow \Delta \cup \{\phi\}}{\Gamma \cup \{\neg\,\phi\} \Rightarrow \Delta}$.
4. $\dfrac{\Gamma \cup \{\phi\} \Rightarrow \Delta}{\Gamma \Rightarrow \Delta \cup \{\neg\,\phi\}}$.
5. $\dfrac{\Gamma \Rightarrow \Delta \cup \{\phi\} \qquad \Gamma \cup \{\psi\} \Rightarrow \Delta}{\Gamma \cup \{\phi \rightarrow \psi\} \Rightarrow \Delta}$.
6. $\dfrac{\Gamma \cup \{\phi\} \Rightarrow \Delta \cup \{\psi\}}{\Gamma \Rightarrow \Delta \cup \{\phi \rightarrow \psi\}}$.
7. $\dfrac{\Gamma \cup \left\{\phi\binom{v}{\bar{u}}\right\} \Rightarrow \Delta}{\Gamma \cup \{\exists v\ \phi\} \Rightarrow \Delta}$ *, if u does not occur free in the lower sequent and $\bar{u}$ is free for v in ϕ.*[1.16]

8. $\dfrac{\Gamma \Rightarrow \Delta \cup \{\phi\binom{v}{\tau}\}}{\Gamma \Rightarrow \Delta \cup \{\exists v\ \phi\}}$, *if τ is free for v in ϕ.*

9. $\dfrac{\Gamma \Rightarrow \Delta \cup \{\phi\} \qquad \Gamma \cup \{\phi\} \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$.

Rules 1 and 2 are structural rules*, specifically* weakening *rules. Rules 3–9 are* logical *rules. In Rules 3–8 the* principal formulas *are those "created" in the lower sequent, i.e., $\neg\,\phi$, $\phi \to \psi$, and $\exists v\,\phi$, respectively. Rule 9 is the* cut *rule; ϕ is the* cut formula*. An* axiom $\overset{\text{def}}{=}$ *a sequent of the form $\{\phi\} \Rightarrow \{\phi\}$.*

These rules are linked to form proofs in the same way as for the natural system. For example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\phi\} \Rightarrow \{\phi\}}{\{\phi\} \Rightarrow \{\phi, \psi\}} \qquad \dfrac{\{\psi\} \Rightarrow \{\psi\}}{\{\phi, \psi\} \Rightarrow \{\psi\}}}{\{\phi, \phi \to \psi\} \Rightarrow \{\psi\}}}{\{\phi, \phi \to \psi, \neg\,\psi\} \Rightarrow 0}}{\{\phi \to \psi, \neg\,\psi\} \Rightarrow \{\neg\,\phi\}}$$

is a proof of the final sequent $\{\phi \to \psi, \neg\,\psi\} \Rightarrow \{\neg\,\phi\}$. To construct this proof tree we have used (starting from the bottom) Rules 4; 3; 5; and 2 and 1; and each branch (of this tree growing upward) terminates in an axiom.

In general, a proof is a finite tree that grows upward, starting from the sequent to be proved, branching with each application of Rule 5 or 9, with each branch terminating in an axiom. Since a given sequent may occur in more than one place in such a tree, formally we must regard each node of the tree as the sequence of sequents leading up to it.

(2.144) **Definition** [C^0] *A* proof *(in any sequent calculus) is a nonempty tree T of finite sequences of sequents, ordered by inclusion, i.e., $S \leqslant S' \leftrightarrow S \subseteq S'$, with the following properties:*

1. *T has exactly one member, say $\langle I \rangle$, of length 1. T is a* proof of *I.*
2. *Suppose $S \in T$ and $|S| = n > 0$. Let $s = S_{n^-}$, the last (top) item of S. Then exactly one of the following must obtain.*
   1. *S has no extension in T, and s is an axiom.*
   2. *S has exactly one immediate extension $S^\frown\langle s' \rangle$ in T, and $\dfrac{s'}{s}$ is an instance of Rule 1, 2, 3, 4, 6, 7, or 8.*
   3. *S has exactly two immediate extensions $S^\frown\langle s' \rangle, S^\frown\langle s'' \rangle$ in T, and $\dfrac{s' \qquad s''}{s}$ is an instance of Rule 5 or 9.*

Given a theory Θ and a sentence σ, $T \vdash^{\mathbf{LK}} \sigma \overset{\text{def}}{\iff}$ there is a proof of $\Sigma \Rightarrow \{\sigma\}$ for some finite $\Sigma \subseteq \Theta$.

It is not difficult to show that $\vdash^{\mathbf{LK}}$ is equivalent to $\vdash^{\mathbf{ND}}$, where '$\vdash^{\mathbf{ND}}$' denotes the natural deduction predicate previously referred to simply as '$\vdash$'.[2.27] This may be done directly by justifying each inference rule of either system in terms of the rules of the other system. Recall that **ND** uses only sentences, whereas **LK** uses arbitrary formulas, but the introduction of a new constant c in the rule (2.27.4) is formally equivalent to the introduction of a new free variable u in the rule (2.143.7).

It is perhaps more natural, however, to establish the equivalence of these systems via the equivalence of each to the semantic notion of entailment, i.e. the completeness theorem, which is, after all, the *raison d'être* of each of them. Since the completeness theorem for either system is provable in C^0, and the conclusion that the systems are equivalent is a purely set-theoretical statement, it follows that this equivalence can be proved in S^0.[32]

2.6.3 Cut-elimination and the subformula property

An important feature of the inference rules 2.143 is that, with the exception of (2.143.9), every formula occurring in the upper sequent(s) is a subformula of a formula in the lower sequent or is obtained by substitution of a term in a subformula of a formula in the lower sequent. This is the *subformula property.*

(2.143.9) is the *cut rule*, and it is a remarkable fact that it is not essential to the system. In other words, any **LK**-proof may be replaced by a proof that makes no use of the cut rule. This is the *cut elimination property* of **LK**. The presence or absence of cut elimination in various related deductive systems—for logics other than classical first-order predicate logic—is an important consideration in proof theory. The *cut-elimination theorem* for **LK**, Gentzen's *Hauptsatz*, has a simple infinitary proof and a more complicated finitary proof, both of which are of interest to us.

(2.145) **Definition** $[\mathsf{C}^0]$ $\mathbf{LK}^- \stackrel{\text{def}}{=} \mathbf{LK}$ *without the cut rule.*

By extension of the comment made above, $\mathbf{LK}^-$ has the *subformula property*, viz., every formula occurring in a proof of a sequent I is obtained by substitution of terms in a subformula of a formula occurring in I (where a substitution of terms may be the null substitution, which does not alter a formula).

2.6.4 Completeness of LK and $\mathbf{LK}^-$

We now adapt the Henkin construction to provide finitary proofs of a limited version of the completeness theorem for $\mathbf{LK}^-$ and the full completeness theorem for **LK**. From the former it is easy to obtain the full completeness theorem for $\mathbf{LK}^-$ using $\mathsf{Infinity}$. This immediately yields the cut-elimination theorem for **LK**, albeit by the infinitary route. We then give a finitary proof of cut elimination, which provides a finitary proof of the full completeness theorem for $\mathbf{LK}^-$. As noted above, for our purposes, this is important because $\mathbf{LK}^-$ has the subformula property.

The completeness theorem for a sequent calculus is easily seen to be equivalent to the statement that any valid[2.142.3] sequent is provable; equivalently, if a sequent $I = (\Gamma \Rightarrow \Delta)$ is not provable then there is an interpretation that falsifies it, i.e., a structure $\mathfrak{S}$, an $\mathfrak{S}$-assignment A to the free variables of I, and a valuation function S for $\mathfrak{S}$ such that

1. for every $\phi \in \Gamma$, $S\langle \phi, A\rangle = 1$; and
2. for every $\phi \in \Delta$, $S\langle \phi, A\rangle = 0$.

Note that we have left 'valuation function' unqualified; we will use a very particular sort of valuation function in our first proof of completeness for $\mathbf{LK}^-$, which we call

[32] This material is not used in the proof of Theorem 2.183, so it is not within the scope of the program (2.38).

for the nonce a *subvaluation function*, defined by a modification of the definition[1.58] of *valuation function*. The salient difference is that in order that $\langle \phi, A\rangle$ be in the domain of a subvaluation function S with a given value $S\langle \phi, A\rangle$, it is not necessary that $\langle \phi', A'\rangle$ be in $\operatorname{dom} S$ for all subformulas ϕ' of ϕ and appropriate extensions A' of the assignment A; instead we only require that enough of these be in $\operatorname{dom} S$ (with appropriate values) to uniquely determine $S\langle \phi, A\rangle$.

(2.146) **Definition** [C^0] *Suppose* $\mathfrak{S}$ *is a* ρ*-structure. A* subvaluation function for $\mathfrak{S}$ $\stackrel{\text{def}}{=}$ *a function* S *such that* $\operatorname{dom} S$ *consists of* 2*-sequences* $\langle \epsilon, A\rangle$ *such that* ϵ *is a* ρ*-expression and* A *is an* $\mathfrak{S}$*-assignment for* ϵ*; if* ϵ *is a formula (as opposed to a term) then* $\operatorname{im} S \subseteq 2$ $(= \{0, 1\})$*; and for any* $\langle \epsilon, A\rangle \in \operatorname{dom} S$

1. *if* $\epsilon = \bar{v}$ *then* $S\langle \epsilon, A\rangle = Av$*;*
2. *if* $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-}\rangle$*, where* X *is an* n*-ary operation index (so* $\tau_0, \ldots, \tau_{n^-}$ *are terms), then* $\langle \tau_m, A\rangle \in \operatorname{dom} S$ *for all* $m \in n$*, and*
$$S\langle \epsilon, A\rangle = X^{\mathfrak{S}}\langle S\langle \tau_0, A\rangle, \ldots, S\langle \tau_{n^-}, A\rangle\rangle;$$
3. *if* $\epsilon = \tilde{X}\langle \tau_0, \ldots, \tau_{n^-}\rangle$*, where* X *is an* n*-ary predicate index, then* $\langle \tau_m, A\rangle \in \operatorname{dom} S$ *for all* $m \in n$*, and*
$$S\langle \phi, A\rangle = 1 \leftrightarrow \big\langle S\langle \tau_0, A\rangle, \ldots, S\langle \tau_{n^-}, A\rangle\big\rangle \in X^{\mathfrak{S}};$$
4. *if* $\epsilon = \neg\, \psi$ *then* $\langle \psi, A\rangle \in \operatorname{dom} S$ *and* $S\langle \epsilon, A\rangle = 1 \leftrightarrow S\langle \psi, A\rangle = 0$*;*
5. *if* $\epsilon = \psi_0 \vee \psi_1$ *then*
    1. *if* $S\langle \epsilon, A\rangle = 1$ *then either*
        1. $\langle \psi_0, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_0, A\rangle = 1$*; or*
        2. $\langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_1, A\rangle = 1$*; and*
    2. *if* $S\langle \epsilon, A\rangle = 0$ *then* $\langle \psi_0, A\rangle, \langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_0, A\rangle = S\langle \psi_1, A\rangle = 0$*;*
6. *if* $\epsilon = \psi_0 \wedge \psi_1$ *then*
    1. *if* $S\langle \epsilon, A\rangle = 1$ *then* $\langle \psi_0, A\rangle, \langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_0, A\rangle = S\langle \psi_1, A\rangle = 1$*; and*
    2. *if* $S\langle \epsilon, A\rangle = 0$ *then either*
        1. $\langle \psi_0, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_0, A\rangle = 0$*; or*
        2. $\langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_1, A\rangle = 0$*;*
7. *if* $\epsilon = \psi_0 \rightarrow \psi_1$ *then*
    1. *if* $S\langle \epsilon, A\rangle = 1$ *then either*
        1. $\langle \psi_0, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_0, A\rangle = 0$*; or*
        2. $\langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \psi_1, A\rangle = 1$*; and*
    2. *if* $S\langle \epsilon, A\rangle = 0$ *then* $\langle \psi_0, A\rangle, \langle \psi_1, A\rangle \in \operatorname{dom} S$*,* $S\langle \psi_0, A\rangle = 1$ *and* $S\langle \psi_1, A\rangle = 0$*;*
8. *if* $\epsilon = \psi_0 \leftrightarrow \psi_1$ *then* $\langle \psi_0, A\rangle, \langle \psi_1, A\rangle \in \operatorname{dom} S$ *and* $S\langle \epsilon, A\rangle = 1 \leftrightarrow (S\langle \psi_0, A\rangle = 1 \leftrightarrow S\langle \psi_1, A\rangle = 1)$*;*
9. *if* $\epsilon = \exists v\, \psi$ *then*
    1. *if* $S\langle \epsilon, A\rangle = 1$ *then for some* $a \in |\mathfrak{S}|$*, letting* $A' = A\big\langle {v \atop a} \big\rangle$*,* $\langle \psi, A'\rangle \in \operatorname{dom} S$ *and* $S\langle \psi, A'\rangle = 1$*; and*

   2. *if* $S\langle\epsilon, A\rangle = 0$ *then for all* $a \in |\mathfrak{S}|$*, letting* $A' = A\langle{v \atop a}\rangle$*,* $\langle\psi, A'\rangle \in \operatorname{dom} S$ *and* $S\langle\psi, A'\rangle = 0$*; and*

10. *if* $\epsilon = \forall v\, \psi$ *then*
   1. *if* $S\langle\epsilon, A\rangle = 1$ *then for all* $a \in |\mathfrak{S}|$*, letting* $A' = A\langle{v \atop a}\rangle$*,* $\langle\psi, A'\rangle \in \operatorname{dom} S$ *and* $S\langle\psi, A'\rangle = 1$*; and*
   2. *if* $S\langle\epsilon, A\rangle = 0$ *then for some* $a \in |\mathfrak{S}|$*, letting* $A' = A\langle{v \atop a}\rangle$*,* $\langle\psi, A'\rangle \in \operatorname{dom} S$ *and* $S\langle\psi, A'\rangle = 0$.

In the following discussion, for simplicity we will restrict our attention to languages with just the negation and implication propositional connectives and the existential quantifier.

(2.147) **Theorem** [C^0] *Suppose* $I = (\Gamma \Rightarrow \Delta)$ *is a sequent in a signature* ρ *without identity that is not* **LK**$^-$*-provable. Then there exists a* ρ*-structure* $\mathfrak{S}$*, an* $\mathfrak{S}$*-assignment* A *to the free variables of* I*, and a subvaluation* S *for* $\mathfrak{S}$ *such that for each* $\phi \in \Gamma \cup \Delta$*,*

1. $\langle\phi, A\rangle \in \operatorname{dom} S$*;*
2. $\phi \in \Gamma \rightarrow S\langle\phi, A\rangle = 1$*; and*
3. $\phi \in \Delta \rightarrow S\langle\phi, A\rangle = 0$.

Proof The proof is a straightforward adaptation of the Henkin construction and is relegated to Note 10.9. □

(2.148) **Theorem: Completeness of LK** [C^0] *Suppose* $I = (\Gamma \Rightarrow \Delta)$ *is a sequent in a signature* ρ *without identity that is not* **LK***-provable. Then there exists a satisfactory* ρ*-structure* $\mathfrak{S}$ *and an* $\mathfrak{S}$*-assignment* A *to the free variables of* I *such that for each* $\phi \in \Gamma \cup \Delta$*,*

1. *if* $\phi \in \Gamma$ *then* $\mathfrak{S} \models \phi[A]$*; and*
2. *if* $\phi \in \Delta$ *then* $\mathfrak{S} \models \neg\,\phi[A]$.

Proof See Note 10.10. □

(2.149) **Theorem: Cut elimination** [C^0] *Suppose* $I = (\Gamma \Rightarrow \Delta)$ *is a sequent in a signature* ρ *without identity. If* I *is* **LK***-provable then* I *is* **LK**$^-$*-provable.*

Remark An infinitary proof is easy. Suppose I is not **LK**$^-$-provable. Then[2.147] there exist a ρ-structure $\mathfrak{S}$, an $\mathfrak{S}$-assignment A to the free variables of I, and a subvaluation S for $\mathfrak{S}$ such that for each $\phi \in \Gamma \cup \Delta$,

1. $\langle\phi, A\rangle \in \operatorname{dom} S$;
2. $\phi \in \Gamma \rightarrow S\langle\phi, A\rangle = 1$; and
3. $\phi \in \Delta \rightarrow S\langle\phi, A\rangle = 0$.

Since we have $\mathsf{Infinity}$, we know that $\mathfrak{S}$ is satisfactory. By induction on logical complexity, it is easily shown that the full valuation function for $\mathfrak{S}$ includes any subvaluation, so for each $\phi \in \Gamma \cup \Delta$,

1. if $\phi \in \Gamma$ then $\mathfrak{S} \models \phi[A]$; and

2. if $\phi \in \Delta$ then $\mathfrak{S} \models \neg\,\phi[A]$.

Thus, $\mathfrak{S}$ and A falsify $\Gamma \Rightarrow \Delta$. Since **LK** is a sound system of deduction, $\Gamma \Rightarrow \Delta$ is not **LK**-provable.

Proof See Note 10.11 for a finitary (C^0-)proof. □

As noted above, (2.149) and (2.148) together provide a finitary proof of the completeness theorem for $\mathbf{LK}^-$.

2.7 Model theory

In this section we present some basic elements of the theory of models. The axiom of infinity is essential for any reasonable theory of models, and with the axiom of infinity, the universe of sets is large enough that proper classes are not as important as in the finitary theory, so the language of pure set theory is sufficiently expressive for most of our purposes. The axiom of choice is required for many natural constructions involving uncountable sets, and we usually assume it.

For the purpose of this section, unless otherwise indicated,

1. *our theory of membership is* ZFC*; and*
2. *languages and structures are sets.*

Note that all structures are therefore satisfactory.

Heretofore in this chapter we have assumed as a convenience that the signatures under consideration are countable, so the corresponding languages and theories are countable, and countable structures suffice as models. Nevertheless, essentially everything we have done applies to signatures of any size.

Suppose κ is an infinite cardinal. It follows from the definition of indexed families and the definition of signatures as indexed families that a signature ρ has cardinality κ iff it has κ indices. It is an easy exercise in cardinal arithmetic to show that $|\mathcal{L}^\rho| = \max\{\omega, |\rho|\}$. The cardinality of a structure $\mathfrak{S}$ is understood to be $||\mathfrak{S}||$, i.e., the cardinality of $|\mathfrak{S}|$.[33]

2.7.1 Completeness

The definitions of proof and consistency are the same for uncountable as for countable theories. Clearly, if Θ is a theory and Θ has a model then Θ is consistent. The converse, i.e., the completeness theorem, also holds in this setting.

(2.150) **Theorem** [ZFC] *Suppose Θ is a consistent theory. Then there is a model of Θ.*

Proof We use the Henkin procedure[§ 2.12] as we did for countable theories. Let ρ be the signature of Θ, and $\kappa \geqslant |\rho|$ be an infinite cardinal. Let $\langle c_\alpha \mid \alpha < \kappa\rangle$ be distinct sets that are not ρ-indices, and let ρ' be the extension of ρ by the addition of the c_αs as constant indices. Then $|\mathcal{L}^{\rho'}| = |\rho'| = \kappa$. Let $\langle \sigma_\alpha \mid \alpha \in \kappa\rangle$ be an enumeration of the ρ'-sentences. Let $\langle \Theta_\alpha \mid \alpha \leqslant \kappa\rangle$ be the sequence of consistent ρ'-theories such that

[33] Apology for the dual use of '$|\cdot|$'.

1. $\Theta_0 = \Theta$;
2. for any $\alpha < \kappa$, letting $\Theta' = \Theta_\alpha \cup \{\sigma_\alpha\}$,
   1. if Θ' is inconsistent then $\Theta_{\alpha+1} = \Theta_\alpha \cup \{\neg\,\sigma_\alpha\}$; and
   2. if Θ' is consistent then
      1. if σ_α is not an existential sentence then $\Theta_{\alpha+1} = \Theta'$; and
      2. if $\sigma_\alpha = \exists u\,\theta$ then $\Theta_{\alpha+1} = \Theta' \cup \left\{\theta\left(\begin{smallmatrix}u\\ \bar{c}\end{smallmatrix}\right)\right\}$, where c is the first item in $\langle c_\alpha \mid \alpha < \kappa\rangle$ that does not occur in Θ'; and
3. for any limit $\alpha \leqslant \kappa$, $\Theta_\alpha = \bigcup_{\beta<\alpha} \Theta_\beta$.

The existence of this sequence is proved as in Section 2.2.1, which is really just the case $\kappa = \omega$. Note that at each successor step in the construction, we introduce only finitely many of the new constants: those that occur in σ_α and one more if σ_α is existential. Since the union of fewer than κ finite sets is smaller than κ (κ being infinite), at any stage in the construction we have used fewer than κ new constants, which allows Step 2.2.2 to be taken, if it is called for.

Θ_κ is a maximal consistent extension of Θ with witnesses, and we construct the model $\mathfrak{H}^{\Theta_\kappa}$ as before.[§ 2.2.2] Recall that $|\mathfrak{H}^{\Theta_\kappa}|$ is the set $\mathcal{K}$ of variable-free ρ'-terms. If ρ has the identity predicate, we let $\mathfrak{G} = \mathfrak{H}^{\Theta_\kappa}/\equiv$, where $\equiv$ is the equivalence relation on $\mathcal{K}$ given by

$$\tau_0 \equiv \tau_1 \leftrightarrow (\tau_0 = \tau_1) \in \Theta_\kappa.$$

Otherwise we just let $\mathfrak{G} = \mathfrak{H}^{\Theta_\kappa}$.

It is straightforward to show that $\mathfrak{G} \models \Theta$. □[2.150]

2.7.2 Elementary substructures and embeddings

In this section we will work in GB and allow languages and structures to be proper classes. Recall[1.58] that if Φ is a class of ρ-formulas then $\overline{\Phi}$ is the class of subformulas of members of Φ.

(2.151) **Definition** [GB] *Suppose $\mathfrak{A}$ and $\mathfrak{A}'$ are structures.*

1. *Suppose Φ is a class of ρ-formulas. Then* $\mathfrak{A}'$ is a Φ-elementary substructure *or simply* Φ-substructure of $\mathfrak{A}$ $\overset{\text{def}}{\iff}$ $\mathfrak{A}' \prec^{\Phi} \mathfrak{A}$ $\overset{\text{def}}{\iff}$
   1. *$\mathfrak{A}'$ is a substructure[2.113] of $\mathfrak{A}$; and*
   2. *for every $\phi \in \overline{\Phi}$ and every $\{\phi\}$-satisfaction relation S for $\mathfrak{A}$,*

      $$\{\langle\psi, A\rangle \in S \mid \operatorname{im} A \subseteq |\mathfrak{A}'|\}$$

      is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}'$.
2. $\mathfrak{A}'$ is an elementary substructure of $\mathfrak{A}$ $\overset{\text{def}}{\iff}$ $\mathfrak{A}' \prec \mathfrak{A}$ $\overset{\text{def}}{\iff}$ $\mathfrak{A}' \prec^{\mathcal{F}^\rho} \mathfrak{A}$.
3. *j is a* Φ-elementary embedding *of a structure $\mathfrak{B}$ in a structure $\mathfrak{A}$* $\overset{\text{def}}{\iff}$ $j : \mathfrak{B} \prec^{\Phi} \mathfrak{A}$ $\overset{\text{def}}{\iff}$ *j is an isomorphism of $\mathfrak{B}$ with a Φ-elementary substructure of $\mathfrak{A}$.*

Elementary in this definition refers to the fact that it deals with the elementary, i.e., first-order, theories of the structures.

Note that if Φ consists of quantifier-free formulas then $\mathfrak{A}' \prec^{\Phi} \mathfrak{A}$ iff $\mathfrak{A}'$ is a substructure of $\mathfrak{A}'$. In particular, if $\Phi = 0$ this is the case, and we will use '$\prec^0$' to represent the substructure relation.

(2.152) *Note that if ϕ is a ρ-formula then*[2.151.2] $\mathfrak{A}' \prec^{\{\phi\}} \mathfrak{A}$ *iff*

1. *$\mathfrak{A}'$ is a substructure of $\mathfrak{A}$; and*
2. *for every $\{\phi\}$-satisfaction relation S for $\mathfrak{A}$,*

$$\{\langle \psi, A\rangle \in S \mid \operatorname{im} A \subseteq |\mathfrak{A}'|\}$$

is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}'$.

Hence, if there is no $\{\phi\}$-satisfaction relation for $\mathfrak{A}$ then every substructure of $\mathfrak{A}$ is trivially a $\{\phi\}$-elementary substructure.

Note that to assert elementarity there is no need to state a condition on values of terms like the condition (2.151.2.2) on the values of formulas, as it would automatically be satisfied for substructures. Indeed, the essence of elementarity is what is says about quantification over $|\mathfrak{A}|$ vs $|\mathfrak{A}'|$, as indicated by the following theorem, known as the *Tarski-Vaught criterion*.

(2.153) **Theorem** [GB] *Suppose $\mathfrak{A}$ is a ρ-structure and $\mathfrak{A}'$ is a substructure of $\mathfrak{A}$.*

1. *Suppose the satisfaction relation for $\mathfrak{A}$ exists (as it does, for example, if $\mathfrak{A}$ is a set). Then $\mathfrak{A}' \prec \mathfrak{A}$ iff for every ρ-formula ϕ and variable u, and every $\mathfrak{A}'$-assignment A for $\exists u\, \phi$,*

$$\exists x \in |\mathfrak{A}|\ \mathfrak{A} \models \phi\big[A\big\langle{u \atop x}\big\rangle\big] \to \exists x \in |\mathfrak{A}'|\ \mathfrak{A} \models \phi\big[A\big\langle{u \atop x}\big\rangle\big].$$

2. *More generally, suppose Φ is a class of ρ-formulas. Then $\mathfrak{A}' \prec^{\Phi} \mathfrak{A}$ iff for every formula $\phi \in \overline{\Phi}$ and variable u,*
   1. *if $\exists u\, \phi \in \overline{\Phi}$, then for every $\{\phi\}$-satisfaction relation S for $\mathfrak{A}$ and $\mathfrak{A}'$-assignment A for $\exists u\, \phi$,*

   $$\exists x \in |\mathfrak{A}|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S \to \exists x \in |\mathfrak{A}'|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S;$$

   2. *and if $\forall u\, \phi \in \overline{\Phi}$ then for every $\{\phi\}$-satisfaction relation S for $\mathfrak{A}$ and $\mathfrak{A}'$-assignment A for $\forall u\, \phi$,*

   $$\forall x \in |\mathfrak{A}'|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S \to \forall x \in |\mathfrak{A}|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S.$$

Remark (2.153.1) is the standard form of the criterion. We use the general form when dealing with proper classes.

Proof Suppose $\mathfrak{A}' \prec^{\Phi} \mathfrak{A}$. Suppose $\exists u\, \phi \in \overline{\Phi}$, S is a $\{\phi\}$-satisfaction relation for $\mathfrak{A}$, A is an $\mathfrak{A}'$-assignment for $\exists u\, \phi$, and $\exists x \in |\mathfrak{A}|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S$.

We must show that $\exists x \in |\mathfrak{A}'|\ \big\langle \phi, A\big\langle{u \atop x}\big\rangle\big\rangle \in S$. Let S_1 be the extension of S to the $\{\exists u\, \phi\}$-satisfaction relation for $\mathfrak{A}$ $\big($by adding $\langle \exists u\, \phi, A'\rangle$ to S just in case $\exists x \in |\mathfrak{A}|\ \big\langle \phi, A'\big\langle{u \atop x}\big\rangle\big\rangle \in S\big)$. Then

$$\langle \exists u\, \phi, A\rangle \in S_1.$$

Let

$$S' = \{\langle \psi, A' \rangle \in S \mid \operatorname{im} A' \subseteq |\mathfrak{A}'|\}$$
$$S_1' = \{\langle \psi, A' \rangle \in S_1 \mid \operatorname{im} A' \subseteq |\mathfrak{A}'|\}.$$

Then[2.151.2.2] S' is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}'$, and S_1' is the $\{\exists u\, \phi\}$-satisfaction relation for $\mathfrak{A}'$.

Since $\langle \exists u\, \phi, A \rangle \in S_1$, $\langle \exists u\, \phi, A \rangle \in S_1'$, so for some $x \in |\mathfrak{A}'|$, $\langle \phi, A\langle{u \atop x}\rangle \rangle \in S_1'$. By the uniqueness of satisfaction, $\langle \phi, A\langle{u \atop x}\rangle \rangle \in S$; hence $\exists x \in |\mathfrak{A}'|\ \langle \phi, A\langle{u \atop x}\rangle \rangle \in S$, as claimed.

The universal quantifier is handled analogously.

Conversely, suppose for every $\phi \in \overline{\Phi}$ and variable u, (2.153.2.1, 2) hold.

(2.154) **Claim** *Suppose $\phi \in \overline{\Phi}$, and S is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}$. Then*

$$\{\langle \psi, A \rangle \in S \mid \operatorname{im} A \subseteq |\mathfrak{A}'|\}$$

is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}'$.

Proof By induction on the logical complexity of $\phi \in \overline{\Phi}$. For atomic formulas, this follows from the fact that $\mathfrak{A}' \prec^0 \mathfrak{A}$. The induction steps corresponding to propositional connectives are trivial. Suppose now that the claim holds for ϕ; $\exists u\, \phi \in \overline{\Phi}$; and S is the $\{\exists u\, \phi\}$-satisfaction relation for $\mathfrak{A}$. Let

$$S_1 = \{\langle \psi, A \rangle \in S \mid \psi \in \overline{\{\phi\}}\}.$$

Then S_1 is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}$, so by induction hypothesis

$$S_1' = \{\langle \psi, A \rangle \in S_1 \mid \operatorname{im} A \subseteq |\mathfrak{A}'|\}$$

is the $\{\phi\}$-satisfaction relation for $\mathfrak{A}'$. Let S' be the extension of S_1' to the $\{\exists u\, \phi\}$-satisfaction relation for $\mathfrak{A}'$ (by adding $\langle \exists u\, \phi, A \rangle$ to S_1' just in case $\exists x \in |\mathfrak{A}'|\ \langle \phi, A\langle{u \atop x}\rangle \rangle \in S_1'$, as before). Using the hypothesis (2.153.2.1), it is easy to check that

$$S' = \{\langle \psi, A \rangle \in S \mid \operatorname{im} A \subseteq |\mathfrak{A}'|\},$$

as claimed.

The induction step for universal quantification is analogous. □[2.154]

This completes the proof of (2.153.2). (2.153.1) is the special case when Φ consists of all ρ-formulas. There is no need to deal explicitly with both quantifiers when Φ is closed under both quantification operations and negation. □[2.153]

2.7.3 Elementary directed families

It is easy to see that for any signature ρ and class Φ of ρ-formulas, the relation $\prec^\Phi$ is transitive.

(2.155) **Theorem** [GB] *Suppose D is a set of ρ-structures, Φ is some class of ρ-formulas, and the relation $\prec^\Phi$ on D, which is necessarily a partial order, is directed,*[3.208.1] *i.e.,*

$$\forall \mathfrak{A}, \mathfrak{B} \in D\, \exists \mathfrak{C} \in D\, (\mathfrak{A} \prec^\Phi \mathfrak{C} \wedge \mathfrak{B} \prec^\Phi \mathfrak{C}).$$

1. *There exists a unique ρ-structure $\mathfrak{D}$ such that*
   1. $|\mathfrak{D}| = \bigcup_{\mathfrak{A} \in D} |\mathfrak{A}|$*; and*
   2. *for every ρ-index X, $X^{\mathfrak{D}} = \bigcup_{\mathfrak{A} \in D} X^{\mathfrak{A}}$.*
2. $\forall \mathfrak{A} \in D\ \mathfrak{A} \prec^{\Phi} \mathfrak{D}$.

Remark We refer to $\mathfrak{D}$ loosely as the *union of D*.

Proof 1 It is easy to check that (2.155.1.1) and (2.155.1.2) define a ρ-structure.

2 It is also easy to check that (2.155.2) holds using the Tarski-Vaught criterion.[2.153]
$\square$[2.155]

A more flexible version of this construction deals with directed systems of elementary embeddings.

(2.156) **Definition** [GB] *Suppose $(D; \leqslant)$ is a directed partial order, $[\mathfrak{M}_a \mid a \in D]$ is a D-indexed family of ρ-structures, and $[i_{ab} \mid a, b \in D \wedge a \leqslant b]$ is a system of Φ-elementary embeddings $i_{ab} : \mathfrak{M}_a \prec^{\Phi} \mathfrak{M}_b$. Let $M_a = |\mathfrak{M}_a|$ for each $a \in D$, and let M, together with the maps $i_a : M_a \overset{\text{inj}}{\to} M$ be the direct limit of the corresponding system.*[3.209] *Clearly there is a unique ρ-structure $\mathfrak{M}$ with universe M such that each i_a is a ρ-embedding of M_a in M. We define the* direct limit *of $[[\mathfrak{M}_a \mid a \in D], [i_{ab} \mid a, b \in D \wedge a \leqslant b]]$ to be $\mathfrak{M}$ together with $i_a : M_a \overset{\text{inj}}{\to} M$.*

(2.157) **Theorem** [GB] *Under the conditions of Definition 2.156, the maps i_a are Φ-elementary.*

Proof This is nothing more than Theorem 2.155 applied to $[\operatorname{im} i_a \mid a \in D]$, which by hypothesis is a directed partial order under the relation $\prec^{\Phi}$ with union $\mathfrak{M}$.$\square$[2.157]

2.7.4 Löwenheim-Skolem theorems

(2.158) *In this section, to avoid gratuitous complications, we will suppose that all signatures are with identity.*

(2.159) **Theorem: Löwenheim-Skolem** [ZFC] *Suppose $\mathfrak{A}$ is a ρ-structure and $\kappa \geqslant |\rho|$ is an infinite cardinal.*

1. **Downward** *Suppose $X \subseteq |\mathfrak{A}|$ with $|X| \leqslant \kappa$. Then there exists $\mathfrak{B} \prec \mathfrak{A}$ with $||\mathfrak{B}|| \leqslant \kappa$ such that $X \subseteq |\mathfrak{B}|$ and*
2. **Upward** *Suppose $\omega \leqslant ||\mathfrak{A}|| \leqslant \kappa$. Then there exists $\mathfrak{B} \succ \mathfrak{A}$ with $||\mathfrak{B}|| = \kappa$.*

Proof 1 Let $<$ be a wellordering of $|\mathfrak{A}|$. Define $\langle X_n \mid n \in \omega \rangle$ as follows. Let $X_0 = X$. Given X_n, let X_{n+1} consist of all $x \in |\mathfrak{A}|$ such that for some ρ-formula ϕ, variable u, and X_n-assignment A for $\operatorname{Free} \phi \backslash \{u\}$,

$$\mathfrak{A} \models \phi\big[A\big\langle{u \atop x}\big\rangle\big]$$

and x is the $<$-least member of $|\mathfrak{A}|$ for which this is true. Let $X_\omega = \bigcup_{n \in \omega} X_n$.

Given a ρ-formula ϕ, variable u, and X_ω-assignment A for $\operatorname{Free}\phi\backslash\{u\}$, let $n \in \omega$ be such that $\operatorname{im} A \subseteq X_n$. By construction,

$$\begin{aligned}\exists x \in |\mathfrak{A}|\ \mathfrak{A} \models \phi\big[A\big\langle{}^{u}_{x}\big\rangle\big] &\rightarrow \exists x \in X_{n+1}\ \mathfrak{A} \models \phi\big[A\big\langle{}^{u}_{x}\big\rangle\big] \\ &\rightarrow \exists x \in X_\omega\ \mathfrak{A} \models \phi\big[A\big\langle{}^{u}_{x}\big\rangle\big].\end{aligned}$$

Note that, in particular, that if τ is a ρ-term, u is a variable not in $\operatorname{Free}\tau$, and A is an X_ω-assignment for τ, then, letting $x_0 = \operatorname{Val}^{\mathfrak{A}} \tau[A]$,

$$\mathfrak{A} \models (u = \tau)\big[A\big\langle{}^{\ u}_{x_0}\big\rangle\big],$$

so

$$\exists x \in X_\omega\ \mathfrak{A} \models \phi\big[A\big\langle{}^{u}_{x}\big\rangle\big],$$

so $x_0 \in X_\omega$.[34]

Since X_ω is closed under all operations of $\mathfrak{A}$, it defines a substructure $\mathfrak{B}$, i.e., $|\mathfrak{B}| = X_\omega$. By construction, $\mathfrak{B}$ satisfies the Tarski-Vaught criterion,[2.153.1] so $\mathfrak{B} \prec \mathfrak{A}$.

Since $|\rho| \leqslant \kappa$, $|X| \leqslant \kappa$, and κ is infinite, it follows by a straightforward induction that for each $n \in \omega$, $|X_n| \leqslant \kappa$.[35] It follows that $|X_\omega| \leqslant \kappa$, as desired.

2 Given a ρ-structure $\mathfrak{A}$ with $\omega \leqslant ||\mathfrak{A}|| \leqslant \kappa$, let ρ' be an expansion of ρ by the addition of a distinct constant c_a for each $a \in |\mathfrak{A}|$. Then $|\rho'| \leqslant \kappa$. Let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ to a ρ'-structure obtained by letting $c_a^{\mathfrak{A}'} = a$ for each $a \in |\mathfrak{A}|$. Let $\Theta' = \operatorname{Th}\mathfrak{A}'$, which is called the *diagram of* $\mathfrak{A}$ (*the* diagram because it is unique up to homologic equivalence). Recall[2.158] that ρ is assumed to be with identity, and note that for $a, b \in |\mathfrak{A}|$, if $a \neq b$ then $\neg\, \bar{c}_a = \bar{c}_b \in \Theta'$. Now let ρ'' be the expansion of ρ' by the addition of new constants $d \in D$, with $|D| = \kappa$. Let Θ'' be the extension of Θ' by the addition of $\neg\, \bar{d} = \bar{d}'$ for all $d, d' \in D$ with $d \neq d'$. Note that $|\rho''| = \kappa$.

If $\Sigma \subseteq \Theta''$ is finite, then $\mathfrak{A}'$ may be expanded to a model of Σ by letting the distinct members of D that occur in Σ have arbitrary distinct denotations in $|\mathfrak{A}|$. This is possible, as we have assumed that $|\mathfrak{A}|$ is infinite. Thus, Θ'' is consistent, and by the completeness theorem[2.150] it has a model $\mathfrak{C}''$. By taking an isomorph if necessary, we may assume that $|\mathfrak{A}| \subseteq |\mathfrak{C}''|$ and moreover that for every $a \in |\mathfrak{A}|$, $c_a^{\mathfrak{C}''} = a$.

Using Part 1 of the theorem, let $\mathfrak{B}'' \prec \mathfrak{C}''$ be such that $||\mathfrak{B}''|| \leqslant \kappa$ and $|\mathfrak{A}| \subseteq |\mathfrak{B}''|$. For any $d, d' \in D$, if $d \neq d'$ then $d^{\mathfrak{B}''} \neq d'^{\mathfrak{B}''}$, so $||\mathfrak{B}''|| \geqslant \kappa$. Hence $||\mathfrak{B}''|| = \kappa$. Note that $\operatorname{Th}\mathfrak{B}'' = \operatorname{Th}\mathfrak{C}'' \supseteq \Theta''$. Let $\mathfrak{B}'$ be the contraction of $\mathfrak{B}''$ to ρ' (dropping the constants $d \in D$). Then $\operatorname{Th}\mathfrak{B}' = \Theta' = \operatorname{Th}\mathfrak{A}'$. It follows directly that $\mathfrak{A}' \prec \mathfrak{B}'$. Let $\mathfrak{B}$ be the contraction of $\mathfrak{B}'$ to ρ. Then $\mathfrak{A} \prec \mathfrak{B}$, and $||\mathfrak{B}|| = \kappa$, as desired. □[2.159]

It is often useful to formalize the above construction for the downward Löwenheim-Skolem theorem in terms of *Skolem functions*, which are the model-theoretic analog of the Skolem operations discussed[§ 2.4.3] above.

(2.160) **Definition** [ZF] *Suppose* $\mathfrak{A}$ *is a* ρ*-structure.*

[34] We could have avoided assuming[2.158] that ρ is with identity by directly arranging that X_{n+1} contain the values of all terms with arguments in X_n.

[35] Note that this calculation relies on the fact that for each ρ-formula ϕ, variable u, and X_n-assignment A for $\operatorname{Free}\phi\backslash\{u\}$, we have added at most one element to X_{n+1}. We used the axiom of choice, via the wellordering principle, to achieve this.

1. *Suppose $\phi = \exists u\ \psi$ is an existential ρ-formula with n free variables. f is a* Skolem function *for ϕ for $\mathfrak{A}$* $\overset{\text{def}}{\iff}$ *$f : {}^{n}|\mathfrak{A}| \to |\mathfrak{A}|$ and for all $x_0, \dots, x_{n^-} \in |\mathfrak{A}|$, letting $x = f\langle x_0, \dots, x_{n^-}\rangle$,*

$$\mathfrak{A} \models \phi\begin{bmatrix} u_0 & \cdots & u_{n^-} \\ x_0 & \cdots & x_{n^-} \end{bmatrix} \to \mathfrak{A} \models \psi\begin{bmatrix} u & u_0 & \cdots & u_{n^-} \\ x & x_0 & \cdots & x_{n^-} \end{bmatrix},$$

where $\langle u_0, \dots, u_{n^-}\rangle$ is the enumeration of Free ϕ *in increasing standard order.*

2. *F is a* complete set of Skolem functions for $\mathfrak{A}$ $\overset{\text{def}}{\iff}$
   1. *for each $f \in F$ there exists $n \in \omega$ such that $f : {}^{n}|\mathfrak{A}| \to |\mathfrak{A}|$;*
   2. *F contains a Skolem function for each existential ρ-formula for $\mathfrak{A}$;*
   3. *F contains $X^{\mathfrak{A}}$ for every operation index X of ρ;*
   4. *F contains the quasi-identity function $\{(\langle x\rangle, x) \mid x \in |\mathfrak{A}|\}$; and*
   5. *F is closed under composition, i.e., F contains every function $g : {}^{n}|\mathfrak{A}| \to |\mathfrak{A}|$ such that for all $x_0, \dots, x_{n^-} \in |\mathfrak{A}|$,*

$$\begin{aligned} &g\langle x_0, \dots, x_{n^-}\rangle \\ &\qquad = f\Big\langle f_0\langle x_{i^0_0}, \dots, x_{i^0_{l_0^-}}\rangle, \dots, f_{m^-}\langle x_{i^{m^-}_0}, \dots, x_{i^{m^-}_{(l_{m^-})^-}}\rangle\Big\rangle, \end{aligned}$$

   where $f, f_0, \dots, f_{m^-} \in F$ and $n = \{i^j_k \mid j \in m \wedge k \in l_j\}$.

(2.161)

1. *Note that if ρ is a signature with identity then (2.160.2.3) and (2.160.2.4) follow from (2.160.2.2), since $X^{\mathfrak{A}}$ is the only Skolem function for $\exists u\ \bar{u} = \tilde{X}\langle \bar{u}_0, \dots, \bar{u}_{m^-}\rangle$, where m is the arity of X; and the quasi-identity is the only Skolem function for $\exists u\ \bar{u} = \bar{v}$.*
2. *Note also that by virtue of (2.160.2.5), F is closed under substitution, i.e., for any m-ary $f \in F$ and $\pi : m \to n$, the n-ary function g such that for all $x_0, \dots, x_{n^-} \in |\mathfrak{A}|$,*

$$g\langle x_0, \dots, x_{n^-}\rangle = f\langle x_{\pi 0}, \dots, x_{\pi(m^-)}\rangle,$$

   is in F.

A complete set of Skolem functions may be constructed (from a suitable choice function or wellordering) by the same sort of ω-sequence of closure operations as used in the proof of (2.159.1), and the construction of elementary substructures is correspondingly more direct:

(2.162) **Theorem** [ZFC] *Suppose $\mathfrak{A}$ is a ρ-structure, $X \subseteq |\mathfrak{A}|$, and F is a complete set of Skolem functions for $\mathfrak{A}$. For each $f \in F$, let k_f be the arity of f, i.e.,* $\operatorname{dom} f = {}^{k_f}|\mathfrak{A}|$. *Let $B = \{f\langle x_0, \dots, x_{k_f^-}\rangle \mid f \in F \wedge x_0, \dots, x_{k_f^-} \in X\}$, and let $\mathfrak{B}$ be the corresponding substructure of $\mathfrak{A}$.*[36] *Then*

1. *$X \subseteq B$;*
2. *$|B| \leqslant \max\{\omega, |X|\}$; and*
3. *$\mathfrak{B} \prec \mathfrak{A}$.*

Proof Straightforward. $\square$[2.162]

[36] I.e., $|\mathfrak{B}| = B$. Note that this does define a substructure, since every ρ-operation symbol has a corresponding Skolem function.[2.160.2.3]

2.7.5 Ultraproducts

(2.163) **Definition** [ZF] *Suppose X is a nonempty set, U is an ultrafilter on $\mathcal{P}X$, ρ is a relational signature, and for each $x \in X$, $\mathfrak{A}_x$ is a ρ-structure that is a set. Recall that $\prod_{x\in X}|\mathfrak{A}_x|$ is the set of functions f such that $\operatorname{dom} f = X$ and $\forall x \in X\ fx \in |\mathfrak{A}_x|$. Let $\equiv^U$ be the equivalence relation on $\prod_{x\in X}|\mathfrak{A}_x|$ such that $f \equiv^U f' \leftrightarrow \{x \mid fx = f'x\} \in U$. Let $A = \prod_{x\in X}|\mathfrak{A}_x|/\equiv^U$ be the set $\equiv^U$-equivalence classes. The* ultraproduct *of $\langle \mathfrak{A}_x \mid x \in X\rangle$ mod U $\overset{\text{def}}{=}$*

$$\prod_{x\in X} \mathfrak{A}_x/U$$

$\overset{\text{def}}{=}$ *the structure $\mathfrak{A}$ with signature ρ defined by:*

1. *$|\mathfrak{A}| = A$;*
2. *for each $n \in \omega$, n-ary ρ-predicate index R, and $a_0, \ldots, a_{n^-} \in A$,*

$$R^{\mathfrak{A}}\langle a_0, \ldots, a_{n^-}\rangle \leftrightarrow \left\{x \in X \mid R^{\mathfrak{A}_x}\langle f_0x, \ldots, f_{n^-}x\rangle\right\} \in U,$$

for some (equivalently for all) $f_0, \ldots, f_{n^-}$ such that $\forall m \in n\ f_m \in a_m$.[37]

(2.164) **Łoś's theorem** [ZFC] *In the setting of Definition 2.163, letting $\mathfrak{A} = \prod\langle \mathfrak{A}_x \mid x \in X\rangle/U$, suppose ϕ is a ρ-formula, $\langle v_0, \ldots, v_{n^-}\rangle$ enumerates* $\operatorname{Free}\phi$*, and $a_0, \ldots, a_{n^-} \in A$. Then*

$$\begin{aligned}
&\mathfrak{A} \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ a_0 & \cdots & a_{n^-}\end{smallmatrix}\right] \\
&\quad \leftrightarrow \exists f_0 \in a_0 \cdots \exists f_{n^-} \in a_{n^-}\ \left\{x \in X \mid \mathfrak{A}_x \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0x & \cdots & f_{n^-}x\end{smallmatrix}\right]\right\} \in U \\
&\quad \leftrightarrow \forall f_0 \in a_0 \cdots \forall f_{n^-} \in a_{n^-}\ \left\{x \in X \mid \mathfrak{A}_x \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0x & \cdots & f_{n^-}x\end{smallmatrix}\right]\right\} \in U.
\end{aligned}$$

In particular, if ϕ is a sentence then

$$\mathfrak{A} \models \phi \leftrightarrow \{x \in X \mid \mathfrak{A}_x \models \phi\} \in U.$$

Proof By induction on the complexity of formulas. AC is invoked in the induction step corresponding to quantification. Suppose ϕ has free variables $v, v_0, \ldots, v_{n^-}$ and $\left\{x \in X \mid \mathfrak{A}_x \models (\exists v\, \phi)\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0x & \cdots & f_{n^-}x\end{smallmatrix}\right]\right\} \in U$. By AC there is a function f such that $\left\{x \in X \mid \mathfrak{A}_x \models \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ fx & f_0x & \cdots & f_{n^-}x\end{smallmatrix}\right]\right\} \in U$, so by induction hypothesis $\mathfrak{A} \models \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ [f] & [f_0] & \cdots & [f_{n^-}]\end{smallmatrix}\right]$, and therefore $\mathfrak{A} \models (\exists v\, \phi)\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ [f_0] & \cdots & [f_{n^-}]\end{smallmatrix}\right]$.

(2.165) **Definition** [ZF] *Suppose $\mathfrak{A}$ is a ρ-structure, X is a nonempty set, and U is an ultrafilter on X. Then the* ultrapower of $\mathfrak{A}$ mod U *is*

$${}^X\mathfrak{A}/U \overset{\text{def}}{=} \prod\langle \mathfrak{A}_x \mid x \in X\rangle/U,$$

where $\mathfrak{A}_x = \mathfrak{A}$ for every $x \in X$.

(2.166) **Theorem** [ZFC] *For $a \in |\mathfrak{A}|$, let $\bar{a}$ be the constant function with domain X and value a. The map $a \mapsto \bar{a}$ is an elementary embedding*[2.151.4] *of $\mathfrak{A}$ into ${}^X\mathfrak{A}/U$.*

Proof Immediate from (2.164). $\square$[2.166]

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[37] If $f_m, f'_m \in a_m$ for each $m \in n$ then $\{x \in X \mid f_mx = f'_mx\} \in U$ for each $m \in n$, so $\{x \in X \mid \forall m \in n\ f_mx = f'_mx\} \in U$; hence, when working "mod U", it doesn't matter which representatives we use from any finite set of U-equivalence classes.

2.7.5.1 Example: Compactness theorem via ultraproducts

The ultrafilter construction will become extremely important in Chapter 9. The following proof of the compactness theorem of first-order predicate logic (in ZFC) is a foreshadowing.[9.50] We first observe that, assuming AC, any filter on a set X may be extended to an ultrafilter. For suppose F is a filter on X. Posit a fixed wellordering of $\mathcal{P}X$. Construct an ordinal sequence $\langle F_\alpha \mid \alpha \leqslant \delta \rangle$ of filters F_α on X, such that

1. $F_0 = F$;
2. for all $\beta < \alpha \leqslant \delta$, $F_\beta \subsetneqq F_\alpha$;
3. for all limit $\alpha \leqslant \delta$, $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$; and
4. $\forall Y \subseteq X\, (Y \in F_\delta \vee (X \backslash Y) \in F_\delta)$, i.e., F_δ is an ultrafilter on X.

To obtain F_α for α a successor ordinal, say $\alpha = \beta + 1$, assuming F_β is not an ultrafilter, let Y be the first subset of X (in the sense of the fixed wellordering of $\mathcal{P}X$ posited above) such that $Y \notin F_\beta$ and $X \backslash Y \notin F_\beta$. Let

$$F_\alpha = \{Z \subseteq X \mid \exists W \in F_\beta\; W \cap Y \subseteq Z\}.$$

Clearly F_α is a filter on X that contains Y, so $F_\alpha \supsetneqq F_\beta$.

To obtain F_α for limit α, let $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$. The construction eventually ends with an ultrafilter extending F.

Now suppose Θ is a ρ-theory. Assume for simplicity that ρ is purely relational, so (2.164) applies as written. Suppose every finite subset of Θ has a model. We wish to obtain a model of Θ. For each finite $t \subseteq \Theta$, let $\mathfrak{A}_t$ be a ρ-model of t (using AC). Let $X = [\Theta]^{<\omega}$, the set of finite subsets of Θ. For each $t \in X$, let $X_t = \{t' \in X \mid t' \supseteq t\}$. Let

$$F = \{Y \subseteq X \mid \exists t \in X\; Y \supseteq X_t\}.$$

Note that if $Y_0 \supseteq X_{t_0}$ and $Y_1 \supseteq X_{t_1}$, then $Y_0 \cap Y_1 \supseteq X_{t_0 \cup t_1}$, so F is a filter. Let $U \supseteq F$ be an ultrafilter on X. Let

$$\mathfrak{A} = \prod_{t \in X} \mathfrak{A}_t / U.$$

We claim that $\mathfrak{A} \models \Theta$. For suppose $\theta \in \Theta$. Then[38]

$$\mathfrak{A} \models \theta \leftrightarrow \{t \in X \mid \mathfrak{A}_t \models \theta\} \in U.$$

By construction, $X_{\{\theta\}} \in F$, so $X_{\{\theta\}} \in U$. Since

$$\{t \in X \mid \mathfrak{A}_t \models \theta\} \supseteq \{t \in X \mid \theta \in t\} = X_{\{\theta\}},$$

$\mathfrak{A} \models \theta$.

[38] We use AC also here, as Theorem 2.164, which we are invoking, depends on AC.

2.7.6 Ultrapowers of proper classes

Suppose X is a nonempty set, U is an ultrafilter on $\mathcal{P}X$, and A is a class. If A is a proper class the canonical representative (mod U) of $f \in {}^{X}A$ for the purpose of the ultrapower construction cannot be taken to be the U-equivalence class $[f]_U$ of f, because this is also a proper class. Instead we use

$$[f]^*_U \overset{\text{def}}{=} \{f' \in [f]_U \mid \forall f'' \in [f]_U \ \operatorname{rk} f' \leqslant \operatorname{rk} f''\}. \tag{2.167}$$

By virtue of the restriction on rank, $[f]^*_U$ is a set. When U is understood we may omit the subscript.

When A is a proper class,

$${}^{X}A/U \overset{\text{def}}{=} \{[f]^*_U \mid f \in {}^{X}A\}. \tag{2.168}$$

(2.169) **Definition** [GB] *Suppose X is a nonempty set, U is an ultrafilter on $\mathcal{P}X$, ρ is a relational signature, and $\mathfrak{A}$ is a ρ-structure, which may be a proper class. Let $A' = {}^{X}|\mathfrak{A}|/U$. We define ${}^{X}\mathfrak{A}/U$ to be the structure $\mathfrak{A}'$ with signature ρ defined by:*

1. $|\mathfrak{A}'| = A'$;
2. *for each $n \in \omega$, n-ary ρ-predicate index R, and $a_0, \dots, a_{n^-} \in A'$,*

$$R^{\mathfrak{A}'}\langle a_0, \dots, a_{n^-}\rangle \leftrightarrow \big\{x \in X \mid R^{\mathfrak{A}}\langle f_0 x, \dots, f_{n^-} x\rangle\big\} \in U,$$

for some (equivalently for all) $f_0, \dots, f_{n^-}$ such that $\forall m \in n\ [f_m]^* = a_m$.

In generalizing (2.164) we labor under the usual limitations on the demonstrability of the existence of satisfaction relations for proper class structures.

(2.170) **Theorem** [GBC] *In the setting of Definition 2.169, letting $\mathfrak{A}' = {}^{X}\mathfrak{A}/U$, suppose ϕ is a ρ-formula and the $\{\phi\}^{\mathfrak{A}}$-satisfaction relation exists. Then the $\{\phi\}^{\mathfrak{A}'}$-satisfaction relation exists, and if $\langle v_0, \dots, v_{n^-}\rangle$ enumerates* Free ϕ *and $a_0, \dots, a_{n^-} \in |\mathfrak{A}'|$ then*

$$\mathfrak{A}' \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ a_0 & \cdots & a_{n^-}\end{smallmatrix}\big] \leftrightarrow \big\{x \in X \mid \mathfrak{A} \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0 x & \cdots & f_{n^-} x\end{smallmatrix}\big]\big\} \in U$$

for some (equivalently for all) $f_0, \dots, f_{n^-}$ such that $\forall m \in n\ [f_m]^* = a_m$.

Proof Note that we allow $\mathfrak{A}$ to be a proper class. As previously discussed, neither the assumption nor the conclusion regarding the existence of $\{\phi\}$-satisfaction relations for $\mathfrak{A}$ and $\mathfrak{A}'$ can therefore be established independently by induction in GBC.

Suppose, therefore, that ϕ is ρ-formula and S is the $\{\phi\}^{\mathfrak{A}}$-satisfaction relation.

(2.171) *Let S' be the class of $\langle \psi, A\rangle$ such that ψ is a subformula of ϕ, $A = \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ a_0 & \cdots & a_{n^-}\end{smallmatrix}\big\rangle$ is an $\mathfrak{A}'$-assignment for ψ, and*

$$\big\{x \in X \mid \big\langle \psi, \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0 x & \cdots & f_{n^-} x\end{smallmatrix}\big\rangle\big\rangle \in S\big\} \in U \tag{2.172}$$

for some $f_0, \dots, f_{n^-}$ such that $\forall m \in n\ [f_m]^* = a_m$.

Since U is an ultrafilter, (2.172) holds for some $f_0, \ldots, f_{n^-}$ such that $\forall m \in n\ [f_m]^* = a_m$ iff it holds for all such $f_0, \ldots, f_{n^-}$.

It is straightforward to show that S' is the $\{\phi\}^{\mathfrak{A}'}$-satisfaction relation. This is just a matter of verifying that for each subformula ψ of ϕ the appropriate clause in the recursive definition of satisfaction is satisfied. The proof is analogous to the proof of (2.164). For example, suppose ψ has free variables $v, v_0, \ldots, v_{n^-}$ and $\exists v\, \psi$ is a subformula of ϕ. Then

$$\begin{aligned}
\left\langle \exists v\, \psi, \left\langle {}^{v_0}_{a_0} {}^{\cdots}_{\cdots} {}^{v_{n^-}}_{a_{n^-}} \right\rangle \right\rangle \in S' &\leftrightarrow \exists f_0, \ldots, f_{n^-} \big([f_0]^* = a_0 \wedge \cdots \wedge [f_{n^-}]^* = a_{n^-} \\
&\qquad \wedge \left\{ x \in X \mid \left\langle \exists v\, \psi, \left\langle {}^{v_0}_{f_0 x} {}^{\cdots}_{\cdots} {}^{v_{n^-}}_{f_{n^-} x} \right\rangle \right\rangle \in S \right\} \in U \big) \\
&\leftrightarrow \exists f, f_0, \ldots, f_{n^-} \big([f_0]^* = a_0 \wedge \cdots \wedge [f_{n^-}]^* = a_{n^-} \\
&\qquad \wedge \left\{ x \in X \mid \left\langle \psi, \left\langle {}^{v}_{f x} {}^{v_0}_{f_0 x} {}^{\cdots}_{\cdots} {}^{v_{n^-}}_{f_{n^-} x} \right\rangle \right\rangle \in S \right\} \in U \big) \\
&\leftrightarrow \exists a \in |\mathfrak{A}'|\ \left\langle \psi, \left\langle {}^{v}_{a} {}^{v_0}_{a_0} {}^{\cdots}_{\cdots} {}^{v_{n^-}}_{a_{n^-}} \right\rangle \right\rangle \in S'.
\end{aligned}$$

Now that we know that the $\{\phi\}^{\mathfrak{A}'}$-satisfaction relation exists and is given by (2.171), the conclusion of the theorem follows immediately. $\square^{2.170}$

Theorem 2.166 also holds for proper class structures:

(2.173) **Theorem** [GBC] *Letting $\bar{a}$ be the constant function with domain X and value a, the map $a \mapsto \bar{a}$ is an elementary embedding of $\mathfrak{A}$ into ${}^X\mathfrak{A}/U$.*

2.8 Satisfactoriness and logic

The principal theorem relating satisfactoriness and logic is the completeness theorem, which states that a consistent ρ-theory Θ has a satisfactory model, i.e., there is a satisfactory ρ-structure $\mathfrak{S}$ such that $\mathfrak{S} \models \Theta$. For the converse weak satisfactoriness suffices.

(2.174) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a weakly satisfactory ρ-structure, Θ is a ρ-theory, and $\mathfrak{S} \models \Theta$.*

1. *Θ is consistent.*
2. *If θ is a ρ-sentence and $\Theta \vdash \theta$ then $\mathfrak{S} \models \theta$.*

Proof 1 Suppose toward a contradiction that Θ is inconsistent. Let π be a proof of $\theta \wedge \neg\,\theta$ from Θ, for some ρ-sentence θ, using the natural deduction system **ND**. Let Φ be the set of formulas in π, and let S be the Φ-satisfaction relation for $\mathfrak{S}$. It is easy to show by induction on position in π that for every sequent $\Sigma \Rightarrow \sigma$ in π,

$$\big(\forall \eta \in \Sigma\ \ \models^S \eta\big) \rightarrow\, \models^S \sigma.$$

Since the sequent $\Sigma \Rightarrow \{\theta \wedge \neg\,\theta\}$ occurs in π for some $\Sigma \subseteq \Theta$, $\models^S \theta \wedge \neg\,\theta$, which is impossible.

2 Suppose $\mathfrak{S} \not\models \theta$. Since $\mathfrak{S}$ is weakly satisfactory, there is a $\{\neg\,\theta\}$-satisfaction relation S for $\mathfrak{S}$, and $\models^S \neg\,\theta$. Thus, $\mathfrak{S} \models \Theta \cup \{\neg\,\theta\}$. But if $\Theta \vdash \theta$ then $\Theta \cup \{\neg\,\theta\}$ is inconsistent, contradicting (2.174.1). $\square^{2.174}$

2.8.1 Satisfaction of logical validities

Recall that a ρ-sentence σ is valid $\overset{\text{def}}{\iff}$ for every satisfactory ρ-structure $\mathfrak{S}$, $\mathfrak{S} \models \sigma$. By the completeness theorem, σ is valid iff $\vdash \sigma$, i.e., σ is provable from the empty set of premises. The following theorem shows that the condition of satisfactoriness may be dropped. We prove the theorem first in GB and then in C^0, to highlight the issues involved in the finitary proof.

(2.175) **Theorem** [GB] *Suppose $\mathfrak{S}$ is a ρ-structure, σ is a ρ-sentence, and $\vdash \sigma$, i.e., σ is valid. Then $\mathfrak{S} \models \sigma$.*

Proof If $\mathfrak{S}$ is a set then the theorem is just the soundness property of the system of deduction in terms of which $\vdash$ is defined.[39] Suppose therefore that $\mathfrak{S}$ is a proper class, and suppose toward a contradiction that S is a $\{\sigma\}$-satisfaction relation for $\mathfrak{S}$ and $\not\models^S \sigma$. In a variation on the construction used in the proof of the downward Löwenheim-Skolem theorem,[2.159.1] define $\langle \alpha_n \mid n \in \omega \rangle$ as follows.

Let α_0 be the least ordinal α such that $|\mathfrak{S}| \cap V_\alpha \neq 0$. Given α_n, let α_{n+1} be the least ordinal α such that

1. for every ρ-operation index F, letting m be the arity of F, for any $x_0, \ldots, x_{m\text{-}} \in |\mathfrak{S}| \cap V_{\alpha_n}$, $F^{\mathfrak{S}}\langle x_0, \ldots, x_{m\text{-}} \rangle \in V_\alpha$; and
2. for every subformula ϕ of σ and $(|\mathfrak{S}| \cap V_{\alpha_n})$-assignment A for ϕ,
    1. if $\phi = \exists u\, \psi$ and $\exists x \in |\mathfrak{S}| \ \models^S \psi\big[A\big\langle{u \atop x}\big\rangle\big]$ then
    $$\exists x \in \big(|\mathfrak{S}| \cap V_\alpha\big) \ \models^S \psi\big[A\big\langle{u \atop x}\big\rangle\big];$$
    and
    2. if $\phi = \forall u\, \psi$ and $\forall x \in \big(|\mathfrak{S}| \cap V_\alpha\big) \ \models^S \psi\big[A\big\langle{u \atop x}\big\rangle\big]$ then
    $$\forall x \in |\mathfrak{S}| \ \models^S \psi\big[A\big\langle{u \atop x}\big\rangle\big].$$

Let $\alpha = \bigcup_{n \in \omega} \alpha_n$.[40] Let $\mathfrak{S}'$ be the substructure of $\mathfrak{S}$ with $|\mathfrak{S}'| = |\mathfrak{S}| \cap V_\alpha$. Clearly, $\mathfrak{S}' \prec^{\{\sigma\}} \mathfrak{S}$. Let S' be the (full) satisfaction relation for $\mathfrak{S}'$, which exists because $\mathfrak{S}'$ is a set.

Since $\mathfrak{S}' \prec^{\{\sigma\}} \mathfrak{S}$, $\not\models^{S'} \sigma$, which contradicts the fact that, as noted above, any set structure satisfies all logical validities. $\square$[2.175]

(2.176) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a ρ-structure, σ is a ρ-sentence, and $\vdash \sigma$, i.e., σ is valid. Then $\mathfrak{S} \models \sigma$.*

Proof Suppose first that ρ is without identity. Let π be a proof of σ in the system $\mathbf{LK}^-$[2.145] (from the empty set of premises). Recall that π is a finite tree of sequents with $0 \Rightarrow \{\sigma\}$ as the final sequent. Suppose S is the $\{\sigma\}$-satisfaction relation for $\mathfrak{S}$. Recall that $\mathbf{LK}^-$ has the subformula property, i.e., all formulas occurring in π are obtained from subformulas of σ by substitution of terms for free variables. It is

[39] The essential property of a set structure for this purpose is the fact that its full satisfaction relation exists.

[40] This is where we use the fact that we are working in GB, as opposed to C^0, for example.

straightforward to show by induction on the position of sequents in π that for every $\Gamma \Rightarrow \Delta$ in π and $\mathfrak{S}$-assignment A for $\Gamma \cup \Delta$,

$$\big(\forall\gamma \in \Gamma \ \models^S \gamma[A]\big) \rightarrow \big(\exists\delta \in \Delta \ \models^S \delta[A]\big).$$

Since $0 \Rightarrow \{\sigma\}$ is the final sequent in π, $\models^S \sigma$. Hence, $\mathfrak{S} \models \sigma$.

Now suppose ρ is with identity. The deductive system $\mathbf{LK}^-$ for ρ does not have the subformula property, inasmuch as identity axioms may occur in a proof that do not derive from subformulas of the final sequent. To put it another way, a sentence σ is valid in logic with identity iff it is provable from identity axioms in logic without identity. Either way, suppose as before that π is a proof of σ and S is the $\{\sigma\}$-satisfaction relation for $\mathfrak{S}$. Let E be the set of axioms of identity that occur in π. Note that the deductive system $\mathbf{LK}^-$ does not allow for the introduction of new constants, so the proof of (2.80) from Axioms 2.79 is not available, and we therefore posit the following additional axioms of identity.

(2.177)

1. $\forall \mathrm{v}_0\ \bar{\mathrm{v}}_0 = \bar{\mathrm{v}}_0$.
2. $\forall \mathrm{v}_0, \mathrm{v}_1\, (\bar{\mathrm{v}}_0 = \bar{\mathrm{v}}_1 \rightarrow \bar{\mathrm{v}}_1 = \bar{\mathrm{v}}_0)$.
3. $\forall \mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\, (\bar{\mathrm{v}}_0 = \bar{\mathrm{v}}_1 \wedge \bar{\mathrm{v}}_1 = \bar{\mathrm{v}}_2 \rightarrow \bar{\mathrm{v}}_1 = \bar{\mathrm{v}}_2)$.

Let S' be the extension of S to the $(\{\sigma\} \cup E)$-satisfaction relation for $\mathfrak{S}$ (with the identity predicate interpreted as the identity relation on $|\mathfrak{S}|$, of course). (The existence of S' follows by a simple argument in C^0.) Then $\models^{S'} E$ and for every $\Gamma \Rightarrow \Delta$ in π,

$$\big(\forall\gamma \in \Gamma \ \models^{S'} \gamma\big) \rightarrow \big(\exists\delta \in \Delta \ \models^{S'} \delta\big),$$

so $\models^{S'} \sigma$, whence $\mathfrak{S} \models \sigma$. $\square$[2.176]

2.9 A comparison of metatheories

We have been using the theory C^0 as our working theory for the discussion of language, structure, and logic. In C^0 one has infinite classes but not necessarily infinite sets. Since linguistic expressions are finitary objects, they are representable as sets in this theory. Any reasonable definition of interpretation, on the other hand, must allow for the existence of infinite structures, which may be proper classes (i.e., not sets). The ability of C^0 to demonstrate the existence of structures and satisfaction relations is sufficient to prove the completeness theorem, which is the basis for our definition of first-order predicate logic. Obviously the corresponding pure set theory S^0 is insufficient for this purpose, as it lacks an axiom of infinity, so it cannot prove the existence of infinite structures and cannot prove the completeness theorem. Nevertheless, taking first-order predicate logic as given, proofs are finitary objects, and S^0 is sufficient to serve as a metatheory for the syntactical aspects of language and logic.

In Section 2.9.1 we will show that S^0 is in fact just as good as C^0 for this purpose, inasmuch as it is capable of proving every theorem of C^0 that refers only to hereditarily finitary (HF) sets. In particular, everything we can prove in C^0 about provability in S^0 is provable in S^0. The usefulness of a theory that can act as its own metatheory has been indicated in Section 1.1.16 in connection with Gödel's famous incompleteness theorems.

We do not need to have the above-mentioned conservative extension result to carry out Gödel's program for S^0, but it makes the work considerably easier by virtue of the greater expressivity of C^0. It is similarly useful in the development of the theory of membership *per se*.

Traditionally, as indicated in Section 1.1.16, it is some version of arithmetic that is the favored metatheory for Gödel's theorems and related work in proof theory. We show in Section 2.9.2 that Peano arithmetic is equi-interpretable with S^0, which allows us to transfer essentially any metatheorem about S^0 to PA. Again, this saves work, and allows a more intuitive development, but it is not essential: we could work exclusively in PA.

We should mention that the weaker theory *primitive recursive arithmetic* (PRA) is sufficient for this purpose and is the proper base theory for a fine analysis of provability in the finitary context, but we do not pursue this line of inquiry in this book. PRA essentially comprises the methodology known as *finitistic*. We have effectively defined *finitary* to refer to the methodology embodied in PA, S^0, and C^0 (all equivalent for this purpose).

2.9.1 Predicative class theory conservatively extends pure set theory

As noted above, we have so far been using the theory C^0 as a surrogate for the finitary theory of membership, despite the fact that it is not entirely finitary, inasmuch as it asserts the existence of infinite classes as long as they are definable with quantification restricted to sets. The strictly finitary theory of membership is the theory S^0.[41] In this section we justify our use of C^0 by showing that C^0 is a conservative extension of S^0. It follows easily from this that C is a conservative extension of S, GB of ZF, etc.

To express this conservative extension result we have to say how a sentence σ of the language $\mathcal{L}^{\mathsf{s}}$ of pure set theory is to be understood as a sentence of the language $\mathcal{L}^{\mathsf{c}}$ of the theory of classes. As we have formulated C^0, S' is the predicate symbol for "setness". Thus, σ^S is σ with all quantification restricted to sets, and it is the c-sentence corresponding to an s-sentence σ. When we say that C^0 conservatively extends S^0 we therefore mean that if σ is an s-sentence and $\mathsf{C}^0 \vdash \sigma^S$, then $\mathsf{S}^0 \vdash \sigma$. This is Theorem 2.183.

As we will see below, this theorem is easily proved in a fully infinitary theory like ZF. If all we are interested in is the endpoint and we believe that ZF is true—at least in what it says about hereditarily finite sets—then we may be satisfied with this, but it is epistemologically more satisfying to prove it finitarily, and by this we do not mean to prove it in C^0, because it is only by virtue of Theorem 2.183 that we know that C^0 is essentially finitary. We will therefore prove the theorem in S^0.

It is convenient for this purpose to reformulate C^0 as multisorted theory C^1 without identity. Recall[§ 10.1] that a multisorted signature has a class Δ of sort or domain indices. Any variable or term is of a particular sort. The signature c^1 of C^1 has two domain indices, i_C and i_S, for *classes* and *sets*. We may indicate typographically that a variable is of set sort by affixing the subscript '$_S$' to a quantifier introducing the variable. Similarly, we may indicate that a variable is of class sort by affixing the subscript '$_C$' to a quantifier. When no such subscript is

[41] The fact that S^0 allows the the existence of infinite sets is immaterial. If we add an axiom of finiteness, we obtain a stronger theory, but it does not prove anything more about hereditarily finite sets.

used we allow for the possibility that the quantified variable is of either set or class sort. Thus, in the formula

$$\forall_S u \, \forall_C v \, \forall w \ \phi, \tag{2.178}$$

u is of set sort and v is of class sort. w must be of one sort or the other, but we have not specified which it is. As a schema, therefore, (2.178) comprises

$$\forall_S u \, \forall_C v \, \forall_S w \ \phi$$

and

$$\forall_S u \, \forall_C v \, \forall_C w \ \phi.$$

There is one binary predicate index, for membership. Its type is

$$\{\langle \mathrm{i}_S, \mathrm{i}_S \rangle, \langle \mathrm{i}_S, \mathrm{i}_C \rangle\},$$

which is to say, a formula $\tau \in \tau'$ is well formed just in case τ is of set sort; τ' may be of either set or class sort.

The defining feature of the membership relation—embodied in the axiom of extension—is that classes with the same members are identical:

$$\forall_C u_0, u_1 \left(\forall_S v \, (\bar{v} \in \bar{u}_0 \leftrightarrow \bar{v} \in \bar{u}_1) \rightarrow \bar{u}_0 = \bar{u}_1\right). \tag{2.179}$$

To eliminate the identity predicate from a theory of membership T, we suppose that identity has been incorporated in T axiomatically[2.79] rather than via inference rules. The following axioms suffice:

(2.180)

1. $\tau = \tau$, *for any term* τ *of class or set sort.*
2. $(\tau_0 = \tau_0' \wedge \tau_1 = \tau_1') \rightarrow (\tau_0 = \tau_1 \leftrightarrow \tau_0' = \tau_1')$, *for any terms* $\tau_0, \tau_0', \tau_1, \tau_1'$ *independently of either class or set sort.*
3. $(\tau_0 = \tau_0' \wedge \tau_1 = \tau_1') \rightarrow (\tau_0 \in \tau_1 \leftrightarrow \tau_0' \in \tau_1')$, *for any terms* τ_0, τ_0' *of set sort and* τ_1, τ_1' *of either class or set sort.*

Note that a term here is either a variable or a constant introduced in the course of a proof.

Now replace any occurrence of $\gamma_0 = \gamma_1$, where γ_0 and γ_1 may be independently of either class or set sort, by

$$\forall v \, (\bar{v} \in \gamma_0 \leftrightarrow \bar{v} \in \gamma_1), \tag{2.181}$$

where v is any set variable that does not occur in γ_0 or γ_1.[42]

Note that this substitution makes the axiom of extension a logical validity, so a theory of membership formulated without the identity predicate does not have an axiom of extension *per se*. Similarly, the first two axioms of identity[2.180] become validities. (2.180.3) follows from

$$\forall_C V \, \forall_S u, u' \left(\forall_S v \, (\bar{v} \in \bar{u} \leftrightarrow \bar{v} \in \bar{u}') \rightarrow (\bar{u} \in \bar{V} \leftrightarrow \bar{u}' \in \bar{V})\right). \tag{2.182}$$

Note that the comprehension schema of C^0 contains

$$\forall_S v \, \exists_C V \, \forall_S w \, (\bar{w} \in \bar{v} \leftrightarrow \bar{w} \in \bar{V}),$$

[42] In signatures such as ρ and ρ', which are without operations, v occurs in a term γ iff $\gamma = \bar{v}$.

so (2.182) implies

$$\forall_S u, u' \left(\forall_S v\,(\bar v \in \bar u \leftrightarrow \bar v \in \bar u') \rightarrow (\bar u \in \tau \leftrightarrow \bar u' \in \tau)\right),$$

where τ is a variable or constant of either class or set sort. The semantic content of (2.179) in the setting of (2.180) is therefore transferred to (2.182), and we may regard this as the axiom of extension for theories of membership without an identity predicate.

(2.183) **Theorem** $[\mathsf{S}^0]$ C^0 *is a conservative extension of* S^0.

Remark As noted at the beginning of this section, this is easily proved infinitarily (in ZF, for example). Suppose $\mathsf{C}^0 \vdash \sigma$, where σ is a ρ'-sentence, i.e., with the identification established above, a ρ-sentence with no class variables. To show that $\mathsf{S}^0 \vdash \sigma$, we suppose that $\mathfrak{S}$ is a set model of S^0, and we show that $\mathfrak{S} \models \sigma$.

(2.184) *To avoid trivial complications, we will suppose that for any $b \in |\mathfrak{S}|$, if $b \subseteq |\mathfrak{S}|$ then b is coextensive with some $c \in |\mathfrak{S}|$, i.e., for all $a \in |\mathfrak{S}|$*

$$a \in b \leftrightarrow \langle a, c\rangle \in \mathrm{i}_\in^{\mathfrak{S}}.$$ [43]

Let $\mathfrak{S}'$ be the ρ-structure defined as follows.

1. $\mathrm{i}_S^{\mathfrak{S}'} = |\mathfrak{S}|$;
2. $|\mathfrak{S}'| = |\mathfrak{S}| \cup \mathrm{PC}^{\mathfrak{S}}$, where $\mathrm{PC}^{\mathfrak{S}} \overset{\text{def}}{=}$ the set of $X \subseteq |\mathfrak{S}|$ such that
    1. for some ρ'-formula ϕ, variables $v, v_0, \ldots, v_{n^-}$ with

       $$\mathrm{Free}\,\phi \subseteq \{v, v_0, \ldots, v_{n^-}\},$$

       and $a_0, \ldots, a_{n^-} \in |\mathfrak{S}|$, for all $a \in |\mathfrak{S}|$,

       $$a \in X \leftrightarrow \mathfrak{S} \models \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ a & a_0 & \cdots & a_{n^-}\end{smallmatrix}\right];$$

    2. X is not coextensive with any element of $|\mathfrak{S}|$, i.e., for all $b \in |\mathfrak{S}|$, $X \neq \left\{a \in |\mathfrak{S}| \,\middle|\, \langle a, b\rangle \in \mathrm{i}_\in^{\mathfrak{S}}\right\}$;
3. $\mathrm{i}_\in^{\mathfrak{S}'} = \mathrm{i}_\in^{\mathfrak{S}} \cup \{\langle a, b\rangle \mid b \in \mathrm{PC}^{\mathfrak{S}} \wedge a \in b\}$.

Note that (2.184) guarantees that $\mathrm{PC}^{\mathfrak{S}} \cap |\mathfrak{S}| = 0$. The members of $\mathrm{PC}^{\mathfrak{S}}$ are the proper classes of $\mathfrak{S}'$.

It is straightforward to verify that $\mathfrak{S}' \models \mathsf{C}^0$. Since $\mathsf{C}^0 \vdash \sigma$, it follows that $\mathfrak{S}' \models \sigma$. By construction, therefore, $\mathfrak{S} \models \sigma$, as claimed.

As in previous examples, an axiom of infinity is required to show that we may suppose that $\mathfrak{S}$ is a set, from which it follows that $\mathfrak{S}'$ is a set structure and therefore has a satisfaction relation. In C^0, all we may suppose is that $\mathfrak{S}$ is satisfactory, and it may be a proper class, in which case $\mathfrak{S}'$ cannot be defined simply by adjoining definable subclasses of $|\mathfrak{S}|$, as these may be proper classes and therefore not suitable candidates to be members of anything. We can get around this by coding these classes in terms of their defining formulas and parameters, but we would still have

[43] c may well be b itself, as it is for every $b \in |\mathfrak{S}|$ in the case that $|\mathfrak{S}|$ is a transitive set and $\mathrm{i}_\in^{\mathfrak{S}} = \{\langle a, b\rangle \mid a, b \in |\mathfrak{S}| \wedge a \in b\}$. Note that in this case every $b \in |\mathfrak{S}|$ is a subset of $|\mathfrak{S}|$ (by the definition of transitivity).

the problem of showing that the satisfaction relation exists for the resulting structure. In S^0 we cannot refer to models at all, and we must argue proof-theoretically; for all practical purposes, this is the case in C^0 as well.

The first finitary proof of (2.183) was given by Joseph Shoenfield[22]. Like Shoenfield's proof, the proof to be given here is constructive in that it provides a natural effective procedure to convert a C^0-proof of a ρ'-sentence to an S^0-proof.[44] It does not require the axiom of infinity and can be carried out in S^0.

Proof See Note 10.12. □

Theorem 2.183 generalizes readily to show that for any s-sentence θ, $\mathsf{C}^0 + \theta$ is a conservative extension of $\mathsf{S}^0 + \theta$, since for any s-sentence σ,

$$\begin{aligned} \mathsf{C}^0 + \theta \vdash \sigma &\leftrightarrow \mathsf{C}^0 \vdash \theta \to \sigma \\ &\leftrightarrow \mathsf{S}^0 \vdash \theta \to \sigma \\ &\leftrightarrow \mathsf{S}^0 + \theta \vdash \sigma. \end{aligned} \tag{2.185}$$

This simple observation does not apply directly to the case of C vs S, which are not primarily defined from C^0 and S^0 respectively by the addition of identical s-sentences. Instead, to obtain C from C^0 we add the $\mathsf{Foundation}$ axiom

$$\forall \mathrm{v}_0\,(\exists \mathrm{v}_1 \in \bar{\mathrm{v}}_0 \to \exists \mathrm{v}_1 \in \bar{\mathrm{v}}_0\, \forall \mathrm{v}_2 \in \bar{\mathrm{v}}_1\ \bar{\mathrm{v}}_2 \notin \bar{\mathrm{v}}_0),$$

in which v_0, being unconstrained, ranges over classes; whereas to obtain S from S^0 we add the $\mathsf{Foundation}$ schema consisting of the axioms[3.8]

$$\forall v_0, \ldots, v_{n^-}\,\left(\exists v\,\phi \to \exists v\,\left(\phi \wedge \forall u \in \bar{v}\ \neg\,\phi\big(\tbinom{v}{\bar{u}}\big)\right)\right),$$

where ϕ is any s-formula, and $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that Free $\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$, and $\bar{u}$ is free for v in ϕ.

There are two ways to deal with this. One is to examine the proof of Theorem 2.183 and show that it generalizes to incorporate $\mathsf{Foundation}$. This is straightforward for both the infinitary proof given above and the finitary proof given in Note 10.12.

It is simpler, however, to use the fact that S may be axiomatized over S^0 by the addition of the $\mathsf{Foundation}$ instances (3.111) and (3.112). The argument used in the proof of Theorem 3.116 also shows that C is axiomatized over C^0 by the same two sentences, interpreted as s-expressions, i.e., with all variables restricted to sets; i.e., we get the existence of an $\in$-minimal element in an arbitrary class from the existence of an $\in$-minimal element in an arbitrary set and the existence of transitive closures. Thus, the general argument (2.185) is applicable.

Either way, we have the following theorem.

Theorem $[\mathsf{S}^0]$ C *is a conservative extension of* S.

2.9.2 Peano arithmetic is equi-interpretable with S^0

The standard model $\mathfrak{A}$ of arithmetic is the natural numbers with the constant 0, the (unary) successor operation S, and the (binary) operations of addition and multiplication. The signature a of arithmetic accordingly has, in addition to the identity

[44] Naturally, there is always an unnatural effective procedure, which is to enumerate S^0-proofs until one comes across a proof of the sentence under consideration.

predicate, a nulary, a unary, and two binary operation indices. The corresponding operations on a-expressions are $\mathbf{0}$, $\mathbf{S}$, $+$, and $\cdot$. We use the usual grouping priority conventions governing addition and multiplication, e.g., $k \cdot m + n = (k \cdot m) + n$.

Axioms of PA

PA1 Succession and zero

$$\forall \mathrm{v}_0 \left(\mathbf{S}\, \bar{\mathrm{v}}_0 \neq \mathbf{0} \wedge \forall \mathrm{v}_1 \left(\mathbf{S}\, \bar{\mathrm{v}}_0 = \mathbf{S}\, \bar{\mathrm{v}}_1 \rightarrow \bar{\mathrm{v}}_0 = \bar{\mathrm{v}}_1 \right) \right).$$

PA2 Addition

$$\forall \mathrm{v}_0 \left(\bar{\mathrm{v}}_0 + \mathbf{0} = \bar{\mathrm{v}}_0 \wedge \forall \mathrm{v}_1 \left(\bar{\mathrm{v}}_0 + \mathbf{S}\, \bar{\mathrm{v}}_1 = \mathbf{S}(\bar{\mathrm{v}}_0 + \bar{\mathrm{v}}_1) \right) \right).$$

PA3 Multiplication

$$\forall \mathrm{v}_0 \left(\bar{\mathrm{v}}_0 \cdot \mathbf{0} = \mathbf{0} \wedge \forall \mathrm{v}_1 \left(\bar{\mathrm{v}}_0 \cdot \mathbf{S}\, \bar{\mathrm{v}}_1 = \bar{\mathrm{v}}_0 \cdot \bar{\mathrm{v}}_1 + \bar{\mathrm{v}}_0 \right) \right).$$

PA4 Induction

$$\forall v_0, \ldots, v_{n^-} \left(\phi\binom{v}{\mathbf{0}} \wedge \forall v \left(\phi \rightarrow \phi\binom{v}{\mathbf{S}\,\bar{v}} \right) \rightarrow \forall v\ \phi \right),$$

where ϕ is any a-formula, and $v, v_0, \ldots, v_{n^-}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.

Note that PA4 corresponds to the Foundation schema of S, which is omitted from S^0. Given the central role played by PA4 in PA, it may appear surprising that it can be omitted in S^0 without sacrificing interpretability. The explanation lies in the fact that the interpretation of PA in S^0 involves the association of numbers in the sense of PA with numbers in the sense of S^0, i.e., finite ordinals, which by the definition of *ordinal* are wellordered by $\in$ (individually and collectively). S^0 conveniently supplies all the machinery required to define *number* and the order relation on numbers, and to show that it is a wellorder—without assuming Foundation. The corresponding constructions in a theory of arithmetic would involve a predicate true of just those numbers whose predecessors are wellordered, which essentially requires an interpretation of all this machinery in PA—pretty much an interpretation of S^0 in PA in the formal sense that we are about to define. It is hard enough to do this with PA4, and it is all the harder to do it without.

We will define in PA a predicate E such that

$$\mathrm{E}^{\mathfrak{A}} = \{\langle m, n\rangle \mid \vec{B}\, m \in \vec{B}\, n\},$$

where $\vec{B} : \omega \xrightarrow{\text{sur}} \mathrm{HF}$[45] is the canonical enumeration of the hereditarily finite sets,[3.211] and showing that each axiom of S^0 becomes a theorem of PA when ⌜$\in$⌝ is replaced by ⌜E⌝.

Recall that $\vec{B}\, m \in \vec{B}\, n$ just in case 2^m is a summand in the (unique) representation of n as a sum of distinct powers of 2. To represent this in arithmetic we first provide an arithmetical definition of ⌜is a power of 2⌝. To do this we observe that n is a power of a prime p iff every factor of n other than 1 is divisible by p. Next we provide an arithmetical definition of ⌜(m) is a summand in the representation of (n) as a sum of distinct powers of 2⌝. To do this we observe that, assuming

[45] Note that HF is defined in S^0 by (3.95), rather than, in effect, (3.96.3).

m is a power of 2, this is equivalent to n/m being odd, where 'n/m' denotes integer division, the fractional part being discarded.[46] Then we have to provide an arithmetical definition of the relation $\ulcorner(m) = 2^{(k)}\urcorner$.

The details of this are not relevant to the remainder of our discussion, and we simply state the result here and relegate its proof to a note.

(2.186) **Theorem** $[\mathsf{S}^0]$ S^0 *and* PA *are equi-interpretable.*

Proof See Note 10.13. □

As pointed out at the end of Note 10.13, the interpretation given there of S^0 in PA also interprets S in PA, and also F in PA. Since F is stronger than S^0, of course, the interpretation given of PA in S^0 is also an interpretation of PA in F. These interpretations are conservative in the sense that a set-theoretical sentence θ is provable in F iff its interpretation in PA is provable in PA, and *vice versa*, so PA and F are for all practical purposes equivalent theories.

It is worth noting that we could spare ourselves quite a bit of work if we defined PA to include an additional binary operation $\uparrow$ with an axiom defining it as exponentiation:

PA5 Exponentiation

$$\forall \mathrm{v}_0 \left(\bar{\mathrm{v}}_0 \uparrow \mathbf{0} = \mathbf{S}\,\mathbf{0} \wedge \forall \mathrm{v}_1\, (\bar{\mathrm{v}}_0 \uparrow (\mathbf{S}\,\bar{\mathrm{v}}_1) = (\bar{\mathrm{v}}_0 \uparrow \bar{\mathrm{v}}_1) \cdot \bar{\mathrm{v}}_0)\right).$$

PA is, however, traditionally defined without exponentiation.

It is also worth noting that the development given here is itself not traditional. Gödel's original demonstration of the interpretability of S^0 in PA (which was not stated quite this way) relied on the representation of finite sequences of numbers by individual numbers via the Chinese remainder theorem of arithmetic.

2.10 Summary

In the previous chapter we presented language and structure as the essence of mathematics, to which it is natural to reply that surely logic is just as fundamental. In the present chapter, however, we show that logic is actually a derivative concept, the syntactic equivalent of the semantic notion of entailment, as established by the completeness theorem of Gödel. Indeed, our proof of the completeness theorem consists of positing rules of deduction with the explicit goal of defining a model for any deductively consistent theory. Proofs being finitary objects, we obtain the compactness theorem as a corollary of the completeness theorem.

We preface our discussion of logic *per se* with remarks concerning the role of infinitary objects: infinite classes being essential to the statement of the completeness theorem but not to the analysis of the deductive system that follows from it. We reiterate that the real threshold of infinitarity is the Infinity axiom, and that the class theory C^0, which is sufficient to prove the completeness theorem, is essentially a finitary theory. Pending our proof of this fact, however, we take pains to present our discussion of logic in the framework of the pure set theory S^0.[2.38]

The rules of deduction can be formulated in many ways. To describe one alternative we first define *propositional languages* and prove the corresponding completeness and compactness theorems; and we then define a system of deduction for

[46] This follows from the fact that the sum of the powers of 2 less than m is less than m ($\sum_{i=0}^{j^-} 2^i = 2^j - 1$), $m/m = 1$, and every power of 2 greater than m is divisible by m with even divisor.

first-order logic based on the application of propositional logic to theories extended by *witness sequences* and *instance classes*.

Up to this point we have restricted our attention to languages with just the two propositional connectives of negation and implication, and only the existential quantifier. The remaining common propositional connectives and the universal quantifier are now introduced with appropriate extensions to the deductive system to achieve completeness.

We define *logical equivalence* of formulas, both absolutely and relative to a theory, as well as equivalence of formulas over a structure; and we show that a change of (bound) variables maintains logical equivalence.

We then expand our scope to languages with *identity* and the interpretation convention peculiar to that predicate. Again, the deductive system is extended appropriately to achieve completeness. We define equivalence of terms analogously to equivalence of formulas, and we prove that the substitution of equivalent expressions in an expression leads to an equivalent expression. We also demonstrate the existence of various useful normal representatives within these equivalence classes.

We then turn our attention to properties of classes of sentences, i.e., *theories*, defining *deductive closure*, *extension*, and *conservative extension*, and introducing the operation of *skolemization*. We then prove Herbrand's theorem on the consistency of universal sentences. We present the easy infinitary proof, but resist the temptation to leave it at that for several reasons. First, from the standpoint of the foundations of mathematics it is important to know whether infinitarity is required. Second, it is important that the novice learn (how) to attend to this sort of issue, particularly inasmuch as once we have crossed the infinitarity threshold, beginning in Chapter 5, the same issues will arise, and there will be no option of assuming some sort of "superinfinitarity" to bail us out. Third, in proving Gödel's second incompleteness theorem we are going to make use of the fact that we have used only finitary methods up to that point. In line with the program (2.38), we present the finitary proof explicitly in S^0 (as opposed to C^0).

We then use Herbrand's theorem and the theorem on skolemization to show that logic with identity conservatively extends logic without. This fundamental theorem also has a simple infinitary proof, but we provide an S^0-proof, for reasons already given.

We then define the extension of a theory by definition, which is—from a practical standpoint—essential to the mathematical enterprise. There is an easy infinitary proof that such extensions are conservative; we provide a finitary proof.

We define the notion of *substructure*, the related syntactical operation of *relativization*, and the notion of *interpretation* of a language or theory in a theory.

At this point we digress to illustrate the foregoing ideas in the setting of geometry. We clarify the role of the axiomatic method in euclidean geometry, and we spend a fair amount of time examining metatheoretical aspects of projective geometry, which is particularly suitable for this purpose.

We have so far concentrated on the natural deduction system that we obtained from the proof of the completeness theorem. Other systems are useful for certain purposes. We briefly describe *Hilbert systems* and then focus our attention on *Gentzen systems*, which have a pleasing symmetry and which are particularly useful for the fine analysis of the structure of proofs. In particular, we prove the *cut elimination theorem*, which validates the use of the cut-free Gentzen system $\mathbf{LK}^-$ (i.e., establishes the completeness of this system), which has the key *subformula* property.

We then present some model-theoretic ideas, particularly the notion of *elementary substructure*, the method of directed elementary systems for obtaining such structures, and the celebrated Löwenheim-Skolem theorems. For this purpose we assume the Infinity axiom, so we work in the theory GB, which stands in the same relation to ZF as C^0 does to S^0. We also make use of axioms of choice as necessary, particularly the full axiom of choice AC. We introduce the method of ultrapowers, which will be particularly important in Chapter 9.

The completeness theorem asserts that every consistent theory has a satisfactory model. We now present the basic theorems relating the logical notions of provability and validity to satisfaction in structures that may not be satisfactory or even weakly satisfactory.

We conclude the chapter with a comparison of several theories that we either have used or may use as metatheories. First, we show that C^0 is a conservative extension of S^0. Again there is a simple infinitary proof, but here there is more reason than ever to provide a *bona fide* finitary proof, i.e., an S^0-proof, and we do this. Finally, we prove the equi-interpretability of S^0 and Peano arithmetic PA. We do this, not out of necessity, but rather because PA is the traditional finitary metatheory for mathematical logic—for historical reasons, not because it is particularly well suited to the task.

Chapter 3

The Theory of Membership

Contents

3.1 Introduction **169**
3.1.1 Notational conventions 170
3.1.2 Elements, classes, and sets 171
3.2 Basic theories of membership **172**
3.2.1 Cumulative hierarchies 173
3.2.2 Basic (pure) set theory 175
3.2.3 Basic theory of membership with (possibly proper) classes 178
3.2.4 Basic theory of membership with (possibly proper) elements and classes 181
3.3 Basic membership-theoretic constructs **183**
3.3.1 Abstraction terms 183
3.3.2 Union, intersection, and difference 184
3.3.3 The power operation 185
3.3.4 Functions 185
3.3.5 Ordinals 189
3.3.6 Finiteness 192
3.3.7 Finite sets and sequences 193
3.3.8 Relations 195
3.3.9 Equivalence relations 196
3.3.10 Order relations 198
3.4 Wellfoundedness **200**
3.4.1 Wellfounded relations and wellorders 200
3.4.2 Recursion 201
3.4.3 Rank and the cumulative hierarchy 202
3.5 Relativization **205**
3.5.1 Inner models 207
3.5.2 $\in$-induction and $\in$-recursion 210
3.5.3 Transitivity 212
3.5.4 Ordinal arithmetic 214
3.5.5 Ordinal sequences 215
3.5.6 Choice functions and wellorderability 215
3.6 Size **216**

3.6.1 Cardinality 216

3.6.2 Infinity 220

3.7 Extended theories of membership 222

3.8 Principles of choice and wellordering 223

3.8.1 Restricted axioms of choice 225

3.8.2 Countability 225

3.8.3 Cardinals and cofinality 226

3.9 Cardinal arithmetic 228

3.9.1 Cardinal exponentiation 231

3.9.2 The singular cardinals problem 233

3.10 More basic constructs 233

3.10.1 Lattices 233

3.10.2 Boolean algebras 235

3.10.3 Stationarity 238

3.10.4 Normality 240

3.10.5 Trees 242

3.11 Topology 245

3.11.1 Subspaces and relative topologies 246

3.11.2 Generating topologies 246

3.11.3 Separation and countability properties 247

3.11.4 Continuity 248

3.11.5 Homeomorphism 248

3.11.6 Product topology 248

3.11.7 Convergence 250

3.11.8 Compactness 251

3.11.9 Directed sets and direct limits 252

3.12 Finitary set theory 253

3.12.1 A canonical enumeration of HF 254

3.13 Inner models 255

3.13.1 The incompleteness of S, etc. 257

3.14 Summary 259

> Classes and concepts may, however, also be conceived as real objects, namely classes as 'pluralities of things' or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions. It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions...
>
> Kurt Gödel
> *Russell's Mathematical Logic* in *The Philosophy of Bertrand Russell* (1944), Vol. 1, 137, P. A. Schilpp (ed.)

> After we came out of the church, we stood talking for some time together of Bishop Berkeley's ingenious sophistry to prove the non-existence of matter, and that every thing in the universe is merely ideal. I observed, that though we are satisfied his doctrine is not true, it is impossible to refute it. I never shall forget the alacrity with which Johnson answered, striking his foot with mighty force against a large stone, till he rebounded from it, 'I refute it thus.'
>
> *Life of Samuel Johnson*
> by James Boswell

[Thomas Jech's *Set Theory*[12] is an excellent source for all aspects of the modern theory of membership.]

3.1 Introduction

Mathematics talks about many different sorts of entities—e.g., curves, transformation groups, fiber bundles, you name it. Many a branch of mathematics is defined by the objects it discusses: the theory of groups, the theory of polynomials, etc. Certain sorts of objects, however, occur in almost every mathematical discussion. Among these are natural numbers, sequences, functions, relations, and sets, to name several. For quite a while, these objects were used uncritically, even in areas of mathematics like geometry, whose logical structure had been a object of scrutiny for millennia.[§ 2.5] This approach worked well enough as long as the applications of these ideas were relatively simple. It was only after Cantor's analysis of Fourier series led him to consider iterative constructions that continued *ad infinitum* and then kept on going that the necessity of a formal development of the theory of membership was recognized. This development eventually showed, among other things, that all of the basic constructions alluded to above, such as natural numbers, sequences, etc., can be defined in terms of membership. The theory of membership is thus the *lingua franca* of mathematics, and any branch of mathematics implicitly includes some part of this theory.[1]

Our goal in this book is a critical examination of language, logic, and the foundations of mathematics. The objects of our interest are therefore such linguistic entities as symbols, terms, sentences; such logical entities as proofs and theories;

[1] Other very general mathematical theories, such as the theory of categories and functors, may be used instead of the theory of membership as a universal framework for mathematics. Each has its merits, but the concept of membership is so simple that it has emerged as the consensus choice for this purpose.

and such mathematical entities as groups, rings, manifolds, etc., all of which we treat generically as *structures*. Our approach to this task is the same as that of any mathematician dealing with any other subject, and we have the same need for membership-theoretic concepts.

Our present concerns, however, go to the heart of the mathematical enterprise: the meaning of mathematical statements, the validity of mathematical deduction, the consistency of mathematical theories, etc. If we are to have faith in our conclusions—indeed, if we are to understand the logical status of our analysis—we must make the framework of that analysis explicit.

The basic relations of the theory of membership are those of *membership* and *identity*. In typographical languages we use the symbol '$\in$' to indicate membership: '$x \in y$' means 'x is a member of y' or, equivalently, 'x belongs to y' or 'y contains x'. As for any theory, in typographical languages we use the symbol '$=$' to indicate identity: '$x = y$' means 'x is y'. Note that identity is a more fundamental notion than membership, one that is often regarded as belonging to logic and which is essentially ubiquitous, not only in mathematics but also in ordinary discourse.

3.1.1 Notational conventions

We will tolerate a somewhat greater degree of notational imprecision in this chapter than in the previous chapters. For example, we may use the name of a variable, say 'v' to denote the term formed from the variable, i.e., $\bar{v}$ $(= \tilde{v}0)$, as in the following paragraph, which introduces another convention.

In general, substitution of terms for variables in a formula ϕ may be indicated as follows. An occurrence of 'ϕ' followed by a series of terms flanked by round brackets is understood to stand for the formula ϕ with the displayed terms substituted for its free variables in some fixed order. The free variables are typically listed in order when the name, 'ϕ' in this case, is first introduced. For example, if we say that ϕ is $\phi(v, a, v_0, \ldots, v_{n^-})$,[1.28] *then*

$$\phi(x, y, z_0, \ldots, z_{n^-}) = \phi\binom{v\ a\ v_0\ \cdots\ v_{n^-}}{x\ y\ z_0\ \cdots\ z_{n^-}}.$$

The following abbreviations are frequently useful.

(3.1)

1. *If τ is a term, u is a variable that does not occur in τ, and ϕ is a formula,*
   1. $\exists u \in \tau\, \phi \stackrel{\text{def}}{=} \exists u\, (u \in \tau \wedge \phi)$
   2. $\forall u \in \tau\, \phi \stackrel{\text{def}}{=} \forall u\, (u \in \tau \to \phi)$
2. *Similarly, the use of a unary predicate symbol as a subscript on a quantifier indicates restriction to entities satisfying the predicate. For example, $\ulcorner \exists_P x \ldots \urcorner$ means $\ulcorner \exists x\, (Px \wedge \ldots) \urcorner$, and $\ulcorner \forall_P x \ldots \urcorner$ means $\ulcorner \forall x\, (Px \to \ldots) \urcorner$.*
3. *For any terms τ_0, τ_1, τ_2,*

$$\tau_0 \in \tau_1 \in \tau_2 \stackrel{\text{def}}{=} (\tau_0 \in \tau_1 \wedge \tau_1 \in \tau_2).$$

4. *Generalizations of (3) to chains of any length of any binary relation.*
5.
   1. $\tau_0 \neq \tau_1 \stackrel{\text{def}}{=} \neg\, \tau_0 = \tau_1$.
   2. $\tau_0 \notin \tau_1 \stackrel{\text{def}}{=} \neg\, \tau_0 \in \tau_1$.
   3. *Strikeout may be used similarly to negate any binary relation.*

3.1.2 Elements, classes, and sets

In a theory of membership we distinguish several sorts of entities.

(3.2) **Definition** *Something is an* element $\overset{\text{def}}{\iff}$ *it is a member of something.*

(3.3) *If something has a member we say it is a* class.

(3.4) *A defining feature of the notions of membership and class is that a class is uniquely determined by its members, i.e., if x and y are classes, and if, for all elements z, $z \in x \iff z \in y$, then $x = y$.*

Note that the converse is a theorem of pure logic: if $x = y$, i.e., if x is y, then anything is true of x iff it is true of y, in particular, for any z, $z \in x$ iff $z \in y$.

Note that (3.3) does not say that every class has a member. On the contrary, in a theory of membership it is very handy and natural to require the existence of at least one class that has no members; to do otherwise simply entails a great deal of circumlocution that serves no purpose. Note that by virtue of (3.4) there is only one empty class.

Definition *A* proper class *is a class that is not an element. A* proper element *or* urelement *or* atom *is an element that is not a class. A* set *is something that is both an element and a class.*

In a theory of membership we may disregard any entity that has no members and is not a member of anything, as the existence of such an object does not imply anything about the membership relation other than the simple fact that something exists that does not participate in it.

In a theory of membership we therefore suppose that everything is either an element[3.2] *or a class*[3.3] *(or both, i.e., a set). We use 'E', 'C', and 'S' as necessary to refer to elements, classes, and sets, respectively. In particular, we use these symbols as subscripts on quantifiers and as predicates. For example, '$\forall_E$' means 'for every element', and '$C(x)$' means 'x is a class'.*

In any reasonable theory of membership, sets exist; indeed, in any reasonable theory of membership, the membership relation has a rich structure when restricted to sets, and in general sets (as opposed to proper elements and proper classes) are the most useful objects in a theory of membership.

By *pure set theory* we mean a theory in which everything is a set. Pure set theory is a mathematical subject in its own right. Pure set theory may also be regarded as subsuming all of mathematics, inasmuch as all mathematical constructions may be regarded set-theoretically, and there is no need to suppose that there exists anything that is not a set.

Alternatively, one may allow for the existence of things other than sets. There is no *a priori* restriction on what the "other things" might be. Apropos the discussion of the preceding chapters, they might be typographical expressions; in physics, they might be physical systems, states of systems, operations on states, etc.; in sociology, they might be people.

On the other hand, at first blush it is hard to see why we should ever have a need for proper classes. Given a class C, can't we always form a class that contains C? Not necessarily. One of the early discoveries in the history of the theory of membership is that there must be some restriction on the formation of classes. The simplest example is that of Russell. Suppose there is a class A whose members are exactly the classes x such that $x \notin x$, i.e., x is not a member of x. If $A \in A$, then, by the definition of A, $A \notin A$. On the other hand, if $A \notin A$, then, again by the definition of A, $A \in A$. This is a logical contradiction.

Thus, any theory of membership that implies the existence of a class whose members are exactly those classes that are not members of themselves is inconsistent and useless for any purpose. There are several responses to this realization. One is to throw up ones hands in despair and forget about constructing any consistent theory of membership. This is not acceptable.

To arrive at a satisfactory theory of membership we must impose some restriction on the *comprehension* operation, which forms the class consisting of all sets satisfying a given condition: the class *comprehended* by the condition (in the Russell paradox the condition is $u \notin u$). In general, a comprehension axiom asserts that there exists a class whose members are the elements satisfying a given condition, and it may also assert that this class is an element, in which case it is a set. In a pure set theory, every class is a set, so the burden of avoiding paradox falls entirely on limiting class comprehension. The Zermelo-Fraenkel theory ZF accomplishes this by using comprehension only to assert the existence of classes that are included in sets. Alternatively, Quine's New Foundation theory NF imposes the restriction that a membership condition must be stratifiable, i.e., each of its variables u can be assigned a rank $fu \in \omega$ such that in any subformula $u \in v$, $fv = fu + 1$, and in any subformula $u = v$, $fv = fu$[19]. Note that $\bar{u} \in \bar{u}$ cannot be stratified, so the Russell paradox is avoided. The Zermelo-Fraenkel approach is standard, and we will not consider NF any further in this book.

In a class theory, we may be more generous with class comprehension, as long as we are sufficiently restrictive as to when a class may be asserted to be a set. In such a theory, if we define A as the class of sets x such that $x \notin x$, then if we suppose A is a set we can derive a contradiction as before. We may therefore conclude that A is a proper class. In standard theories of membership—including all those in which the membership relation is wellfounded in a sense to be made precise—A as just defined consists of all sets. In theories of this type, classes like A exist, but they are "too big" to be sets.

(3.5) *In light of the preceding discussion we have four types of theories of membership:*

1. *Everything is a set, i.e., there are no proper elements or proper classes. This is* pure set theory*, of which we take* ZF *as the paradigm, as remarked above.*
2. *There are proper elements, but no proper classes. This is standard set theory with atoms (urelements), e.g.,* ZFA.
3. *There are proper classes, but no proper elements. The Gödel-Bernays and Morse-Kelly theories are of this type.*
4. *There are proper elements and proper classes.*

'Set theory' is used loosely to refer to any theory of membership.

3.2 Basic theories of membership

In the following discussion of theories of membership we will use the concepts and conventions of Chapters 1 and 2. As discussed there, this chapter and those are interdependent, inasmuch as those chapters employ the basic theory of membership, while this chapter requires some understanding of the nature of language and logic.

A language appropriate to a theory of membership has a binary predicate symbol '$\in$', in addition to the identity symbol '$=$'. We could naturally define the language of an all-purpose theory of membership as multi-sorted, with domains of elements, sets, and classes (together with a universal domain encompassing those), but in the interest of simplicity (and, in the long run, flexibility), we will instead posit three unary predicates with the symbols 'E', 'S', and 'C', respectively.

(3.6) **Definition** *For the sake of definiteness, and for future reference, we define the core similarity type for theories of membership to be* $\mathsf{s} = [\Pi^{\mathsf{s}}, \Phi^{\mathsf{s}}, T^{\mathsf{s}}]$,[1.29] *where*

1. $\Pi^{\mathsf{s}} = \{0, 1\}$*;*
2. $\Phi^{\mathsf{s}} = 0$*; and*
3. $T^{\mathsf{s}}(0) = T^{\mathsf{s}}(1) = 2$*.*

0 indexes the identity relation, as always, and 1 indexes the membership relation.

The standard s-language $\mathcal{L}^{\mathsf{s}}$ is defined as in Section 1.3.1. We grant ourselves the privilege of introducing additional predicates and operations *ad libitum* in the usual way of mathematics. To allow the introduction of operations by definition we have not defined s as a purely relational similarity type.[1.29] The class Φ of operation indices is empty now, but it is allowed to grow.

$\mathcal{L}^{\mathsf{s}}$, with the provision of urelements and the axioms to be given, is capable of discussing virtually anything mathematicians discuss. For example, we might let the urelements be the real numbers. We could let the sets be all collections of real numbers, collections of collections of real numbers, etc. To explain what we mean by 'etc.' here, suppose we assign rank 0 to any collection of real numbers, rank 1 to any collection of collections of real numbers, and, in general, rank $\alpha + 1$ to any collection of collections of rank α. Then 'etc.' could comprehend all collections of finite rank. These, we have said, would be the sets. The elements would be the real numbers and the sets. As for the classes, we could simply stipulate that all classes are sets. Alternatively, we could posit the existence of proper classes, which would be collections (of elements) other than the sets as just defined. The collection consisting of all the sets is one example of such a collection.

3.2.1 Cumulative hierarchies

The preceding example indicates the general sort of structure that set theory is intended to describe. We begin with a collection of urelements (proper elements, i.e., things that are not sets), which may be empty. Let V_0 be this collection. Let V_1 consist of all elements of V_0 and all subcollections of V_0 that are sets. It may be that all subcollections of V_0 are sets, or it may be that only some are. For example, we may wish to require that sets be finite, whereas V_0 may be infinite. We will use the phrase 'subset' (of a collection) to refer to a subcollection that happens to be a set. Thus, V_1 is V_0 together with all subsets of V_0. Let V_2 be V_1 together with all subsets of V_1. Similarly, let V_3 be V_2 together with all subsets of V_2, and so on.

The result of this construction is a *cumulative hierarchy*—specifically, the *von Neumann hierarchy*: $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$, where $V_{\alpha+1}$ consists of all sets included in V_α, along with the urelements if there are any. 'Cumulative' is used to indicate that each level (or stage) of the hierarchy includes each of the preceding levels.

Often, each level also contains[3.7] each of the preceding levels, but not always, e.g., when there are infinitely many atoms and sets are all finite. We define the *rank* of a set to be α if it first appears in level $V_{\alpha+1}$. Note that this is also the least α such that V_α includes the set.[2]

(3.7) *This is a good place to establish the following convention:*

1. a contains b *iff a is a class and b is a member of a.*
2. a includes b *iff a and b are classes and every member of b is a member of a.*

The reader is warned that many authors use 'contain' for both 'contain' and 'include', with the expected attendant confusion, which is addressed by saying 'contains as a member' when 'contain' is intended and the context does not permit the reader to immediately infer this.

We have indicated how the levels V_α are generated for natural numbers α, but the hierarchy may continue beyond this. For example, we may have a set with members of arbitrarily high (finite) rank. We assign such a set rank ω, where ω is the order type of the natural numbers. (We will make all this official in due course. The purpose of this discussion is to provide a conceptual framework for the formal presentation of the theory.) We define V_ω to be the union of the V_α's for finite indices α, and $V_{\omega+1}$ to be V_ω together with the collection of subsets of V_ω. Note that no set first appears in V_ω, because any set in V_ω is in V_α for some finite α. As above, the sets of rank ω are those in $V_{\omega+1} \backslash V_\omega$.

We may continue in this way as long as we wish—up to $\omega \cdot 2 \stackrel{\text{def}}{=} \omega + \omega$, $\omega \cdot 3$, $\omega \cdot \omega$, ω^ω, ... You might wish to take a moment to try to imagine how long this could go on. The limit of human imagination is puny compared to the theoretical limit of any sentient being's imagination, which is in turn puny compared to the limit of just the countable stages of this process, which is called 'ω_1', ω being called for this purpose 'ω_0'. Far beyond this is ω_2, then $\omega_3, \ldots, \omega_\omega, \ldots, \omega_{\omega_1}, \ldots$. Beginning to get the picture?[3]

We will refer to the order type of the entire hierarchy as 'Ω',[4] and to the collection consisting of all the elements (proper elements and sets) as 'V_Ω' or simply as 'V'. The only subcollections of V that we regard as sets are those that are in some V_α, with $\alpha < \Omega$. There are therefore no sets of rank Ω and there is no level $V_{\Omega+1}$. If we wish to admit proper classes, they are subcollections of V that are not in V.

If we admit only finite sets, $\Omega = \omega$, the order type of the natural numbers. In the standard axiomatization of set theory with infinite sets, however, Ω is stupendously large, as we indicated above. Note that in set theory without proper classes, 'Ω' and 'V' are not terms of the theory: they don't denote sets (or urelements), so

[2]An alternative to the von Neumann hierarchy is to construct an object of type $\alpha + 1$ for each collection of objects of type α. Objects of different types are regarded as different even if they have the same members. This approach has no advantage, and it has the disadvantage of distinguishing entities that are better identified—for example, there is no utility in distinguishing the empty collection of objects of one type from the empty collection of objects of any other type. This method was used by Russell and Whitehead in their *Principia Mathematica*[21] as a way of developing a theory of set-like objects that avoided the inconsistency of early set theories, but it is now a cumbrous historical artifact.

[3]Note, however, that if, as suggested above, we wish to restrict our attention to finite sets, then the cumulative hierarchy consists of just the finite levels—any finite subset of V_ω is included in V_α for some finite α and is therefore in $V_{\alpha+1}$. Thus we cannot obtain any new sets by taking finite subsets of what we already have in V_ω.

[4]After we have defined 'ordinal', we will refer to Ω as 'Ord', freeing 'Ω' for other, less specific, uses.

they don't denote anything. In this case they are to be viewed as figures of speech, i.e., expressions containing them are interpreted as standing for expressions without them.

We are thus presented with the following all-purpose picture of a model of set theory (by which we mean a theory of membership generally[3.5]). At the bottom is a collection V_0, consisting of urelements. If there are no urelements, then V_0 is the empty collection, and we will impose axioms to ensure that this is a set, i.e., that there exists a set with no members. Including V_0 and extending "above" it is a collection V_1. The additional elements are the subsets of V_0. Note that if there are no urelements, then V_0 is the empty set, and $V_1 \backslash V_0 = V_1$ contains just one thing—the empty set. Next we have $V_2 \subseteq V_3 \subseteq \cdots$, as described above. This goes on for Ω steps. Finally, if we wish, we may have a stratum at the very top consisting of the proper classes, which are subclasses of $V \overset{\text{def}}{=} V_\Omega$. By definition, proper classes are not elements, and they cannot be members of anything, so the hierarchy does not continue beyond this point, and there is in particular no $V_{\Omega+1}$. All proper classes are included in V_Ω but in no smaller V_α, so they naturally may be considered to have rank Ω.

3.2.2 Basic (pure) set theory

In the interest of simplicity, we will begin with a theory S that does not have proper elements or proper classes, i.e., everything is a set. The axioms asserting the existence of sets are only those that are essential for a useful theory.[5]

(3.8) **Axioms of S**

S1. Extension

$$\forall \mathrm{v}_0, \mathrm{v}_1 \,(\forall \mathrm{v}_2 \,(\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1) \rightarrow \mathrm{v}_0 = \mathrm{v}_1)$$

S2. Comprehension This is actually an infinite collection of axioms, all of the same form, i.e., an *axiom schema.*

$$\forall v_0, \ldots, v_{n^-} \,\forall u \,\exists w \,\forall v \,(v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is any s-formula, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such Free $\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.[6]

S3. Existence

$$\exists \mathrm{v}_0 \,\forall \mathrm{v}_1 \;\; \mathrm{v}_1 \notin \mathrm{v}_0$$

S4. Pair

$$\forall \mathrm{v}_0, \mathrm{v}_1 \,\exists \mathrm{v}_2 \,(\mathrm{v}_0 \in \mathrm{v}_2 \wedge \mathrm{v}_1 \in \mathrm{v}_2)$$

S5. Collection This is also an axiom schema.

$$\forall v_0, \ldots, v_{n^-} \,\forall u \,\big(\forall v \in u \,\exists w \,\forall a \,(\phi \rightarrow a \in w) \rightarrow \exists w \,\forall v \in u \,\forall a \,(\phi \rightarrow a \in w)\big),$$

where ϕ is any s-formula, and $a, u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that Free $\phi \subseteq \{a, v, v_0, \ldots, v_{n^-}\}$.

[5] S is the standard *Zermelo-Fraenkel theory* ZF with the axioms of powerset and infinity omitted.

[6] If $n = 0$, the initial quantifier string indicated by '$\forall v_0, \ldots, v_{n^-}$' is the empty string, so it is—in effect—not there.

S6. Foundation Another schema.

$$\forall v_0, \ldots, v_{n^-} \left(\exists v\, \phi \rightarrow \exists v \left(\phi \wedge \forall u \in v\ \neg\, \phi\tbinom{v}{u}\right)\right),$$

where ϕ is any s-formula, $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that Free $\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$, and u is free for v in ϕ.

We have included Foundation among the basic axioms of the theory of membership, but it is of a different character, inasmuch as it is restrictive, rather than expansive, and for the time being we will develop the theory without it.

(3.9) **Definition** S^0 *is the theory with axioms* S1–5*, i.e., omitting* Foundation.

Let's examine these axioms with particular attention to our intuitive image of a cumulative hierarchy.

S1 expresses the essential nature of the membership relation, viz., that the identity of a set depends exclusively on what is "in" it.[7] There is no possibility of distinct sets with the same members. Note that the converse of S1, by which we mean the statement $\ulcorner \forall x, y\, (x = y \rightarrow \forall z\, (z \in x \leftrightarrow z \in y))\urcorner$, is a validity of predicate logic with identity.

S2, also known as the Aussonderungsaxiom (*selection* axiom) or the axiom of subsets, says that for any formula ϕ and distinct variables u and v, for any values of u and the remaining free variables of ϕ other than v, the collection denoted by the abstraction term $\ulcorner\{(v) \in (u) \mid (\phi)\}\urcorner$, is a set.[8] In a cumulative hierarchy, it is reasonable to require that whenever a set x is created, all subcollections of x are also created as sets. We have, however, no way of saying directly that all subcollections of x are sets. If we try to do so with a statement like 'for all y, if y is a subcollection of x then there exists z such that $z = y$', we have merely uttered a tautology: clearly, for all y, no matter what condition we put on y, there exists z such that $z = y$, viz., y. The problem is that in set theory we cannot "quantify" over all "collections", only over all sets.[9] The expressive constraints of our language allow us to say only that all *definable* subcollections of x are sets, and this is what S2 does (or rather, as the British would say, 'what S2 *do*';[10] remember that S2 is a schema—a collection of sentences with a common form).

Note that the justification of S2 by means of the cumulative hierarchy picture requires the presence of the condition $v \in u$, i.e., we do not say that any definable collection is a set, only that any definable subcollection of a set is a set. Leaving aside the issue of justification in terms of cumulative hierarchies, why should we not omit this condition and simply posit that *any* abstraction term defines a set? There are two reasons. The first is that we may wish to specialize the theory to sets with some characteristic, e.g., finiteness. The second reason is more fundamental. As we have already seen, the abstraction term $\ulcorner\{x \mid x \notin x\}\urcorner$ cannot denote a set.

S3 gets the cumulative hierarchy started. This is the only axiom that is not a universal sentence, and is therefore the only axiom that requires that something exist. We have written it so that it mandates the existence of specific set, viz., the

[7] This is true of classes in general. Remember that for the present, we are supposing that all classes are sets.

[8] The use of abstraction terms is probably familiar to the reader. We will treat them formally in Section 3.3.1.

[9] This is true for pure set theory, which is the present context. In an appropriate theory of classes, we can replace all the instances of S2 by a single sentence involving quantification over classes. We don't necessarily get a stronger axiom in this way.

[10] Cf., 'the royal family *do* their best to preserve the dignity of the monarchy'.

empty set, and it is also called the Empty set axiom; but we could have made it less specific—for example, $\exists \mathrm{v}_0\, \mathrm{v}_0 = \mathrm{v}_0$. Note that the existence of an empty set follows by S2 from the existence of any set. The empty set has rank 0.

By virtue of S1, there is only one empty set so the following definition is legitimate.

(3.10) **Definition** [S^0] $0 \stackrel{\text{def}}{=}$ *the* empty set.

This is an example of extension by definition,[2.107.2] i.e., we add a new constant symbol '0' and a new axiom

$$\ulcorner \forall a\, a \notin 0 \urcorner$$

to our language and theory, respectively. The new theory is a conservative extension of the original.[2.108.2]

Another way to regard the definition of '0' is to view any formula involving '0' as an abbreviation for a formula without '0'. Specifically, we would regard a formula ϕ containing one or more occurrences of '0' to be an abbreviation for a formula $\exists u(\forall v(v \notin u) \wedge \psi)$, where u is a variable that does not occur in ϕ, and ψ is the result of replacing '0' everywhere by u.[11] For metatheoretical purposes this approach to definitions is sometimes preferable, so that we may consider S^0—or any other theory—as a fixed theory in a fixed language. There is, however, no need to maintain this stance throughout: the general theory of definition[§ 2.4.5] provides an effective procedure to eliminate defined symbols from any theory.

S4 says that, given any sets x and y, there is a set that contains x and y. S2 allows us to show that there is a set that contains x and y and nothing else, i.e.,

$$\forall a, b\, \exists c\, \forall x(x \in c \leftrightarrow (x = a \vee x = b)). \tag{3.11}$$

Another way to put it is: 'for any a and b, $\{x \mid x = a \vee x = b\}$ exists (i.e., is a set)'. By S1, c is uniquely determined in (3.11), so the following definition is well made.

(3.12) **Definition** [S^0] $\{a, b\} \stackrel{\text{def}}{=}$ *the* pair of a and b $\stackrel{\text{def}}{=}$ *the set that contains a and b and nothing else.*

The following special case is useful:

Definition [S^0] $\{a\} \stackrel{\text{def}}{=} \{a, a\}$. *We call this* the singleton of a *or just* singleton a.

In terms of a cumulative hierarchy, $\{x, y\}$ will appear in the next level after both x and y have appeared. Again, we have control over which collections to admit as sets, and no higher law compels us to accept all pairs of sets as sets, but we don't get a very useful concept of set if we don't. Hence we adopt S4.

S5 is a schema, like S2. It says the following: $\ulcorner$Let $(v_0, \ldots, v_{n^-}, u)$ be given. Suppose that for every $(v) \in (u)$ there is a set that includes the collection $\{(a) \mid (\phi)\}$.[12] Then there is a set that includes the collection $\{(a) \mid (\phi)\}$ for every $(v) \in (u)$.$\urcorner$

Let us say for the nonce that any collection included in a set is *set-sized*. For fixed $v_0, \ldots, v_{n^-}$, the collections $\{a \mid \phi(v, a, v_0, \ldots, v_{n^-})\}$ form a family indexed by v. Thus, under the hypothesis of S5, as v ranges over the set u, we have—if you will—*set-many* set-sized collections defined by ϕ, and each instance of S5 says that the

[11] It does not matter what variables we choose for u and v, but if we wanted to provide an explicit rule we could.

[12] Note that this is only interesting if a and v occur free in ϕ.

union of a certain definable family of set-many set-sized collections is set-sized. As in the case of S2, the expressive limitations of our language restrict us to definable families. We cannot talk about arbitrary families of collections of sets.

Foundation, also known as the axiom of regularity, is the best we can do to ensure that the universe of sets that we're talking about really is a hierarchy such as we've described. First let's persuade ourselves that it's true of such a universe of sets. Suppose $\phi(v, v_0, \ldots, v_{n^-})$ is a formula with the free variables shown. Let $v_0, \ldots, v_{n^-}$ have fixed values, and let X be the collection of sets x such that ϕ holds with $v = x$. Suppose X is nonempty. Let α_0 be the least index α such that V_α contains a member of X. Let x be any member of X in V_{α_0}. Then for any $y \in x$, y is in some V_α with $\alpha < \alpha_0$, so y cannot be in X. This justifies the axiom. Note that this "proof" depends on the levels being wellordered, i.e., in any nonempty collection of levels there is a least, or first, one.

Note that axioms of S^0 make sense in the absence of Foundation, and we may imagine a universe V of sets larger than that of the cumulative hierarchy in which S^0 holds but Foundation may fail. Within V we may define V_Ω and show that S^0 and Foundation hold within it. In the context of S^0, we refer to V_Ω as an *inner model* of S, i.e., S^0 + Foundation. We will use this relatively easy exercise later in this chapter[3.103] as our introduction to the *metatheory* of membership.

3.2.2.1 Ordered pairs

Before moving on to the consideration of proper classes, we pause to define another useful operation.

(3.13) **Definition** [S^0] *For sets x and y, the* ordered pair of x and y $\stackrel{\text{def}}{=} (x, y)$ $\stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$.

(3.14) **Theorem** [S^0] $(x, y) = (x', y') \leftrightarrow (x = x' \wedge y = y')$.

Proof The $\leftarrow$ direction is a validity of logic with identity. To prove the $\rightarrow$ direction, suppose $(x, y) = (x', y')$, i.e.,

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

We first observe that $\{x'\} = \{x\}$ or $\{x'\} = \{x, y\}$. In the first case $x' = x$, and in the second case $x' = x = y$; in either event, $x' = x$. Next we note that $\{x', y'\} = \{x\}$ or $\{x', y'\} = \{x, y\}$. Since $x' = x$, either $\{x, y'\} = \{x\}$ or $\{x, y'\} = \{x, y\}$. In the second case, $y' = y$, and we are finished. In the first case, $y' = x$, so now, since $x' = x$,

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} = \{\{x\}, \{x, x\}\} = \{\{x\}\},$$

from which it follows that $\{x, y\} = \{x\}$, so $y = x$, so $y = y'$. □[3.14]

3.2.3 Basic theory of membership with (possibly proper) classes

We next define a theory C that extends S by permitting the existence of proper classes.[13] Proper elements are still excluded; thus, everything is a class. A class is a set iff it is a member of something.

[13] C is the *Gödel-Bernays theory* GB with the axioms of powerset and infinity omitted.

(3.15) **Definition** *We define* c *to be* s *with one additional unary predicate, represented typographically by 'S' and signifying that its argument is a set (as opposed to a proper class).*

Definition *Quantification* restricted to sets *or* to S *is frequently used.*

$$\exists_S v\,\phi \overset{\text{def}}{=} \exists v\,(\boldsymbol{S}v \wedge \phi)$$
$$\forall_S v\,\phi \overset{\text{def}}{=} \forall v\,(\boldsymbol{S}v \to \phi).$$

Axiom C0 states that the predicate S is true of exactly the sets.

(3.16) **Axioms of** C

C0.

$$\forall \mathrm{v}_0\,(\boldsymbol{S}\mathrm{v}_0 \leftrightarrow \exists \mathrm{v}_1\ \mathrm{v}_0 \in \mathrm{v}_1)$$

C1. Extension

$$\forall \mathrm{v}_0, \mathrm{v}_1\,(\forall \mathrm{v}_2\,(\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1)) \to \mathrm{v}_0 = \mathrm{v}_1)$$

C2a. (Predicative) Comprehension

$$\forall v_0 \cdots \forall v_{n^-}\,\exists w\,\forall_S v\,(v \in w \leftrightarrow \phi),$$

where ϕ is any c-formula with all quantifiers restricted to S, and $v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.

C2b. Separation

$$\forall_S \mathrm{v}_0\,\forall \mathrm{v}_1\,\big(\forall \mathrm{v}_2\,(\mathrm{v}_2 \in \mathrm{v}_1 \to \mathrm{v}_2 \in \mathrm{v}_0) \to \boldsymbol{S}\mathrm{v}_1\big)$$

C3. Existence

$$\exists_S \mathrm{v}_0\,\forall \mathrm{v}_1\ \mathrm{v}_1 \notin \mathrm{v}_0$$

C4. Pair

$$\forall_S \mathrm{v}_0, \mathrm{v}_1\,\exists_S \mathrm{v}_2\,(\mathrm{v}_0 \in \mathrm{v}_2 \wedge \mathrm{v}_1 \in \mathrm{v}_2)$$

C5. Collection

$$\forall \mathrm{v}_0\,\forall_S \mathrm{v}_1\big(\forall \mathrm{v}_2 \in \mathrm{v}_1\,\exists_S \mathrm{v}_3\,\forall \mathrm{v}_4\,\big((\mathrm{v}_2, \mathrm{v}_4) \in \mathrm{v}_0 \to \mathrm{v}_4 \in \mathrm{v}_3\big) \\ \to \exists_S \mathrm{v}_3\,\forall \mathrm{v}_2 \in \mathrm{v}_1\,\forall \mathrm{v}_4\,\big((\mathrm{v}_2, \mathrm{v}_4) \in \mathrm{v}_0 \to \mathrm{v}_4 \in \mathrm{v}_3\big)\big).$$

C6. Foundation

$$\forall \mathrm{v}_0\,(\exists \mathrm{v}_1 \in \mathrm{v}_0 \to \exists \mathrm{v}_1 \in \mathrm{v}_0\,\forall \mathrm{v}_2 \in \mathrm{v}_1\ \mathrm{v}_2 \notin \mathrm{v}_0)$$

As already noted, C0 is the definition of 'S'. Indeed, c could be regarded as an extension-by-definition of s, with C0 treated as a definition, rather than an axiom.

(3.17) *We will proceed for the time being on the basis of the theory* C^0, *which we define as* C0–5,[3.9] *deferring the incorporation of* C6 *until we have formally defined the inner model* V_Ω.

C1 is the same extension axiom as for sets: a class is uniquely determined by its members.

C2a is similar to S2 in that it says that formulas define classes with certain restrictions. In the case of S2 the restriction was that the elements satisfying the formula all had to be members of some pre-existing set. This prevented us from asserting the existence of the Russell set $\{x \mid x \notin x\}$, for example. In C2a we remove this restriction without risk of resurrecting the Russell paradox, as C2a only says that $\{x \mid x \notin x\}$ is a *class*, the class of all *sets* that are not members of themselves (which happens to be the class of all sets if we assume Foundation, but no matter). It is not a set, so the fact that it is not a member of itself does not imply that it is in $\{x \mid x \notin x\}$, and there is no contradiction.

The Gödel-Bernays version of the class comprehension schema is *predicative* in that it imposes the restriction on ϕ that its quantified variables are restricted to S. (Note that this does not apply to the "parameters" $v_0, \ldots, v_{n^-}$.) This version is sufficient for our purposes and has the advantage that it gives a theory of classes that is a conservative extension of S^0.[2.183] If we drop this restriction we have the *Morse-Kelly theory of classes*.

(3.18) *We call a theory of membership a* class theory $\overset{\text{def}}{\iff}$ *it admits proper classes. A class theory is* predicative $\overset{\text{def}}{\iff}$ *its class comprehension axiom is* C2a*.*

Given C2a*, abstraction terms*[§ 3.3.1] *become even more useful: as long as ϕ has only set-restricted quantification,*

$$\mathsf{C}^0 \vdash \ulcorner \forall_S (v) \, ((v) \in \{(v) \mid (\phi)\} \leftrightarrow (\phi))\urcorner.$$

Note that there is a class that contains all sets.

(3.19) **Definition** [C^0] *The* set-theoretical universe $\overset{\text{def}}{=} V \overset{\text{def}}{=}$ *the class of all sets.*

The use of 'V' for this purpose is traditional. In a class theory we often use the same symbol for a unary predicate applicable to sets and for the class of sets that satisfy it. By this convention, we would use 'S' for the universe of sets, or we would use 'V' as the predicate symbol for setness, but in the interest of clarity we choose to keep these notations distinct.

Given that C2a only asserts the existence of classes, we need another axiom to obtain the set-existence content of S2. C2b suffices for this.[14] C2a and C2b together subsume S2, as the following proof schema shows:

$\ulcorner$Given a set u, to show that $\exists_S w \, \forall v \, (v \in w \leftrightarrow (v \in u \wedge (\phi)))$, let $w = \{v \mid v \in u \wedge (\phi)\}$ (using C2a), and then use C2b to conclude that since the class w is included in the set u, w is a set.$\urcorner$

C2a is the only axiom (schema) of C^0 that asserts the existence of a class without also asserting that it is a set. Note that C4 and C5 would have no force if they omitted the requirement that the class whose existence is asserted be a set; otherwise V, the class of all sets,[3.19] would trivially satisfy the condition.

[14] We have called C2b the Separation axiom, but it is really only in conjunction with C2a that it is this, as it then says that any collection "separated" from a set by means of a formula (with quantification restricted to S) is a set. Standing alone, C2b could be called the Setness axiom, as it simply says that a class that is "small enough to fit in a set" is a set. S2 could also be called (and often is called) the Separation schema: in pure set theory, one cannot make the assertion of comprehension except in the context of separation.

Note that $\mathsf{C5}$ is a single axiom. The direct translation of the $\mathsf{Collection}$ schema of S^0 would consist of the sentences

$$\forall v_0 \cdots \forall v_{n\text{-}}\, \forall_S u \,\big(\forall v \in u\, \exists_S w\, \forall a\, (\phi \to a \in w)) \\ \to \exists_S w\, \forall v \in u\, \forall a\, (\phi \to a \in w)\big),$$

where ϕ is any c-formula with all quantifiers restricted to S, and a, u, v, w, v_0, …, $v_{n\text{-}}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{a, v, v_0, \ldots, v_{n\text{-}}\}$. Any instance of this schema can be obtained from $\mathsf{C5}$ by the following argument, where R is a variable distinct from all the others:

$\ulcorner$Let $(v_0), \ldots, (v_{n\text{-}}), (u)$ be given, and suppose $(\forall v \in u\, \exists_S w\, \forall a\, (\phi \to a \in w))$. Using $\mathsf{C2a}$, let $(R) = \{((v), (a)) \mid (\phi)\}$.[15] Then

$$(\forall v \in u\, \exists_S w\, \forall a\, ((v, a) \in R \to a \in w)).$$

Hence using $\mathsf{C5}$, $(\exists_S w\, \forall v \in u\, \forall a\, ((v, a) \in R \to a \in w))$. Thus,

$$(\exists_S w\, \forall v \in u\, \forall a\, (\phi \to a \in w)).^{\urcorner}$$

Note that $\mathsf{Foundation}$ is also a single axiom in C replacing a schema in S. The direct translation of the $\mathsf{Foundation}$ schema of S would consist of sentences

$$\forall v_0, \ldots, v_{n\text{-}}\, \big(\exists_S v\, \phi \to \exists_S v\, \big(\phi \wedge \forall u \in v\, \neg\, \phi\tbinom{v}{u}\big)\big).$$

Any instance of this schema can be obtained from $\mathsf{C6}$ by an argument similar to the preceding, using $\mathsf{C2a}$ to infer the existence of a class defined from ϕ in a suitable way.

C^0 is a conservative extension[2.183] of S^0 in the sense that if σ is an s-sentence then $\mathsf{S}^0 \vdash \sigma$ iff $\mathsf{C}^0 \vdash \sigma^S$, where σ^S is the relativization[2.112] of σ to the predicate S. (C is also a conservative extension of S.) In developing pure set theory, we may therefore innocently refer to proper classes defined by formulas as though they actually exist.

3.2.4 Basic theory of membership with (possibly proper) elements and classes

For the sake of completeness we give here a system of axioms for the basic theory of membership with the possibility of proper elements as well as proper classes.

Definition *We define* ec *to be* s *with three additional unary predicates, represented typographically by '$\boldsymbol{E}$', '$\boldsymbol{C}$', and '$\boldsymbol{S}$', signifying that their arguments are respectively elements, classes, or sets (the last being equivalent to the conjunction of the first and second).*

We will not at this point separately define a theory allowing proper elements but no proper classes, i.e., a set theory with atoms; the reader can easily supply such a definition.

Axioms of CA

$\mathsf{CA0}$.

$$\forall \mathrm{v}_0\, \forall \mathrm{v}_1\, (\mathrm{v}_0 \in \mathrm{v}_1 \to (\boldsymbol{E}\mathrm{v}_0 \wedge \boldsymbol{C}\mathrm{v}_1)) \\ \wedge\, \forall \mathrm{v}_0\, \big((\boldsymbol{E}\mathrm{v}_0 \vee \boldsymbol{C}\mathrm{v}_0) \wedge \big(\boldsymbol{S}\mathrm{v}_0 \leftrightarrow (\boldsymbol{E}\mathrm{v}_0 \wedge \boldsymbol{C}\mathrm{v}_0)\big)\big) \\ \wedge\, \exists \mathrm{v}_0\, (\boldsymbol{S}\mathrm{v}_0 \wedge \forall \mathrm{v}_1\, (\boldsymbol{E}\mathrm{v}_1 \wedge \neg\, \boldsymbol{C}\mathrm{v}_1 \to \mathrm{v}_1 \in \mathrm{v}_0))$$

[15] To use $\mathsf{C2a}$ directly, we let $(R) = \{x \mid \exists (v), (a)\ x = ((v), (a)) \wedge (\phi)\}$.

CA1. Extension

$$\forall_C \mathrm{v}_0\, \forall_C \mathrm{v}_1\, (\forall_E \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1)) \rightarrow \mathrm{v}_0 = \mathrm{v}_1)$$

CA2a. (Predicative) Comprehension

$$\forall v_0 \cdots \forall v_{n^-}\, \exists_C w\, \forall_E v\, (v \in w \leftrightarrow \phi),$$

where ϕ is any ec*-formula with all quantifiers restricted to E, and $v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that* $\mathrm{Free}\,\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.

CA2b. Separation

$$\forall_S \mathrm{v}_0\, \forall_C \mathrm{v}_1\, \big(\forall \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_1 \rightarrow \mathrm{v}_2 \in \mathrm{v}_0) \rightarrow \boldsymbol{S}\mathrm{v}_1\big)$$

CA3. Existence

$$\exists_S \mathrm{v}_0\, \forall \mathrm{v}_1\;\; \mathrm{v}_1 \notin \mathrm{v}_0$$

CA4. Pair

$$\forall_E \mathrm{v}_0, \mathrm{v}_1\, \exists_S \mathrm{v}_2\, (\mathrm{v}_0 \in \mathrm{v}_2 \wedge \mathrm{v}_1 \in \mathrm{v}_2)$$

CA5. Collection

$$\forall v_0 \cdots \forall v_{n^-}\, \forall_S u\, \big(\forall v \in u\, \exists_S w\, \forall a\, (\phi \rightarrow a \in w) \\ \rightarrow \exists_S w\, \forall v \in u\, \forall a\, (\phi \rightarrow a \in w)\big),$$

where ϕ is any sec*-formula with all quantifiers restricted to E, and a, u, v, w, v_0, $\ldots$, v_{n^-} are distinct variables such that* $\mathrm{Free}\,\phi \subseteq \{a, v, v_0, \ldots, v_{n^-}\}$.

CA6. Foundation

$$\forall_C \mathrm{v}_0\, (\exists \mathrm{v}_1 \in \mathrm{v}_0 \rightarrow \exists \mathrm{v}_1 \in \mathrm{v}_0\, \forall \mathrm{v}_2 \in \mathrm{v}_1\;\; \mathrm{v}_2 \notin \mathrm{v}_0)$$

CA0 largely defines E, C, and S. It states that everything that is a member of something is an element, but it leaves open whether every element is a member of something. This is settled in the affirmative by CA4. It also states that everything that contains something is a class, but it leaves open the possibility that a class might contain nothing, and CA3 states that there is indeed such a class (and that it is in fact a set, i.e., an element as well as a class). The empty set differs from an urelement in only one respect: it is declared to be a class. There is nothing deep about this: it would just require a lot of annoying circumlocution to maintain the position that every class must have a member. Finally, CA0 states that there is a set that contains every proper element. Since CA2a tells us that the proper elements form a class, CA2b permits us to infer that the proper elements form a set. With this restriction, CA is still sufficiently general for all our purposes.

Note that CA2a implies that there is a class that contains all elements and one that contains exactly the sets. As is customary in class theories, we press the unary predicate symbols 'E' and 'S' into service as constant symbols to denote these classes.

3.3 Basic membership-theoretic constructs

This section is given over mainly to simple definitions of useful objects, with proofs of their basic properties, which go a long way toward demonstrating how set theory[16] may be regarded as subsuming all of mathematics. The basic principles are adequately demonstrated without consideration of proper elements. Proper classes, on the other hand, are quite convenient, particularly in the setting of a predicative theory.[3.18] Thus, we will work primarily in C^0.

3.3.1 Abstraction terms

A very useful construction is the following. Suppose ϕ is a formula; $v, u_0, \ldots, u_{n^-}$ are distinct variables; and

$$\operatorname{Free}\phi \subseteq \{v, u_0, \ldots, u_{n^-}\}. \tag{3.20}$$

Then

$$\mathsf{C}^0 \vdash \forall u_0, \ldots, u_{n^-}\, \exists! w\, \forall_S v\, (v \in w \leftrightarrow \phi),$$

where w is a variable not in $\{v, u_0, \ldots, u_{n^-}\}$, so the following sentence is a legitimate definition in C^0 of a new n-ary operation index F:

$$w = \tilde{F}\langle u_0, \ldots, u_{n^-}\rangle \leftrightarrow \forall_S v\, (v \in w \leftrightarrow \phi). \tag{3.21}$$

In a typographical language, for mnemonic purposes, we incorporate ϕ into the terms of the form $\tilde{F}\langle \tau_0, \ldots, \tau_{n^-}\rangle$. In the simplest case we let

$$\tilde{F}\langle u_0, \ldots, u_{n^-}\rangle = \underline{\{}\, v \,\underline{|}\, \phi \,\underline{\}}.$$

We call these *abstraction terms.*

Note that abstraction in this sense is a form of quantification, which may be seen most easily if define $\mathbf{\Gamma} v\, \phi$ to be the term $\underline{\{}\, v \,\underline{|}\, \phi \,\underline{\}}$. The value of $\mathbf{\Gamma} v\, \phi$ is the class of v such that ϕ. Any other quantifier may be defined in terms of $\mathbf{\Gamma}$. For example, $\exists v\, \phi$ is true iff $\mathbf{\Gamma} v\, \phi$ is nonempty; $\forall v\, \phi$ is true iff $\mathbf{\Gamma} v\, \phi$ is the entire universe of elements; $\exists! v\, \phi$ is true iff $\mathbf{\Gamma} v\, \phi$ has exactly one member; etc. The variable v is *bound* by the abstraction operation, just as it is bound by the other quantification operations.

Note that in order that the substitution of a term τ for a variable u_m in $\underline{\{}\, v \,\underline{|}\, \phi \,\underline{\}}$ have the desired meaning, with ϕ as above,[3.20] τ must be free for u_m in $\underline{\{}\, v \,\underline{|}\, \phi \,\underline{\}}$, i.e.,[1.16.1] no variable free in τ may bound in $\underline{\{}\, v \,\underline{|}\, \phi \,\underline{\}}$; thus, τ must be free for u_m in ϕ, and v must not occur free in τ. All of this is quite obvious and intuitive in practice.

In a theory such as S^0 without proper classes we must qualify abstraction terms to make sure they denote sets. We may use our standard rule (in a theory of membership) that a term that is otherwise "undefined" has the value 0. In other words, we replace (3.21) by

$$\begin{aligned} w = \tilde{F}\langle u_0, \ldots, u_{n^-}\rangle \leftrightarrow \big(\exists w\, \forall v\, (v \in w \leftrightarrow \phi) \wedge \forall v\, (v \in w \leftrightarrow \phi)\big) \\ \vee \big(\neg \exists w\, \forall v\, (v \in w \leftrightarrow \phi) \wedge w = \mathbf{0}\big). \end{aligned}$$

[16] Remember that we use 'set theory' loosely to refer to all theories of membership.

We may also use the following variation:

$$\underline{\{}\, v \mathbin{\underline{\in}} \tau \,\underline{|}\, \phi \,\underline{\}},$$

where τ is a term in which v does not occur. In the context of a pure set theory, τ is necessarily a set, so S2 applies.

Another variation on the notation is

$$\underline{\{}\, \tau \,\underline{|}\, \phi \,\underline{\}},$$

where τ is a term (with the proviso, in theory with proper classes, that its value is always a set). Unless otherwise stated, the corresponding definition[3.21] is

$$w \mathbin{\underline{=}} \underline{\{}\, \tau \,\underline{|}\, \phi \,\underline{\}} \leftrightarrow \forall_S v \,(v \in w \leftrightarrow \exists v_0, \ldots, v_{k^-} \,(v = \tau \wedge \phi)),$$

where $\mathrm{Free}\,\tau = \{v_0, \ldots, v_{k^-}\}$. Thus, all the variables in $\mathrm{Free}\,\tau$ are typically bound in the abstraction term, but this need not be so. For example, $\{(x, y) \mid x \in y\}$ ordinarily is the entire membership relation regarded as a class of ordered pairs, but we might also say 'for any given y, the class $\{(x, y) \mid x \in y\}$', by which we would mean the same as if we said 'for any given y, the class $\{(x, y') \mid x \in y' \wedge y' = y\}$', in which x and y' are bound, but y remains free. Again, rest assured, all this is quite obvious in practice.

3.3.2 Union, intersection, and difference

Definition [C^0]

1. *The* union *of x and y* $\stackrel{\text{def}}{=} x \cup y \stackrel{\text{def}}{=} \{z \mid z \in x \vee z \in y\}$.
2. *The* intersection *of x and y* $\stackrel{\text{def}}{=} x \cap y \stackrel{\text{def}}{=} \{z \mid z \in x \wedge z \in y\}$.
3. *The* union of *x* $\stackrel{\text{def}}{=} \bigcup x \stackrel{\text{def}}{=} \{y \mid \exists z \in x\ y \in z\}$.
4. *The* intersection of *x* $\stackrel{\text{def}}{=} \bigcap x \stackrel{\text{def}}{=} \{y \mid \forall z \in x\ y \in z\}$.

$\cup$ and $\cap$ are obviously associative operations, so grouping is immaterial in expressions like $\ulcorner x \cup y \cup z \urcorner$. Indeed, $x \cup y \cup z = \bigcup\{x, y, z\}$.

$\ulcorner \bigcup \urcorner$ and $\ulcorner \bigcap \urcorner$ are also used similarly to $\ulcorner \sum \urcorner$(and $\ulcorner \prod \urcorner$). Thus, $\bigcup_{y \in x} Fy = \bigcup\{Fy \mid y \in x\}$.

(3.22) **Theorem** [C^0] *Suppose x is a set. then $\bigcup x$ is a set.*

Proof Let $R = \{(z, y) \mid z \in x \wedge y \in z\}$. Then for all $z \in x$ there exists a set w such that $\forall y\ \big((z, y) \in R \to y \in w\big)$. For example, let $w = z$. Hence, by Collection there exists a set w such that $\forall z \in x\, \forall y\ \big((z, y) \in R \to y \in w\big)$. Then $\bigcup x \subseteq w$, so by Separation $\bigcup x$ is a set. □[3.22]

(3.23) *(3.22) is often stated as the* Union *axiom. Note that in* C^0*,* Union *states that the union of a set is a set, whereas in* S^0*,* Union *simply states that $\bigcup x$ exists.*

3.3.3 The power operation

(3.24) **Definition** [C^0] *Suppose x is a class. The* powerclass of x $\stackrel{\text{def}}{=} \mathcal{P}x \stackrel{\text{def}}{=} \{y \mid y \subseteq x\}$.

Thus, $\mathcal{P}x$ consists of all subsets of x. If x is a set, then $\mathcal{P}x$ consists of all subclasses of x, because any subclass of a set is a set. If x is a proper class, of course, it has subclasses that are not sets, e.g., x itself, and these are not in $\mathcal{P}x$ (by definition, being proper classes, they are not in any class).

It is easy to show that if x is a proper class then $\mathcal{P}x$ is also a proper class. The interesting question is whether for every set x, $\mathcal{P}x$ is a set. As we will see, this question is not settled by C.

The Power *axiom, as formulated in C^0, states that for any set x, $\mathcal{P}x$ is a set. In S^0,* Power *simply states, in effect, that $\mathcal{P}x$ exists, i.e., that for every x there exists y such that for all z, $z \subseteq x \leftrightarrow z \in y$.*

Although we do not at this time adopt Power as an axiom, the power operation is still useful in the context of C^0.

3.3.4 Functions

(3.25) **Definition** [C^0]

1. F *is* prefunction $\stackrel{\text{def}}{\iff}$ *F is a class of ordered pairs.*
2. F *is* function $\stackrel{\text{def}}{\iff} \operatorname{Fcn} F \stackrel{\text{def}}{\iff}$ *F is a prefunction such that for all x there exists at most one y such that $(x, y) \in F$.*

What we have called a prefunction, viz., a class of ordered pairs, is often referred to as a *relation*, but we will define *relation* to mean a class of finite sequences.[3.62] Our use of *prefunction* is unorthodox, but it is justified in the context of this book, as we make little use of classes of ordered pairs other than as functions or precursors thereof.

(3.26) **Definition** [C^0] *Suppose R is a prefunction and X, Y are classes.*

1. *The* domain *of* $R \stackrel{\text{def}}{=} \operatorname{dom} R \stackrel{\text{def}}{=} \{x \mid \exists y\ (x, y) \in R\}$.
2. *The* image *of* $R \stackrel{\text{def}}{=} \operatorname{im} R \stackrel{\text{def}}{=} \{y \mid \exists x\ (x, y) \in R\}$.
3. $R \restriction X \stackrel{\text{def}}{=} \{(x, y) \in R \mid x \in X\}$.
4. *The* image of X by R $\stackrel{\text{def}}{=} R^{\rightarrow}X \stackrel{\text{def}}{=} \{y \mid \exists x \in X\ (x, y) \in R\}$ $\big(= \operatorname{im}(R \restriction X)\big)$.
5. *The* inverse of R $\stackrel{\text{def}}{=} R^{-1} \stackrel{\text{def}}{=} \{(x, y) \mid (y, x) \in R\}$.
6. *The* inverse image *or* preimage of Y by R $\stackrel{\text{def}}{=} R^{\leftarrow}Y \stackrel{\text{def}}{=} \{x \mid \exists y \in Y\ (x, y) \in R\}$.

Definition [C^0]

1. *If F is a function and $x \in \operatorname{dom} F$ then* the value of F at $x \stackrel{\text{def}}{=} Fx \stackrel{\text{def}}{=} F(x) \stackrel{\text{def}}{=} F_x \stackrel{\text{def}}{=}$ *that (unique) element y such that $(x, y) \in F$.*[17]
2. *Note that we may indicate the argument of a function by simple juxtaposition; round brackets are only necessary to resolve ambiguity.*[18]

(3.27) **Theorem** [C^0] *Suppose F is a function and $\operatorname{dom} F$ is a set.*

1. $\operatorname{im} F$ *is a set.*
2. F *is a set.*

Proof With an appropriate change of variables, instantiating to $\mathrm{v}_0 = F$ and $\mathrm{v}_1 = \operatorname{dom} F$ (which is justified, because $\operatorname{dom} F$ is assumed to be a set), C5 says that if

$$\forall x \in \operatorname{dom} F\, \exists_S y\, \forall z\, \big((x, z) \in F \rightarrow z \in y\big), \tag{3.28}$$

then

$$\exists_S y\, \forall x \in \operatorname{dom} F\, \forall z\, \big((x, z) \in F \rightarrow z \in y\big). \tag{3.29}$$

Clearly, (3.28) holds, because we can let $y = \{Fx\}$. Hence (3.29) holds, i.e., there is a set y such that $\operatorname{im} F \subseteq y$. By C2b, therefore, $\operatorname{im} F$ is a set.

To show that F is a set, let $R = \{(x, (x, y)) \mid x \in \operatorname{dom} F \wedge (x, y) \in F\}$.[19] R is a function, so by (3.27.1) $\operatorname{im} R$ is a set. Since $F = \operatorname{im} R$, F is a set. $\square_{3.27}$

(3.30) *(3.27.1) is often stated as the* Replacement *axiom. Clearly it may be stated as a schema in an axiomatization of* S^0.

The theory C^0 may be formulated with Union and Replacement in place of Collection. We have already established one direction of this equivalence with (3.22) and (3.27). The following theorem establishes the converse.

Theorem [C0–4]

(3.31) Suppose

1. (Union) *for every set x, $\bigcup x$ is a set; and*
2. (Replacement) *for every function F, if $\operatorname{dom} F$ is a set then $\operatorname{im} F$ is a set.*

Suppose x_0 is a class, x_1 is a set, and

$$\forall x_2 \in x_1\, \exists_S x_3\, \forall x_4\, \big((x_2, x_4) \in x_0 \rightarrow x_4 \in x_3\big).$$

Then

$$\exists_S x_3\, \forall x_2 \in x_1\, \forall x_4\, \big((x_2, x_4) \in x_0 \rightarrow x_4 \in x_3\big).$$

Proof Let F consist of all ordered pairs (x_2, y) such that

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[17] Note that ⌜the value of · at ·⌝ is a binary operation symbol. As always, because every term must have a value, if the arguments are not appropriate—i.e., if it is not the case that the first argument is a function and the second is in its domain—then it is given the default value 0. In practice, we don't allow this to happen, so the choice of default value is immaterial.

[18] Like most of our notations, these perform multiple services—in particular, subscripts are often used in other ways than to indicate an argument of a function.

[19] $R = \{w \mid \exists_S x, y, z\, (z = (x, y) \wedge w = (x, z) \wedge x \in \operatorname{dom} F \wedge z \in F)\}$, which is a definition involving only quantification restricted to S, so R exists by C2a.

1. $x_2 \in x_1$; and
2. $\forall x_4\,(x_4 \in y \leftrightarrow (x_2, x_4) \in x_0)$.

(3.32) **Claim** *F is a function with domain x_2.*

Proof We must show that, given $x_2 \in x_1$, there exists a unique y such that $(x_2, y) \in F$. Given $x_2 \in x_1$, by hypothesis there exists x_3 such that $\forall x_4\,\big((x_2, x_4) \in x_0 \rightarrow x_4 \in x_3\big)$. Let $y = \{x_4 \mid (x_2, x_4) \in x_0\}$. Since $y \subseteq x_3$, by Separation, y is a set. Thus, $(x_2, y) \in F$. The uniqueness of y follows from Extension. $\square$[3.32]

Since $\operatorname{dom} F = x_2$ is a set, $\operatorname{im} F$ is a set,[3.31.1] so $\bigcup \operatorname{im} F$ is a set.[3.31.2] Let $x_3 = \bigcup \operatorname{im} F = \{x_4 \mid \exists x_2 \in x_1\ (x_2, x_4) \in x_0\}$. Then clearly $\forall x_2 \in x_1\, \forall x_4\,\big((x_2, x_4) \in x_0 \rightarrow x_4 \in x_3\big)$, as desired. $\square$[3.31]

(3.33) **Definition** [C^0]

1. F is a function from X to Y $\overset{\text{def}}{\iff}$ $F : X \to Y$ $\overset{\text{def}}{\iff}$ *F is a function,* $\operatorname{dom} F = X$, *and* $\operatorname{im} F \subseteq Y$.
2. F is a partial function from X to Y $\overset{\text{def}}{\iff}$ $F : X \rightharpoonup Y$ $\overset{\text{def}}{\iff}$ *F is a function,* $\operatorname{dom} F \subseteq X$, *and* $\operatorname{im} F \subseteq Y$.[20] *Note that if $F : X \to Y$ then $F : X \rightharpoonup Y$. In a discussion of partial functions from a fixed class X,* total *functions are those whose domain is X.*
3. *A function F is* injective *(or* one-one *or* 1-1*)* $\overset{\text{def}}{\iff}$ $\forall a, b \in \operatorname{dom} F\,(a \neq b \rightarrow Fa \neq Fb)$. *We write $F : X \overset{\text{inj}}{\to} Y$ or $F : X \overset{\text{inj}}{\rightharpoonup} Y$ in this case.*
4. *A function F is* surjective to Y *(or is* onto Y*)* $\overset{\text{def}}{\iff}$ $\operatorname{im} F = Y$. *When the target class Y is clearly understood, we simply say that F is* surjective *or* onto. *We write $F : X \overset{\text{sur}}{\to} Y$ or $F : X \overset{\text{sur}}{\rightharpoonup} Y$ in this case.*
5. *A function F is* bijective from X to Y $\overset{\text{def}}{\iff}$ $F : X \overset{\text{bij}}{\to} Y$ $\overset{\text{def}}{\iff}$ $F : X \overset{\text{inj}}{\to} Y$ *and* $F : X \overset{\text{sur}}{\to} Y$. *A bijection from X to Y is also called a* 1-1 correspondence of X with Y. *Note that* $\ulcorner \overset{\text{bij}}{\rightharpoonup} \urcorner$ *is not defined.*

(3.34) **Theorem** [C^0] *Suppose $F : X \overset{\text{inj}}{\rightharpoonup} Y$. Then F^{-1}*[3.63.1] *is a function. Moreover,*

1. $F^{-1} : Y \overset{\text{inj}}{\rightharpoonup} X$;
2. *if* $F : X \overset{\text{sur}}{\rightharpoonup} Y$ *then* $F^{-1} : Y \overset{\text{inj}}{\to} X$;
3. *if* $F : X \overset{\text{inj}}{\to} Y$ *then* $F^{-1} : Y \overset{\text{sur}}{\rightharpoonup} X$; *and*
4. *if* $F : X \overset{\text{bij}}{\to} Y$ *then* $F^{-1} : Y \overset{\text{bij}}{\to} X$.[21]

Proof Straightforward. $\square$[3.34]

[20] It would be more proper to say 'F is a function from a part of X to Y' rather than 'F is a partial function from X to Y', as 'partial' really characterizes $\operatorname{dom} F$ *vis-à-vis* X, not F itself.

[21] Recall[3.33.5] that in order that F be a bijection from X to Y it is not enough that $F : X \overset{\text{inj}}{\rightharpoonup} Y$ and $F : X \overset{\text{sur}}{\rightharpoonup} Y$—it must also be the case that $F : X \to Y$, i.e., F must be "total" on X, so that F^{-1} is surjective.

(3.35) **Definition** [C^0] *Suppose X is a class and Y is a set.* X pre $Y \overset{\text{def}}{=} {}^Y X \overset{\text{def}}{=} \{f \mid f : Y \to X\}$.

Given that Y is a set, every $f : Y \to X$ is a set, so ${}^Y X$ contains all (and only) the functions from Y to X.

3.3.4.1 Functional abstraction terms

The notation of abstraction terms[§ 3.3.1] has a specific adaptation to functions. Suppose τ is a term, ϕ is a formula, and u is a variable.

(3.36) $\underline{\langle}\, \tau \,\underline{|}\, \phi \,\underline{\rangle}_u$ *is synonymous with* $\underline{\{}\underline{(}u \underline{,} \tau\underline{)} \,\underline{|}\, \phi\, \underline{\}}$ *and denotes a function f whose domain is $\{u \mid \phi\}$ and whose value at any x in its domain is the value of τ with x for u.*

Obviously, in order that this be useful it is necessary that $u \in \operatorname{Free}\tau \cap \operatorname{Free}\phi$. When it is clear from the context (or immaterial to the discussion) which of the free variables of τ and ϕ is to be taken for u, the specifying subscript may be omitted. Often $\operatorname{Free}\tau \cap \operatorname{Free}\phi$ consists of a single variable. For example,

$$\underline{\langle \{x\} \mid x \in X \cap Y \rangle}$$

is clearly intended to denote the function

$$\{(x, \{x\}) \mid x \in X \cap Y\}.$$

3.3.4.2 Indexed families of classes

Note that—although a function f may be a proper class—the value $f x$ of f at some $x \in \operatorname{dom} f$ is necessarily a set. The following definition describes function-like objects whose values may be proper classes.

(3.37) **Definition** [C^0]

1. *A is a* family $\overset{\text{def}}{\iff}$ *A is a class of ordered pairs and for all $(i, a) \in A$, either*
   1. $a = 0$*; or*
   2. $\forall c\,\big((i, c) \in A \to \exists d\, c = \{d\}\big)$.
2. *Suppose A is a family and $i \in \operatorname{dom} A$. The class* indexed by i in A $\overset{\text{def}}{=}$

$$A_{[i]} \overset{\text{def}}{=} \begin{cases} 0 & \textit{if } (i, 0) \in A \\ \{d \mid (i, \{d\}) \in A\} & \textit{otherwise.}^{22} \end{cases}$$

This notation corresponds to the use of $\ulcorner f_x \urcorner$ in the case of a function f.

Analogously to (3.36), $\underline{[}\, \tau \,\underline{|}\, \phi \,\underline{]}_u$ *denotes the family A whose domain is the class of u such that ϕ and whose value $A_{[u]}$, for any u in its domain, is τ. The subscript is usually superfluous.*

[22] It would not do to simply let a family A be a class of ordered pairs, and to let $A_{[i]} = \{d \mid (i, d) \in A\}$, because then $A_{[i]} = 0 \leftrightarrow \forall a\ (i, a) \notin A$, which is an awkward and inflexible convention.

3.3.5 Ordinals

(3.38) **Definition** $[\mathsf{C}^0]$ x *is* transitive $\overset{\text{def}}{\iff}$ $\mathrm{Tran}\,x$ $\overset{\text{def}}{\iff}$ $\forall y, z\,(z \in y \in x \to z \in x)$.

(3.39) **Definition** $[\mathsf{C}^0]$ X is an ordinal $\overset{\text{def}}{\iff}$ $\mathrm{Ord}\,X$ $\overset{\text{def}}{\iff}$ X *is a transitive set that is wellordered by* $\in$*, i.e.,*

1. $\forall x, y \in X\,(x \in y \vee x = y \vee y \in x)$*; and*
2. *every nonempty* $Y \subseteq X$ *has an* $\in$*-minimal member, i.e.,* $\forall Y \subseteq X\,(Y \neq 0 \to \exists x \in Y\, \forall y \in x\; y \notin Y)$.[23]

$\mathrm{Ord} \overset{\text{def}}{=} \{x \mid \mathrm{Ord}\,x\}$.

(3.40) **Theorem** $[\mathsf{C}^0]$

1. $\mathrm{Ord}\,0$.
2. *Suppose* $\mathrm{Ord}\,x$ *and* $y \in x$. *Then* $\mathrm{Ord}\,y$.
3. *Suppose* $\mathrm{Ord}\,x$, $\mathrm{Ord}\,y$, *and* $x \subseteq y$. *Then* $x = y$ *or* $x \in y$.

Proof 1, 2 Straightforward. $\square^{3.40.1,\,2}$

3 Suppose $x \neq y$. Then $y \nsubseteq x$, so $A = \{y' \in y \mid y' \notin x\}$ is nonempty. Let[3.39.2] z be an $\in$-minimal member of A, i.e., $z \in A$ and $\forall w \in z\; w \notin A$. Since $z \in y$ and y is transitive, $\forall w \in z\; w \in y$. Hence, $\forall w \in z\; w \in x$, i.e., $z \subseteq x$. Since $z \in A$, $z \notin x$, i.e., $\forall w \in x\; z \neq w$. Since x is transitive, $\forall w \in x\; z \notin w$. Since y is totally ordered by $\in$,[3.39.1] $\forall w \in x\; w \in z$, i.e., $x \subseteq z$. Thus, $z = x$. Therefore, $x \in y$. $\square^{3.40}$

We will generally follow the common practice of using lowercase Greek letters for ordinals.

(3.41) **Theorem** $[\mathsf{C}^0]$ $\forall_{\mathrm{Ord}} \alpha, \beta\,(\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$.

Proof Suppose $\alpha, \beta \in \mathrm{Ord}$. Clearly $\gamma = \alpha \cap \beta$ is transitive and wellordered by $\in$, so it is an ordinal. Since $\gamma \subseteq \alpha, \beta$, by (3.40.3) $\gamma = \alpha$ or $\gamma \in \alpha$, and also $\gamma = \beta$ or $\gamma \in \beta$. If $\gamma \in \alpha$ and $\gamma \in \beta$ then $\gamma \in \alpha \cap \beta = \gamma$, in which case $\{\gamma\}$ is a nonempty subset of γ that does not have an $\in$-minimal member, contradicting (3.39.2). So $\gamma = \alpha$ or $\gamma = \beta$. Therefore, either $\alpha = \gamma = \beta$ or $\alpha = \gamma \in \beta$ or $\beta = \gamma \in \alpha$. $\square^{3.41}$

(3.42) **Theorem** $[\mathsf{C}^0]$ Ord *is transitive and is wellordered by* $\in$*, i.e.,* $\forall X \subseteq \mathrm{Ord}\,(X \neq 0 \to \exists x \in X\, \forall y \in x\; y \notin X)$.

Proof We have already shown that Ord is transitive[3.40.2] and is totally ordered by $\in$.[3.41] To show that it is wellordered by $\in$, suppose $X \subseteq \mathrm{Ord}$ is nonempty. Suppose $\alpha \in X$. If α is $\in$-minimal in X we are finished; otherwise, let $Y = X \cap \alpha$, and let β be $\in$-minimal in Y. Then β is $\in$-minimal in X. $\square^{3.42}$

Ord would therefore be an ordinal but for the fact that it is not a set:

(3.43) **Theorem** $[\mathsf{C}^0]$ Ord *is not a set.*

---

[23]Note that (3.39.2) is an instance of Foundation.[3.16] Since we are working in C^0 this must be separately stipulated. Note also that by virtue of (3.39.2), the disjuncts in (3.39.1) are mutually exclusive.

Proof Suppose Ord is a set. Then Ord is an ordinal, so Ord $\in$ Ord, in which case $\{\text{Ord}\}$ is a nonempty subset of Ord that does not have an $\in$-minimal member.□[3.43]

Since[3.41] Ord is totally ordered by membership, it is a common practice, which we often follow, to use '$<$' for '$\in$' when dealing with ordinals; likewise,[3.40.3] we use '$\leqslant$' for '$\subseteq$'.

(3.44) **Theorem** [C^0] *Suppose x is a set of ordinals. Then $\bigcup x$ is an ordinal and is the least upper bound of x.*

Proof Straightforward. □[3.44]

The following definition simply emphasizes the role of $\bigcup x$ as the least upper bound (supremum) of a set of ordinals x.

Definition [C^0] *Suppose x is a set of ordinals.* $\sup x \stackrel{\text{def}}{=} \bigcup x$.

The following definition is most useful as applied to ordinals, but it is meaningful in general.

(3.45) **Definition** [C^0]

1. *For any set x,* $x^+ \stackrel{\text{def}}{=} x \cup \{x\}$.
2. *For any ordinal α,* $\alpha + 1 \stackrel{\text{def}}{=} \alpha^+$.

The following theorem is the rationale behind (3.45.2).

(3.46) **Theorem** [C^0] *Suppose α is an ordinal. Then α^+ is an ordinal and is the (immediate) successor of α in* Ord, *i.e.,* $\forall_{\text{Ord}}\beta\,\big(\alpha < \beta \leftrightarrow (\alpha^+) \leqslant \beta\big)$.

Proof Straightforward. □[3.46]

Since 0 is included in any set and 0 is an ordinal, $0 \leqslant \alpha$ for any ordinal α, i.e., 0 is the least ordinal. The immediate successor of 0 is $0^+ = \{0\}$, and every other ordinal is greater than this. We call this ordinal '1'. Similarly, 1 has a successor, viz., $1^+ = \{0, 1\} = \{0, \{0\}\}$, which we call '2'. Every natural number may in this way be identified with an ordinal, and we use some of the terminology normally associated with numbers. In particular, we use $\ulcorner (\tau) + 1 \urcorner$ to mean $\ulcorner (\tau) \cup \{(\tau)\} \urcorner$, for terms τ regarded as denoting ordinals.

To summarize, the following sequence is an initial segment of the ordinals

(3.47) **Definition** [C^0]

$$\begin{aligned}
&0\\
&1 \stackrel{\text{def}}{=} 0 + 1 = 0 \cup \{0\} = \{0\}\\
&2 \stackrel{\text{def}}{=} 1 + 1 = 1 \cup \{1\} = \{0, 1\} = \{0, \{0\}\}\\
&3 \stackrel{\text{def}}{=} 2 + 1 = 2 \cup \{2\} = \{0, 1, 2\} = \{0, \{0\}, \{0, \{0\}\}\}\\
&\ \ \vdots
\end{aligned}$$

Note that every ordinal n in this list other than 0 has an immediate predecessor, which we denote by $\ulcorner n - 1 \urcorner$ or $\ulcorner n^- \urcorner$,[1.28] but this is not necessarily true for every ordinal.

(3.48) **Definition** $[\mathsf{C}^0]$

1. α *is a* successor ordinal $\overset{\text{def}}{\iff} \operatorname{Suc}\alpha \overset{\text{def}}{\iff} \operatorname{Ord}\alpha \wedge \exists\beta\ \alpha = \beta + 1$. *Note that* β *is uniquely determined by* α*, and in this case* $\alpha - 1 \overset{\text{def}}{=} \alpha^{-} \overset{\text{def}}{=} \beta$.
2. α *is a* limit ordinal $\overset{\text{def}}{\iff} \operatorname{Lim}\alpha \overset{\text{def}}{\iff} \operatorname{Ord}\alpha \wedge \alpha \neq 0 \wedge \neg \operatorname{Suc}\alpha$.
3. α *is a* natural number *or simply* number $\overset{\text{def}}{\iff} \operatorname{Num}\alpha \overset{\text{def}}{\iff} \operatorname{Ord}\alpha \wedge \forall\beta \leqslant \alpha\,(\beta = 0 \vee \operatorname{Suc}\beta)$.
4. $\omega \overset{\text{def}}{=} \{\alpha \mid \operatorname{Num}\alpha\}$.

In keeping with our convention that any predicate symbol applicable to sets may be used to represent the corresponding class, we could use 'Num' instead of 'ω',[3.48.4] but the latter notation is standard.

(3.49) **Theorem** $[\mathsf{C}^0]$ *ω is transitive and wellordered by $\in$. Hence, either $\omega = \operatorname{Ord}$ or $\omega \in \operatorname{Ord}$.*

Proof Straightforward. □[3.49]

3.3.5.1 Induction on ordinals

(3.50) **Theorem: Induction on ordinals** $[\mathsf{C}^0]$ *Suppose $X \subseteq \operatorname{Ord}$ and $\forall_{\mathrm{Ord}}\alpha\,(\alpha \subseteq X \rightarrow \alpha \in X)$. Then $X = \operatorname{Ord}$.*

Proof Suppose toward a contradiction that there exists something in $\operatorname{Ord}\setminus X$, and let α be the least such.[3.42] Then any $\beta \in \alpha$ is in X, i.e., $\alpha \subseteq X$, so $\alpha \in X$. □[3.50]

(3.50) establishes the principle of induction on ordinals. In the typical application, one has an ordinal-indexed family $[x_{[\alpha]} \mid \alpha \in \operatorname{Ord}]$ of which it has been shown that for every $\alpha \in \operatorname{Ord}$, if $x_{[\beta]}$ has a certain property for every $\beta < \alpha$ then $x_{[\alpha]}$ has the property. Let X be the class of ordinals α for which $x_{[\alpha]}$ has the property under consideration (assuming set quantification is adequate to describe X). Then (3.50) implies that $X = \operatorname{Ord}$, i.e., $x_{[\alpha]}$ has the property for all α.

(3.78) is the general statement of this principle. The following theorem is a variation that exploits the fact that by definition every nonzero ordinal in ω (i.e., every number $n > 0$) has an immediate predecessor. It is a basic principle of arithmetic and goes by the name of *mathematical induction*.

(3.51) **Theorem: Mathematical induction** $[\mathsf{C}^0]$ *Suppose X is a class, $0 \in X$ and $\forall n \in \omega\,(n \in X \rightarrow n + 1 \in X)$. Then $\omega \subseteq X$.*

Remark In theories, such as S^0, in which ω may not exist, we state this as a metatheorem:

$$\mathsf{S}^0 \vdash \phi\binom{n}{\mathbf{0}} \wedge \forall_{\mathrm{Num}} m\,\Big(\phi\binom{n}{m} \rightarrow \phi\binom{n}{m+\mathbf{1}}\Big) \rightarrow \forall_{\mathrm{Num}} n\ \phi,$$

for any s-formula ϕ with $m \notin \operatorname{Free}\phi$.

Proof If $\omega \nsubseteq X$, let n be the least member of $\omega\setminus X$. Since $0 \in X$, $n \neq 0$, so n is a successor, say $n = m + 1$. Then $m \in X$, so $n \in X$. □[3.51]

3.3.6 Finiteness

(3.52) **Definition** [C^0] *x is* finite $\overset{\text{def}}{\Longleftrightarrow}$ *there exist a number n and a function f such that $f : n \overset{\text{bij}}{\to} x$; otherwise, x is* infinite.

Note that a finite class is necessarily a set.[3.27.1]

Since every finite class is a set, for the sake of emphasis we present the following theorem, summarizing some of the basic properties of finiteness, in S^0.

(3.53) **Theorem** [S^0] *Suppose x is finite.*

1. *Suppose $f : x \overset{\text{inj}}{\to} x$. Then $\operatorname{im} f = x$.*
2. *Suppose $y \subseteq x$. Then y is finite.*
3. *Suppose every member of x is finite. Then $\bigcup x$ is finite.*

Proof 1 Since any finite set is bijective with a number, it suffices to prove this for the case that x is a number. We do this by induction. Since we are working in S^0, we cannot quote (3.51), which would otherwise apply; instead, we proceed as follows. Suppose toward a contradiction that there exist a number n and $f : n \overset{\text{inj}}{\to} n$ such that $\operatorname{im} f \neq n$. Let n be the least such number, and suppose $f : n \overset{\text{inj}}{\to} n$ and $\operatorname{im} f \neq n$. Clearly, n is not 0. n is therefore a successor ordinal, say $n = m + 1$, and

(3.54) *for every $h : m \overset{\text{inj}}{\to} m$, $\operatorname{im} h = m$.*

Let $g = f \upharpoonright m$, and let $l = fm$. If $l = m$ then g maps m injectively to a proper subset of m, contradicting (3.54). Hence, $l \in m$. If $m \notin \operatorname{im} f$ then $g : m \overset{\text{inj}}{\to} m \backslash \{l\}$, again contradicting (3.54). Hence, we may let k be such that $fk = m$. Note that $k \neq m$, so $k \in m$ and $gk = m$; also, since $\operatorname{im} f \neq n$ and m and l are both in $\operatorname{im} f$, there exists $p \in m$ other than l such that $p \notin \operatorname{im} f$. Let

$$h = \{(i, j) \in g \mid i \neq k\} \cup \{(k, l)\}.$$

In other words, h is g redefined at k to have the value l. Then $h : m \overset{\text{inj}}{\to} m$, and $p \notin \operatorname{im} h$, contradicting (3.54).

2 Again, it suffices to prove this for the case that x is a number, and we may use induction. It is enough to show that if every subset of a number m is finite then every subset of $m + 1$ is finite. Thus, suppose $y \subset m + 1$, and every subset of m is finite. Let $y' = y \cap m$. Then y' is finite. If $m \notin y$ then $y = y'$, so y is finite; otherwise, let $f : N \overset{\text{bij}}{\to} y'$, where N is a number, and let $g = f \cup \{(N, m)\}$. Then $g : N + 1 \overset{\text{bij}}{\to} y$.

3 First show that the union of a finite set and a 1-element set is finite, which we have essentially just done. Then show by induction on m that the union of a finite set and an m-element set is finite; hence the union of two finite sets is finite. Then show by induction on m that the union of m finite sets is finite; hence, the union of a finite set of finite sets is finite. □[3.53]

3.3.7 Finite sets and sequences

As noted previously, finite sequences are ubiquitous constructions in mathematics. There are multiple ways to represent a sequence—specifically a finite sequence—as a set. In this book we preferentially use the following.

(3.55) **Definition** $[\mathsf{C}^0]$

1. *s is a* finite sequence*, or simply a* sequence*,* $\overset{\text{def}}{\iff} \operatorname{Seq} s \overset{\text{def}}{\iff}$ *s is a function and* $\operatorname{dom} s$ *is a number, i.e.,* $\operatorname{dom} s \in \omega$*.*
2. *Suppose $n \in \omega$. s is an n-*sequence $\overset{\text{def}}{\iff}$ *s is a function with domain n.*

The use of angle brackets for functions[§ 3.3.4.1] may be adapted to the finite case as follows.

(3.56) **Definition** $[\mathsf{C}^0]$

$$\left\langle {a_0 \cdots a_{n^-} \atop b_0 \cdots b_{n^-}} \right\rangle \overset{\text{def}}{=} \{(a_m, b_m) \mid m \in n \wedge \forall m' \in n \backslash m \; a_{m'} \neq a_m\}.$$

Note that if $a = a_m$ for more than one $m \in n$, then the value assigned to a by $\left\langle {a_0 \cdots a_{n^-} \atop b_0 \cdots b_{n^-}} \right\rangle$ is b_m, where m is the greatest $m' \in n$ such that $a_{m'} = a$.

An n-sequence s is $\left\langle {0 \cdots n^- \atop s_0 \cdots s_{n^-}} \right\rangle$ in this notation, where we have followed the common practice of indicating the argument of a sequence as a subscript.

(3.57) **Definition** $[\mathsf{C}^0]$

1. *Omitting the top line, we have the handy notation*

$$\langle s_0, \ldots, s_{n^-} \rangle$$

   for any n-sequence s.
2. *We use a similar notation for families of (possibly proper) classes indexed by a number n, so that $[A_0, \ldots, A_{n^-}]$ is the family A such that* $\operatorname{dom} A = n$ *and $A_{[m]} = A_m$ for each $m \in n$.*

3.3.7.1 Ordered n-tuples

The following is another approach to the representation of finite sequences. Recall[3.13] that $(x, y) = \{\{x\}, \{x, y\}\}$.

(3.58) **Definition** $[\mathsf{C}^0]$

1. *For any set x, $(x) \overset{\text{def}}{=} x$.*
2. *Note that with this definition, for any sets x, y, $(x, y) = ((x), y)$.*
3. *For any sets x, y, z, $(x, y, z) \overset{\text{def}}{=} ((x, y), z)$.*
4. *For any sets x, y, z, w, $(x, y, z, w) \overset{\text{def}}{=} ((x, y, z), w)$.*
5. *In general, $(x_0, \ldots, x_n) \overset{\text{def}}{=} ((x_0, \ldots, x_{n^-}), x_n)$.*

Note that an n-tuple is also an m-tuple for any nonzero $m < n$. Note also that any set is a 1-tuple, because $x = (x)$. There are no 0-tuples. Note also that (3.58.5) is really a schema, since we have not defined $\ulcorner n\text{-tuple}\urcorner$.

For these and other reasons, the n-tuple construction is awkward in comparison with the n-sequence construction. In fact, the simplest way to say that a set x is an n-tuple, where $n > 0$ is a number, is to say that there exist n-sequences s, t such that

1. $t_0 = s_0$;
2. for each $m < n - 1$, $t_{m+1} = (t_m, s_{m+1})$; and
3. $x = t_{n^-}$.

Note that

$$\begin{aligned} t_0 &= s_0 = (s_0) \\ t_1 &= (t_0, s_1) = (s_0, s_1) \\ t_2 &= (t_1, s_2) = ((s_0, s_1), s_2) = (s_0, s_1, s_2) \\ &\vdots \\ x = t_{n^-} &= (s_0, s_1, \ldots, s_{n^-}). \end{aligned}$$

t is uniquely determined by s, so this construction establishes the bijection

$$\langle s_0, \ldots, s_{n^-} \rangle \mapsto (s_0, \ldots, s_{n^-})$$

between n-sequences and n-tuples for any $n > 0$.[24]

n-sequences have additional advantages over n-tuples, including the ease with which substitution, truncation and concatenation operations are specified, and their generalizability to infinite (wellordered) sequences.

(3.59) **Theorem** [C^0] *Suppose X is a set and Y is finite. Then ${}^YX$[3.35] is a set.*

Proof Let X be given. Since Y is finite, there exist $n \in \omega$ and $f : n \xrightarrow{\text{bij}} Y$. It is easy to use f to define a bijection between ${}^YX$ and ${}^nX$. By Replacement[3.30] ${}^YX$ is a set if ${}^nX$ is a set. It is therefore sufficient to show that ${}^nX$ is a set for each $n \in \omega$.

(3.60) *Suppose toward a contradiction that this is not the case. Let*[3.49] *n be $\in$-minimal such that $n \in \omega$ and ${}^nX$ is not a set.*

Since ${}^0X = \{0\}$, $n \neq 0$. Let m be such that $n = m + 1$.[3.48.3] Since $m \in \omega$[3.49] and $m \in n$, ${}^mX$ is a set.[3.60]

Given $s = \langle s_0, \ldots, s_{m^-} \rangle \in {}^mX$, the function $\{(x, \langle s_0, \ldots, s_{m^-}, x \rangle) \mid x \in X\}$ has domain X, which is a set by hypothesis, so its image is a set. This is $\{t \in {}^nX \mid t \restriction m = s\}$. Let

$$a = \{(s, t) \mid s \in {}^mX \wedge t \in {}^nX \wedge s = t \restriction m\}.$$

Then, as we have just shown,

$$\forall s \in {}^mX \, \exists_S b \, \forall t \big((s, t) \in a \rightarrow t \in b\big),$$

[24] Remember that there are no 0-tuples. There is, on the other hand, a (unique) 0-sequence, viz., 0.

so Collection applies to show that there exists a set b such that

$$\forall s \in {}^{m}X\, \forall t\big((s,t) \in a \rightarrow t \in b\big).$$

Clearly ${}^{n}X \subseteq b$, so ${}^{n}X$ is a set, contrary to (3.60). $\square^{3.59}$

${}^{n}X$ may be regarded as a special case of the following product operation.

(3.61) **Definition** [C^0]

1. *Suppose $n \in \omega$, and $X_0, \ldots, X_{n^-}$ are classes.*

$$X_0 \times \cdots \times X_{n^-} \stackrel{\text{def}}{=} \{f \mid \operatorname{Fcn} f \wedge \operatorname{dom} f = n \wedge \forall m \in n\ fm \in X_m\}.$$

2. *More generally, suppose I is a set and for each $i \in I$, X_i is a class.*

$$\bigtimes_{i \in I} X_i \stackrel{\text{def}}{=} \{f \mid \operatorname{Fcn} f \wedge \operatorname{dom} f = I \wedge \forall i \in I\ fi \in X_i\}.$$

(Thus, $X_0 \times \cdots \times X_{n^-} = \bigtimes_{i\in n} X_i$.)

We may define homologous constructs using n-tuples. Thus

Definition [C^0]

1. $X_0 \dot\times \cdots \dot\times X_{n^-} \stackrel{\text{def}}{=} \{(x_0, \ldots, x_{n^-}) \mid \forall i \in n\ x_i \in X_i\}$.
2. $X^n \stackrel{\text{def}}{=} \underbrace{X \dot\times \cdots \dot\times X}_{n \text{ factors}}$.

3.3.8 Relations

(3.62) **Definition** [C^0]

1. *R is an n-*ary relation $\stackrel{\text{def}}{\iff}$ *$n \in \omega$ and R is a class of n-sequences.*
2. *R is a* relation $\stackrel{\text{def}}{\iff}$ *R is an n-ary relation for some n.*
3. *F is an n-*ary function $\stackrel{\text{def}}{\iff}$ *F is a function and* $\operatorname{dom} F$ *is an n-ary relation.*

We use *nulary*, *unary*, *binary*, *ternary*, ..., for 0-*ary*, 1-*ary*, 2-*ary*, 3-*ary*, ... (Since the only 0-sequence is 0, the only nulary relations are 0 and $\{0\} = 1$.)

relation is often defined in terms of tuples instead of sequences. This has the virtue of simplicity in the binary case, as it makes use of the primary notion of ordered pair in place of the more elaborate notion of 2-sequence, but it is otherwise disadvantageous.

Recall that the interpretation of an n-ary predicate or operation index in a structure is respectively an n-ary relation or function in the above sense, and the same notation may be used:

1. *If R is an n-ary relation then for any $x_0, \ldots, x_{n^-}$*

$$R(x_0, \ldots, x_{n^-}) \stackrel{\text{def}}{\iff} \langle x_0, \ldots, x_{n^-}\rangle \in R.$$

2. *If F is an n-ary function then*

$$F(x_0, \dots, x_{n^-}) \overset{\text{def}}{=} F\langle x_0, \dots, x_{n^-}\rangle \ \big(= F(\langle x_0, \dots, x_{n^-}\rangle)\big).$$[25]

Binary relations are of particular importance, and many definitions are particular to this case. We extend to binary relations the familiar practice of placing a binary predicate symbol between its arguments.

Definition $[\mathsf{C}^0]$ *Suppose R is a binary relation. Then for any x, y, $x \mathrel{R} y \overset{\text{def}}{\iff} \langle x, y\rangle \in R$.*

Some of the definitions given above[3.26] for prefunctions are applicable to binary relations.

(3.63) **Definition** $[\mathsf{C}^0]$ *Suppose R is a binary relation and X, Y are classes.*

1. *The* domain *of R* $\overset{\text{def}}{=} \operatorname{dom} R \overset{\text{def}}{=} \{x \mid \exists y\ x \mathrel{R} y\}$.
2. *The* image *of R* $\overset{\text{def}}{=} \operatorname{im} R \overset{\text{def}}{=} \{y \mid \exists x\ x \mathrel{R} y\}$.
3. $R \restriction X \overset{\text{def}}{=} \{\langle x, y\rangle \in R \mid x \in X\}$.
4. *The* image of X by R $\overset{\text{def}}{=} R^{\rightarrow}X \overset{\text{def}}{=} \{y \mid \exists x \in X\ x \mathrel{R} y\}$.
5. *The* inverse of R $\overset{\text{def}}{=} R^{-1} \overset{\text{def}}{=} \{\langle x, y\rangle \mid y \mathrel{R} x\}$.
6. *The* inverse image *or* preimage of Y by R $\overset{\text{def}}{=} R^{\leftarrow}Y \overset{\text{def}}{=} \{x \mid \exists y \in Y\ x \mathrel{R} y\}$. *Thus, $R^{\leftarrow}Y = (R^{-1})^{\rightarrow}Y$.*
7. *The* field *of R* $\overset{\text{def}}{=} \operatorname{fld} R \overset{\text{def}}{=} \operatorname{dom} R \cup \operatorname{im} R$.
8. *R is a* relation on X $\overset{\text{def}}{\iff} \operatorname{fld} R \subseteq X$.

(3.64) **Theorem** $[\mathsf{C}^0]$ *Suppose R is a prefunction or binary relation and R is a set. Then $\operatorname{dom} R$, $\operatorname{im} R$, $\operatorname{fld} R$, and R^{-1} are sets; and for any class X, $R \restriction X$, $R^{\rightarrow}X$, and $R^{\leftarrow}X$ are sets.*

Proof Straightforward. □[3.64]

3.3.9 Equivalence relations

(3.65) **Definition** $[\mathsf{C}^0]$ *Suppose R is a binary relation.*

1. *R is* reflexive $\overset{\text{def}}{\iff} \forall x \in \operatorname{fld} R\, (x \mathrel{R} x)$.
2. *R is* irreflexive $\overset{\text{def}}{\iff} \forall x\ \neg\, x \mathrel{R} x$.
3. *R is* symmetric $\overset{\text{def}}{\iff} \forall x, y\, (x \mathrel{R} y \rightarrow y \mathrel{R} x)$.
4. *R is* asymmetric $\overset{\text{def}}{\iff} \forall x, y\, (x \mathrel{R} y \rightarrow \neg\, y \mathrel{R} x)$.[26]

[25] In Chapters 1 and 2 we were careful to use the latter notation in keeping with the greater level of formality of those chapters. Note that round brackets are used here as generic grouping symbols. If we used tuples instead of sequences to define n-arity, we could interpret the round brackets in their specific sense[3.58] without any further explanation.

[26] Note that 'asymmetric' does not mean 'not symmetric'. A relation is asymmetric iff it is both antisymmetric and irreflexive. 'asymmetric' in this sense is not as well entrenched in the literature as 'antisymmetric', but it is useful for our purposes.

5. R *is* antisymmetric $\overset{\text{def}}{\iff} \forall x, y\,(x\,R\,y \wedge y\,R\,x \rightarrow x = y)$.
6. R *is* transitive $\overset{\text{def}}{\iff} \forall x, y, z\,(x\,R\,y \wedge y\,R\,z \rightarrow x\,R\,z)$.

Definition $[\mathsf{C}^0]$

1. *R is an* equivalence relation $\overset{\text{def}}{\iff}$ *R is a reflexive, symmetric, transitive binary relation.*
2. *R is an* equivalence relation on X $\overset{\text{def}}{\iff}$ *R is an equivalence relation and $X =$* $\operatorname{dom} R$ *(equivalently, $X = \operatorname{im} R$ or $X = \operatorname{fld} R$, since R is reflexive).*[27]
3. *If R is an equivalence relation on X and $x \in X$, the R*-equivalence class of x $\overset{\text{def}}{=} [x]_R \overset{\text{def}}{=} \{y \mid y\,R\,x\}$.[28]

(3.66) **Theorem** $[\mathsf{C}^0]$ *Suppose E is an equivalence relation on X. The E-equivalence classes are disjoint, and their union is X.*

Proof Straightforward. $\square^{3.66}$

(3.67) **Definition** $[\mathsf{C}^0]$

1. *Suppose E is an equivalence relation on X, and R is an n-ary relation on X.* R is E-invariant $\overset{\text{def}}{\iff}$ *for all $x_0, \ldots, x_{n^-}, y_0, \ldots, y_{n^-} \in X$, if $x_i\,E\,y_i$ for all $i \in n$, then*
$$R(x_0, \ldots, x_{n^-}) \leftrightarrow R(y_0, \ldots, y_{n^-}).$$
2. *Suppose R is an n-ary relation on X. Define the binary relation $\equiv^R$ on X by the condition that for all $x, y \in X$, $x \equiv^R y$ $\overset{\text{def}}{\iff}$ for any $m \in n$ and any $\langle x_0, \ldots, x_{n^-}\rangle$ and $\langle y_0, \ldots, y_{n^-}\rangle$ in ${}^nX$, if $x_i = y_i$ for all $i \neq m$, $x_m = x$, and $y_m = y$, then*
$$R(x_0, \ldots, x_{n^-}) \leftrightarrow R(y_0, \ldots, y_{n^-}).$$

Clearly $\equiv^R$ is the largest equivalence relation E on X such that R is E-invariant.

Various notions of E-invariance are useful for functions, and it is best to provide *ad hoc* definitions of these as circumstances arise.

(3.68) **Definition** $[\mathsf{C}^0]$ *Suppose the equivalence classes of an equivalence relation E on a class X are all sets (as they are if X is a set, for example). Let $[x] = [x]_E$ for $x \in X$.*

1. X modulo *or* mod E $\overset{\text{def}}{=} X/E \overset{\text{def}}{=} \{[x] \mid x \in X\}$. *This is also called the* quotient *of X mod E.*
2. *Suppose S is an n-ary relation on X that is E-invariant.*[3.67] *Then* S mod E $\overset{\text{def}}{=} S/E \overset{\text{def}}{=}$
$$\{\langle [x_0], \ldots, [x_{n^-}]\rangle \mid S(x_0, \ldots, x_{n^-})\}.$$

[27] It is not enough that R be an equivalence relation and be a relation on X in the sense of (3.63.8).

[28] The subscript 'R' is often omitted when there is only one relevant equivalence relation.

3. *Similarly, if* $F : {}^nX \rightharpoonup Y$ *is* E*-invariant in the sense that for all* $\langle x_0, \dots, x_{n^-}\rangle$ *and* $\langle y_0, \dots, y_{n^-}\rangle$ *in* $\operatorname{dom} F$*, if* $x_i \mathrel{E} y_i$ *for* $i = 1, \dots, n$*, then*

$$F(x_0, \dots, x_{n^-}) = F(y_0, \dots, y_{n^-}),$$

   then we may define F/E *as the function with domain*

$$\{\langle [x_0], \dots, [x_{n^-}]\rangle \mid \langle x_0, \dots, x_{n^-}\rangle \in \operatorname{dom} F\}$$

   such that

$$(F/E)([x_0], \dots, [x_{n^-}]) = F(x_0, \dots, x_{n^-})$$

   for any $\langle x_0, \dots, x_{n^-}\rangle \in \operatorname{dom} F$*.*

4. *If* $Y = X$*, then we may proceed differently. If we suppose* F *is* E*-invariant in the weaker sense that for all* $\langle x_0, \dots, x_{n^-}\rangle$ *and* $\langle y_0, \dots, y_{n^-}\rangle$ *in* $\operatorname{dom} F$*, if* $x_i \mathrel{E} y_i$ *for* $i = 1, \dots, n$*, then*

$$F(x_0, \dots, x_{n^-}) \mathrel{E} F(y_0, \dots, y_{n^-}),$$

   then we may define F/E *as the function with domain*

$$\{\langle [x_0], \dots, [x_{n^-}]\rangle \mid \langle x_0, \dots, x_{n^-}\rangle \in \operatorname{dom} F\}$$

   such that

$$(F/E)([x_0], \dots, [x_{n^-}]) = [F(x_0, \dots, x_{n^-})]$$

   for any $\langle x_0, \dots, x_{n^-}\rangle \in \operatorname{dom} F$*.*

Definition $[\mathsf{C}^0]$ *Suppose* X *is a class.* $\boldsymbol{=}^X \overset{\text{def}}{=} \{\langle x, x\rangle \mid x \in X\}$*, i.e.,* $\boldsymbol{=}^X$ *is the identity relation on* X*.*[29]

The following observation is trivial.

Suppose E *is an equivalence relation on a class* X*. Then* E *is* E*-invariant. If the equivalence classes of* E *are sets then* $E/E = \boldsymbol{=}^{X/E}$*.*

3.3.10 Order relations

(3.69) **Definition** $[\mathsf{C}^0]$ *Suppose* R *is a relation and* X *is a class.*

1. R preorders $X \overset{\text{def}}{\iff}$ *R is transitive and reflexive and* $X = \operatorname{fld} R$*.*
2. R totally preorders $X \overset{\text{def}}{\iff}$ *R preorders X and* $\forall x, y \in X\,(x \mathrel{R} y \vee y \mathrel{R} x)$*. We use 'linear' interchangeably with 'total' in describing orders.*

(3.70) *Note that if* R *is a preorder then* $\equiv^R = R \cap R^{-1}$*.*[3.67.2]

These notions of order can be modified by adding the condition of antisymmetry.

(3.71) **Definition** $[\mathsf{C}^0]$ *Suppose* R *is a relation and* X *is a class.*

1. R partially orders $X \overset{\text{def}}{\iff}$ *R preorders X and R is antisymmetric.*

[29] The use of the bold symbol '$\boldsymbol{=}$' here is not mandated by our notational conventions; its purpose is merely to avoid confusion with the more general use of '$=$'.

2. R totally orders $X \overset{\text{def}}{\iff}$ *R totally preorders X and R is antisymmetric.*

The use of the prefix 'pre' is explained as follows. Suppose R preorders X. Let $\equiv$ be $\equiv^R$ $(= R \cap R^{-1})$.[3.70] Suppose for each $x \in X$, the $\equiv$-equivalence class $[x] = [x]_\equiv$ of x is a set. Then $R/\equiv$[3.68] partially orders $X/\equiv$. If R totally preorders X then $R/\equiv$ totally orders $X/\equiv$.

Note also that a preorder R on X is a partial order iff $\equiv^R = =^X$.

(3.69.1) and (3.71) may also be modified by imposing the condition of irreflexivity in lieu of reflexivity and hence asymmetry in addition to antisymmetry.[3.65.4] Keep in mind that a relation R is asymmetric iff it is both antisymmetric and irreflexive; and if R is transitive and irreflexive then it is asymmetric.

Definition $[\mathsf{C}^0]$ R strongly preorders $X \overset{\text{def}}{\iff}$ *R is transitive and irreflexive and $X = \operatorname{fld} R$. We use 'irreflexive' and 'strict' interchangeably with 'strong' in describing orders. We use 'weak' and 'reflexive' to describe the previous notions of order*[3.69, 3.71] *by way of distinction or emphasis.*

(3.72) **Theorem** $[\mathsf{C}^0]$ *Suppose R and S are respectively a (weak) preorder and a strong preorder on a class X.*

1. $R \setminus \equiv^R$ $(= R \setminus R^{-1})$[3.70] *is a strong preorder on X.*
2. $S \cup \equiv^S$ *is a (weak) preorder on X.*
3. *The following are equivalent:*
   1. $S = R \setminus \equiv^R$
   2. $R = S \cup \equiv^S$

Proof Straightforward. □[3.72]

Definition $[\mathsf{C}^0]$ *Suppose R and S are respectively a (weak) preorder and a strong preorder on a class X. R and S* correspond $\overset{\text{def}}{\iff}$ *$S = R \setminus \equiv^R$ (iff $R = S \cup \equiv^S$).*[3.72.3] *Note that if R and S correspond, then R uniquely determines S and* vice versa*.*

Definition $[\mathsf{C}^0]$ *Suppose S is a strong preorder. Then S is a* strong total preorder*,* strong partial order*, or* strong total order*, according as the corresponding weak preorder is a total preorder, partial order, or total order, respectively.*

(3.73)

1. *It is commonplace to use '$<$' and related symbols, such as '$\prec$', to represent a strong order and '$\leqslant$', etc., to represent the corresponding weak order.*
2. *For specificity, we refer to the order relation in conjunction with its field as a structure, often using the informal convention that $(X; <)$, for example, is the structure with domain X and predicate $<$.*[30] *If the predicate $<$ is understood from context, we may refer to the structure by the name of its domain.*

The symbol '$\leqslant$' is intended to represent the disjunction of $<$ and $=$, as indicated by the usual reading 'less than or equal'. If $\leqslant$ and $<$ are corresponding weak and strong partial orders on X, then $\equiv^{\leqslant} = \equiv^{<} = =^X$, so this reading is appropriate.[3.72.3] In the more general case of preorders, the reading 'less than or equivalent' would be more appropriate.

[30] Thus, $(X; <)$ and $(Y; <)$ may have order relations that are completely unrelated.

3.4 Wellfoundedness

3.4.1 Wellfounded relations and wellorders

(3.74) **Definition** [C^0] *Suppose R is a binary relation on a class X, $X' \subseteq X$, and $x \in X'$.*

1. x *is* R-minimal in X' $\overset{\text{def}}{\Longleftrightarrow} \forall x' \in X' \, (x' \, R \, x \to x \equiv^R x')$.
2. x *is* R-maximal in X' $\overset{\text{def}}{\Longleftrightarrow} \forall x' \in X' \, (x \, R \, x' \to x \equiv^R x')$.
3. x *is* R-minimum in X' $\overset{\text{def}}{\Longleftrightarrow} \forall x' \in X' \, (x' \neq x \to x \, R \, x' \wedge \neg \, x' \, R \, x)$.
4. x *is* R-maximum in X' $\overset{\text{def}}{\Longleftrightarrow} \forall x' \in X' \, (x' \neq x \to x' \, R \, x \wedge \neg \, x \, R \, x')$.

If $X' = X$ we may omit the phrase 'in X''.

Note that a class $X' \subseteq X$ can have at most one R-minimum and at most one R-maximum. (3.74) is most often useful when R has some order-like features, but it does not have to be transitive.

(3.75) *Note that if R is irreflexive and x is R-minimal in X' then $\forall x' \in X' \; \neg(x' \, R \, x)$. (If $x' \, R \, x$ implies $x' \equiv^R x$ then by the definition of $\equiv^R$, $x' \, R \, x$ implies $x \, R \, x$, contradicting the irreflexivity of R.) Similarly, if x is R-maximal in X' then $\forall x' \in X' \; \neg(x \, R \, x')$.*

(3.76) **Definition** [C^0] *Suppose R is a binary relation on a class X.*

1. R *is* wellfounded $\overset{\text{def}}{\Longleftrightarrow}$
$$\forall X' \subseteq X \, (X' \neq 0 \to \exists x \in X' \, \forall x' \in X' \, (x' \, R \, x \to x' \equiv^R x)),$$
i.e., x is R-minimal in X'.
2. R *is a* strong *or* weak prewellorder $\overset{\text{def}}{\Longleftrightarrow}$ *R is a respectively strong or weak wellfounded total preorder.*
3. R *is a* strong *or* weak wellorder $\overset{\text{def}}{\Longleftrightarrow}$ *R is a respectively strong or weak wellfounded total order.*

(3.76.1) is a variation on the more common definition:

$$\forall X' \subseteq X \, (X' \neq 0 \to \exists x \in X' \, \forall x' \in X' \; \neg(x' \, R \, x)). \tag{3.77}$$

Note that by this definition a reflexive relation (on a nonempty class) is not wellfounded. The definition we have given[3.76] is, however, still substantive in this case. Note also that (3.76.1) and (3.77) coincide for irreflexive relations.[3.75]

The following theorem is just a restatement of wellfoundedness in the form of an induction principle. (3.50) is a special case.

(3.78) **Theorem: Induction** [C^0] *Suppose R is a wellfounded irreflexive relation on a class X, and suppose $Y \subseteq X$ is such that $\forall x \in X \, (R^{\leftarrow}\{x\} \subseteq Y \to x \in Y)$. Then $Y = X$.*

Proof Let $X' = X \backslash Y$, and suppose toward a contradiction that $X' \neq 0$. Using the wellfoundedness and irreflexivity of R,[3.75] let $x \in X'$ be such that $\forall x' \in X' \; \neg(x' \, R \, x)$. Then $R^{\leftarrow}\{x\} \subseteq Y$, so $x \in Y$ by hypothesis, so $x \notin X'$. □[3.78]

Note that $\mathsf{Foundation}$[3.16] essentially states that $\in$ is wellfounded and irreflexive.

3.4.2 Recursion

(3.79) **Definition** [C^0] *A relation R is* setlike $\overset{\text{def}}{\iff}$ *for all sets x, $R^{\leftarrow}\{x\}$ is a set.*

This use of 'setlike' derives from the fact that $\in$ has this property.

(3.80) **Theorem: Definition by recursion** [C^0] *Suppose R is an irreflexive wellfounded setlike relation on a class X, and suppose G is a function such that* $\operatorname{dom} G$ *consists of all $\langle x, f\rangle$ such that $x \in X$, f is a function, and* $\operatorname{dom} f = R^{\leftarrow}\{x\}$. *Then there exists a unique function F such that*

1. $\operatorname{dom} F = X$; *and*
2. $\forall x \in X \;\; Fx = G\langle x, F \restriction (R^{\leftarrow}\{x\})\rangle$.

Proof

(3.81) *For the nonce, say that f is* acceptable *iff*

1. *f is a nonempty set that is a function;*
2. $\operatorname{dom} f \subseteq X$*;*
3. $R^{\leftarrow} \operatorname{dom} f \subseteq \operatorname{dom} f$*, i.e., if $x \in \operatorname{dom} f$ and $y \, R \, x$ then $y \in \operatorname{dom} f$; and*
4. $\forall x \in \operatorname{dom} f \;\; fx = G\langle x, f \restriction (R^{\leftarrow}\{x\}\rangle$.

(3.82) **Claim** *If f and f' are acceptable and $x \in \operatorname{dom} f \cap \operatorname{dom} f'$, then $fx = f'x$.*

Proof Let $X' = \{x \in \operatorname{dom} f \cap \operatorname{dom} f' \mid fx \neq fx'\}$. We claim that $X' = 0$. Suppose toward a contradiction that $X' \neq 0$. Since R is wellfounded, let $x \in X'$ be R-minimal in X', so that, since R is irreflexive,[3.75] for all y, if $y \, R \, x$ then $y \notin X'$. Since $x \in \operatorname{dom} f \cap \operatorname{dom} f'$ and f, f' are acceptable,[3.81.3] for any y, $y \, R \, x \rightarrow y \in \operatorname{dom} f \cap \operatorname{dom} f'$, so if $y \notin X'$ then $fy = f'y$; hence,

$$f \restriction (R^{\leftarrow}\{x\}) = f' \restriction (R^{\leftarrow}\{x\}),$$

so $fx = f'x$, which is inconsistent with the membership of x in X'. □[3.82]

(3.83) **Claim** *Suppose $x \in X$ and there exists an acceptable f with $x \in \operatorname{dom} f$. Then there exists a $\subseteq$-minimum such f.*

Proof Let f_0 be the class of (y, a) such that for every acceptable f, if $x \in \operatorname{dom} f$ then $fy = a$, i.e., f_0 is the intersection of all acceptable f with $x \in \operatorname{dom} f$. Such a class exists because acceptability implies set-ness and is defined using only set-restricted quantifiers. f_0 is a set because it is included in a set, viz., any acceptable f with $x \in \operatorname{dom} f$, of which at least one exists by hypothesis. f_0 is nonempty because all acceptable functions agree on their common domain, so, letting a be the common value at x of all acceptable functions defined at x, $(x, a) \in f_0$. Thus f_0 satisfies (3.81.1). It is easy to check that it also satisfies (3.81.2–4), so it is acceptable and it is therefore the $\subseteq$-minimum acceptable f with $x \in \operatorname{dom} f$. □[3.83]

Let H be the class of (x, f) such that $x \in X$ and f is the $\subseteq$-minimum acceptable f with $x \in \operatorname{dom} f$. Note that H is a function, and we will use functional notation, writing $\ulcorner Hx = f \urcorner$ for $\ulcorner (x, f) \in H \urcorner$.

(3.84) *By (3.83)* $\operatorname{dom} H$ *is the class of $x \in X$ such that there exists an acceptable f with $x \in \operatorname{dom} f$.*

(3.85) **Claim** $\operatorname{dom} H = X$.

Proof Let $X' = X \setminus \operatorname{dom} H$. The claim is that $X' = 0$, so suppose toward a contradiction that $X' \neq 0$. Let x be R-minimal in X'. Then $R^{\leftarrow}\{x\} \subseteq \operatorname{dom} H$. Since R is setlike by hypothesis, $R^{\leftarrow}\{x\}$ is a set, and by Replacement[3.30] and Union,[3.23] $f = \bigcup \big(H^{\rightarrow}(R^{\leftarrow}\{x\})\big)$ is a set. By (3.82) f is a function, so it satisfies (3.81.1). It is easy to check that it also satisfies (3.81.2–4), so f is acceptable. Since $x \in \operatorname{dom} f$, $x \in \operatorname{dom} H$,[3.84] contrary to our supposition. □[3.85]

Let $F = \bigcup \operatorname{im} H$. It is readily shown that F has properties 3.80.1–2. The proof of Claim 3.82 is readily adapted to show that F is unique with these properties. □[3.80]

Before we leave the topic of recursion we point out that we may say that $x \in X$ and a is the value at x of the function defined recursively from R, X, and G as in Theorem 3.80, using only set-restricted quantification.

For example, we may say $\ulcorner x \in X$ and there exists an f that is acceptable *vis-à-vis* R, X, and G, such that $(x, a) \in f\urcorner$ or $\ulcorner x \in X$ and for every f that is acceptable *vis-à-vis* R, X, and G, $(x, a) \in f\urcorner$. In the next chapter we will have occasion to examine more closely the definability of functions defined by recursion, and the existence of the two forms just given will be seen to be critical; but for now, it suffices to note that they employ only set-restricted quantification.

As we have noted above, Foundation implies that $\in$ is wellfounded and irreflexive, and, as also noted, $\in$ is setlike, so Foundation justifies the use of definition by $\in$-recursion, as well as the method of proof by $\in$-induction, which is essentially the content of Theorem 3.78.

3.4.3 Rank and the cumulative hierarchy

(3.86) **Definition** [C^0] *Suppose R is an irreflexive wellfounded setlike relation. The R-*rank function $\stackrel{\text{def}}{=} \operatorname{rk}^R \stackrel{\text{def}}{=}$ *the function on* $\operatorname{fld} R$ *defined recursively*[3.80] *by the condition that*

$$\operatorname{rk}^R x = \bigcup_{y\,R\,x} (\operatorname{rk}^R y)^+.^{3.45}$$

Note that it follows from the definition of z^+ as $z \cup \{z\}$ that $y\,R\,x$ implies $\operatorname{rk}^R y \in \operatorname{rk}^R x$.

(3.87) **Theorem** [C^0] *Suppose R is an irreflexive wellfounded setlike relation. Then* $\operatorname{im} \operatorname{rk}^R$ *is an initial segment of* Ord *(therefore either an ordinal or* Ord*).*

Proof Let $X = \operatorname{fld} R$ and let $F = \operatorname{rk}^R$. We use R-induction[3.78] to show that $\forall x \in X \; \operatorname{rk}^R x \in \mathrm{Ord}$. It suffices[3.78] to show that for every $x \in X$, if $\forall y \in R^{\leftarrow}\{x\} \; \operatorname{rk}^R y \in \mathrm{Ord}$ then $\operatorname{rk}^R x \in \mathrm{Ord}$. This follows immediately from the definition of $\operatorname{rk}^R x$ as $\bigcup_{y\,R\,x} (\operatorname{rk}^R y)^+$.[3.46 3.44]

Thus, $\operatorname{im} \operatorname{rk}^R \subseteq \mathrm{Ord}$. If $\operatorname{im} \operatorname{rk}^R = \mathrm{Ord}$ we are finished; otherwise, let α be the least ordinal not in $\operatorname{im} \operatorname{rk}^R$. We will show that $\operatorname{im} \operatorname{rk}^R = \alpha$, and then we will be finished. Suppose toward a contradiction that this is not the case. Let $Y = \{x \in X \mid \operatorname{rk}^R x \geqslant \alpha\}$. Then $Y \neq 0$. Let x be an R-minimal element of Y. As noted above, for any $y \in R^{\leftarrow}\{x\}$, $\operatorname{rk}^R y < \operatorname{rk}^R x$ (since $<$ is $\in$ for ordinals). By the R-minimality of x in Y, for any $y \in R^{\leftarrow}\{x\}$, $\operatorname{rk}^R y < \alpha$, whence $(\operatorname{rk}^R y)^+ \leqslant \alpha$.

Thus, $\mathrm{rk}^R x \leqslant \alpha$. Since $\mathrm{rk}^R x \geqslant \alpha$, $\mathrm{rk}^R x = \alpha$. Thus, $\alpha \in \mathrm{im}\,\mathrm{rk}^R$; contradiction. $\square^{3.87}$

(3.88) **Definition** [C^0] *Suppose R is an irreflexive wellfounded setlike relation. The* rank of $R \overset{\text{def}}{=} \mathrm{rk}\, R \overset{\text{def}}{=} \mathrm{im}\,\mathrm{rk}^R$.

Of particular interest are rank functions for the membership relation when it is wellfounded.

Definition [C^0] *Suppose X is a class.* $\in^X \overset{\text{def}}{=} \{\langle x, y\rangle \in X \times X \mid x \in y\}$.

It is easy to see that if α is an ordinal then $\mathrm{rk}^{\in^\alpha}$ is the identity function on α and $\mathrm{rk} \in^\alpha = \alpha$.

(3.89) **Theorem** [C^0]

1. *Suppose x and y are transitive classes, $x \subseteq y$, and $\in^y$ is wellfounded. Then $\in^y$ is irreflexive (and of course setlike) and* $\mathrm{rk}^{\in^x} = \mathrm{rk}^{\in^y} \restriction x$.
2. *Suppose x and y are transitive classes and $\in^x$ and $\in^y$ are wellfounded. Let $z = x \cap y$. Then z is transitive and* $\mathrm{rk}^{\in^x} \restriction z = \mathrm{rk}^{\in^z} = \mathrm{rk}^{\in^y} \restriction z$.

Proof Straightforward $\in$-inductions. $\square^{3.89}$

(3.90) **Definition** [C^0] *Suppose x is a set.*

1. *x is* subtransitive $\overset{\text{def}}{\iff}$ *there exists a transitive set $y \supseteq x$.*
2. *Suppose x is subtransitive. Then the* transitive closure of $x \overset{\text{def}}{=} \mathrm{tc}\, x \overset{\text{def}}{=}$ *the intersection of all transitive sets $y \supseteq x$. Note that if x is subtransitive then $\mathrm{tc}\, x$ is transitive and is the $\subseteq$-least transitive set $y \supseteq x$.*
3. *x is* ranked $\overset{\text{def}}{\iff}$ *x is subtransitive and $\in^{\mathrm{tc}\, x}$ is wellfounded.*
4. *If x is ranked the* rank of $x \overset{\text{def}}{=} \mathrm{rk} \in^{\mathrm{tc}\, x}$.

We are now in a position to define the cumulative hierarchy.

Definition [C^0]

1. *Suppose* $\alpha \in \mathrm{Ord}$. $V_\alpha \overset{\text{def}}{=} \{x \mid x \text{ is ranked} \wedge \mathrm{rk}\, x < \alpha\}$.
2. $V_\Omega \overset{\text{def}}{=}$ *the class of all ranked sets.*

NB: *The classes V_α are not necessarily sets.*

(3.91) **Theorem** [C^0]

1. *V_Ω is transitive.*
2. *$\in^{V_\Omega}$ is wellfounded.*
3. *Any subset of V_Ω is in V_Ω.*

Proof Obviously, if $x \in y$ then

1. if y is subtransitive then x is subtransitive; and

2. if y is ranked then x is ranked.

Hence, V_Ω is transitive. Similarly, the wellfoundedness of $\in^{V_\Omega}$ follows from the wellfoundedness of $\in^{\mathrm{tc}\,x}$ for every $x \in V_\Omega$.

Now suppose x is a set and $x \subseteq V_\Omega$. Let $y = \bigcup_{z\in x} \mathrm{tc}\{z\}$. Then y is transitive and includes x; in fact, $y = \mathrm{tc}\,x$. For all $z \in x$, $z \in V_\Omega$, so $\in^{\mathrm{tc}\,z}$ is wellfounded, from which it is easy to see that $\in^{\mathrm{tc}\{z\}}$ is wellfounded, from which it is easy to see that $\in^{y}$ is wellfounded, so $x \in V_\Omega$. □[3.91]

The following theorem summarizes the main features of the classes V_α ($\alpha \in \mathrm{Ord}$).

(3.92) **Theorem** [C^0]

1. *For any ordinal α, V_α is transitive.*
2. *If $\beta \leqslant \alpha$ then $V_\beta \subseteq V_\alpha$.*
3. $V_0 = 0$.
4. *For any $\alpha \in \mathrm{Ord}$, $V_{\alpha+1} = \mathcal{P}\,V_\alpha$.*[3.24]
5. *For any limit ordinal α, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.*
6. $V_\Omega = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$.

Proof 1, 2, 3, 5, 6 Straightforward.

4 Suppose $x \in V_{\alpha+1}$. Then $\mathrm{rk}\,x \leqslant \alpha$, so $\forall y \in x\ \mathrm{rk}\,y < \alpha$. Hence $x \subseteq V_\alpha$, i.e., $x \in \mathcal{P}\,V_\alpha$. On the other hand, suppose $x \in \mathcal{P}\,V_\alpha$. Then for any $y \in x$, $y \in V_\alpha$, so $\mathrm{rk}\,y < \alpha$. Hence, $\mathrm{rk}\,x \leqslant \alpha < \alpha + 1$, so $x \in V_{\alpha+1}$. □[3.92]

(3.93) *Thus, the classes V_α correspond precisely to the cumulative hierarchy as defined informally at the beginning of this chapter.*[§ 3.2.1]

3.4.3.1 Hereditarity

(3.94) *For any property of sets, we say that a set x has that property* hereditarily $\overset{\text{def}}{\Longleftrightarrow}$ *x is ranked and every set in $\mathrm{tc}\{x\}$*[3.90.2] *has the property.*

Of particular interest is the notion of hereditary finiteness, and we make the following definition.

(3.95) **Definition** [S^0] *A set x is* hereditarily finite $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{HF}\,x$ $\overset{\text{def}}{\Longleftrightarrow}$ *x is ranked and for all $y \in \mathrm{tc}\{x\}$, y is finite.*

As usual, in a class theory, the class $\mathrm{HF} \overset{\text{def}}{=} \{x \mid \mathrm{HF}\,x\}$.

(3.96) **Theorem** [C^0]

1. *Suppose x is a finite set. Then $\mathcal{P}\,x$ is a finite set.*
2. *Every set in V_ω is finite.*
3. $V_\omega = \mathrm{HF}$.

Proof 1 It suffices to show that for each $m \in \omega$, $\mathcal{P}m$ is a finite set. This we do by induction on $m \in \omega$. It is trivially true for $m = 0$, since $\mathcal{P}0 = 1$. Suppose therefore that $m \in \omega$ and $\mathcal{P}m$ is finite. We will show that $\mathcal{P}(m+1)$ is finite. To this end $f : \mathcal{P}m \overset{\text{bij}}{\to} N$, where $N \in \omega$. Let $g : N + 1 \to \omega$ be defined recursively by the conditions

$$g(0) = N$$
$$g(n+1) = g(n) + 1.$$

It follows easily by induction on $n \in N + 1$ that $g(n) \in \omega$. In particular, $g(N) \in \omega$. (In other words, if N is finite then twice N is finite.)

Let $h : \mathcal{P}(m+1) \to \omega$ be such that

$$h(X) = \begin{cases} f(X) & \text{if } m \notin X \\ g(f(X \backslash \{m\})) & \text{if } m \in X. \end{cases}$$

Then $h : \mathcal{P}(m+1) \overset{\text{bij}}{\to} g(N)$.

2 Since every set in V_ω is included in V_n for some $n \in \omega$, it is enough to show that each V_n is finite.[3.53.2] Since $V_{n+1} = \mathcal{P}V_n$,[3.92.4] this follows from (3.96.1) by induction on $n \in \omega$.

3 If $x \in V_\omega$ then $\operatorname{tc}\{x\} \subseteq V_\omega$,[3.92.1] so[3.96.2] x is hereditarily finite. On the other hand, if x is HF then it is easy to show by $\in$-induction that $\operatorname{rk} y$ is finite for every $y \in \operatorname{tc}\{x\}$ (since by (3.53.3) the supremum (i.e.,[3.44] union) of a finite set of finite ordinals is finite); hence, in particular, $\operatorname{rk} x$ is finite, i.e., $x \in V_\omega$. □[3.96]

In view of (3.96.3) we will use 'HF' and 'V_ω' more or less interchangeably.

3.5 Relativization

We now consider a specialization of the notion of relativization,[2.112] defined in Chapter 2, that is particularly useful in the theory of membership. Recall[2.111] the use of the superscript '$^+$' to indicate extension by definition. In the case at hand, s^+ is an expansion of the basic signature s of pure set theory by the addition of predicate and operation indices whose definitions are added to S^0 to create the extension S^{0+}. C^{0+} is corresponding extension of C^0. These are conservative extensions of their base theories, and are conventionally not explicitly recognized. In the interest of notational brevity, in our development of the theory of membership we generally follow this convention, but it is sometimes advantageous to use the more precise notation; and this is one of those times.

(3.97) **Definition** [S^0] *Suppose ϕ is an s-formula, τ is an s^+-term, and no variable occurs in both ϕ and τ. The* relativization of ϕ to τ $\overset{\text{def}}{=}$ ϕ^τ $\overset{\text{def}}{=}$ *the result of restricting each quantification in ϕ to τ.*[31]

Note that ϕ is required to be an s-formula, employing only the membership and identity predicates, whereas τ is permitted to have additional defined predicates and operations.

[31] Technically this is defined for a given term τ by recursion on complexity of formulas ϕ.

Theorem 3.98 states the obvious relationship between an s-formula ϕ relativized to a class M[32] To set this in the framework of (3.97) and the statement that ϕ holds in M, i.e., in the structure $(M; \in)$. (3.98) states that for any s-formula ϕ, a certain sentence is a theorem of C^{0+}. This sentence both uses and mentions ϕ, i.e., it uses both ϕ and a name for ϕ. Since (3.98) refers to an arbitrary s-formula ϕ, it is advantageous to be able to refer to a particular name for any given formula. In our general discussion of language in Chapter 1 we have defined specific expression-building operations, viz., $\neg$, $\vee$, etc., applicable to any language. We have also defined the operation $=$, applicable to any language with identity; and we have defined $\in$, applicable to $\mathcal{L}^{\mathsf{s}}$ and any of its expansions, such as $\mathcal{L}^{\mathsf{c}}$, $\mathcal{L}^{\mathsf{s}^+}$, etc. For the present purpose, the signatures s^+ and c^+ are presumed to have indices for these operations, and the theories S^{0+} and C^{0+} are presumed to contain their definitions.

It follows from the unique readability theorem[1.39] that for any model $\mathfrak{M}$ of C^{0+} and any $\phi \in |\mathfrak{M}|$ such that $\mathfrak{M} \models \ulcorner [\phi]$ is an s-formula$\urcorner$, there is a unique s^+-term τ formed from the standard expression-building operations such that $\mathfrak{M} \models \ulcorner (\tau) = [\phi]\urcorner$. We call this the *standard* s^+*-term for* ϕ, and we define $\hat{\phi}$ to be this term.

Note that (3.98) is formulated and proved in the pure set theory S^{0+}, which is appropriate since it deals only with linguistic expressions and proofs, all of which are hereditarily finite sets. The reference to the theory C^{0+}, which is infinite and therefore potentially a proper class, is only a convenience of notation: the notion of a C^{0+}-proof may be formuated in terms of the definition of C^{0+}.

(3.98) **Theorem** $[\mathsf{S}^0]$ *Suppose ϕ is an* s*-formula, $\langle v_0, \ldots, v_{n^-}\rangle$ enumerates* Free ϕ, *and $C, c_0, \ldots, c_{n^-}$ are distinct constants (presumed to be in the signature* c^+*).*

1. *Let S be the substitution $\left\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ c_0 & \cdots & c_{n^-}\end{smallmatrix}\right\rangle$. Then $\mathsf{C}^{0+} \vdash \ulcorner$Suppose $(c_0), \ldots, (c_{n^-}) \in (C)$, and A is the assignment $\left\langle\begin{smallmatrix} (\hat{v}_0) & \cdots & (\hat{v}_{n^-}) \\ (c_0) & \cdots & (c_{n^-})\end{smallmatrix}\right\rangle$. Then*

$$(\phi^C(S)) \leftrightarrow (C) \models (\hat{\phi})[A].\urcorner.$$ [33]

2. *In particular, if ϕ is a sentence then*

$$\mathsf{C}^{0+} \vdash \ulcorner(\phi^C) \leftrightarrow (C) \models (\hat{\phi})\urcorner.$$

Equivalently,

$$\mathsf{C}^{0+} \vdash \ulcorner(\phi^C) \leftrightarrow (C) \models \ulcorner(\phi)\urcorner\urcorner.$$ [34]

Remark This theorem is a good example of a simple fact that stubbornly resists a simple statement. The following example will serve to illuminate its meaning. Suppose $\phi = \exists v\ \bar{u} \in \bar{v}$ (so that $n = 1$ and $v_0 = \ulcorner u\urcorner$), $C = \ulcorner M\urcorner$, $c_0 = \ulcorner x\urcorner$, and $v = \ulcorner y\urcorner$. Then (3.98) yields the following theorem, where we have substituted the typographical terms '$\exists v\ \bar{u} \in \bar{v}$' and '$u$' for the standard terms $\hat{\phi}$ and $\hat{v}_0$ that denote the same respective objects.

[32] ϕ is, of course, not relativized to a class M *per se*, but rather to a term regarded as defining M.

[33] As is customary, when there is no opportunity for confusion, if C denotes a class then C is also regarded as denoting the structure $(C; \in)$.

[34] The inner pair of corner quotes around ϕ may be taken to indicate either the standard name $\hat{\phi}$ for ϕ or any term that, within the prevailing context, refers unambiguously to ϕ.

(3.99) $[\mathsf{C}^{0+}]$ *Suppose $x \in M$. Then*

$$(\exists y \in M\ x \in y) \leftrightarrow M \models (\exists v\ \bar{u} \in \bar{v})\big[{}^{u}_{x}\big].$$

Using "in-stream assignment",[1.62] we may state this as follows.

(3.100) $[\mathsf{C}^{0+}]$ *Suppose $x \in M$. Then*

$$(\exists y \in M\ x \in y) \leftrightarrow M \models \ulcorner \exists y\ \boxed{x} \in y \urcorner.$$

Proof The theorem is proved by a straightforward argument by induction on the complexity of ϕ. This is legitimate because the assertion of the theorem *vis-à-vis* ϕ is that for any appropriate $C, c_0, \ldots, c_{n^-}$, there exists a C^{0+}-proof of a particular sentence, say $\theta_{\phi,\ldots}$, so it is expressible using only set quantification. It doesn't matter that if C denotes a proper class then $\theta_{\phi,\ldots}$ involves proper class quantification (given that $\ulcorner M \models \psi[A] \urcorner$ means that for every partial valuation function S for $(M; \in)$ that is adequate for ψ, $S\langle\psi, A\rangle = 1$.) $\square$[3.98]

(3.98) may be regarded as a theorem schema or "metatheorem"—a description of an infinite class of C^0-theorems.[35] When we invoke (3.98) in the course of a C^0-proof, we mean that a proof of the relevant instance of (3.98) is to be supposed inserted at that point. As we have previously discussed, practically every proof presented in mathematics is really just a sketch of a proof, and theorem schemas may be used in this context. To flesh out a proof sketch to a formal proof would in general require

1. presenting a formal version of each statement in the sketch;
2. filling in the gaps left by the author (which any sufficiently sophisticated reader is expected to be able to fill);
3. inserting proofs of the previously published theorems that are quoted; and
4. inserting proofs asserted to exist by theorems such as (3.98).

Note that the inductive proof of (3.98) provides an effective procedure to create the C^0-proof whose existence is asserted.

3.5.1 Inner models

Definition $[\mathsf{C}^0]$ *Suppose Θ is an* s*-theory. M is an* inner model of Θ $\overset{\text{def}}{\iff}$ *M is a transitive proper class and $(M; \in) \models \Theta$.*

We will show that V_Ω is an inner model of S, i.e., S^0 ($=$ $\mathsf{S1}$–$\mathsf{5}$) plus $\mathsf{S6}$, the $\mathsf{Foundation}$ schema.

(3.101) **Definition** $[\mathsf{C}^0]$ *A class M is* almost universal $\overset{\text{def}}{\iff}$ $\forall_S x \subseteq M\, \exists y \in M\ x \subseteq y$.

Note that an almost universal class is necessarily nonempty.

(3.102) **Theorem** $[\mathsf{C}^0]$ *Suppose M is transitive and almost universal. Then M satisfies axioms* $\mathsf{S1}$, $\mathsf{S4}$, *and* $\mathsf{S5}$ *of* S *as listed in (3.8).*

[35] The theorem having been precisely stated, we now relax our standards and revert to the practice of using 'C^0' for 'C^{0+}'.

Proof We will make use of the fact that since M is transitive, for any $y \in M$, $\ulcorner \exists x \in y\,(\dots)\urcorner$, i.e., $\ulcorner \exists x\,(x \in y \wedge \dots)\urcorner$ is equivalent to $\ulcorner \exists x \in M\,(x \in y \wedge \dots)\urcorner$; and likewise for bounded universal quantification. Remember that to show that $M \models \sigma$ we must show that for every $\{\sigma\}$-valuation function (or satisfaction relation) S for $(M; \in)$, $\models^S \sigma$.

S1. Extension Suppose

$$\sigma = \forall \mathrm{v}_0, \mathrm{v}_1\,(\forall \mathrm{v}_2\,(\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1) \rightarrow \mathrm{v}_0 = \mathrm{v}_1),$$

and suppose S is a $\{\sigma\}$-valuation function for $(M; \in)$. We must show that $\models^S \sigma$, i.e., $S\langle \sigma, 0\rangle = 1$. Remember that a $\{\sigma\}$-valuation function is also a ϵ-valuation function for any subexpression ϵ of σ.

By the definition of valuation,

$$S\Big\langle \mathrm{v}_2 \in \mathrm{v}_0, \Big\langle {\mathrm{v}_2 \ \mathrm{v}_0 \atop y_2 \ y_0} \Big\rangle\Big\rangle = 1 \leftrightarrow y_2 \in y_0,$$

$$S\Big\langle \mathrm{v}_2 \in \mathrm{v}_1, \Big\langle {\mathrm{v}_2 \ \mathrm{v}_1 \atop y_2 \ y_1} \Big\rangle\Big\rangle = 1 \leftrightarrow y_2 \in y_1,$$

$$S\Big\langle \mathrm{v}_0 = \mathrm{v}_1, \Big\langle {\mathrm{v}_0 \ \mathrm{v}_1 \atop y_0 \ y_1} \Big\rangle\Big\rangle = 1 \leftrightarrow y_0 = y_1,$$

so we must show that

$$\forall y_0, y_1 \in M\,(\forall y_2 \in M\,(y_2 \in y_0 \leftrightarrow y_2 \in y_1) \rightarrow y_0 = y_1).$$

This follows from the fact that M is transitive.

Before continuing, we note that the above argument may be abbreviated by invoking Theorem 3.98, which informs us that

$$\mathsf{C}^0 \vdash \ulcorner (M \models \sigma) \leftrightarrow \forall y_0, y_1 \in M\,(\forall y_2 \in M\,(y_2 \in y_0 \leftrightarrow y_2 \in y_1) \rightarrow y_0 = y_1)\urcorner.$$

In effect, (3.98) provides us with an infinite collection of C^0-theorems, of which this is one. Since these theorems and their individual proofs are so obvious, we will often invoke them without explicit recognition, starting now.

S4. Pair Suppose

$$\sigma = \forall \mathrm{v}_0, \mathrm{v}_1\,\exists \mathrm{v}_2\,(\mathrm{v}_0 \in \mathrm{v}_2 \wedge \mathrm{v}_1 \in \mathrm{v}_2).$$

Suppose $x, y \in M$. Then $\{x, y\} \subseteq M$, so by almost universality $\exists z \in M\,(x \in z \wedge y \in z)$.

S5. Collection Suppose

$$\sigma = \forall v_0, \dots, v_{n^-}\,\forall u\,\big(\forall v \in u\,\exists w\,\forall a\,(\phi \rightarrow a \in w) \\ \rightarrow \exists w\,\forall v \in u\,\forall a\,(\phi \rightarrow a \in w)\big),$$

where ϕ is an s-formula, and $a, u, v, w, v_0, \dots, v_{n^-}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{a, v, v_0, \dots, v_{n^-}\}$.

σ is not a specific sentence, so we cannot conveniently invoke (3.98). Suppose therefore that S is a $\{\sigma\}$-valuation function for $(M; \in)$. We must show that $\models^S \sigma$. Suppose $y_0, \dots, y_{n^-}, x \in M$ and

$$\forall y \in x\,\exists z \in M\,\forall r \in M\,\big(S\Big\langle \phi, \Big\langle {v_0 \ \cdots \ v_{n^-} \ v \ a \atop y_0 \ \cdots \ y_{n^-} \ y \ r} \Big\rangle\Big\rangle = 1 \rightarrow r \in z\big).$$

By the collection axiom of C^0, there is a set z such that

$$\forall y \in x\, \forall r \in M\ \big(S\big\langle \phi, \big\langle {}^{v_0\ \cdots\ v_{n^-}\ v\ a}_{y_0\ \cdots\ y_{n^-}\ y\ r} \big\rangle\big\rangle = 1 \rightarrow r \in z\big).$$

$z \cap M$ is a set, so by almost universality there exists $z' \in M$ such that $z \cap M \subseteq z'$, so

$$\forall y \in x\, \forall r \in M\ \big(S\big\langle \phi, \big\langle {}^{v_0\ \cdots\ v_{n^-}\ v\ a}_{y_0\ \cdots\ y_{n^-}\ y\ r} \big\rangle\big\rangle = 1 \rightarrow r \in z'\big).$$ [36]

$\square$[3.102]

(3.103) **Theorem** [C^0] $V_\Omega \models \mathsf{S}$.

Proof Refer to (3.8) for a listing of S. V_Ω is transitive[3.91.1] and any subset of V_Ω is in V_Ω[3.91.3] so V_Ω is almost universal.[3.101] Hence, by Theorem 3.102 $V_\Omega \models \mathsf{S1}$, $\mathsf{S4}$, and $\mathsf{S5}$. Clearly $0 \in V_\Omega$, so $V_\Omega \models \mathsf{S3}$. Only $\mathsf{S2}$ and $\mathsf{S6}$ remain.

S2. Comprehension Suppose θ is

$$\forall v_0, \ldots, v_{n^-}\, \forall u\, \exists w\, \forall v\, (v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is an s-formula, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$; and suppose S is a $\{\theta\}$-satisfaction relation for V_Ω. It follows from the definition of satisfaction that we must show that

$$\forall y_0, \ldots, y_{n^-} \in V_\Omega\, \forall x \in V_\Omega\, \exists z \in V_\Omega\, \forall y \in V_\Omega \\ \big(y \in z \leftrightarrow \big(y \in x \wedge \models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{y\ y_0\ \cdots\ y_{n^-}}\big]\big)\big).$$

Suppose $y_0, \ldots, y_{n^-}, x \in V_\Omega$. It follows from $\mathsf{C2}$[37] (Comprehension for C^0) that there exists a set z' such that

$$\forall y \in V_\Omega\, \big(y \in z' \leftrightarrow \big(y \in x \wedge \models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{y\ y_0\ \cdots\ y_{n^-}}\big]\big)\big).$$

Let $z = z' \cap V_\Omega$. Then[3.91] $z \in V_\Omega$, and

$$\forall y \in V_\Omega\, \big(y \in z \leftrightarrow \big(y \in x \wedge \models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{y\ y_0\ \cdots\ y_{n^-}}\big]\big)\big).$$

S6. **Foundation** Suppose θ is

$$\forall v_0, \ldots, v_{n^-}\, \big(\exists v\, \phi \rightarrow \exists v\, \big(\phi \wedge \forall u \in v\, \neg\, \phi\big({}^{v}_{u}\big)\big)\big),$$

where ϕ is an s-formula, $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$, and u is free for v in ϕ; and suppose S is a $\{\theta\}$-satisfaction relation for V_Ω. We must show that

$$\forall y_0, \ldots, y_{n^-} \in V_\Omega \big(\exists y \in V_\Omega\ \models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{y\ y_0\ \cdots\ y_{n^-}}\big] \\ \rightarrow \exists y \in V_\Omega\, \big(\models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{y\ y_0\ \cdots\ y_{n^-}}\big] \wedge \forall x \in y\, \neg \models^S \phi\big[{}^{v\ v_0\ \cdots\ v_{n^-}}_{x\ y_0\ \cdots\ y_{n^-}}\big]\big)\big).$$ [38]

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[36] Note that this is not a proof schema, with one proof for each formula ϕ. It is a single proof of a single theorem, in which 'ϕ' occurs as a variable.

[37] Note that a single instance of $\mathsf{C2}$ suffices, taking $\phi, y_0, \ldots, y_{n^-}, x$ as parameters. We do not use a separate instance of $\mathsf{C2}$ for each ϕ.

Suppose $y_0, \ldots, y_{n^-} \in V_\Omega$ and $\exists y \in V_\Omega \models^S \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right]$. Let

$$Y = \left\{ y \in V_\Omega \,\middle|\, \models^S \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right] \right\}.$$

Since Y is nonempty and $\in^{V_\Omega}$ is wellfounded, there exists $y \in Y$ such that $\forall x \in y\ x \notin Y$. Since V_Ω is transitive, $y \in V_\Omega$ and

$$\models^S \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right] \wedge \forall x \in y\ \neg \models^S \phi\left[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ x & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right]),$$

as desired. $\square$[3.103]

Theorem 3.103 is often formulated in the following way.

(3.104) **Theorem** [S^0] *Suppose θ is an axiom of S. Let θ^{V_Ω} be the relativization*[2.112] *of θ to (an s-formula defining) V_Ω. Then $\mathsf{S}^0 \vdash \theta^{V_\Omega}$.*

Proof We will show that $\mathsf{C}^0 \vdash \theta^{V_\Omega}$. Since C^0 is a conservative extension of S^0, it follows that $\mathsf{S}^0 \vdash \theta^{V_\Omega}$. We know[3.103] that $\mathsf{C}^0 \vdash$ $\ulcorner$for all x, if x is an axiom of S then $V_\Omega \models x\urcorner$. Let $\hat\theta$ be the standard s^+-term for θ. Then $\mathsf{C}^0 \vdash \ulcorner(\hat\theta)$ is an axiom of $\mathsf{S}\urcorner$, so $\mathsf{C}^0 \vdash V_\Omega \models (\hat\theta)$. Hence,[3.98.2] $\mathsf{C}^0 \vdash \theta^{V_\Omega}$. $\square$[3.104]

(3.104) states that each θ^{V_Ω} is a theorem of S^0, so although (3.104) is itself a single theorem, it may also be viewed as a theorem schema, and a proof of it may be given as a proof schema, i.e., as a recipe for constructing an S^0-proof of $\ulcorner(\theta^{V_\Omega})\urcorner$ for any axiom θ of S. Of course, the proof just given is a proof schema of sorts, but its description of an S^0-proof of $\ulcorner(\theta^{V_\Omega})\urcorner$ for a given θ is rather indirect. It is not difficult to remodel it as a schema dealing exclusively with S^0-proofs—without the detour through C^0—but nothing is gained thereby, except perhaps a little more confidence in the conclusion. Do it if you must.[39]

Having ascertained the status of Foundation in the axiomatic framework of the theory of membership, we henceforward proceed on the basis of the systems $\mathsf{S} = \mathsf{S}^0 + \mathsf{Foundation}$ and $\mathsf{C} = \mathsf{C}^0 + \mathsf{Foundation}$.[40] This is not to say that everything to follow depends on Foundation; we simply choose not to distinguish what does from what doesn't, as we have no further interest in its omission. In particular, S^0 suffices for the entire elementary theory of structure, language, and logic, since all the relevant definitions are by recursion on ω, which is explicitly defined as wellfounded. Thus, S^0 suffices as a metatheory; nevertheless, in the interest of notational simplicity, we will generally state metatheorems as theorems of S.

Some of the early material that follows is repetitive of work already done, but is re-presented for the sake of continuity or to illustrate a difference of approach.

3.5.2 $\in$-induction and $\in$-recursion

(3.105) **Theorem** [S]

1. *Every set is ranked; hence,*
2. [C] $V = V_\Omega$.

[38] We have made use of the fact that V_Ω is transitive.

[39] Oh, ye of little faith.

[40] Recall that we have formulated Foundation somewhat differently for the pure set theory[3.8] and the class theory.[3.16]

Proof We now have Foundation, so if there is a set that is not ranked then there exists an $\in$-minimal such set x. Every $y \in x$ is ranked and is *a fortiori* subtransitive, so it has a transitive closure $\operatorname{tc} y$. Clearly, $x \cup \bigcup_{y \in x} \operatorname{tc} y$ is the transitive closure of x. Given that $\in^{\operatorname{tc} y}$ is by hypothesis wellfounded for each $y \in x$, it is easy to show that $\in^{\operatorname{tc} x}$ is wellfounded, so x is ranked.

In C we may state the same thing in terms of the proper classes V and V_Ω.□[3.105]

Thus, in the presence of Foundation the cumulative hierarchy V_Ω comprises the entire set-theoretical universe. $\in$ is now the paradigm of an irreflexive wellfounded setlike relation, and *proof by induction on the membership relation* or *$\in$-induction* is an important special case of the general method of proof by induction on a wellfounded relation:[3.78]

(3.106) **Theorem: $\in$-induction** [S] *Suppose ϕ is an* s*-formula, u and v are variables with $v \in \operatorname{Free}\phi$ and $u \notin \operatorname{Free}\phi$, and u free for v in ϕ. Then*

$$\mathsf{S} \vdash \forall v\left(\forall u \in v\, \phi\tbinom{v}{u} \to \phi\right) \to \forall v\, \phi.$$

Remark Like (3.104), this is a metatheorem stating that certain sentences are theorems of S. As in that case, it is naturally a theorem of S^0. As discussed above, since we have, in effect, adopted Foundation into the canon, we simply state it as a theorem of S. That the "metatheory" S is also the "object theory" here has no particular significance. S is a suitable theory for proving many metatheorems, often dealing with provability in theories that have nothing to do with the theory of membership.

Proof This is just the schema of contrapositives of the Foundation axioms of S,[3.9] with ϕ replaced by $\neg\phi$. □[3.106]

The contrapositive of the class Foundation axiom[3.16] is the principle of $\in$-induction for C. Since we have the class comprehension axiom C2a, we do not need to state this as a schema.

(3.107) **Theorem: $\in$-induction** [C] *Suppose $\forall_S x\,(x \subseteq X \to x \in X)$. Then $\forall_S x\ x \in X$.*

Proof Suppose toward a contradiction that $\forall x \in X\,(x \subseteq X \to x \in X)$, and $\exists_S x\ x \notin X$. Then by Foundation[3.16] (with $V \backslash X$ for v_0), there exists x_1 such that $x_1 \notin X$ and $\forall x_2 \in x_1\ x_2 \in X$, contrary to hypothesis. □[3.107]

Similarly, definition by $\in$-recursion is an important special case of recursion on a wellfounded relation:

(3.108) **Theorem: $\in$-recursion** [C] *Suppose G is a function such that $\operatorname{dom} G$ contains every function in V,*[3.19] *and suppose X is a class. Then there exists a unique function F such that*

1. $\operatorname{dom} F = X$;
2. $\forall x \in X\ Fx = G(F \restriction x)$.

The definition[3.86, 3.90.4] of rank is simply formulated:

(3.109) **Definition** [C] *The* rank *of a set* $x \overset{\text{def}}{=} \operatorname{rk} x$, *is defined recursively by:*

$$\operatorname{rk} x = \bigcup_{y \in x} (\operatorname{rk} y)^+.$$ [3.45]

The following is essentially (3.87) for $\in$.

Theorem [C] *The rank of any set is an ordinal and it is the least ordinal greater than the ranks of all the sets in it.*

3.5.3 Transitivity

(3.110) **Theorem** [C]

1. *Suppose x is a set. Then there exists a unique set y such that*
    1. $\mathrm{Tran}\, y$,
    2. $x \subseteq y$, *and*
    3. $\forall z \left((\mathrm{Tran}\, z \wedge x \subseteq z) \rightarrow y \subseteq z\right)$.
2. *For any class x there exists a unique class y satisfying 1.1–3* vis-à-vis *x.*

Proof See the proof of (3.105). $\square$[3.110]

Given Theorem 3.110 the following definition is legitimate.

Definition [C] *Suppose x is a class. The* transitive closure of x $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{tc}\, x$ $\overset{\text{def}}{\Longleftrightarrow}$ *the $\subseteq$-minimum transitive class including x.*

3.5.3.1 A finite axiomatization of Foundation in S

Recall that we have stated Foundation as an infinite schema in S. Making use of transitivity, we can replace it by two of its instances. Let $\phi = \ulcorner x$ is not subtransitive$\urcorner$.[3.90.1] Consider the following two instances of Foundation:

(3.111) $$\ulcorner \exists x\, (x \text{ is not subtransitive}) \\ \rightarrow \exists x \left(x \text{ is not subtransitive} \wedge \forall y \in x\, (y \text{ is subtransitive})\right)\urcorner$$

and

(3.112) $$\ulcorner \forall x\, (\exists y \in x \rightarrow \exists y \in x\, \forall z \in y\ z \notin x)\urcorner.$$

In S^0 we can show that if a set y is subtransitive then $\mathrm{tc}\, y$ exists.[3.90.2] We can also show that if $\mathrm{tc}\, y$ exists for every $y \in x$, then $x \cup \bigcup_{y \in x} \mathrm{tc}\, y$ is transitive and includes x, so x is subtransitive. Thus, $\mathsf{S}^0 +$ (3.111) $\vdash$

(3.113) $$\ulcorner \forall x\, \exists y\, (\mathrm{Tran}\, y \wedge x \subseteq y)\urcorner.$$

Now consider an arbitrary instance of Foundation:

(3.114) $$\forall v_0, \ldots, v_{n^-} \left(\exists v\, \phi \rightarrow \exists v \left(\phi \wedge \forall u \in v\ \neg\, \phi\tbinom{v}{u}\right)\right),$$

where ϕ is an s-formula, $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that $\mathrm{Free}\, \phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$ and u is free for v in ϕ. Let v', w, w' be distinct variables not in $\{u, v, v_0, \ldots, v_{n^-}\}$. The following is a (sketch of a) proof of (3.114) from (3.113) and (3.112). (Uses of theorems of pure logic asserting the equivalence of sentences related by a change of variables are among the omitted steps.)

$\ulcorner$Suppose $(v_0, \ldots, v_{n^-})$ are given, and suppose $(\exists v\, \phi)$. We wish to show that

$$\left(\exists v \left(\phi \wedge \forall u \in v\ \neg\, \phi\tbinom{v}{u}\right)\right).$$

To this end, let (v) be such that (ϕ). Using (3.113), let (w) be transitive such that $\{(v)\} \subseteq (w)$, i.e. $(v) \in (w)$. Let

$$(w') = \{(u) \in (w) \mid (\phi\tbinom{v}{u})\}. \tag{3.115}$$

Note that $(v) \in (w')$; hence,[3.112] there exists (v') such that

$$(v') \in (w') \wedge \forall (u) \in (v')\ (u) \notin (w').$$

Since (w) is transitive and $(v') \in (w') \subseteq (w)$, any $(u) \in (v')$ is in (w). Thus,

$$(v') \in (w') \wedge \forall (u) \in (v')\ (\neg\, \phi\tbinom{v}{u}).$$

Since[3.115] $(\phi\tbinom{v}{v'})$, it follows that

$$\exists (v')\ \big((\phi\tbinom{v}{v'}) \wedge \forall (u) \in (v')\ (\neg\, \phi\tbinom{v}{u})\big),$$

equivalently,

$$\exists (v)\ \big((\phi) \wedge \forall (u) \in (v)\ (\neg\, \phi\tbinom{v}{u})\big),$$

as claimed.[41] ⌝

Thus we have the following theorem of S (or, for that matter, of S^0, but no matter).

(3.116) **Theorem** [S] S^0 + (3.111) + (3.112) $\vdash \mathsf{S}$.

3.5.3.2 Transitive collapse

We have noted above that a wellfounded irreflexive setlike relation R resembles $\in$ in that it permits inductive proofs and recursive constructions. The following theorem extends the analogy by showing that such a relation R on a class X is actually homomorphic to $\in$ on a transitive class M. R is *iso*morphic to $\in$ just in case R resembles $\in$ is one more particular:

Definition [C] *A relation R on a class X is* extensional $\overset{\text{def}}{\iff} \forall x, x' \in X\ \big(\forall y \in X\ (y\, R\, x \leftrightarrow y\, R\, x') \rightarrow x = x'\big)$.

(3.117) **Theorem** [C] *Suppose R is wellfounded irreflexive setlike relation on a class X. There is a unique transitive class M and $F : X \overset{\text{sur}}{\to} M$ such that $\forall x, y \in X\ (y\, R\, x \leftrightarrow Fy \in Fx)$. F is injective (hence, bijective) iff R is extensional.*

Proof Define F recursively[3.79] so that

1. $\operatorname{dom} F = X$; and
2. $\forall x \in X\ \ Fx = F^{\rightarrow}(R^{\leftarrow}\{x\})$.

Let $M = \operatorname{im} F$. Then $F : X \overset{\text{sur}}{\to} M$. To show that M is transitive, suppose $a \in M$ and $b \in a$. Let $x \in X$ be such that $Fx = a$. Then $a = F^{\rightarrow}(R^{\leftarrow}\{x\})$, so $\exists y \in R^{\leftarrow}\{x\}\ Fy = b$; hence, $b \in \operatorname{im} F = M$. Given $x, y \in X$,

$$y\, R\, x \leftrightarrow y \in R^{\leftarrow}\{x\} \leftrightarrow Fy \in F^{\rightarrow}(R^{\leftarrow}\{x\}) = Fx.$$

---

[41] $(\exists v\ (\phi \wedge \forall u \in v\ \neg\, \phi\tbinom{v}{u}))$ is certainly a suitable value for $\ulcorner \exists (v)\ \big((\phi) \wedge \forall (u) \in (v)\ (\neg\, \phi\tbinom{v}{u})\big)\urcorner$.

So there exist M and F as specified. We now show that they are unique. To this end, suppose M' is transitive, $F' : X \overset{\text{sur}}{\to} M'$, and $\forall x, y \in X\,(y\,R\,x \leftrightarrow F'y \in F'x)$. Then for any $x \in X$

$$\begin{aligned} F'x &= \{b \in M' \mid b \in F'x\} = \{F'y \mid F'y \in F'x\} \\ &= \{F'y \mid y\,R\,x\} = \{F'y \mid y \in R^{\leftarrow}\{x\}\} \\ &= F'^{\rightarrow}(R^{\leftarrow}\{x\}), \end{aligned}$$

so F' satisfies the same recursive definition as F and therefore $F' = F$,[3.80] so $M' = \operatorname{im} F' = \operatorname{im} F = M$, as well.

It is clear that F is injective iff R is extensional. □[3.117]

Definition [C] *Suppose R, X, M, and F are as in (3.117). The* transitive collapse *of R (i.e., of the structure $(X;R)$)* $\overset{\text{def}}{=}$ *M (i.e., $(M;\in)$), and the* transitive collapsing map $\overset{\text{def}}{=}$ *F.*[42]

(3.118) **Theorem** [C] *Suppose R is a strict wellordering of a set X. Then there is a unique ordinal α and $f : \alpha \overset{\text{bij}}{\to} X$ such that $\forall \beta, \gamma \in \alpha\,\big(\beta \in \gamma \leftrightarrow (f\beta)\,R\,(f\gamma)\big)$.*

Proof R is wellfounded by definition, it is setlike because X is a set, and it is easily seen to be extensional, so Theorem 3.117 applies to yield the existence of a unique transitive class α and function $g : X \overset{\text{bij}}{\to} \alpha$ such that $\forall x, y \in X\,(y\,R\,x \leftrightarrow y \in x)$. Since X is a set, by Replacement, α is a set. Since R is a (strong) total order by definition, $\in$ totally orders α, so $\in$ wellorders α, so α is an ordinal.[3.39] Let $f = g^{-1}$. □[3.118]

(3.119) **Definition** [C] *Suppose R and α are as in Theorem 3.118. The* order type *of R* $\overset{\text{def}}{=}$ *α.*[43]

Note that the order type of a wellorder is just its rank as a wellfounded relation.[3.88]

It is tempting to suppose that if we drop the requirement that X be a set, Theorem 3.118 would hold it we allowed α to be either an ordinal or the class Ord of all ordinals, but this is true only if R is setlike, which it need not be. For example, let $X = \text{Ord}$ and let R be such that for distinct $\alpha, \beta \in X$, $\alpha\,R\,\beta$ iff

$$\beta = 0 \vee (\alpha \neq 0 \wedge \alpha \in \beta).$$

Note that we have taken 0 from the first position and made it the last, so the order type of R is—if you will—Ord +1.

3.5.4 Ordinal arithmetic

(3.120) **Definition** [C] *Lower case Greek letters represent ordinals. The usual grouping precedence rules apply.*

[42] The transitive collapse is also called the *Mostowski collapse* after Andrzej Mostowski.

[43] In the general sense, an order type is an isomorphism type of orders. As discussed following (1.31), this notion is difficult to manage rigorously. It is nevertheless so useful in the case of wellorders that in early versions of set theory, wellorder types were sometimes introduced as a distinct sort of entity. The von Neumann ordinals now elegantly provide this service.

1. *$\alpha + \beta$ is defined for any α by recursion on β as follows.*
   1. $\alpha + 0 \stackrel{\text{def}}{=} \alpha$.
   2. $\alpha + (\beta + 1) \stackrel{\text{def}}{=} (\alpha + \beta) + 1$.
   3. $\alpha + \eta \stackrel{\text{def}}{=} \bigcup_{\beta \in \eta} (\alpha + \beta)$ *for limit* η.
2. *$\alpha \cdot \beta$ is defined for any α by recursion on β as follows.*[44]
   1. $\alpha \cdot 0 \stackrel{\text{def}}{=} 0$.
   2. $\alpha \cdot (\beta + 1) \stackrel{\text{def}}{=} \alpha \cdot \beta + \alpha$.
   3. $\alpha \cdot \eta \stackrel{\text{def}}{=} \bigcup_{\beta \in \eta} \alpha \cdot \beta$ *for limit* η.
3. *α^β is defined for any α by recursion on β as follows.*
   1. $\alpha^0 \stackrel{\text{def}}{=} 1$.
   2. $\alpha^{\beta+1} \stackrel{\text{def}}{=} \alpha^\beta \cdot \alpha$.
   3. $\alpha^\eta \stackrel{\text{def}}{=} \bigcup_{\beta \in \eta} \alpha^\beta$ *for limit* η.

The separate treatment of 0, successor ordinals, and limit ordinals is very common in ordinal recursion.

3.5.5 Ordinal sequences

Definition [C] *Suppose* $\operatorname{Ord} \alpha$. *An α-sequence is a function with domain α.*

We often use an expression such as $\ulcorner\langle c_\beta \mid \beta \in \alpha\rangle\urcorner$, or, equivalently, $\ulcorner\langle c_\beta \mid \beta < \alpha\rangle\urcorner$ for an α-sequence. Finite sequences[3.55] are, of course, examples of ordinal sequences.

Definition [C] *Suppose* $c = \langle c_\beta \mid \beta < \alpha\rangle$ *and* $d = \langle d_\beta \mid \beta < \alpha'\rangle$ *are ordinal sequences. Then* c concatenate $d \stackrel{\text{def}}{=}$ the result of concatenating d to c $\stackrel{\text{def}}{=} c ^\frown d$ $\stackrel{\text{def}}{=}$ *the sequence* $e = \langle e_\beta \mid \beta < \alpha + \alpha'\rangle$*, where*

1. *for all* $\beta < \alpha$, $e_\beta = c_\beta$; *and*
2. *for all* $\beta < \alpha'$, $e_{\alpha+\beta} = d_\beta$.

3.5.6 Choice functions and wellorderability

Definition [C]

1. f *is a* choice function $\stackrel{\text{def}}{\Longleftrightarrow}$ f *is a function and* $\forall y \in \operatorname{dom} f\, (y \neq 0 \rightarrow f y \in y)$.
2. f *is* choice function for x $\stackrel{\text{def}}{\Longleftrightarrow}$ f *is a choice function and* $x \backslash \{0\} \subseteq \operatorname{dom} f$.

In other words, a choice function chooses a member of each nonempty member of its domain, and a choice function for x chooses a member of each nonempty member of x. For convenience, we permit the domain of a choice function to contain 0, but we do not require that the domain of a choice function *for* x contain 0, even if x contains 0.

Definition [C] x can be wellordered *or* is wellorderable $\stackrel{\text{def}}{\Longleftrightarrow}$ *there exists a wellordering of* x*, i.e., a wellorder* R *such that* $\operatorname{fld} R = x$.

[44] This is read ‘α times β’ but it clearly means ‘α [repeated] β times’.

The existence of wellorderings is closely tied to the existence of choice functions.

(3.121) **Theorem** [C] *Suppose x is a set. x is wellorderable iff there exists a choice function for $\mathcal{P}x$.*

Proof $\rightarrow$ Suppose R is a wellordering of x. Let f be the class of (y, z) such that y is a nonempty subset of x and z is the R-least member of y. Then f is a choice function for $\mathcal{P}x$.

$\leftarrow$ Suppose f is a choice function for $\mathcal{P}x$. Let G be the function such that

1. $\operatorname{dom} G$ consists of all functions in V; and
2. for each $g \in \operatorname{dom} G$,

$$Gg = \begin{cases} f(x \backslash \operatorname{im} g) & \text{if } x \nsubseteq \operatorname{im} g \\ 0 & \text{otherwise.}^{45} \end{cases}$$

Let[3.108] F be the function with domain Ord such that $\forall \alpha \in \text{Ord}\ \ F\alpha = G(F \restriction \alpha)$. Let

$$A = \{\alpha \in \text{Ord} \mid x \nsubseteq (F^{\rightarrow}\alpha)\}. \tag{3.122}$$

By construction, for any $\alpha \in A$, $F\alpha \in x\backslash(F^{\rightarrow}\alpha)$. Let $F' = F \restriction A$. Then $F' : A \overset{\text{inj}}{\rightarrow} x$. $\operatorname{im} F'$ is a subclass of the set x, so it is a set. Hence F'^{-1} maps a set onto A, so by Replacement A is a set. Since A is obviously an initial segment of Ord, A is an ordinal.

Since $A \notin A$, $x \subseteq (F^{\rightarrow}A)$,[3.122] so $F' : A \overset{\text{bij}}{\rightarrow} x$. Thus, $\{\langle F\alpha, F\beta\rangle \mid \alpha \in \beta \in A\}$ is a wellordering of x. $\square$[3.121]

The axiom of choice, AC, states that every set has a choice function.

We may use 'Choice' to refer specifically to AC or generically to choice principles possibly weaker than AC. The status of AC as a membership-theoretic axiom is one of the most important issues to be dealt with in the foundations of mathematics.

3.6 Size

3.6.1 Cardinality

Perhaps the most basic attribute of a set is its size—this is, at any rate, a very important and interesting concept in set theory. There seems to be only one reasonable way to formalize the fundamental notion of sets having the same size.

(3.123) **Definition** [C] *Sets x and y* have the same size *or cardinality* $\overset{\text{def}}{\iff}$ *x and y are* equipollent *(have the same power)* $\overset{\text{def}}{\iff}$ $x \sim y$ $\overset{\text{def}}{\iff}$ *there exists $f : x \overset{\text{bij}}{\rightarrow} y$.*

[45] The choice of 0 for the fallback value of G is arbitrary.

Clearly:

Theorem [C] $\sim$ *is an equivalence relation.*

Note that (3.123) only defines the predicate ⌜· and · have the same size⌝; it does not define an operation ⌜the size of ·⌝. The most straightforward way to define the size of a set x is to let it be the equivalence class $[x]_\sim$ of x *vis-à-vis* $\sim$. The problem with this is that, unless $x = 0$, $[x]_\sim$ is a proper class. A simple way to repair this defect is to take the subset of $[x]_\sim$ consisting of its members with minimum rank.[46]

If x is wellorderable then it is equipollent with an ordinal,[3.118] and in this case we can let the size of x be the least ordinal equipollent with x. The axiom of choice implies that every set can be wellordered,[3.121] and in theories with AC, this is the preferred definition. The following definition is written so as to be applicable in the absence of choice but to reduce to the latter definition in the presence of choice.

(3.124) **Definition** [C]

1. *Suppose x is a set. The* size *or* cardinality of x $\overset{\text{def}}{=} |x| \overset{\text{def}}{=}$
   1. *the least ordinal equipollent with x if there is one;*
   2. *otherwise, the set of all sets y such that $y \sim x$ and $\forall y' \, (y' \sim x \to \operatorname{rk} y' \geqslant \operatorname{rk} y)$.*
2. *c is a* cardinality $\overset{\text{def}}{\iff}$ *$c = |x|$ for some set x.*
3. *c is a* cardinal $\overset{\text{def}}{\iff}$ *c is a cardinality and an ordinal. A cardinal in this sense may also be called an* initial ordinal *by way of emphasis.*
4. Card $\overset{\text{def}}{=}$ *the class of cardinals.*

Note that the cardinality of an ordinal is always a cardinal, and no ordinal is a cardinality in the sense (3.124.1.2): 1 is the only nonempty ordinal all of whose members have the same size (viz., 0), so it would be a cardinality in the sense (3.124.1.2), but for the fact that there is an ordinal equipollent with 0, viz., 0.

Obviously,

(3.125) *κ is a cardinal iff κ is an ordinal that is not equipollent with any ordinal $\alpha < \kappa$.*

There at least two ways to formalize the concept of a set y being at least as large as a set x. We may say that there exists $f : y \xrightarrow{\text{sur}} x$ (y is big enough to "cover" x), or, alternatively, that there exists $f : x \xrightarrow{\text{inj}} y$ (y is big enough to "hold" x).[3.33] The latter implies the former,[3.127] and we will take it as our definition.[47]

(3.126) **Definition** [C]

1. *For sets x and y, $x \preccurlyeq y$ $\overset{\text{def}}{\iff}$ x is equipollent with a subset of y, i.e., $\exists f : x \xrightarrow{\text{inj}} y$.*
2. *Correspondingly, if a and b are cardinalities, $a \leqslant b$ $\overset{\text{def}}{\iff}$ $A \preccurlyeq B$ for some (equivalently, for all) sets A, B such that $|A| = a$ and $|B| = b$.*

[46] This is, in fact, a general method of defining the quotient of a class by an equivalence relation whose equivalence classes are not all sets. Notice that we rely on Foundation for this.

[47] In the absence of the axiom of choice, it is consistent that the converse fails.

3. *As usual,* $\prec$ *and* $<$ *are the strong order relations corresponding to* $\preccurlyeq$ *and* $\leqslant$*, respectively.*

(3.127) **Theorem** [C] *Suppose* $x \preccurlyeq y$ *and* $x \neq 0$*. Then there exists* $g : y \overset{\text{sur}}{\to} x$*.*

Proof Since $x \preccurlyeq y$, by definition there exists $f : x \overset{\text{inj}}{\to} y$. Since $x \neq 0$, there exists $z \in x$. Let g be the function with domain y such that for all $w \in y$

$$gw = \begin{cases} f^{-1}w & \text{if } w \in \operatorname{im} f \\ z & \text{otherwise.} \end{cases}$$

Clearly $g : y \overset{\text{sur}}{\to} x$. □3.127

The following two theorems were among the earliest indications that there might be a substantive theory of sets *per se*. The first is due to Cantor.

(3.128) **Theorem** [C] *Suppose* x *is a set and* $\mathcal{P}x$ *is a set. Then* $x \prec \mathcal{P}x$*. Hence* $|x| < |\mathcal{P}x|$*.*

Proof Clearly $x \preccurlyeq \mathcal{P}x$, as witnessed by the function $\{(y, \{y\}) \mid y \in x\}$. Thus, we only have to show that $\mathcal{P}x \not\preccurlyeq x$.

(3.129) *It suffices*[3.127] *to show that there does not exist* $f : x \overset{\text{sur}}{\to} \mathcal{P}x$*. Suppose toward a contradiction that* $f : x \overset{\text{sur}}{\to} \mathcal{P}x$*.*

Let $Y = \{y \in x \mid y \notin fy\}$. Let $y \in x$ be such that $fy = Y$. Then

$$y \in Y \leftrightarrow y \notin fy \leftrightarrow y \notin Y.$$

□3.128

The proof of (3.128) is often referred to as a *diagonal* argument, and it is a device that would later be adapted by Gödel and others to prove fundamental results in logic. The "diagonality" of the argument can be seen as follows. Identify $\mathcal{P}x$ with ${}^{x}2$ in the usual way with characteristic functions, i.e., given $x' \subseteq x$, let $\chi^{x'} : x \to 2$ be given by

$$\chi^{x'}y = \begin{cases} 1 & \text{if } y \in x' \\ 0 & \text{if } y \notin x'. \end{cases}$$

Imagine the set x to be linearly ordered,[48] and imagine $x \times x$ as a two-dimensional array. Suppose $f : x \to {}^{x}2$. For each $y \in x$, let the yth row of this array be the characteristic function of fy. In effect, letting g represent the array,

$$g(y, y') = (fy)y'$$

Define $h : x \to 2$ as the complement of the diagonal of this array:

$$hy = 1 - g(y, y).$$

Then for every $y \in x$, $hy \neq (fy)y$, so $h \neq fy$. f is therefore not surjective. Thus, x cannot be mapped onto $\mathcal{P}x$, so[3.127] $\mathcal{P}x \not\preccurlyeq x$.

[48] It is not essential to the argument that x be linearly ordered—this is just to motivate the use of 'diagonal' to describe the method.

(3.130) **Schröder-Bernstein theorem** [C] *For any sets x and y, $(x \preccurlyeq y \wedge y \preccurlyeq x) \to x \sim y$.*

Proof Suppose $f : x \xrightarrow{\text{inj}} y$ and $g : y \xrightarrow{\text{inj}} x$. Define by recursion a sequence $\langle x_n \mid n \in \omega \rangle$ as follows:

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= g^{\to}(f^{\to}x_n). \end{aligned}$$

Similarly, let $\langle y_n \mid n \in \omega \rangle$ be such that

$$\begin{aligned} y_0 &= y \\ y_{n+1} &= f^{\to}(g^{\to}y_n). \end{aligned}$$

By induction on n, for any $n \in \omega$,

$$\begin{aligned} x_n &\supseteq g^{\to}y_n \supseteq x_{n+1} \\ y_n &\supseteq f^{\to}x_n \supseteq y_{n+1}. \end{aligned} \tag{3.131}$$

For $n \in \omega$, let

$$\begin{aligned} x'_n &= x_n \backslash g^{\to}y_n & y'_n &= y_n \backslash f^{\to}x_n \\ x''_n &= g^{\to}y_n \backslash x_{n+1} & y''_n &= f^{\to}x_n \backslash y_{n+1} \\ x''' &= \bigcap_{n\in\omega} x_n & y''' &= \bigcap_{n\in\omega} y_n. \end{aligned}$$

Clearly,[3.131] the sets $x''', x'_0, x''_0, x'_1, x''_1, \ldots,$ are pairwise disjoint and their union is x; likewise, *mutatis mutandis*, for y. Since f is injective,

$$f \restriction x'_n : x'_n \xrightarrow{\text{bij}} y''_n.$$

Since g is injective,

$$g \restriction y'_n : y'_n \xrightarrow{\text{bij}} x''_n,$$

so

$$g^{-1} \restriction x''_n \xrightarrow{\text{bij}} y'_n.$$

Finally,

$$f \restriction x''' : x''' \xrightarrow{\text{bij}} y'''.$$

Let

$$h = \bigcup_{n\in\omega} f \restriction x'_n \cup \bigcup_{n\in\omega} g^{-1} \restriction x''_n \cup f \restriction x'''.$$

Then $h : x \xrightarrow{\text{bij}} y$, so $x \sim y$. □[3.130]

(3.132) **Theorem** [C]

1. *$\preccurlyeq$ is a preorder.*
2. *$\sim$ is $\equiv^{\preccurlyeq}$.*
3. *$\leqslant$ is a partial order.*

Proof Trivially, for any sets x, y, z, $x \preccurlyeq x$ and $(x \preccurlyeq y \wedge y \preccurlyeq z) \to x \preccurlyeq z$, so $\preccurlyeq$ is a preorder. As for any preorder, $\equiv^{\preccurlyeq}$ is $\preccurlyeq \cap \preccurlyeq^{-1}$, which by (3.130) is $\sim$. The corresponding relation on cardinalities is identity, so $\leqslant$ is a partial order. □[3.132]

3.6.2 Infinity

Naïvely one might think that if a set x is a proper subset of a set y—i.e., $x \subseteq y \wedge x \neq y$—then x is smaller than y—i.e., $x \preccurlyeq y \wedge \neg x \sim y$—but this is not true in general. The parable of the *Hotel of Many Rooms* illustrates this. The Hotel of Many Rooms has a room for each natural number. A traveler arrives one evening and is informed that the hotel is full. She is surprised when the desk clerk tells her that there is nevertheless no difficulty in accommodating her. He will simply ask the occupant of Room 0 to move into Room 1, the occupant of Room 1 to move into Room 2, *ad infinitum*. Room 0 is thus freed up for the newcomer.

The critical property of the Hotel of Many Rooms is of course that it has *infinitely* many rooms; specifically, for the purpose of the parable, the set of rooms is *Dedekind-infinite*, i.e., not *Dedekind-finite*. Dedekind-finiteness is just one of several reasonable notions of finiteness.

(3.133)

1. *A set x is* Dedekind-finite $\overset{\text{def}}{\iff}$ *it is not equipollent with a proper subset of itself.*[49]
2. *Alternatively, we may say that x is* finite $\overset{\text{def}}{\iff}$ *it is not the image of a proper subset of itself via some function.*
3. *Finally, we may say that a set x is* finite $\overset{\text{def}}{\iff}$ *it is equipollent with a number.*

In the absence of Choice, these characterizations of finiteness are not equivalent. The last is the strongest, and we have already taken it our official definition of finiteness.[3.52] Note that by this definition, a finite set is equipollent with a number, so its cardinality is a cardinal.[3.124]

(3.134) **Theorem** [C] *Suppose x is a set.*

1. *Suppose x is finite,*[3.52] *i.e., finite in sense (3.133.3). Then x is finite in sense (3.133.2).*
2. *Suppose x is finite in sense (3.133.2). Then x is finite in sense (3.133.1), i.e., Dedekind-finite.*

Remark This is a refinement of (3.53.1), which states that any finite set is Dedekind-finite.

Proof 1 If x is finite there exists $n \in \omega$ and $h : x \overset{\text{bij}}{\to} n$. Any counterexample to (3.133.2) for x may be transferred to n by h, so it is enough to show that for any $n \in \omega$, $y \subsetneqq n$ and $f : y \to n$, f is not surjective. This we do by induction.[3.51] It is clearly true for $n = 0$, for then there exists no $y \subsetneqq n$. Suppose it is true for n. We will show that it is true for $n + 1$.

To this end, suppose toward a contradiction that $y \subsetneqq n + 1$ and $f : y \overset{\text{sur}}{\to} n + 1$. Construct $y' \subsetneqq n$ and $f' : y' \overset{\text{sur}}{\to} n$ as follows.

1. If $(n, n) \in f$, let $y' = y \backslash \{n\}$ and $f' = f \backslash \{(n, n)\}$.
2. Otherwise, there exists $m \in n$ such that $(m, n) \in f$.
    1. If $n \notin y$, let $y' = y \backslash \{m\}$ and $f' = f \backslash \{(m, n)\}$.

[49] In the parable, we use the fact that $\{0, 1, 2, \ldots\}$ is equipollent with $\{1, 2, \ldots\}$ via $n \mapsto n + 1$.

2. Otherwise, let $y' = y \backslash \{n\}$ and $f' = f \backslash \{(m,n),(n,fn)\} \cup \{(m,fn)\}$.

$\square^{3.134.1}$

2 Suppose there do not exist $y \subsetneqq x$ and $f : y \overset{\text{sur}}{\to} x$, and suppose toward a contradiction that $y' \subsetneqq x$ and $g : x \overset{\text{inj}}{\to} y'$. Let $y = g^{\to}x$ and $f = g^{-1}$. Then $y \subsetneqq x$ and $f : y \overset{\text{sur}}{\to} x$, contrary to hypothesis. $\square^{3.134.2}$ $\square^{3.134}$

(3.135) **Theorem** [C]

1. *Suppose $m, n \in \omega$. Then $m \prec n$ iff $m \in n$. In particular, n is a cardinal.*
2. *ω is not a finite set. If ω is a set, it is a cardinal.*

Proof 1 Suppose $m, n \in \omega$, and suppose $m \in n$. Then $m \preccurlyeq n$ (via id^{m}[3.126]). Since $m \subsetneqq n$ and n is finite by definition, either (3.134.1) or (3.134.2) implies that there does not exist $f : m \overset{\text{bij}}{\to} n$, so $m \not\approx n$. Hence $m \prec n$. In particular, n is not equipollent with any preceding ordinal so n is a cardinal.[3.125]

Conversely, suppose $m \prec n$. Then $m \not\approx n$, so $m \neq n$; and there exists $f : m \overset{\text{inj}}{\to} n$, so $n \notin m$ (otherwise $n \subsetneqq m$, so m is Dedekind-infinite,[3.133.1] hence infinite[3.134]). Thus, $m \in n$. $\square^{3.135.1}$

2 Suppose ω is a set (i.e., not a proper class). Suppose toward a contradiction that $f : \omega \overset{\text{bij}}{\to} n$ for some f and $n \in \omega$. Then $f \restriction (n+1) : n+1 \overset{\text{inj}}{\to} n$, contradicting the fact that $n+1$ is a number and is therefore finite. Since ω is not equipollent with an preceding ordinal, it is a cardinal. $\square^{3.135}$

(3.136) **Theorem** [C] *If there exists an infinite set then ω is a set.*

Proof Suppose x is an infinite set. Let $\alpha \in \mathrm{Ord}$ be such that $x \in V_\alpha$. By (3.135.5) $\alpha > \omega$, so $\omega \in \alpha$, so ω is a set. $\square^{3.136}$

(3.137) *The* Infinity *axiom states that there exists an infinite set, i.e., there exists a set that is not equipollent with a number. In* C *this is equivalent*[3.136] *to the statement that ω is a set. In* S *it is equivalent to the statement that ω exists.*

We have previously shown[3.116] how to derive the entire Foundation schema of S from two of its instances, viz., (3.111) and (3.112). Assuming Infinity we may eliminate the former by providing an alternative derivation of the existence of transitive closures. Recall that ordinals are by definition wellordered by $\in$, so Foundation is not required to justify definition by ordinal recursion. Thus, we may argue from S^0 + Infinity as follows.

Suppose x is an arbitrary set. Let f be the function with domain ω such that $f(0) = x$ and for any $n \in \omega$, $f(n+1) = \bigcup f(n) = \{y \mid \exists z \in f(n)\ y \in z\}$. Let $X = \bigcup_{n \in \omega} f(n)$. Then X is clearly the smallest transitive set that includes x.

Thus, every instance of Foundation is derivable from S^0, Infinity, and the single instance (3.112), which we state here in the fashion of (3.8).

Foundation'

$$\forall v_0\,(\exists v_1 \in v_0 \rightarrow \exists v_1 \in v_0\, \forall v_2 \in v_1\;\; v_2 \notin v_0).$$

Foundation' is the usual form of the axiom of foundation for a pure set theory in the presence of Infinity.

3.7 Extended theories of membership

As we will soon see, the basic theories S and C do not mandate the existence of an infinite set. As we will also see, without infinite sets, it is impossible to model even so basic a mathematical concept as that of a geometrical point, or real number. Indeed, the theory S is bi-interpretable with Peano arithmetic, which is a very meager portion of mathematics as we know it. Infinity is therefore a standard axiom of membership.

In this setting, two other statements that we have presented above as potential axioms of membership become vitally important: Power and AC. As we will show presently, these are not important axioms in an explicitly finitary theory, as[3.210]

$$\mathsf{S} + \neg\,\mathsf{Infinity} \vdash \mathsf{Power} \wedge \mathsf{AC},$$

but in the theory S + Infinity they are powerful set existence principles.

Power is often regarded as an axiom in the traditional sense that it is clearly true—like Euclid's axioms of plane geometry with the possible exception of the parallel postulate. The standard axiomatizations of the theory of membership therefore include it. AC is often regarded as not quite as obviously true—very like the parallel postulate in geometry.

Definition

1. $\mathsf{ZF} \overset{\text{def}}{=} \mathsf{S} + \mathsf{Infinity} + \mathsf{Power}$. *This is the* Zermelo-Fraenkel *theory.*
2. $\mathsf{GB} \overset{\text{def}}{=} \mathsf{C} + \mathsf{Infinity} + \mathsf{Power}$. *This is the* Gödel-Bernays *theory, also called the* von Neumann-Bernays-Gödel *theory.*
3. $\mathsf{ZF}^- \overset{\text{def}}{=} \mathsf{S} + \mathsf{Infinity}$.
4. $\mathsf{GB}^- \overset{\text{def}}{=} \mathsf{C} + \mathsf{Infinity}$.
5. $\mathsf{ZFC} \overset{\text{def}}{=} \mathsf{ZF} + \mathsf{AC}$.
6. $\mathsf{GBC} \overset{\text{def}}{=} \mathsf{GB} + \mathsf{AC}$.
7. $\mathsf{ZFC}^- \overset{\text{def}}{=} \mathsf{ZF}^- + \mathsf{AC}$.
8. $\mathsf{GBC}^- \overset{\text{def}}{=} \mathsf{GB}^- + \mathsf{AC}$.

Since C is a conservative extension of S, each of the above GB-type theories is a conservative extension of the corresponding ZF-type theory. For the convenience of reference to proper classes, we will continue to prefer to work in GB-type theories.

Just as 'set theory' is traditionally used to refer to theories of membership in general, 'ZF' is often used to refer loosely to any any of the above theories in statements that apply to all of them.

For ease of reference, we present here a standard list of axioms for ZF. In anticipation of certain uses, we list Union explicitly, even though it follows from Collection and Comprehension.

(3.138) **Axioms of** ZF

1. Extension
$$\forall \mathrm{v}_0, \mathrm{v}_1\, (\forall \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1) \rightarrow \mathrm{v}_0 = \mathrm{v}_1)$$

2. Comprehension
$$\forall v_0, \ldots, v_{n^-}\, \forall u\, \exists w\, \forall v\, (v \in w \leftrightarrow (v \in u \wedge \phi)),$$
where ϕ is any s*-formula, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such* $\mathrm{Free}\, \phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.

3. Existence
$$\exists \mathrm{v}_0\, \forall \mathrm{v}_1\;\; \mathrm{v}_1 \notin \mathrm{v}_0$$

4. Pair
$$\forall \mathrm{v}_0, \mathrm{v}_1\, \exists \mathrm{v}_2\, (\mathrm{v}_0 \in \mathrm{v}_2 \wedge \mathrm{v}_1 \in \mathrm{v}_2)$$

5. Collection
$$\forall v_0, \ldots, v_{n^-}\, \forall u\, \big(\forall v \in u\, \exists w\, \forall a\, (\phi \rightarrow a \in w) \rightarrow \exists w\, \forall v \in u\, \forall a\, (\phi \rightarrow a \in w)\big),$$
where ϕ is any s*-formula, $a, u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that* $\mathrm{Free}\, \phi \subseteq \{a, v, v_0, \ldots, v_{n^-}\}$*, and u is free for v in ϕ.*

6. Foundation
$$\forall v_0, \ldots, v_{n^-}\, \big(\exists v\, \phi \rightarrow \exists v\, \big(\phi \wedge \forall u \in v\; \neg\, \phi\tbinom{v}{u}\big)\big),$$
where ϕ is any s*-formula, $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that* $\mathrm{Free}\, \phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$*, and u is free for v in ϕ.*

7. Union
$$\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2, \mathrm{v}_3\, (\mathrm{v}_2 \in \mathrm{v}_3 \in \mathrm{v}_0 \rightarrow \mathrm{v}_2 \in \mathrm{v}_1)$$

8. Power
$$\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, \big(\forall \mathrm{v}_1\, (\mathrm{v}_1 \in \mathrm{v}_2 \rightarrow \mathrm{v}_1 \in \mathrm{v}_0) \rightarrow \mathrm{v}_2 \in \mathrm{v}_1\big)$$

9. Infinity
$$\exists \mathrm{v}_0\, (\exists \mathrm{v}_1\;\; \mathrm{v}_1 \in \mathrm{v}_0 \wedge \forall \mathrm{v}_1 \in \mathrm{v}_0\, \exists \mathrm{v}_2 \in \mathrm{v}_0\, \mathrm{v}_1 \in \mathrm{v}_2)$$

In the presence of (3.138.1–6), (3.138.9) is easily seen to be equivalent to the statement that an infinite set exists.

3.8 Principles of choice and wellordering

> The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?
>
> Jerry Lloyd Bona

AC differs from the other set existence axioms in that it does not say that a set exists that satisfies a certain definition, but rather that a set exists that has a certain property, viz., that of being a choice function for a given set.[50] AC nevertheless resembles the other existence axioms in that it contributes to the expression of the

[50] If x is a set and there is a rule by which we can choose a member of each nonempty set $y \in x$, then we do not need Choice to show that a choice function exists—we can use Replacement instead.

idea that each level of the cumulative hierarchy should consist of *all* subsets of the previous level.

As noted above,[3.121] AC is equivalent modulo ZF (or GB) to the principle that every set can be wellordered. Historically, in diverse branches of mathematics, a number of similar principles were formulated, which seemed plausible and were useful in proofs, but which were themselves resistant to all attempts at formal justification. Essentially all of these have been shown to follow from AC, and many are equivalent to AC. We will mention here just one of these, perhaps the best known: Zorn's lemma.

Definition [C] *Suppose* $<$ *is a partial order on a set* P, $X \subseteq P$, *and* $p \in P$.

1. X *is a* chain $\overset{\text{def}}{\iff}$ $<$ *totally orders* X.
2. p *is an* upper bound *for* X $\overset{\text{def}}{\iff}$ $\forall x \in X\ x \leqslant p$.

(3.139) **Theorem** [ZF] *The following are equivalent:*

1. *(Axiom of choice) Every set has a choice function.*
2. *(Wellordering principle) Every set can be wellordered.*
3. *(Zorn's lemma) Suppose* $<$ *is a partial order on a nonempty set* P, *and suppose every chain in* P *has an upper bound. Then there is a maximal element in* P.

Proof 1↔2 See (3.121). Note that Power is used for this, as to wellorder a set x we use a choice function for $\mathcal{P}x$. □

2→3 Assume the wellordering principle. Suppose $<$ is a partial order on a nonempty set P, and suppose every chain in P has an upper bound. Let R be a wellordering of P. By recursion on ordinals, define a function F with domain Ord such that $\operatorname{im} F \subseteq P$, and for every $\alpha \in \mathrm{Ord}$

1. if there exists $p \in P$ such that $F\beta < p$ for all $\beta < \alpha$, then $F\alpha$ is the R-least such p;
2. otherwise, $F\alpha = P$.[51]

Let $A = \{\alpha \in \mathrm{Ord} \mid F\alpha \neq P\}$. It is easy to show by $\in$-induction that A is an initial segment of Ord and that $F \restriction A$ is an order-preserving map of A into P. Hence, $F^{\to}A$ is a chain in $(P; <)$. Also, since P is a set, A is a set and is therefore an ordinal. So $A \in \operatorname{dom} F$. By the definition of A, $fA = P$. By hypothesis, $F^{\to}A$ has an upper bound, say p, but no strict upper bound (with which it could be extended), so p is $<$-maximal. □

3→1 Assume Zorn's lemma. Suppose x is a set. Let P be the set of partial choice functions for x, i.e., $\operatorname{dom} f \subseteq x$ and for all $y \in x$ if $y \neq 0$ then $fy \in y$. P is partially ordered by the inclusion relation. Suppose $C \subseteq P$ is $\subsetneqq$-chain. Then $\bigcup C$ is an upper bound for C. Thus $(P; \subsetneqq)$ satisfies the condition of Zorn's lemma. Therefore let f be a maximal element. Then f is a choice function for x, i.e., $\operatorname{dom} f = x$. For if not, suppose $y \in x \setminus \operatorname{dom} f$. If $y \neq 0$, let z be any element of y; if $y = 0$, let z be any element. Then $f \subsetneqq f \cup \{(y, z)\} \in P$. □ □[3.139]

[51] P is chosen simply as a fixed set not in P.

3.8.1 Restricted axioms of choice

(3.140) **Definition** [ZF^-]

1. $\mathsf{AC}_A \overset{\text{def}}{\iff}$ *for every function f with domain A, there exists g such that* $\operatorname{dom} g = \{a \in A \mid fa \neq 0\}$ *and* $\forall a \in \operatorname{dom} g\ ga \in fa$. *$g$ is a* choice function for f.
2. $\mathsf{AC}_A(B) \overset{\text{def}}{\iff}$ *for every function f with domain A such that $\forall a \in A\ fa \subseteq B$, there exists a choice function for f.*
3. $\mathsf{DC} \overset{\text{def}}{\iff}$ *for any binary relation R on a nonempty set A, if $\forall a \in A\, \exists b\ a\,R\,b$, then for any $a \in A$, there exists a function g with domain ω such that $g0 = a$ and $\forall n \in \omega\ g(n)\,R\,g(n+1)$. g is a* dependent choice function for R; *and this is the* Dependent Choice *axiom.*

3.8.2 Countability

Definition [ZF^-] x *is* countable *or* denumerable $\overset{\text{def}}{\iff} x \sim \omega$.

Countable sets are of interest for several reasons. In the first place, they are particularly amenable to recursive constructions and inductive proofs, as these can often be performed on enumerations in order type ω, obviating consideration of limit cases. Second, countable sets are—to the average mathematician, at least—easily grasped as completed infinitudes. Third, class theories like C, while they do not mandate the existence of infinite sets, do mandate the existence of countably infinite classes, viz., V_ω and its infinite subclasses, and any countable structure has an isomorph of this type.

Recall[3.94] that a set x is said to have a property hereditarily just in case every set in $\operatorname{tc}\{x\}$ has the property. For countability we make the following special definition. It is equivalent to the generic definition in the presence of AC.

(3.141) **Definition** [ZF^-]

1. x *is* hereditarily countable $\overset{\text{def}}{\iff} \mathrm{HC}(x) \overset{\text{def}}{\iff} \operatorname{tc}\{x\}$ *is countable.*
2. [GB^-] $\mathrm{HC} \overset{\text{def}}{=}$ *the class of* HC *sets.*

Note that if $\operatorname{tc} x$ is countable then $\operatorname{tc}\{x\}$, which is just $\{x\} \cup \operatorname{tc} x$, is also countable. Using AC it is easy to show that any countable set of HC sets is HC, and that HC is in fact the smallest class that contains all its countable subsets.[52] It is not hard to show that $\mathsf{AC}_\omega(\mathcal{P}\omega)$ suffices for this.[53]

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[52] Remember that finite sets, and 0 in particular, are countable, so any class that contains all its countable subsets includes $V_{\omega+1}$.

[53] In GB^- we define ω_1 as the class of countable ordinals.[3.147] The smallest class C that contains all its countable subsets may be defined as $\bigcup_{\alpha \in \omega_1} C_\alpha$, where

1. $C_0 = 0$;
2. for each $\alpha \in \omega_1$, $C_{\alpha+1}$ is the set of countable subsets of C_α; and
3. for each limit $\alpha \in \omega_1$, $C_\alpha = \bigcup_{\beta \in \alpha} C_\beta$.

Note that $C_\alpha \subseteq V_\alpha$. Without Choice we can show that $\mathrm{HC} \subseteq V_{\omega_1}$ and that for each $\alpha \in \omega_1$, $\mathrm{HC} \cap V_\alpha \subseteq C_\alpha$. To show the reverse inclusion, we observe that if A is a countable transitive set then $(A; \in)$ is isomorphic to a binary relation on ω; and given the relation on ω, the isomorphism with $(A; \in)$ is unique. Given a sequence $\langle A_n \mid n \in \omega \rangle$ of countable transitive sets, $\mathsf{AC}_\omega(\mathcal{P}\omega)$ therefore implies the existence of a sequence $\langle f_n \mid n \in \omega \rangle$ such that for each $n \in \omega$, $f_n : \omega \overset{\text{sur}}{\to} A_n$, which allows us to enumerate $\bigcup_{n \in \omega} A_n$.

3.8.3 Cardinals and cofinality

(3.142) **Definition** [C] *Suppose* $X \in \mathrm{Ord}$ *or* $X = \mathrm{Ord}$*, and suppose* $Y \subseteq X$*.*

1. Y *is* unbounded in X $\overset{\text{def}}{\iff} \forall \alpha \in X\, \exists \beta \in Y\ \beta > \alpha$*. Note that* X *cannot be a successor ordinal.*
2. Y *is* closed in X $\overset{\text{def}}{\iff}$ *for all* $\alpha \in X$*, if* $Y \cap \alpha$ *is unbounded in* α *then* $\alpha \in Y$*.*
3. Y *is* club in X $\overset{\text{def}}{\iff}$ Y *is* closed unbounded in X $\overset{\text{def}}{\iff}$ Y *is closed in* X *and unbounded in* X*.*
4. *Suppose* $f : X \to \mathrm{Ord}$*.*
   1. f *is* increasing $\overset{\text{def}}{\iff} \forall \alpha, \beta \in X\, (\alpha < \beta \to f\alpha < f\beta)$*.*
   2. *Suppose* f *is increasing. Then* f *is* continuous $\overset{\text{def}}{\iff}$ *for all* $\alpha \in X$*, if* $\alpha \in \mathrm{Lim}$ *then* $f^{\to}\alpha$ *is unbounded in* $f\alpha$*.*
   3. *The standard topology*[3.185] *on* Ord *is generated from the intervals*

$$(\alpha, \beta) \overset{\text{def}}{=} \{\gamma \mid \alpha < \gamma < \beta\}$$

as a base.[§ 3.11.2] *The use of the terms 'limit', 'closed', and 'continuous' in the context of ordinals derives from this topology.*

We may omit explicit reference to X when $X = \mathrm{Ord}$ or when its identity may be inferred from the context.

(3.143) **Theorem** [C] *Suppose* $\alpha, \beta \in \mathrm{Ord}$*,* $f : \alpha \overset{\text{sur}}{\to} \beta$*, and* f *is increasing.*[3.142.4.1] *Then* $\alpha = \beta$ *and* $f = \mathrm{id}^{\alpha}$*, the identity function on* α*.*

Proof We show by induction on $\gamma \in \alpha$ that $f\gamma = \gamma$, i.e., we suppose toward a contradiction that $\gamma \in \alpha$ is least such that $f\gamma \neq \gamma$. Then $\forall \gamma' < \gamma\ f\gamma' = \gamma'$. Since f is increasing, $f\gamma \geqslant \gamma$, so $f\gamma > \gamma$ and $\beta > f\gamma > \gamma$. Since f is order-preserving, $\forall \gamma' \in \alpha\, (\gamma' \geqslant \gamma \to f\gamma' \geqslant f\gamma > \gamma)$, so $\gamma \notin \mathrm{im}\, f$, contradicting the fact that $\beta > \gamma$ and the assumption that f maps onto β.

Since $f = \mathrm{id}^{\alpha}$ and f is surjective to β, $\beta = \alpha$. □[3.143]

We have noted above[3.135.1] that if α and β are distinct finite ordinals, then $\alpha \nsim \beta$. This is not true in general for infinite ordinals. For example $\omega + 1 \sim \omega$, as shown by the function $f : \omega + 1 \overset{\text{bij}}{\to} \omega$, given by

$$f\alpha = \begin{cases} 0 & \text{if } \alpha = \omega \\ \alpha + 1 & \text{if } \alpha \in \omega. \end{cases}$$

With a little ingenuity it is possible to define bijections between ω and ordinals considerably beyond ω. Nevertheless, there are ordinals larger than ω. Indeed, for any ordinal α there is a larger ordinal.

These assertions are a consequence of the Power axiom, which we now begin to use regularly.

(3.144) **Theorem** [ZF] *Suppose* X *is a set. Then there exists an ordinal* α *such that* $\alpha \not\preccurlyeq X$*.*

Proof A binary relation on a subset of X is a subset of $X \times X$ and hence is in $\mathcal{P}(X \times X)$. Let W be the subset of $\mathcal{P}(X \times X)$ consisting of the wellorders of subsets of X. Let $F : W \to \mathrm{Ord}$ be the order-type map, i.e., FR is the order type[3.119] of R for $R \in W$.

Suppose α is an ordinal and $\alpha \preccurlyeq X$, say $f : \alpha \overset{\text{inj}}{\to} X$. Let $R = \{\langle f\alpha', f\alpha''\rangle \mid \alpha' < \alpha'' < \alpha\}$. Then $R \in W$, and $\alpha = FR$, so $\alpha \in \operatorname{im} F$.

Since F is a function and $\operatorname{dom} F = W$ is a set, $\operatorname{im} F$ is a set of ordinals, so $\bigcup \operatorname{im} F$ is an ordinal, say α. Clearly, $\alpha \notin \operatorname{im} F$, so $\alpha \not\preccurlyeq X$. □[3.144]

(3.145) **Theorem** [GB] Card[3.124.4] *is closed and unbounded.*

Proof Theorem 3.144 shows that Card is unbounded. To show that Card is closed, suppose $\alpha \in \mathrm{Ord}$ and $\mathrm{Card} \cap \alpha$ is unbounded in α. Then for any $\beta \in \alpha$, there exists $\kappa \in \mathrm{Card} \cap \alpha$ such that $\beta < \kappa$; and since $\kappa \in \mathrm{Card}$, $\beta \prec \kappa$, so $\beta \prec \alpha$. Hence, $\alpha \in \mathrm{Card}$. □[3.145]

(3.146) **Definition** [ZF] *Suppose α is an ordinal. Recall*[3.124.1.1] *that $|\alpha|$ is the least cardinal κ such that $\alpha \sim \kappa$. Note that $|\alpha|$ is the greatest cardinal $\leqslant \alpha$.*

1. $\alpha^+ \overset{\text{def}}{=}$ *the least cardinal greater than α. Note that if $\kappa = \alpha^+$ then $\kappa = |\alpha|^+$.*[54]
2. *Suppose κ is a cardinal.*
    1. *κ is a* successor cardinal $\overset{\text{def}}{\iff}$ *$\kappa = \lambda^+$ for some λ (the* predecessor of κ *as a cardinal).*
    2. *κ is a* limit cardinal $\overset{\text{def}}{\iff}$ *the cardinals in κ form an unbounded subset of κ.*

Note that every cardinal other than 0 is either a successor or a limit cardinal, but not both. Note also that every infinite cardinal is a limit ordinal.[55]

(3.147) **Definition** [ZF]

1. *We define ω_α by recursion on ordinals α as follows.*
    1. $\omega_0 \overset{\text{def}}{=} \omega$.
    2. $\omega_{\alpha+1} \overset{\text{def}}{=} \omega_\alpha^+$.
    3. *If η is a limit ordinal,* $\omega_\eta \overset{\text{def}}{=} \bigcup_{\alpha\in\eta} \omega_\alpha$.
2. $\aleph_\alpha$ (read: 'aleph-α') $\overset{\text{def}}{=} \omega_\alpha$.

$\omega_\alpha \mapsto \alpha$ is evidently the transitive collapsing map for the class of infinite cardinals, and any infinite cardinal is ω_α for some $\alpha \in \mathrm{Ord}$. As a matter of usage, the '$\aleph$' notation is often used to name a cardinal when only its size is of interest—i.e., when any other set of the same size would do as well—whereas the 'ω' notation is used when the order type is also of interest; nevertheless, the two notations are strictly synonymous, and we will generally use the latter.

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[54] Context should prevent confusion of the successor operation on cardinals with the successor operation on ordinals.[3.45]

[55] 'limit' as an adjective applied to ordinals has multiple meanings. When the ordinal in question is a cardinal, it is usually the sense (3.146.2.2) that is intended. The construction used above to show that $\omega + 1 \sim \omega$ is easily adapted to show that $\alpha + 1 \sim \alpha$ for any infinite ordinal α, so no infinite successor ordinal is a cardinal.

Definition [ZF] *Suppose R is a total order on a set X that has no maximum element, and α is an ordinal.*

1. $f : \alpha \overset{\text{cof}}{\to} X \overset{\text{def}}{\iff} f$ *is* cofinal *in X (with respect to R)* $\overset{\text{def}}{\iff} f : \alpha \to X$ *and* $\operatorname{im} f$ *is R-unbounded.*
2. *The* cofinality of $R \overset{\text{def}}{=} \operatorname{cf} R \overset{\text{def}}{=}$ *the least ordinal α such that there exists $f : \alpha \overset{\text{cof}}{\to} X$ (with respect to R).*
3. *The terms 'cofinal' and 'cofinality' as applied to ordinals X refer to the $\in$-ordering. We are primarily interested in this setting.*

Definition [ZF] *Suppose κ is a cardinal.*

1. κ *is* regular $\overset{\text{def}}{\iff} \kappa$ *is infinite and* $\operatorname{cf} \kappa = \kappa$.
2. κ *is* singular $\overset{\text{def}}{\iff} \kappa$ *is infinite and* $\operatorname{cf} \kappa < \kappa$.

(3.148) **Theorem** [ZF] *Suppose R is a total order on a set X with no maximum. Then $\operatorname{cf} R$ is a regular cardinal.*

Proof Straightforward. $\square$[3.148]

Theorem [ZFC] *Every successor cardinal is regular.*

Remark Choice is necessary for this. It is consistent with ZF that ω_1 is singular.

Proof Suppose $\kappa = \lambda^+$, and suppose toward a contradiction that κ is singular. Then $\operatorname{cf} \kappa \leqslant \lambda$, so there exists $f : \lambda \overset{\text{cof}}{\to} \kappa$. Using AC, for each $\alpha < \lambda$, let $g_\alpha : \lambda \overset{\text{sur}}{\to} f\alpha$. Let $g = \{((\alpha, \beta), g_\alpha \beta) \mid \alpha, \beta \in \lambda\}$. Then $g : \lambda \dot{\times} \lambda \overset{\text{sur}}{\to} \bigcup_{\alpha \in \lambda} f\alpha = \kappa$. So $|\kappa| \leqslant |\lambda \dot{\times} \lambda| = \lambda$, contrary to the assumption that κ is a cardinal. $\square$[3.152]

3.9 Cardinal arithmetic

We have previously defined operations of addition, multiplication, and exponentiation on ordinals, which generalize these operations as applied to numbers. There are also versions of these operations applicable to cardinalities, which generalize the number operations in a different way. We will use the same symbols for these as for the ordinal-specific operations; unless otherwise stated, it is usually the cardinality-specific operations that we have in mind.

We first note that AC implies the equivalence of the various definitions of finiteness (3.133).

(3.149) **Theorem** [ZFC] *If x is Dedekind-finite then x is finite. Hence,*[3.134] *(3.133.1, 2, 3) are equivalent.*

Proof We will prove the contrapositive. Suppose x is infinite. Let F be a choice function for $\mathcal{P}x$. Define x_n by recursion on $n \in \omega$ so that $x_n = F(x \backslash \{x_m \mid m \in n\})$. For each $n \in \omega$, since x is not equipollent with n, $x \neq \{x_0, \ldots, x_{n^-}\}$, so $x \backslash \{x_m \mid m \in n\}$ is nonempty, and $x_n \in x \backslash \{x_m \mid m \in n\}$. $n \mapsto x_n$ is therefore injective. Let $y = x \backslash \{x_n \mid n \in \omega\}$, and let

$$g = \{(a, a) \mid a \in y\} \cup \{(x_n, x_{n+1}) \mid n \in \omega\}.$$

Then g is a bijection of x with a proper subset of x (viz., $x\backslash\{x_0\}$), so x is Dedekind-infinite. $\square$[3.149]

(3.150) **Definition** [ZF] *Suppose a and b are cardinalities.*

1. $a + b \stackrel{\text{def}}{=} |A \cup B|$, *where A, B are any disjoint sets such that $|A| = a$ and $|B| = b$.*
2. $a \cdot b \stackrel{\text{def}}{=} |A \times B|$, *where A, B are any sets such that $|A| = a$ and $|B| = b$.*
3. $a^b \stackrel{\text{def}}{=} |{}^B A|$, *where A, B are any sets such that $|A| = a$ and $|B| = b$.*

To validate this definition, of course, we must show that for any sets A, A', B, B', if $A \sim A'$, $B \sim B'$, then

1. if $A \cap B = A' \cap B' = 0$ then $A \cup B \sim A' \cup B'$;
2. $A \times B \sim A' \times B'$; and
3. ${}^B A \sim {}^{B'} A'$.

This is straightforward.

The power of AC quickly becomes apparent when we try to analyze these and other operations and relations that are invariant under equipollence, and we henceforth in this section assume this axiom unless otherwise stated.

It follows immediately from the wellordering principle that every set x is equipollent with a cardinal, viz., the least ordinal that is the order type of a wellordering of x, so every cardinality is a cardinal, and the arithmetical operations defined previously for cardinalities[3.150] are operations on the cardinals.

With AC we can define sums and products of arbitrary sets of cardinals (i.e., not just pairs of cardinals, as in (3.150)):

Definition [ZFC] *Suppose I is a set and $\langle \kappa_i \mid i \in I \rangle$ is a cardinal-valued function on I.*

1. $\sum_{i \in I} \kappa_i \stackrel{\text{def}}{=} |\bigcup_{i \in I} A_i|$, *where $\langle A_i \mid i \in I \rangle$ is any function on I such that $\forall i \in I\ |A_i| = \kappa_i$ and $\forall i, j \in I\,(i \neq j \rightarrow A_i \cap A_j = 0)$.*
2. $\prod_{i \in I} \kappa_i \stackrel{\text{def}}{=} |\bigtimes_{i \in I} A_i|$,[3.61.2] *where $\langle A_i \mid i \in I \rangle$ is any function on I such that $\forall i \in I\ |A_i| = \kappa_i$.*

To show that this definition is valid we must show that for any $\langle A_i \mid i \in I \rangle$ and $\langle A'_i \mid i \in I \rangle$, if $\forall i \in I\ A_i \sim A'_i$ then $\bigtimes_{i \in I} A_i \sim \bigtimes_{i \in I} A'_i$; and if $\forall i, j \in I\,(i \neq j \rightarrow A_i \cap A_j = A'_i \cap A'_j = 0)$ then $\bigcup_{i \in I} A_i \sim \bigcup_{i \in I} A'_i$. To do so we first invoke AC to obtain $\langle f_i \mid i \in I \rangle$ such that $\forall i \in I\ f : A_i \stackrel{\text{bij}}{\rightarrow} A'_i$. The desired equivalences easily follow.

It is easy to see that cardinal addition and multiplication are commutative (unlike ordinal addition and multiplication) in the most general sense: Cardinal-valued functions $\langle \kappa_i \mid i \in I \rangle$ and $\langle \lambda_j \mid j \in J \rangle$ have the same sum and product if there exists $f : I \stackrel{\text{bij}}{\rightarrow} J$ such that $\forall i \in I\ \lambda_{f\,i} = \kappa_i$. It is also easy to see that cardinal multiplication and cardinal exponentiation are special cases of generalized addition and multiplication, respectively.

Note that ${}^{\lambda}\kappa$ is by definition the set of functions $f : \lambda \to \kappa$, whereas κ^{λ} is an ordinal, viz., the cardinality of ${}^{\lambda}\kappa$. The reader is warned that the 'κ^{λ}' often appears in the literature meaning '${}^{\lambda}\kappa$'. Note that the ordinary exponential notation also does double duty in this book, in that for $n \in \omega$, we have used 'κ^{n}' to denote the set $\underbrace{\kappa \dot{\times} \cdots \dot{\times} \kappa}_{n \text{ times}}$ of n-tuples (as opposed to the set ${}^{n}\kappa$ of n-sequences). The cardinality of this set is of course the same as that of ${}^{n}\kappa$, viz., κ^{n} in the cardinal-arithmetic sense.

(3.151)

1. *We have already mentioned that cardinal addition and multiplication are commutative.*
2. *They are obviously also associative: If $\langle I_j \mid j \in J\rangle$ is a partition of I, i.e., the I_js are pairwise disjoint and $\bigcup_{j\in J} I_j = I$, then*

$$\sum_{i\in I}\kappa_i = \sum_{j\in J}\sum_{i\in I_j}\kappa_i$$
$$\prod_{i\in I}\kappa_i = \prod_{j\in J}\prod_{i\in I_j}\kappa_i.$$

3. *Familiar distributive laws also hold:*

$$\lambda\cdot\sum_{i\in I}\kappa_i = \sum_{i\in I}(\lambda\cdot\kappa_i)$$
$$\Big(\prod_{i\in I}\kappa_i\Big)^{\lambda} = \prod_{i\in I}\kappa_i^{\lambda}$$
$$\mu^{\kappa+\lambda} = \mu^{\kappa}\cdot\mu^{\lambda}$$
$$\mu^{\kappa\cdot\lambda} = \big(\mu^{\kappa}\big)^{\lambda}.$$

Addition and multiplication of infinite cardinals are easily characterized.

(3.152) **Theorem** [ZFC] *Suppose κ and λ are cardinals. If either κ or λ is infinite then $\kappa+\lambda = \kappa\cdot\lambda = \max\{\kappa,\lambda\}$.*

Proof It is enough to show that if κ is an infinite cardinal then $\kappa\cdot\kappa = \kappa$. To this end let R be the binary relation on $\mathrm{Ord}\times\mathrm{Ord}$ such that $\langle\alpha,\beta\rangle\, R\,\langle\alpha',\beta'\rangle$ iff

1. $\max\{\alpha,\beta\} < \max\{\alpha',\beta'\}$; or
2. $\max\{\alpha,\beta\} = \max\{\alpha',\beta'\}$ and $\alpha < \alpha'$; or
3. $\max\{\alpha,\beta\} = \max\{\alpha',\beta'\}$ and $\alpha = \alpha'$ and $\beta < \beta'$.

Clearly, R is a strict wellordering. Let $\Gamma : \mathrm{Ord}\times\mathrm{Ord} \to \mathrm{Ord}$ be such that for any $\alpha,\beta \in \mathrm{Ord}$, $\Gamma\langle\alpha,\beta\rangle$ is the order type of $\{\langle\alpha',\beta'\rangle \mid \langle\alpha',\beta'\rangle\, R\,\langle\alpha,\beta\rangle\}$. Γ is the transitive collapsing map for $(\mathrm{Ord}\times\mathrm{Ord}; R)$, so $\Gamma : \mathrm{Ord}\times\mathrm{Ord} \overset{\text{bij}}{\to} \mathrm{Ord}$. Clearly,

(3.153)

1. Γ *is injective;*
2. *for any ordinal α, $\Gamma^{\to}(\alpha\times\alpha)$ is an initial segment of* Ord, *i.e., an ordinal;*

3. $\Gamma^{\to}(\alpha \times \alpha) \geqslant \alpha$; *and*
4. *for any limit ordinal* α, $\Gamma^{\to}(\alpha \times \alpha) = \sup_{\beta<\alpha} \Gamma^{\to}(\beta \times \beta)$.

(3.154) **Claim** *For every infinite cardinal* κ, $\Gamma^{\to}(\kappa \times \kappa) = \kappa$; *hence,*[3.153.1] $|\kappa \times \kappa| = \kappa$.

Proof For each $n < \omega$, $n \times n$ is finite, so $\Gamma^{\to}(n \times n) < \omega$. Thus,[3.153.4]

$$\Gamma^{\to}(\omega \times \omega) = \sup_{n<\omega} \Gamma^{\to}(n \times n) = \omega.$$

Now suppose toward a contradiction that the theorem fails, and let κ be the least cardinal for which it fails. Then $\kappa > \omega$, and $\kappa < \Gamma\langle\kappa, \kappa\rangle$.[3.153.3] Hence,[3.153] for some $\alpha < \kappa$, $\kappa \leqslant \Gamma\langle\alpha, \alpha\rangle$, so $\kappa \subseteq \Gamma^{\to}(\alpha \times \alpha)$. Hence, $\kappa = |\kappa| \leqslant |\alpha \times \alpha| \leqslant |\lambda \times \lambda|$, where $\lambda = |\alpha| < \kappa$. Since λ is necessarily infinite, by the minimality of κ, $|\lambda \times \lambda| = \lambda$, so $\kappa \leqslant \lambda$, a contradiction. $\square$[3.154]

Hence $\kappa \cdot \kappa = |\kappa \times \kappa| = \kappa$ for all infinite cardinals κ, which, as noted above, suffices to prove the theorem. $\square$[3.152]

3.9.1 Cardinal exponentiation

(3.152) neatly characterizes (infinite) cardinal addition and multiplication.[3.152] Cardinal exponentiation, on the other hand, has been a major theme of fundamental research in set theory. We have already essentially proved the following theorem.

(3.155) **Theorem** [ZFC] *For any cardinal* κ, $\kappa < 2^{\kappa}$.

Proof Considering characteristic functions, we see that $2^{\kappa} = |{}^{\kappa}2| = |\mathcal{P}\kappa|$.[3.128] $\square$[3.155]

The following theorem generalizes Cantor's diagonal argument and may be used to prove Theorem 3.157, which states two of the few simple general properties of cardinal exponentiation in addition to (3.155) that are provable in ZFC.

(3.156) **Theorem** [ZFC] **König's lemma** *Suppose* $\langle\kappa_i \mid i \in I\rangle$ *and* $\langle\lambda_i \mid i \in I\rangle$ *are cardinal-valued functions, and suppose* $\forall\beta \in \alpha\ \kappa_\beta < \lambda_\beta$. *Then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof Trivially, $\sum_{i\in I} \kappa_i \leqslant \prod_{i\in I} \lambda_i$. We have to show that $\prod_{i\in I} \lambda_i \nleqslant \sum_{i\in I} \kappa_i$. Let

$$A = \{(i, \alpha) \mid i \in I \wedge \alpha \in \kappa_i\}$$
$$B = \{f \mid \mathrm{Fcn}\, f \wedge \mathrm{dom}\, f = I \wedge \forall i \in I\ fi \in \lambda_i\}.$$

Then $\sum_{i\in I} \kappa_i = |A|$, and $\prod_{i\in I} \lambda_i = |B|$. Suppose $f : A \to B$.

We must show that f is not surjective. Let $g \in B$ be such that for each $i \in I$, gi is the least member of λ_i not in $\{f(i, \alpha)(i) \mid \alpha \in \kappa_i\}$. Since $\kappa_i < \lambda_i$ for each $i \in I$, gi is well defined, and $g \neq f(i, \alpha)$ for any $i \in I$ and $\alpha \in \kappa_i$, so $g \notin \mathrm{im}\, f$. $\square$[3.128]

Note that (3.155) follows from (3.157) by letting $\kappa_\alpha = 1$ and $\lambda_\alpha = 2$ for $\alpha < \kappa$.

(3.157) **Theorem** [ZFC] *Suppose* κ *is an infinite cardinal.*

1. $\kappa < \mathrm{cf}\, 2^{\kappa}$.
2. $\kappa < \kappa^{\mathrm{cf}\,\kappa}$.

Proof 1 Suppose $f : \kappa \to 2^\kappa$. We must show that f is not cofinal in 2^κ. For this it suffices to show that $\sum_{\alpha<\kappa} \kappa_\alpha < 2^\kappa$, where $\kappa_\alpha = |f\alpha|$ for each $\alpha < \kappa$. This follows from (3.156) by letting $\lambda_\alpha = 2^\kappa$ for every $\alpha < \kappa$, using the fact that $\prod_{\alpha<\kappa} \lambda_\alpha = (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa$. $\square$[3.157.1]

2 By the definition of cofinality, there is a sequence $\langle \kappa_\alpha \mid \alpha < \mathrm{cf}\,\kappa \rangle$ such that $\kappa_\alpha < \kappa$ for all $\alpha < \kappa$, and $\sum_{\alpha<\mathrm{cf}\,\kappa} \kappa_\alpha = \kappa$. By (3.156) $\kappa = \sum_{\alpha<\mathrm{cf}\,\kappa} \kappa_\alpha < \prod_{\alpha<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}$.
$\square$[3.157.2] $\square$[3.157]

The following theorem completely characterizes κ^λ for infinite cardinals in terms of $\kappa^{\mathrm{cf}\,\kappa}$ and μ^λ for $\mu < \kappa$.

(3.158) **Theorem** [ZFC] *Suppose κ, λ are infinite cardinals.*

1. *If $\kappa \leqslant \lambda$ then $\kappa^\lambda = 2^\lambda$.*
2. *If $\mu < \kappa$ and $\mu^\lambda \geqslant \kappa$ then $\kappa^\lambda = \mu^\lambda$.*
3. *If $\kappa > \lambda$ and $\forall \mu < \kappa\ \mu^\lambda < \kappa$ then*
    1. *if $\mathrm{cf}\,\kappa > \lambda$ then $\kappa^\lambda = \kappa$; and*
    2. *if $\mathrm{cf}\,\kappa \leqslant \lambda$ then $\kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$.*

Proof 1 $2^\lambda \leqslant \kappa^\lambda \leqslant (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda$.

2 $\mu^\lambda \leqslant \kappa^\lambda \leqslant (\mu^\lambda)^\lambda = \mu^{\lambda\cdot\lambda} = \mu^\lambda$.

3.1 Every function from λ to κ is bounded below κ, so $\kappa^\lambda \leqslant \sum_{\alpha<\kappa} |\alpha|^\lambda \leqslant \kappa\cdot\kappa = \kappa$.

3.2 κ is singular, so there exists a strictly increasing sequence $\langle \mu_\alpha \mid \alpha < \mathrm{cf}\,\kappa \rangle$ of cardinals $< \kappa$ such that $\kappa = \sum_{\alpha<\mathrm{cf}\,\kappa} \mu_\alpha$. Thus[3.156] $\kappa = \sum_{\alpha<\mathrm{cf}\,\kappa} \mu_\alpha < \prod_{\alpha<\mathrm{cf}\,\kappa} \mu_{\alpha+1} = \prod_{\alpha<\mathrm{cf}\,\kappa} \mu_\alpha$. Hence, $\kappa^\lambda \leqslant \prod_{\alpha<\mathrm{cf}\,\kappa} \mu_\alpha^\lambda \leqslant \prod_{\alpha<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa} \leqslant \kappa^\lambda$. $\square$[3.158]

3.9.1.1 The continuum hypothesis

Given (3.155), it is natural to wonder just how big 2^κ is relative to κ. The following is a statement of the simplest possibility.

1. **Continuum hypothesis** (CH) $2^\omega = \omega_1$.
2. **Generalized continuum hypothesis**(GCH)*: For every infinite cardinal κ, $2^\kappa = \kappa^+$.*

The origin of the name 'continuum hypothesis' is Cantor's observation that 2^ω is the cardinality of the set of real numbers, also known as *the continuum*. The function $\kappa \mapsto 2^\kappa$ is also known as the *continuum function*; however, in the generalized continuum hypothesis it is the hypothesis that is generalized, not the continuum.

3.9.2 The singular cardinals problem

(3.158) demonstrates the central role played by the function $\kappa \mapsto \kappa^{\operatorname{cf}\kappa}$ in the characterization of cardinal exponentiation. Note that if κ is regular then $\operatorname{cf}\kappa = \kappa$, so $\kappa^{\operatorname{cf}\kappa} = \kappa^\kappa = 2^\kappa$.[3.158.1] We will show[8.9 7.4] in later chapters that ZFC provides no information about the size of 2^κ for regular infinite cardinals κ beyond that given by (3.157) and the obvious fact of monotonicity: $\kappa \leqslant \lambda \rightarrow 2^\kappa \leqslant 2^\lambda$. These results have the form of *relative consistency statements*. Specifically, working in the finitary theory S, we show that if ZF is consistent (which is not provable in ZF if ZF is consistent) then so is ZFC plus essentially any statement as to the sizes of the cardinals 2^κ for regular cardinals κ, compatible with (3.157) and the monotonicity condition.

Thus, the investigation of cardinal exponentiation reduces largely to the examination of the behavior of $\kappa^{\operatorname{cf}\kappa}$ at singular cardinals κ. The *singular cardinal hypothesis* is in effect what remains of the generalized continuum hypothesis if its restriction on the size of 2^κ for regular cardinals κ is deleted.

Singular cardinals hypothesis (SCH) *For every singular cardinal κ, if $2^{\operatorname{cf}\kappa} < \kappa$ then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.*

Note that SCH follows easily from GCH. Note also that if $2^{\operatorname{cf}\kappa} \geqslant \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$, whereas if $2^{\operatorname{cf}\kappa} < \kappa$ then by (3.157.2) $\kappa^{\operatorname{cf}\kappa} \geqslant \kappa^+$, so SCH simply states that $\kappa^{\operatorname{cf}\kappa}$ has the least possible value for singular κ.

The consistency of ZFC + ¬ SCH—unlike that of ZFC + ¬ GCH—does not follow from Con ZF. In fact, Con(ZFC + ¬ SCH) is equivalent (over S) to the consistency of the existence of a particular sort of large cardinal (viz., a measurable cardinal κ of Mitchell order κ^{++}).

On the other hand, the singular cardinals problem also differs from the regular cardinals (non-)problem inasmuch as in some instances the size of 2^κ for a singular cardinal κ may be inferred from the behavior of the continuum function below κ. The first theorem along this line was the celebrated result of Silver, which we state and prove as Theorem 8.216, that if κ is a singular cardinal of uncountable cofinality, and $\{\lambda < \kappa \mid 2^\lambda = \lambda^+\}$ is stationary in κ, then $2^\kappa = \kappa^+$. (Silver proved more than this, but this statement gives the spirit of it.) Note that this is a theorem of ZFC. Prior to Silver's work it was widely supposed that no such theorem could be proved in ZFC. Soon after it, Galvin and Hajnal proved (in ZFC) that if ω_α is a strong limit singular cardinal of uncountable cofinality then $2^{\omega_\alpha} < \omega_{(2^{|\alpha|})^+}$. More is now known about the arithmetic of singular cardinals and related issues. Countable cofinality remains rather stubborn, but we have the remarkable theorem of Shelah that if ω_ω is a strong limit then $2^{\omega_\omega} < \omega_{\omega_4}$.

3.10 More basic constructs

3.10.1 Lattices

Definition [C] *Suppose $(X; \leqslant)$ is a partial order, $Y \subseteq X$, and $x \in X$.*

1. *x is a* lower (upper) bound *of Y $\overset{\text{def}}{\iff}$ for all $y \in Y$, $x \leqslant y$ ($x \geqslant y$).*
2. *x is* the greatest lower (least upper) bound *of Y $\overset{\text{def}}{\iff}$ x is a lower (upper) bound of Y and is an upper (lower) bound of the lower (upper) bounds of Y. In this context,* join (meet) *and* supremum (infimum) *are synonyms for* least

upper (greatest lower) bound. *Note that since $\leqslant$ is antisymmetric,*[3.71.1, 3.65.5] *if a join (meet) exists, it is unique, so the definite article is justified.*

Definition [C] *Suppose $\leqslant$ is a binary relation on a class X.*

1. $(X;\leqslant)$ *is an* upper (lower) semilattice $\stackrel{\text{def}}{=}$ $\leqslant$ *is a partial order and for all $x, x' \in X$, the join (meet) of $\{x, x'\}$ exists.*
2. $(X;\leqslant)$ *is a* lattice $\stackrel{\text{def}}{=}$ *it is both an upper and lower semilattice.*

Definition [C] *Suppose $(X;\leqslant)$ is an upper (lower) semilattice, and $x, x' \in X$. $x \vee x'$ $(x \wedge x')$ $\stackrel{\text{def}}{=}$ the join (meet) of $\{x, x'\}$.*

It is easy to show that in an upper (lower) semilattice, joins (meets) of finite sets exist, and they are definable in terms of $\vee$ $(\wedge)$. For example, the join of $\{x, y, z\}$ is both $x \vee (y \vee z)$ and $(x \vee y) \vee z$. Note that this implies the associative law for join. The commutative law obviously also holds. The following theorem lists these along with a third key identity, idempotence; and for lattices the remaining key identity, absorption.

(3.159) **Theorem** [C]

1. *Suppose $(X,\leqslant)$ is an upper semilattice and $x, y, z \in X$.*
   1. $x \vee (y \vee z) = (x \vee y) \vee z$.
   2. $x \vee y = y \vee x$.
   3. $x \vee x = x$.
2. *The same holds for $\wedge$ when $(X;\leqslant)$ is a lower semilattice.*
3. *If $(X;\leqslant)$ is a lattice then for any $x, y \in X$, $x \vee (x \wedge y) = x \wedge (x \vee y) = x$.*

Proof Straightforward. □[3.159]

We may use join and/or meet to define (semi)lattices as algebraic structures.

(3.160) **Definition** [C] *Suppose X is a class.*

1. *Suppose $\cdot$ is a binary operation on X. $(X;\cdot)$ is a* semilattice $\stackrel{\text{def}}{\iff}$ *for all $x, y, z \in X$*
   1. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
   2. $x \cdot y = y \cdot x$; *and*
   3. $x \cdot x = x$.
2. *Suppose $\vee$ and $\wedge$ are binary operations on X. $(X;\vee,\wedge)$ is a* lattice $\stackrel{\text{def}}{\iff}$ $(X;\vee)$ *and $(X;\wedge)$ are semilattices and the* absorption law[3.159.3] *holds, i.e., for all $x, y \in X$*
$$x \vee (x \wedge y) = x \wedge (x \vee y) = x.$$

(3.161) **Theorem** [C]

1. *Suppose $(X;\cdot)$ is a semilattice (in the sense of (3.160)).*

1. *Let*
$$\leqslant = \{\langle x, y\rangle \in {}^2X \mid x \cdot y = y\}.$$
Then $(X; \leqslant)$ *is an upper semilattice.*
2. *Let*
$$\leqslant = \{\langle x, y\rangle \in {}^2X \mid x \cdot y = x\}.$$
Then $(X; \leqslant)$ *is a lower semilattice.*

2. *Suppose* $(X; \vee, \wedge)$ *is a lattice.*
   1. $\forall x, y \in X\ x \vee y = y \leftrightarrow x \wedge y = x$.
   2. *Let*
$$\leqslant = \{\langle x, y\rangle \in {}^2X \mid x \vee y = y\} = \{\langle x, y\rangle \in {}^2X \mid x \wedge y = x\}.$$
Then $(X; \leqslant)$ *is a lattice.*

Proof Straightforward. $\square$3.161

3.10.2 Boolean algebras

Definition [C] *Suppose* $\mathbb{X} = (X; \vee, \wedge)$ *is a lattice.*

1. $\mathbb{X}$ *is* distributive $\overset{\text{def}}{\iff}$ *for all* $x, y, z \in X$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$$
equivalently,[56] *for all* $x, y, z \in X$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$
2. $\mathbb{X}$ *is* bounded $\overset{\text{def}}{\iff}$ *it has both a greatest and a least element. In this case* $\mathbf{1} \overset{\text{def}}{=}$ *the greatest and* $\mathbf{0} \overset{\text{def}}{=}$ *the least element.*
3. *Suppose* $\mathbb{X}$ *is bounded. Elements* $x, y \in X$ *are* complementary $\overset{\text{def}}{\iff} x \vee y = \mathbf{1}$ *and* $x \wedge y = \mathbf{0}$. *We also say that each* is complementary to, complements, *or* is a complement of *the other.*
4. $\mathbb{X}$ *is* complemented $\overset{\text{def}}{\iff} \mathbb{X}$ *is bounded and for all* x *there exists a unique* y *that complements* x.
5. *Suppose* $\mathbb{X}$ *is complemented. Then for each* $x \in X$, the complement of $x \overset{\text{def}}{=} \neg x \overset{\text{def}}{=}$ *the (unique) element complementary to* x.
6. $\mathbb{X}$ *is a* boolean algebra $\overset{\text{def}}{\iff} \mathbb{X}$ *is distributive and complemented.*

---

[56] Suppose for all $x, y, z \in X$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
and suppose $x, y, z \in X$. Then
$$\begin{aligned} x \vee (y \wedge z) &= \big(x \vee (x \wedge z)\big) \vee (y \wedge z) \\ &= x \vee \big((x \wedge z) \vee (y \wedge z)\big) = \big((x \vee y) \wedge x\big) \vee \big((x \vee y) \wedge z\big) \\ &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

The following definition summarizes the features of a boolean algebra as an operational structure.

Definition [C] *A* boolean algebra *is a structure* $\mathfrak{A} = (A; \vee, \wedge, \mathbf{1}, \mathbf{0})$ *with the following properties:*

1. *For all* $x, y, z \in A$

$$\begin{aligned} x \vee x &= x, & x \wedge x &= x, \\ x \vee y &= y \vee x, & x \wedge y &= y \wedge x, \\ x \vee (y \vee z) &= (x \vee y) \vee z, & x \wedge (y \wedge z) &= (x \wedge y) \wedge z, \\ (x \vee y) \wedge z &= (x \wedge z) \vee (y \wedge z), & (x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z). \end{aligned}$$

2. *For each* $x \in A$ *there exists a unique element* $\neg x \in A$ *such that*

$$x \vee \neg x = \mathbf{1} \qquad \textit{and} \qquad x \wedge \neg x = \mathbf{0}\,.$$

(3.162) **Theorem** [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra and* $x, y \in |\mathfrak{A}|$.

1. $\neg \mathbf{1} = \mathbf{0}$ *and* $\neg \mathbf{0} = \mathbf{1}$*;*
2. $\neg(x \vee y) = \neg x \wedge \neg y$ *and* $\neg(x \wedge y) = \neg x \vee \neg y$*; and*
3. $\neg \neg x = x$.

Proof Straightforward. □3.162

(3.162) shows that $\neg$ is an isomorphism of $\mathfrak{A}$ with the structure $(|\mathfrak{A}|; \wedge, \vee, \mathbf{0}, \mathbf{1})$, which is therefore also a boolean algebra.

(3.163) *We say that* $\vee$ *and* $\wedge$ *are* dual *to each other, as are* $\mathbf{1}$ *and* $\mathbf{0}$. $\neg$ *is dual to itself. We refer to such correspondences in general as* duality.

The corresponding partial order[3.161.2.2] is given by

$$x \leqslant y \leftrightarrow x \vee y = y \leftrightarrow x \wedge y = x.$$

The combination rules for boolean algebras were originally propounded as applying to logical operations. This was extensively developed by George Boole, hence the name given these structures. It was Boole's intention that the individuals of an algebra $\mathfrak{A}$ be *properties* regarded *extensionally*, i.e., each property being identified with the collection of all things with that property (as opposed to *intentionally*, i.e., with reference to the inherent *meaning* of the property—the meaning of the assertion that a thing has that property).

From this extensional viewpoint the paradigm of a boolean algebra is an algebra of subclasses of a given class.

Definition [C]

1. *A boolean algebra* $(X; \leqslant)$ *is a* set-algebra $\overset{\text{def}}{\iff} \leqslant$ *is the relation* $\subseteq$.
2. *Suppose* U *is a set. The* set-algebra of $U \overset{\text{def}}{=} (\mathcal{P}U; \subseteq)$. *Note that in this case,*
   1. $\mathbf{1} = U$*;*
   2. $\mathbf{0} = 0$*;*

3. *for any* $x \subseteq U$, $\neg\, x = U\backslash x$; *and*
4. *for any* $x, y \subseteq U$, $x \vee y = x \cup y$ *and* $x \wedge y = x \cap y$.

In fact,[3.167] this paradigm is general: every boolean algebra is isomorphic to a set-algebra.

(3.164) **Definition** [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra. An* ideal[57] on $\mathfrak{A}$ *is a subset* I *of* $\mathfrak{A}$ *such that*

1. $\forall x \in I\, \forall y \in \mathfrak{A}\, (x \wedge y \in I)$, *i.e.*, $\forall x \in I\, \forall y \leqslant x\ y \in I$,
2. $\forall x, y \in I\ x \vee y \in I$, *and*
3. $\mathbf{1} \notin I$.

Definition [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra and* $\mathbf{1} \neq x \in \mathfrak{A}$. *The* principal ideal of $x \stackrel{\text{def}}{=}$

$$\lceil x \rceil \stackrel{\text{def}}{=} \{y \in \mathfrak{A} \mid y \leqslant x\}.$$

The dual[3.163] notion to *ideal* is *filter*.

Definition [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra. A* filter on $\mathfrak{A}$ *is a subclass* F *of* $|\mathfrak{A}|$ *such that*

1. $\forall x \in F\, \forall y \geqslant x\ y \in F$,
2. $\forall x, y \in F\ x \wedge y \in F$, *and*
3. $\mathbf{0} \notin F$.

Definition [C] *Suppose* $X \subseteq |\mathfrak{A}|$ *is an ideal (a filter) on a boolean algebra* $\mathfrak{A}$. *The* dual filter (dual ideal) of $X \stackrel{\text{def}}{=} X^* \stackrel{\text{def}}{=} \{\neg\, x \mid x \in X\}$.

[57] The notion of an ideal was first introduced by Ernst Kummer in the theory of rings, and he used them to prove certain special cases of Fermat's last theorem. An ideal in a ring $\mathfrak{R}$ is a subset I of $\mathfrak{R}$ such that

1. $\forall x \in I\, \forall y \in |\mathfrak{R}|\ x \cdot y \in I$, *and*
2. $\forall x, y \in I\, x + y \in I$.

For any nonzero $x \in \mathfrak{R}$ the set $[x]$ of multiples of x is clearly an ideal. An ideal formed in this way is *principal*.

Suppose $\mathfrak{R}'$ is a ring extension of $\mathfrak{R}$ and let x' be a any nonzero element of $\mathfrak{R}'$. The set $[x'] \cap \mathfrak{R}$ of multiples of x' that lie in $\mathfrak{R}$ is clearly an ideal in $\mathfrak{R}$, which may or may not be principal. The multiplicative properties of $\mathfrak{R}'$ are reflected in the properties of these ideals. In particular, we can define operations on ideals in $\mathfrak{R}$ that correspond to the operations of multiplication and greatest common divisor in $\mathfrak{R}'$.[58] The utility of ideals, of course, does not lie in this trivial circumlocution, but rather in the fact that ideals in a ring $\mathfrak{R}$ behave in general as though they were formed from elements of an extension of $\mathfrak{R}$, even when no such extension exists. Ideals, in other words, correspond to "ideal" elements of (an extension of) $\mathfrak{R}$, whence the name.

[58] We define the product IJ of ideals I and J by

$$IJ \stackrel{\text{def}}{=} \{x \cdot y \mid x \in I \wedge y \in J\};$$

and we define the greatest common divisor of I and J by

$$\gcd(I, J) \stackrel{\text{def}}{=} \{x + y \mid x \in I \wedge y \in J\}.$$

For $x \in |\mathfrak{A}| \backslash \{\mathbf{0}\}$, the *principal filter of* $x \overset{\text{def}}{=}$

$$\lfloor x \rfloor \overset{\text{def}}{=} \{x \in \mathfrak{A} \mid x \geqslant a\}.$$

Definition [GB] *An ideal (filter) on a boolean algebra* $\mathfrak{A}$ *is* maximal $\overset{\text{def}}{\iff}$ *it is not included in any larger ideal (filter). A maximal filter is also called an* ultrafilter.

(3.165) **Theorem** [GB] *A filter* F *on a boolean algebra* $\mathfrak{A}$ *is an ultrafilter iff for every* $a \in |\mathfrak{A}|$, *either* $a \in F$ *or* $\neg a \in F$.

Proof Straightforward. □3.165

(3.166) **Definition** [GB] *Suppose* $\mathfrak{A}$ *is a boolean algebra.*

1. $\mathfrak{A}$ *is* complete $\overset{\text{def}}{\iff}$ *every class* $X \subseteq |\mathfrak{A}|$ *has a least upper bound, denoted* $\bigvee X$, *which is also called the* join *of* X. *Clearly this is equivalent to the existence of greatest lower bounds, also called* meets.
2. *Suppose* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}$. $\mathfrak{B}$ *is a* complete subalgebra *of* $\mathfrak{A}$ $\overset{\text{def}}{\iff}$ *for every* $X \subseteq |\mathfrak{B}|$, $\bigvee X$ *as computed in* $\mathfrak{A}$ *is in* $\mathfrak{B}$. *(Note that it is not enough that* $\mathfrak{B}$ *be a complete algebra in its own right.)*

(3.167) **Theorem: Stone representation theorem** [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra and* $\mathfrak{A}$ *is a set.* $\mathfrak{A}$ *is isomorphic to an algebra of sets of ultrafilters via the correspondence* $a \mapsto$ *the set of ultrafilters* F *on* $\mathfrak{A}$ *such that* $a \in F$.

Proof This is a straightforward exercise in boolean algebra. □3.167

3.10.3 Stationarity

(3.168) **Definition** [ZF]

1. *A function* f *from ordinals to ordinals is* regressive $\overset{\text{def}}{\iff}$

   $$\forall \alpha \in \operatorname{dom} f\,(\alpha > 0 \to f\alpha < \alpha).$$

   We also say that f presses down *in this case.*[59]
2. *A subset* S *of a limit ordinal* α *is* stationary $\overset{\text{def}}{\iff}$ *it intersects every closed unbounded*3.142.3 *subset of* α.
3. *Suppose* $\gamma \in \mathrm{Ord}$ *and* $\forall \alpha \in \gamma\ A_\alpha \subseteq \gamma$. *The* diagonal intersection of $\langle A_\alpha \mid \alpha \in \gamma \rangle \overset{\text{def}}{=} \Delta\langle A_\alpha \mid \alpha \in \gamma\rangle \overset{\text{def}}{=} \{\beta \in \gamma \mid \forall \alpha \in \beta\ \beta \in A_\alpha\}$.

Note that if $\operatorname{cf} \alpha = \omega$, a subset A of α is stationary iff $\alpha \backslash A$ is bounded below α, so the notion of stationarity is interesting only for (subsets of) ordinals of uncountable cofinality.

We should point out that there are important variations on the definitions of *closed unbounded* and *stationary*. The definitions given above are historically primary.

[59] Note that *regressive function* is just a synonym of *choice function* specific to this application.

(3.169) **Theorem** [ZF] *Suppose α is a limit ordinal of uncountable cofinality.*

1. *Suppose C, C' are closed unbounded in α. Then $C \cap C'$ is closed unbounded in α.*
2. *Suppose C is closed unbounded and S is stationary in α. Then $S \cap C$ is stationary in α.*

Proof 1 $C \cap C'$ is obviously closed, so it suffices to show that it is unbounded. Suppose $\beta_0 < \alpha$. We must show that there exists $\beta \in C \cap C'$ such that $\beta > \beta_0$. To this end define a sequence $\beta_0 < \beta_1 < \cdots$, such that for each $n \in \omega$,

1. β_{2n+1} is the least member of C above β_{2n}; and
2. β_{2n+2} is the least member of C' above β_{2n+1}.

That these exist follows from the fact that C and C' are unbounded. Let $\beta = \sup_{n\in\omega} \beta_n$. Since $\mathrm{cf}\,\alpha$ is uncountable, $\beta \in \alpha$. Since C and C' are closed in α, $\beta \in C \cap C'$. $\square^{3.169.1}$

2 This follows immediately from (3.169.1). $\square^{3.169.2}$ $\square^{3.169}$

Theorem 3.169.2 justifies the following definition.

(3.170) **Definition** [ZF] *Suppose α is a limit ordinal of uncountable cofinality. The* closed unbounded filter over α *is the set of $X \subseteq \alpha$ such that X includes a closed unbounded subset of α.*

The dual of the closed unbounded filter over α is the *nonstationary ideal* over α, i.e., the set of nonstationary subsets of α.

The closed unbounded filter and nonstationary ideal *over* α are respectively a filter and an ideal *on* $\mathcal{P}\,\alpha$. This reflects the following terminological convention.

(3.171) *We generally refer to an ideal or filter or other similar object as being* on *a boolean algebra $\mathfrak{A}$. If the algebra is the full subset algebra of a set S then we will also refer to the object as being* over *S.*

Suppose κ is an uncountable regular cardinal and α is a limit ordinal of cofinality κ. Then there is a closed unbounded subset C of α of order type κ. By (3.169) every closed unbounded subset of α includes a closed unbounded subset of C, and every stationary subset of α includes a stationary subset of C; and these correspond respectively to closed unbounded and stationary subsets of κ. It follows that for all practical purposes the discussion of closed unbounded and stationary subsets of ordinals of cofinality κ may be restricted to subsets of κ *per se*. We will therefore henceforth focus particularly on this case.

(3.172) **Theorem** [ZF] *Suppose κ is an uncountable regular cardinal.*

1. *Suppose $\beta < \kappa$ and for each $\alpha < \beta$, C_α is closed unbounded in κ. Then $\bigcap_{\alpha<\beta} C_\alpha$ is closed unbounded in κ.*
2. *Suppose for each $\alpha < \kappa$, C_α is closed unbounded in κ. Then $\Delta_{\alpha<\kappa} C_\alpha$ is closed unbounded in κ.*

Proof 1 Obviously $\bigcap_{\alpha<\beta} C_\alpha$ is closed, so we have only to show that it is unbounded. Let $\gamma_{0,0}$ be an arbitrary element of κ. For $n \in \omega$ and $\alpha < \beta$, define $\gamma_{n,\alpha}$ as follows:

1. If $\alpha > 0$ then $\gamma_{n,\alpha}$ is the least member of C_α that exceeds $\gamma_{n,\delta}$ for all $\delta < \alpha$.

2. If $n > 0$ then $\gamma_{n,0}$ is the least member of C_0 that exceeds $\gamma_{m,\delta}$ for every $m < n$ and $\delta < \beta$.

Since κ is regular, $\beta < \kappa$, and every C_α is unbounded, this construction is legitimate. Let $\gamma = \sup_{n\in\omega,\alpha<\beta} \gamma_{n,\alpha}$. Note that $\gamma = \sup_{n\in\omega} \gamma_{n,\alpha}$ for every $\alpha < \beta$. Since $\mathrm{cf}\,\kappa = \kappa$ is uncountable, $\gamma \in \kappa$, and since each C_α is closed, $\gamma \in C_\alpha$ for each $\alpha < \beta$.

2 Let $C = \Delta_{\alpha<\kappa} C_\alpha$. We first show that C is closed. Suppose $\beta < \kappa$ and $C \cap \beta$ is unbounded in β. We will show that $\beta \in C$, i.e., for all $\alpha < \beta$, $\beta \in C_\alpha$. To this end, suppose $\alpha < \beta$. By definition, for any $\gamma \in C\backslash(\alpha + 1)$, $\gamma \in C_\alpha$. Thus, $C_\alpha \cap \beta$ is unbounded in β, hence $\beta \in C_\alpha$.

Next we show that C is unbounded. Suppose $\gamma_0 < \kappa$. For $n \in \omega$, let γ_{n+1} be the least element of $\bigcap_{\alpha<\gamma_n} C_\alpha$ above γ_n. Let $\gamma = \sup_{n\in\omega} \gamma_n$. Then $\gamma \in C_\alpha$ for every $\alpha < \gamma$, so $\gamma \in C$. $\square$3.172

(3.173) **Theorem: Fodor's lemma** [ZFC] *Suppose κ is a limit ordinal of uncountable cofinality, S is a stationary subset of κ, and $f : S \to \kappa$ is regressive. Then f is constant on a stationary subset of κ, i.e., for some $\alpha \in \kappa$, $f^{\leftarrow}\{\alpha\}$ is stationary in κ.*

Proof Suppose toward a contradiction that f is not constant on any stationary set. For each $\alpha \in \kappa$, let C_α be a closed unbounded subset of κ such that $\forall\beta \in (S \cap C_\alpha)\ f\beta \neq \alpha$. Let $C = \Delta_{\alpha<\kappa} C_\alpha$. Then C is closed unbounded, so there exists $\beta \in S \cap C\backslash\{0\}$. Let $\alpha = f\beta$. Then $\alpha < \beta$, since f is assumed to be regressive. Since $\beta \in C$ and $\alpha < \beta$, $\beta \in C_\alpha$, so $f\beta \neq \alpha$; contradiction. $\square$3.173

3.10.4 Normality

The closed unbounded filter and the nonstationary ideal over a cardinal κ, together with Fodor's lemma, are instances of the important phenomenon of *normality*.

(3.174) **Definition** [ZF] *Suppose H is an ideal (filter) over a set S.*

1. *The* dual *filter (ideal) to H* $\overset{\text{def}}{=} H^* \overset{\text{def}}{=} \{X \subseteq S \mid S\backslash X \in H\}$.
2. $H^+ \overset{\text{def}}{=} (\mathcal{P}\,S)\backslash H$ $\big((\mathcal{P}\,S)\backslash H^*\big)$.
3. *The following terminology is used, often with a prefix such as 'H-' for specificity.*
   1. *The sets in an ideal H are said to be* small.
   2. *The sets in a filter H are said to be* large.
   3. *The sets in H^+ are said to be* nonsmall *or* stationary.
   4. *We also use* almost *in the expected way, e.g., subsets of S are* almost equal *iff their symmetric difference is small.*

(3.175) **Definition** [ZFC] *Suppose F is a filter over a cardinal κ (i.e., a filter on $\mathcal{P}\kappa$).*

1. *F is* principal $\stackrel{\text{def}}{\Longleftrightarrow}$ *there exists $x \in S$ such that $\forall X \in F\ x \in X$.*
2. *Suppose λ is a cardinal. F is* λ-complete $\stackrel{\text{def}}{\Longleftrightarrow}$ *for every $\gamma < \lambda$ and $f : \gamma \to F$, $\bigcap_{\alpha<\gamma} f\alpha \in F$. ($F$ is closed under intersections of fewer than λ of its elements.)*
3. *F is* normal $\stackrel{\text{def}}{\Longleftrightarrow}$ *F is nonprincipal, κ-complete, and closed under diagonal intersections of κ sequences.*

Suppose I is an ideal over κ. Then I is principal, λ-complete, or normal according as its dual filter I^* is.

(3.176) **Theorem** [ZF] *Suppose F is a filter over a cardinal κ.*

1. *If F is nonprincipal then every set in F is infinite. Hence, κ is infinite.*
2. *If F is nonprincipal and κ-complete then κ is regular.*
3. *If F is normal then κ is uncountable (and regular).*

Proof 1 For every $\alpha \in \kappa$, $\kappa \backslash \{\alpha\} \in F$. If $X \in F$ and X is finite then $0 = X \cap \bigcap_{\alpha \in X} (\kappa \backslash \{\alpha\}) \in F$; contradiction.

2 By (3.176.1) κ is infinite. For each $\alpha < \kappa$, $\kappa \backslash \alpha = \bigcap_{\beta<\alpha} \kappa \backslash \{\beta\} \in F$, since F is nonprincipal and κ-complete. Suppose $\kappa = \bigcup \{\lambda_\alpha \mid \alpha < \mu\}$ for some $\mu < \kappa$. As we have just shown, $\forall \alpha < \mu\ (\kappa \backslash \lambda_\alpha) \in F$, so by κ-completeness, $0 = \bigcap_{\alpha<\mu} (\kappa \backslash \lambda_\alpha) \in F$; contradiction.

3 It suffices to show that $\kappa \neq \omega$. Suppose the contrary. For each $n \in \omega$ let $X_n = \omega \backslash (n+2)$. Since F is nonprincipal, each $X_n \in F$. Let $X = \Delta_{n \in \omega} X_n$. Then $m \in X$ iff $\forall k < m\ m \in X_k$. But if $m > 0$ then $m \notin X_{m^-}$, so $X = \{0\}$, which is not in F because F is nonprincipal. $\square$[3.176]

(3.177) **Theorem** [ZFC] *Suppose I is a nonprincipal κ-complete ideal over a cardinal κ. Then I is normal iff for every $X \in I^+$ and $f : X \to \kappa$, if f is regressive then $\exists \alpha \in \kappa\ f^{\leftarrow}\{\alpha\} \in I^+$ (i.e., if f is regressive on an I-stationary set then f is constant on an I-stationary set.)*

Proof Let $F = I^*$, the filter dual to I.

$\rightarrow$ Suppose I is normal and f is regressive on $X \in I^+$. Suppose toward a contradiction that $\forall \alpha \in \kappa\ f^{\leftarrow}\{\alpha\} \in I$. As in the proof of (3.173) for each $\alpha < \kappa$ let $Y_\alpha = \kappa \backslash f^{\leftarrow}\{\alpha\} \in F$. Let $Y = \Delta_{\alpha<\kappa} Y_\alpha$ and let $Y' = Y \backslash \{0\}$. By normality, $Y' \in F$, so $Y' \cap X \neq 0$. Suppose $\alpha \in Y' \cap X$. Let $\beta = f\alpha < \alpha$. Then $\alpha \in Y_\beta = \kappa \backslash f^{\leftarrow}\{\beta\}$, so $\beta \neq f\alpha$; contradiction.

$\leftarrow$ Suppose for every $X \in I^+$ and $f : X \to \kappa$, if f is regressive then $\exists \alpha \in \kappa\ f^{\leftarrow}\{\alpha\} \in I^+$. To show that I is normal, suppose for each $\alpha < \kappa$, $X_\alpha \in F$, let $X = \Delta_{\alpha<\kappa} X_\alpha$, and suppose toward a contradiction that $X \notin F$, i.e., $Y = \kappa \backslash X \in I^+$. For each $\alpha \in Y$ let $f\alpha$ be the least β such that $\alpha \notin X_\beta$. Then f is regressive on Y, so by hypothesis there exists $\beta \in \kappa$ and $Y' \in I^+$ such that $Y' \subseteq Y$ and $\forall \alpha \in Y'\ f\alpha = \beta$. Then $Y' \cap X_\beta = 0$. Since $X_\beta \in F$, $Y' \in I$, contradicting the fact that $Y' \in I^+$.$\square$[3.177]

(3.178) **Theorem** [ZFC] *A normal filter (over an uncountable regular cardinal) contains only stationary sets.*

Proof Suppose is F is a normal filter on the uncountable cardinal κ. Suppose toward a contradiction that $A \in F$ is nonstationary. Let $C \subseteq \kappa$ be closed unbounded in κ and disjoint from A, so $C \in F^*$. For $\alpha \in \kappa$ let

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \in C \\ \sup\{\alpha \cap C\} & \text{if } \alpha \notin C. \end{cases}$$

Since C is closed, if $\alpha \notin C$ then $C \cap \alpha$ is bounded below α, so $\sup\{\alpha \cap C\} < \alpha$, and f is therefore regressive. Since F is normal, f is constant on a set $B \in F^+$. The constant value of f on B cannot be any $\beta > 0$, because for all $\alpha < \kappa$, $f(\alpha) = \beta \to \alpha < \inf(C \backslash (\beta + 1))$. Since $|\inf(C \backslash (\beta + 1))| < \kappa$, and F is nonprincipal and κ-complete, this set is not in F^+ (because its complement is in F). So the constant value of f on B is 0; whence, $B \subseteq C \cup \inf C$. In other words, $B \backslash \inf C \subseteq C$. Since F is nonprincipal and κ-complete, $\inf C \in F^*$, so $B \backslash \inf C \in F^+$; hence, $C \in F^+$, contradicting the fact that $C \in F^*$. $\square$[3.178]

3.10.5 Trees

In the interest of efficiency, we collect here some basic definitions and theorems relating to trees, which are structures that arise naturally in several contexts in the remainder of this book. We will restrict our attention to sets for convenience.

(3.179) **Definition** [C]

1. *A* tree $\overset{\text{def}}{=}$ *a partial order* $(T; <)$[3.73] *such that T is a set, and for each $a \in T$, the set $\{b \in T \mid b < a\}$ of $<$-predecessors of a is wellordered by $<$.*
2. *Suppose $(T; <)$ is a tree.*
   1. *For $a \in T$, the* order of a $\big(\text{in } (T; <)\big) \overset{\text{def}}{=} \operatorname{o} a \overset{\text{def}}{=}$ *the order type of $\{b \in T \mid b < a\}$ with respect to $<$.*
   2. *The* height *of $(T; <)$* $\overset{\text{def}}{=} \operatorname{ht}(T; <) \overset{\text{def}}{=} \{\operatorname{o} a \mid a \in T\}$. *Note that $\operatorname{ht}(T; <)$ is an ordinal.*
   3. *A* branch *of $(T; <)$ is a subset of $(T; <)$ that is linearly ordered by $<$ and is maximal with respect to this condition.*
   4. *The members of T are often referred to as* nodes.

Since any wellordered set is uniquely bijective with an ordinal, we may identify $a \in T$ with the unique order-preserving bijection $s : \operatorname{o} a \overset{\text{bij}}{\to} \{b \in T \mid b < a\}$. The following definition describes trees from this point of view.

(3.180) **Definition** [C]

1. *Suppose s is an ordinal sequence and $\alpha \leqslant \operatorname{dom} s$ (i.e., $\alpha \in \operatorname{dom} s$ or $\alpha = \operatorname{dom} s$, since* dom s *is an ordinal). Then the* initial segment of s of length $\alpha \overset{\text{def}}{=} s \restriction \alpha$. $s \restriction \alpha$ *is a* proper *initial segment of s* $\overset{\text{def}}{\iff}$ *it is a proper subset of s, i.e., $\alpha < \operatorname{dom} s$.*
2. *T is a* tree $\overset{\text{def}}{\iff}$ *T is a set of ordinal sequences and $\forall t \in T\, \forall \alpha \in \operatorname{dom} s\ s \restriction \alpha \in T$, i.e., T is closed under the formation of initial segments.*
3. *Suppose T is a tree.*
   1. *T is a tree* on M $\overset{\text{def}}{\iff} \forall s \in T\ \operatorname{im} s \subseteq M$.
   2. *b is a* branch *of T* $\overset{\text{def}}{\iff}$ *b is an ordinal sequence all of whose proper initial segments are in T, but $b \notin T$.*
   3. *The* height of $T \overset{\text{def}}{=} \operatorname{ht} T \overset{\text{def}}{=} \bigcup_{s \in T} \operatorname{dom} s$.

Given a tree in the second sense,[3.180] we obtain a tree in the first sense,[3.179] by letting $s \leqslant t \leftrightarrow s \subseteq t$. (As always, '$<$' and '$\leqslant$' refer respectively to corresponding strong and weak order relations.)

Trees occur in a variety of contexts, in which special conventions may be apply. The following definition is such a case.

(3.181) **Definition** [C] $(T; \leqslant)$ *is a* sequence tree on *a set M* $\overset{\text{def}}{=}$ *T is a tree of finite sequences on M in the sense of (3.180), and $\leqslant$ is the relation of* reverse inclusion, *i.e., $s \leqslant^T t \leftrightarrow t \subseteq s$.*

We may describe the order relation on a sequence tree $(T; \leqslant)$ by saying that T *grows downward*. The rationale for this choice is given by the following theorem.

(3.182) **Theorem** [ZF]

1. *Suppose M is a wellordered set and $(T; <)$ is a sequence tree on M. Then T has an infinite branch iff $<$ is not wellfounded.*
2. [DC] *Suppose M is an arbitrary set and $(T; <)$ is a sequence tree on M. Then T has an infinite branch iff $<$ is not wellfounded.*

Proof 1 Let W be a wellordering of M. Suppose b is an infinite branch of T. Let $B = \{b \restriction n \mid n \in \omega\}$. Then $B \subseteq T$, and B has no $<$-minimal member, so $<$ is not wellfounded. Conversely, suppose $<$ is not wellfounded, and let $X \subseteq T$ be nonempty with no $<$-minimal member. Let $s^0 \in X$ be arbitrary, and use recursion to define a sequence $\langle s^n \mid n \in \omega \rangle$ such that for each $n \in \omega$, $s^n \in X$ and s^{n+1} is the W-least element of the set $\{s \in X \mid s < s^n\rangle$. Then $\bigcup_{n \in \omega} s^n$ is an infinite branch of T. $\square$[3.182.1]

2 We modify the preceding proof to use DC.[3.140.4] Suppose $X \subseteq T$ is nonempty and has no $<$-minimal member. Let $s^0 \in X$ be arbitrary. By DC there exists a function $\langle s^n \mid n \in \omega \rangle$ such that for each $n \in \omega$, $s^n \in X$ and $s^{n+1} < s^n$. $\bigcup_{n \in \omega} s^n$ is an infinite branch of T. $\square$[3.182.2] $\square$[3.182]

Theorem 3.182.2, equating wellfoundedness to the nonexistence of infinite descending sequences, is the most important consequence of DC and the reason for its frequent inclusion as an axiom in situations where AC is not available.

(3.183) **Definition** [ZF]

1. *Suppose* $s, t \in \mathrm{Seq}$.
   1. t extends $s \overset{\text{def}}{\iff} s \subseteq t$.
   2. *Suppose t extends s.*
      1. t properly extends $s \overset{\text{def}}{\iff} |t| > |s|$.
      2. t immediately extends $s \overset{\text{def}}{\iff} |t| = |s| + 1$.
2. *Suppose T is a sequence tree.*
   1. *Suppose $s \in T$.*
      1. $T_s \overset{\text{def}}{=} \{t \in \mathrm{Seq} \mid s ^\frown t \in T\}$.
      2. $T_{(s)} \overset{\text{def}}{=} \{t \in T \mid t \subseteq s \vee s \subseteq t\} = \{t \in T \mid t \subseteq s\} \cup (s ^\frown T_s)$.
   2. *Suppose $n \in \omega$. $T|n \overset{\text{def}}{=} \{s \in T \mid |s| \leqslant n\}$. If $\mathcal{T}$ is a set of trees then $\mathcal{T}|n \overset{\text{def}}{=} \{T|n \mid T \in \mathcal{T}\}$.*
   3. $[T] \overset{\text{def}}{=}$ *the set of infinite branches of T.*
3. *We extend the definition of the concatenation operation on sequences to apply to a sequence s and a set S of sequences:*

$$s ^\frown S \overset{\text{def}}{=} \{s ^\frown t \mid t \in S\}.$$

(3.184) **Theorem** [ZF] *Suppose S and T are sequence trees on a wellordered set M.*

1. *If S and T are wellfounded then there exists an order-preserving function $f : S \to T$ iff $\mathrm{rk}\, S \leqslant \mathrm{rk}\, T$.*[60]
2. *If S is wellfounded and T is not wellfounded then there exists an order-preserving function $f : S \to T$.*
3. *If S is not wellfounded and T is wellfounded then there does not exist an order-preserving function $f : S \to T$.*

Proof In general for a sequence tree R on a set M, define the rank of s in R to be $\mathrm{rk}\, R_s$ if R_s is wellfounded. R is wellfounded iff $R_{\langle m\rangle}$ is wellfounded for every $\langle m\rangle \in R$, in which case, by definition, $\mathrm{rk}\, R = \bigcup\{\mathrm{rk}\, R_{\langle m\rangle} + 1 \mid \langle m\rangle \in R\}$, which is the least ordinal greater than $\mathrm{rk}\, R_{\langle m\rangle}$ for every $\langle m\rangle \in R$. Therefore also, $\mathrm{rk}\, R_s = \bigcup\{\mathrm{rk}\, R_{s ^\frown \langle m\rangle} + 1 \mid s ^\frown \langle m\rangle \in R\}$.

1 Suppose S and T are wellfounded. Suppose $f : S \to T$ is order-preserving. Then it is easy to show by $\supseteq$-induction (from the bottom up on these trees that "grow downward") that for all $s \in S$, $\mathrm{rk}\, S_s \leqslant \mathrm{rk}\, T_{fs}$, so $\mathrm{rk}\, S \leqslant \mathrm{rk}\, T$.

Conversely, suppose $\mathrm{rk}\, S \leqslant \mathrm{rk}\, T$. We will define an order-preserving $f : S \to T$ be by $\subseteq$-recursion (from the top down) so that $\forall s \in S\ \mathrm{rk}\, S_s \leqslant \mathrm{rk}\, T_{fs}$. Let $f0 = 0$ (0 being the highest node of any sequence tree). Now suppose fs has been generated. Let $t = fs$. By construction, $\bigcup\{\mathrm{rk}\, S_{s ^\frown \langle m\rangle} + 1 \mid \langle s ^\frown \langle m\rangle \in S\} = \mathrm{rk}\, S_s \leqslant \mathrm{rk}\, T_t = \bigcup\{\mathrm{rk}\, T_{t ^\frown \langle m\rangle} + 1 \mid t ^\frown \langle m\rangle \in T\}$. Hence, for every $s ^\frown \langle m\rangle \in S$ there exists $t ^\frown \langle n\rangle \in T$ such that $\mathrm{rk}\, S_{s ^\frown \langle m\rangle} \leqslant \mathrm{rk}\, T_{t ^\frown \langle n\rangle}$, and we let $f(s ^\frown \langle m\rangle) = t ^\frown \langle n\rangle$, where n is the first such $n \in M$. $\square$3.184.1

[60] f is order-preserving $\overset{\text{def}}{\iff}$ for any $s \subsetneq t \in S$, $fs \subsetneq ft$.

2 We will define an order-preserving $f : S \to T$ by $\subseteq$-recursion so that $\forall s \in S$, T_{fs} is not wellfounded. Again we let $f0 = 0$. Now supposing fs has been generated such that T_{fs} is not wellfounded, we let n be the first element of M such that $T_{(fs)^\frown\langle n\rangle}$ is not wellfounded, and let $f(s^\frown\langle m\rangle) = (fs)^\frown\langle n\rangle$ for all $s^\frown\langle m\rangle \in S$. $\square$3.184.2

3 Note that what we have just done is to map S order-preservingly into a single infinite branch of T—the canonical infinite branch, if you will, as defined by the fixed wellordering of M. If S is not wellfounded, let $x \in {}^\omega M$ be the similarly defined infinite branch of S. If $f : S \to T$ is order-preserving, then $\bigcup\{f(x \restriction n) \mid n \in \omega\}$ is an infinite branch of T, which contradicts the wellfoundedness of T. $\square$3.184

3.11 Topology

(3.185) **Definition** [ZF] *A* topology *is a set $\mathcal{T}$ of sets that is closed under finite intersections and arbitrary unions.*

Note that $\bigcup\mathcal{T}$ is necessarily a member of $\mathcal{T}$, because $\mathcal{T}$ is closed under arbitrary unions. Similarly, $0 \in \mathcal{T}$, because $\bigcup 0 = 0$. If X is a set and $\mathcal{T}$ is a topology and $X = \bigcup\mathcal{T}$, we say that $\mathcal{T}$ is a topology *for* or *on* X. The pair $\mathcal{X} = \langle X, \mathcal{T}\rangle$ is said to be a *topological space*. Since X and $\mathcal{X}$ are definable from $\mathcal{T}$, they are essentially superfluous entities. We accord them independent recognition primarily because a given set X may be given different topologies; indeed, it may be a constituent of other sorts of mathematical structures as well. We routinely substitute 'X' for '$\mathcal{X}$' when it is clear which topology is intended. In the context of a particular topological space $\langle X, \mathcal{T}\rangle$, we refer to the elements of X as *points*, ans to subsets of X as *pointsets*, or simply *sets*.

Definition [ZF] *Suppose $\mathcal{X} = \langle X, \mathcal{T}\rangle$ is a topological space. Then*

1. *A pointset A is* open $\overset{\text{def}}{\iff} A \in \mathcal{T}$.
2. *A pointset A is* closed $\overset{\text{def}}{\iff} X\backslash A$ *is open.*
3. *If $x \in X$ and $A \subseteq X$, then A is a* neighborhood *of x* $\overset{\text{def}}{\iff}$ *there is an open set B such that $x \in B \subseteq A$.*

The following theorem gives a good feeling for the meaning of openness.

(3.186) **Theorem** [ZF] *Suppose $\langle X, \mathcal{T}\rangle$ is a topological space, and $A \subseteq X$. A is open iff A includes a neighborhood of each of its points iff A is a neighborhood of each of its points.*

Proof Straightforward. $\square$3.186

Intuitively, an open set is one that "surrounds" each of its points, or—to put it another way—one that contains every point "sufficiently close to" each of its points, or—to put it yet another way—one that does not "end abruptly". The following theorem is in effect a restatement of the definition[3.185] of 'topology' in terms of closedness as opposed to openness:

(3.187) **Theorem** [ZF] *The set of closed sets in a topological space is closed under finite unions and arbitrary intersections.*

Proof Straightforward. $\square^{3.187}$

Definition [ZF] *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space, $x \in X$, and $A \subseteq X$. x is a* limit point *(also* accumulation *or* cluster *point) of A $\overset{\text{def}}{\Longleftrightarrow}$ every neighborhood of x contains a point of A other than x.*

Intuitively, x is a limit point of A iff there are points in A, other than x, that are "arbitrarily close to" x. The following theorem, whose proof is straightforward, gives an intuitively useful characterization of closedness in terms of limit points.

(3.188) **Theorem** [ZF] *A pointset is closed iff it contains all of its limit points.*

Proof Straightforward. $\square^{3.188}$

Definition [ZF] *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space and $A \subseteq X$.*

1. *The* interior $A \overset{\text{def}}{=} A^o \overset{\text{def}}{=} \bigcup\{G \subseteq A \mid G \text{ is open}\}$.
2. *The* closure *of* $A \overset{\text{def}}{=} A^c \overset{\text{def}}{=} \bigcap\{F \supseteq A \mid F \text{ is closed}\}$.
3. *The* boundary *of* $A \overset{\text{def}}{=} \partial A \overset{\text{def}}{=} A^c \backslash A^o$.

Note that A^o and A^c are respectively open and closed by virtue of Definition 3.185 and Theorem 3.187, respectively. ∂A may also be characterized as the intersection of A^c and $(X \backslash A)^c$. A^c is also denoted by '$\bar{A}$'.

Definition [ZF] *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space and $A \subseteq X$. A is* dense $\overset{\text{def}}{\Longleftrightarrow}$ *every nonempty open set intersects A, i.e., $\forall G \in \mathcal{T}\,(G \neq 0 \to G \cap A \neq 0)$.*

For example, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.

3.11.1 Subspaces and relative topologies

(3.189) **Definition** [ZF]

1. *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space and $Y \subseteq X$. The* relative topology *on $Y \overset{\text{def}}{=} \{A \cap Y \mid A \in \mathcal{T}\}$. (It is trivial to verify that this is a topology.)*
2. *If $\mathcal{U}$ is any topology on Y, then $\langle Y, \mathcal{U} \rangle$ is a* subspace of $\langle X, \mathcal{T} \rangle$ $\overset{\text{def}}{\Longleftrightarrow}$ *$\mathcal{U}$ is the relative topology on Y.*

3.11.2 Generating topologies

Definition [ZF]

1. *A* base *for a topological space $\langle X, \mathcal{T} \rangle \overset{\text{def}}{=}$ a set $\mathcal{B}$ of open sets such that every open set is a union of sets in $\mathcal{B}$.*[61]
2. *A* neighborhood base *for a point $x \in X$ is a collection $\mathcal{B}$ of neighborhoods of x such that every neighborhood of x includes a member of $\mathcal{B}$.*

[61] By definition the union of 0 is 0, so the empty subset of X is the union of the empty subset of $\mathcal{B}$, and need not be a member of $\mathcal{B}$.

(3.190) **Theorem** [ZF] *A collection $\mathcal{B}$ of open sets is a base for $\langle X, \mathcal{T}\rangle$ iff for every $x \in X$, $\{B \in \mathcal{B} \mid x \in B\}$ is a neighborhood base for x.*

Proof Straightforward. $\square$[3.190]

Note that not every collection $\mathcal{B}$ of subsets of a set X is the base for a topology on X. The following fact is obvious:

(3.191) **Theorem** [ZF] *In order that a topology be generated by closing $\mathcal{B}$ under unions, it is necessary and sufficient that the intersection of any two members of $\mathcal{B}$ be a union of members of $\mathcal{B}$, and for this it is necessary and sufficient that for any $A, B \in \mathcal{B}$, for any $x \in A \cap B$, there is a set $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

Proof Straightforward. $\square$[3.191]

On the other hand, given any collection $\mathcal{B}$ of subsets of X, if we close $\mathcal{B}$ under the operations of finite intersection and arbitrary union, we always obtain a topology on X. This motivate the following definition:

Definition [ZF] *Suppose $\langle X, \mathcal{T}\rangle$ is a topological space.*

1. *A* subbase *for $\langle X, \mathcal{T}\rangle$ is a collection of open sets such that every set in $\mathcal{T}$ is a union of finite intersections of sets in $\mathcal{B}$.*
2. *A* neighborhood subbase *for $\langle X, \mathcal{T}\rangle$ is a collection $\mathcal{B}$ of open sets such that the collection of finite intersections of sets in $\mathcal{B}$ constitutes a neighborhood base for $\mathcal{T}$.*

Clearly, if $\mathcal{B}$ is a subbase for $\mathcal{T}$, then $\mathcal{T}$ is the minimum topology that includes $\mathcal{B}$. That is to say, any topology that includes $\mathcal{B}$ includes $\mathcal{T}$. We also say that $\mathcal{T}$ is the *smallest*, *weakest*, or *coarsest* topology that includes $\mathcal{B}$. This observation is worth stating as a theorem:

(3.192) **Theorem** [ZF] *Suppose $\mathcal{B}$ is a collection of subsets of a set X. Then there is a minimum (smallest, weakest, coarsest) topology on X that includes $\mathcal{B}$, and $\mathcal{B}$ is a subbase for this topology.*

3.11.3 Separation and countability properties

A *separation property* is a principle of the form: If A_1 and A_2 are disjoint sets in the classes Γ_1 and Γ_2, respectively, then there exist disjoint sets B_1 and B_2 in the classes Σ_1 and Σ_2, respectively, such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. Obviously, many variations may be played on this theme. The most important of these in topology is the *Hausdorff property*:

Definition [ZF] *A topological space is* Hausdorff $\overset{\text{def}}{\iff}$ *for any distinct points x and y there are disjoint open sets X and Y such that $x \in X$ and $y \in Y$.*

Non-Hausdorff topologies are uncommon in most applications, so much so that some authors make the Hausdorff property part of the definition of a topology. Note that any subspace of a Hausdorff space is Hausdorff.

Most interesting topological spaces are uncountable, but these often have certain important aspects of countability. The following definition singles out some important classes of topologies in terms of countability:

(3.193) **Definition** [ZF] *A topology is:*

1. first countable $\overset{\text{def}}{\iff}$ *every point has a countable neighborhood base;*
2. second countable $\overset{\text{def}}{\iff}$ *there is a countable base for the topology;*
3. separable $\overset{\text{def}}{\iff}$ *there is a countable dense set.*

3.11.4 Continuity

Definition [ZF] *Suppose* $\langle X, \mathcal{S}\rangle$ *and* $\langle Y, \mathcal{T}\rangle$ *are topological spaces and* $f : X \to Y$. *Then* f *is* continuous *iff for every* $B \in \mathcal{T}$, $f^{\leftarrow}B \in \mathcal{S}$. f *is* continuous at *a point* $x \in X$ *iff for every neighborhood* B *of* fx, *there is a neighborhood* A *of* x *such that* $f^{\rightarrow}A \subseteq B$.

The following theorem shows that these definitions of 'continuous' and 'continuous at a point' have the appropriate relationship:

(3.194) **Theorem** [ZF] *A function* $f : X \to Y$ *is continuous iff for all* $x \in X$, f *is continuous at* x.

Proof Suppose $f : X \to Y$ is continuous. Let $x \in X$ be given. We claim that f is continuous at x. Suppose therefore that B is a neighborhood of fx. Let B' be an open subneighborhood of fx, and let $A = f^{\leftarrow}B'$. Then $x \in A$, and A is open by virtue of the continuity of f. By construction, $f^{\rightarrow}A \subseteq B' \subseteq B$ as desired.

Now suppose that f is continuous at every point in X. Suppose $H \subseteq Y$ is an arbitrary open set, and let $G = f^{\leftarrow}H$. We claim that G is open. It is enough to show that for any $x \in G$ there is a neighborhood of x included in G. Suppose therefore that $x \in G$. Then H is a neighborhood of fx, so by the continuity of f at x there is a neighborhood A of x such that $x' \in A \to fx' \in H$. But this means that $x' \in A \to x' \in f^{\leftarrow}H = G$, so $A \subseteq G$, as desired. □[3.194]

3.11.5 Homeomorphism

Definition [ZF] *A* homeomorphism *of topological spaces* $\langle X, \mathcal{T}\rangle$ *and* $\langle Y, \mathcal{U}\rangle \overset{\text{def}}{=}$ *a bijection* $f : X \to Y$ *such that both* f *and* f^{-1} *are continuous. We say that such an* f *is* bicontinuous. $\langle X, \mathcal{T}\rangle$ *and* $\langle Y, \mathcal{U}\rangle$ *are* homeomorphic $\overset{\text{def}}{\iff}$ *there exists a homeomorphism of them.*

3.11.6 Product topology

We have previously discussed one way to create a new topology from a given one, viz., the process of relativization.[3.189] We now introduce another powerful such technique: the formation of a product topology. Suppose $\langle X, \mathcal{T}\rangle$ and $\langle Y, \mathcal{U}\rangle$ are topological spaces. There is a natural topology on the product $X \times Y = \{\langle x, y\rangle \mid x \in X \wedge y \in Y\}$, known as the *product topology.* We will presently give a direct definition of the product topology, but an understanding of its true significance depends on an analysis in terms of the *projection mappings*, which we now define.

Definition [ZF] *Suppose* X *and* Y *are sets. The* projection *mappings* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$ *are defined by:*

$$\pi_X\langle x, y\rangle = x \qquad \textit{and} \qquad \pi_Y\langle x, y\rangle = y.$$

The product topology on $X \times Y$ is designed so that these projection mappings are continuous. Specifically, it is the minimum topology on $X \times Y$ with respect to which the projection mappings are continuous, and we may take this as the definition of the product topology; but if we do so it is incumbent upon us to show that there is a minimum such topology. For this, Theorem 3.192 is useful. Let's see how this is done. π_X is continuous iff for any open set $A \subseteq X$, $\pi_X{}^{\leftarrow}A$ is open. In other words,

(3.195) $\{\langle x, y\rangle \mid x \in A\}$ *is open.*

Similarly, π_Y is continuous iff for every open $B \subseteq Y$,

(3.196) $\{\langle x, y\rangle \mid y \in B\}$ *is open.*

The minimum topology on $X \times Y$ that contains all sets of the form (3.195) and (3.196) consists of arbitrary unions of finite intersections of these sets.[3.192] A little reflection reveals that this topology consists of arbitrary unions of sets $A \times B$ ($A \in \mathcal{T}, B \in \mathcal{U}$). Intuitively, these are (greatly) generalized "open rectangles" in $X \times Y$. We therefore make the definition:

(3.197) **Definition** [ZF] *The* product *of topological spaces* $\langle X, \mathcal{T}\rangle$ *and* $\langle Y, \mathcal{U}\rangle \stackrel{\text{def}}{=}$ *the space* $\langle X \times Y, \mathcal{V}\rangle$, *where the* product topology $\mathcal{V}$ *has as a base* $\{A \times B \mid A \in \mathcal{T} \wedge B \in \mathcal{U}\}$.

According to the discussion leading up to this definition, we have the equivalent definition in terms of the continuity of the projection mappings:

(3.198) **Theorem** [ZF] *The product topology is the weakest topology on the cartesian product with respect to which the projection mappings are continuous.*

By iteration of the product operation applied to pairs of spaces we can form the product of any finite set of spaces. A little work shows that any arrangement of the factor spaces leads to the same product space up to homeomorphic equivalence. We can also form the product of infinitely many spaces. The universe of the product space is the cartesian product

$$\prod_{i \in I} X_i \stackrel{\text{def}}{=} \big\{\langle x_i \mid i \in I\rangle \,\big|\, \forall i \in I\ x_i \in X_i\big\}.$$

The projection mappings π_i are defined by:

$$\pi_i x = x_i,$$

where $x = \langle x_i \mid i \in I\rangle$. The product topology is the minimum topology with respect to which the projections are continuous, i.e., the minimum topology containing the sets

$$\big\{\langle x_i \mid i \in I\rangle \,\big|\, x_{i_0} \in A\big\}, \tag{3.199}$$

with $i_0 \in I$ and $A \in \mathcal{T}_{i_0}$.

In other words, the sets (3.199) constitute a subbase for the product topology. We therefore have:

Definition [ZF] *Suppose* $\langle X_i, \mathcal{T}_i\rangle$ $(i \in I)$ *are topological spaces. The* product $\prod_{i \in I}\langle X_i, \mathcal{T}_i\rangle \stackrel{\text{def}}{=}$ *the topological space* $\langle X, \mathcal{T}\rangle$, *where* $X = \prod_{i \in I} X_i$ *and a base for* $\mathcal{T}$ *consists of the sets*

$$\big\{\langle x_i \mid i \in I\rangle \,\big|\, \forall k \in n\ x_{i_k} \in A_k\big\},$$

where $n \in \omega$, $\forall k \in n\ i_k \in I$, *and* $\forall k \in n\ A_k \in \mathcal{T}_{i_k}$.

In other words, the product topology is generated by basic open sets that are cartesian products of open subsets of the factor spaces all but finitely many of whose factors are the entire respective factor space.

(3.200) **Theorem** [ZF] *A product of two (and therefore of finitely many) Hausdorff spaces is Hausdorff. A product of countably many first-countable spaces is first-countable.*

Proof Straightforward. $\square^{3.200}$

3.11.7 Convergence

Any topology may be characterized in terms of convergence, if the latter notion is understood in a suitably general sense. For us it will suffice to consider convergence in the usual sense of ω-sequences. This works well for first-countable Hausdorff spaces, and we will limit our consideration to this case.

(3.201) **Theorem** [ZF] *If x is a point in a first-countable space,*[3.193.1] *then there is a sequence $\langle A_n \mid n \in \omega \rangle$ of neighborhoods of x that constitutes a neighborhood base for x such that $A_0 \supseteq A_1 \supseteq \ldots$.*

Proof Let $\langle B_n \mid n \in \omega \rangle$ be an enumeration of a countable neighborhood base for x. Define $A_n = \bigcap_{m \leqslant n} B_m$. Then $\langle A_n \mid n \in \omega \rangle$ is as desired. $\square^{3.201}$

Definition [ZF] *A sequence $\boldsymbol{s} = \langle s_n \mid n \in \omega \rangle$* converges *to a point x $\overset{\text{def}}{\iff}$ for every neighborhood A of x, $\boldsymbol{s}$ is eventually in A, i.e., $\exists N \in \omega\, \forall n > N\ s_n \in A$.*

(3.202) **Theorem** [ZFC] *Suppose A is a set in a first-countable topological space. The following are equivalent:*

1. *A is open.*
2. *Every sequence that converges to a point in A is eventually in A.*
3. *Every sequence that converges to a point in A is sometime in A.*

Proof **1 → 2** Let A be an open set. Suppose $\boldsymbol{s}$ is a sequence that converges to a point $x \in A$. Then, by the definition of convergence, since A is a neighborhood of x, $\boldsymbol{s}$ is eventually in A, as claimed.

2 → 3 This is immediate.

3 → 1 We prove the contrapositive. Suppose A is not open. Let $x \in A$ be such that no neighborhood of x is included in A. We will define a sequence that converges to x that is never in A. Let $\{A_n \mid n \in \omega\}$ be a neighborhood base for x as in (3.201), so that $A_0 \supseteq A_1 \supseteq \ldots$. Using AC, for each $n \in \omega$ let s_n be a point in $A_n \backslash A$. Let $\boldsymbol{s} = \langle s_n \mid n \in \omega \rangle$. By construction, $\boldsymbol{s}$ is never in A. We need only show that $\boldsymbol{s}$ converges to x, i.e., for any neighborhood B of x, $\boldsymbol{s}$ is eventually in B. So let B be a neighborhood of x. Let $N \in \omega$ be such that $A_N \subseteq B$. Then for all $n \geqslant N$, $s_n \in A_n \subseteq A_N \subseteq B$, so $\boldsymbol{s}$ is eventually in B, as claimed. $\square^{3.202}$

In terms of closed sets we have the dual statement:

Theorem [ZFC] *Suppose A is a set in a first-countable space. The following are equivalent:*

1. *A is closed.*
2. *If a sequence that is frequently in A (i.e., is not eventually not in A) converges to a point x, then $x \in A$.*
3. *If a sequence in A converges to a point x, then $x \in A$.*

(3.203) **Theorem** [ZF] *In a Hausdorff space no sequence converges to more than one point.*

Proof Suppose toward a contradiction that $\boldsymbol{s}$ is a sequence that converges to both x and y, with $x \neq y$. Let A and B be disjoint neighborhoods of x and y, respectively. Then $\boldsymbol{s}$ is eventually in A and eventually in B, which is impossible, since A and B are disjoint. $\square^{3.203}$

Definition [ZF] *Suppose $\langle x_n \mid n \in \omega \rangle$ is a convergent sequence in a Hausdorff space $\langle X, \mathcal{T} \rangle$. Then $\lim_{n\to\infty} x_n \overset{\text{def}}{=}$ that unique $x \in X$ such that $\langle x_n \mid n \in \omega \rangle$ converges to x.*

3.11.8 Compactness

Definition [ZF] *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space and $A \subseteq X$.*

1. *An* open cover *of $A \overset{\text{def}}{=}$ a set $\Gamma \subseteq \mathcal{T}$ such that $A \subseteq \bigcup \Gamma$.*
2. *A is* compact $\overset{\text{def}}{\iff}$ *every open cover of A has a finite subcover, i.e., for every $\Gamma \subseteq \mathcal{T}$, if $A \subseteq \bigcup \Gamma$ then there exists a finite $\Gamma' \subseteq \Gamma$ such that $A \subseteq \bigcup \Gamma'$.*

(3.204) **Theorem** [ZF] *Suppose $\langle X, \mathcal{T} \rangle$ is a Hausdorff space and $A \subseteq X$ is compact. Then A is closed.*

Proof Suppose toward a contradiction that A is not closed. We will show that A is not compact. Since A is not closed, $X \backslash A$ is not open. Thus[3.186] there exists $x \in X \backslash A$, such that $X \backslash A$ does not include a neighborhood of x, that is:

(3.205) *Every neighborhood of x intersects A.*

Let Γ consist of all open sets that are disjoint from a neighborhood of x. Since $\mathcal{T}$ is Hausdorff, $A \subseteq \bigcup \Gamma$, i.e., Γ is an open cover of A.

We claim that Γ has no finite subcover of A. Suppose toward a contradiction that $\Gamma' \subseteq \Gamma$ is finite and $A \subseteq \bigcup \Gamma'$. For each $G \in \Gamma'$ let B_G be an open neighborhood of x disjoint from G.[62] Then $\bigcap_{G \in \Gamma'} B_G$ is an open neighborhood of x that is disjoint from $\bigcup \Gamma'$ and therefore from A, since Γ' covers A, contradicting (3.205).

Thus, Γ is an open cover of A with no finite subcover, so A is not compact, contrary to hypothesis. $\square^{3.204}$

(3.206) **Theorem** [ZF] *A closed subset of a compact set is compact.*

[62] Choice is not required because Γ' is finite, and we can prove by induction on $n \in \omega$ that choice functions exist for n-element sets of sets.

Proof Suppose A is compact and $B \subseteq A$ is closed. Suppose Γ is an open cover of B. Then $\Gamma \cup \{X \backslash B\}$ is an open cover of A. Let $\Delta \subseteq (\Gamma \cup \{X \backslash B\})$ be a finite subcover of A. Clearly, $\Delta \cap \Gamma$ covers B. Thus, every open cover of B has a finite subcover, and B is compact, as claimed. $\square^{3.206}$

(3.207) **Theorem** [ZF] *The image of a compact set under a continuous function is compact.*

Proof Straightforward. $\square^{3.207}$

3.11.9 Directed sets and direct limits

(3.208) **Definition** [GB]

1. *A partial order $\leqslant$ on a set D is* directed $\overset{\text{def}}{\iff} \forall a, b \in D\, \exists c \in D\, (a \leqslant c \wedge b \leqslant c)$. *We say that $(D; \leqslant)$ is a* directed set.
2. *Suppose $(D; \leqslant)$ is a directed set. A $(D; \leqslant)$*-system, *or simply a D*-system *if $\leqslant$ is understood, $\overset{\text{def}}{=}$ a family $[M_a \mid a \in D]$ of classes together with a family $[i_{ab} \mid a \leqslant b]$, such that for all $a \leqslant b \leqslant c$,*
   1. $i_{ab} : M_a \overset{\text{inj}}{\to} M_b$;
   2. $i_{ac} = i_{bc} \circ i_{ab}$.

   A directed system $\overset{\text{def}}{=}$ *a $(D; \leqslant)$-system for some directed set $(D; \leqslant)$.*

Suppose $\mathcal{M} = (M; i_{ab})_{a \leqslant b \in D}$ is a directed system. Let $S = \{\langle a, x\rangle \mid a \in D \wedge x \in M_{[a]}\}$. For $\langle a, x\rangle, \langle b, y\rangle \in S$, let

$$\langle a, x\rangle \equiv \langle b, y\rangle \leftrightarrow \exists c \in D\, (a, b \leqslant c \wedge i_{ac} x = i_{bc} y).$$

$\equiv$ is an equivalence relation on S. For each $\langle a, x\rangle \in S$ let $[\langle a, x\rangle]$ be its $\equiv$-equivalence class (which is a set, since D is a set). Let $\tilde{M}$ be the class of these equivalence classes. For each $a \in D$ and $x \in M_a$, let $i_a x = [\langle a, x\rangle]$.

(3.209)

1. $\forall a \in D\, (i_a : M_a \overset{\text{inj}}{\to} \tilde{M})$;
2. $\forall a, b \in D\, (a \leqslant b \to i_a = i_b \circ i_{ab})$; *and*
3. $\tilde{M} = \bigcup_{a \in D} i_a{}^{\to} M_a$.

Note that for any system $(M'; i'_a)_{a \in D}$, where $i'_a : M_a \overset{\text{inj}}{\to} M'$ for each $a \in D$, and (3.209.1, 2) holds, there exists a unique $j : \tilde{M} \overset{\text{inj}}{\to} M'$ such that $\forall a \in D\, (i'_a = j \circ i_a)$. The system $\tilde{\mathcal{M}} = (\tilde{M}; i_a)_{a \in D}$ is therefore universal among such systems and is a *direct limit* in the categorical sense, which is specified up to isomorphic equivalence.

Let

1. $D^+ = D \cup \{D\}$;
2. let $\leqslant^+ = \leqslant \cup \{\langle a, D\rangle \mid a \in D^+\}$;
3. let M^+ be the family with domain D^+ such that $M^+_{[a]} = M_{[a]}$ for $a \in D$ and $M^+_{[D]} = \tilde{M}$; and

4. let $i_{aD} = i_a$, for every $a \in D$, and let i_{DD} be the identity on $\tilde{M}$.

Then $(D^+; \leqslant^+)$ is a directed set that extends $(D; \leqslant)$ with one additional element at the top, and $\mathcal{M}^+ \stackrel{\text{def}}{=} (M^+; i_{ab})_{a \leqslant b \in D^+}$ is the canonical extension of $(M; i_{ab})_{a \leqslant b \in D}$ to a $(D^+; \leqslant^+)$-system.

3.12 Finitary set theory

1. F *is the theory* $\mathsf{S} + \neg\,\mathsf{Infinity}$.
2. G *is the theory* $\mathsf{C} + \neg\,\mathsf{Infinity}$.

Recall that $\mathsf{Infinity}$ states that there exists an infinite set; hence, $\neg\,\mathsf{Infinity}$ states that every set is finite, i.e., equipollent with a number.

F and G both say that all sets are finite. F is a pure set theory, i.e., all things are sets. G is the corresponding theory that admits proper classes with the predicative comprehension axiom. In particular, G recognizes the existence of ω and V_ω.

The following theorem is an adaptation of (3.135) to F.

(3.210) **Theorem** [F]

1. *Every ordinal is a number.*
2. Power.
3. *For every ordinal n, V_n exists, i.e., there exists a set containing exactly the sets of rank $< n$.*
4. $\forall x\, \exists_{\mathrm{Num}} n\ x \in V_n$.
5. AC.

Proof 1 Suppose α is an ordinal that is not a number. Then $\forall_{\mathrm{Num}} n\ n \in \alpha$. As in the proof of (3.135.2), suppose $f : \alpha \overset{\text{bij}}{\to} n$ for some number n. Then $n + 1 \subseteq \alpha$, and $f\restriction(n+1) : n+1 \overset{\text{inj}}{\to} n$, contradicting the finiteness of n.

2 Since every set is equipollent with a number, it suffices to show that for each number m, $\mathcal{P}m$ exists. This we do by induction on m, essentially as in the proof of (3.135.4).

3 This follows by induction on n, since $V_{n+1} = \mathcal{P}V_n$.

4 Given x, let $n = \operatorname{rk} x$. $x \in V_{n+1}$.

5 The wellordering principle follows immediately from the fact that any set is equipollent with an ordinal. $\square$[3.210]

3.12.1 A canonical enumeration of HF

We begin by describing an effective enumeration of all hereditarily finite[3.95] sets. A standard such enumeration is as follows:

$$\begin{aligned} x_0 &= \{\,\} \\ x_1 &= \{x_0\} \\ x_2 &= \{x_1\} \\ x_3 &= \{x_1, x_0\} \\ x_4 &= \{x_2\} \\ x_5 &= \{x_2, x_0\} \\ x_6 &= \{x_2, x_1\} \\ x_7 &= \{x_2, x_1, x_0\} \\ x_8 &= \{x_3\} \\ &\vdots \end{aligned}$$

Note that $x_n = \{x_m \mid \text{there is a '1' in the } m\text{th position in the binary representation of } n\}$, numbering from the right, starting with $m = 0$.

To formally define this enumeration we first note that the binary representation of a number[63] may be identified with the set of numbers indexing the positions of the '1's, e.g., 25 has the binary representation '11001', which corresponds to the set $\{4, 3, 0\}$. We therefore first define the operation bin, which enumerates all finite sets of numbers; we then define the operation Bin, which enumerates all hereditarily finite sets.

(3.211) **Definition** [S]

1. *The operation* bin *on numbers is defined recursively as follows.*
   1. $\operatorname{bin} 0 = 0$.
   2. $\operatorname{bin}(n+1) \stackrel{\text{def}}{=} (\operatorname{bin} n \cup \{m\}) \setminus m$, *where m is the least number not in* $\operatorname{bin} n$ *(equivalently, the largest number included in* $\operatorname{bin} n$*).*
2. *The operation* Bin *on numbers is defined recursively as follows.*

$$\operatorname{Bin} n \stackrel{\text{def}}{=} \{\operatorname{Bin} m \mid m \in \operatorname{bin} n\}.$$

3. x precedes y (in the canonical enumeration of hereditarily finite sets) $\stackrel{\text{def}}{\iff} x \prec y \stackrel{\text{def}}{\iff} \exists_{\text{Num}} m, n\, (x = \operatorname{Bin} m \wedge y = \operatorname{Bin} n \wedge m < n)$.

We leave the proof of the following theorem as a pleasant exercise.

(3.212) **Theorem** [S]

1. $\forall_{\text{HF}} x \big((\forall y \in x\ \text{Num}\, y) \to \exists !_{\text{Num}} n\, x = \operatorname{bin} n\big)$.
2. $\forall_{\text{HF}} x\, \exists !_{\text{Num}} n\ x = \operatorname{Bin} n$.
3. $\forall_{\text{HF}} x, y\, (\operatorname{rk} x < \operatorname{rk} y \to x \prec y)$.
4. *For each number n, V_n is an initial segment of $\prec$.*

---

[63] Recall that a number is a finite ordinal.

Note in particular that $\mathrm{Bin} : \omega \overset{\text{bij}}{\to} \mathrm{HF}$.[3.212.2] Thus Bin^{-1} is well defined, and $\mathrm{Bin}^{-1} : \mathrm{HF} \overset{\text{bij}}{\to} \omega$. Not surprisingly, Bin^{-1} is useful just as often as Bin, and for the sake of notational efficiency we make the following definition.

(3.213) **Definition** [S]

1. *Suppose* $\mathrm{Num}\, n$. $\vec{B}\, n \overset{\text{def}}{=} \mathrm{Bin}\, n$.
2. *Suppose* $\mathrm{HF}\, x$. $\overleftarrow{B}\, x \overset{\text{def}}{=} \mathrm{Bin}^{-1}\, x$.

3.13 Inner models

The method of inner models was introduced in Section 3.5.1 in connection with the class V_Ω of ranked sets, which C^0 proves is an inner model of S. There we used the theorem (3.102) of C^0 on almost universal classes. We now establish a slight variant of this theorem in the setting of C.

(3.214) **Theorem** [C] *Suppose M is transitive and almost universal. Then M satisfies axioms all the axioms of* S[3.8] *with the possible exception of* Comprehension. *Assuming* Power *we may conclude that* $M \models$ Power. *Assuming* Infinity *we may conclude that* $M \models$ Infinity.

Proof S1, S4, S5 These follow from Theorem 3.102.

S3 Since we are now working in C, we know that since M is nonempty and transitive, $0 \in M$. (0 is the unique $\in$-minimal element of any nonempty transitive class.) So $M \models$ S3.

Foundation Suppose

$$\sigma = \forall v_0, \ldots, v_{n^-} \left(\exists v\, \phi \to \exists v \left(\phi \wedge \forall u \in v\, \neg\, \phi\tbinom{v}{u}\right)\right),$$

where ϕ is an s-formula, $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$, and u is free for v in ϕ. Suppose S is a $\{\phi\}$-valuation function for $(M; \in)$. We must show that

$$\forall y_0, \ldots, y_{n^-} \in M \left(\exists y \in M\, S\Big\langle \phi, \Big\langle\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-}\end{smallmatrix}\Big\rangle\Big\rangle = 1 \right.$$
$$\left. \to \exists y \in M \left(S\Big\langle \phi, \Big\langle\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-}\end{smallmatrix}\Big\rangle\Big\rangle = 1 \wedge \forall x \in y\, S\Big\langle \phi, \Big\langle\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ x & y_0 & \cdots & y_{n^-}\end{smallmatrix}\Big\rangle\Big\rangle = 0\right)\right).$$

Let $y_0, \ldots, y_{n^-} \in M$ be given. Let

$$X = \left\{ y \in M \,\middle|\, S\Big\langle \phi, \Big\langle\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ y & y_0 & \cdots & y_{n^-}\end{smallmatrix}\Big\rangle\Big\rangle = 1 \right\}.$$

We must show that

$$\exists y \in X \to \exists y \in X\, \forall x \in y\; x \notin X,$$

which is just the Foundation axiom of C.

Power Assume Power. Suppose

$$\sigma = \forall \mathrm{v}_0 \,\exists \mathrm{v}_1 \,\forall \mathrm{v}_2 \,\big(\forall \mathrm{v}_1 \,(\mathrm{v}_1 \in \mathrm{v}_2 \to \mathrm{v}_1 \in \mathrm{v}_0) \to \mathrm{v}_2 \in \mathrm{v}_1\big).$$

Suppose $x \in M$. $\mathcal{P}\,x$ is a set, and $\{z \in M \mid z \subseteq x\}$ is included in $\mathcal{P}\,x$, so it is a set, and by almost universality $\exists y \in M \,\forall z \in M \,(z \subseteq x \to z \in y)$.

Infinity If M is a set then by the definition of almost universality, since $M \subseteq M$, there exists $x \in M$ such that $M \subseteq x$, so $x \in x$, which violates Foundation. Hence M is a proper class.

It is a general fact that for any transitive class M, $\mathrm{rk}^{\to}M$ is a transitive class of ordinals. For otherwise, let α be the least ordinal such that $\alpha \in \mathrm{rk}^{\to}M$ and $\exists \beta < \alpha \;\beta \notin \mathrm{rk}^{\to}M$, and suppose $x \in M$ and $\mathrm{rk}\,x = \alpha$. Since $\exists \beta \;\beta < \alpha$, $\alpha \neq 0$. If α is a successor then there exists $y \in x$ such that $\mathrm{rk}\,y = \alpha - 1$. Since M is transitive, $y \in M$, so this contradicts the minimality of α. Hence, α is a limit ordinal. But then $\mathrm{rk}^{\to}x$ is cofinal in α, which again leads to a contradiction with the minimality of α.

Thus, $\mathrm{rk}^{\to}M$ is a transitive class of ordinals, which must be Ord, since otherwise $M \subseteq V_\alpha$ for some α. In particular, $\omega \in \mathrm{rk}^{\to}M$. Let $x \in M$ be such that $\mathrm{rk}\,x = \omega$. The rank of any finite set is a successor ordinal, so x is infinite. It follows that $M \models$ Infinity.[3.137] $\square$[3.214]

(3.215) **Theorem** [C]

1. $V \models$ S.
2. *If* Infinity *then* $V \models$ ZF$^-$.
3. *If* Infinity *and* Power *then* $V \models$ ZF.
4. $V_\omega \models$ S.

Proof 1–3 Since V is almost universal, Theorem 3.214 gives us everything except Comprehension.

Suppose therefore that

$$\sigma = \forall v_0, \ldots, v_{n^-} \,\forall u \,\exists w \,\forall v \,(v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is an s-formula, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{u, v, v_0, \ldots, v_{n^-}\}$. Suppose S is a $\{\sigma\}$-valuation function for V.

We must show that

$$\forall y_0, \ldots, y_{n^-} \in V \,\forall x \in V \,\exists z \in V \,\forall y \in V$$
$$\Big(y \in z \leftrightarrow \Big(y \in x \wedge S\Big\langle \phi, \Big\langle {u \; v \; v_0 \; \cdots \; v_{n^-} \atop x \; y \; y_0 \; \cdots \; y_{n^-}} \Big\rangle\Big\rangle = 1\Big)\Big).$$

Let $x, y_0, \ldots, y_{n^-} \in V$ be given, and let

$$X = \Big\{ y \in V \,\Big|\, S\Big\langle \phi, \Big\langle {u \; v \; v_0 \; \cdots \; v_{n^-} \atop x \; y \; y_0 \; \cdots \; y_{n^-}} \Big\rangle\Big\rangle = 1 \Big\}.$$

Then the separation axiom of C implies that

$$\exists z \in V \,\forall y \in V (y \in z \leftrightarrow (y \in x \wedge y \in X),$$

as claimed.

4 A straightforward modification of the proof of (3.214) and (3.215.1). $\square^{3.215}$

3.13.1 The incompleteness of S, etc.

In Section 1.5 we showed that the satisfaction relation for $(V_\omega;\in)$ is not definable over $(V_\omega;\in)$ (i.e., not the extension of an s-formula interpreted in $(V_\omega;\in)$). We now use that result to show that the existence of a satisfaction relation for $(V_\omega;\in)$ is not provable in C. The version of this result given here is not the sharpest, since it is obtained as a theorem of GB^-, which is C extended by the axiom of infinity. It would be more satisfying to use C itself, but note that the result—that C does not imply something—implies that C is consistent. As we will see, if C is consistent then C does not imply that C is consistent, so the result does not follow from C alone—assuming C is consistent—but we will see that it does follow from C plus the assumption that C is consistent.

(3.216) **Theorem** [GB^-] C *does not imply the existence of a satisfaction relation for* $(V_\omega;\in)$.

Remark Let $\mathfrak{V}_\omega = (V_\omega;\in)$. As noted above, we work in GB^-, which allows us to show that satisfaction relations exist for the infinite structures $\mathfrak{V}_\omega$ and $\mathfrak{V}'$ used in the proof, since they are sets.

Proof Let F be the set of s-formulas with one free variable. For $\phi \in F$, let

$$\hat{\phi} = \{x \in V_\omega \mid \mathfrak{V}_\omega \models \phi[\tfrac{v}{x}]\},$$

where $\mathrm{Free}\,\phi = \{v\}$. Let $V' = \{\hat{\phi} \mid \phi \in F\}$. Let $\mathfrak{V}'$ be the c-structure with domain V' and the usual membership relation. Note that $V_\omega \subseteq V' \subseteq V_{\omega+1}$. It is easy to see that $\mathfrak{V}'$ is a model of C (the minimum model, in fact).[64] Suppose A is a member of V'. It is easy to show that if $\mathfrak{V}'$ says A satisfies the definition of the satisfaction relation for $\mathfrak{V}_\omega$, then it actually does satisfy the definition, so it is the satisfaction relation. Since A is by construction definable over $\mathfrak{V}_\omega$ by an s-formula, by Theorem 1.73 it is not the satisfaction relation for $\mathfrak{V}_\omega$. Therefore, $\mathfrak{V}'$ says there is no satisfaction relation for $\mathfrak{V}_\omega$. Since $\mathfrak{V}' \models \mathsf{C}$, C does not imply the existence of a satisfaction relation for $\mathfrak{V}_\omega$. $\square^{3.216}$

The following theorem states the semantic incompleteness of S. The proof is carried out in C with the additional hypothesis of the weak satisfactoriness of $(V;\in)$. Note that V is the full universe of sets—if there are no infinite sets, it is V_ω.

(3.217) **Gödel's first incompleteness theorem** [C]

1. *Suppose* $(V;\in)$ *is weakly satisfactory. Then there is a true* s*-sentence* σ *that is not* S*-provable, i.e., there exists an* s*-sentence* σ *such that* $(V;\in) \models \sigma$ *and* $\mathsf{S} \not\models \sigma$.[65]
2. *Similarly, if* $(V_\omega;\in)$ *is weakly satisfactory then there is an* s*-sentence* σ *such that* $(V_\omega;\in) \models \sigma$ *and* $\mathsf{S} \not\models \sigma$.

---

[64] It is not necessary to allow for parameters from V_ω in the formulas defining the elements of V', as any element of V_ω is definable within it.

[65] Keep in mind that $(V;\in) \models \mathsf{S}$.[3.215]

Remark The theorem asserts the semantic incompleteness of S, in that there is an s-sentence σ that is true in the *standard* model $(V;\in)$ (or the finitary model $(V_\omega;\in)$) of S, such that $\mathsf{S}\nvdash\sigma$. Note that this does not contradict the completeness theorem, which states that for any s-sentence σ, if σ is true in *every* model $\mathfrak{M}$ of S then $\mathsf{S}\vdash\sigma$. If σ is an s-sentence as in (3.217) then σ is false in some *nonstandard* model of S.

Proof 1 Let **Pbl** be an s-formula with one free variable u that expresses S-provability, so that for any s-sentence σ, $(V;\in)\models\mathbf{Pbl}\left[\begin{smallmatrix}u\\ \sigma\end{smallmatrix}\right]\leftrightarrow\mathsf{S}\vdash\sigma$.

Suppose toward a contradiction that every true s*-sentence* σ *is* S*-provable, i.e.,*

$$(V;\in)\models\sigma\rightarrow\mathsf{S}\vdash\sigma. \tag{3.218}$$

Since $(V;\in)\models\mathsf{S}$[3.215] and $(V;\in)$ is assumed weakly satisfactory, it follows[2.174.2] that for every s sentence σ,

$$\mathsf{S}\vdash\sigma\rightarrow(V;\in)\models\sigma.$$

Hence,[3.218] for every s-sentence σ,

$$\mathsf{S}\vdash\sigma\leftrightarrow(V;\in)\models\sigma. \tag{3.219}$$

Define by $\prec$-recursion[3.211] for $a\in V_\omega$, the *canonical* s*-definition of* $a\overset{\text{def}}{=}\mathbf{Def}_a$ so that

1. $\mathbf{Def}_0=\forall \mathrm{v}_1\ \neg\,\mathrm{v}_1\in\mathrm{v}_0$;
2. if $a\neq 0$, let $\langle b_0,\dots,b_k\rangle$ be the enumeration of a in increasing $\prec$-order (or decreasing order—it doesn't matter, but the former choice is closer to a definition we will make subsequently[4.51]), and let

$$\mathbf{Def}_a=\forall\mathrm{v}_1\,\Big(\mathrm{v}_1\in\mathrm{v}_0\leftrightarrow\Big(\mathbf{Def}_{b_0}(\mathrm{v}_1)\vee\Big(\mathbf{Def}_{b_1}(\mathrm{v}_1)\vee \\ \cdots\vee\Big(\mathbf{Def}_{b_{k\text{-}}}(\mathrm{v}_1)\vee\mathbf{Def}_{b_k}(\mathrm{v}_1)\Big)\cdots\Big)\Big)\Big).$$

It is straightforward to show that for all $a,b\in V_\omega$, $(V;\in)\models\mathbf{Def}_a\left[\begin{smallmatrix}\mathrm{v}_0\\ b\end{smallmatrix}\right]$ iff $(V_\omega;\in)\models\mathbf{Def}_b\left[\begin{smallmatrix}\mathrm{v}_0\\ a\end{smallmatrix}\right]$ iff $b=a$.

Let T be the class of 2-sequences $\langle\psi,a\rangle$ such that ψ is an s-formula with the single free variable v_0 and

$$\mathsf{S}\vdash\exists\mathrm{v}_0\,\big(\mathbf{Def}_a\wedge\psi\big).$$

Then[3.219] for any s-formula ψ with the single free variable v_0, and any $a\in V_\omega$,

$$\begin{aligned}\langle\psi,a\rangle\in T&\leftrightarrow\mathsf{S}\vdash\exists\mathrm{v}_0\,(\mathbf{Def}_a\wedge\psi)\\ &\leftrightarrow(V;\in)\models\ \exists\mathrm{v}_0\,(\mathbf{Def}_a\wedge\psi)\\ &\leftrightarrow(V;\in)\models\psi\left[\begin{smallmatrix}\mathrm{v}_0\\ a\end{smallmatrix}\right]\\ &\leftrightarrow(V_\omega;\in)\models\psi\left[\begin{smallmatrix}\mathrm{v}_0\\ a\end{smallmatrix}\right].\end{aligned} \tag{3.220}$$

Let ϕ be an s-formula with two free variables $\mathrm{v}_0,\mathrm{v}_1$ that expresses the definition of T, so for all $\psi,a\in V_\omega$,

$$\langle\psi,a\rangle\in T\leftrightarrow(V_\omega;\in)\models\phi\left[\begin{smallmatrix}\mathrm{v}_0&\mathrm{v}_1\\ \psi&a\end{smallmatrix}\right].$$

Then[3.220] for any s-formula ψ with the single free variable v_0, and any $a \in V_\omega$,

$$(V_\omega;\in) \models \phi\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \psi & a\end{smallmatrix}\right] \leftrightarrow (V_\omega;\in) \models \psi\left[\begin{smallmatrix}\mathrm{v}_0\\ a\end{smallmatrix}\right].$$

According to Theorem 1.73 (see (1.75)), this is impossible. □[3.217.1]

2 Entirely analogous. □[3.217]

3.14 Summary

The notion of membership is nearly ubiquitous in mathematics, and the theory of membership must inform any discussion of the foundations of mathematics. In this regard, the present chapter plays two roles. On the one hand, it provides tools necessary for the definition and analysis of basic concepts having to do with structure, language, and logic. On the other hand, it begins the development of the theory of membership *per se*, which is to say it begins the development of mathematics itself, and formulates some of the major issues in the foundations of mathematics, such as infinitarity and choice principles.

In the theory of membership we distinguish several sorts of objects. A *class* is an object that is nothing but the collection of its members. Thus, for classes we posit the $\mathsf{Extension}$ axiom: classes with the same members are identical. In the interest of clarity, and without any loss of utility, we stipulate that only classes have members. We define an *element* as something that is a member of a some class. We allow for the possibility of classes that are not elements and elements that are not classes, but we ignore objects that are not either elements or classes as irrelevant to a theory of membership. An object that is both an element and a class is a *set*. An element is *proper* iff it is not a class, and a class is *proper* iff it is not an element. Sets are present in any theory of membership; proper elements and classes may be included or not (independently).

The basic axioms of membership follow from our intuitive understanding of the meaning of 'membership'. If we exclude proper elements and classes we have basic pure set theory S^0. Allowing proper classes we obtain the theory C^0. The essentials of the theory of membership do not require consideration of proper elements, although it is useful for special purposes to admit them. That said, in the application of the theory of membership to anything outside of itself, like the physical world, proper elements are precisely the objects of primary interest: classes are merely tools. The theory of membership is often called 'set theory', and it is often presented as a pure set theory, with proper elements and classes excluded. In this book we generally allow for proper classes, and we distinguish the pure theory of sets from the general theory of membership.

We begin with the basic set theory S^0. This includes $\mathsf{Extension}$, which—as noted above—essentially states that $\in$ is the membership relation and not something else, and that proper elements are excluded. Then there are several axioms that assert the existence of sets relating in some way to given sets. Note that Pair, for example, implies that everything is a member of something, so there are no proper classes.

We define a few basic constructs in the context of S^0 and then move to the corresponding class theory C^0. An essential feature of C^0 is that its $\mathsf{Comprehension}$ axiom asserts the existence of a class (of sets) that is coextensive with any property of sets that may be expressed by a formula in which all quantification is over sets, although it may have proper classes as parameters. This is the *predicative* Comprehension schema. By virtue of the restriction to predicative comprehension, C^0 is a conservative extension of S^0.

We continue with the definition of a quite a few constructs of general utility. We devote considerable attention to wellfounded relations and wellorders; and we define the von Neumann ordinals as sets that are wellordered by the membership relation. S^0 and, more conveniently, C^0 are sufficient to handle the "mathematics" of the first two chapters, dealing with structure, language and logic. In the other direction, some of the ideas developed in those chapters are used in this chapter. For example, it is necessary to have a theory of language in order to define an axiom schema, or the notion of an abstraction term.[66]

By this time we have more than enough tools to define the class V_Ω of *ranked* sets, and we show that these constitute a cumulative hierarchy of the form with which we began the discussion. We show that the Foundation axiom is equivalent to the assertion that all sets are ranked, and that $V_\Omega \models \mathsf{S}^0 + \mathsf{Foundation}$. With this result in hand, we formally add Foundation to S^0 and C^0 to obtain the theories S and C. After a few results giving the lay of the land under the new regime, we define the axiom of choice AC and the equivalent *wellordering principle* and *Zorn's lemma*. We then introduce the subject of *size* of sets. We analyze various notions of finiteness and point out that with AC they are all equivalent, but promise to show that without AC they may not be. We define the axiom of Infinity.

Up to this point, although we have used the power operation $\mathcal{P}$, we have not posited the Power axiom, which now begins to be quite useful. In particular, in conjunction with Infinity it implies the existence of a vast universe of sets of increasing size. As we will see in the sequel, this is not a puerile indulgence in megalomania: these sets are useful. To keep track of the possibilities we define a number of set and class theories that incorporate Foundation and Infinity, but differ in their inclusion of Power and AC. All are important.

We develop the theory of cardinals and cofinality, and we derive some basic properties of cardinal exponentiation and present the (generalized) continuum hypothesis and the singular cardinals hypothesis (or problem), which have informed some of the major advances in set theory.

After developing some more mathematics, including boolean algebras and topological spaces, we turn our attention to metatheoretical topics, finishing with several metatheorems building on the undefinability of the satisfaction relation for $(V_\omega; \in)$ by a formula interpreted in $(V_\omega; \in)$, which is a theorem of C.[1.73]. We show that C does not imply the existence of a satisfaction relation for $(V_\omega; \in)$; and we show the semantic incompleteness of S as a theory of either $(V; \in)$ or $(V_\omega; \in)$.

[66] Although this carries the risk of introducing an invalid circularity of argument—of "begging the question", in the proper meaning of that phrase—it is pretty clear that this has not happened.

Chapter 4

Definability, Computability, Provability

Contents

4.1 **Paradox** **261**
4.2 **Decidability** **262**
4.3 **The science of computation** **263**
4.4 **Set-theoretic complexity** **264**
4.5 **Evaluation and satisfaction** **272**
4.6 **Satisfaction in the finitary interpretation** **275**
4.6.1 Complexity of classes 279
4.6.2 The complexity of satisfaction 283
4.7 **Computability** **283**
4.7.1 The Church-Turing thesis 289
4.7.2 Undecidability of predicate logic 290
4.8 **Recursive functions** **292**
4.8.1 Σ_1-uniformization 293
4.8.2 Complete and universal sets 294
4.8.3 A universal partial recursive function 295
4.9 **The unsolvability of the halting problem** **297**
4.10 **The recursion theorem** **298**
4.11 **Consistency and incompleteness** **299**
4.11.1 The first incompleteness theorem 300
4.11.2 The second incompleteness theorem 305
4.11.3 The first incompleteness theorem, Rosser's improvement 305
4.12 **Relative definability over** HF **306**
4.13 **Summary** **311**

4.1 Paradox

All Cretans are liars.

Epimenides the Cretan.

The above statement is often presented as an ancient example of a logical paradox, known as the *Liar Paradox*. A more self-contained and genuinely problematic version is the sentence 'I am lying', or simply:

(4.1) *This sentence is false.*

Clearly, if Sentence 4.1 is true, then it is false; and if it is false, then it is true. So it can be neither true nor false. We can twist ourselves into pretzels trying to make sense of this sentence or we can conclude that it is meaningless. As logicians (I hope you're starting to feel like a logician by now), we should be very reluctant to take the latter step. After all, we have expended a lot of energy and ink defining the predicates '... is true' and '... is false'. To accept now that it is meaningless to apply them to a particular sentence would seem to imperil all our hard work. Nevertheless, it is meaningless, and I encourage you to try to formalize Sentence 4.1 using the apparatus of Chapter 1 to see where the breakdown occurs.

We will return to the liar paradox in a moment, But first, some other paradoxes to delight and mystify. Consider the *Richard paradox*. This purports to prove:

(4.2) *Every natural number is definable by an English phrase consisting of no more than thirty-three syllables.*

The proof goes as follows. Suppose that the claim is false. Then there is a smallest natural number, say N, that is not definable by an English phrase consisting of no more than thirty-three syllables. But then the following is a definition of N: 'the smallest natural number that is not definable by an English phrase consisting of no more than thirty-three syllables'. This is an English phrase consisting of no more than thirty-three syllables, whose existence contradicts the definition of N. Thus, the assumption that the claim is false has led to a contradiction, and we are forced to conclude that every natural number is definable by an English phrase consisting of no more than thirty-three syllables. Since the English language has only finitely many syllables, only finitely many numbers can be defined by phrases of any fixed length, and the natural numbers constitute an infinite collection. So (4.2) is a genuine paradox. Again, you are encouraged to try to resolve this paradox. The proof must be invalid ... mustn't it? Where is the error?

Our final example is the *Russell paradox*.

Suppose we allow that any definable collection of sets is a set. Then there is a set A that consists of all sets that are not members of themselves.

Is A a member of A? By the definition of A, A belongs to A if and only if A is not a member of A, so '$A \in A$' can be neither true nor false, and this is the Russell paradox.[1]

4.2 Decidability

One of the great achievements of 19th and early 20th century mathematics was to establish the adequacy of the formal framework of Chapters 1–3 for the pursuit of mathematics. From the narrowly formal point of view, mathematics is the business of listing the axioms of a theory and then finding out what those axioms entail.

[1]Russell's paradox is actually a simplification of Cantor's paradox, which followed from his proof that the powerset $\mathcal{P}A$ of a set A is not a surjective image of A. Letting A be the universe of all sets, $\mathcal{P}A = A$, so the identity map is a surjection.

Of course, an arbitrary set of axioms is not likely to be worth studying, nor is an arbitrary deduction from a set of axioms likely to be worth deducing, even if the axioms themselves describe an important class of structures. Of the vast terrain that is accessible to mathematical formalism, only a minuscule portion is ever visited by mathematicians, but this does not diminish the importance of the formal framework. It does not comprehend the soul of mathematics, but it does constrain its body: by current standards, an idea is mathematical only if it can be expressed formally, and an argument is mathematical only if it can be carried out in the system of deduction described in Chapter 2.

As previously discussed,[§ 2.2.5.2] the completeness theorem provides us with a mechanical method of generating all the sentences that are entailed by a given theory. A mathematician of course applies this mechanism very selectively in creating a proof of a particular theorem, but, in principle, one could program a computer to generate all the theorems of a given theory. It now becomes very important to know whether there also exists a mechanical method of generating all the sentences that are *not* entailed by a given theory, because, if there is, then there is a mechanical method for determining whether any given sentence σ is entailed by a given theory T: simply start generating all the theorems of T while simultaneously generating all the sentences that are not theorems of T. The given sentence σ will eventually appear on one list or the other, and then we know whether σ is a theorem. Let us call such a procedure a *decision procedure* for T. A decision procedure would not have to operate in precisely the above fashion—i.e., by generating the lists of theorems and non-theorems simultaneously—which we know would be very time-consuming, and, indeed, when decision procedures do exist, they can often be made much more efficient than this. The only requirement we impose on a decision procedure for a theory T is that it be *effective*—by which we mean that it is the deterministic execution of a finite program—and that it always terminate after finitely many steps with the correct answer to the question: 'Is σ a theorem of T?'. We say that T is *decidable* if and only if there is an effective decision procedure for T.

An efficient (as well as effective) decision procedure for a theory such as ZF, which is capable of formalizing all of mathematics, would revolutionize the practice of mathematics. There would still be room for creativity in the definition of new entities and the formulation of conjectures, but the proofs could all be left to the procedure. Note that 'efficient' is a key word here—an inefficient procedure might be of no use at all.

4.3 The science of computation

To show that there exists a decision procedure for a theory T one may describe an effective procedure and show by any satisfactory means that it is a decision procedure for T. We do not need to say what an effective procedure is in general in order to be satisfied that a particular process is one, nor do we necessarily require a formal theory of procedures in order to be satisfied that a particular one performs as claimed.

To show that there does not exist a decision procedure for T, on the other hand, we must say—if not what an effective procedure is—at least what an effective procedure does. That is, we have to have a characterization of the class of operations that can be effected by effective procedures, and a body of knowledge relating to this class, sufficient to show that the operation in question is not in it. We need—in short—a *science of computation*.

We refer here to the science of computation much as we refer to (euclidean) geometry as the science of mensuration or to arithmetic as the science of numeration. In all these cases the objects of study are regarded as having an existence sufficiently objective that their properties are to be *discovered*, not *defined*.

In the case of geometry, this attitude applies to the subject as it was originally conceived, as the science of physical space. Although we now know that physical space is not euclidean, we have a common intuitive concept of euclidean space, which may be taken as the source of the axioms of the mathematical theory of geometry. The science of numeration is an even better model for the science of computation: It is really a rather fierce nihilist who would deny the objectivity of natural numbers with the operations of addition and multiplication, or the objective truth value of sentences in the first-order predicate language of this structure.

The notion of computation is similarly robust. Theories of computation have been developed in diverse frameworks, and all have been shown to be equivalent. For our purposes the most convenient framework is that of hereditarily finite (HF) sets[3.95] with the membership predicate. For example, the expressions of standard languages with HF signatures are HF sets, and we have convenient set-theoretical constructs for such basic combinatorial concepts as number, ordered pair, finite sequence, etc.; so a theory of computation on HF sets is immediately applicable to decision problems for mathematical theories.

We should point out that the notion of computability is relevant not just to the question of decidability of theories. The word problem for finitely presented groups and the solvability of diophantine equations are examples.

4.4 Set-theoretic complexity

We will present the theory of computation as part of a more general theory of set-theoretic complexity, which is of interest in its own right. In this endeavor, the object of our attention is set theory, where 'set theory' is construed as referring generically to theories of membership. Set theory is also—as always—our metatheory. Given the opportunities for confusion afforded by this conjunction, it behoves us to pay close attention to the distinction between use and mention of set-theoretic expressions. To this end we recapitulate our discussion so far of language, logic, and theories of membership.

In Chapter 1 we presented the basic concepts of structure and language—including the structure of language, i.e., syntax, and the meaning of language, i.e., semantics. Meaning is formalized by the notion of valuation of terms and formulas, with the attendant notions of satisfaction, satisfiability, and entailment. In Chapter 2 we obtained syntactic equivalents of satisfiability and entailment, viz., consistency and provability, respectively. In Chapter 3 we developed the theory of membership enough to formalize Chapters 1 and 2. We defined several theories that may be used for this purpose, including S and C, that differ in their admission of proper classes. Of these, S is the simpler, and as it is sufficient for the purpose, it would be the theory of choice, but it is much easier if we allow proper classes. We have shown[§ 2.9.1] that C is a conservative extension of S, so we may be comfortable that everything of a finitary nature that we prove in C may be proved finitarily, i.e., in S.[2]

In the setting of C, HF exists as a (potentially proper) class, and HF $= V_\omega$.

[2] Of note, our proof of the conservative extension result was done in S.

We regard linguistic expressions and proofs as hereditarily finite sets, i.e., members of V_ω. Theories are subclasses of V_ω. The models that arise in the proof of the completeness theorem are also subclasses of V_ω, as are the satisfaction relations for these models, which are complete extensions of the initial theories. These are definable by formulas with set-restricted quantification, so predicative comprehension axioms (as in C) suffice to prove their existence. To address more subtle issues, such as decidability, we need to look more closely at the complexity of these defining formulas, as this determines the complexity of the corresponding classes.

For reasons that will become clear in Section 4.11 we will keep unusually close track of the languages and theories involved. Recall[3.15] that s is the signature of pure set theory, with two predicate indices, for membership and identity. c has an additional unary predicate for "setness". The extension of a language and theory by the addition of defined predicates and operations will be indicated by the superscript '+'.[2.111] Any such theory is a conservative extension of the base theory.[2.108]

In particular, S^+ is an extension by definition of S, with signature s^+. Likewise, C^+ is an extension by definition of C, with signature c^+. In any particular instance we will suppose that C^+ is the result of adding the predicative class comprehension axiom to S^+. Thus, since C is, in effect, a conservative extension of S, C^+ is in the same sense a conservative extension of S^+. Our use of these extended theories is merely a convenience, as is our use of C. The entire discussion could be formulated in s and carried out in S.

Nevertheless, for this chapter, unless otherwise noted, we regard C^+ as our metatheory—the theory in which the discussion is taking place. Theories S^+ are typically object theories. We begin with a classification of s-formulas in terms of logical complexity in an appropriate sense. In the context of any signature s^+ and theory S^+ that extends S by definition, any s^+-formula is S^+-provably equivalent to an s-formula, so we can apply the s-classification to s^+-formulas, and in an appropriate sense to s^+-terms as well.

(4.3) *To facilitate discussing the defined constructs as features of an object language, we extend the conventions (1.45.3, 4) whereby a bold version of a symbol of our typographical realization of S^+ is used to name the corresponding operation on s^+-expressions. For example, given Definition 3.12, for any s^+-terms τ, τ', $\boldsymbol{\{}\tau, \tau'\boldsymbol{\}}$ is the corresponding s^+-term.*[3] *If the bolding convention is impractical, we may use an underline for the same purpose.*

In the interest of clarity we will adhere rather strictly to the rules for representing s^+-expressions. For example, we observe the distinction between a variable v and the corresponding term $\bar{v}$, which we often ignored in Chapter 3.

The following very useful classification of set-theoretic formulas is known as the *Levy hierarchy* after Azriel Levy.

[3] Formally,

1. we expand s^+ by the addition of a 2-ary operation index O;
2. we define $\boldsymbol{\{}\tau, \tau'\boldsymbol{\}} \stackrel{\text{def}}{=} \tilde{O}\langle\tau, \tau'\rangle$ for any s^+-terms τ, τ'; and
3. for some distinct variables a, b, x, y, we extend S^+ by the addition of the sentence

$$\forall a, b, x\, \Big(\bar{x} = \boldsymbol{\{}\bar{a}, \bar{b}\boldsymbol{\}} \leftrightarrow \big(\bar{a} \in \bar{x} \wedge \bar{b} \in \bar{x} \wedge \forall y \in \bar{x}\, (\bar{y} = \bar{a} \vee \bar{y} = \bar{b})\big)\Big).$$

Note that once we have introduced the operation $\tau, \tau' \mapsto \boldsymbol{\{}\tau, \tau'\boldsymbol{\}}$, we need never mention O again, and we typically never mention it at all.

(4.4) **Definition** [C^+] *Recall the definition*[3.1.1] *of the operations of* bounded quantification*. Suppose ρ is a signature that expands the signature* s *of pure set theory.*

1. *ϕ is Δ_0^ρ $\overset{\text{def}}{\iff}$ $\Delta_0^\rho\,\phi$ $\overset{\text{def}}{\iff}$ ϕ is a ρ-formula, and for every subformula ψ of ϕ:*
   1. *if $\psi = \exists u\,\theta$ for some variable u and formula θ, then $\psi = \exists u \in \tau\,\sigma$ for some formula σ and some term τ not containing u, i.e., $\theta = u \in \tau \wedge \sigma$;*
   2. *if $\psi = \forall u\,\theta$ for some variable u and formula θ, then $\psi = \forall u \in \tau\,\sigma$ for some formula σ and some term τ not containing u, i.e., $\theta = u \in \tau \rightarrow \sigma$.*
2. *We also refer to Δ_0^ρ formulas as* bounded*.*
3. *$\Sigma_0^\rho\phi \overset{\text{def}}{=} \Pi_0^\rho\phi \overset{\text{def}}{=} \Delta_0^\rho\phi$. The reason for this notational redundancy will become clear presently.*
4. *$\Delta_0\phi \overset{\text{def}}{\iff} \Delta_0^{\mathsf{s}}\phi$.*

We follow the usual practice of using the same notation for a defined predicate applicable to sets and the class defined by the predicate, as long as the definition employs only set-restricted quantification. For example, $\Delta_0^\rho \overset{\text{def}}{=}$ the class of Δ_0^ρ formulas. In general, when 'Δ', 'Σ', and 'Π' are used as above without a superscript indicating signature, the implicit signature is s.

(4.5) **Definition** [C^+] *Suppose ρ is a signature that expands s. We define Σ_n^ρ and Π_n^ρ for numbers $n > 0$ recursively by stipulating that for any number n:*

1. *ϕ is Σ_{n+1}^ρ $\overset{\text{def}}{\iff}$ $\Sigma_{n+1}^\rho\phi$ $\overset{\text{def}}{\iff}$ $\phi = \exists u_1, \ldots, u_m\,\psi$ for some m, where ψ is Π_n^ρ.*
2. *ϕ is Π_{n+1}^ρ $\overset{\text{def}}{\iff}$ $\Pi_{n+1}^\rho\phi$ $\overset{\text{def}}{\iff}$ $\phi = \forall u_1, \ldots, u_m\,\psi$ for some m, where ψ is Σ_n^ρ.*
3. *In particular,*
   1. *ϕ is Σ_1^ρ $\overset{\text{def}}{\iff}$ $\Sigma_1^\rho\phi$ $\overset{\text{def}}{\iff}$ $\phi = \exists u_1, \ldots, u_m\,\psi$ for some m, where ψ is Δ_0^ρ.*
   2. *ϕ is Π_1^ρ $\overset{\text{def}}{\iff}$ $\Pi_1^\rho\phi$ $\overset{\text{def}}{\iff}$ $\phi = \forall u_1, \ldots, u_m\,\psi$ for some m, where ψ is Δ_0^ρ.*

Here we have used the standard conventions that $\mathsf{Q}u_1, \ldots, u_m$ is $\mathsf{Q}u_1 \cdots \mathsf{Q}u_m$, where Q is $\exists$ or $\forall$. m may be 0, in which case the initial quantifier sequence is empty, and $\phi = \psi$. Note that for each $n \in \omega$, Σ_n^ρ and Π_n^ρ include Δ_0^ρ, Σ_m^ρ, and Π_m^ρ for each $m < n$.

(4.6) *In the following discussion, we suppose that T, T', etc., are s^+-theories that extend S^+ (not necessarily conservatively).*

Recall that for any theory Θ,

$$\phi \overset{\Theta}{\equiv} \psi \overset{\text{def}}{\iff} \Theta \vdash \phi \leftrightarrow \psi.$$

Note that if $\mathsf{T}' \supseteq \mathsf{T}$ and $\phi \overset{\mathsf{T}}{\equiv} \psi$, then $\phi \overset{\mathsf{T}'}{\equiv} \psi$.

(4.7) **Definition** [C^+] *An s^+-formula ϕ is Σ_n (Π_n)* relative to T $\overset{\text{def}}{\iff}$ *ϕ is Σ_n^{T} (Π_n^{T}) $\overset{\text{def}}{\iff}$ there exists a Σ_n (Π_n) formula ϕ' such that $\phi \overset{\mathsf{T}}{\equiv} \phi'$. ϕ is Δ_n^{T} $\overset{\text{def}}{\iff}$ ϕ is both Σ_n^{T} and Π_n^{T}.*[4]

[4] Note that for $n > 0$, Δ_n is strictly a semantical notion, and we do not define syntactical Δ_n-classes. Trivially, a formula that is both Σ_1^ρ and Π_1^ρ is Δ_0^ρ, since these are defined in terms of grammatical structure, rather than (T-provable) semantic equivalence. Similarly, a formula that is both Σ_2^ρ and Π_2^ρ is either Σ_1^ρ or Π_1^ρ.

(4.8) *It is important to note that the superscript to 'Δ', etc., in (4.7) denotes a theory extending* S, *rather than a signature expanding* s *as in (4.4). By way of clarification we observe that the typical scenario involves one or more of the following elements:*

1. *a theory* S^+ *that is an extension by definition of the pure set theory* S *with signature* s^+*;*
2. *a theory* T *with signature* s^+ *that extends* S^+*;*
3. *a signature* ρ *that includes* s *and is included in* s^+*.*

Using Δ_0 *as a typical example:*

1. $\Delta_0 = \Delta_0^{\mathsf{s}}$ *is the class of bounded* s*-formulas.*
2. Δ_0^{ρ} *is the class of bounded* ρ*-formulas.*
3. Δ_0^{T} *is the class of* s^+*-formulas that are* T*-equivalent to a* Δ_0 *formula.*[5]

(4.9) **Theorem** [C^+] *Suppose* ϕ *is* Σ_1^{T} (Π_1^{T})*. Then there exists a* Δ_0 *formula* ψ *with a free variable* u *such that* $\phi \overset{\mathsf{T}}{\equiv} \exists u\, \psi$ $\big(\phi \overset{\mathsf{T}}{\equiv} \forall u\, \psi\big)$*.*[4.6]

Proof Suppose ϕ is Σ_1^{T}. By definition, $\phi \overset{\mathsf{T}}{\equiv} \exists u_1 \cdots \exists u_n\, \theta$ for some Δ_0 formula θ and variables $u_1, \ldots, u_n$. Let $\psi = \exists u_1 \in \bar{u} \cdots \exists u_n \in \bar{u}\, \theta$, where u is a variable that is not in $\{u_1, \ldots, u_n\}$ and does not occur free in θ. Notice that ψ is Δ_0. $\mathsf{T} \supseteq \mathsf{S}$,[4.6] so it is straightforward to show that $\exists u_1 \cdots \exists u_n\, \theta \overset{\mathsf{T}}{\equiv} \exists u\, \psi$. Since $\overset{\mathsf{T}}{\equiv}$ is an equivalence relation, $\phi \overset{\mathsf{T}}{\equiv} \exists u\, \psi$.

Similarly, if ϕ is T-equivalent to $\forall u_1 \cdots \forall u_n\, \theta$, then it is T-equivalent to $\forall u\, \psi$, where $\psi = \forall u_1 \in \bar{u} \cdots \forall u_n \in \bar{u}\, \theta$, if $u \notin \{u_1, \ldots, u_n\} \cup \operatorname{Free} \theta$. □[4.9]

(4.10) **Theorem** [C^+]

1. *Suppose* ϕ *and* ψ *are both* Σ_1^{T} (Π_1^{T})*. Then* $\phi \wedge \psi$ *and* $\phi \vee \psi$ *are* Σ_1^{T} (Π_1^{T})*.*
2. *Suppose* ϕ *is* Σ_1^{T}*. Then*
   1. $\neg\, \phi$ *is* Π_1^{T}*;*
   2. *for any variable* u*,* $\exists u\, \phi$ *is* Σ_1^{T}*;*
   3. *for any variables* u, v*,* $\exists u \in \bar{v}\, \phi$ *and* $\forall u \in \bar{v}\, \phi$ *are* Σ_1^{T}*.*
3. *Suppose* ϕ *is* Π_1^{T}*. Then*
   1. $\neg\, \phi$ *is* Σ_1^{T}*;*
   2. *for any variable* u*,* $\forall u\, \phi$ *is* Π_1^{T}*;*
   3. *for any variables* u, v*,* $\exists u \in \bar{v}\, \phi$ *and* $\forall u \in \bar{v}\, \phi$ *are* Π_1^{T}*.*

Proof 1 Suppose ϕ and ψ are Σ_1^{T}; specifically, suppose they are respectively T-equivalent to $\exists u\, \phi'$ and $\exists v\, \psi'$, where ϕ' and ψ' are Δ_0.[4.9] By a change of bound variables, if necessary, we may arrange that u is not free in ψ' and v is not free in ϕ'. Then $(\phi \wedge \psi) \overset{\mathsf{T}}{\equiv} \exists u\, \exists v\, (\phi' \wedge \psi')$. The proof is straightforward. Since $\phi' \wedge \psi'$ is Δ_0, we are done. The proofs for the cases of $\vee$ with Σ_1 and of $\wedge$ and $\vee$ with Π_1 are similar. □[4.10.1]

[5] Recall that when 'Δ' is used without a superscript, it is 'Δ^{s}' that is intended; likewise for 'Σ' and 'Π'.

2 Suppose ϕ is Σ_1^{T} and u, v are variables. Specifically, suppose $\phi \overset{\mathsf{T}}{\equiv} \exists w\, \phi'$, where ϕ' is Δ_0 and $w \notin \{u, v\}$, which we may arrange by a change of bound variables in $\exists w\, \phi'$.

2.1 Then $\neg\, \phi \overset{\mathsf{T}}{\equiv} \neg\, \exists w\, \phi' \overset{\mathsf{T}}{\equiv} \forall w\, \neg\, \phi'$. $\neg\, \phi'$ is Δ_0, so $\neg\, \phi$ is Π_1^{T}.

2.2 $\exists u\, \phi \overset{\mathsf{T}}{\equiv} \exists u\, \exists w\, \phi'$, which is Σ_1.

2.3 By definition, $\exists u \in \bar{v}\, \phi$ is $\exists u\, (\bar{u} \in \bar{v} \wedge \phi)$. $\bar{u} \in \bar{v}$ is Σ_1^{T}, so by (4.10.1), $\bar{u} \in \bar{v} \wedge \phi$ is Σ_1^{T}, say $\bar{u} \in \bar{v} \wedge \phi \overset{\mathsf{T}}{\equiv} \exists w\, \theta$ for some Δ_0 formula θ. Then $\exists u \in \bar{v}\, \phi \overset{\mathsf{T}}{\equiv} \exists u\, \exists w\, \theta$, which is Σ_1, so $\exists u \in \bar{v}\, \phi$ is Σ_1^{T}.

Let W be a variable not in $\{u, v, w\} \cup \operatorname{Free} \phi'$. Then

$$\forall u \in \bar{v}\, \exists w\, \phi' \rightarrow \exists W\, \forall u \in \bar{v}\, \exists w \in \bar{W}\, \phi'$$

is an instance of the collection schema of S (so it is in T), and

$$\exists W\, \forall u \in \bar{v}\, \exists w \in \bar{W}\, \phi' \rightarrow \forall u \in \bar{v}\, \exists w\, \phi'$$

is a theorem of pure logic, so

$$\forall u \in \bar{v}\, \exists w\, \phi' \overset{\mathsf{T}}{\equiv} \exists W\, \forall u \in \bar{v}\, \exists w \in \bar{W}\, \phi'.$$

Since $\forall u \in \bar{v}\, \exists w \in \bar{W}\, \phi'$ is Δ_0, $\forall u \in \bar{v}\, \phi$ is Σ_1^{T}. $\square^{4.10.2}$

3 The proof of (4.10.2) applies *mutatis mutandis*. $\square^{4.10.3}$ $\square^{4.10}$

(4.11) **Theorem** [C^+] *Suppose ϕ and ψ are Δ_1^{T}. Then $\neg\, \phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$, $\exists u \in \bar{v}\, \phi$, and $\forall u \in \bar{v}\, \phi$ are Δ_1^{T}. $\exists u\, \phi$ and $\forall u\, \phi$ are respectively Σ_1^{T} and Π_1^{T}.*

Proof This is a straightforward application of Theorem 4.10. $\square^{4.11}$

Definition [C^+]

1. *An s^+- term τ is Σ_1^{T}, Π_1^{T}, or Δ_1^{T} according as the formula $\bar{u} = \tau$ is respectively Σ_1^{T}, Π_1^{T}, or Δ_1^{T}, where u is any variable not occurring in τ.*
2. *An n-ary predicate index P of s^+ is Σ_1^{T}, Π_1^{T}, or Δ_1^{T} according as the formula $\tilde{P}\langle \bar{\mathrm{v}}_0, \ldots, \bar{\mathrm{v}}_{n\text{-}} \rangle$ is respectively Σ_1^{T}, Π_1^{T}, or Δ_1^{T}.*
3. *An n-ary operation index O of s^+ is Σ_1^{T}, Π_1^{T}, or Δ_1^{T} according as the term $\tilde{O}\langle \bar{\mathrm{v}}_0, \ldots, \bar{\mathrm{v}}_{n\text{-}} \rangle$ is respectively Σ_1^{T}, Π_1^{T}, or Δ_1^{T}.*

(4.12) **Theorem** [C^+] *Suppose τ is a Σ_1^{T} term. Then τ is also Π_1^{T} and therefore Δ_1^{T}.*

Proof Let u, v be distinct variables not occurring in τ. Clearly,

$$\bar{u} = \tau \overset{\mathsf{T}}{\equiv} \forall v\, (\bar{v} \neq \bar{u} \rightarrow \bar{v} \neq \tau),$$

and the latter formula is easily seen to be Π_1^{T}. $\square^{4.12}$

(4.13) **Theorem** [C^+]

1. *Suppose u is a variable. Then $\bar{u}$ is Δ^{T}_0.*
2. *Suppose τ, τ' are Δ^{T}_1 terms, and and τ' is free for u in τ. Then $\tau\binom{u}{\tau'}$ is Δ^{T}_1.*

Proof The first assertion is obvious. For the second, let v, w be variables not occurring in τ or τ'. Then

$$\bar{v} = \tau\tbinom{u}{\tau'} \overset{\mathsf{T}}{\equiv} \exists w \left(\bar{w} = \tau' \wedge \bar{v} = \tau\tbinom{u}{\bar{w}}\right),$$

so $\tau\binom{u}{\tau'}$ is Σ^{T}_1; hence,[4.12] it is Δ^{T}_1. $\square$[4.13]

(4.14) **Theorem** [C^+] *Suppose ϕ and ψ are Δ^{T}_1 formulas, τ is a Δ^{T}_1 term, and u is a variable not occurring in τ. Then $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$, $\exists u \in \tau\, \phi$, and $\forall u \in \tau\, \phi$ are Δ^{T}_1.*

Proof The only new thing here[4.11] is quantification bounded by a Δ^{T}_1 term. It is easy to see that $\exists u \in \tau\, \phi$ is T-equivalent to $\exists v\, (\bar{v} = \tau \wedge \exists u \in \bar{v}\, \phi)$ and to $\forall v\, (\bar{v} = \tau \to \exists u \in \bar{v}\, \phi)$, so it is Δ^{T}_1; and $\forall u \in \tau\, \phi$ is handled analogously. $\square$[4.14]

(4.15) **Theorem** [C^+] *In the scenario (4.8) suppose all the predicate and operation indices of ρ are Δ^{T}_1.*

1. *Every ρ-term is Δ^{T}_1.*
2. *Every Δ^{ρ}_0 formula is Δ^{T}_1.*

Proof It follows easily from (4.13) by induction on complexity that every ρ-term is Δ^{T}_1.

Suppose P is an n-ary predicate index of ρ.[6] By hypothesis, $\tilde{P}\langle \bar{\mathrm{v}}_0, \ldots, \bar{\mathrm{v}}_{n\text{-}}\rangle$ is Δ^{T}_1. Suppose $\tau_0, \ldots, \tau_{n\text{-}}$ are Δ^{T}_1 terms. Then

$$\begin{aligned}\tilde{P}\langle \tau_0, \ldots, \tau_{n\text{-}}\rangle &\overset{\mathsf{T}}{\equiv} \exists v_0, \ldots, v_{n\text{-}} \left(\left(\bigwedge\nolimits_{m<n} (\bar{v}_m = \tau_m)\right) \wedge \tilde{P}\langle v_0, \ldots, v_{n\text{-}}\rangle\right)\\ &\overset{\mathsf{T}}{\equiv} \forall v_0, \ldots, v_{n\text{-}} \left(\left(\bigwedge\nolimits_{m<n} (\bar{v}_m = \tau_m)\right) \to \tilde{P}\langle v_0, \ldots, v_{n\text{-}}\rangle\right),\end{aligned}$$

so every atomic ρ-formula is Δ^{T}_1.

Now we use (4.14) to show by induction on complexity that every Δ^{ρ}_0-formula is Δ^{T}_1. $\square$[4.15]

The following theorem provides a few complexity classifications, looking forward to Theorem 4.17. We remind the reader of the conventions regarding metalanguage names for operations (including 0-ary, or constant, operations) on object-language expressions.[4.3] In the case of expressions defined originally in the context of theories of membership with proper classes or proper elements, it is an equivalent definition in the context of pure set theory that is intended here.

(4.16) **Theorem** [C^+] *The following s^+-expressions are $\Delta^{\mathsf{S}^+}_1$. Distinct (metalanguage) names for variables are presumed to denote distinct variables.*

[6] P may be the binary index 0, which by our convention is interpreted as identity.

1. $\mathbf{0}$.
2. $\bigcup \bar{u}$, $\bigcap \bar{u}$, $\bar{u} \cup \bar{v}$, $\bar{u} \cap \bar{v}$, $\bar{u} \cup \bar{v} \cup \bar{w}$, *etc.*
3. $\{\bar{u}\}$, $\{\bar{u}, \bar{v}\}$, *etc.*
4. $(\bar{u}, \bar{v})$.
5. $\ulcorner(\bar{u})$ *is an ordered pair*$\urcorner$, *which we take to be* $\exists v, w\ \bar{u} = (\bar{v}, \bar{w})$, *or any* S^+-*equivalent formula.*
6. $\mathbf{Fcn}\, \bar{u}$.
7. $\mathbf{dom}\, \bar{u}$.
8. $\mathbf{im}\, \bar{u}$.
9. $\bar{u}(\bar{v})$, *regarded as a term with variables u and v and the definition*

$$\bar{w} = \bar{u}(\bar{v}) \leftrightarrow (\mathbf{Fcn}\, \bar{u} \wedge \bar{v} \in \mathbf{dom}\, \bar{u} \wedge (\bar{v}, \bar{w}) \in \bar{u}) \\ \vee (\neg(\mathbf{Fcn}\, \bar{u} \wedge \bar{v} \in \mathbf{dom}\, \bar{u}) \wedge \bar{w} = \mathbf{0}).$$

10. $\bar{u} \restriction \bar{v}$.

Proof We provide only a few proofs by way of illustration. As in the statement of the theorem, distinct names for variables are presumed to denote distinct variables unless otherwise stated.

1 $\bar{u} = \mathbf{0} \overset{\mathsf{T}}{\equiv} \forall v \in \bar{u}\ \bar{v} \neq \bar{v}$.

2 $\bar{v} = \bigcup \bar{u} \overset{\mathsf{T}}{\equiv} \forall w \in \bar{v}\, \exists v_0 \in \bar{u}\ \bar{w} \in \bar{v}_0 \wedge \forall v_0 \in \bar{u}\, \forall w \in \bar{v}_0\ \bar{w} \in \bar{v}$, etc.

3 $\bar{w} = \{\bar{u}\} \overset{\mathsf{T}}{\equiv} (\bar{u} \in \bar{w} \wedge \forall v \in \bar{w}\ \bar{v} = \bar{u})$, etc.

4 $\bar{w} = (\bar{u}, \bar{v}) \overset{\mathsf{T}}{\equiv} \bar{w} = \{\{\bar{u}\}, \{\bar{u}, \bar{v}\}\}$, which is Δ_1^{T} by (3) and (4.15.1).

5 $\exists v, w\ \bar{u} = (\bar{v}, \bar{w}) \overset{\mathsf{T}}{\equiv} \exists v, w \in \bigcup\bigcup \bar{u}\ \bar{u} = (\bar{v}, \bar{w})$, which is Δ_1^{T} by (2), (4) and (4.11).

6 $\mathbf{Fcn}\, \bar{u} \overset{\mathsf{T}}{\equiv} \ulcorner \forall x \in (\bar{u})\ (x \text{ is an ordered pair}) \wedge \forall y, z, z' \in \bigcup\bigcup (\bar{u})\ \big((y, z) \in (\bar{u}) \wedge (y, z') \in (\bar{u})\big) \to z = z'\big)\urcorner$. □4.16

Theorem 4.16 gives some indication of the abundance of Δ_1^{T}, but this is only fully revealed in the following theorem, which states that Δ_1^{T} is closed under recursive definitions.

(4.17) **Theorem** [C^+] *Suppose τ and η are Δ_1^{T} terms, and u, v, w, f are distinct variables not occurring in τ or η. (See the following remark for explanation.) Let*

$$\psi = \forall u \left(\exists v\ \bar{v} \in \bar{u} \to \exists v \in \bar{u}\, \forall w \in \bar{u}\ \bar{w} \notin \tau\big(\begin{smallmatrix} u_0 \\ \bar{v} \end{smallmatrix}\big)\right),$$

let

$$\theta = \mathbf{Fcn}\, \bar{f} \wedge \bar{u} \in \mathbf{dom}\, \bar{f} \wedge \forall v \in \mathbf{dom}\, \bar{f}\ \forall w \in \tau\big(\begin{smallmatrix} u_0 \\ \bar{v} \end{smallmatrix}\big)\ \bar{w} \in \mathbf{dom}\, \bar{f} \\ \wedge\ \forall v \in \mathbf{dom}\, \bar{f}\ \bar{f}(\bar{v}) = \eta\Big(\begin{smallmatrix} v_0 \\ \bar{f} \restriction \tau\left(\begin{smallmatrix} u_0 \\ \bar{v} \end{smallmatrix}\right) \end{smallmatrix}\Big),$$

let

$$\begin{aligned} \zeta &= \psi \to \exists! v\, \exists f\, \big(\theta \wedge \bar{v} = \bar{f}(\bar{u})\big) \\ \phi_\Sigma &= \exists f\, \big(\theta \wedge \bar{v} = \bar{f}(\bar{u})\big) \\ \phi_\Pi &= \forall f\, \big(\theta \to \bar{v} = \bar{f}(\bar{u})\big). \end{aligned}$$

Then

1. ϕ_Σ *is* Σ_1^{T}*;*
2. ϕ_Π *is* Π_1^{T}*; and*
3. $\mathsf{T} \vdash \zeta$*; hence,*
4. *if* $\mathsf{T} \vdash \psi$ *then* $\mathsf{T} \vdash \phi_\Sigma \leftrightarrow \phi_\Pi$*, so* ϕ_Σ *and* ϕ_Π *are* Δ_1^{T}*.*

Remark Speaking informally, τ is regarded as a function of one of its variables, u_0; the remaining variables of τ are "parameters". Similarly, η is regarded as a function of v_0. Let $\prec$ be a binary predicate with the definition $\bar{w} \prec \bar{v} \leftrightarrow \bar{w} \in \tau\binom{u_0}{\bar{v}}$, where $\bar{v}$ is free for u_0 in τ, i.e, $\tau(x) = \{y \mid y \prec x\}$. ψ says that that $\prec$ is irreflexive and wellfounded. Since T is a pure set theory, $\tau(x)$ is a set, so $\prec$ is setlike. $\prec$ is therefore a suitable substrate for definition by recursion.[3.80] η defines the value of a function f at x recursively in terms of $f \restriction \tau(x)$, i.e., in terms of the $\prec$-predecessors of x. θ says that f is a function satisfying the recursive definition and u is in its domain. ζ says that if $\prec$ is irreflexive and wellfounded then there is a function f satisfying the recursive definition given by η, with $u \in \operatorname{dom} f$, and $f(u)$ is the same for all such functions. ϕ_Σ and ϕ_Π are respectively Σ_1^{T} and Π_1^{T} ways of saying that v is the (unique) value assigned at u by the recursion.

Proof 1, 2 The complexity calculations leading to (1) and (2) are straightforward.

3 Theorem 3.80 tells us that $\mathsf{CA}^+ \vdash \zeta$. We could use conservative extension results to show that $\mathsf{S}^+ \vdash \zeta$, but it is more direct to display an S^+-proof of ζ by simple modification of the proof of (3.80).

4 Straightforward. □[4.17]

The following theorem provides some more useful complexity classifications.

(4.18) **Theorem** $[\mathsf{C}^+]$

1. *The following are* $\Delta_1^{\mathsf{S}^+}$*.*
   1. **Tran** $\bar{u}$*.*
   2. **Ord** $\bar{u}$*.*
   3. **Suc** $\bar{u}$*.*
   4. **Lim** $\bar{u}$*.*
   5. **Num** $\bar{u}$*.*
   6. $\bar{u}+1$*. Recall that* $x + 1 \overset{\text{def}}{=} x^+ \overset{\text{def}}{=} x \cup \{x\}$*, the successor of* x *if* $\operatorname{Ord} x$*.*
   7. $\bar{u}-1$*. Recall that* $x - 1 \overset{\text{def}}{=} x^- \overset{\text{def}}{=}$ *the predecessor of* x *if* $\operatorname{Suc} x$*; otherwise* 0*.*
   8. $\ulcorner 1 \urcorner$, $\ulcorner 2 \urcorner$, $\ulcorner 3 \urcorner$, *etc.*.[3.47]

9. **Seq** $\bar{u}$. *Recall that* $\mathrm{Seq}\, x \overset{\text{def}}{\iff} \mathrm{Fcn}\, x \wedge \mathrm{Num}(\mathrm{dom}\, x)$.
10. $\langle\,\rangle$, $\langle\bar{u}\rangle$, $\langle\bar{u}, \bar{v}\rangle$, *etc.*
11. $\ulcorner x \subseteq y \urcorner$.
12. $\ulcorner$*A is a family*$\urcorner$.3.37[7]
13. $\ulcorner A_{[i]} \urcorner$.3.37
14. $\ulcorner [\,] \urcorner$, $\ulcorner [x] \urcorner$, $\ulcorner [x, y] \urcorner$, *etc.*3.57
15. $\ulcorner \mathrm{rk}\, x \urcorner$.3.109
16. $\ulcorner \mathrm{tc}\, x \urcorner$. *[Use recursive definition.]*
17. $\ulcorner \mathrm{HF}\, x \urcorner$.

2. *The following are* $\Sigma_1^{\mathsf{S}^+}$.
   1. $\ulcorner x \sim y \urcorner$.3.123
3. *The following are* $\Pi_1^{\mathsf{S}^+}$.
   1. $\ulcorner \mathcal{P}\, x \urcorner$.
   2. $\ulcorner {}^{Y}X \urcorner$.3.35
   3. $\ulcorner$*R is wellfounded*$\urcorner$.3.76

4.5 Evaluation and satisfaction

(4.19) **Definition**

1. *A* transitive interpretation *of* s^+ *is an interpretation* $\mathfrak{I}$ *for which the domain of individuals is a transitive collection of sets,* $\ulcorner \in \urcorner$ *is interpreted as the membership relation, and any other symbols of* s^+ *are interpreted according to their* S^+*-definitions.*
2. *The* standard interpretation *of* s^+ *is the transitive interpretation whose domain of individuals contains every set.*
3. *The* finitary interpretation *of* s^+ *is the transitive interpretation whose domain of individuals contains exactly the sets of finite rank.*

We use 'interpretation' rather than 'structure' here so that we do not have to suppose that the domain of individuals exists. When working in C^+, this distinction is not important, and we may reasonably require that the domain of a transitive interpretation be a class; when working in set theories without proper classes, however, this requirement would be excessively restrictive, as it would exclude, for example, the standard interpretation. Note that S^+ admits the possibility that all sets have finite rank, in which case the finitary interpretation is the standard interpretation.[8]

The value of a term (in a transitive interpretation, for an assignment of its variables) is simply the set it denotes in the ordinary sense. We regard the value of a formula to be either true(ness) or false(ness). We let 1 ($= \{0\}$) be the value of a true formula and 0 the value of a false formula (for a given assignment), and we will use the following defined constants and operations:

[7]Recall that a family is in effect a class-valued function, and the concept of a family is only necessary in the presence of proper classes; otherwise an ordinary function will serve. Thus the notion of a family is not needed in the context of S^+; nevertheless, as we have made this and related notions integral to our definitions of signatures and structures, which we wish to discuss in S^+, we need to analyze their set-theoretic complexity.

[8]To allow for its use in diverse contexts, we do not formulate Definition 4.19 within a particular theory.

(4.20) **Definition** [C^+] $\mathrm{T} \overset{\text{def}}{=} 1$ *and* $\mathrm{F} \overset{\text{def}}{=} 0$. *We define operations on truth values corresponding to the propositional connectives in the expected way:* $\dot{\neg}\,\mathrm{T} = \mathrm{F}$, $\dot{\neg}\,\mathrm{F} = \mathrm{T}$, $\mathrm{T} \mathbin{\dot{\wedge}} \mathrm{T} = \mathrm{T}$, *etc.*

Since a Δ_0 formula ϕ is an s-formula involving only bounded quantification, the truth value of ϕ at an assignment A of its free variables is the same for any transitive interpretation whose domain of individuals includes $\operatorname{im} A$. Hence the following definition makes no reference to interpretation.

(4.21) **Definition** [C^+] *F is a* Δ_0-valuation function $\overset{\text{def}}{\Longleftrightarrow}$

1. *F is a set and is a function whose domain consists of sequences $\langle \eta, A \rangle$, where η is a Δ_0 expression and A is an assignment for η, i.e., A is a finite function and* $\operatorname{Free}\eta \subseteq \operatorname{dom} A$*; and*
2. *for all* $\langle \eta, A \rangle \in \operatorname{dom} F$,
   1. *if* $\eta = \bar{v}$ *for some variable v, then* $F\langle \eta, A \rangle = Av$*;*
   2. *if* $\eta = \tau = \tau'$ *for some terms* τ, τ'[9] *then, letting* $B = A \restriction \operatorname{Free}\tau$ *and* $B' = A \restriction \operatorname{Free}\tau'$,
      1. $\langle \tau, B \rangle \in \operatorname{dom} F$ *and* $\langle \tau', B' \rangle \in \operatorname{dom} F$, *and*
      2. $F\langle \eta, A \rangle = \begin{cases} \mathrm{T} & \textit{if } F\langle \tau, B \rangle = F\langle \tau', B' \rangle \\ \mathrm{F} & \textit{if } F\langle \tau, B \rangle \neq F\langle \tau', B' \rangle; \end{cases}$
   3. *if* $\eta = \tau \in \tau'$ *for some terms* τ, τ' *then, letting* $B = A \restriction \operatorname{Free}\tau$ *and* $B' = A \restriction \operatorname{Free}\tau'$,
      1. $\langle \tau, B \rangle \in \operatorname{dom} F$ *and* $\langle \tau', B' \rangle \in \operatorname{dom} F$, *and*
      2. $F\langle \eta, A \rangle = \begin{cases} \mathrm{T} & \textit{if } F\langle \tau, B \rangle \in F\langle \tau', B' \rangle \\ \mathrm{F} & \textit{if } F\langle \tau, B \rangle \notin F\langle \tau', B' \rangle; \end{cases}$
   4. *if* $\eta = \neg\phi$ *for some formula ϕ, then* $\langle \phi, A \rangle \in \operatorname{dom} F$ *and*
      $$F\langle \eta, A \rangle = \dot{\neg}\, F\langle \phi, A \rangle;$$
   5. *if* $\eta = \phi \vee \phi'$, $\phi \wedge \phi'$, $\phi \rightarrow \phi'$, *or* $\phi \leftrightarrow \phi'$ *for some formulas* ϕ, ϕ' *then, letting* $B = A \restriction \operatorname{Free}\phi$ *and* $B' = A \restriction \operatorname{Free}\phi'$,
      1. $\langle \phi, B \rangle \in \operatorname{dom} F$ *and* $\langle \phi', B' \rangle \in \operatorname{dom} F$, *and*
      2. $F\langle \eta, A \rangle = \begin{cases} F\langle \phi, B \rangle \mathbin{\dot{\vee}} F\langle \phi', B' \rangle & \textit{if } \eta = \phi \vee \phi' \\ F\langle \phi, B \rangle \mathbin{\dot{\wedge}} F\langle \phi', B' \rangle & \textit{if } \eta = \phi \wedge \phi' \\ F\langle \phi, B \rangle \mathbin{\dot{\rightarrow}} F\langle \phi', B' \rangle & \textit{if } \eta = \phi \rightarrow \phi' \\ F\langle \phi, B \rangle \mathbin{\dot{\leftrightarrow}} F\langle \phi', B' \rangle & \textit{if } \eta = \phi \leftrightarrow \phi'; \end{cases}$
   6. *if* $\eta = \exists v \in \tau\, \phi$ *for some variable v, term τ with* $v \notin \operatorname{Free}\tau$, *and formula ϕ, then, letting* $A' = A \restriction \operatorname{Free}\tau$ *and* $B = A \restriction \operatorname{Free}\phi$,
      1. $\langle \tau, A' \rangle \in \operatorname{dom} F$,
      2. $\forall x \in F\langle \tau, A' \rangle\ \langle \phi, B\langle {}^{v}_{x} \rangle \rangle \in \operatorname{dom} F$, *and*
      3. $F\langle \eta, A \rangle = \begin{cases} \mathrm{T} & \textit{if } \exists x \in F\langle \tau, A' \rangle\, F\big\langle \phi, B\langle {}^{v}_{x} \rangle \big\rangle = \mathrm{T} \\ \mathrm{F} & \textit{if } \forall x \in F\langle \tau, A' \rangle\, F\big\langle \phi, B\langle {}^{v}_{x} \rangle \big\rangle = \mathrm{F}; \end{cases}$

[9] In the case of s, of course, any term τ is $\bar{v}$ for some variable v, as there are no operation indices in s, and $\operatorname{Free}\tau = \{v\}$. The definition given here is designed to be readily adaptable to languages with operation indices.

7. *if* $\eta = \forall v \in \tau\, \phi$ *for some variable* v, *term* τ *with* $v \notin \operatorname{Free}\tau$, *and formula* ϕ, *then, letting* $A' = A \restriction \operatorname{Free}\tau$ *and* $B = A \restriction \operatorname{Free}\phi$,
   1. $\langle \tau, A' \rangle \in \operatorname{dom} F$,
   2. $\forall x \in F\langle \tau, A' \rangle \big\langle \phi, B\big\langle {v \atop x} \big\rangle \big\rangle \in \operatorname{dom} F$, *and*
   3. $F\langle \eta, A \rangle = \begin{cases} \mathrm{T} & \textit{if } \forall x \in F\langle \tau, A' \rangle\, F\big\langle \phi, B\big\langle {v \atop x} \big\rangle \big\rangle = \mathrm{T} \\ \mathrm{F} & \textit{if } \exists x \in F\langle \tau, A' \rangle\, F\big\langle \phi, B\big\langle {v \atop x} \big\rangle \big\rangle = \mathrm{F}. \end{cases}$

Note that the restriction to Δ_0 is essential to the existence of valuation functions: if η were, say, $\exists v\, \phi$, then the condition corresponding to (4.21.2.6) would require $\big\langle \phi, B\big\langle {v \atop x} \big\rangle \big\rangle \in \operatorname{dom} F$ for *all* x, not just for x in the value of some term (i.e., the value assigned to some variable, as we are dealing with the signature s, which has no operation indices). F could not then be a set, as the definition mandates.[4.21.1]

(4.22) **Theorem** $[\mathsf{C}^+]$ *Suppose* ϵ *is a* Δ_0 *expression and* A *is an assignment for* ϵ. *Then there is a unique set* x *such that there exists a* Δ_0*-valuation function* F *such that* $(\langle \epsilon, A \rangle, x) \in F$.

Proof We do not choose to apply our general theorem[3.80] on recursive definition, as that would require us to work with the relation $\ulcorner\epsilon$ is a subexpression of ϵ' and A is an assignment for ϵ that occurs in the evaluation of $\epsilon'[A']\urcorner$, which is more trouble than it's worth. It's easier instead to use the subexpression relation *per se*, with appropriate modifications of the proof of (3.80) by insertions of quantification over assignments.

(4.23) **Claim** *Any two* Δ_0*-valuation functions* F, F' *agree on their common domain.*

Proof Suppose toward a contradiction that this is not true for F, F'. Let η have minimal complexity such that for some assignment B for η, $\langle \eta, B \rangle \in \operatorname{dom} F \cap \operatorname{dom} F'$ and $F(\eta, B) \neq F'(\eta, B)$. It is straightforward to derive a contradiction by examining the pertinent case in Definition 4.21. $\square$[4.23]

(4.24) **Claim** *The union of any set of* Δ_0*-valuation functions is a* Δ_0*-valuation function.*

Proof The union is a function by virtue of (4.23), and it is straightforward to show that it is a Δ_0-valuation function. $\square$[4.24]

(4.25) **Claim** *Suppose* ϵ *is a* Δ_0 *expression and* A *is an assignment for* ϵ. *Then there is a* Δ_0*-valuation function* F *such that* $\langle \epsilon, A \rangle \in \operatorname{dom} F$.

Proof Suppose toward a contradiction that this is not true. Let ϵ have minimal complexity such that for some assignment A for ϵ, there is no Δ_0-valuation function F such that $\langle \epsilon, A \rangle \in \operatorname{dom} F$. (Since Δ_0-valuation functions are by definition sets, only set quantification is involved.) It is straightforward to derive a contradiction by forming the union G of Δ_0-valuation functions for the immediate subexpressions of ϵ paired with the relevant assignment(s) derived from A, and then extending G to $\langle \epsilon, A \rangle$ with the value given by the definition. $\square$[4.25]

Given ϵ and A as in the statement of the theorem, we invoke Claim 4.25 to conclude that there exists a Δ_0-valuation function F with $\langle \epsilon, A \rangle \in \operatorname{dom} F$ and

Claim 4.23 to conclude that any two such functions have the same value at $\langle\epsilon, A\rangle$, i.e., there is a unique set x such that $(\langle\epsilon, A\rangle, x) \in F$, as claimed, $\square$[4.22]

Note that the proof of Claim 4.25 depends on the fact that one takes the union of a set (not a proper class) of valuation functions "below" $\langle\epsilon, A\rangle$ to obtain a valuation function "at" $\langle\epsilon, A\rangle$, which follows from the fact that only bounded quantification is allowed.

We now define Val_0, which is the valuation operation restricted to Δ_0 expressions.

(4.26) **Definition** $[\mathsf{C}^+]$ *Suppose ϵ is a Δ_0 expression and A is an assignment for ϵ. $\mathrm{Val}_0\,\epsilon[A] \overset{\text{def}}{=}$ the unique set x such that there is a Δ_0-valuation function F such that $\langle\epsilon, A\rangle \in \operatorname{dom} F$ and $F\langle\epsilon, A\rangle = x$. If ϵ is a term, which is automatically Δ_0, we may omit the subscript, and we define $\mathrm{Val}\,\epsilon[A] \overset{\text{def}}{=} \mathrm{Val}_0\,\epsilon[A]$.*[10]

By Theorem 4.22 this is a legitimate C^+-definition.

(4.27) **Theorem** $[\mathsf{C}^+]$ $\ulcorner \mathrm{Val}_0 \urcorner$ *is* $\Delta_1^{\mathsf{S}^+}$.

Proof Straightforward using (4.11), (4.14), and previous classification results such as (4.16) and (4.18). First show that $\ulcorner F$ is a Δ_0-valuation function$\urcorner$ is $\Delta_1^{\mathsf{S}^+}$. $\square$[4.27]

4.6 Satisfaction in the finitary interpretation

Recall[4.19.3] that the finitary interpretation of s^+ is the transitive interpretation whose individuals are the hereditarily finite sets.[3.95] Recall[3.135.6] that the class HF of hereditarily finite sets is the class V_ω of sets of finite rank.

(4.28) *For the duration of this chapter, as a convenience we impose the condition on signatures that they be hereditarily finite. Note that this applies to the generic signatures s^+ extending s.*

Thus, s^+-expressions are HF sets, just like the objects assigned to their variables in the finitary interpretation, which is to say, among the things that s^+-expressions "talk about" are s^+-expressions. In this situation use and mention are easily conflated, and the interest of clarity necessitates a notation that may at first appear excessively elaborate, but which will stand us in good stead.

(4.29) **Theorem** $[\mathsf{C}^+]$

1. $\ulcorner \mathrm{HF}\,x \urcorner$ *is* $\Delta_1^{\mathsf{S}^+}$.
2. $\ulcorner \mathrm{bin}\,\alpha \urcorner$, $\ulcorner \mathrm{Bin}\,\alpha \urcorner$, $\ulcorner x \prec y \urcorner$ *are* $\Delta_1^{\mathsf{S}^+}$.[3.211]
3. *The predicates and operations relating to signatures defined in Section 1.2.3 are $\Delta_1^{\mathsf{S}^+}$. Given that we have restricted our attention to* HF *signatures*[4.28] *the references to V, the class of all sets, in (1.29.1.2) and (1.29.2.2) may be omitted. Keep in mind also that in the context of S^+, 'class' and 'set' are synonymous.*
4. *The predicates and operations relating to structures in Section 1.2.4 are $\Delta_1^{\mathsf{S}^+}$.*

[10] The preceding footnote is applicable here: The s-terms are just the variables, and the valuation function is trivial for these; we are looking ahead to languages with operation indices.

5. *The predicates and operations relating to standard languages in Section 1.3.1, i.e., the languages $\mathcal{L}^\rho$ for an* HF *signature ρ, are $\Delta_1^{\mathsf{S}^+}$. For this purpose, we replace the classes $\mathcal{V}$, $\mathcal{Q}$, etc., by predicates, and we replace statements like $\ulcorner x \in \mathcal{V} \urcorner$, expressing membership in such a class, by $\ulcorner \mathcal{V}x \urcorner$, i.e., asserting of x that it has the property $\mathcal{V}$. Arguments for some of these predicates are written as superscripts, as in $\ulcorner \mathcal{E}^\rho x \urcorner$, which has two free variables, $\ulcorner \rho \urcorner$ and $\ulcorner x \urcorner$.*

6. *The predicates and operations relating to interpretation of ρ-expressions in ρ-structures in Section 1.4 are $\Delta_1^{\mathsf{S}^+}$.*

Proof These computations are fairly straightforward using (4.18). For example, $\mathsf{S}^+ \vdash \ulcorner \mathrm{HF}\, x \leftrightarrow \mathrm{Num}(\mathrm{rk}\, x) \urcorner$. $\square$[4.29]

Since $\ulcorner\mathrm{HF}\urcorner$ is $\Delta_1^{\mathsf{S}^+}$, if ϕ is $\Sigma_1^{\mathsf{S}^+}$ then $\exists_{\mathrm{HF}}\, u\ \phi$ is $\Sigma_1^{\mathsf{S}^+}$; if ϕ is $\Pi_1^{\mathsf{S}^+}$ then $\forall_{\mathrm{HF}}\, u\ \phi$ is $\Pi_1^{\mathsf{S}^+}$.

It is customary to describe the valuation operation for formulas in terms of the *satisfaction relation.* As already noted, there is no s^+-formula that S^+-provably defines the satisfaction relation for the full universe of sets. Since it is consistent with S^+ that all sets are HF, there is also no s^+-formula that S^+-provably defines the satisfaction relation for the finitary interpretation. We can, however, define satisfaction for limited classes of formulas in any transitive interpretation.

(4.30) **Definition** $[\mathsf{C}^+]$

1. $\mathrm{Sat}_0\, \epsilon[A] \overset{\mathrm{def}}{\iff}$
   1. *ϵ is a Δ_0 formula,*
   2. *A is an* HF*-assignment for ϵ, and*
   3. $\mathrm{Val}_0\, \epsilon[A] = \mathrm{T}$.
2. $\mathrm{Sat}_1^\Sigma\, \epsilon[A] \overset{\mathrm{def}}{\iff}$
   1. *ϵ is a Σ_1 formula, say $\epsilon = \exists v_0 \cdots \exists v_{k^-}\, \epsilon'$, where ϵ' is Δ_0,*
   2. *A is an* HF*-assignment for ϵ, and*
   3. *there exists $\langle x_0, \ldots, x_{k^-} \rangle \in {}^k\,\mathrm{HF}$ such that*
   $$\mathrm{Sat}_0\, \epsilon'\big[A\big\langle\begin{smallmatrix} v_0 & \cdots & v_{k^-} \\ x_0 & \cdots & x_{k^-} \end{smallmatrix}\big\rangle\big].$$
3. $\mathrm{Sat}_1^\Pi(\epsilon, A) \overset{\mathrm{def}}{\iff}$
   1. *ϵ is a Π_1 formula, say $\epsilon = \forall v_0 \cdots \forall v_{k^-}\, \epsilon'$, where ϵ' is Δ_0,*
   2. *A is an* HF*-assignment for ϵ, and*
   3. *for every $\langle x_0, \ldots, x_{k^-} \rangle \in {}^k\,\mathrm{HF}$*
   $$\mathrm{Sat}_0\, \epsilon'\big[A\big\langle\begin{smallmatrix} v_0 & \cdots & v_{k^-} \\ x_0 & \cdots & x_{k^-} \end{smallmatrix}\big\rangle\big].$$

Nothing prevents us from defining similar satisfaction predicates for the standard (not necessarily finitary) interpretation or any other transitive interpretation with a defined domain; nevertheless, the symbols just defined are specific to the finitary interpretation.

(4.31) **Theorem** $[\mathsf{C}^+]$

1. $\ulcorner \mathrm{Sat}_0 \urcorner$ *is* $\Delta_1^{\mathsf{S}^+}$.

2. $\ulcorner \mathrm{Sat}_1^\Sigma \urcorner$ *is* $\Sigma_1^{\mathsf{S}^+}$.
3. $\ulcorner \mathrm{Sat}_1^\Pi \urcorner$ *is* $\Pi_1^{\mathsf{S}^+}$.

Proof Straightforward. □[4.31]

Note that Sat_1^Σ and Sat_1^Π give essentially the same information, as for any Σ_1 formula ϕ and HF-assignment A for ϕ, $\mathrm{Sat}_1^\Sigma(\phi, A) \leftrightarrow \mathrm{Sat}_1^\Pi(\phi', A)$, where ϕ' is the standard Π_1 formula that is S-equivalent to $\neg\,\phi$ (obtained by bringing the negation sign inside the existential quantifiers while changing them to universal quantifiers). It is for this reason that we have split the indicators 'Σ' and 'Π' from the subscript '1' to suggest that Sat_1^Σ and Sat_1^Π are respectively Σ and Π versions of the satisfaction relation at level 1. We could, of course, define similar satisfaction predicates for higher levels of the set-theoretic complexity hierarchy,[4.5] but we do not need these.

Note that Sat_0, Sat_1^Σ and Sat_1^Π agree on Δ_0 formulas, and we often omit the qualifying sub- and superscripts in this context.

(4.32) **Theorem** $[\mathsf{C}^+]$ *Suppose* ϕ *and* ψ *are* Δ_0 *formulas and* A *is an* HF*-assignment to* $\mathrm{Free}\,\phi \cup \mathrm{Free}\,\psi$*. Then*

1. $\mathrm{Sat}(\neg\,\phi, A) \leftrightarrow \neg\,\mathrm{Sat}(\phi, A)$.
2. $\mathrm{Sat}(\phi \vee \psi, A) \leftrightarrow \big(\mathrm{Sat}(\phi, A) \vee \mathrm{Sat}(\psi, A)\big)$.
3. $\mathrm{Sat}(\phi \wedge \psi, A) \leftrightarrow \big(\mathrm{Sat}(\phi, A) \wedge \mathrm{Sat}(\psi, A)\big)$.
4. $\mathrm{Sat}(\phi \rightarrow \psi, A) \leftrightarrow \big(\mathrm{Sat}(\phi, A) \rightarrow \mathrm{Sat}(\psi, A)\big)$.
5. $\mathrm{Sat}(\phi \leftrightarrow \psi, A) \leftrightarrow \big(\mathrm{Sat}(\phi, A) \leftrightarrow \mathrm{Sat}(\psi, A)\big)$.

Proof Essentially immediate. □[4.32]

(4.33) **Definition** $[\mathsf{C}^+]$ $x \text{ add } y \overset{\text{def}}{=} x \frown y \overset{\text{def}}{=} x \cup \{y\}$.

In other words, $x \frown y$ is obtained from x by adding y (as a member). $\frown$ is the *adjunction* operation. The special case $x \mapsto x \frown x$ is the *successor* operation, which agrees with the definition we have already made for ordinals.[3.46]

Any HF set can be obtained by a composition of the 0-ary operation $\mapsto 0$ and the binary operation $x, y \mapsto x \frown y = x \cup \{y\}$, working "from the ground up"; and all sets obtained in this way are HF. We will combine these operations to create canonical names for hereditarily finite sets in much the same way as we use the zero and successor operations in arithmetic to name numbers (using the terms $\mathbf{0}$, $\boldsymbol{S}(\mathbf{0})$, $\boldsymbol{S}(\boldsymbol{S}(\mathbf{0}))$, ...).

By stating Definition 4.33 we have implicitly declared that s^+ has an index for $\ulcorner \frown \urcorner$, and that C^+ contains its definition; and we have previously[3.10] declared the same for the 0 operation. We now formalize this by assigning specific HF sets as the indices and specific formulas as the definitions for these operations, and we define s', c', S', and C' to be the resulting signatures and theories. Recall the definition[1.29.2] of 'unisorted signature', and recall[3.6] that s has been defined with 1 as the index for the membership predicate.[11]

[11] 0, as always, is the index for the identity predicate.

(4.34) **Definition** $[\mathsf{C}^+]$

$$\begin{aligned}\mathsf{s}' &= \mathsf{s} \cup \{(1,\{2\}),(1,\{3\})\}\\ &\quad \cup \{(2,\{(2,0)\}),(2,\{(3,2)\})\}\\ \mathsf{c}' &= \mathsf{c} \cup \{(1,\{2\}),(1,\{3\})\}\\ &\quad \cup \{(2,\{(2,0)\}),(2,\{(3,2)\})\}\end{aligned}$$

In other words, we have defined s' to be $[\Pi', \Phi', T']$, where

1. $\Pi' = \{0, 1\}$, the set of predicate indices, already in s;
2. $\Phi' = \{2, 3\}$, the set of operation indices; and
3. $T' = \{(0, 2), (1, 2), (2, 0), (3, 2)\}$, the arity function.

In the interest of clarity we will typically refer to the indices 0,1,2,3 by the (metalanguage) names '$\mathrm{i}_=$', '$\mathrm{i}_\in$', 'i_0', '$\mathrm{i}_\frown$', respectively.

(4.35) **Definition** $[\mathsf{C}^+]$

1. Δ_0', Σ_n', *and* Π_n' $\stackrel{\text{def}}{=}$ $\Delta_0^{\mathsf{s}'}$, $\Sigma_n^{\mathsf{s}'}$, *and* $\Pi_n^{\mathsf{s}'}$, *respectively.*
2. Val', Val_0', Sat_0', $\mathrm{Sat}_1^{\Sigma'}$, *and* $\mathrm{Sat}_1^{\Pi'}$ *are defined for* s'*-expressions by the obvious modifications of the definitions*[4.21, 4.26, 4.30] *of the unprimed symbols. In particular, we require of a* Δ_0'*-valuation function* F *that if* $\langle \eta, A\rangle \in \operatorname{dom} F$ *then, in addition to (4.21),*
   1. *if* $\eta = \mathbf{0}$ *then* $F\langle \eta, A\rangle = 0$*; and*
   2. *if* $\eta = \tau \frown \tau'$ *for some terms* τ, τ' *then, letting* $B = A \restriction \mathrm{Free}\,\tau$ *and* $B' = A \restriction \mathrm{Free}\,\tau'$, $\langle \tau, B\rangle$, $\langle \tau', B'\rangle \in \operatorname{dom} F$, *and*

$$F\langle \eta, A\rangle = (F\langle \tau, B\rangle)^\frown(F\langle \tau', B'\rangle).$$

As we now have operation symbols, for future reference we explicitly list the definitional properties relating to Val' below.

(4.36) **Theorem** $[\mathsf{C}^+]$ *Suppose* u *is a variable and* A *is an appropriate* HF*-assignment.*

1. $\mathrm{Val}'\,\bar{u}[A] = Au$.
2. $\mathrm{Val}'\,\mathbf{0}[A] = 0$.
3. $\mathrm{Val}'(\tau \frown \tau')[A] = (\mathrm{Val}'\,\tau[A])^\frown(\mathrm{Val}'\,\tau'[A])$.
4. $\mathrm{Sat}_0'(\bar{u} \in \tau)[A] \leftrightarrow Au \in \mathrm{Val}'\,\tau[A]$.
5. $\mathrm{Sat}_0'(\bar{u} = \tau)[A] \leftrightarrow Au = \mathrm{Val}'\,\tau[A]$.

Proof Straightforward. $\square$[4.36]

(4.37) **Theorem** $[\mathsf{C}^+]$

1. $\ulcorner \mathrm{Sat}_0' \urcorner$ *is* $\Delta_1^{\mathsf{S}^+}$.
2. $\ulcorner \mathrm{Sat}_1^{\Sigma'} \urcorner$ *is* $\Sigma_1^{\mathsf{S}^+}$.
3. $\ulcorner \mathrm{Sat}_1^{\Pi'} \urcorner$ *is* $\Pi_1^{\mathsf{S}^+}$.

Proof Straightforward. $\square^{4.37}$

For convenience we combine $\ulcorner \mathrm{Sat}_1^{\Sigma} \urcorner$ and $\ulcorner \mathrm{Sat}_1^{\Pi} \urcorner$ in a single predicate $\ulcorner \mathrm{Sat}_1 \urcorner$.

Definition [C^+] *Suppose ϕ is an s'-formula and A is an* HF*-assignment for ϕ. Then*

1. $\mathrm{Sat}_1'\,\phi[A] \overset{\text{def}}{\iff}$
   1. *ϕ is Σ_1' and $\mathrm{Sat}_1^{\Sigma'}\,\phi[A]$, or*
   2. *ϕ is Π_1' and $\mathrm{Sat}_1^{\Pi'}\,\phi[A]$.*
2. $\mathrm{Sat}\,\phi[A] \overset{\text{def}}{\iff}$ *ϕ is an s-formula and $\mathrm{Sat}_1'\,\phi[A]$.*

Note that $\mathrm{Sat}_1^{\Sigma'}$, $\mathrm{Sat}_1^{\Pi'}$, and Sat_1' extend Sat_1^{Σ}, Sat_1^{Π}, and Sat_1, respectively, i.e., if ϕ is an s-formula and A is an HF-assignment for ϕ, then $\mathrm{Sat}_1\,\phi[A] \leftrightarrow \mathrm{Sat}_1'\,\phi[A]$.

4.6.1 Complexity of classes

Having defined satisfaction predicates, we can now define the the complexity of definable classes in terms of the complexity of definitions.

(4.38) **Definition** [C^+] *Suppose $X \subseteq$ HF.*

1. *X is $\Delta_0 \overset{\text{def}}{=}$ there is a Δ_0 formula ϕ with one free variable u such that $\forall_{\mathrm{HF}}\, x\, \big(x \in X \leftrightarrow \mathrm{Sat}_0\,\phi\big[{u \atop x}\big]\big)$.*
2. *X is $\Sigma_1 \overset{\text{def}}{=}$ there is a Σ_1 formula ϕ with one free variable u such that $\forall_{\mathrm{HF}}\, x\, \big(x \in X \leftrightarrow \mathrm{Sat}_1^{\Sigma}\,\phi\big[{u \atop x}\big]\big)$.*
3. *X is $\Pi_1 \overset{\text{def}}{=}$ there is a Π_1 formula ϕ with one free variable u such that $\forall_{\mathrm{HF}}\, x\, \big(x \in X \leftrightarrow \mathrm{Sat}_1^{\Pi}\,\phi\big[{u \atop x}\big]\big)$.*
4. *X is $\Delta_1 \overset{\text{def}}{=} X$ is Σ_1 and Π_1.*

Definition [C^+] *Suppose ϕ and ψ are s'-formulas that are individually either Σ_1' or Π_1'. Then ϕ and ψ are* HF-equivalent $\overset{\text{def}}{\iff} \mathrm{Free}\,\phi = \mathrm{Free}\,\psi$ *and for all* HF-*assignments A for ϕ and ψ, $\mathrm{Sat}_1'\,\phi[A] \leftrightarrow \mathrm{Sat}_1'\,\psi[A]$.*

(4.39) **Theorem** [C^+]

1. *Suppose ϕ is Δ_0'. Then are Σ_1 and Π_1 formulas that are* HF*-equivalent to ϕ.*
2. *Suppose ϕ is Σ_1' (Π_1'). Then there is a Σ_1 (Π_1) formula that is* HF*-equivalent to ϕ.*

Remark Note the similarity of this theorem to (4.15), which applied here would say that for any Δ_0' formula ϕ there are T-equivalent Σ_1 and Π_1 formulas. The present theorem asserts that these equivalences are *true* (in the finitary interpretation, to which Sat and Sat$'$ refer), as opposed to *provable*. In general, asserting that a sentence σ is provable (in a given theory) is not the same as asserting that σ is true (in a given interpretation).[12] The proof is of course organized along similar inductive lines.

[12] And neither of these is the same as asserting σ.

Proof We will show by induction on complexity that for any Δ_0' formula ϕ there exist Δ_0 formulas σ, π, and variables u, v, such that $\exists u\, \sigma$ and $\forall v\, \pi$ are HF-equivalent to ϕ. This obviously implies the theorem. As usual, variables are assumed throughout to be chosen so as to be distinct from one another and from variables already present unless otherwise indicated.

(4.40) **Claim** *Suppose τ is an s'-term and $\phi = \bar{u} = \tau$. Then there exists a Δ_0 formula σ and a variable v such that $\exists v\, \sigma$ is* HF*-equivalent to ϕ.*

Remark Note that for this special case, it is sufficient to establish HF-equivalence with a Σ_1 formula.

Proof By induction on the complexity of τ. The case of $\tau = \bar{v}$, for a variable v, is trivial.

Suppose $\phi = \bar{u} = \mathbf{0}$. Let $\sigma = \forall v \in \bar{u}\ \neg\, \bar{v} = \bar{v}$. Then $\exists w\, \sigma$ is HF-equivalent to ϕ, where w is any variable other than u.

Next suppose $\phi = \bar{u} = \tau_0 \mathbin{\frown} \tau_1$, and σ_0, σ_1 are Δ_0 such that $\exists u_0\, \sigma_0$ and $\exists u_1\, \sigma_1$ are HF-equivalent to $\bar{v}_0 = \tau_0$ and $\bar{v}_1 = \tau_1$, respectively. Let

$$\sigma = \exists v_0, v_1, u_0, u_1 \in \bar{w}\, \big(\sigma_0 \wedge \sigma_1 \\ \wedge\, \bar{v}_1 \in \bar{u} \wedge \forall v \in \bar{v}_0\ \bar{v} \in \bar{u} \wedge \forall v \in \bar{u}\, (\bar{v} \in \bar{v}_0 \vee \bar{v} = \bar{v}_1)\big).$$

Then $\exists w\, \sigma$ is clearly HF-equivalent to ϕ. The claim follows by induction. $\square$[4.40]

Now suppose τ_0, τ_1 are s'-terms, and σ_0, σ_1 are Δ_0 such that $\exists v_0\, \sigma_0$ and $\exists v_1\, \sigma_1$ are HF-equivalent to $\bar{u}_0 = \tau_0$ and $\bar{u}_1 = \tau_1$, respectively.

Suppose $\phi = \tau_0 = \tau_1$. Let $\sigma = \exists u_0, u_1, v_0, v_1 \in \bar{v}\, (\sigma_0 \wedge \sigma_1 \wedge \bar{u}_0 = \bar{u}_1)$. Then $\exists v\, \sigma$ is HF-equivalent to ϕ. Let $\pi = \neg \exists u_0, u_1, v_0, v_1 \in \bar{v}\, (\sigma_0 \wedge \sigma_1 \wedge \neg(\bar{u}_0 = \bar{u}_1))$. Then $\forall v\, \pi$ is HF-equivalent to ϕ.

Now suppose $\phi = \tau_0 \in \tau_1$. Let $\sigma = \exists u_0, u_1, v_0, v_1 \in \bar{v}\, (\sigma_0 \wedge \sigma_1 \wedge \bar{u}_0 \in \bar{u}_1)$. Then $\exists v\, \sigma$ is HF-equivalent to ϕ. Let $\pi = \neg \exists u_0, u_1, v_0, v_1 \in \bar{v}\, (\sigma_0 \wedge \sigma_1 \wedge \neg(\bar{u}_0 \in \bar{u}_1))$. Then $\forall v\, \pi$ is HF-equivalent to ϕ.

Thus, the theorem holds for all atomic s'-formulas. The induction steps corresponding to propositional connectives are straightforward.

Lastly, we deal with bounded quantification. Suppose ψ is Δ_0'; σ and π are Δ_0 such that ψ is HF-equivalent to $\exists u\, \sigma$ and to $\forall u\, \pi$; and σ_0 is Δ_0 such that $\bar{u}_1 = \tau$ is HF-equivalent to $\exists u_0\, \sigma_0$.

Suppose $\phi = \exists v \in \tau\ \psi$. Let $\sigma' = \exists u_0, u_1 \in \bar{w}\, (\sigma_0 \wedge \exists v \in \bar{u}_1\, \exists u \in \bar{w}\ \sigma)$. Then ϕ is HF-equivalent to $\exists w\, \sigma'$.

Suppose $\phi = \forall v \in \tau\ \psi$. Let $\sigma' = \exists u_0, u_1 \in \bar{w}\, (\sigma_0 \wedge \forall v \in \bar{u}_1\, \exists u \in \bar{w}\ \sigma)$. Then ϕ is HF-equivalent to $\exists w\, \sigma'$. Note that this uses the $\mathsf{Collection}$ property of HF, i.e., for any $x_1 \in \mathrm{HF}$,

$$\forall y \in x_1\, \exists_{\mathrm{HF}} x\ \ \mathrm{Sat}_0\, \sigma\big[A\big\langle {u \atop x} \big\rangle\big] \leftrightarrow \exists_{\mathrm{HF}} z\, \forall y \in x_1\, \exists x \in z\ \ \mathrm{Sat}_0\, \sigma\big[A\big\langle {u \atop x} \big\rangle\big].$$

We can DeMorgan the preceding constructions to obtain Π_1 equivalents. Thus, $\exists v \in \tau\ \psi$ is HF-equivalent to $\forall w\, \forall u_0, u_1 \in \bar{w}\, (\sigma_0 \to \exists v \in \bar{u}_1\, \forall u \in \bar{w}\ \pi)$. And $\forall v \in \tau\ \psi$ is HF-equivalent to $\forall w\, \forall u_0, u_1 \in \bar{w}\, (\sigma_0 \to \forall v \in \bar{u}_1 \forall u \in \bar{w}\ \pi)$. $\square$[4.39]

The assertions we have previously made concerning the complexity of formulas and terms (*vis-à-vis* a theory), as in (4.16), (4.17), and (4.18), are of course applicable to the classes they define—usually with some degree of simplification, as it is not necessary to deal with linguistic expressions *per se* and their provable equivalence. We will not give a long list of these and similar complexity assessments, as the calculations are for the most part quite straightforward; however, several warrant particular attention.

The following theorem on recursive definition is essentially (4.17) expressed in terms of definable classes. It is also a refinement of (3.80), taking complexity considerations into account, and we will present it in this format.

(4.41) **Theorem** [C^+] *Suppose R is an irreflexive wellfounded relation on a Δ_1 class $X \subseteq \mathrm{HF}$, which is setlike in the sense of* HF, *i.e., for all $x \in X$, $R^{\leftarrow}\{x\} \in \mathrm{HF}$. Suppose further that $x \mapsto R^{\leftarrow}\{x\}$ is Δ_1. Suppose G is a Δ_1 function such that* $\operatorname{dom} G$ *consists of all $\langle x, f\rangle$ such that $x \in X$, f is a function, and* $\operatorname{dom} f = R^{\leftarrow}\{x\}$. *Let F be the (unique) function such that*

1. $\operatorname{dom} F = X$, *and*
2. $\forall x \in X \; Fx = G\langle x, F \restriction (R^{\leftarrow}\{x\})\rangle$,

as guaranteed by (3.80). Then F is Δ_1.

Proof Referring to the proof of (3.80), we first show that the class of acceptable sets f is Δ_1. This is a somewhat lengthy but straightforward exercise, left to the reader. The proofs of Claims 3.82, 3.83, and 3.85 do not change. These allow us to define F in two ways: either as the class of ordered pairs (x, y) such that $x \in X$ and there exists an acceptable f such that $(x, y) \in f$, or as the class of ordered pairs (x, y) such that $x \in X$ and for all acceptable f such that $x \in \operatorname{dom} f$, $(x, y) \in f$. The former shows that F is Σ_1 and the latter shows that F is Π_1, so F is Δ_1. $\square$[4.41]

The following definition provides Δ_0 formulas θ_x and ν_x, for each $x \in \mathrm{HF}$, that define respectively the classes $\{x\}$ and x.

(4.42) **Definition** [C^+] *For each $x \in \mathrm{HF}$,*

1. *let $\langle a^x_m \mid m \in |x|\rangle$ enumerate x in decreasing $\prec$-order;*[3.211.3]
2. *let $u_x = \mathrm{v}_{n+2}$, where $n = \overleftarrow{B}\, x$; and*
3. *let $\theta_x \overset{\text{def}}{=}$ the Δ_0 formula with* $\operatorname{Free}\theta_x = \{\mathrm{v}_0\}$ *given recursively by*
   1. $\theta_0 = \forall \mathrm{v}_1 \in \bar{\mathrm{v}}_0 \; \bar{\mathrm{v}}_1 \neq \bar{\mathrm{v}}_1$; *and*
   2. *for $x \neq 0$,*

$$\theta_x = \exists u_{a^x_0} \in \bar{\mathrm{v}}_0 \cdots \exists u_{a^x_{|x|^-}} \in \bar{\mathrm{v}}_0 \left(\theta_{a^x_0}\binom{\mathrm{v}_0}{\bar{u}_{a^x_0}} \wedge \cdots \wedge \theta_{a^x_{|x|^-}}\binom{\mathrm{v}_0}{\bar{u}_{a^x_{|x|^-}}}\right.$$
$$\left.\wedge\, \forall \mathrm{v}_1 \in \bar{\mathrm{v}}_0 \left(\bar{\mathrm{v}}_1 = \bar{u}_{a^x_0} \vee \cdots \vee \bar{\mathrm{v}}_1 = \bar{u}_{a^x_{|x|^-}}\right)\right).$$

Now define ν_x for $x \in \mathrm{HF}$ as follows. $\nu_0 \overset{\text{def}}{=} \bar{\mathrm{v}}_0 \neq \bar{\mathrm{v}}_0$; if $x \neq 0$, let $\nu_x \overset{\text{def}}{=} \theta_{a^x_0} \vee \cdots \vee \theta_{a^x_{|x|^-}}$. (In particular, $\nu_{\{x\}} = \theta_x$.)

(4.43) **Theorem** [C^+] *Suppose $x, y \in \mathrm{HF}$.*

1. $y = x$ *iff* $\operatorname{Sat}_0 \theta_x\left[\substack{\mathrm{v}_0\\ y}\right]$.
2. $y \in x$ *iff* $\operatorname{Sat}_0 \nu_x\left[\substack{\mathrm{v}_0\\ y}\right]$.

Hence, x is Δ_0.

Proof It is straightforward to show by $\in$-induction that for all $x \in \mathrm{HF}$,

$$\forall_{\mathrm{HF}}\, y \left(\mathrm{Sat}_0\, \theta_x \left[\begin{smallmatrix} \mathrm{v}_0 \\ y \end{smallmatrix}\right] \leftrightarrow y = x \right),$$

and from this it follows that for all $y \in \mathrm{HF}$, $y \in x$ iff $\mathrm{Sat}_0\, \nu_x \left[\begin{smallmatrix} \mathrm{v}_0 \\ y \end{smallmatrix}\right]$. $\square^{4.43}$

The following theorem is the principal application of the θ_xs.

(4.44) **Theorem** $[\mathsf{C}^+]$ *Suppose* $A \subseteq \mathrm{HF} \times \mathrm{HF}$ *is* Σ_1 (Π_1), *and* $a \in \mathrm{HF}$. *Then* $\{b \mid \langle a, b\rangle \in A\}$ *is* Σ_1 (Π_1).

Proof Let $B = \{b \mid \langle a, b\rangle \in A\}$. Suppose A is Σ_1. Then there exists a Σ_1 formula ϕ with $\mathrm{Free}\,\phi = \{\mathrm{v}_0, \mathrm{v}_1\}$, such that

$$A = \left\{\langle x, y\rangle \,\middle|\, \mathrm{Sat}_1\, \phi\left[\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ x & y \end{smallmatrix}\right]\right\},$$

as can easily be verified using previous complexity computations (for $\langle\cdot,\cdot\rangle$, etc.). Without loss of generality, suppose $\phi = \exists \mathrm{v}_2\, \phi'$, where ϕ' is Δ_0. Let $\psi = \exists \mathrm{v}_0\, \exists \mathrm{v}_2\, (\theta_a \wedge \phi')$. Then ψ is Σ_1, $\mathrm{Free}\,\psi = \{\mathrm{v}_1\}$, and

$$B = \left\{b \,\middle|\, \mathrm{Sat}_1\, \psi\left[\begin{smallmatrix} \mathrm{v}_1 \\ b \end{smallmatrix}\right]\right\}.$$

Similarly, suppose A is Π_1 and suppose

$$A = \left\{\langle x, y\rangle \,\middle|\, \mathrm{Sat}_1\, \phi\left[\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ x & y \end{smallmatrix}\right]\right\},$$

where $\phi = \forall \mathrm{v}_2\, \phi'$, ϕ' is Δ_0, and $\mathrm{Free}\,\phi' = \{\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\}$. Let $\psi = \forall \mathrm{v}_0\, \forall \mathrm{v}_2\, (\theta_a \rightarrow \phi')$. Then ψ is Π_1, $\mathrm{Free}\,\psi = \{\mathrm{v}_1\}$, and

$$B = \left\{b \,\middle|\, \mathrm{Sat}_1\, \psi\left[\begin{smallmatrix} \mathrm{v}_1 \\ b \end{smallmatrix}\right]\right\}.$$

$\square^{4.44}$

(4.44) is more general than it appears at first. For example, suppose we have a Σ_1 set A of objects of the form $(\langle a, b\rangle, c)$. Then for fixed $b \in \mathrm{HF}$, $\{\langle a, c\rangle \mid (\langle a, b\rangle, c) \in A\}$ is Σ_1. To show this, let $A' = \{\langle b, \langle a, c\rangle\rangle \mid (\langle a, b\rangle, c) \in A\}$. Then apply (4.44) with A' for A and b for a. Thus, argument specification rather generally preserves membership in Σ_1 (Π_1).

(4.45) **Theorem** $[\mathsf{C}^+]$ *The functions* $x \mapsto \theta_x$ *and* $x \mapsto \nu_x$ *are* Δ_1.

Remark This has nothing to do with the fact that θ_x and ν_x are Δ_0^{s} formulas (for each $x \in \mathrm{HF}$). Here we are talking about the two functions $x \mapsto \theta_x$ and $x \mapsto \nu_x$, and the theorem states that these are Δ_1 as subclasses of HF, i.e., each is defined by a Σ_1^{s} and by a Π_1^{s} formula (in the finitary interpretation).

Proof The proof is straightforward. Note that we use (4.41) to show that the recursive definition of θ_x leads to a Δ_1 function. $\square^{4.45}$

4.6.2 The complexity of satisfaction

By convention, we use 'Sat_1^Σ' and 'Sat_1^Π' to denote the subclasses of HF defined by the corresponding s^+-formulas. These are partial satisfaction relations for (HF; ∈), and by Theorem 4.31 they are respectively Σ_1 and Π_1. We can similarly define Sat_2^Σ, Sat_2^Π, Sat_3^Σ, Sat_3^Π, etc.; and we can show that these are respectively Σ_2, Π_2, Σ_3, Π_3, etc. By Theorem 1.73 the full satisfaction relation for (HF; ∈) is not definable over (HF; ∈), so it is not Σ_n (or Π_n) for any n. It is natural to inquire whether the above complexity classifications are the best possible.

(4.46) **Theorem** [C^+] Sat_1^Σ *is not* Π_1. *Likewise,* Sat_1^Π *is not* Σ_1.

Proof Suppose toward a contradiction that Sat_1^Σ is Π_1, so its complement is Σ_1. In other words $\{\langle\psi, A\rangle \mid \neg\,\mathrm{Sat}_1^\Sigma\,\psi[A]\}$ is Σ_1, so there is a Σ_1 formula ϕ with two free variables, u and v, such that for every Σ_1 formula ψ and HF-assignment A for ψ,

$$\mathrm{Sat}_1^\Sigma\,\phi\big[\begin{smallmatrix} u & v \\ \psi & A\end{smallmatrix}\big] \leftrightarrow \neg\,\mathrm{Sat}_1^\Sigma\,\psi[A].$$

Specializing to the case of formulas ψ with one free variable, by a simple modification of ϕ we obtain a Σ_1 formula ϕ' such that for every Σ_1 formula ψ with one free variable w and every $a \in \mathrm{HF}$,

$$\mathrm{Sat}_1^\Sigma\,\phi'\big[\begin{smallmatrix} u & v \\ \psi & a\end{smallmatrix}\big] \leftrightarrow \neg\,\mathrm{Sat}_1^\Sigma\,\psi\big[\begin{smallmatrix} w \\ a\end{smallmatrix}\big].$$

Let $\psi = \phi'\big(\begin{smallmatrix} u & v \\ \bar{w} & \bar{w}\end{smallmatrix}\big)$. Then

$$\mathrm{Sat}_1^\Sigma\,\phi'\big[\begin{smallmatrix} u & v \\ \psi & \psi\end{smallmatrix}\big] \leftrightarrow \neg\,\mathrm{Sat}_1^\Sigma\,\psi\big[\begin{smallmatrix} w \\ \psi\end{smallmatrix}\big] \leftrightarrow \neg\,\mathrm{Sat}_1^\Sigma\,\phi'\big[\begin{smallmatrix} u & v \\ \psi & \psi\end{smallmatrix}\big],$$

a contradiction.

Hence, Sat_1^Σ is not Π_1, and it follows immediately that Sat_1^Π is not Σ_1. □4.46

Clearly, we can easily show also that Sat_2^Σ is not Π_2, etc. With the axiom of infinity, we can prove the general theorem that for every $n > 0$, Sat_n^Σ is not Π_n. Without it we can show that for every $n > 0$, if Sat_n^Σ exists, it is not Π_n. Note that for any numeral $\boldsymbol{n}$, $\mathsf{C}^+ \vdash \ulcorner\mathrm{Sat}_{(\boldsymbol{n})}^\Sigma$ exists$\urcorner$. For example, $\mathsf{C}^+ \vdash \ulcorner\mathrm{Sat}_{365}^\Sigma$ exists$\urcorner$.

4.7 Computability

As discussed in Section 4.3, our attitude toward computability is that it is an intrinsic property of functions, and that the science of computation begins with a useful characterization of this property. We have chosen to do this in the framework of HF. We will first show that any Δ_1 function $f : \mathrm{HF} \to \mathrm{HF}$ is computable, by the direct method of exhibiting an effective procedure. We then claim that any effective procedure can be naturally modeled as a Δ_1 function. This claim is buttressed by the fact that all notions of computability that have ever been defined have been shown to be equivalent (to one another and therefore to Δ_1). We will therefore define *computable* to mean Δ_1, and proceed from there.

We call a class $X \subseteq \mathrm{HF}$ computable iff its characteristic function $\{(x,1) \mid x \in X\} \cup \{(x,0) \mid x \in \mathrm{HF} \setminus X\}$ is computable.

(4.47) Sat_0' *is computable; i.e., there is an effective procedure that will determine, for any* Δ_0' *sentence* σ, *whether* $\mathrm{Sat}_0'\,\sigma$.

Remark Since s'-expressions are HF and Sat'_0 is Δ_1, we are showing that a particular Δ_1 class is computable. We will later use this to show that any Δ_1 class is computable.

Demonstration Let s'' be s' with one additional binary predicate index, with the intended interpretation as the inclusion relation. Let $\tau, \tau' \mapsto \tau \subseteq \tau'$ be the corresponding operation on s'-terms. Let Val''_0 be the valuation operation for $\Delta_0^{\mathsf{s}''}$. Then[13]

$$
\begin{aligned}
\mathrm{Val}''_0(\tau = \tau') &= \mathrm{Val}''_0(\tau \subseteq \tau') \mathbin{\dot\wedge} \mathrm{Val}''_0(\tau' \subseteq \tau) \\
\mathrm{Val}''_0(\tau \in \mathbf{0}) &= 0 \\
\mathrm{Val}''_0(\tau \in \tau_0 \frown \tau_1) &= \mathrm{Val}''_0(\tau \in \tau_0) \mathbin{\dot\vee} \mathrm{Val}''_0(\tau = \tau_1) \\
\mathrm{Val}''_0(\mathbf{0} \subseteq \tau) &= 1 \\
\mathrm{Val}''_0(\tau_0 \frown \tau_1 \subseteq \tau) &= \mathrm{Val}''_0(\tau_0 \subseteq \tau) \mathbin{\dot\wedge} \mathrm{Val}''_0(\tau_1 \in \tau) \\
\mathrm{Val}''_0(\neg\, \phi) &= \dot\neg\, \mathrm{Val}''_0(\phi) \\
\mathrm{Val}''_0(\phi \vee \phi') &= \mathrm{Val}''_0(\phi) \mathbin{\dot\vee} \mathrm{Val}''_0(\phi') \\
\mathrm{Val}''_0(\phi \wedge \phi') &= \mathrm{Val}''_0(\phi) \mathbin{\dot\wedge} \mathrm{Val}''_0(\phi') \\
\mathrm{Val}''_0(\phi \rightarrow \phi') &= \mathrm{Val}''_0(\phi) \mathbin{\dot\rightarrow} \mathrm{Val}''_0(\phi') \\
\mathrm{Val}''_0(\phi \leftrightarrow \phi') &= \mathrm{Val}''_0(\phi) \mathbin{\dot\leftrightarrow} \mathrm{Val}''_0(\phi') \\
\mathrm{Val}''_0(\exists u \in \mathbf{0}\ \ \phi) &= 0 \\
\mathrm{Val}''_0(\exists u \in \tau_0 \frown \tau_1\ \ \phi) &= \mathrm{Val}''_0(\exists u \in \tau_0\ \ \phi) \mathbin{\dot\vee} \mathrm{Val}''_0\left(\phi\tbinom{u}{\tau_1}\right) \\
\mathrm{Val}''_0(\forall u \in \mathbf{0}\ \ \phi) &= 1 \\
\mathrm{Val}''_0(\forall u \in \tau_0 \frown \tau_1\ \ \phi) &= \mathrm{Val}''_0(\forall u \in \tau_0\ \ \phi) \mathbin{\dot\wedge} \mathrm{Val}''_0\left(\phi\tbinom{u}{\tau_1}\right).
\end{aligned}
\tag{4.48}
$$

Any $\Delta_0^{\mathsf{s}''}$ formula is uniquely of one of the forms occurring on the left side in (4.48) and may be evaluated using the right side of the corresponding equation. The second, fourth, eleventh and thirteenth of these provide the value directly; the remainder require evaluation of one or two $\Delta_0^{\mathsf{s}''}$ formulas. We therefore have a recursive (self-calling) procedure for computing Val''_0. Since every $\Delta_0^{\mathsf{s}'}$ formula is $\Delta_0^{\mathsf{s}''}$, we have the desired procedure, as long as we can show that it always halts, i.e., terminates.

(4.49) **Definition** *To do so, we define for the nonce the* complexity *of an* s''*-formula* ϕ *to be the 3-sequence* $\langle a, b, c\rangle$*, where* a *is the number of occurrences of a quantifier or propositional connective sign in* ϕ*,* b *is the number of occurrences of* $\mathrm{i}_\frown$ *in* ϕ*, and* c *is the number of occurrences of* $\mathrm{i}_=$ *in* ϕ*. Say that a formula* ϕ *with complexity* $\langle a, b, c\rangle$ *is* simpler than *a formula* ϕ' *with complexity* $\langle a', b', c'\rangle \overset{\text{def}}{\iff}$

1. $a < a'$*, or*
2. $a = a'$ *and* $b < b'$*, or*
3. $a = a'$ *and* $b = b'$ *and* $c < c'$*.*

By this definition, any formula on the right side in (4.48) is simpler than the formula on the left. Since the order just defined is a wellorder, the procedure halts for any input.

[13] Recall[4.20] the use of '$\dot\wedge$', etc., to denote operations on truth values. Here we'll just use 1 and 0 for T and F, respectively.

Since we are presenting this demonstration as an exercise in common sense, as opposed to a formal proof, we need not invoke any abstract set-theoretic principles, and we may argue directly as follows: By the construction of our procedure the evaluation of any s''-sentence σ is accomplished by evaluating either zero, one, or two simpler sentences. If the procedure fails to halt for σ then it fails to halt for one of the latter sentences. We may define σ^* in this case to be the first of these in the order given in (4.48). Now

(4.50) *suppose toward a contradiction that the procedure fails to halt for some s''-sentence σ_0.*

Let $\sigma_1 = \sigma_0^*$, $\sigma_2 = \sigma_1^*$, ... Let $\langle a_n, b_n c_n\rangle$ be the complexity of σ_n. Since the complexity is strictly decreasing, a_n cannot increase, so for some n_0, for all $n \geqslant n_0$, $a_n = a_{n_0}$. For $n \geqslant n_0$, b_n cannot increase, so for some $n_1 \geqslant n_0$, for all $n \geqslant n_1$, $\langle a_n, b_n\rangle = \langle a_{n_1}, b_{n_1}\rangle$. Finally, for $n \geqslant n_1$, c_n cannot increase, so for some $n_2 \geqslant n_1$, for all $n \geqslant n_2$, $\langle a_n, b_n, c_n\rangle = \langle a_{n_2}, b_{n_2}, c_{n_2}\rangle$. Since the complexity is strictly decreasing, this is a contradiction, and the supposition (4.50) is untenable. $\square$[4.47]

We have described the procedure as a manipulation of s''-expressions without specifying exactly what these expressions are. They may be sequences of symbols, handwritten or imprinted on a tape as in a Turing machine; sequences of memory bits in an electronic computer; hereditarily finite sets; .... The essential thing is to recognize that any general model of computation must allow the implementation of the foregoing procedure.

To apply (4.47) to HF sets in general, we use the fact that every HF set is $\mathrm{Val}'\,\tau$ for some constant (variable-free) s'-term τ, so operations on HF sets may be represented by operations on constant s'-terms. It is useful to have a unique name for each HF set, and for this purpose we use the operation $\vec{B} : \omega \overset{\text{bij}}{\to} \mathrm{HF}$ and the associated ordering $\prec$ of HF in order type ω.[3.211 3.213]

Recall that any s'-term s_0 is either $\mathbf{0}$ or is uniquely of the form $s_1 \leftharpoonup t_0$, where s_1, t_0 are s'-terms. s_1 in turn is either $\mathbf{0}$ or $s_2 \leftharpoonup t_1$, s_2 is either $\mathbf{0}$ or $s_3 \leftharpoonup t_2$, etc. Decomposing a term in this way we eventually arrive at $s_n = \mathbf{0}$. For each $m \in n$, $\mathrm{Val}'\,s_m = \mathrm{Val}'\,s_{m+1} \cup \{\mathrm{Val}'\,t_m\}$, so

$$\begin{aligned}
\mathrm{Val}'\,s_0 &= \mathrm{Val}'\,s_1 \cup \{\mathrm{Val}'\,t_0\}\\
&= \mathrm{Val}'\,s_2 \cup \{\mathrm{Val}'\,t_1, \mathrm{Val}'\,t_0\}\\
&\;\;\vdots\\
&= \mathrm{Val}'\,s_n \cup \{\mathrm{Val}'\,t_{n^-}, \dots, \mathrm{Val}'\,t_0\}\\
&= 0 \cup \{\mathrm{Val}'\,t_{n^-}, \dots, \mathrm{Val}'\,t_0\}\\
&= \{\mathrm{Val}'\,t_{n^-}, \dots, \mathrm{Val}'\,t_0\}.
\end{aligned}$$

The following definition is therefore legitimate.

Definition $[\mathsf{S}^+]$ *Suppose τ is a constant s'-term. s^τ and $t^\tau \overset{\text{def}}{=}$ the (unique) finite sequences such that, letting $n = |t^\tau|$,*

1. $|s^\tau| = n + 1$;
2. $s_0^\tau = \tau$;
3. $\forall m \in n\,(s_m^\tau = s_{m+1}^\tau \leftharpoonup t_m^\tau)$; *and*

4. $s^\tau_n = \mathbf{0}$.

(4.51) **Definition** $[\mathsf{S}^+]$ *An* s'*-term* τ *is* canonical $\overset{\text{def}}{\iff}$ τ *is constant and for every subterm* τ' *of* τ*, letting* $n' = |t^{\tau'}|$,

$$\mathrm{Val}'\, t^{\tau'}_0 \prec \mathrm{Val}'\, t^{\tau'}_1 \prec \cdots \prec \mathrm{Val}'\, t^{\tau'}_{n'^{-}}.$$

(4.52) **Theorem** $[\mathsf{S}^+]$ *For every* HF *set* x *there is a unique canonical* s'*-term* τ *such that* $\mathrm{Val}'\,\tau = x$.

Proof By $\in$-induction. If $x = 0$, then $\mathbf{0}$ is the unique constant s'-term τ such that $\mathrm{Val}'\,\tau = x$, and $\mathbf{0}$ is trivially canonical.

Suppose now that x is HF and for every $y \in x$ there is a unique canonical term τ_y such that $\mathrm{Val}'\,\tau_y = y$. Let $n = |x|$, and let $y_0, \ldots, y_{n^-}$ be such that $y_0 \prec y_1 \prec \cdots \prec y_{n^-}$ and $x = \{y_m \mid m \in n\}$. For each $m \in n$, let t_m be the (unique) canonical s'-term such that $\mathrm{Val}'\, t_m = y_m$. Let

$$\begin{aligned} s_n &= \mathbf{0} \\ s_{n^-} &= s_n \mathbin{\overset{\frown}{}} t_{n^-} \\ &\vdots \\ s_1 &= s_2 \mathbin{\overset{\frown}{}} t_1 \\ s_0 &= s_1 \mathbin{\overset{\frown}{}} t_0. \end{aligned}$$

(4.53) **Claim** *For each* $k \leqslant n$*,* s_k *is canonical.*

Proof By reverse induction on k, i.e., we start with $k = n$ and work down to $k = 0$. If $k = n$ then $s_k = s_n = \mathbf{0}$, which is canonical. Suppose $k < n$ and s_{k+1} is canonical. Then

$$t^{s_k} = \langle t_k, t_{k+1}, \ldots, t_{n^-} \rangle,$$

and $\mathrm{Val}'\, t_k \prec \mathrm{Val}'\, t_{k+1} \prec \cdots \prec \mathrm{Val}'\, t_{n^-}$, so s_k is canonical iff every proper subterm of s_k is canonical. The proper subterms of s_k are s_{k+1}, t_k, and proper subterms of these, all of which are canonical by hypothesis. $\square$[4.53]

In particular, s_0 is canonical,[4.53] and $\mathrm{Val}'\, s_0 = x$, which establishes existence. To establish uniqueness, suppose τ is a canonical s'-term with $\mathrm{Val}'\,\tau = x$. Then $|t^\tau| = n$ and for each $m \in n$, $\mathrm{Val}'\, t^\tau_m = y_m$. By induction hypothesis, for each $m \in n$, $t^\tau_m = t_m$, so $\tau = s_0$. Thus, there is a unique canonical s'-term τ with $\mathrm{Val}'\,\tau = x$. $\square$[4.52]

By Theorem 4.52 the following definition is legitimate

(4.54) **Definition** $[\mathsf{S}^+]$ *Suppose* x *is an* HF *set. The* canonical name of x $\overset{\text{def}}{=}$ $\hat{x}$ $\overset{\text{def}}{=}$ $\mathrm{Nm}\, x$ $\overset{\text{def}}{=}$ *the (unique) canonical* s'*-term* τ *such that* $\mathrm{Val}'\,\tau = x$.

In this chapter it will be convenient to use the form '$\mathrm{Nm}\, x$'; later we will use the more efficient '$\hat{x}$'.

(4.55) *Let* S *be the function that assigns to each canonical* s'*-term* τ *the canonical* s'*-term* τ' *such that* $\mathrm{Val}'\,\tau'$ *is the* $\prec$*-successor of* $\mathrm{Val}'\,\tau$*, i.e.,* $\overleftarrow{B}\,\mathrm{Val}'\,\tau' = (\overleftarrow{B}\,\mathrm{Val}'\,\tau) + 1$.[3.213] S *is computable.*

Demonstration For the purpose of describing the procedure, let τ_n be the canonical term for $\vec{B}\,n$. The first few τ_ns are these:

$$
\begin{aligned}
\tau_0 &= \mathbf{0} \\
\tau_1 &= \mathbf{0} \curvearrowleft \tau_0 \\
\tau_2 &= \mathbf{0} \curvearrowleft \tau_1 \\
\tau_3 &= (\mathbf{0} \curvearrowleft \tau_1) \curvearrowleft \tau_0 \\
\tau_4 &= \mathbf{0} \curvearrowleft \tau_2 \\
\tau_5 &= (\mathbf{0} \curvearrowleft \tau_2) \curvearrowleft \tau_0 \\
\tau_6 &= (\mathbf{0} \curvearrowleft \tau_2) \curvearrowleft \tau_1 \\
\tau_7 &= ((\mathbf{0} \curvearrowleft \tau_2) \curvearrowleft \tau_1) \curvearrowleft \tau_0 \\
\tau_8 &= \mathbf{0} \curvearrowleft \tau_3 \\
\tau_9 &= (\mathbf{0} \curvearrowleft \tau_3) \curvearrowleft \tau_0 \\
\tau_{10} &= (\mathbf{0} \curvearrowleft \tau_3) \curvearrowleft \tau_1 .
\end{aligned} \tag{4.56}
$$

Observe that

$$\tau_n = (\cdots((\mathbf{0} \curvearrowleft \tau_{k_{l-1}}) \curvearrowleft \tau_{k_{l-2}}) \curvearrowleft \cdots) \curvearrowleft \tau_{k_0}$$

where $k_0 < \cdots < k_{l^-}$, and $n = \sum_{m \in l} 2^{k_m}$. In other words, $k_0, \ldots, k_{l^-}$ are the positions where '1' occurs in the binary representation of n, where positions are numbered from the right starting with 0.

Thus, an effective procedure for S may be obtained from an effective procedure for generating the next binary numeral. It is convenient for this purpose to imagine an ω-sequence of bins extending to the left (i.e., bin 0 is the rightmost bin). A bin may be full or empty. The full bins correspond to '1's in a binary numeral, the empty bins to '0'. Given a numeral in this way, the next numeral is obtained by finding the rightmost empty bin, filling it, and emptying all bins to its right.

Explicitly in terms of canonical terms, this procedure may be described as follows. As is familiar from computer programming, we start with the first line, after executing a line go to the next line unless instructed otherwise, and proceed until instructed to halt.

1. let $\zeta = \mathbf{0}$;
2. if $\tau = \mathbf{0}$ output $\mathbf{0} \curvearrowleft \zeta$ and halt;
3. let τ', ζ' be such that $\tau = \tau' \curvearrowleft \zeta'$;
4. if $\zeta' \neq \zeta$ then output $\tau \curvearrowleft \zeta$ and halt;
5. let $\tau = \tau'$ and $\zeta = S\zeta'$ and go to line 2.

Note that, like the program described by (4.48), the current program is recursive in that it calls itself (in line 5). It is easy to show by induction on the complexity of terms that it halts for any input s'-term.

Applied to canonical terms, the procedure successively removes the last-added term (i.e. removes ζ' from $\tau = \tau' \curvearrowleft \zeta'$, leaving a reduced *core* τ'), compares it first to $\mathbf{0}$ and thereafter to the canonical term for the successor of the term previously removed until it finds a gap in the sequence (an empty bin) or it has removed all the added terms and the core has been reduced to $\mathbf{0}$. At this point it adds the successor of the last-removed term to the remaining core, producing the canonical term for the $\prec$-successor of the set denoted by the term with which it began. $\square$[4.55]

(4.57) *Suppose* $X \subseteq$ HF *is* Σ_1. *Then* X *is computably enumerable, i.e., there is an effective procedure that creates a list* $x_0, x_1, \ldots,$ *of* HF *sets such that* $X = \{x_n \mid n \in \omega\}$.

Remark To put a set on the list the procedure must describe it somehow, and we will suppose that this procedure—like the previous ones—manipulates expressions and that the output of the procedure is a list of canonical s'-terms. In general, we will suppose that all effective procedures dealing with HF sets do so via their canonical names in some implementation of s'.

Demonstration To describe such a procedure, let[4.38.1] ϕ be Σ_1 with one free variable u such that $\forall_{\mathrm{HF}}\, x \left(x \in X \leftrightarrow \mathrm{Sat}_1^{\Sigma}\, \phi\left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right]\right)$. For simplicity, suppose $\phi = \exists v\ \psi$, where ψ is Δ_0.[14] Note that if $\mathrm{Val}'\,\tau = x$ and $\mathrm{Val}'\,\eta = y$, then[15]

$$\mathrm{Sat}_0\, \psi\left[\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right] \leftrightarrow \mathrm{Sat}_0'\, \psi\left(\begin{smallmatrix} u & v \\ \tau & \eta \end{smallmatrix}\right).$$

We therefore want to enumerate the canonical terms τ such that for some canonical term η, $\mathrm{Sat}_0'\, \psi\left(\begin{smallmatrix} u & v \\ \tau & \eta \end{smallmatrix}\right)$. We use the procedure described for (4.55) to generate all pairs of canonical terms in some simple order, e.g.[4.56] $\langle \tau_0, \tau_0 \rangle$, $\langle \tau_1, \tau_0 \rangle$, $\langle \tau_1, \tau_1 \rangle$, $\langle \tau_2, \tau_0 \rangle$, $\langle \tau_2, \tau_1 \rangle$, $\langle \tau_2, \tau_2 \rangle$, $\ldots$, $\langle \tau_m, \tau_n \rangle$, $\ldots$. For each of these we use the procedure for (4.47) to check whether $\mathrm{Sat}_0'\, \psi\left(\begin{smallmatrix} u & v \\ \tau_m & \tau_n \end{smallmatrix}\right)$. If this is the case we output τ_m. $\square$[4.57]

(4.58) **Definition** *Notice that we have described two sorts of effective procedures, which we will call* terminable *and* interminable.

1. *A* terminable *procedure accepts a finitary input and then either computes for a while and halts, producing a finitary output; or computes forever, never halting and never producing any output.*
2. *An* interminable *procedure accepts no input. It simply starts and never stops. From time to time it outputs a finitary object. Note that the sequence of outputs may have infinite length or finite length, possibly 0.*

Note that a terminable procedure might not terminate, i.e., halt; it is simply a member of the class of procedures that by design may halt. The output of an interminable procedure may be considered in the singular, viz., as the entire sequence finitary objects that it generates. Generally it will be evident from context which sort of procedure is meant, and we simply say 'effective procedure' or just 'procedure'.

(4.59) *Suppose* $X \subseteq$ HF *is* Δ_1. *Then* X *is computable.*

Remark By this, of course, we mean that the characteristic function of X is a computable function.

[14] Any Σ_1 formula is HF-equivalent to such a formula, since for any Δ_0 formula ψ,

$$\mathrm{Sat}_1^{\Sigma}\, \exists v_1, \ldots, v_n\ \psi\left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right] \leftrightarrow \mathrm{Sat}_1^{\Sigma}\, \exists v\, \exists v_1 \in \bar{v}, \ldots, \exists v_n \in \bar{v}\ \psi\left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right].$$

[15] Note that the expression on the left refers to an assignment of sets to variables, whereas the expression on the right refers to a substitution of terms for variables.

Demonstration Since both X and $\mathrm{HF} \setminus X$ are Σ_1, both are computably enumerable.[4.57] To ascertain whether $x \in X$, we run procedures that enumerate X and $\mathrm{HF} \setminus X$ concurrently; keeping with the notion of a computation as a linear sequence of steps, we can *dovetail* the computations, doing first a step for X, then a step for $\mathrm{HF} \setminus X$, then another step for X, etc. Eventually x (actually $\mathrm{Nm}\, x$, the canonical term for x) will appear in one of the lists, and then we have the answer to our question.□[4.59]

(4.60) **Definition**

1. *We say that* $f : \mathrm{HF} \to \mathrm{HF}$ *is* computable *iff there is an effective (terminable) procedure that for any* HF *input* x *halts with output* fx*.*
2. *We say that* $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ *is* partial computable *iff there is an effective procedure that for any input* $x \in \mathrm{HF} \setminus \mathrm{dom}\, f$ *fails to halt, and for any input* $x \in \mathrm{dom}\, f$ *halts with output* fx*.*

Note that the computable functions are exactly the partial computable functions f *with* $\mathrm{dom}\, f = \mathrm{HF}$*. For emphasis in the setting of partial computable functions, we may refer to a computable function as* total computable*.*

(4.61) *Suppose* $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ *is* Σ_1*. Then* f *is partial computable.*

Demonstration To compute fx in the sense of (4.60.2) we may computably enumerate f;[4.57] if and when this enumeration produces a pair (x, y), we output y and halt. If $x \notin \mathrm{dom}\, f$, this procedure with input x does not halt. □[4.61]

4.7.1 The Church-Turing thesis

We now assert that any model of computation that corresponds to the conventional understanding of that concept involves a finitary entity accepting a finitary *input*, undergoing a programmed sequence of changes of (finitary) state, and either *halting* with a finitary *output* or continuing forever without halting. For any given effective procedure, any completed computation (i.e., one that has halted) may be represented by an HF set, and we may reasonably state that the class of HF sets that represent completed computations is Δ_1. If the given procedure computes a function $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ in the sense (4.60.2), then $x \in \mathrm{dom}\, f \wedge fx = y$ iff there is an HF set z that represents a completed computation with input x and output y. Hence f is Σ_1.

Conversely, as we have shown,[4.61] any Σ_1 function $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ is computable in a reasonable sense, viz., the sense we have developed in (4.47), (4.55), (4.57), (4.59), and finally (4.61).

We conclude that any reasonable definition of computability is equivalent to the one we have developed here and therefore all such definitions are equivalent to one another. As noted above, for many definitions these equivalences have been rigorously proved, which lends support to the general thesis, which is known as the Church-Turing thesis*.*

It is important to note that we do not *use* the Church-Turing thesis. Any time we claim that a function f is Σ_1 because it is computable, it will be a straightforward exercise to exhibit a Σ_1 definition of f. To eliminate any possibility of misunderstanding we now simply *define* computable to mean Σ_1.

Definition $[\mathsf{C}^+]$ $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ *is* computable $\overset{\text{def}}{\iff}$ f *is* Σ_1.

Of course, this essentially renders the Church-Turing thesis tautological, which is the intention.

4.7.2 Undecidability of predicate logic

We now address the question that was our original motivation for the development of the theory of computability: that of the decidability of predicate logic.

Definition $[\mathsf{C}^+]$ *Let* R *be the fragment of* S *consisting of the following sentences:*

$$\forall \mathrm{v}_0, \mathrm{v}_1\, (\forall \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \in \mathrm{v}_1) \rightarrow \mathrm{v}_0 = \mathrm{v}_1)$$
$$\exists \mathrm{v}_0\, \forall \mathrm{v}_1\;\; \neg\, \mathrm{v}_1 \in \mathrm{v}_0$$
$$\forall \mathrm{v}_0, \mathrm{v}_1\, \exists \mathrm{v}_2\, \forall \mathrm{v}_3\, (\mathrm{v}_3 \in \mathrm{v}_2 \leftrightarrow \mathrm{v}_3 \in \mathrm{v}_0 \vee \mathrm{v}_3 = \mathrm{v}_1).$$

Recall[4.42] that for $x \in \mathrm{HF}$, θ_x is a Δ_0 formula with free variable v_0 such that for all $y \in \mathrm{HF}$, $\mathrm{Sat}_0\, \theta_x\big[{\mathrm{v}_0 \atop y}\big] \leftrightarrow y = x$. It is easy to show by $\in$-induction on x that for all $x \in \mathrm{HF}$

$$\mathsf{R} \vdash \exists!\mathrm{v}_0\;\; \theta_x.$$

Suppose ψ is Δ_0, $\langle w_0, \ldots, w_{n^-}\rangle$ enumerates $\mathrm{Free}\, \psi$, $x_0, \ldots, x_{n^-} \in \mathrm{HF}$, and $u_0, \ldots, u_{n^-}$ are distinct variables not occurring in ψ or in θ_{x_m} for any $m \in n$. It is straightforward to show by induction on the complexity of ψ that if

$$\mathrm{Sat}_0\, \psi\big[{w_0 \cdots w_{n^-} \atop x_0 \cdots x_{n^-}}\big]$$

then

$$\mathsf{R} \vdash \theta_{x_0}(u_0) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \psi\big({w_0 \cdots w_{n^-} \atop u_0 \cdots u_{n^-}}\big).$$

Suppose $l \leqslant n$ and let $\phi = \exists w_0, \ldots, w_{l^-}\;\psi$. Then for any $x_l, \ldots, x_{n^-} \in \mathrm{HF}$, if

$$\mathrm{Sat}_1^{\Sigma}\, \phi\big[{w_l \cdots w_{n^-} \atop x_l \cdots x_{n^-}}\big]$$

then there exist $x_0, \ldots, x_{l^-} \in \mathrm{HF}$ such that

$$\mathrm{Sat}_0\, \psi\big[{w_0 \cdots w_{n^-} \atop x_0 \cdots x_{n^-}}\big],$$

whence

$$\mathsf{R} \vdash \theta_{x_0}(u_0) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \psi\big({w_0 \cdots w_{n^-} \atop u_0 \cdots u_{n^-}}\big);$$

hence, since $\mathsf{R} \vdash \exists \mathrm{v}_0\;\theta_{x_m}$ for each $m \in l$,

$$\mathsf{R} \vdash \theta_{x_l}(u_l) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \exists w_0, \ldots, w_{l^-}\;\; \psi\big({w_l \cdots w_{n^-} \atop u_l \cdots u_{n^-}}\big),$$

i.e.,

$$\mathsf{R} \vdash \theta_{x_l}(u_l) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \phi\big({w_l \cdots w_{n^-} \atop u_l \cdots u_{n^-}}\big).$$

In summary:

(4.62) **Theorem** $[\mathsf{C}^+]$ *If* ϕ *is* Σ_1, $\langle v_0, \ldots, v_{n^-}\rangle$ *enumerates* $\mathrm{Free}\,\phi$, $x_0, \ldots, x_{n^-} \in \mathrm{HF}$, *and* $u_0, \ldots, u_{n^-}$ *are distinct variables not occurring in* ϕ *or in* θ_{x_m} *for any* $m \in n$ *then*

$$\mathrm{Sat}_1^{\Sigma}\, \phi\big[{v_0 \cdots v_{n^-} \atop x_0 \cdots x_{n^-}}\big] \rightarrow \mathsf{R} \vdash \theta_{x_0}(u_0) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \phi\big({v_0 \cdots v_{n^-} \atop u_0 \cdots u_{n^-}}\big).$$

In this sense, R proves every true Σ_1 (true in the sense of Sat, i.e., true in the finitary interpretation). Now suppose that—in the same sense—R does not prove any false Σ_1:

(4.63) *If ϕ is Σ_1, $\langle v_0, \ldots, v_{n^-}\rangle$ enumerates* $\operatorname{Free}\phi$, *$x_0, \ldots, x_{n^-} \in \mathrm{HF}$, and $u_0, \ldots, u_{n^-}$ are distinct variables not occurring in ϕ or in θ_{x_m} for any $m \in n$ then*

$$\neg\, \mathrm{Sat}_1^{\Sigma}\, \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ x_0 & \cdots & x_{n^-}\end{smallmatrix}\big] \rightarrow \mathsf{R} \nvdash \theta_{x_0}(u_0) \wedge \cdots \wedge \theta_{x_{n^-}}(u_{n^-}) \rightarrow \phi\big(\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ u_0 & \cdots & u_{n^-}\end{smallmatrix}\big).$$

(4.64) *This sort of condition is referred to generically as* ω-consistency.[16]

(4.63) clearly follows from the existence of the satisfaction relation for $(\mathrm{HF}; \in)$, because in this case,

$$(\mathrm{HF}; \in) \models \mathsf{R},$$

i.e., R is true, so

$$(\mathrm{HF}; \in) \models \sigma$$

for every σ such that $\mathsf{R} \vdash \sigma$.

(4.65) **Theorem** $[\mathsf{C}^+]$ *Suppose (4.63). Then first-order predicate logic is undecidable.*

Proof Specifically, we will show that the class of s-validities is not recursive. Suppose toward a contradiction that it is recursive. We will show that every Σ_1 class is Δ_1.

For suppose $X \subseteq \mathrm{HF}$ is Σ_1. Let ϕ be a Σ_1 formula with $\operatorname{Free}\phi = \{\mathrm{v}_0\}$ such that $X = \big\{x \in \mathrm{HF} \;\big|\; \mathrm{Sat}_1^{\Sigma}\, \phi\big[\begin{smallmatrix}\mathrm{v}_0 \\ x\end{smallmatrix}\big]\big\}$. Let ρ be a conjunction of the (three) axioms of R. Let $x \in \mathrm{HF}$ be given. If $x \in X$ then[4.62]

$$\vdash \rho \wedge \theta_x \rightarrow \phi,$$

whereas if $x \notin X$ then[4.63]

$$\nvdash \rho \wedge \theta_x \rightarrow \phi.$$

We apply the putative decision procedure for $\vdash$ to ascertain which of these is the case, and then we know whether $x \in X$. X is therefore decidable, i.e., Δ_1.

Since we know[4.46] that not every Σ_1 is Δ_1, this is a contradiction. □[4.65]

A few remarks are in order. We have shown that predicate logic with identity and one other binary predicate is undecidable. Since logic with identity is a conservative extension of logic without, we could modify the proof to work for logic with one binary predicate, without identity. Since our proof of the conservative extension result was finitary (carried out in C^+) we can simply invoke this result and still have a proof of the undecidability result from C^+ plus the assumption of ω-consistency of (the theory corresponding to) R.

It is worth noting that propositional logic and predicate logic with only unary predicates (and no operations) are decidable.

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[16] In the context of theories of arithmetic, with the intended interpretation $(\omega; +, \cdot)$, a theory T is ω-consistent iff it is not the case that there is a formula ϕ with $\operatorname{Free}\phi = \{u\}$, such that $\mathsf{T} \vdash \exists u\ \phi$, but for every $n \in \omega$, $\mathsf{T} \vdash \neg\, \phi\big(\begin{smallmatrix} u \\ \tau_n\end{smallmatrix}\big)$, where τ_n denotes n in the intended interpretation. 'HF-consistency' would be more appropriate than 'ω-consistency' in our setting, but the essential idea is the same. Note that ω-consistency trivially implies consistency, because if T is inconsistent then T proves everything. The converse need not hold.

As to the assumption of ω-consistency of R, there really can be no serious doubt as to its correctness; as noted above, it follows from the existence of a satisfaction relation for (HF; $\in$) (which in turn follows from the Infinity axiom). Our purpose here is not to obtain the undecidability result with minimal assumptions, but simply to obtain the undecidability result (as a significant discovery in the science of computation—if you will). Surely no one who understands the above argument will be tempted to try to design a decision procedure for predicate logic.

4.8 Recursive functions

We regard the discussion of the preceding section as establishing the foundation of the science of computation, and from now on we may refer to the *theory* of computation, which is the theory of $\Sigma_1(\mathrm{HF})$.

We are primarily interested in using this theory to address metamathematical issues such as decidability of theories, but its value extends well beyond this. It is also a rich and beautiful theory in its own right, and we will provide a glimpse of this.

The reader will have noticed that many of the procedures we have described have been recursive (self-calling). The science of computability may be developed with the notion of algorithmic recursion having a central role, and 'recursive' is often used synonymously with 'computable'.

The theory of computation is therefore also referred to as the theory of recursive functions*.*

Although the terms are synonymous, to emphasize the change in point of view, we will use 'recursive' preferentially from this point on. Thus, 'computable' is for us an intuitive concept, for which we have succeeded in giving a precise mathematical definition; whereas 'recursive' is a term introduced into the discussion by mathematical definition.

The following definition contains no essentially new ideas, but it reflects the conventional terminology as just described.

(4.66) **Definition** $[\mathsf{C}^+]$

1. *Suppose* $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$.
   1. f *is* partial recursive $\overset{\text{def}}{\iff} f$ *is* p.r. $\overset{\text{def}}{\iff} f$ *is* Σ_1.[4.60.2]
   2. f *is* total recursive *or simply* recursive $\overset{\text{def}}{\iff} f$ *is* rec $\overset{\text{def}}{\iff} f$ *is* p.r. *and* $\operatorname{dom} f = \mathrm{HF}$.[4.60.1]
2. *Suppose* $X \subseteq \mathrm{HF}$.
   1. X *is* semirecursive $\overset{\text{def}}{\iff} X$ *is* recursively enumerable $\overset{\text{def}}{\iff} X$ *is* r.e. $\overset{\text{def}}{\iff} X$ *is* Σ_1.[4.57]
   2. X *is* recursive $\overset{\text{def}}{\iff} X$ *is* Δ_1.[4.59]

We have used HF as a convenient domain over which to develop the science of computation. Using the Δ_1 bijection $\vec{B} : \omega \overset{\text{bij}}{\to} \mathrm{HF}$, all the relevant notions may also be defined over the domain of natural numbers, i.e., in terms of subsets of ω and functions from ω to ω. This is often conceptually advantageous, and it is how the theory is usually presented, often in conjunction with other convenient domains, principally ${}^n\omega$ ($n \in \omega$). The various domains used for this purpose are always infinite Δ_1 subsets of HF. When we thus restrict our attention to functions with domain included in a set D, the partial recursive functions f are those such that f is Σ_1 and $f : D \rightharpoonup \mathrm{HF}$; such a function f is (total) recursive iff $\operatorname{dom} f = D$.

In this connection the following theorem is relevant.

(4.67) **Theorem** [C^+] *Suppose* $f : \mathrm{HF} \rightharpoonup \mathrm{HF}$ *is* Σ_1. *Then* $\operatorname{dom} f$ *is* Σ_1. *If* $\operatorname{dom} f$ *is* Δ_1 *then* f *is* Δ_1.

Proof Straightforward. $\square$[4.67]

In light of this theorem, and in keeping with the preceding discussion, we will often refer to any partial recursive function with a recursive domain as recursive (on its domain).

4.8.1 Σ_1-uniformization

Definition [C^+] *Suppose* R *is a prefunction*[3.25] *and* F *is a function.* F uniformizes $R \overset{\text{def}}{\iff} F \subseteq R$ *and* $\operatorname{dom} F = \operatorname{dom} R$.

Note that F is a choice function for the function $G = \{(x, R^{\to}\{x\}) \mid x \in \operatorname{dom} R\}$, assuming $R^{\to}\{x\}$ is a set for every $x \in \operatorname{dom} R$.

A key property of Σ_1 is that any Σ_1 prefunction on HF is uniformized by some Σ_1 function. We refer to this as Σ_1-*uniformization*, and we say that Σ_1 has the *uniformization property*.

If we wanted to define a uniformizing function for R and did not care about its complexity, we could use $H = \{(x, y) \in R \mid \forall y' \prec y\ (x, y') \notin R\}$, but this is not in general Σ_1 for a Σ_1 prefunction R because the complement of R is not in general Σ_1. So instead of using the $\prec$-least y such that $(x, y) \in R$, we use the $\prec$-least pair $\langle y, z\rangle$ such that z witnesses that $(x, y) \in R$.

(4.68) **Theorem** [C^+] *Suppose* R *is a* Σ_1 *prefunction on* HF. *Then there is a* Σ_1 *function* F *that uniformizes* R.

Proof Since R is Σ_1 there is a Δ_0 class $S \subseteq {}^3\mathrm{HF}$ such that

$$R = \{(x, y) \mid \exists_{\mathrm{HF}}\, z\ \langle x, y, z\rangle \in S\}.$$

Let F be the class of ordered pairs (x, y) such that there exists $s \in {}^2\mathrm{HF}$ such that

1. $\langle x, s_0, s_1\rangle \in S$;
2. for every HF 2-sequence $s' \prec s$, $\langle x, s'_0, s'_1\rangle \notin S$; and
3. $y = s_0$.

Clearly F uniformizes R. To show that F is Σ_1, we just have to show that Clause 2 in its definition may be expressed in a Σ_1 way. To do this, note that $\ulcorner \forall s' \prec s \ldots \urcorner$ is equivalent to $\ulcorner \exists X, n\,(\operatorname{Ord} n \wedge X = V_n \wedge s \in X \wedge \forall s' \in X\,(s' \prec s \rightarrow \ldots))\urcorner$.[3.212.4 4.29.2]
$\square$[4.68]

4.8.2 Complete and universal sets

(4.69) **Theorem** [C^+] *Suppose* $f : \mathrm{HF} \to \mathrm{HF}$ *is recursive and* $A \subseteq \mathrm{HF}$. *If* A *is* Σ_1 *or* Π_1 *then* $f^{\leftarrow}A$ *is respectively* Σ_1 *or* Π_1.

Proof Straightforward. $\square$[4.69]

(4.70) **Definition** [C^+] *A set* $A \subseteq \mathrm{HF}$ *is a* complete Σ_1 (Π_1) set $\overset{\text{def}}{\Longleftrightarrow}$ *A is* Σ_1 (Π_1) *and for any* Σ_1 (Π_1) *set* $B \subseteq \mathrm{HF}$ *there exists a recursive* $f : \mathrm{HF} \to \mathrm{HF}$ *such that* $B = f^{\leftarrow}A$.

Obviously, if A is a complete Σ_1 set then $\neg A$ is a complete Π_1 set, and *vice versa*. The question of the existence of complete sets is of interest.

(4.44) may be viewed as a special case of (4.69), since $b \mapsto \langle a, b\rangle$ is a recursive function for any fixed $a \in \mathrm{HF}$.[17] The following definition is to (4.44) as (4.70) is to (4.69).

Definition [C^+] $B \subseteq \mathrm{HF} \times \mathrm{HF}$ *is a* universal Σ_1 set $\overset{\text{def}}{\Longleftrightarrow}$

1. B *is* Σ_1; *and*
2. *for every* Σ_1 $A \subseteq \mathrm{HF}$, $\exists_{\mathrm{HF}}\, a\ \forall_{\mathrm{HF}}\, b\, (b \in A \leftrightarrow \langle a, b\rangle \in B)$.

Clearly, a universal set is a special sort of complete set. We now settle the question of existence.

(4.71) *Let* $U \overset{\text{def}}{=}$ *the set of* $\langle a, b\rangle \in \mathrm{HF} \times \mathrm{HF}$ *such that*

1. a *is a* Σ_1 *formula and* $\mathrm{Free}\, a = \{\mathrm{v}_0\}$; *and*
2. $\mathrm{Sat}_1^{\Sigma}\, a\left[{\mathrm{v}_0 \atop b}\right]$.

(4.72) **Theorem** [C^+] *U is universal* Σ_1 *set.*

Proof It is straightforward to verify that U is Σ_1; and it is clearly universal, as any Σ_1 set $A \subseteq \mathrm{HF}$ is defined by some Σ_1 formula with the single free variable v_0. $\square$[4.72]

If we let $U_a = \{b \mid \langle a, b\rangle \in U\}$ *then* a *may be regarded as an index of the* Σ_1 *set* U_a, *specifically,* a *is said to be a* Turing index.

The corresponding indexing of Σ_1 subsets of ω, i.e., r.e. sets, is traditionally represented by 'W'. Thus,

(4.73) **Definition** [C^+] *For* $n \in \omega$, $W_n \overset{\text{def}}{=} \{m \in \omega \mid \langle\, \vec{B}\, n,\, \vec{B}\, m\rangle \in U\}$.

[17] That $b \mapsto \langle a, b\rangle$ is recursive seems so obvious as hardly to require proof, but we should recognize that in stating this so casually we are actually invoking the Church-Turing thesis. It is a good exercise to construct a formal proof, essentially recapitulating the proof of (4.44)—nothing is free.

4.8.3 A universal partial recursive function

To name Σ_1 functions in a similarly effective way is not as straightforward, because the class of Σ_1 formulas ϕ with two free variables, say v_0 and v_1, that define a function, i.e., $\forall x \in \mathrm{HF}\ \exists! y \in \mathrm{HF}\ \mathrm{Sat}_1^\Sigma\, \phi\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ x & y\end{smallmatrix}\bigr]$, is not Σ_1.[18] Recall, however, the Σ_1-uniformization principle,[4.68] which states that every Σ_1 prefunction R is uniformized by a Σ_1 function F. A glance at the proof of Theorem 4.68 makes it clear that a Σ_1 formula for F may be obtained computably from a Σ_1 formula for R.

The following definition gives a specific way of doing this modeled on the proof of Theorem 4.68.

(4.74) **Definition** [C^+]

1. *Let* $\Phi \overset{\text{def}}{=}$ *the set of* $(\langle a, b\rangle, c) \in \mathrm{HF}$ *such that*
   1. a *is a* Δ_0 *formula with* $\mathrm{Free}\, a = \{\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\}$*; and*
   2. *there exists* $d \in \mathrm{HF}$ *such that*
      1. $\mathrm{Sat}_0\, a\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ b & c & d\end{smallmatrix}\bigr]$*; and*
      2. *for every* $\langle c', d'\rangle \in \mathrm{HF} \times \mathrm{HF}$*, if* $\langle c', d'\rangle \prec \langle c, d\rangle$ *then* $\neg\, \mathrm{Sat}_0\, a\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ b & c' & d'\end{smallmatrix}\bigr]$.
2. *Suppose* $a \in \mathrm{HF}$*. The* p.r. function with index $a \overset{\text{def}}{=} \varphi_a \overset{\text{def}}{=} \{(b, c) \mid (\langle a, b\rangle, c) \in \Phi\}$.

As in the proof of (4.68), Φ is Σ_1, and $\Phi : \mathrm{HF} \times \mathrm{HF} \rightharpoonup \mathrm{HF}$. It is therefore a p.r. function. Each φ_a is likewise partial recursive.

To see that Φ is in fact universal, suppose $\varphi : \mathrm{HF} \rightharpoonup \mathrm{HF}$ is partial recursive. Let a be a Δ_0 formula with $\mathrm{Free}\, a = \{\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\}$ such that for all $b, c \in \mathrm{HF}$, $(b, c) \in \varphi$ iff

$$\exists_{\mathrm{HF}}\, d\ \ \mathrm{Sat}_0\, a\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ b & c & d\end{smallmatrix}\bigr]. \tag{4.75}$$

φ is a function, so for all $b \in \mathrm{HF}$ there exists at most one $c \in \mathrm{HF}$ such that (4.75) holds, from which it follows that for all $b, c \in \mathrm{HF}$, $(b, c) \in \varphi$ iff $(\langle a, b\rangle, c) \in \Phi$.

As above, a is said to be a *Turing index* for φ_a. Also as discussed above, a universal p.r. function Φ and corresponding indexed functions φ_a are usually defined primarily over the domain ω and secondarily over the domains ${}^n\omega$; and Turing indices are also taken to be numbers.

The following definition expresses an important feature of the universal objects we have defined above.

(4.76) **Definition** [C^+]

1. $U \subseteq \mathrm{HF} \times \mathrm{HF}$ *is a* good universal Σ_1 set $\overset{\text{def}}{\Longleftrightarrow}$
   1. U *is a universal* Σ_1 *set; and*
   2. *there is a recursive* $s : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ *such that for all* $a, b, c \in \mathrm{HF}$

$$\langle a, \langle b, c\rangle\rangle \in U \leftrightarrow \langle s\langle a, b\rangle, c\rangle \in U.$$

[18]It is easily seen to be Π_1: $\forall_{\mathrm{HF}}\, x, y, y'\ \bigl(\neg\, \mathrm{Sat}_1^\Sigma\, \phi\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ x & y\end{smallmatrix}\bigr] \vee \neg\, \mathrm{Sat}_1^\Sigma\, \phi\bigl[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ x & y'\end{smallmatrix}\bigr] \vee y = y'\bigr)$. We will soon have the tools to show that it is not Σ_1.

2. $\Phi : \mathrm{HF} \times \mathrm{HF} \rightharpoonup \mathrm{HF}$ *is a* good universal p.r. function $\overset{\text{def}}{\iff}$
   1. Φ *is a universal* p.r. *function; and*
   2. *there is a recursive* $s : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ *such that for all* $a, b, c, d \in \mathrm{HF}$

$$(\langle a, \langle b, c\rangle\rangle, d) \in \Phi \leftrightarrow (\langle s\langle a, b\rangle, c\rangle, d) \in \Phi.$$

The common name of the following theorem derives from the particular statement of it in Kleene's original presentation, where, *mutatis mutandis*, $s^m_n : {}^{m+1}\mathrm{HF} \to \mathrm{HF}$ is recursive such that for all $e, a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathrm{HF}$

$$\langle e, a_1, \ldots, a_m, b_1, \ldots, b_n\rangle \in U^{m+n} \leftrightarrow \langle s^m_n\langle e, a_1, \ldots, a_m\rangle, b_1, \ldots, b_n\rangle \in U^n,$$

U^{m+n} and U^n being universal for Σ_1 subsets of ${}^{m+n}\omega$ and ${}^n\omega$, respectively, in the obvious sense.

(4.77) **Theorem: s-m-n** $[\mathsf{C}^+]$

1. *The set* U *defined above*[4.71] *is a good universal* Σ_1 *set.*
2. *The function* Φ[4.74] *is a good universal* p.r. *function.*

Proof 1 Since U is Σ_1, $\{\langle a, b, c\rangle \mid \langle a, \langle b, c\rangle\rangle \in U\}$ is Σ_1. Let θ be a Σ_1 formula with free variables $\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2$ such that for all $a, b, c \in \mathrm{HF}$,

$$\langle a, \langle b, c\rangle\rangle \in U \leftrightarrow \mathrm{Sat}^\Sigma_1\, \theta\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ c & a & b\end{smallmatrix}\big].$$

Recall the definition[4.42.3] of the Δ_0 formula θ_x that defines $\{x\}$, and recall[4.45] that $x \mapsto \theta_x$ is recursive. Define $s : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ so that for all $a, b \in \mathrm{HF}$

$$s\langle a, b\rangle = \exists u_1\, \exists u_2\, \big(\theta_a(\bar{u}_1) \wedge \theta_b(\bar{u}_2) \wedge \theta\big(\begin{smallmatrix}\mathrm{v}_1 & \mathrm{v}_2\\ \bar{u}_1 & \bar{u}_2\end{smallmatrix}\big)\big),$$

where u_1 and u_2 are the first two variables that do not occur in θ_a, θ_b or θ. Then s is recursive, and for all $a, b, c \in \mathrm{HF}$

$$\begin{aligned}\langle a, \langle b, c\rangle\rangle \in U &\leftrightarrow \mathrm{Sat}^\Sigma_1\, \theta\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ c & a & b\end{smallmatrix}\big] \leftrightarrow \mathrm{Sat}^\Sigma_1\, s\langle a, b\rangle\big[\begin{smallmatrix}\mathrm{v}_0\\ c\end{smallmatrix}\big]\\ &\leftrightarrow \langle s\langle a, b\rangle, c\rangle \in U.\end{aligned}$$

2 Since Φ is Σ_1, $\{\langle a, b, c, d\rangle \mid (\langle a, \langle b, c\rangle\rangle, d) \in \Phi\}$ is Σ_1. Let θ be a Δ_0 formula with free variables $\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_5$ such that for all $a, b, c, d \in \mathrm{HF}$,

$$(\langle a, \langle b, c\rangle\rangle, d) \in \Phi \leftrightarrow \exists_{\mathrm{HF}} f\ \ \mathrm{Sat}_0\, \theta\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_3 & \mathrm{v}_4 & \mathrm{v}_5\\ c & d & a & b & f\end{smallmatrix}\big].$$

Let

$$\theta' = \bar{\mathrm{v}}_3 \in \bar{\mathrm{v}}_2 \wedge \bar{\mathrm{v}}_4 \in \bar{\mathrm{v}}_2 \wedge \exists \mathrm{v}_5 \in \bar{\mathrm{v}}_2\ \theta.$$

Then for all $a, b, c, d \in \mathrm{HF}$,

$$(\langle a, \langle b, c\rangle\rangle, d) \in \Phi \leftrightarrow \exists_{\mathrm{HF}} e\ \ \mathrm{Sat}_0\, \theta'\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 & \mathrm{v}_4\\ c & d & e & a & b\end{smallmatrix}\big].$$

Since Φ is a function, $(\langle a, \langle b, c\rangle\rangle, d) \in \Phi$ iff there exists $e \in \mathrm{HF}$ such that $\mathrm{Sat}_0\, \theta'\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 & \mathrm{v}_4\\ c & d & e & a & b\end{smallmatrix}\big]$ and $\langle d, e\rangle$ is the $\prec$-least pair with this property.

Now define $s : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ so that for all $a, b \in \mathrm{HF}$

$$s\langle a, b\rangle = \exists u_0 \in \bar{\mathrm{v}}_2\, \exists u_1 \in \bar{\mathrm{v}}_2\, \big(\theta_a(\bar{u}_0) \wedge \theta_b(\bar{u}_1) \wedge \theta'\big(\begin{smallmatrix}\mathrm{v}_3 & \mathrm{v}_4\\ \bar{u}_0 & \bar{u}_1\end{smallmatrix}\big)\big),$$

where u_0 and u_1 are the first two variables that do not occur in θ_a, θ_b or θ'. Then s is recursive, and for all $a, b, c, d, e \in \mathrm{HF}$

$$\mathrm{Sat}_0\, \theta'\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 & \mathrm{v}_4\\ c & d & e & a & b\end{smallmatrix}\big] \leftrightarrow \mathrm{Sat}_0\, s\langle a, b\rangle\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ c & d & e\end{smallmatrix}\big]$$

It follows that for any $a, b, c, d \in \mathrm{HF}$, $(\langle a, \langle b, c\rangle\rangle, d) \in \Phi$ iff there exists $e \in \mathrm{HF}$ such that $\mathrm{Sat}_0\, s\langle a, b\rangle\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ c & d & e\end{smallmatrix}\big]$ and $\langle d, e\rangle$ is the $\prec$-least pair with this property.

By the definition of Φ, therefore, $(\langle a, \langle b, c\rangle\rangle, d) \in \Phi$ iff $(\langle s\langle a, b\rangle, c\rangle, d) \in \Phi$, as desired. $\square$ 4.77

The following corollary of the s-m-n theorem shows its versatility.

Theorem $[\mathsf{C}^+]$ *Suppose $U \subseteq \mathrm{HF} \times \mathrm{HF}$ is a good universal Σ_1 set. Then U is* effectively universal *in the sense that there is a recursive $t : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ such that*

$$\forall_{\mathrm{HF}}\, a, b, c\, \big(\langle b, \varphi_a c\rangle \in U \leftrightarrow \langle t\langle a, b\rangle, c\rangle \in U\big).$$

Remark In other words, $\varphi_a{}^{\leftarrow} U_b = U_{t\langle a,b\rangle}$. Thus, the operation of forming a recursive preimage $\varphi^{\leftarrow} A$ of a Σ_1 set A may be represented effectively (i.e., recursively) in terms of U-codes (Turing indices). The reason to state this in terms of recursive preimages is that this is a very general way of deriving one Σ_1 set from another. Φ is similarly effectively universal.

Proof For all $a, b, c \in \mathrm{HF}$,

$$\begin{aligned}\langle b, \varphi_a c\rangle \in U &\leftrightarrow \exists_{\mathrm{HF}}\, d\, \big((c, d) \in \varphi_a \wedge (b, d) \in U\big)\\ &\leftrightarrow \exists_{\mathrm{HF}}\, d\, \big((\langle a, c\rangle, d) \in \Phi \wedge (b, d) \in U\big)\end{aligned}$$

Thus, $\{\langle\langle b, a\rangle, c\rangle \mid \langle b, \varphi_a c\rangle \in U\}$ is Σ_1, so it is U_d for some d, and for all $a, b, c \in \mathrm{HF}$

$$\langle d, \langle\langle b, a\rangle, c\rangle\rangle \in U \leftrightarrow \langle b, \varphi_a c\rangle \in U.$$

Let $t : \mathrm{HF} \times \mathrm{HF} \to \mathrm{HF}$ be such that

$$t\langle a, b\rangle = s\langle d, \langle b, a\rangle\rangle,$$

with s as in (4.76.2). t is Σ_1 by virtue of (4.44) and the subsequent remarks. As it is a total function, it is recursive. Then

$$\langle b, \varphi_a c\rangle \in U \leftrightarrow \langle d, \langle\langle b, a\rangle, c\rangle\rangle \in U \leftrightarrow \langle s\langle d, \langle b, a\rangle\rangle, c\rangle \in U \leftrightarrow \langle t\langle a, b\rangle, c\rangle \in U,$$

as required. $\square$ 4.8.3

4.9 The unsolvability of the halting problem

A major issue in the early development of the theory of computability was the problem of determining whether a given effective procedure with a given input halts; in particular, the question is whether there is an effective procedure that will make this determination—i.e., is the so-called *halting problem* solvable, or *decidable*?

In our terminology, this becomes:

Let $H = \{\langle a, b\rangle \mid b \in \operatorname{dom} \varphi_a\}$. *Is* H Δ_1?

H is easily seen to be Σ_1, so the question is whether H is Π_1. If it is, then for every $a \in \mathrm{HF}$, $\operatorname{dom} \varphi_a$ is Π_1. But every Σ_1 set X is the domain of a p.r. function, e.g., $\{(a, 0) \mid a \in X\}$, so every Σ_1 set would be Π_1. This we have shown not to be the case.[4.46] Thus, we have proved the following theorem.

(4.78) **Theorem** [C^+] *The halting problem is unsolvable.*

4.10 The recursion theorem

If an effective procedure $\mathcal{P}$ calls an effective procedure $\mathcal{P}'$, then a Turing index for $\mathcal{P}$ is computable from a Turing index for $\mathcal{P}'$, i.e., there is a total recursive function f such that for any $a \in \mathrm{HF}$, if $\mathcal{P}' = \varphi_a$ then $\mathcal{P} = \varphi_{f(a)}$. In the special case that $\mathcal{P}'$ and $\mathcal{P}$ are the same procedure—i.e., $\mathcal{P}$ calls itself—and n is a Turing index for $\mathcal{P}$, then $\varphi_n = \varphi_{f(n)}$. In this case, the nature of f—which is in effect a description of $\mathcal{P}$ with an indication of where $\mathcal{P}'$ is to be inserted—makes it obvious that there exists n such that $\varphi_n = \varphi_{f(n)}$. What is not so obvious is that a *fixed point* in this sense exists for every total recursive f. The statement that it does is known as the *recursion theorem* of Kleene.

(4.79) **Theorem** [C^+] *Suppose* $f : \mathrm{HF} \to \mathrm{HF}$ *is recursive. Then there exists* $n \in \mathrm{HF}$ *such that* $\varphi_{f(n)} = \varphi_n$.

Proof Given $a \in \mathrm{HF}$, let $g_a : \mathrm{HF} \rightharpoonup \mathrm{HF}$ be the p.r. function with the following description: To compute $g_a m$, first compute $\varphi_a a$. If and when this computation halts, say with output b, let $c = f(b)$. Then compute $\varphi_c m$. If and when this computation halts, $g_a m$ is the output.

Note that if $a \notin \operatorname{dom} \varphi_a$ then $g_a = 0$, the empty function. If $a \in \operatorname{dom} \varphi_a$ then $g_a = \varphi_{f(\varphi_a a)}$. Either way, g_a is a p.r. function, and a Turing index for g_a is easily obtained from a. Let h be a total recursive function such that for any $a \in \mathrm{HF}$, $h(a)$ is a Turing index for g_a. Let b be a Turing index for h. Then

$$\varphi_{\varphi_b a} = \begin{cases} \varphi_{f(\varphi_a a)} & \text{if } a \in \operatorname{dom} \varphi_a \\ 0 & \text{otherwise.} \end{cases}$$

In particular, letting $a = b$, since $\varphi_b = h$ is total,

$$\varphi_{\varphi_b b} = \varphi_{f(\varphi_b b)}.$$

Let $n = \varphi_b b$. Then $\varphi_{f(n)} = \varphi_n$, as desired.

The above argument is typical in its implicit invocation of the Church-Turing thesis. More formally, we may proceed as follows. The function $\langle a, m\rangle \mapsto \varphi_{f(\varphi_a a)} m$ is clearly Σ_1. Let e be a Turing index for it, so that $\varphi_e\langle a, m\rangle = \varphi_{f(\varphi_a a)} m$. Using the recursive function s from (4.76.2.2), $\varphi_{s\langle e,a\rangle} m = \varphi_{f(\varphi_a a)} m$. Let b be such that $\varphi_b a = s\langle e, a\rangle$ for any a (e being fixed).[4.44] Then $\varphi_{\varphi_b a} m = \varphi_{f(\varphi_a a)} m$. Letting $n = \varphi_b b$, $\varphi_n m = \varphi_{f(n)} m$. □[4.79]

We now state the recursion theorem in a form more convenient for later generalization.

(4.80) **Theorem** [C^+] *Suppose* $A \subseteq \mathrm{HF} \times \mathrm{HF}$ *is* Σ_1. *Then there exists* $a \in HF$ *such that* $A_a = U_a$.

Proof Let s be as in (4.76.1.2). $\{\langle b,c\rangle \mid \langle s\langle b,b\rangle,c\rangle \in A\}$ is Σ_1, so there exists $d \in \mathrm{HF}$ such that for all $b,c \in \mathrm{HF}$

$$\langle d,\langle b,c\rangle\rangle \in U \leftrightarrow \langle s\langle b,b\rangle,c\rangle \in A.$$

Let $a = s\langle d,d\rangle$. Then for all $c \in \mathrm{HF}$

$$\begin{aligned}\langle a,c\rangle \in U &\leftrightarrow \langle s\langle d,d\rangle,c\rangle \in U \leftrightarrow \langle d,\langle d,c\rangle\rangle \in U\\ &\leftrightarrow \langle s\langle d,d\rangle,c\rangle \in A\\ &\leftrightarrow \langle a,c\rangle \in A,\end{aligned}$$

i.e., $A_a = U_a$. $\square$[4.80]

4.11 Consistency and incompleteness

We have previously shown[3.217] in C, on the hypothesis that $(\mathrm{HF};\in)$ is weakly satisfactory, that S is semantically incomplete, i.e., there is an s-sentence σ such that $(\mathrm{HF};\in) \models \sigma$ but $\mathsf{S} \nvdash \sigma$. The proof was indirect, in that it did not specify any such sentence: It simply argued that since $(\mathrm{HF};\in) \models \mathsf{S}$, no $(\mathrm{HF};\in)$-false sentence is S-provable, so if every $(\mathrm{HF};\in)$-true sentence is S-provable then $(\mathrm{HF};\in)$-truth is equivalent to S-provability; since S-provability is definable over $(\mathrm{HF};\in)$, this implies that $(\mathrm{HF};\in)$-satisfaction is definable over $(\mathrm{HF};\in)$, which we know[1.73] is not the case.

In this section we present a direct argument that a specific sentence σ is true but not provable. σ is designed to say that σ is not S-provable. To show that σ is not S-provable, it suffices to assume that S is consistent. To express that σ is $(\mathrm{HF};\in)$-true, we only need a $\{\sigma\}$-satisfaction relation for $(\mathrm{HF};\in)$, and this is achievable in C. We therefore have the semantic incompleteness of S derived in C from the consistency of S, a weaker hypothesis than that of (3.217).

Note that the expressive power of C is only required to formulate the statement that σ is true. Considerations of provability may be discussed in S. Since C is a conservative extension of S, part of what we have done in the proof sketched in the preceding paragraph is to prove in S that if S is consistent then σ is not S-provable. Since σ "says" that σ is not S-provable, we have in fact proved σ itself from the same hypothesis, i.e., the consistency of S. It follows that if S proves the consistency of S, then S proves σ. But, as described in the preceding paragraph, if S is consistent then S does not prove σ. It follows that if S is consistent then S does not prove the consistency of S. This is the celebrated second incompleteness theorem of Gödel.

Recall Theorem 4.62, which says, in effect, that any true Σ_1 statement about HF sets is R-provable, where R is a finite fragment of S. The following theorem and the next extend this to s'-formulas.

(4.81) **Theorem** $[\mathsf{C}^+]$ *Suppose ϕ is a Δ_0' sentence and* $\mathrm{Sat}_0'\,\phi$. *Then* $\mathsf{S}' \vdash \phi$.

Proof We adapt the argument by which we established (4.47). We temporarily extend S' by definition of the predicate ⌜is included in⌝ to the theory S'' in the signature s''. We now use the identities represented in (4.48) to show by induction on an appropriate notion of complexity[4.49] that for any Δ_0''-sentence ϕ, if $\mathrm{Sat}_0''\,\phi$ then $\mathsf{S}'' \vdash \phi$, and if $\mathrm{Sat}_0''\neg\phi$ then $\mathsf{S}'' \vdash \neg\phi$.

For example, suppose $\phi = \tau = \tau'$ and the claim is true for $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$. Suppose $\mathrm{Sat}_0''\,\phi$. Then $\mathrm{Sat}_0''\,\tau \subseteq \tau'$ and $\mathrm{Sat}_0''\,\tau' \subseteq \tau$, so $\mathsf{S}'' \vdash \tau \subseteq \tau'$ and $\mathsf{S}'' \vdash \tau' \subseteq \tau$, whence

$\mathsf{S}'' \vdash \phi$. On the other hand, if $\mathrm{Sat}_0'' \neg \phi$ then either $\mathrm{Sat}_0'' \neg \tau \subseteq \tau'$ or $\mathrm{Sat}_0'' \neg \tau' \subseteq \tau$, so $\mathsf{S}'' \vdash \neg \tau \subseteq \tau'$ or $\mathsf{S}'' \vdash \neg \tau' \subseteq \tau$, whence $\mathsf{S}'' \vdash \neg \phi$.

The remaining identities in (4.48) are adapted similarly. Now we can use the fact that S'' is a conservative extension of S' to reach the conclusion of the theorem. Alternatively, we could have arranged the preceding argument to take place in S'.
$\square^{4.81}$

(4.82) **Theorem** $[\mathsf{C}^+]$ *Suppose ϕ is a Σ_1' s'-formula, A is an* HF*-assignment for ϕ, S is a substitution of constant s'-terms for its free variables such that $\forall v \in \mathrm{Free}\,\phi\ \mathrm{Val}'\, S(v) = A(v)$, and $\mathrm{Sat}_1^{\Sigma'}\,\phi[A]$. Then $\mathsf{S}' \vdash \phi(S)$.*

Proof Suppose $\phi = \exists u_1, \ldots, u_n\, \psi$, where ψ is Δ_0', suppose A and S are as specified, and suppose $\mathrm{Sat}_1^{\Sigma'}\,\phi[A]$. Then for some $x_1, \ldots, x_n \in \mathrm{HF}$, $\mathrm{Sat}_0'\,\psi[A']$, where $A' = A\left\langle\begin{smallmatrix} u_1 & \cdots & u_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right\rangle$. Let $\tau_1, \ldots, \tau_n$ be constant s'-terms such that $\mathrm{Val}'\,\tau_m = x_m$, $m = 1, \ldots, n$, and let $S' = S\left\langle\begin{smallmatrix} u_1 & \cdots & u_n \\ \tau_1 & \cdots & \tau_n \end{smallmatrix}\right\rangle$. Then $\mathrm{Sat}_0'\,\psi(S')$, so, by (4.81), $\mathsf{S}' \vdash \psi(S')$. Therefore $\mathsf{S}' \vdash \exists u_1, \ldots, u_n\, \psi(S)$, i.e., $\mathsf{S}' \vdash \phi(S)$, as claimed. $\square^{4.82}$

4.11.1 The first incompleteness theorem

The following definability results are easily derived.

1. *The class of s'-sentences that are S'-provable, i.e. for which there exists an S'-proof, is Σ_1.*
2. *The class of s'-formulas with one free variable v_0 is Δ_1.*
3. *The class of $\langle x_0, x_1\rangle \in \mathrm{HF}$ such that $x_0 = \mathrm{Nm}\, x_1$ is Σ_1.*
4. *The class of $\langle x_0, x_1, x_2\rangle \in \mathrm{HF}$ such that*
    1. *if x_1 is an s'-formula with one free variable and x_2 is an s'-term then x_0 is the result of substituting x_2 for the free variable in x_1;*
    2. *otherwise, $x_0 = 0$,*

    is Σ_1.

For the present purpose, we are not interested in these results *per se* but rather in certain s-formulas whose existence they implicitly assert. The formulas in question are Σ_1 and Π_1, and we need to know that they have the correct meaning, as defined by Sat_1^{Σ} and Sat_1^{Π}, and also that certain sentences derived from them are provable in S. The following theorem is an exact statement of what we need. Note that the formulas whose existence is asserted are s-formulas that talk about s'-formulas and S'-proofs.

(4.83) **Theorem** $[\mathsf{C}^+]$

1. *There exists a Σ_1 formula Pbl' with one free variable v_0 that expresses S'-provability, i.e., for every $x \in \mathrm{HF}$, $\mathrm{Sat}_1^{\Sigma}\,\mathrm{Pbl}'\left[\begin{smallmatrix}\mathrm{v}_0 \\ x\end{smallmatrix}\right]$ iff x is an S'-provable sentence.*
2. *There exist a Σ_1 formula $\overset{\Sigma}{\mathrm{Form}}{}_1'$ and a Π_1 formula $\overset{\Pi}{\mathrm{Form}}{}_1'$, each with one free variable v_0, such that*

1. *for every* $x \in \mathrm{HF}$, $\mathrm{Sat}_1^\Sigma\ \overset{\Sigma}{\mathrm{Form}'_1}\left[\begin{smallmatrix}\mathrm{v}_0\\ x\end{smallmatrix}\right]$ *iff* $\mathrm{Sat}_1^\Pi\ \overset{\Pi}{\mathrm{Form}'_1}\left[\begin{smallmatrix}\mathrm{v}_0\\ x\end{smallmatrix}\right]$ *iff* x *is an* s'*-formula with one free variable.*[19]
2. $\mathsf{S} \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_0\, (\overset{\Sigma}{\mathrm{Form}'_1} \leftrightarrow \overset{\Pi}{\mathrm{Form}'_1})$.

3. *There exists a* Σ_1 *formula* $\overset{\Sigma}{\mathrm{Nm}'}$ *with two free variables,* v_0 *and* v_1*, such that*

   1. *for every* $x_0, x_1 \in \mathrm{HF}$, $\mathrm{Sat}_1^\Sigma\ \overset{\Sigma}{\mathrm{Nm}'}\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ x_0 & x_1\end{smallmatrix}\right]$ *iff* $x_0 = \mathrm{Nm}\, x_1$*, i.e.,* x_0 *is the canonical* s'*-term for* x_1*; and*
   2. $\mathsf{S} \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_1\ \exists_{\mathrm{HF}}!\mathrm{v}_0\ \overset{\Sigma}{\mathrm{Nm}'}$.

4. *There exists a* Σ_1 *formula* $\overset{\Sigma}{\mathrm{Sub}'_1}$ *with three free variables,* v_0*,* v_1*, and* v_2*, such that*

   1. *for every* $x_0, x_1, x_2 \in \mathrm{HF}$, $\mathrm{Sat}_1^\Sigma\ \overset{\Sigma}{\mathrm{Sub}'_1}\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ x_0 & x_1 & x_2\end{smallmatrix}\right]$ *iff*
      1. *if* x_1 *is an* s'*-formula with one free variable and* x_2 *is an* s'*-term then* x_0 *is the result of substituting* x_2 *for the free variable in* x_1*,*
      2. *otherwise,* $x_0 = 0$*;*

      and
   2. $\mathsf{S} \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_1, \mathrm{v}_2\ \exists_{\mathrm{HF}}!\mathrm{v}_0\ \overset{\Sigma}{\mathrm{Sub}'_1}$.

Proof Straightforward. □4.83

(4.84)

1. *Let* Pbl', $\overset{\Sigma}{\mathrm{Form}'_1}$, $\overset{\Pi}{\mathrm{Form}'_1}$, $\overset{\Sigma}{\mathrm{Nm}'}$, *and* $\overset{\Sigma}{\mathrm{Sub}'_1}$ *be as in (4.83).*
2. *Let* $\delta = \overset{\Pi}{\mathrm{Form}'_1} \wedge \neg\, \exists_{\mathrm{HF}}\, \mathrm{v}_1, \mathrm{v}_2 \left(\overset{\Sigma}{\mathrm{Nm}'}\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \bar{\mathrm{v}}_1 & \bar{\mathrm{v}}_0\end{smallmatrix}\right) \wedge \overset{\Sigma}{\mathrm{Sub}'_1}\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \bar{\mathrm{v}}_0 & \bar{\mathrm{v}}_1\end{smallmatrix}\right) \wedge \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \bar{\mathrm{v}}_2\end{smallmatrix}\right)\right)$.
3. *Let* $\boldsymbol{\delta} = \mathrm{Nm}\,\delta$*. Note that* $\mathrm{Sat}_1^\Sigma\ \overset{\Sigma}{\mathrm{Nm}'}\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \boldsymbol{\delta} & \delta\end{smallmatrix}\right]$.[4.83.3.1]
4. *Let* $\sigma = \delta\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\delta}\end{smallmatrix}\right)$.
5. *Let* $\boldsymbol{\sigma} = \mathrm{Nm}\,\sigma$.

(4.85) **Theorem** $[\mathsf{C}^+]$ $\mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)\right)$.

Remark In other words, σ S'-provably says 'I am not S'-provable'.

To make sure we understand (4.85), note that

1. σ is an s'-sentence;
2. $\boldsymbol{\sigma}$ is the canonical s'-term such that $\mathrm{Val}'\,\boldsymbol{\sigma} = \sigma$;
3. Pbl' is an s-formula with one free variable v_0;
4. $\mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)$ is the s'-sentence obtained by substituting the s'-term $\boldsymbol{\sigma}$ for v_0 in Pbl';
5. $\sigma \leftrightarrow \neg\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)$ is an s'-sentence;

[19] When naming a formula that defines a Δ_1 class we may use an overset 'Σ' or 'Π' to indicate that it is Σ_1 or Π_1, respectively.

6. $\ulcorner \sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}} \urcorner$ is an s^+-term that denotes $\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}$;
7. $\ulcorner \mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\right) \urcorner$ is an s^+-sentence that asserts that $\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}$ is an S'-theorem; and
8. $\ulcorner$**Theorem** $[\mathsf{C}^+]$ $\mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\right).\urcorner$ is an s^+-sentence that asserts that $\ulcorner \mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\right) \urcorner$ is a C^+-theorem.

The proof that follows justifies $\ulcorner$**Theorem** $[\mathsf{C}^+]$ $\mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\right).\urcorner$, as it is a C^+-proof of $\ulcorner \mathsf{S}' \vdash \left(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\right) \urcorner$, which is to say, a C^+-proof that there is an S'-proof of $\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}$.

Proof Let $\hat{\boldsymbol{\delta}} = \mathrm{Nm}\,\boldsymbol{\delta}$. Then $\mathrm{Val}'\,\boldsymbol{\delta} = \delta$, $\mathrm{Val}'\,\hat{\boldsymbol{\delta}} = \boldsymbol{\delta}$, and $\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Nm}}'\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \boldsymbol{\delta} & \delta\end{smallmatrix}\right]$,[4.84.3] so[4.82]

$$\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Nm}}'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \hat{\boldsymbol{\delta}} & \boldsymbol{\delta}\end{smallmatrix}\right),$$

and hence[4.82]

$$\mathsf{S}' \vdash \overset{\Sigma}{\mathrm{Nm}}'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \hat{\boldsymbol{\delta}} & \boldsymbol{\delta}\end{smallmatrix}\right). \tag{4.86}$$

Since $\mathsf{S}' \vdash \forall_{\mathrm{HF}}\,\mathrm{v}_1\ \exists_{\mathrm{HF}}!\mathrm{v}_0\overset{\Sigma}{\mathrm{Nm}}'$,[4.83.3.2]

$$\mathsf{S}' \vdash \forall_{\mathrm{HF}}\,\mathrm{v}_0\left(\overset{\Sigma}{\mathrm{Nm}}'\binom{\mathrm{v}_1}{\boldsymbol{\delta}} \rightarrow \mathrm{v}_0 = \hat{\boldsymbol{\delta}}\right). \tag{4.87}$$

Since $\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Sub}}{}_1'\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \sigma & \delta & \boldsymbol{\delta}\end{smallmatrix}\right]$ and $\mathrm{Val}'\,\boldsymbol{\sigma} = \sigma$, $\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Sub}}{}_1'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \boldsymbol{\sigma} & \boldsymbol{\delta} & \hat{\boldsymbol{\delta}}\end{smallmatrix}\right)$, and hence

$$\mathsf{S}' \vdash \overset{\Sigma}{\mathrm{Sub}}{}_1'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \boldsymbol{\sigma} & \boldsymbol{\delta} & \hat{\boldsymbol{\delta}}\end{smallmatrix}\right). \tag{4.88}$$

Since $\mathsf{S}' \vdash \forall_{\mathrm{HF}}\,\mathrm{v}_1, \mathrm{v}_2\ \exists_{\mathrm{HF}}!\mathrm{v}_0\overset{\Sigma}{\mathrm{Sub}}{}_1'$,[4.83.4.2]

$$\mathsf{S}' \vdash \forall_{\mathrm{HF}}\,\mathrm{v}_0\left(\overset{\Sigma}{\mathrm{Sub}}{}_1'\left(\begin{smallmatrix}\mathrm{v}_1 & \mathrm{v}_2\\ \boldsymbol{\delta} & \hat{\boldsymbol{\delta}}\end{smallmatrix}\right) \rightarrow \mathrm{v}_0 = \boldsymbol{\sigma}\right). \tag{4.89}$$

Observe that[4.84.4, 2]

$$\sigma = \overset{\Pi}{\mathrm{Form}}{}_1'\binom{\mathrm{v}_0}{\boldsymbol{\delta}} \wedge \neg\,\exists_{\mathrm{HF}}\,\mathrm{v}_1, \mathrm{v}_2\left(\overset{\Sigma}{\mathrm{Nm}}'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \bar{\mathrm{v}}_1 & \boldsymbol{\delta}\end{smallmatrix}\right) \wedge \overset{\Sigma}{\mathrm{Sub}}{}_1'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \boldsymbol{\delta} & \bar{\mathrm{v}}_1\end{smallmatrix}\right) \wedge \mathrm{Pbl}'\binom{\mathrm{v}_0}{\bar{\mathrm{v}}_2}\right). \tag{4.90}$$

Since $\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Form}}'\left[\begin{smallmatrix}\mathrm{v}_0\\ \delta\end{smallmatrix}\right]$ and $\mathrm{Val}'\,\boldsymbol{\delta} = \delta$,

$$\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Form}}'\binom{\mathrm{v}_0}{\boldsymbol{\delta}},$$

so

$$\mathsf{S}' \vdash \overset{\Sigma}{\mathrm{Form}}'\binom{\mathrm{v}_0}{\boldsymbol{\delta}}.$$

Since $\mathsf{S}' \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_0\, \big(\overset{\Pi}{\mathrm{Form}}{}' \leftrightarrow \overset{\Sigma}{\mathrm{Form}}{}'\big)$,[4.83.2.2]

$$\mathsf{S}' \vdash \overset{\Pi}{\mathrm{Form}}{}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\delta}\end{smallmatrix}\right).$$

Hence

$$\mathsf{S}' \vdash \Big(\sigma \leftrightarrow \neg\, \exists_{\mathrm{HF}}\, \mathrm{v}_1, \mathrm{v}_2\, \big(\overset{\Sigma}{\mathrm{Nm}}{}'\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \bar{\mathrm{v}}_1 & \boldsymbol{\delta}\end{smallmatrix}\right) \wedge \overset{\Sigma}{\mathrm{Sub}}{}'_1\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \boldsymbol{\delta} & \bar{\mathrm{v}}_1\end{smallmatrix}\right) \wedge \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \bar{\mathrm{v}}_2\end{smallmatrix}\right)\big)\Big).$$

Using (4.86) and (4.87), we see that

$$\mathsf{S}' \vdash \Big(\sigma \leftrightarrow \neg\, \exists_{\mathrm{HF}}\, \mathrm{v}_2\, \big(\overset{\Sigma}{\mathrm{Sub}}{}'_1\left(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \boldsymbol{\delta} & \hat{\boldsymbol{\delta}}\end{smallmatrix}\right) \wedge \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \bar{\mathrm{v}}_2\end{smallmatrix}\right)\big)\Big).$$

Using (4.88) and (4.89), we see that

$$\mathsf{S}' \vdash \sigma \leftrightarrow \neg\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right),$$

as claimed. □[4.85]

(4.91) **Theorem** [C^+] *Suppose* S' *is consistent. Then*

1. $\mathsf{S}' \nvdash \sigma$.
2. σ *is true in the sense that there is a* $\{\sigma\}$*-satisfaction relation* $\mathrm{Sat}^{\{\sigma\}}$ *for* $(\mathrm{HF}; \in)$, *and* $\mathrm{Sat}^{\{\sigma\}}\, \sigma$.

Proof 1 Suppose S' is consistent, and

(4.92) *suppose toward a contradiction that* $\mathsf{S}' \vdash \sigma$.

Then[4.83.1] $\mathrm{Sat}_1^{\Sigma}\, \mathrm{Pbl}'\left[\begin{smallmatrix}\mathrm{v}_0\\ \sigma\end{smallmatrix}\right]$, so $\mathrm{Sat}_1^{\Sigma}\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)$, and therefore

$$\mathsf{S}' \vdash \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right). \tag{4.93}$$

But $\mathsf{S}' \vdash \big(\sigma \to \neg\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)\big)$.[4.85] Hence[4.92] $\mathsf{S}' \vdash \neg\, \mathrm{Pbl}'\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\sigma}\end{smallmatrix}\right)$. Hence[4.93] S' is inconsistent. □[4.91.1]

2 Since σ is a specific s'-expression one can define a $\{\sigma\}$-satisfaction relation $\mathrm{Sat}^{\{\sigma\}}$ by extending Definitions 4.30 and 4.35, so as to cover all subformulas of σ.[4.90] $\mathrm{Sat}^{\{\sigma\}}$ of course extends $\mathrm{Sat}_1^{\Sigma'}$ and $\mathrm{Sat}_1^{\Pi'}$. Since $\mathrm{Sat}_1^{\Pi'}\, \overset{\Pi}{\mathrm{Form}}{}'_1\left[\begin{smallmatrix}\mathrm{v}_0\\ \delta\end{smallmatrix}\right]$ and $\mathrm{Val}'\, \boldsymbol{\delta} = \delta$, $\mathrm{Sat}_1^{\Pi'}\, \overset{\Pi}{\mathrm{Form}}{}'_1\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\delta}\end{smallmatrix}\right)$, so

$$\mathrm{Sat}^{\{\sigma\}}\, \overset{\Pi}{\mathrm{Form}}{}'_1\left(\begin{smallmatrix}\mathrm{v}_0\\ \boldsymbol{\delta}\end{smallmatrix}\right). \tag{4.94}$$

Since

1. $\boldsymbol{\delta}$ is the unique x such that $\mathrm{Sat}_1^{\Sigma'}\, \big(\overset{\Sigma}{\mathrm{Nm}}{}'\left(\begin{smallmatrix}\mathrm{v}_1\\ \boldsymbol{\delta}\end{smallmatrix}\right)\big)\left[\begin{smallmatrix}\mathrm{v}_0\\ x\end{smallmatrix}\right]$,

2. σ is the unique x such that $\mathrm{Sat}_1^{\Sigma'}\big(\overset{\Sigma}{\mathrm{Sub}'_1}\binom{\mathrm{v}_1}{\boldsymbol{\delta}}\big)\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_2\\ x & \boldsymbol{\delta}\end{smallmatrix}\big]$, and

3. $\mathsf{S}' \nvdash \sigma$,[4.91.1] so $\neg\,\mathrm{Sat}_1^{\Sigma'}\,\overset{\Sigma}{\mathrm{Pbl}'}\big[\begin{smallmatrix}\mathrm{v}_0\\ \sigma\end{smallmatrix}\big]$,[4.83.1]

it follows that

$$\neg\,\exists_{\mathrm{HF}}\, x_1, x_2\left(\mathrm{Sat}^{\{\sigma\}}\big(\overset{\Sigma}{\mathrm{Nm}'}\binom{\mathrm{v}_1}{\boldsymbol{\delta}}\big)\big[\begin{smallmatrix}\mathrm{v}_0\\ x_1\end{smallmatrix}\big] \wedge \mathrm{Sat}^{\{\sigma\}}\big(\overset{\Sigma}{\mathrm{Sub}'_1}\binom{\mathrm{v}_1}{\boldsymbol{\delta}}\big)\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_2\\ x_2 & x_1\end{smallmatrix}\big] \wedge \mathrm{Sat}^{\{\sigma\}}\big[\begin{smallmatrix}\mathrm{v}_0\\ x_2\end{smallmatrix}\big]\right),$$

so

$$\mathrm{Sat}^{\{\sigma\}}\left(\neg\,\exists_{\mathrm{HF}}\,\mathrm{v}_1,\mathrm{v}_2\,\big(\overset{\Sigma}{\mathrm{Nm}'}\big(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \bar{\mathrm{v}}_1 & \boldsymbol{\delta}\end{smallmatrix}\big) \wedge \overset{\Sigma}{\mathrm{Sub}'_1}\big(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \boldsymbol{\delta} & \bar{\mathrm{v}}_1\end{smallmatrix}\big) \wedge \mathrm{Pbl}'\binom{\mathrm{v}_0}{\bar{\mathrm{v}}_2}\big)\right).$$

Thus,[4.94]

$$\mathrm{Sat}^{\{\sigma\}}\left(\overset{\Pi}{\mathrm{Form}'_1}\binom{\mathrm{v}_0}{\boldsymbol{\delta}} \wedge \neg\,\exists_{\mathrm{HF}}\,\mathrm{v}_1,\mathrm{v}_2\,\big(\overset{\Sigma}{\mathrm{Nm}'}\big(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ \bar{\mathrm{v}}_1 & \boldsymbol{\delta}\end{smallmatrix}\big) \wedge \overset{\Sigma}{\mathrm{Sub}'_1}\big(\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ \bar{\mathrm{v}}_2 & \boldsymbol{\delta} & \bar{\mathrm{v}}_1\end{smallmatrix}\big) \wedge \mathrm{Pbl}'\binom{\mathrm{v}_0}{\bar{\mathrm{v}}_2}\big)\right).$$

i.e.,[4.90] $\mathrm{Sat}^{\{\sigma\}}\,\sigma$. □[4.91]

(4.91) is the promised sharp form of Gödel's first incompleteness theorem. It improves on (3.217) in two ways. First, it provides a specific s'-sentence σ that is true in $(\mathrm{HF};\in,0,\leftharpoondown)$ but is not provable in S. Second, it proves this result in C^+ on the assumption that S is consistent, whereas (3.217) assumes $(\mathrm{HF};\in)$ is weakly satisfactory. Recall that this means that for every s-sentence θ there is a $\{\theta\}$-satisfaction relation for $(\mathrm{HF};\in)$.[20] For (4.91) we only need a $\{\sigma\}$-satisfaction relation, and the existence of this is provable in C^+ (by explicit definition). Not only is the consistency of S' a weaker assumption than the weak satisfactoriness of $(\mathrm{HF};\in)$, it is a purely syntactical statement, which may be formulated in a pure set theory without $\mathsf{Infinity}$, such as S; whereas to say anything about models of S we must allow either infinite sets or proper classes.

Let Con' *be a natural* Π'_1 *formula that expresses the consistency of* S'*:* ⌜*for every* π*,* π *is not an* S'*-proof of a sentence of the form* $\theta \wedge \neg\,\theta$⌝.

s^+ is an expansion of s', so Pbl', Con', and σ are s^+-expressions, and the C^+-proofs of (4.85) and (4.91.1), which we have sketched, witness that $\mathsf{C}^+ \vdash \big(\sigma \leftrightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\big)$ and $\mathsf{C}^+ \vdash \big(\mathrm{Con}' \rightarrow \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\big)$, so

$$\mathsf{C}^+ \vdash (\mathrm{Con}' \rightarrow \sigma). \tag{4.95}$$

Since C^+ is a conservative extension of S',

$$\mathsf{S}' \vdash (\mathrm{Con}' \rightarrow \sigma). \tag{4.96}$$

As usual, the entire preceding discussion has been carried out in C^+, so we have the following version of the first incompleteness theorem.

(4.97) **Gödel's first incompleteness theorem** $[\mathsf{C}^+]$

$$\mathsf{S}' \vdash \Big(\mathrm{Con}' \rightarrow \big(\sigma \wedge \neg\,\mathrm{Pbl}'\binom{\mathrm{v}_0}{\boldsymbol{\sigma}}\big)\Big).$$

Note that the semantic element has been eliminated in this version.

[20] S and S' are interchangeable for this discussion, as are $(\mathrm{HF};\in)$ and $(\mathrm{HF};\in,0,\leftharpoondown)$. The weak satisfactoriness of $(\mathrm{HF};\in)$ implies the consistency of S (as a theorem of C). The completeness theorem tells us that if S is consistent then it has a (strongly) satisfactory model, but we cannot infer from this that $(\mathrm{HF};\in)$ is a model of S.

4.11.2 The second incompleteness theorem

The sentence σ that we have exhibited to witness the incompleteness of S' is clearly constructed for this purpose and is not otherwise of any interest. The presence of Con' as a hypothesis in the proof whose existence is asserted in (4.97), however, points the way to a sentence whose demonstrable unprovability is of great interest.

We have just shown[4.96] that $\mathsf{S}' \vdash (\mathrm{Con}' \to \sigma)$. If, therefore, $\mathsf{S}' \vdash \mathrm{Con}'$ then $\mathsf{S}' \vdash \sigma$. But we have also shown that if S' is consistent then $\mathsf{S}' \nvdash \sigma$.[4.91.1] We therefore have

(4.98) **Gödel's second incompleteness theorem** $[\mathsf{C}^+]$ *If* S' *is consistent then*

$$\mathsf{S}' \nvdash \mathrm{Con}'.$$

In other words, if S' *is consistent, its consistency cannot be demonstrated in* S'.

Since C^+ and S' are conservative extensions of S, the incompleteness theorems are also true *mutatis mutandis* with C^+ and S' replaced by S. The only difficulty is the technical one that we do not have s-terms other than variables, so we cannot substitute anything for $\ulcorner\sigma\urcorner$, $\ulcorner\boldsymbol{\sigma}\urcorner$, or $\ulcorner\mathrm{Con}'\urcorner$, and we must resort to circumlocution. It is nevertheless straightforward to express the sense of ⌜if there is no S-proof of an inconsistency then the natural s-sentence that expresses this is not S-provable⌝ as an s-sentence; and this sentence is a theorem of S. Alternatively, we could take the expansive view that defined predicates and operations are an intrinsic part of any theory, so that (4.98) *is* Gödel's second incompleteness theorem for S; but the austerity of S, with its single binary relation of membership (even the identity predicate may be eliminated by virtue of the comprehension axiom) has a certain appeal. We will often use S as a stand-in for S' or S^+ or any other conservative extension of S, relying on the reader to make the necessary adjustments.

The key features of S that permit the statement and proof of the incompleteness theorems are that the notion of S-provability is Σ_1 and that S is capable of formulating and proving the pertinent facts surrounding this notion. Suppose T is a recursively enumerable theory (in the standard language for a recursively enumerable signature $\rho \subseteq \mathrm{HF}$) in which S is interpretable. Then T has both of the features just named, and the arguments just applied to S are applicable to T. Thus we have the following general theorem.

(4.99) **Theorem** $[\mathsf{C}^+]$ *Suppose* T *is a recursively enumerable ρ-theory in which* S *is interpretable. Let* $\mathrm{Con}\,\mathsf{T}$ *be a ρ-sentence that naturally expresses the consistency of* T. *If* T *is consistent then* $\mathsf{T} \nvdash \mathrm{Con}\,\mathsf{T}$.

4.11.3 The first incompleteness theorem, Rosser's improvement

Note that the first incompleteness theorem[3.217 4.97] does not state the *syntactic* incompleteness of S', which is the assertion that there is an s'-sentence θ such that neither θ nor $\neg\,\theta$ is S'-provable. In other words, the possibility is left open that S' is consistent but that nevertheless $\mathsf{S}' \vdash \neg\,\sigma$, i.e., $\mathsf{S}' \vdash \exists_{\mathrm{HF}}\, \mathrm{v}_0\, \mathrm{Prf}'\binom{\mathrm{v}_1}{\boldsymbol{\sigma}}$, where Prf' is a Σ_1' formula with free variables $\mathrm{v}_0, \mathrm{v}_1$ that says v_0 is an S'-proof of v_1. By the first incompleteness theorem, assuming S is consistent, for every $x \in \mathrm{HF}$, x is not an S'-proof of σ, and in fact, $\mathsf{S}' \vdash \neg\,\mathrm{Prf}'\binom{\mathrm{v}_0\ \ \mathrm{v}_1}{\mathrm{Nm}\,x\ \ \boldsymbol{\sigma}}$.

Definition $[\mathsf{C}^+]$ *Recall*[4.64] *that a theory* T *that extends* S' *is* ω-consistent $\overset{\text{def}}{\iff}$ T *is consistent and for any* s'*-formula* ϕ *with one free variable* v*, it is not the case that* $\mathsf{T} \vdash \exists_{\mathrm{HF}}\, v\, \phi$ *and for all* HF *sets* x, $\mathsf{T} \vdash \neg\,\phi\binom{v}{\mathrm{Nm}\,x}$.

Given the preceding remarks, it is clear that if we assume (working in C^+) that S' is ω-consistent, then we may conclude that it is syntactically incomplete, as it can prove neither σ nor $\neg\,\sigma$.

Note that the preceding remarks do not bear on the second incompleteness theorem. Its hypothesis is the simple consistency of S.

We can prove the syntactic incompleteness of S from the hypothesis of consistency—as opposed to ω-consistency—if we substitute for σ the *Rosser sentence* ρ, which says 'for any proof of me, there is a shorter proof of my negation', where, for the nonce, we say that the length of a proof π is the number of occurrences of 0 and $\backsim$ in $\mathrm{Nm}\,\pi$. The essential thing is that there are only finitely many proofs of any given length. Let Sh' be an s'-formula with two free variables, v_0 and v_1, that says that v_0 is shorter than v_1. ρ is designed so that, letting $\tau = \mathrm{Nm}(\rho)$ and $\tau' = \mathrm{Nm}(\neg\,\rho)$,

$$\mathsf{S}' \vdash \rho \leftrightarrow \forall_{\mathrm{HF}}\, \mathrm{v}_2 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_2 & \tau \end{smallmatrix}\right) \rightarrow \exists_{\mathrm{HF}}\, \mathrm{v}_3 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \tau' \end{smallmatrix}\right) \wedge \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \bar{\mathrm{v}}_2 \end{smallmatrix}\right)\right)\right). \tag{4.100}$$

We have the following slight improvement of (4.91).

(4.101) **First incompleteness theorem, Rosser's improvement** $[\mathsf{C}^+]$ *If* S *is consistent, then* S' *neither proves nor disproves* ρ*, so* S *is syntactically incomplete.*

Proof See Note 10.14.

4.12 Relative definability over HF

[In this section we will work in ZF unless otherwise noted.]

Suppose $A \subseteq \mathrm{HF}$. We characterize definability *relative to* A by supposing that information regarding membership in A is available. Let ρ be an expansion of s by the addition of one unary predicate index. The A-interpretation of ρ is the standard interpretation of the membership and identity predicates in $(\mathrm{HF};\in)$, with the new unary predicate interpreted as A. The classes Δ_0, Σ_n, and Π_n of ρ-formulas are defined as for s-formulas, and the classes Σ_n^A, Π_n^A, and Δ_n^A are defined with respect to the A-interpretation of these formulas.

Of particular interest is the case of Σ_1^A. We have shown in (4.47), (4.60), and (4.61) how to construct an effective procedure to evaluate a Σ_1 formula. We may model the evaluation of a Σ_1^A formula the same way, with one additional feature: Given an s'-term τ, we may obtain the answer to the question whether (the set denoted by) τ is in A. We refer to this as *consulting an oracle* for A, and we may also refer to A itself as an oracle. We naturally use the same terminology in connection with Π_1^A and Δ_1^A.

For the rest of this section we will adopt the traditional point of view described in Section 4.8, using primarily ω in place of HF, and subsets of ω or functions $f : \omega \rightharpoonup \omega$ in place of subsets of HF.

(4.102) **Definition** $[\mathsf{ZF}]$ *Suppose* $A, B \subseteq \omega$.

1. A *is* Turing reducible *to* $B \overset{\text{def}}{\iff} A$ *is* recursive in $B \overset{\text{def}}{\iff} A$ *is* recursive relative to $B \overset{\text{def}}{\iff} A \leqslant_{\mathrm{T}} B \overset{\text{def}}{\iff} A$ *is* Δ_1^B.
2. $A <_{\mathrm{T}} B \overset{\text{def}}{\iff} (A \leqslant_{\mathrm{T}} B \wedge B \nleqslant_{\mathrm{T}} A)$.
3. $A \equiv_{\mathrm{T}} B \overset{\text{def}}{\iff} (A \leqslant_{\mathrm{T}} B \wedge B \leqslant_{\mathrm{T}} A)$.

4. *A* Turing degree *is an equivalence class of* $\equiv_{\mathrm{T}}$.
5. $\mathcal{D} \stackrel{\text{def}}{=}$ *the set of Turing degrees.*
6. *Suppose* $d_0, d_1' \in \mathcal{D}$. $d_0 \leqslant d_1 \stackrel{\text{def}}{\Longleftrightarrow}$ *for some (equivalently, for any)* $A_0 \in d_0$ *and* $A_1 \in d_1$, $A_0 \leqslant_{\mathrm{T}} A_1$.

Clearly, $\leqslant_{\mathrm{T}}$ is a preordering of $\mathcal{P}\omega$, and $\leqslant$ is a partial ordering of $\mathcal{D}$.

The structure $(\mathcal{D}; \leqslant)$ fascinates in much the same way as does $(\omega; \leqslant)$. Both evoke a sense of inevitability and simplicity that lends a peculiar appeal to their study. In both cases, many questions that are simple to state are very difficult—but not impossible—to answer.

Definition [ZF] *Suppose* $A \subseteq \omega$. *The* partial recursive function relative to A with index $n \stackrel{\text{def}}{=} \varphi_n^A \stackrel{\text{def}}{=}$ *the function defined as in (4.74) with* A *allowed as a predicate (*$\vec{B}\,n$ *is required to be a* Δ_0 *formula in a signature with one additional unary predicate, interpreted as* A*).*

The halting problem for computations relative to any given oracle A is unsolvable relative to A by the same argument as in the absolute case.[4.78] This leads to the following *jump* operation on $\mathcal{D}$, which is critical element of its structure.

(4.103) **Definition** [C^+] *Suppose* $d \in \mathcal{D}$ *is a Turing degree. The* jump of $d \stackrel{\text{def}}{=} d'$ $\stackrel{\text{def}}{=}$ *the degree of the halting problem for computations relative to* A*, where* A *is any member of* d*, i.e.,* d' *is the degree of the set* $\{\langle m, n\rangle \mid m \in \operatorname{dom} \varphi_n^A\}$.

Since there are only countably many effective procedures, each Turing degree is countable. Since $\mathcal{P}\omega$ is uncountable, $\mathcal{D}$ is uncountable. A closer analogy to $(\omega; \leqslant)$ is the structure $(\mathcal{R}; \leqslant)$, where $\mathcal{R}$ is the set of recursively enumerable degrees, i.e., the degrees of Σ_1 sets.[21] Since there are only countably many effective procedures, Σ_1 is countable, so $\mathcal{R}$ is countable.

Of course, in stating that $\mathcal{R}$ is countable, we leave open the possibility that it is finite, or even that its cardinality is 2. Its cardinality cannot be 1, because, as we have shown,[4.78] the set H corresponding to the halting problem is r.e. but not recursive. It turns out that $\mathcal{R}$ is infinite, and $(\mathcal{R}; \leqslant)$ is in fact a very rich structure.

Since H is Σ_1-complete, its degree, $0'$,[4.103] is the maximum member of $\mathcal{R}$. We will content ourselves with a proof just one property of $(\mathcal{R}; \leqslant)$ that gives a hint of the richness of its theory, viz., the existence of recursively incomparable degrees in $\mathcal{R}$.[4.104] It is worth noting that the relative difficulty of the proof derives from the requirement that the degrees contain r.e. sets. If we merely require that they be recursive in $0'$ a simpler argument will serve,[22] and if we impose no requirement on their absolute complexity, there is an even simpler proof.

It is convenient to formalize the notion of relative recursiveness in terms of characteristic functions as follows.[23] Given $e \in \omega$ and $F : \omega \to 2$ we let $\varphi_e^F : \omega \rightharpoonup 2$

[21] Note that the sets in a recursively enumerable degree are not necessarily all recursively enumerable. In fact, if $A \subseteq \omega$ is Σ_1 but not Δ_1, then $\omega \backslash A$ is in the same Turing degree as A, but it is not Σ_1 (otherwise A would be Δ_1).

[22] It is worth pointing out that there exist degrees $d \leqslant 0'$ such that $d \notin \mathcal{R}$, i.e. there is no r.e. set in d. (The proof is not trivial.)

[23] Recall that the characteristic function $\chi_A : \omega \to 2$ of $A \subseteq \omega$ is defined by the condition that

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

be the function computed by the effective procedure with Turing index e using F as an oracle in the sense that φ_e may ask from time to time for the value of F at some number $k \in \omega$. If the computation of $\varphi_e^F(n)$ halts with an output other than 0 or 1, we regard it as not halting. Thus, $A \leqslant_{\mathrm{T}} B$ iff for some $e \in \omega$, $\varphi_e^{\chi_A} = \chi_B$.

We now expand on this notion to define φ_e^F, where $F : \omega \rightharpoonup 2$ may not be total. In this case, if the computation of $\varphi_e^F(n)$ ever asks for the value of F at some number not in $\operatorname{dom} F$, we regard the computation as not halting. Since computations are finitary, for any $e \in \omega$, $F : \omega \rightharpoonup 2$, and $n \in \omega$, if the computation $\varphi_e^F(n)$ halts then $\varphi_e^{F'}(n)$ halts for some finite $F' \subseteq F$; and if $\varphi_e^{F'}(n)$ halts for some $F' \subseteq F$ then $\varphi_e^F(n)$ halts with the same output.

(4.104) **Theorem** [ZF] **Friedberg-Muchnik** *There exist incomparable r.e. degrees.*

Proof We will describe an effective procedure $\mathcal{P}$ that enumerates sets $A^0, A^1 \subseteq \omega$ such that for every $e \in \omega$, $\varphi_e^{\chi_{A^0}} \neq \chi_{A^1}$ and $\varphi_e^{\chi_{A^1}} \neq \chi_{A^0}$. Let $\mathcal{C}$ be a process that generates all sequences $\langle e, F, n\rangle$ such that $e, n \in \omega$, $F : \omega \rightharpoonup 2$ is finite, and $\varphi_e^F(n)$ halts with output 0: we simply dovetail all computations of the above sort, and whenever one halts with value 0 we append the appropriate item to the list. The list is clearly infinite, and we let $\langle C_k \mid k \in \omega\rangle$ be the list in order of enumeration.

We will generate recursive sequences

$$A_0^i \subseteq A_1^i \subseteq \cdots \quad (i \in 2)$$

of finite subsets of ω, ultimately letting $A^i = \bigcup_{s\in\omega} A_s^i$. For each $\langle e, i\rangle \in \omega \times 2$, we must satisfy the *requirement* $\mathcal{R}_e^i$ that $\chi_{A^i} \neq \varphi_e^{\chi_{A^{\bar{\imath}}}}$, i.e., for some $w \in \omega$, $\varphi_e^{\chi_{A^{\bar{\imath}}}}(w)$ either fails to halt or it halts with output other than $\chi_{A^i}(w)$. We say that w is a *witness* that $\mathcal{R}_e^i$ is satisfied. At each stage s of the construction we will have a proposed witness $w_e^i(s) \in \omega$ for each requirement $\mathcal{R}_e^i$ such that $e < s$. Thus, $\operatorname{dom} w_e^i = \{s \in \omega \mid s > e\}$. To avoid interference, we arrange that for any $i \in 2$ and $s \in \omega$, $\langle w_e^i(s) \mid e < s\rangle$ is injective.

As the construction proceeds, the satisfaction of one requirement may disrupt the satisfaction of other requirements. To manage this we *prioritize* requirements as follows.

Let $\prec$ be the lexicographic ordering of $\omega \times 2$:

$$\langle e, i\rangle \prec \langle e', i'\rangle \leftrightarrow e < e' \vee (e = e' \wedge i < i').$$

We say that $\mathcal{R}_e^i$ has *higher priority* than $\mathcal{R}_{e'}^{i'} \overset{\text{def}}{\iff} \langle e, i\rangle \prec \langle e', i'\rangle$.

For $i \in 2$, let

$$\bar{\imath} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1. \end{cases}$$

At each stage s we will have a *restriction* $r_e^i(s) \in \omega$ for each $i \in 2$ and $e < s$, whose purpose is to protect $\mathcal{R}_e^i$ from injury due to the addition of something to $A^{\bar{\imath}}$ below $r_e^i(s)$ after stage s in the course of satisfying a requirement of lower priority.

Initially, we let $A_0^0 = A_0^1 = 0$, and we begin the construction at stage 0.

(4.105) *At stage s, we have defined A_s^i $(i \in 2)$, and we have defined $w_e^i(s)$ and $r_e^i(s)$ for $e < s$ and $i \in 2$. For each $i \in 2$, $\langle w_e^i(s) \mid e < s\rangle$ is injective. For each $i \in 2$, let*

b^i be the least $b \in \omega$ that exceeds every element of A^i_s and every $w^i_e(s)$ and $r^{\bar\imath}_e(s)$ for $e < s$.[24] *Let $C = \{C_k \mid k < s\}$, and proceed as follows.*[25]

1. *If there exists $\langle e, i\rangle \in s \times 2$ such that $w^i_e(s) \notin A^i_s$ and for some $F \subseteq \chi_{A^{\bar\imath}_s}$, $\langle e, F, w^i_e(s)\rangle \in C$, then let $\langle e, i\rangle$ be the lexicographically least such. We would like to satisfy $\mathcal{R}^i_e$ by putting $w^i_e(s)$ into A^i, but we first check whether this would violate any restriction of higher priority.*
   1. *Suppose for some $\langle e', \bar\imath\rangle \prec \langle e, i\rangle$, $w^i_e(s) < r^{\bar\imath}_{e'}(s)$ (so that the addition of $w^i_e(s)$ to A^i would violate the restriction imposed by $\mathcal{R}^{i'}_{e'}$ at this stage). Then we say that $\mathcal{R}^i_e$ is* injured *(by $\mathcal{R}^{\bar\imath}_{e'}$, for every such $\langle e', \bar\imath\rangle$).*
      1. *Let $A^{i'}_{s+1} = A^{i'}_s$ ($i' \in 2$).*
      2. *Let $w^i_e(s+1) = b^i$; and let $r^i_e(s+1) = 0$.*
   2. *Suppose the condition of (4.105.1.1) does not apply. Then we let $\mathcal{R}^i_e$* act *(by enumerating $w^i_e(s)$ into A^i).*
      1. *Thus, we let $A^i_{s+1} = A^i_s \cup \{w^i_e(s)\}$, and we let $A^{\bar\imath}(s+1) = A^{\bar\imath}_s$.*
      2. *Let $w^i_e(s+1) = w^i_e(s)$; and let $r^i_e(s+1)$ be the least $r \in \omega$ such that $\operatorname{dom} F \subseteq r$ for some F such that $\langle e, F, w^i_e(s)\rangle \in C$ (which will protect $\mathcal{R}^i_e$ against injury by requirements of lower priority).*
      3. *Suppose for some $\langle e', \bar\imath\rangle \succ \langle e, i\rangle$, $w^i_e(s) < r^{\bar\imath}_{e'}(s)$. Then we say that $\mathcal{R}^i_e$ has* injured *each such $\langle e', \bar\imath\rangle$. For simplicity, we will treat all requirements of lower priority as though they have been injured in this way. Let e_0 be least $e' \leqslant s$ such that $\langle e', \bar\imath\rangle \succ \langle e, i\rangle$. For each $e' \in s \backslash e_0$, let $w^{\bar\imath}_{e'}(s+1) = b^{\bar\imath} + e' - e_0$; and let $r^{\bar\imath}_{e'}(s+1) = 0$.*
2. *For each $e < s$ and $i \in 2$ such that $w^i_e(s+1)$ or $r^i_e(s+1)$ has not been assigned a value thus far, let $w^i_e(s+1) = w^i_e(s)$ and $r^i_e(s+1) = r^i_e(s)$.*
3. *For each $i \in 2$ let $w^i_s(s+1)$ be the least $w \notin A^i_{s+1}$ such that $w \neq w^i_e(s+1)$ for any $e < s$, and let $r^i_s(s+1) = 0$.*

Several points should be emphasized.

1. When a witness for $\mathcal{R}^i_e$ is either first proposed[4.105.3] (at stage $s = e$) or changed due to injury[4.105.1.1.2 or .1.2.3] (at stage s), we require that $w^i_e(s+1) \notin A^i_{s+1}$. At this point r^i_e is set to 0, so it imposes no restriction on the addition of members to $A^{\bar\imath}$.

2. If we later find that there is a computation $\langle e, F, w^i_e\rangle$ that indicates that $\varphi_e^{\chi_{A^{\bar\imath}}}(w^i_e) = 0$, we enumerate w^i_e into A^i,[4.105.1.2.1] to foil φ_e as a reduction of A^i to $A^{\bar\imath}$, thus meeting the requirement $\mathcal{R}^i_e$. At such time we impose a restriction r^i_e to prevent any change to $A^{\bar\imath}$ that would spoil this computation (and thus injure $\mathcal{R}^i_e$). (Note that if we permitted the placement of w^i_e in A^i to be reversed, the process would not be a recursive enumeration of A^i.)

3. There are two ways a requirement may be injured:
   1. $\mathcal{R}^i_e$ may be injured directly when $w^i_e \in A^i$ and $r^i_e > 0$, and a witness $w^{\bar\imath}_{e'}$ is added to $A^{\bar\imath}$ to satisfy a requirement $\mathcal{R}^{\bar\imath}_{e'}$ of higher priority, where $w^{\bar\imath}_{e'} < r^i_e$, thus potentially spoiling the computation on the basis of which we put w^i_e into A^i.

[24] b^i just provides us with a convenient way of defining a set of numbers for consideration as members of A^i that are not hindered by any condition imposed prior to stage s.

[25] Note that at stage 0, only (4.105.3) is effective, with the result that for each $i \in 2$, $A^i_1 = 0$, $w^i_0(1) = 0$, and $r^i_0(1) = 0$.

2. $\mathcal{R}_e^i$ may be injured indirectly when $w_e^i \notin A^i$, and a requirement $\mathcal{R}_{e'}^{\bar{\imath}}$ of higher priority prevents the addition of w_e^i to A^i (because $w_e^i < r_{e'}^{\bar{\imath}}$) when it is necessary to foil the reduction of A^i to $A^{\bar{\imath}}$ by φ_e.

4. Note that requirements $\mathcal{R}_e^i$ and $\mathcal{R}_{e'}^i$ in the same direction cannot injure each other because we have required that their witnesses $w_e^i(s)$ and $w_{e'}^i(s)$ at any stage s are different.

It is fairly easy to show that each requirement acts and is injured only finitely often, so there is always a stage beyond which it neither acts nor is injured. We reason as follows. Let $s_0 < s_1 < \cdots$ be the stages at which $\mathcal{R}_e^i$ acts (adds its current witness to A^i). A requirement $\mathcal{R}_{e'}^{\bar{\imath}}$ of lower priority cannot be injured by $\mathcal{R}_e^i$ in the interval $[0, s_0)$, and it can be injured at most once by $\mathcal{R}_e^i$ in each of the intervals $[s_n, s_{n+1})$. On the other hand, $\mathcal{R}_e^i$ must be injured at least once between any two consecutive stages at which it acts. Since there are only finitely many requirements of higher priority than any given requirement, it follows by induction that any requirement can be injured only finitely often and can act only finitely often.

To show that A^0 and A^1 are $\leqslant_T$-incomparable, suppose toward a contradiction that $\chi_{A^i} = \varphi_e^{\chi_{A^{\bar{\imath}}}}$. Let w be the eventual value of $w_e^i(s)$. Suppose first that $w \in A^i$, so $\varphi_e^{\chi_{A^{\bar{\imath}}}}(w) = 1$. At some stage s after the last stage at which $\mathcal{R}_e^i$ was injured, $\mathcal{R}_e^i$ acted to put w in A^i. Thus

1. $w_e^i(s) = w$;

2. there exists $F \subseteq \chi_{A_s^{\bar{\imath}}}$ and $k < s$ such that $C_k = \langle e, F, w \rangle$; and

3. $\operatorname{dom} F \subseteq r_e^i(s+1)$.

Since $\mathcal{R}_e^i$ is not injured after stage s, $A^{\bar{\imath}}$ receives no new additions below $r_e^i(s+1)$ after stage s, so $A^{\bar{\imath}} \cap \operatorname{dom} F = A_s^{\bar{\imath}} \cap \operatorname{dom} F$. Hence, $F \subseteq \chi_{A^{\bar{\imath}}}$, from which it follows that $\varphi_e^{\chi_{A^{\bar{\imath}}}}(w) = 0$; contradiction.

Suppose, on the other hand, that $w \notin A^i$, so $\varphi_e^{\chi_{A^{\bar{\imath}}}}(w) = 0$. Let $k \in \omega$ be such that $C_k = \langle e, F, w \rangle$, with $F \subseteq \chi_{A^{\bar{\imath}}}$. Let $s > k$ be such that $A_s^{\bar{\imath}} \cap \operatorname{dom} F = A^{\bar{\imath}} \cap \operatorname{dom} F$, and neither $\mathcal{R}_e^i$ nor any requirement of higher priority either acts or is injured after stage $s - 1$. Then $F \subseteq \chi_{A_s^{\bar{\imath}}}$, so the condition of (4.105.1) is satisfied at $\langle e, i \rangle$. It follows that the least $\langle e', i' \rangle$ satisfying this condition is $\leqslant \langle e, i \rangle$. By construction, requirement $\mathcal{R}_{e'}^{i'}$ either acts or is injured at this stage, contrary to assumption. $\square$ 4.104

The structure $(\mathcal{R}; \leqslant)$ of the r.e. degrees is now known to be quite complicated. We content ourselves with listing a few facts. As noted above, $(\mathcal{R}; \leqslant)$ is a partial order with least element 0 and greatest element $0'$. It is easily seen to be an upper semilattice in the sense that any two elements a, b have a least upper bound $a \vee b$. Given $A \in a$ and $B \in b$, let $C = A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $a \vee b$ is the degree of C. Meets do not exist in general, so $(\mathcal{R}; \leqslant)$ is not a lattice.

1. Any countable partial order can be embedded in $(\mathcal{R}; \leqslant)$.

2. Any countable upper semilattice can be embedded in $(\mathcal{R}; \leqslant, \vee)$. (The additional requirement here is that the embedding must commute with the join operation.)

3. $(\mathcal{R}; \leqslant)$ is dense, i.e., $\forall_{\mathcal{R}} a, b\, (a < b \rightarrow \exists_{\mathcal{R}} c\ a < c < b)$.

4. Let S_0 and S_1 be respectively the sets of sentences of the language of $(\omega; +, \cdot)$ (two binary operations) and the language of $(\mathcal{R}; \leqslant)$ (one binary relation). There exist recursive maps $t_0 : S_0 \to S_1$ and $t_1 : S_1 \to S_0$ such that

1. for $\sigma \in S_0$, $(\omega; +, \cdot) \models \sigma \iff (\mathcal{R}; \leqslant) \models t_0(\sigma)$; and
2. for $\sigma \in S_1$, $(\mathcal{R}; \leqslant) \models \sigma \iff (\omega; +, \cdot) \models t_1(\sigma)$.

Hence the first-order theory of $(\mathcal{R}; \leqslant)$ is essentially equivalent to that of the standard model of arithmetic. The same applies to $(\mathcal{D}; \leqslant)$ *vis-à-vis* the standard model of second-order arithmetic, i.e., $(\omega, \mathcal{P}\omega; +, \cdot, \in)$, or, equivalently, $(V_{\omega+1}; \in)$.

4.13 Summary

We begin with some well known paradoxes, all of which signal the presence of some limitation on *definability*—of numbers in Richard's paradox; of sets in Russell's paradox; of truth in Epimenides' paradox. We also mention the issue of *decidability*: there is an effective procedure for generating all the theorems of a finite (indeed, any recursively enumerable) theory, e.g., mindlessly generating them all by the application of the methods of logic; is there also an effective procedure for generating all the nontheorems? If there is, then there is a *decision procedure* for the theory. The existence of such a procedure for set theory—or even for pure logic—would have profound significance for the foundations of mathematics. This is the question of decidability of theories.

The resolution of the paradoxes lies in making precise the notion of definability. Taking advantage of the universality of the theory of membership for the discussion of structure, we define definability in terms of the complexity of set-theoretic formulas. The precise formulation of decidability questions requires a definition of *effective procedure* or *computation*, and we discover (as part of our investigation of the *science of computation*) that the intuitive notion of computation is coextensive with that of a Σ_1 partial function $f : V_\omega \rightharpoonup V_\omega$.

We precisely evaluate the complexity of the notion of satisfaction in the structure $(V_\omega; \in)$ and show that the notion of satisfaction for Σ_1 formulas is Σ_1 but not Π_1. This leads to the celebrated undecidability results for first-order predicate logic (with at least one non-unary predicate other than identity) and for the *halting problem*.

We then apply these methods to provability and consistency. Using S as a suitable theory of finitary objects, we define the famous Gödel s-sentence σ that says 'I am not S-provable', i.e., we show that $\mathsf{S} \vdash \big(\sigma \leftrightarrow \neg \operatorname{Pbl} \boldsymbol{\sigma}\big)$. We then show, as a theorem of C, that if S is consistent then $\mathsf{S} \nvdash \sigma$ and $(V_\omega; \in) \models \sigma$. From this we show that $\mathsf{S} \vdash \big(\operatorname{Con} \mathsf{S} \rightarrow (\sigma \wedge \neg \operatorname{Pbl} \boldsymbol{\sigma})\big)$. From this it follows, as a theorem of S, that if S is consistent then $\mathsf{S} \nvdash \operatorname{Con} \mathsf{S}$.

We conclude with a brief introduction to the theory of relative definability, specifically relative computability, and prove the celebrated Friedberg-Muchnik theorem, illustrating the important *priority* technique.

Chapter 5

Infinitarity

Contents

5.1 Introduction **314**
5.1.1 Countable infinitarity 315
5.1.2 Complexity over type 1 316
5.2 Definability of pointsets **320**
5.2.1 Recursive and semirecursive pointsets 321
5.2.2 Recursive functions 322
5.2.3 Recursive substitution 324
5.2.4 The Kleene pointclasses 326
5.2.5 Universal and complete pointsets 329
5.2.6 A normal form for Σ^1_1/Π^1_1 332
5.2.7 Closure properties of analytical classes 334
5.2.8 Relative definability 335
5.2.9 Topological aspects 336
5.2.10 Boldface universality and completeness 339
5.2.11 Tree representations 340
5.3 Classical descriptive set theory **345**
5.3.1 Introduction 345
5.3.2 Real numbers 345
5.3.3 Dedekind completion 349
5.3.4 Topology of reals 351
5.3.5 Metric spaces 352
5.3.6 Completeness 353
5.3.7 Polish spaces 355
5.3.8 The Borel hierarchy 357
5.3.9 The projective hierarchy 362
5.4 Structural properties of pointclasses **366**
5.4.1 Norms and the prewellordering property 368
5.4.2 Analytic boundedness 371
5.4.3 κ-Borel and κ-Suslin 372
5.4.4 Scales and uniformization 377
5.5 Perfect sets **382**
5.6 Category **385**

5.6.1 Category in product spaces 389
5.7 Measure 391
5.7.1 Extending a measure to a σ-algebra 393
5.7.2 Product measures 395
5.7.3 Measurability 395
5.8 Determinacy 397
5.8.1 Borel determinacy 400
5.9 Suslin's operation and analytic sets 407
5.10 Suslin's hypothesis 410
5.11 Limitations of ZF 411
5.12 Summary 412

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.[1]

Leopold Kronecker

It is well known that the man [Hippasus of Metapontum] who first made public the theory of the irrationals perished in a shipwreck in order that the inexpressible and unimaginable should ever remain veiled... and so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is forever beaten by the waves.

Scholium to EUCLID, *Elementa*, X, 1
Anonymous

[Yiannis Moschovakis's *Descriptive Set Theory*[17] is an excellent reference for the material in this chapter, as is Alexander Kechris's *Classical Descriptive Set Theory*[15], particularly for the classical theory, which begins in Section 5.3.]

5.1 Introduction

Chapter 4 is essentially a discussion of the theory of the finitary objects, which are conveniently represented in various (equivalent) ways, e.g., as hereditarily finite sets, with the natural theory F (the basic theory S of pure sets with an added axiom of finiteness), or as natural numbers, with the natural theory PA (Peano arithmetic). The important foundational issues that arise in this theory are those of definability, computability, and provability relating to the elementary theory of the structures, such as $(\mathrm{HF}; \in)$ and $(\omega; +, \cdot)$, consisting of finitary objects with the basic relations and operations that essentially define them. The objects themselves, although they are abstract, are generally regarded as existing in an absolute sense: We wonder whether Goldbach's conjecture is true or false, but we don't question whether it is meaningful or wonder what it means.[2]

The infinite enters into this discussion only in that there are infinitely many objects under consideration. Basic theories like F and PA do not recognize the existence of infinitary objects, and the use of theories like C, that recognize the existence of infinite classes but do not mandate the existence of infinite sets is, in this context, largely a convenience—simply a way of talking about attributes of

[1] God created the integers, all else is the work of Man.
[2] Well, if you are wondering what it means, it is this: Every even number greater than 2 is the sum of two prime numbers.

finitary objects in terms of the classes of objects they define. When we use such a theory to discuss finitary things, any classes that arise may be eliminated in favor of their definitions. As a conservative extension of S,[2.183] C does not prove any additional theorems about sets.

In declining to assert the existence of a "completed infinitude", C is quite in line with the attitude expressed by Kronecker in the headnote to this chapter. Kronecker's point was well taken, as later investigations into the nature of logic, meaning, truth, provability, decidability, etc.,[§4] have made abundantly clear. Nevertheless, it strikes the typical modern reader as a bit fussy. The problem is that much of mathematics is difficult or impossible to formulate without allowing infinitary objects as members of collections, arguments and values of functions, etc. For example, even so humble and primitive-appearing an object as a geometrical point is—as we will see—essentially infinitary, and the ancient concept of a curve as a "locus of points" calls upon us to accept points a members of classes.

The admission of infinite classes is of course essential for certain purposes. For example, it is not reasonable to discuss structures without allowing them to be infinite, and the completeness theorem is true only if infinite structures are allowed. Nevertheless, since a proper class, by definition, is not a member of anything, it does not exist in the same sense as a set. A full embrace of infinitarity requires the admission of infinite sets. This is accomplished by the addition of Infinity to S, resulting in the theory ZF^-, i.e., ZF with the Power axiom removed.[3] The addition of Infinity to C yields GB^-, i.e., GB$-$Power. Ontologically, C is intermediate between S and ZF^-, and GB^- goes a little beyond ZF^-.

It is consistent with ZF^- that all sets be countable. Likewise, it is consistent with GB^- that all sets be countable; however, it is not consistent that all classes be countable. In particular, just as—in the context of C—the universe V of sets is infinite, so—in the context of GB^-—V is uncountable.

It is a theorem of S that if the powerset $\mathcal{P}x$ of a set x exists, then $\mathcal{P}x$ is larger than x. Thus—in the context of ZF^-—if $\mathcal{P}\omega$ exists then it is uncountable. If $\mathcal{P}\mathcal{P}\omega$ exists, it is larger than $\mathcal{P}\omega$, and so on. The same is true in the context of GB^-, and the corresponding theories in this sequence are interleaved between those of the ZF sequence. In the theories ZF and GB, as we have seen in Chapter 3, the full Power axiom combines with Replacement to generate very large sets.

The purpose of this chapter is to study the fundamental significance of Infinity, and our focus is the countably infinitary. We adopt ZF as our basic theory for the convenience it affords, although we do not make much use of Power.

5.1.1 Countable infinitarity

The simplest sort of infinitary object is countably infinitary, in the sense that a countable amount of information is required to specify it. In the membership-theoretic context, this requires of a set not just that it have only countably many members, but that its members be countable, likewise its members' members, etc. The most general countably infinitary object in this context is therefore an hereditarily countable (HC) set.[3.141] Recall that a set x is hereditarily countable iff its transitive closure $\operatorname{tc} x$ is countable, i.e., there is a bijection $f : \omega \overset{\text{bij}}{\to} \operatorname{tc} x$. The structure $(x; \in)$ is therefore isomorphic to the structure $(\omega; E)$, where E is the binary relation on ω such that $\forall m, n \in \omega\, (\langle m, n\rangle \in E \leftrightarrow fm \in fn)$. E encodes all

[3] In the present context it would be more natural to call this theory 'S with Infinity added', but the use of 'ZF^-' is conventional.

the information in x and is for the present purpose interchangeable with x. Note that $E \in V_{\omega+1}$, so $V_{\omega+1}$ is a sufficiently large domain for the investigation of the countably infinitary.

(5.1) *Finitary and countably infinitary objects are referred to generically as* type-0 *and* type-1*, respectively.*

As the preceding discussion makes clear, type-0 objects may be taken to be members of V_ω, and type-1 objects may be taken to be members of $V_{\omega+1}$.

Our focus in this chapter on the countably infinitary is not as limiting as it might appear. According to the (downward) Löwenheim-Skolem theorem[2.159.1] any structure has a countable elementary substructure, so to some extent HC reflects the structure of the entire set-theoretical universe—no elementary theory can require its models to be uncountable. More interestingly, one of the fascinating aspects of the theory of the countably infinitary is how the assumption of the existence of large sets can answer many natural questions that are unresolvable in the natural theory appropriate to the context of the question, which is typically ZF^- or the extension of ZF^- by a limited Power axiom. In a rare instance, most notably the determinacy of Borel sets,[5.177] ZF suffices where ZF^- fails; more typically, ZF is also inadequate to the task, but so-called *large-cardinal hypotheses* suffice.

5.1.2 Complexity over type 1

The classification of sets by type may be extended beyond types 0 and 1. In general, we may say that a set x has type n or less iff $(\operatorname{tc} x; \in)$ is isomorphic to a structure $(y; E)$ with $y \in V_{\omega+n}$. In general, a set of type-n objects is of type-$(n+1)$; however, a definable set of type-n objects is typically regarded as having more to do with type-n than type-$(n+1)$. Indeed, the "theory of type-n" is largely the theory of definable subsets of type n.

Thus, just as the set-theoretic complexity of relations over type-0 is important to the study of foundational questions regarding finitary objects, the definability of relations on type-1 objects is central to their foundational study. Indeed, the subject of this chapter is commonly known as *descriptive set theory.*

Recall the Levy classification of s-formulas,[§4.4] where s is the signature of set theory:

1. A formula with only bounded quantification is Δ_0, and we define $\Sigma_0 = \Pi_0 = \Delta_0$.
2. If ϕ is Π_n (Σ_n) then $\exists u_0 \cdots \exists u_{m\text{-}}\ \phi$ ($\forall u_0 \cdots \forall u_{m\text{-}}\ \phi$) is Σ_{n+1} (Π_{n+1}).

Given a transitive set M, we may define a Σ_n (Π_n) relation on M to be a set $X \subset {}^m M$ such that for some Σ_n (Π_n) s-formula ϕ with $\langle u_0, \ldots, u_{m\text{-}} \rangle$ an enumeration of its free variables, for all $x = \langle x_0, \ldots, x_{m\text{-}} \rangle \in {}^m M$, $x \in X$ iff $(M, \in) \models \phi\left[\begin{smallmatrix} u_0 & \cdots & u_{m\text{-}} \\ x_0 & \cdots & x_{m\text{-}} \end{smallmatrix}\right]$. We define Δ_n to be $\Sigma_n \cap \Pi_n$.

For the discussion of classes of finitary objects, this classification is quite useful when specialized to $M = V_\omega$. To deal with countably infinitary objects we may let $M = V_{\omega+1}$, but now the distinction between bounded and unbounded quantification is insufficiently discriminating, and we must distinguish three sorts of quantification, based on degree of restriction.

1. Bounded quantification: The quantified variable is restricted to membership in an hereditarily finite set.

2. Type-0 quantification: The quantified variable is restricted to V_ω or some other suitable set of type-0 objects.

3. Type-1 quantification: The quantified variable is restricted to $V_{\omega+1}$ or some other suitable set of type-1 objects.

We will ultimately settle on ω and ${}^\omega\omega$ as canonical type-0 and type-1 domains, respectively, but for the purpose of this introduction, we will use $D_0 \overset{\text{def}}{=} V_\omega$ and $D_1 \overset{\text{def}}{=} {}^{D_0}2$. Note that a set $f \in D_1$ is a function from D_0 into $2 = \{0, 1\}$. It is therefore the characteristic function of a subset of D_0, and we have the bijection $f \mapsto f^{\leftarrow}\{1\} = \{a \in D_0 \mid f(a) = 1\}$ between D_1 and $\mathcal{P}\, D_0 = \mathcal{P}\, V_\omega = V_{\omega+1}$.

For the purpose of sketching the outlines of the theory of complexity of relations among type-1 objects with these domains, we will use the two-sorted structure $\mathfrak{D} = (D_0, D_1; E_0, E_1)$, where E_0 is the usual membership relation restricted to D_0, and E_1 is the relation of membership of a type-0 object in (the set whose characteristic function is) a type-1 object: $a\, E_1\, f \leftrightarrow f\,a = 1$. We use characteristic functions with domain D_0 instead of subsets of D_0 solely for the purpose of rendering D_1 disjoint from D_0 in the interest of clarity.

The discussion in this section should be regarded as introductory. Formal definitions and theorems will follow in Section 5.2 based on the principles arrived at here.

(5.2) *Let s^1 be a fixed signature appropriate to $\mathfrak{D}$ that expands the standard signature s of pure set theory. Thus, if all the variables occurring in ϕ are type-0 then ϕ is an s-formula.*

Suppose ϕ is an s^1-formula with free variables $v_0, \ldots, v_{m^-}$. Then the extension of ϕ $\overset{\text{def}}{=} \hat{\phi} \overset{\text{def}}{=}$ *the relation $R \subseteq D_{i_0} \times \cdots D_{i_{m^-}}$, where for each $k \in m$, i_k is the type $(0$ or $1)$ of the variable v_k, and for every $\langle a_0, \ldots, a_{m^-}\rangle \in D_{i_0} \times \cdots \times D_{i_{m^-}}$, $\langle a_0, \ldots, a_{m^-}\rangle \in R$ iff*

$$\mathfrak{D} \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{m^-} \\ a_0 & \cdots & a_{m^-} \end{smallmatrix}\right].[4]$$

(5.3) *Note that if all the variables occurring in ϕ are type-0 then $\hat{\phi}$ is the relation on $D_0 = V_\omega$ defined by ϕ regarded as an s-formula.*[5.2]

(5.4) *Here, 'ϕ' and 'ψ' refer to s^1-formulas.*

1. *ϕ is Δ^0_0 $\overset{\text{def}}{\iff}$ every quantification in ϕ is bounded, i.e., $\ulcorner \exists u\, E_0\, v \urcorner$ or $\ulcorner \forall u\, E_0\, v \urcorner$, where u and v are (by grammatical necessity) type-0 variables. ϕ is Σ^0_0 or Π^0_0 $\overset{\text{def}}{\iff}$ ϕ is Δ^0_0.*

2. *For all $n \in \omega$, ϕ is respectively Σ^0_{n+1} or Π^0_{n+1} $\overset{\text{def}}{\iff}$ ϕ is respectively $\exists u_0 \cdots \exists u_{m^-}\ \psi$ or $\forall u_0 \cdots \forall u_{m^-}\ \psi$, where $u_0, \ldots, u_{m^-}$ are type-0 variables and ψ is respectively Π^0_n or Σ^0_n.*

Suppose $n \in \omega$.

[4]Obviously, $\hat{\phi}$ depends on the chosen order of the variables $v_0, \ldots, v_{m^-}$, and we have also implicitly supposed that these have been listed without repetition. Properly we should define the extension of ϕ as the set of $\mathfrak{D}$-assignments A for ϕ such that $\operatorname{dom} A = \operatorname{Free}\phi$ and $\mathfrak{D} \models \phi[A]$. $\hat{\phi}$ is then a set of functions with domain $\operatorname{Free}\phi = \{v_0, \ldots, v_{m^-}\}$, rather than a set of functions with domain $m = \{0, \ldots, m^-\}$. In the interest of convenience, we are going to tolerate this little bit of ambiguity.

1. *R is respectively Σ^0_n or Π^0_n $\overset{\text{def}}{\iff}$ $R = \hat{\phi}$ for a formula ϕ that is respectively Σ^0_n or Π^0_n.*
2. *R is Δ^0_n $\overset{\text{def}}{\iff}$ R is both Σ^0_n and Π^0_n.*[5]

For the purpose of this introduction we are interested in Δ^0_1, Σ^0_1, and Π^0_1. Recall that the corresponding classes Δ_1, Σ_1, and Π_1 of relations on V_ω were characterized in Chapter 4 in terms of effective procedures. Briefly, $X \subseteq V_\omega$ is Σ_1 iff there is an effective procedure $\mathcal{P}$ such that for all $x \in V_\omega$, $x \in X$ iff $\mathcal{P}x$ halts. The Σ_1 subsets of V_ω are exactly the semirecursive sets.[6] Accordingly, X is Π_1 iff there is an effective procedure that halts exactly on its complement. X is Δ_1 iff there is an effective procedure $\mathcal{P}$ that always halts, such that for all $x \in V_\omega$, $\mathcal{P}x$ halts with output 1 if $x \in X$ and 0 if $x \notin X$. We say that $\mathcal{P}$ computes X in this case.

As noted above[5.3] these characterizations apply directly to Σ^0_1, Π^0_1, and Δ^0_1 relations (on $D_0 = V_\omega$) defined by s^1-formulas that do not contain any type-1 variables. The procedural point of view is just as useful in the general case of type-1 relations.

Suppose ϕ is Δ^0_0 with free variables $v_0, \ldots, v_{m-}$, and suppose for simplicity that $v_0, \ldots, v_{m'-}$ are type-0, and $v_{m'}, \ldots, v_{m-}$ are type-1. Suppose $a_0, \ldots, a_{m'-} \in D_0$ and $a \in D_0$ is transitive such that $\{a_0, \ldots, a_{m'-}\} \subseteq a$. Then for any $a_{m'}, \ldots, a_{m-} \in D_1$, in the interpretation of

$$\phi\begin{bmatrix} v_0 \cdots v_{m'-}\ v_{m'} \cdots v_{m-} \\ a_0 \cdots a_{m'-}\ a_{m'} \cdots a_{m-} \end{bmatrix}$$

in $\mathfrak{D}$, every quantification is bounded by a member of a,[7] so for each type-1 variable a_k ($k = m', \ldots, m-1$), the interpretation depends only on $a_k \restriction a$.

For $a \in D_0$ and $c \in D_1$, let $c \upharpoonleft a \overset{\text{def}}{=} (c \restriction a)^{\leftarrow}\{1\}$. Then for any $b \in a$, $b\, E_1\, a_k$ iff $b \in (a_k \upharpoonleft a)$. Let ϕ' be obtained from ϕ by replacing each type-1 variable v_k by a type-0 variable v'_k, and each subformula $\ulcorner u\, E_1\, v_k \urcorner$ by $\ulcorner u\, E_0\, v'_k \urcorner$. Then for any $a_{m'}, \ldots, a_{m-} \in D_1$,

$$\mathfrak{D} \models \phi\begin{bmatrix} v_0 \cdots v_{m'-}\ v_{m'} \cdots v_{m-} \\ a_0 \cdots a_{m'-}\ a_{m'} \cdots a_{m-} \end{bmatrix} \leftrightarrow \mathfrak{D} \models \phi'\begin{bmatrix} v_0 \cdots v_{m'-}\ \ v'_{m'} \ \ \cdots v'_{m-} \\ a_0 \cdots a_{m'-}\ (a_{m'} \upharpoonleft a) \cdots (a_{m-} \upharpoonleft a) \end{bmatrix}.$$

Thus, $\langle a_0, \ldots, a_{m-} \rangle \in \hat{\phi}$ iff

$$\text{(5.5)} \quad \exists a \in D_0 \big(\operatorname{Tran} a \wedge \{a_0, \ldots, a_{m'-}\} \subseteq a \\ \wedge \langle a_0, \ldots, a_{m'-}, a_{m'} \upharpoonleft a, \ldots, a_{m-} \upharpoonleft a \rangle \in \hat{\phi}' \big).$$

Also, $\langle a_0, \ldots, a_{m-} \rangle \in \hat{\phi}$ iff

$$\text{(5.6)} \quad \forall a \in D_0 \big((\operatorname{Tran} a \wedge \{a_0, \ldots, a_{m'-}\} \subseteq a) \\ \rightarrow \langle a_0, \ldots, a_{m'-}, a_{m'} \upharpoonleft a, \ldots, a_{m-} \upharpoonleft a \rangle \in \hat{\phi}' \big).$$

ϕ' is Δ^0_0, so $\hat{\phi}'$ is a recursive relation on D_0. Likewise,

$$\langle b^{\leftarrow}\{1\} \mid b \in D_0 \rangle,$$
$$\{b \in D_0 \mid b \text{ is transitive}\},$$

[5] Of course, this is redundant[5.4.1] when $n = 0$.

[6] A set $X \subseteq V_\omega$ is semirecursive iff there is an effective procedure that enumerates X, so semirecursive subsets of V_ω are also called *recursively enumerable*,[4.66.2.1] but this is not the best terminology to use for the extension of this notion to type-1.

[7] Since a is transitive, members of members of a are members of a, etc.

and

$$\{\langle b_0, \dots, b_{m'-}, a\rangle \in {}^{m'+1}D_0 \mid \{a_0, \dots, a_{m'-}\} \subseteq a\}$$

are recursive, so there is a recursive relation $R \subseteq {}^{m+1}D_0$ such that[5.5]

$$\langle a_0, \dots, a_{m-}\rangle \in \hat{\phi} \leftrightarrow \exists a \in D_0\ \langle a_0, \dots, a_{m'-}, a, a_{m'} \restriction a, \dots, a_{m-} \restriction a\rangle \in R. \tag{5.7}$$

Likewise, there is a recursive relation $R' \subseteq {}^{m+1}D_0$ such that[5.6]

$$\langle a_0, \dots, a_{m-}\rangle \in \hat{\phi} \leftrightarrow \forall a \in D_0\ \langle a_0, \dots, a_{m'-}, a, a_{m'} \restriction a, \dots, a_{m-} \restriction a\rangle \in R'.^{8} \tag{5.8}$$

It is not surprising that the Δ^0_0 relation $\hat{\phi}$ should be representable in terms of a recursive relation on D_0. The essential insight afforded by (5.7) and (5.8) is that this relation only looks at a finite part of the type-1 arguments.

In the terminology of Section 4.12 we may view (5.7) and (5.8) as describing effective procedures $\mathcal{P}$ and $\mathcal{P}'$ that take $a_0, \dots, a_{m'-}$ as input and use $a_{m'}, \dots, a_{m-}$ as oracles. In the present context, rather than giving the type-1 objects the special status of oracles, it is natural to regard them as inputs *per se*. The essential thing, as noted above, is that at any stage in a computation only a finite amount of information about the type-1 inputs has been accessed.

$\mathcal{P}$ and $\mathcal{P}'$ look successively at type-0 objects a, ordered in some effective way in an ω-sequence, say as $\vec{B}\,0$, $\vec{B}\,1, \dots$. $\mathcal{P}$ halts if and when it finds that $\langle a_0, \dots, a_{m'-}, a, a_{m'} \restriction a, \dots, a_{m-} \restriction a\rangle \in R$, whereas $\mathcal{P}'$ halts if and when it finds that $\langle a_0, \dots, a_{m'-}, a, a_{m'} \restriction a, \dots, a_{m-} \restriction a\rangle \notin R'$.

Clearly, $\mathcal{P}$ halts just in case $\langle a_0, \dots, a_{m-}\rangle \in \hat{\phi}$, and $\mathcal{P}'$ halts just in case $\langle a_0, \dots, a_{m-}\rangle \notin \hat{\phi}$. We may dovetail these computations to obtain a single effective procedure $\mathcal{P}''$ that always halts with the answer to the question 'is $\langle a_0, \dots, a_{m-}\rangle$ in $\hat{\phi}$?'.

In general, we call a relation on type-0 and type-1 objects *semirecursive* $\overset{\text{def}}{\iff}$ it is the set of inputs on which an effective procedure halts. A relation is *recursive* $\overset{\text{def}}{\iff}$ both it and its complement are semirecursive. Clearly, as in the case just presented, X is recursive just in case there is an effective procedure that halts for every input $\langle a_0, \dots, a_{m-}\rangle$ with the answer to the question 'is $\langle a_0, \dots, a_{m-}\rangle$ in X?'.

Thus, we have shown that Δ^0_0 relations are recursive. We now turn to Σ^0_1 and Π^0_1. Suppose ϕ is Σ^0_1, say $\phi = \exists u_0, \cdots \exists u_{k-}\ \psi$, where $u_0, \dots, u_{k-}$ are type-0 variables, and ψ is Δ^0_0.[5.4.2] Suppose without loss of generality that $u_0, \dots, u_{k-}$ are free in ψ, and let $\operatorname{Free}\psi$ be ordered as $u_0, \dots, u_{k-}, v_0, \dots, v_{m-}$, with $v_0, \dots, v_{m'-}$ of type-0 and $v_{m'}, \dots, v_{m-}$ of type-1 as before. Then for any $a_0, \dots, a_{m'-} \in D_0$ and $a_{m'}, \dots, a_{m-} \in D_1$,

$$\begin{aligned}
&\langle a_0, \dots, a_{m-}\rangle \in \hat{\phi}\\
&\quad \leftrightarrow \exists b_0, \dots, b_{k-} \in D_0\ \langle b_0, \dots, b_{k-}, a_0, \dots, a_{m-}\rangle \in \hat{\psi}\\
&\quad \leftrightarrow \exists b \in D_0\ \exists b_0, \dots, b_{k-} \in b\ \langle b_0, \dots, b_{k-}, a_0, \dots, a_{m-}\rangle \in \hat{\psi}\\
&\quad \leftrightarrow \exists b \in D_0\ \langle b, a_0, \dots, a_{m-}\rangle \in \hat{\psi}',
\end{aligned}$$

where $\psi' = \exists u_0, \dots, u_{k-} \in v\ \psi$, where v is a type-0 variable. Note that ψ' is Δ^0_0.

The point is that we may condense the string $\ulcorner \exists u_0, \cdots \exists u_{k-} \urcorner$ of existential type-0 quantifiers into the single existential type-0 quantifier $\ulcorner \exists u \urcorner$. Hence, without loss of

[8]The easy way to get this, of course, is to let R' be the complement of a relation R chosen to satisfy (5.7) with $\neg\,\phi$ for ϕ.

generality, we may suppose that $\phi = \exists u\ \psi$. Using the normal form (5.7) (with ψ for ϕ) there is a recursive $S \subseteq {}^{m+2}D_0$ such that

$$\begin{aligned}&\langle a_0, \ldots, a_{m^-}\rangle \in \hat{\phi}\\ &\quad \leftrightarrow \exists b \in D_0\, \exists c \in D_0\ \langle b, a_0, \ldots, a_{m'^-}, c, a_{m'} \restriction c, \ldots, a_{m^-} \restriction c\rangle \in S.\end{aligned}$$

Let R be the set of sequences $\langle a_0, \ldots, a_{m'^-}, a, c_{m'}, \ldots, c_{m^-}\rangle \in {}^{m+1}D_0$ such that a is transitive and there exist $b, c \in a$ such that

1. a is transitive,
2. $c_{m'}, \ldots, c_{m^-} \in {}^{a}2$, and
3. there exist $b, c \in a$ such that $\langle b, a_0, \ldots, a_{m'^-}, c, c_{m'} \restriction c, \ldots, c_{m^-} \restriction c\rangle \in S$.

Then R is recursive and

$$\langle a_0, \ldots, a_{m^-}\rangle \in \hat{\phi} \leftrightarrow \exists a \in D_0\ \langle a_0, \ldots, a_{m'^-}, a, a_{m'} \restriction a, \ldots, a_{m^-} \restriction a\rangle \in R. \tag{5.9}$$

Thus the normal form (5.7) which we derived for Δ^0_0 is applicable to Σ^0_1. In procedural terms, therefore, any Σ^0_1 relation T is semirecursive, i.e., there is an effective procedure $\mathcal{P}$ that halts exactly for those inputs that are in T.

Similarly, if ϕ is Π^0_1 with the same variable type sequence as above, there is a recursive $R \subseteq {}^{m+1}D_0$ such that

$$\langle a_0, \ldots, a_{m^-}\rangle \in \hat{\phi} \leftrightarrow \forall a \in D_0\ \langle a_0, \ldots, a_{m'^-}, a, a_{m'} \restriction a, \ldots, a_{m^-} \restriction a\rangle \in R. \tag{5.10}$$

Thus, if T is a Δ^0_1 relation there is an effective procedure that halts for all inputs $\langle a_0, \ldots, a_{m^-}\rangle$ (of the appropriate sequence of types) and answers the question 'is $\langle a_0, \ldots, a_{m^-}\rangle$ in T?', i.e., Δ^0_1 relations are recursive.

5.2 Definability of pointsets

Based on the informal remarks of the preceding section, we now begin the rigorous development of the theory of definability of pointsets. The use of $D_0 = V_\omega$ and $D_1 = {}^{D_0}2$ respectively as domains of type-0 and type-1 objects in the preceding discussion was a temporary choice facilitating the adaptation of the Levy hierarchy of set-theoretic complexity to define the classes Σ^0_1, Π^0_1, and Δ^0_1 and identify their principal structural properties: the normal forms (5.9) and (5.10), and the characterization of Δ^0_1 relations as recursive and Σ^0_1 as semirecursive.[9] With this accomplished we may proceed to develop the theory of classes derived from these by type-0 and type-1 quantification without explicit reference to the grammatical structure of formulas, although logical notation for set-theoretic operations will be natural and useful.

As noted above, ω and ${}^{\omega}\omega$ are the conventional paradigms for the classes of type-0 and type-1 objects, for reasons both historical and practical.

(5.11) **Definition** [ZF]

1. $U_0 \overset{\text{def}}{=} \omega$.
2. $U_1 \overset{\text{def}}{=} {}^{\omega}\omega$.

[9] The class Δ^0_0 has served its purpose and is no longer of any particular interest.

(5.12) **Definition** [ZF]

1. $\mathfrak{s}$ *is a* type $\overset{\text{def}}{\iff} \mathfrak{s} \in {}^{<\omega}2$ *and* $\mathfrak{s} \neq 0$.[10]
    1. $\mathfrak{s}$ *is a* 0-type $\overset{\text{def}}{\iff} \mathfrak{s} \in {}^{<\omega}1$, *i.e.*, $\mathfrak{s} = \langle 0, 0, \ldots, 0 \rangle$.
    2. $\mathfrak{s}$ *is a* 1-type $\overset{\text{def}}{\iff}$ *it is not a 0-type.*
    3. $\mathfrak{s}$ *is a* pure 1-type $\overset{\text{def}}{\iff} \mathfrak{s} \in {}^{<\omega}\{1\}$, *i.e.*, $\mathfrak{s} = \langle 1, 1, \ldots, 1 \rangle$.
2. *Given a type* $\mathfrak{s}$ *of length* m, $U_{\mathfrak{s}} \overset{\text{def}}{=} \times_{k \in m} U_{\mathfrak{s}_k}$.[11]
3. x *is a* point[12] $\overset{\text{def}}{\iff} x \in U_{\mathfrak{s}}$ *for some type* $\mathfrak{s}$.
4. *Suppose* x *is a point and* $k < |x|$. *Then the* kth coordinate of $x \overset{\text{def}}{=} x_k$.
5. U *is a* pointspace $\overset{\text{def}}{\iff} U = U_{\mathfrak{s}}$ *for some type* $\mathfrak{s}$.
6. *The* type *of a point* $x \overset{\text{def}}{=}$ *the (unique*[13]*) type* $\mathfrak{s}$ *such that* $x \in U_{\mathfrak{s}}$.
7. X *is a* pointset $\overset{\text{def}}{\iff} X \subseteq U$ *for some pointspace* U.
8. *The* type *of a nonempty pointset* X *is the (unique) type* $\mathfrak{s}$ *such that* $X \subseteq U_{\mathfrak{s}}$. *The empty set is a pointset of every type.*
9. *Suppose* U *is a pointspace and* $X \subseteq U$. *Then* $\neg X \overset{\text{def}}{=} U \backslash X$.[14]
10. Γ *is a* pointclass $\overset{\text{def}}{\iff} \Gamma$ *is a set of pointsets.*

Note that we do not define the type of a pointclass. The important pointclasses contain pointsets of all types.

For $i \in 2$, there is a natural correspondence $x \mapsto \langle x \rangle$ between U_i and $U_{\langle i \rangle}$, and we will not always maintain this distinction scrupulously.

To indicate the type of a quantified variable we may use the following convention: '$\exists^0$' and '$\forall^0$' introduce variables of type-0, while '$\exists^1$' and '$\forall^1$' introduce variables of type-1.

5.2.1 Recursive and semirecursive pointsets

We will model the definition of Σ^0_1 for pointsets on (5.9). For this purpose we make the following definition.

Definition [ZF] *Suppose* $\mathfrak{s}$ *is a type of length* m, *and* $n \in \omega$.

1. *Suppose* x *is a point of type* $\mathfrak{s}$. *Then the* pointwise restriction of x to $n \overset{\text{def}}{=} x \,|\, n$ $\overset{\text{def}}{=}$ *the sequence* x' *of length* m *such that for all* $k \in m$,
    1. *if* $\mathfrak{s}_k = 0$ *then* $x'_k = x_k$; *and*
    2. *if* $\mathfrak{s}_k = 1$ *then* $x'_k = x_k \restriction n$.

[10] Recall that $2 = \{0, 1\}$, and ${}^{<\omega}2$ is the set of finite sequences of 0s and 1s.

[11] For example, if $\mathfrak{s} = \langle 0, 0, 1 \rangle$ then $U_{\mathfrak{s}} = U_0 \times U_0 \times U_1$. Note that $\mathfrak{s} \neq \mathfrak{s}' \to U_{\mathfrak{s}} \cap U_{\mathfrak{s}'} = 0$.

[12] The appropriateness of the word 'point' for this notion will become obvious. For now let it be said that a member of m-dimensional euclidean space is essentially a point of type $\mathfrak{s}$, where $\mathfrak{s}$ is a pure 1 type of length m (and we take U_1 for the sake of illustration to be the set $\mathbb{R}$ of real numbers).

[13] See the previous footnote but one.

[14] If X is a nonempty pointset then it is a subset of only one pointspace, so the identity of U may be inferred from X, and this definition of $\neg X$ is unambiguous. Obviously $\neg 0$ could be any pointspace. In practice the relevant pointspace is clear from the context.

Thus, pointwise restriction of the point x to n restricts the type-1 coordinates of x to n and leaves the type-0 coordinates alone.

2. $U_{\mathfrak{s}}^n \stackrel{\text{def}}{=} \{x \mid n \mid x \in U_{\mathfrak{s}}\}$.
3. $U_{\mathfrak{s}}^{<\omega} = \bigcup_{n\in\omega} U_{\mathfrak{s}}^n$.
4. *For $x \in U_{\mathfrak{s}}^n$ and $n' \in \omega$ we define $x \mid n'$ in the obvious way so that $(x \mid n) \mid n' = x \mid n''$, where $n'' = \min(n, n')$.*

Note that $U_{\mathfrak{s}}^{<\omega}$ is a recursive subset of V_ω.

(5.13) **Definition** [ZF] *Suppose $\mathfrak{s}$ is a type and $X \subseteq U_{\mathfrak{s}}$.*

1. *X is Σ_1^0 $\stackrel{\text{def}}{\iff}$ there is a recursive $R \subseteq U_{\mathfrak{s}}^{<\omega}$ such that for all $x \in U_{\mathfrak{s}}$,*

$$x \in X \leftrightarrow \exists a \in \omega \; x \mid a \in R.$$

2. *X is Π_1^0 $\stackrel{\text{def}}{\iff}$ $\neg X$ is Σ_1^0, i.e., there is a recursive $R \subseteq U_{\mathfrak{s}}^{<\omega}$ such that for all $x \in U_{\mathfrak{s}}$,*

$$x \in X \leftrightarrow \forall a \in \omega \; x \mid a \in R.$$

3. *x is Δ_1^0 $\stackrel{\text{def}}{\iff}$ x is both Σ_1^0 and Π_1^0.*
4. *X is* recursive $\stackrel{\text{def}}{\iff}$ *X is Δ_1^0, and X is* semirecursive $\stackrel{\text{def}}{\iff}$ *X is Σ_1^0.*

5.2.2 Recursive functions

Note the double use of 'recursive' in (5.13). First we refer to recursive subsets of V_ω in order to define Σ_1^0 and Π_1^0 as applied to pointsets; then we use these to define 'recursive' and 'semirecursive' as applied to pointsets. The rationale for the first definition has been supplied in the introduction to this chapter. Underlying that discussion was the intuitive understanding of effective procedures embodied in the Church-Turing thesis—that the halting sets of effective procedures with type-0 inputs are exactly the sets that are Σ_1-definable over $(V_\omega; \in)$. In that discussion we also had occasion to discuss effective procedures with type-1 inputs—a concept with which we were already familiar as computation with oracles. Specifically, we were concerned with procedures $\mathcal{P}$ that—in the present setting—accept points of any given type as input and produce a type-0 output. If $\mathcal{P}$ is used to define a semirecursive set, its output may be regarded as the mere fact of halting. If it is used to define a recursive set, it always halts, and its output is 0 or 1.

In Chapter 4 we also had occasion to discuss *interminable procedures*,[4.58] whose output is a sequence (of type-0 objects) of finite length or of length ω, thus potentially a type-1 object. We may also allows such a procedure to accept points of any given type as input, and we will refer to this as a *type*-1 *procedure*.

A function $F : U_{\mathfrak{s}} \to U_1$ may reasonably be said to be recursive if there is an effective type-1 procedure $\mathcal{P}$ that, given input $x \in U_{\mathfrak{s}}$, enumerates Fx,[15] and we may say that $\mathcal{P}$ computes F. Note that F is assumed to be total, so for any $x \in U_{\mathfrak{s}}$, $\mathcal{P}x$ enumerates a function, i.e., for all $a \in \omega$ there is a unique $b \in \omega$ such that $\mathcal{P}x$ outputs (a, b) at some point.

[15] Fx is a function from ω to ω, so it is a set of ordered pairs from ω, and to enumerate Fx is to list all $(a, b) \in Fx$, i.e., all (a, b) such that $(Fx)a = b$.

Suppose F is a recursive function in this sense, and let $F^* = \{x^\frown\langle a,b\rangle \mid x \in U_{\mathfrak{s}} \wedge (Fx)a = b\}$. Then F^* is Σ^0_1. To see this let $\mathcal{P}'$ be a procedure with input from $U_{\mathfrak{s}^\frown\langle 0,0\rangle}$ that simulates $\mathcal{P}$ and halts for input $x^\frown\langle a,b\rangle$ if and when $\mathcal{P}x$ outputs (a,b). Then $\mathcal{P}'(x^\frown\langle a,b\rangle)$ halts iff $x^\frown\langle a,b\rangle \in F^*$, so F^* is Σ^0_1.

F^* is also Π^0_1. To see this, let $\mathcal{P}''$ be a procedure with input from $U_{\mathfrak{s}^\frown\langle 1,1\rangle}$ that simulates $\mathcal{P}$ and halts for input $x^\frown\langle a,b\rangle$ if and when $\mathcal{P}x$ outputs (a,b') for some $b' \neq b$. Since F is total and $Fx \in {}^\omega\omega$, for any $a \in \omega$, $\mathcal{P}x$ outputs (a,b') for exactly one b'; hence, $\mathcal{P}''(x^\frown\langle a,b\rangle)$ halts iff $x^\frown\langle a,b\rangle \notin F^*$, so F^* is Π^0_1.

Thus, if F is recursive in the above sense then F^* is Δ^0_1, i.e., recursive as a subset of $U_{\mathfrak{s}^\frown\langle 0,0\rangle}$.[5.13.4] Conversely, if F^* is recursive then a procedure that decides it is easily adapted to a procedure that computes F. This analysis justifies the following definition.

(5.14) **Definition** [ZF]

1. *Suppose* $\mathfrak{s}$ *is a type and* $F : U_{\mathfrak{s}} \to U_0$. F *is* recursive $\overset{\text{def}}{\iff}$ *the relation* $\{x^\frown\langle a\rangle \mid Fx = a\}$ *is recursive.*

2. *Suppose* $\mathfrak{s}$ *is a type and* $F : U_{\mathfrak{s}} \to U_1$. F *is* recursive $\overset{\text{def}}{\iff}$ *the relation* $\{x^\frown\langle a,b\rangle \mid (Fx)a = b\}$ *is recursive.*

3. *Suppose* $\mathfrak{s}, \mathfrak{t}$ *are types and* $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$. F *is* recursive $\overset{\text{def}}{\iff}$ *for each* $k < |\mathfrak{t}|$, *the function* $x \mapsto (Fx)_k$ *is recursive.*[16]

We emphasize that in the consideration of recursive functions F *from* $U_{\mathfrak{s}}$ *to* $U_{\mathfrak{t}}$, *where* $\mathfrak{t}$ *is a 1-type, we require that* F *be total. Note that we do not define 'semirecursive' in this case.*

The following theorem lists some handy properties of recursive functions.

(5.15) **Theorem** [ZF] *Suppose* $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}$ *are types.*

1. *Suppose* $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$. *Then* F *is recursive iff for each* $k < |\mathfrak{t}|$, *the function* F_k *defined by the condition that for each* $x \in U_{\mathfrak{s}}$, $F_k x = (Fx)_k$, *is recursive.*

2. (Projection) *Suppose* $m \leqslant |\mathfrak{s}|$. *Then the function* $F : U_{\mathfrak{s}} \to U_{\mathfrak{s}\restriction m}$ *defined by the condition that for all* $x \in U_{\mathfrak{s}}$, $Fx = x \restriction m$, *is recursive.*

3. (Composition) *Suppose* $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$ *and* $G : U_{\mathfrak{t}} \to U_{\mathfrak{u}}$ *are recursive. Then* $G \circ F$ *is recursive.*

4. (Permutation of coordinates) *Suppose* $\mathfrak{s}$ *is a type and* $\pi : |\mathfrak{s}| \overset{\text{bij}}{\to} |\mathfrak{s}|$. *Then* $\mathfrak{s}' = \mathfrak{s} \circ \pi$ *is also a type, and* $x \mapsto x \circ \pi$ *is a recursive function from* $U_{\mathfrak{s}}$ *to* $U_{\mathfrak{s}'}$.

5. (Recursive constant) *Suppose* $y \in U_{\mathfrak{t}}$ *is such that for every* $k < |\mathfrak{t}|$, *if* $\mathfrak{t}_k = 1$ *then* y_k *is recursive (i.e., a total recursive function). Then the function* $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$ *defined by the condition that for all* $x \in U_{\mathfrak{s}}$, $Fx = y$, *is recursive.*

Proof Straightforward. $\square$[5.15]

[16] We are invoking (5.14.1) or (5.14.2) according as $\mathfrak{t}_k = 0$ or 1.

5.2.3 Recursive substitution

Suppose $\mathfrak{s}, \mathfrak{t}$ are types, $Y \subseteq U_{\mathfrak{t}}$, and $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$ is recursive. Let $X = F^{\leftarrow}Y$. Then X is in a natural sense no more complicated than Y (assuming a measure of complexity such that all recursive relations have minimum complexity). We may say that X is derived from Y by *recursive substitution*, in that if we think of X and Y as predicates then $X(x) \leftrightarrow Y(Fx)$, i.e., X derives from the substitution of the "term" 'Fx' for the "free variable of Y".

Definition [ZF] *Suppose* Γ *is a pointclass.* Γ *is* recursively closed $\overset{\text{def}}{\iff}$ Γ *is closed under recursive substitution, i.e., for any types s and t, any* $Y \in \Gamma$ *of type t, and any recursive* $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$, $F^{\leftarrow}Y \in \Gamma$.

Note that the condition that Γ be recursively closed does not just impose a condition on $\Gamma \cap \mathcal{P}U_{\mathfrak{s}}$ for each type s individually; it also relates $\Gamma \cap \mathcal{P}U_{\mathfrak{s}}$ and $\Gamma \cap \mathcal{P}U_{\mathfrak{t}}$ for distinct types $\mathfrak{s}$ and $\mathfrak{t}$ via recursive functions from $U_{\mathfrak{s}}$ to $U_{\mathfrak{t}}$ and *vice versa*. It will be useful to have standard recursive isomorphisms of the $U_{\mathfrak{s}}$s, which the following definition provides.

Definition [ZF]

1. *Suppose* $m > 0$. $F_m \overset{\text{def}}{=}$ *the* $\prec$*-increasing enumeration of* ${}^m\omega$, *where* $\prec$ *is the canonical* ω*-ordering of* V_ω.[3.211.3] *Thus,* $F : \omega \overset{\text{bij}}{\to} {}^m\omega$. *Suppose* $\mathfrak{s}$ *is a* 0*-type. Then* $F_{\mathfrak{s}} \overset{\text{def}}{=} F_{|\mathfrak{s}|}$. *Note that if* $\mathfrak{s}$ *is a* 0*-type, then* $U_{\mathfrak{s}} = {}^{|\mathfrak{s}|}\omega$, *so* $F_{|\mathfrak{s}|} : \omega \overset{\text{bij}}{\to} U_{\mathfrak{s}}$.
2. *Suppose* $\mathfrak{s}$ *is a* 1*-type. For any* $x \in U_{\mathfrak{s}}$ *let* x^0 *be the sequence of type-*0 *coordinates of* x *in the order in which they occur in* x, *and let* x^1 *be the sequence of the type-*1 *coordinates of* x *in the order in which they occur in* x. *Let* m^0 *and* m^1 *be the respective lengths of* x^0 *and* x^1. *Thus,* $x^0 \in {}^{m^0}\omega$ *and* $x^1 \in {}^{m^1}({}^\omega\omega)$. *(Note that* m^0 *may be* 0, *but* $m^1 > 0$.*) Let* $\hat{x} \in {}^\omega\omega$ *be defined by the condition that for any* $n \in \omega$, $F_{m^1}\hat{x}_n = \langle (x^1_k)_n \mid k < m^1 \rangle$. *Let* $G : U_{\mathfrak{s}} \to {}^\omega\omega$ *be such that for any* $x \in U_{\mathfrak{s}}$, $Gx = x^0 {}^\frown \hat{x}$. *Clearly,* G *is a bijection.* $F_{\mathfrak{s}} \overset{\text{def}}{=} G^{-1}$.

(5.16) **Theorem** [ZF] *For any type* $\mathfrak{s}$, $F_{\mathfrak{s}}$ *and* ${F_{\mathfrak{s}}}^{-1}$ *are recursive; and, as noted in the definition,*

1. *if* $\mathfrak{s}$ *is a 0-type then* $F_{\mathfrak{s}} : \omega \overset{\text{bij}}{\to} U_{\mathfrak{s}}$*; and*
2. *if* $\mathfrak{s}$ *is a 1-type then* $F_{\mathfrak{s}} : {}^\omega\omega \overset{\text{bij}}{\to} U_{\mathfrak{s}}$.

Proof *Straightforward.* □[5.16]

(5.17) **Theorem** [ZF] *A recursively closed pointclass* Γ *is uniquely determined by* $\Gamma \cap \mathcal{P}U_1$.

Proof Since Γ is recursively closed, for any 1-type $\mathfrak{s}$, $\forall X \subseteq U_{\mathfrak{s}}\,(X \in \Gamma \leftrightarrow {F_{\mathfrak{s}}}^{\leftarrow}X \in \Gamma)$, so $\Gamma \cap \mathcal{P}U_1$ uniquely determines $\Gamma \cap \mathcal{P}U_{\mathfrak{s}}$. By a similar argument using the recursive bijections $F_{\mathfrak{s}} : \omega \overset{\text{bij}}{\to} U_{\mathfrak{s}}$ for 0-types $\mathfrak{s}$, $\Gamma \cap \mathcal{P}U_0$ uniquely determines $\Gamma \cap \mathcal{P}U_{\mathfrak{s}}$ for any 0-type $\mathfrak{s}$.

It remains to be shown that $\Gamma \cap \mathcal{P}U_1$ uniquely determines $\Gamma \cap \mathcal{P}U_0$. For this the recursive maps $n \mapsto \langle n, 0, 0, \dots \rangle$ from ω into ${}^\omega\omega$ and $x \mapsto x_0$ from ${}^\omega\omega$ to ω may be used to show that $\forall X \subseteq \omega\,(X \in \Gamma \leftrightarrow \{x \in {}^\omega\omega \mid x_0 \in X\} \in \Gamma)$. □[5.17]

It is certainly not the case that $\Gamma \cap \mathcal{P} U_0$ $(= \Gamma \cap \mathcal{P}\omega)$ uniquely determines a recursively closed pointclass Γ. Indeed, classical descriptive set theory deals exclusively with pointclasses Γ for which $\Gamma \cap \mathcal{P}\omega = \mathcal{P}\omega$.

Recursively closed pointclasses are particularly amenable to the definition of operations corresponding to various logical operations.

(5.18) **Definition** [ZF] *Suppose Γ is a recursively closed pointclass, Q is either $\exists$ or $\forall$, and $i \in 2$.*

1. $\neg\,\Gamma \overset{\text{def}}{=}$ *the class of pointsets X such that for some $Y \in \Gamma$ of type $\mathfrak{s}$, $X = U_{\mathfrak{s}} \backslash Y$.*
2. *The* dual of $\Gamma \overset{\text{def}}{=} \neg\,\Gamma$*. In the interest of notational compression, $\breve{\Gamma} \overset{\text{def}}{=} \neg\,\Gamma$.*
3. $\vee\,\Gamma \overset{\text{def}}{=}$ *the class of pointsets X such that for some $Y, Y' \in \Gamma$ of type $\mathfrak{s}$, $X = Y \cup Y'$.*
4. $\wedge\,\Gamma \overset{\text{def}}{=}$ *the class of pointsets X such that for some $Y, Y' \in \Gamma$ of type $\mathfrak{s}$, $X = Y \cap Y'$.*
5. $\mathsf{Q}^{<}\Gamma \overset{\text{def}}{=}$ *the class of pointsets X such that for some type $\mathfrak{s}$ and $X' \in \Gamma$ of type $\langle 0\rangle^\frown \mathfrak{s}$, $X = \{\langle a\rangle^\frown x \in U_{\langle 0\rangle^\frown \mathfrak{s}} \mid \mathsf{Q} b < a\ (\langle b\rangle^\frown x) \in X'\}$.*
6. $\mathsf{Q}^{i}\Gamma \overset{\text{def}}{=}$ *the class of pointsets X such that for some type $\mathfrak{s}$ and $X' \in \Gamma$ of type $\langle i\rangle^\frown \mathfrak{s}$, $X = \{x \in U_{\mathfrak{s}} \mid \mathsf{Q}^{i} a\ (\langle a\rangle^\frown x) \in X'\}$.*

It is not important that we have defined $\mathsf{Q}^{<}$ and Q^{i} by quantification over first coordinates, as we can use a permutation operation to bring any coordinate to the first position.[5.15.4]

(5.19) **Theorem** [ZF] *Suppose Γ is a recursively closed pointclass. Then $\neg\,\Gamma$ and $\mathsf{Q}^{i}\Gamma$ are recursively closed for $\mathsf{Q} = \exists$ or $\forall$ and $i \in 2$. Σ^0_1 is recursively closed; hence, Π^0_1 and Δ^0_1 are recursively closed.*

Proof Suppose $X \in \neg\,\Gamma$, say $X = U_{\mathfrak{s}} \backslash X'$ for some $X' \in \Gamma$. Suppose $U_{\mathfrak{t}}$ is a pointspace and $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ is recursive. Let $Y = F^{\leftarrow}X$. Then for all $y \in U_{\mathfrak{t}}$

$$\begin{aligned} y \in Y &\leftrightarrow Fy \in X \leftrightarrow Fy \notin X' \\ &\leftrightarrow y \notin F^{\leftarrow}X'. \end{aligned}$$

Since Γ is recursively closed, $F^{\leftarrow}X' \in \Gamma$, so $Y \in \neg\,\Gamma$. Hence $\neg\,\Gamma$ is recursively closed.

Now suppose $X \in \mathsf{Q}^{i}\Gamma$, say $X = \{x \in U_{\mathfrak{s}} \mid \mathsf{Q}^{i} a\ (\langle a\rangle^\frown x) \in X'\}$, where $X' \in \Gamma$. Suppose $U_{\mathfrak{t}}$ is a pointspace and $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ is recursive, and let $F' : U_{\langle i\rangle^\frown t} \to U_{\langle i\rangle^\frown \mathfrak{s}}$ be defined by the condition that for any $\langle a\rangle^\frown y \in U_{\langle i\rangle^\frown \mathfrak{t}}$, $F'(\langle a\rangle^\frown y) = \langle a\rangle^\frown(Fy)$. Note that F' is recursive, so $F'^{\leftarrow}X' \in \Gamma$. Let $Y = F^{\leftarrow}X$. Then for all $y \in U_{\mathfrak{t}}$

$$\begin{aligned} y \in Y &\leftrightarrow Fy \in X \leftrightarrow \mathsf{Q}^{i} a\ (\langle a\rangle^\frown(Fy)) \in X' \leftrightarrow \mathsf{Q}^{i} a\ F'(\langle a\rangle^\frown y) \in X' \\ &\leftrightarrow \mathsf{Q}^{i} a\ (\langle a\rangle^\frown y) \in F'^{\leftarrow}X', \end{aligned}$$

so $Y \in \mathsf{Q}^{i}\Gamma$. Hence $\mathsf{Q}^{i}\Gamma$ is recursively closed.

Finally, suppose X is Σ^0_1, say[5.13.1] $X = \{x \in U_{\mathfrak{s}} \mid \exists^0 a\ x \,|\, a \in R\}$, where $R \subseteq U_{\mathfrak{s}}^{<\omega}$ is recursive. Suppose $U_{\mathfrak{t}}$ is a pointspace and $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ is recursive. Let $Y = F^{\leftarrow}X$. Then for all $y \in U_{\mathfrak{t}}$

$$y \in Y \leftrightarrow Fy \in X \leftrightarrow \exists^0 a\ (Fy) \,|\, a \in R.$$

Let $\mathcal{P}$ be an effective procedure that, given input $y \in U_{\mathfrak{t}}$, enumerates Fy. Let S consist of all sequences of the form $y \,|\, b$ with $y \in U_{\mathfrak{t}}$ and $b \in \omega$, such that $\mathcal{P}$, using only the information contained in $y \,|\, b$, computes an initial segment $(Fy) \,|\, a$ of Fy in fewer than b steps, such that $(Fy) \,|\, a \in R$. S is recursive (because of the limitation on the number of steps), and

$$y \in Y \leftrightarrow \exists^0 a\ (Fy) \,|\, a \in R \leftrightarrow \exists^0 b\ y \,|\, b \in S,$$

so Y is Σ^0_1. Hence Σ^0_1 is recursively closed.

It follows that $\Pi^0_1 = \neg\, \Sigma^0_1$ is recursively closed, and therefore $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$ is as well. □[5.19]

5.2.4 The Kleene pointclasses

With (5.13) as the foundation, we now extend the type-1 definability hierarchy by quantification operations. The classes so defined are the *Kleene pointclasses*, named after Stephen C. Kleene, who laid the foundations and developed much of the original theory of recursion in higher types. We take the classes Σ^0_1 and Π^0_1 already defined[5.13] as a foundation.

(5.20) **Definition: The Kleene pointclasses** [ZF] *Suppose* $n > 0$*.*

1. $\Sigma^0_{n+1} \overset{\text{def}}{=} \exists^0\, \Pi^0_n$.
2. $\Pi^0_{n+1} \overset{\text{def}}{=} \forall^0\, \Sigma^0_n$.
3. Arithmetical $\overset{\text{def}}{=} \bigcup_{n\in\omega} \Sigma^0_n$.
4. $\Sigma^1_1 \overset{\text{def}}{=} \exists^1$ Arithmetical.
5. $\Pi^1_1 \overset{\text{def}}{=} \forall^1$ Arithmetical.
6. $\Sigma^1_{n+1} \overset{\text{def}}{=} \exists^1\, \Pi^1_n$.
7. $\Pi^1_{n+1} \overset{\text{def}}{=} \forall^1\, \Sigma^1_n$.
8. Analytical $\overset{\text{def}}{=} \bigcup_{n\in\omega} \Sigma^1_n$.
9. *For* $i \in 2$, $\Delta^i_n \overset{\text{def}}{=} \Sigma^i_n \cap \Pi^i_n$.

(5.21) **Theorem** [ZF] *The Kleene pointclasses are recursively closed.*

Proof We use Theorem 5.19 to prove this by induction, simultaneously showing that (5.20) is legitimate, the quantification operations only being defined for recursively closed pointclasses.[5.18] □[5.21]

(5.22) **Theorem** [ZF] *Suppose* $n > 0$*.*

1. $\Sigma^0_n = \neg\,\Pi^0_n$.
2. Σ^0_n *and* Π^0_n *are closed under the operations* $\exists^<$ *and* $\forall^<$.
3. Σ^0_n *is closed under* $\exists^0$ *and* Π^0_n *is closed under* $\forall^0$.
4. Σ^0_n *and* Π^0_n *are closed under disjunction and conjunction.*

Proof We proceed by induction on n, so in the following discussion we suppose that either $n = 1$ or that $n > 1$ and the theorem holds for $n - 1$.

1 This follows immediately from the definition (5.13.2) if $n = 1$, and from De Morgan's laws and the induction hypothesis if $n > 1$. $\square$[5.22.1]

2

(5.23) **Claim** $\exists^<\Sigma^0_n \subseteq \Sigma^0_n$.

Proof It suffices to show that for any type $\mathfrak{s} = \mathfrak{t}^\frown\langle 0\rangle$, and any Σ^0_n set $X \subseteq U_\mathfrak{s}$, letting $Y = \{x^\frown\langle a\rangle \in U_\mathfrak{s} \mid \exists b < a\ x^\frown\langle b\rangle \in X\}$, Y is Σ^0_n.

If $n = 1$ then by definition[5.13.1] there is a recursive $R \subseteq U_\mathfrak{s}^{<\omega}$ such that for all $x \in U_\mathfrak{s}$,

$$x \in X \leftrightarrow \exists^0 c\ x \,|\, c \in R.$$

Thus,

$$\begin{aligned} x^\frown\langle a\rangle \in Y &\leftrightarrow \exists b < a\ x^\frown\langle b\rangle \in X \leftrightarrow \exists b < a\,\exists^0 c\ (x\,|\,c)^\frown\langle b\rangle \in R \\ &\leftrightarrow \exists^0 c\,\exists b < a\ (x\,|\,c)^\frown\langle b\rangle \in R \leftrightarrow \exists^0 c\ (x\,|\,c)^\frown\langle a\rangle \in R', \end{aligned}$$

where $R' = \{y^\frown\langle a\rangle \in U^{<\omega}_{\mathfrak{s}^\frown\langle 0\rangle} \mid \exists b < a\ y^\frown\langle b\rangle \in R\}$. R' is recursive, so Y is Σ^0_1.

If $n > 1$ then by definition[5.20.1]

$$x \in X \leftrightarrow \exists^0 c\ x^\frown\langle c\rangle \in S,$$

where $S \subseteq U_{\mathfrak{s}^\frown\langle 0\rangle}$ is $\Pi^0_{n^-}$.

Thus,

$$\begin{aligned} x^\frown\langle a\rangle \in Y &\leftrightarrow \exists b < a\ x^\frown\langle b\rangle \in X \leftrightarrow \exists b < a\,\exists^0 c\ x^\frown\langle b, c\rangle \in S \\ &\leftrightarrow \exists^0 c\,\exists b < a\ x^\frown\langle b, c\rangle \in S \leftrightarrow \exists^0 c\ x^\frown\langle a, c\rangle \in S', \end{aligned}$$

where $S' = \{x^\frown\langle a, c\rangle \in U_{\mathfrak{s}^\frown\langle 0,0\rangle} \mid \exists b < a\ x^\frown\langle b, c\rangle \in S\}$. Since S is $\Pi^0_{n^-}$, by induction hypothesis, S' is $\Pi^0_{n^-}$, so Y is Σ^0_n. $\square$[5.23]

(5.24) **Claim** $\forall^<\Sigma^0_n \subseteq \Sigma^0_n$.

Proof It suffices to show that for any type $\mathfrak{s} = \mathfrak{t}^\frown\langle 0\rangle$, and any Σ^0_n set $X \subseteq U_\mathfrak{s}$, letting $Y = \{x^\frown\langle a\rangle \in U_\mathfrak{s} \mid \forall b < a\ x^\frown\langle b\rangle \in X\}$, Y is Σ^0_n.

We proceed largely as in the proof of (5.23), with the additional feature that we use the collection principle: For any $S \subseteq {}^2\omega$, $\forall a < b\,\exists^0 c\ \langle a, c\rangle \in S \to \exists^0 c'\,\forall a < b\,\exists c < c'\ \langle a, c\rangle \in S$.

Thus, if $n = 1$, with R as above,

$$\begin{aligned} x^\frown\langle a\rangle \in Y &\leftrightarrow \forall b < a\,\exists^0 c\ (x\,|\,c)^\frown\langle b\rangle \in R \leftrightarrow \exists^0 c'\,\forall b < a\,\exists c < c'\ (x\,|\,c)^\frown\langle b\rangle \in R \\ &\leftrightarrow \exists^0 c'\ (x\,|\,c')^\frown\langle a\rangle \in R', \end{aligned}$$

where R' is the set of $y^\frown\langle a\rangle \in U^{<\omega}_{\mathfrak{s}^\frown\langle 0\rangle}$ such that, letting c' be such that $y^\frown\langle a\rangle \in U^{c'}_{\mathfrak{s}^\frown\langle 0\rangle}$, for all $b < a$ there exists $c < c'$ such that $(y\,|\,c)^\frown\langle b\rangle \in R$. R' is recursive, so Y is Σ^0_1.

The case that $n > 1$ employs the collection principle in a similar way. $\square^{5.24}$

The corresponding results for Π^0_n are obtained *mutatis mutandis*, or directly by applying (5.22.1) and De Morgan's laws. $\square^{5.22.2}$

3 Suppose $\mathfrak{t}$ is a type, $\mathfrak{s} = \mathfrak{t}^\frown\langle 0\rangle$, $X \subseteq U_{\mathfrak{s}}$ is Σ^0_1 and $Y = \{y \in U_{\mathfrak{t}} \mid \exists a\; y^\frown\langle a\rangle \in X\}$. Suppose first that X is Σ^0_1, and let $R \subseteq U^{<\omega}_{\mathfrak{s}}$ be recursive such that for all $x \in U_{\mathfrak{t}}$

$$x \in X \leftrightarrow \exists^0 b\; x\,|\,b \in R.$$

Then

$$\begin{aligned} y \in Y &\leftrightarrow \exists^0 a\; y^\frown\langle a\rangle \in X \leftrightarrow \exists^0 a\, \exists^0 b\; (y\,|\,b)^\frown\langle a\rangle \in R \\ &\leftrightarrow \exists^0 c\, \exists a < c\, \exists b < c\; (y\,|\,b)^\frown\langle a\rangle \in R \leftrightarrow \exists^0 c\; (y\,|\,c) \in R', \end{aligned}$$

for some appropriate recursive $R' \subseteq U^{<\omega}_{\mathfrak{s}}$, so Y is Σ^0_1.

The same sort of computation works for Σ^0_n with $n > 0$, using the fact that $\exists^< \Pi^0_{n^-} \subseteq \Pi^0_{n^-}$. The result for Π^0_n may be derived analogously or by a De Morgan argument from the Σ^0_n case. $\square^{5.22.3}$

4 For Σ^0_n, use the equivalences

$$\begin{aligned} &\exists^0 a\; \langle a\rangle^\frown x \in S \vee \exists^0 a\; \langle a\rangle^\frown x \in T \leftrightarrow \exists^0 a\, (\langle a\rangle^\frown x \in S \vee \langle a\rangle^\frown x \in T) \\ &\exists^0 a\; \langle a\rangle^\frown x \in S \wedge \exists^0 a\; \langle a\rangle^\frown x \in T \leftrightarrow \exists^0 a, b\, (\langle a\rangle^\frown x \in S \wedge \langle b\rangle^\frown x \in T) \end{aligned}$$

together with the induction hypothesis for $\Pi^0_{n^-}$ (or the corresponding closure property of the recursive sets if $n = 1$), the definition (5.20.1), and (5.22.3) in the case of conjunction.

For Π^0_n use the dual argument or apply De Morgan's rules to the Σ case. $\square^{5.22.4}$

$\square^{5.22}$

Note that (5.22.4) is more general than closure under union and intersection. For suppose $\mathfrak{s}$ is a type of length m, $j_1 : m_1 \overset{\text{inj}}{\to} m$, and $j_2 : m_2 \overset{\text{inj}}{\to} m$, where $m_1, m_2 \leqslant m$. Let $\mathfrak{s}^1 = \mathfrak{s} \circ j_1$ and $\mathfrak{s}^2 = \mathfrak{s} \circ j_2$. Likewise, for $x \in U_{\mathfrak{s}}$ let $x^1 = x \circ j_1$ and $x^2 = x \circ j_2$. In other words, x^1 and x^2 are "extracts" of x.

We may define a disjunction operation specific to this situation by stipulating that for $X \subseteq U_{\mathfrak{s}^1}$ and $Y \subseteq U_{\mathfrak{s}^2}$, $X \vee Y \subseteq U_{\mathfrak{s}}$ and

$$x \in (X \vee Y) \leftrightarrow (x^1 \in X \vee x^2 \in Y).$$

(5.22.4) is to be understood as stating that Σ^0_n and Π^0_n are closed under such disjunction operations and the analogous conjunction operations.

Note that if we let $X^i = \{x \in U_{\mathfrak{s}} \mid x^i \in X\}$ for $X \subseteq U_{\mathfrak{s}^i}$, $i = 0, 1$, then $X \vee Y = X^1 \cup Y^2$, so we could obtain the general result from the special case of union and intersection using the fact that an extraction operation as above is a permutation followed by a projection and is recursive by (5.15), and the Kleene classes are closed under recursive substitution.

(5.25) **Theorem** [ZF] *Suppose* $n > 0$. Δ^0_n *is closed under* $\neg$, $\exists^<$, *and* $\forall^<$. $\exists^0\,\Delta^0_n \subseteq \Sigma^0_{n+1}$ *and* $\forall^0\,\Delta^0_n \subseteq \Pi^0_{n+1}$.

Proof Immediate from (5.22). $\square^{5.25}$

It is obvious that $\Sigma^i_n \subseteq \Pi^i_{n+1}$ and $\Pi^i_n \subseteq \Sigma^i_{n+1}$. It is easily shown by induction on n that $\Sigma^i_n \subseteq \Sigma^i_{n+1}$ and $\Pi^i_n \subseteq \Pi^i_{n+1}$. Thus we have the following inclusion diagram:

$$\begin{array}{ccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \cdots \text{Arithmetical} \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & &
\end{array}$$

(5.25)

$$\begin{array}{cccccccccc}
 & & & \Sigma^1_1 & & & & \Sigma^1_2 & & \\
 & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\subseteq & \Delta^1_1 & & & & \Delta^1_2 & & & & \cdots \text{Analytical} \\
 & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & & \Pi^1_1 & & & & \Pi^1_2 & &
\end{array}$$

(5.26) *Note in particular that* Arithmetical $= \bigcup_{n=1}^{\infty} \Sigma^0_n = \bigcup_{n=1}^{\infty} \Pi^0_n$.

The obvious questions are whether these inclusions are strict. The first of these—whether Σ^0_1 is Δ^0_1—is the type-1 version of the halting problem, and we answer it in the next section in the same way: in the negative by demonstrating the existence of universal sets.

This technique also allows us to deal with all the other questions with one exception: whether Δ^1_1 is Arithmetical. In fact, it is easy to see that it is not; the more interesting question is whether there is a natural extension of the arithmetical hierarchy that does exhaust Δ^1_1, and the answer is that there is: the *hyperarithmetical hierarchy*. Unfortunately, this topic is beyond the scope of this book; fortunately, the analogous object in classical descriptive set theory, viz., the *Borel hierarchy*, is significantly easier to handle, and we will develop its theory beginning in Section 5.3.

5.2.5 Universal and complete pointsets

(5.27) **Definition** [ZF] *Suppose* Γ *is a recursively closed pointclass,* $\mathfrak{s}$ *is a type, and* $X \subseteq U_{\mathfrak{s}}$.

1. X *is* 0-universal for $\Gamma \cap \mathcal{P}U_{\mathfrak{t}}$ $\overset{\text{def}}{\iff}$ $\mathfrak{s} = \langle 0\rangle^\frown \mathfrak{t}$ *and for every* $Y \in \Gamma \cap \mathcal{P}U_{\mathfrak{t}}$ *there exists* $n \in \omega$ *such that* $Y = \{y \in U_{\mathfrak{t}} \mid \langle n\rangle^\frown y \in X\}$.
2. X *is* recursively Γ-complete $\overset{\text{def}}{\iff}$ $X \in \Gamma$ *and for any type* $\mathfrak{t}$ *and* $Y \subseteq U_{\mathfrak{t}}$, *if* $Y \in \Gamma$ *then there is a recursive* $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ *such that* $Y = F^{\leftarrow}X$.
3. X *is* recursively Γ-0-complete $\overset{\text{def}}{\iff}$ $X \in \Gamma$, $\mathfrak{s}$ *is a* 0*-type, and for any* 0*-type* $\mathfrak{t}$ *and* $Y \subseteq U_{\mathfrak{t}}$, *if* $Y \in \Gamma$ *then there is a recursive* $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ *such that* $Y = F^{\leftarrow}X$.

(5.28) **Theorem** [ZF] *Suppose* Γ *is a recursively closed pointclass.*

1. *Suppose $\mathfrak{s}, \mathfrak{t}$ are both 1-types or both 0-types. Then there is a 0-universal set for $\Gamma \cap \mathcal{P} U_{\mathfrak{s}}$ iff there is a 0-universal set for $\Gamma \cap \mathcal{P} U_{\mathfrak{t}}$.*
2. *Suppose $X \subseteq U_{\langle 0,1\rangle}$ is 0-universal for $\Gamma \cap \mathcal{P} U_{\langle 1\rangle}$. Then X is recursively Γ-complete, as is $\{\langle n\rangle ^\frown x \mid \langle n, x\rangle \in X\} \subseteq {}^\omega\omega$.*
3. *Suppose $X \subseteq U_{\langle 0,0\rangle}$ is 0-universal for $\Gamma \cap \mathcal{P} U_{\langle 0\rangle}$. Then X is recursively Γ-0-complete, as is $\mathrm{Bin}^{\leftharpoonup} X \subseteq \omega$.*

Proof 1 Suppose $X \subseteq U_{\langle 0\rangle ^\frown \mathfrak{s}}$ is 0-universal for $\Gamma \cap \mathcal{P} U_{\mathfrak{s}}$. Let $F = F_{\mathfrak{s}} \circ F_{\mathfrak{t}}^{-1}$. Let $Y = \{\langle n\rangle ^\frown y \mid \langle n\rangle ^\frown Fy \in X\}$. Then $Y \in \Gamma \cap \mathcal{P} U_{\langle 0\rangle ^\frown \mathfrak{t}}$. Suppose $Y' \in \Gamma \cap \mathcal{P} U_{\mathfrak{t}}$. Let $X' = F^{\rightharpoonup} Y'$. Then $X' \in \Gamma \cap \mathcal{P} U_{\mathfrak{s}}$. Since X is universal, for some $n \in \omega$, for all $y \in U_{\mathfrak{t}}$,

$$y \in Y' \leftrightarrow Fy \in X' \leftrightarrow \langle n\rangle ^\frown Fy \in X \leftrightarrow \langle n\rangle ^\frown y \in Y.$$

Hence Y is 0-universal for $\Gamma \cap U_{\mathfrak{t}}$. $\square$[5.28.1]

2 This follows pretty directly from the proof of Theorem 5.17. Given a 1-type $\mathfrak{t}$ and $Y \in \Gamma \cap \mathcal{P} U_{\mathfrak{t}}$, let $Y' = F_{\mathfrak{t}}^{\leftharpoonup} Y \subseteq {}^\omega\omega$. Since Γ is recursively closed, $Y' \in \Gamma$, so there exists $n \in \omega$ such that $Y' = \{x \in {}^\omega\omega \mid \langle n, x\rangle \in X\}$. Let $F : U_{\mathfrak{t}} \to U_{\langle 0,1\rangle}$ be defined by the condition that $Fy = \langle n, F_{\mathfrak{t}}^{-1} y\rangle$. Then F is recursive and $Y = F^{\leftharpoonup} X$.

Given a 0-type $\mathfrak{t}$ and $Y \in \Gamma \cap \mathcal{P} U_{\mathfrak{t}}$, let $Y' = \{y \in {}^\omega\omega \mid y \restriction |\mathfrak{t}| \in Y\}$. Clearly, $Y' \in \Gamma$. Let $n \in \omega$ be such that $Y' = \{x \in {}^\omega\omega \mid \langle n, x\rangle \in X\}$. Let $F : U_{\mathfrak{t}} \to U_{\langle 0,1\rangle}$ be defined by the condition that $Fy = \langle n, y ^\frown \langle 0, 0, \ldots\rangle\rangle$. Then F is recursive and $Y = F^{\leftharpoonup} X$.

Hence X is recursively Γ-complete. Since $\langle n, x\rangle \mapsto \langle a\rangle ^\frown x$ is birecursive, $\{\langle n\rangle ^\frown x \mid \langle n, x\rangle \in X\}$ is also recursively Γ-complete. $\square$[5.28.2]

3 Given a 0-type $\mathfrak{t}$ and $Y \in \Gamma \cap \mathcal{P} U_{\mathfrak{t}}$, let $Y' = \mathrm{Bin}^{\leftharpoonup} Y$. Clearly, $Y' \in \Gamma$. Let $n \in \omega$ be such that $Y' = \{a \in \omega \mid \langle n, a\rangle \in X\}$. Let $F : U_{\mathfrak{t}} \to U_{\langle 0,0\rangle}$ be defined by the condition that $Fy = \langle n, \overleftarrow{B}\, y\rangle$. Then F is recursive and $Y = F^{\leftharpoonup} X$.

Hence X is recursively Γ-0-complete. Since Bin is birecursive, $\mathrm{Bin}^{\leftharpoonup} X$ is also recursively Γ-0-complete. $\square$[5.28.3] $\square$[5.28]

(5.29) **Definition** [ZF] *Suppose Γ is a recursively closed pointclass.*

1. Γ has 0-0-universal sets $\overset{\text{def}}{\iff}$ *there is a 0-universal set for $\Gamma \cap U_{\mathfrak{s}}$ for some (equivalently, for any) 0-type $\mathfrak{s}$.*[5.28.1]
2. Γ has 0-1-universal sets $\overset{\text{def}}{\iff}$ *there is a 0-universal set for $\Gamma \cap U_{\mathfrak{s}}$ for some (equivalently, for any) 1-type $\mathfrak{s}$.*[5.28.1]
3. Γ has 0-universal sets $\overset{\text{def}}{\iff}$ *Γ has 0-0- and 0-1-universal sets.*

(5.30) **Theorem** [ZF] *Suppose Γ is recursively closed and S is 0-i-universal for Γ, $i = 0$ or 1. Then $S \notin \neg\Gamma$.*

Proof Suppose without loss of generality that $S \subseteq U_{\langle 0,0\rangle ^\frown s}$. Let $T = \{\langle a\rangle ^\frown x \mid \langle a, a\rangle ^\frown x \notin S\}$. If $S \in \neg\Gamma$ then $T \in \Gamma$, so for some $b \in \omega$, $T = \{\langle a\rangle ^\frown x \mid \langle b, a\rangle ^\frown x \in S\}$. But then for any $x \in U_{\mathfrak{s}}$,

$$\langle b\rangle ^\frown x \in T \leftrightarrow \langle b, b\rangle ^\frown x \in S \leftrightarrow \langle b\rangle ^\frown x \notin T,$$

which contradiction establishes the theorem. $\square$[5.30]

(5.31) **Theorem** [ZF] Σ^0_1 *has* 0-*universal sets.*

Proof We have already seen an example of 0-universality in our discussion of semirecursive subsets of ω, where we defined the sets W_n, $n \in \omega$,[4.73] such that

1. every semirecursive subset of ω is W_n for some n; and
2. the set $S^0 = \{\langle n, m\rangle \mid n \in \omega \wedge m \in W_n\}$ is semirecursive.

S^0 is therefore 0-0-universal for Σ^0_1.

Now let $S^1 \subseteq \omega \times {}^\omega\omega$ be such that for all $n \in \omega$ and $x \in {}^\omega\omega$,

$$\langle n, x\rangle \in S \leftrightarrow \exists^0 a \ \overleftarrow{B}(x \restriction a) \in W_n.$$

Clearly, S^1 is Σ^0_1, as it is the halting set for a procedure that, given n and x, looks successively at each initial segment $x \restriction a$ of x, computes its numerical code $\overleftarrow{B}(x \restriction a)$ and checks whether it occurs in the listing of W_n, which it is concurrently creating.

Now suppose $X \subseteq {}^\omega\omega$ is Σ^0_1. Let[5.13.1] $R \subseteq {}^{<\omega}\omega$ be recursive such that

$$x \in X \leftrightarrow \exists^0 a \ x \restriction a \in R.$$

Let $n \in \omega$ be such that $W_n = \mathrm{Bin}^{\leftarrow} R$. Then for each $x \in {}^\omega\omega$,

$$\begin{aligned} x \in X \leftrightarrow \exists^0 a \ x \restriction a \in R \leftrightarrow \exists^0 a \ \overleftarrow{B}(x \restriction a) \in W_n \\ \leftrightarrow \langle n, x\rangle \in S^1. \end{aligned}$$

Thus S^1 is 0-1-universal for Σ^0_1. $\square$[5.31]

(5.32) **Theorem** [ZF] Σ^0_n *and* Π^0_n *have* 0-*universal sets, for all* $n > 0$.

Proof Obviously, if S is 0-i-universal for a recursively closed pointclass Γ then $\neg S$ is 0-i-universal for $\neg\,\Gamma$. In the interest of uniformity we will generally deal primarily with Σ pointclasses.

(5.31) provides 0-i-universal sets $S^i \subseteq U_{\langle 0,i\rangle}$ for Σ^0_1.

We now proceed by induction. Suppose $n > 0$ and there is an 0-i-universal set for Σ^0_n (in any i-type[5.29]). Let $S \subseteq U_{\langle 0,0,i\rangle}$ be 0-universal for $\Sigma^0_n \cap \mathcal{P}U_{\langle 0,i\rangle}$. Let $T = \{\langle a\rangle^\frown x \mid \exists^0 b \ \langle a, b\rangle^\frown x \notin S\}$.

(5.33) **Claim** T *is* 0-*universal for* $\Sigma^0_{n+1} \cap \mathcal{P}U_{\langle i\rangle}$.

Proof T is clearly Σ^0_{n+1}. Now suppose $X \subseteq U_{\langle i\rangle}$ is Σ^0_{n+1}. Let $Y \subseteq U_{\langle 0,i\rangle}$ be Π^0_n such that $X = \{x \in U_{\langle i\rangle} \mid \exists^0 b \ \langle b\rangle^\frown x \in Y\}$. Let $a \in \omega$ be such that $Y = \{y \in U_{\langle 0,i\rangle} \mid \langle a\rangle^\frown y \notin S\}$. Then $X = \{x \in U_{\langle i\rangle} \mid \exists^0 b \ \langle a, b\rangle^\frown x \notin S\} = \{x \in U_{\langle i\rangle} \mid \langle a\rangle^\frown x \in T\}$. $\square$[5.33]

By induction, therefore, 0-i-universal sets exist for Σ^0_n and Π^0_n for all $n > 0$, $i = 0, 1$. $\square$[5.32]

(5.34) *It follows*[5.30] *from (5.32) that for every* $n > 0$, $\Sigma^0_n \neq \Pi^0_n$, *so* $\Delta^0_n \subsetneq \Sigma^0_n$ *and* $\Delta^0_n \subsetneq \Pi^0_n$. *Also, since* $\Sigma^0_n, \Pi^0_n \subseteq \Delta^0_{n+1}$, $\Sigma^0_n \subsetneq \Delta^0_{n+1}$ *and* $\Pi^0_n \subsetneq \Delta^0_{n+1}$. *These inequalities also hold individually for every type* $\mathfrak{s}$.

5.2.6 A normal form for Σ^1_1/Π^1_1

The following theorem illustrates the usefulness of functions as canonical type-1 objects. Recall that $U_0 = \omega$ and $U_1 = {}^\omega\omega$.

(5.35) **Theorem** [ZF] *Suppose $X \subseteq U_\mathfrak{s}$ is Σ^1_1. Then there is a Π^0_1 set $S \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_\mathfrak{s}$,*

$$z \in X \leftrightarrow \exists^1 f\ \langle f\rangle^\frown z \in S.$$

Obviously, therefore, if X is Π^1_1 then there is a Σ^0_1 set $S \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_\mathfrak{s}$,

$$z \in X \leftrightarrow \forall^1 f\ \langle f\rangle^\frown z \in S.$$

Proof The following claim is the crux of the matter.

(5.36) **Claim** *Suppose $S \subseteq U_{\langle 0,0\rangle^\frown \mathfrak{s}}$ is Π^0_n, where $n > 0$. Then there exists a Π^0_n set $T \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_\mathfrak{s}$*

$$\forall^0 a\, \exists^0 a'\ \langle a, a'\rangle^\frown z \in S \leftrightarrow \exists^1 f\ \langle f\rangle^\frown z \in T. \tag{5.37}$$

Proof Let

$$T = \{\langle f\rangle^\frown z \in U_{\langle 1\rangle^\frown \mathfrak{s}} \mid \forall^0 a\, \langle a, fa\rangle^\frown z \in S\}.$$

T is Π^0_n because $\langle a, f\rangle^\frown z \mapsto \langle a, fa\rangle^\frown z$ is recursive, and Π^0_n is closed under recursive substitution and $\forall^0$.

To prove (5.37), we first observe that for any $f \in U_1$, if $\forall^0 a\ \langle a, fa\rangle^\frown z \in S$ then $\forall^0 a\, \exists^0 a'\ \langle a, a'\rangle^\frown z \in S$, which establishes the $\leftarrow$ direction.

To prove the $\rightarrow$ direction, suppose $\forall^0 a\, \exists^0 a'\ \langle a, a'\rangle^\frown z \in S$. Let $f \in U_1$ be defined by the condition that for each $a \in U_0$, fa is the least a' in U_0 $(= \omega)$ such that $\langle a, a'\rangle^\frown z \in S$. $\square^{5.36}$

(5.38) **Claim** *Suppose $X \subseteq U_\mathfrak{s}$ is arithmetical. Then there is a Π^0_1 set $S \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_\mathfrak{s}$,*

$$z \in X \leftrightarrow \exists^1 f\ \langle f\rangle^\frown z \in S.$$

Proof Every arithmetical set is Π^0_n for some $n > 0$.[5.26] We will prove the claim for Π^0_n pointsets (simultaneously for all types s) by induction on $n > 0$. For $n = 1$ the result is trivial. Suppose it is true for some $n > 0$ (for all types $\mathfrak{s}$), and suppose $X \subseteq U_\mathfrak{s}$ is Π^0_{n+1}. Then there exists a Δ^0_n set $Y \subseteq U_{\langle 0,0\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_\mathfrak{s}$

$$z \in X \leftrightarrow \forall^0 a\, \exists^0 a'\ \langle a, a'\rangle^\frown z \in Y.$$[17]

Let[5.36] $T \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ be Π^0_n such that for all $z \in U_\mathfrak{s}$

$$z \in X \leftrightarrow \exists^1 f\ \langle f\rangle^\frown z \in T.$$

By the induction hypothesis (for Π^0_n subsets of $U_{\langle 1\rangle^\frown \mathfrak{s}}$) there is a Π^0_1 set $T' \subseteq U_{\langle 1,1\rangle^\frown \mathfrak{s}}$ such that for all $\langle f\rangle^\frown z \in U_{\langle 1\rangle^\frown \mathfrak{s}}$

$$\langle f\rangle^\frown z \in T \leftrightarrow \exists^1 g\ \langle g, f\rangle^\frown z \in T'.$$

[17] If $n > 1$ we could take Y to be $\Pi^0_{n\text{-}}$. Our use of Δ^0_n permits a uniform treatment of all cases, including $n = 1$.

For all $z \in U_{\mathfrak{s}}$, then,

$$z \in X \leftrightarrow \exists^1 f\, \exists^1 g\, \langle g, f\rangle^\frown z \in T'.$$

For $h \in {}^\omega\omega$, let h^0, h^1 be defined as follows. Given $a \in \omega$, if $ha = \overleftarrow{B}\langle b, c\rangle$ for some $b, c \in \omega$, then $h^0 a = b$ and $h^1 a = c$; otherwise $h^0 a = h^1 a = 0$. Clearly, $\langle h\rangle \mapsto \langle h^0, h^1\rangle$ is recursive. Let

$$S = \{\langle h\rangle^\frown z \in U_{\langle 1\rangle^\frown \mathfrak{s}} \mid \langle h^0, h^1\rangle^\frown z \in T'\}.$$

Then S is Π^0_1, and for all $z \in U_{\mathfrak{s}}$

$$z \in X \leftrightarrow \exists^1 h\, \langle h\rangle^\frown z \in S,$$

as desired. □[5.38]

To complete the proof of the theorem, suppose $X \subseteq U_{\mathfrak{s}}$ is Σ^1_1. Let[5.4.4] $W \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ be arithmetical such that for all $z \in U_{\mathfrak{s}}$

$$z \in X \leftrightarrow \exists^1 f\, \langle f\rangle^\frown z \in W.$$

Let[5.38] $S' \subseteq U_{\langle 1,1\rangle^\frown \mathfrak{s}}$ be Π^0_1 such that for all $\langle f\rangle^\frown z \in U_{\langle 1\rangle^\frown \mathfrak{s}}$

$$\langle f\rangle^\frown z \in W \leftrightarrow \exists^1 g\, \langle g, f\rangle^\frown z \in S'.$$

Then for all $z \in U_{\mathfrak{s}}$

$$z \in X \leftrightarrow \exists^1 f\, \exists^1 g\, \langle g, f\rangle^\frown z \in S',$$

and by the argument used in the proof of (5.38) there exists a Π^0_1 set $S \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$ such that for all $z \in U_{\mathfrak{s}}$

$$z \in X \leftrightarrow \exists^1 h\, \langle h\rangle^\frown z \in S,$$

as desired. □[5.35]

(5.39) **Theorem** [ZF] Σ^1_n *and* Π^1_n *have* 0*-universal sets for all* $n > 0$.

Proof Let[5.32] $S \subseteq U_{\langle 0,1\rangle^\frown \mathfrak{s}}$ be 0-universal for $\Pi^0_1 \cap \mathcal{P} U_{\langle 1\rangle^\frown \mathfrak{s}}$. Let

$$T = \{\langle a\rangle^\frown x \in U_{\langle 0\rangle^\frown \mathfrak{s}} \mid \exists^1 f\, \langle a, f\rangle^\frown x \in S\}.$$

T is Σ^1_1.

Suppose $X \subseteq U_{\mathfrak{s}}$ is Σ^1_1. Then[5.35] for some Π^0_1 $Y \subseteq U_{\langle 1\rangle^\frown \mathfrak{s}}$, $X = \{x \mid \exists^1 f\, \langle f\rangle^\frown x \in Y\}$, and for some $a \in \omega$, $Y = \{\langle f\rangle^\frown x \mid \langle a, f\rangle^\frown x \in S\}$, so

$$X = \{x \mid \langle a\rangle^\frown x \in T\}.$$

Hence T is 0-i-universal for Σ^1_1 (assuming $\mathfrak{s}$ is an i-type).

It is now easy to show by induction on $n > 0$ that 0-universal sets exist for Σ^1_n and Π^1_n. □[5.39]

It follows from (5.39) that for every $n > 0$, $\Sigma^1_n \neq \Pi^1_n$, *so* $\Delta^1_n \subsetneq \Sigma^1_n$, $\Delta^1_n \subsetneq \Pi^1_n$, $\Sigma^1_n \subsetneq \Delta^1_{n+1}$ *and* $\Pi^1_n \subsetneq \Delta^1_{n+1}$. *These inequalities also hold individually for every type* $\mathfrak{s}$.

Together with (5.34) we therefore have the following inclusion diagram.

(5.39)

$$\begin{array}{ccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \cdots\text{Arithmetical} \\
 & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & &
\end{array}$$

$$\begin{array}{cccccccccc}
 & & & \Sigma^1_1 & & & & \Sigma^1_2 & & \\
 & & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\subseteq & \Delta^1_1 & & & & \Delta^1_2 & & & & \cdots\text{Analytical} \\
 & & \supsetneq & & \subsetneq & & \supsetneq & & \subsetneq & \\
 & & & \Pi^1_1 & & & & \Pi^1_2 & &
\end{array}$$

5.2.7 Closure properties of analytical classes

The following theorem states closure properties for Σ^1_n and Π^1_n, $n > 0$, analogous to those stated for Σ^0_n and Π^0_n in (5.22). The arrangement of these results in (5.22) was chosen to conform to the order of their derivation. Here we arrange the results according to content, with the order motivated by the following observations.

1. Closure under binary disjunction (conjunction) implies closure under n-ary disjunction (conjunction) for any fixed $n \in \omega$.

2. Bounded existential (universal) quantification may be regarded as finitary disjunction (conjunction) with a variable finite bound.

3. Type-0 existential (universal) quantification may be regarded as ω-ary disjunction (conjunction).

4. Type-1 existential (universal) quantification may be regarded as ${}^{\omega}\omega$-ary disjunction (conjunction).

The proof of the following theorem is the first point in this chapter where a Choice principle is used. We require only a minimal such principle. Specifically, we require choice functions for countable sets of nonempty subsets of $\mathcal{P}V_\omega$. In naming this axiom we look a little bit ahead to the following section, where we define real numbers as members of $\mathcal{P}V_\omega$. Thus, the axiom we need may be described as choice for countable sets of subsets of $\mathbb{R}$, i.e., $\mathsf{AC}_\omega(\mathbb{R})$.[3.140]

(5.40) **Theorem** $[\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})]$ *Suppose* $n > 0$ *and* Γ *is* Σ^1_n (Π^1_n).

1. $\neg\,\Gamma$ *is* Π^1_n (Σ^1_n).
2. Γ *is closed under disjunction, conjunction, and type-0 quantification.*
3. Γ *is closed under type-1 existential (universal) quantification.*

Proof The proof is a fairly straightforward application of principles and techniques already presented. $\mathsf{AC}_\omega(\mathbb{R})$ is used in the proof of closure of Σ^1_n (Π^1_n) under universal (existential) type-0 quantification, to show that if $\forall^0 a\, \exists^1 f\ \langle a, f\rangle \in R$ then

$\exists^1 f\,\forall^0 a\ \langle a, f^a\rangle \in R$, where $f^a \in U_1$ is defined for each $a \in U_0$ by the condition that for all $b \in U_0$,

$$f^a b = \begin{cases} c & \text{if } \vec{B}\,(fb) = \langle a, c\rangle \text{ for some } c \\ 0 & \text{otherwise.} \end{cases}$$

$\square^{5.40}$

5.2.8 Relative definability

It is often useful to consider pointclasses defined by complexity relative to a type-1 object.

(5.41) **Definition** [ZF] *Suppose Γ is a recursively closed pointclass.*

1. *Suppose $z \in U_1$.* Γ relativized to z $\stackrel{\text{def}}{=} \Gamma(z) \stackrel{\text{def}}{=}$ *the class Γ' defined by the condition that for any pointset $X \subseteq U_s$, $X \in \Gamma' \stackrel{\text{def}}{\Longleftrightarrow}$ there exists $Y \subseteq U_{\langle 1\rangle ^\frown \mathfrak{s}}$ such that $Y \in \Gamma$ and $X = \{x \in U_s \mid \langle z\rangle ^\frown x \in Y\}$.*[18]

2. *The* full relativization of Γ *or simply the* relativization of Γ $\stackrel{\text{def}}{=}$

$$\underset{\sim}{\Gamma} \stackrel{\text{def}}{=} \bigcup_{z \in U_1} \Gamma(z).$$

(5.42) **Theorem** [ZF] *Suppose Γ is a recursively closed pointclass. Let $\Gamma' = \underset{\sim}{\Gamma}$. Then $\underset{\sim}{\Gamma}' = \Gamma'$.*

Proof Straightforward. $\square^{5.42}$

Suppose Γ is a recursively closed pointclass. Γ is relativized $\stackrel{\text{def}}{\Longleftrightarrow} \underset{\sim}{\Gamma} = \Gamma$.

In print, we usually use boldface symbols for the classes $\underset{\sim}{\Sigma}^i_n$, etc. For example, $\boldsymbol{\Sigma}^0_1 = \underset{\sim}{\Sigma}^0_1$. As we will see, these classes may be defined directly in a natural way, without reference to the relativization operation. The use of the wavy underline as a component of the name of a pointclass is usually restricted to handwriting, where it may be regarded as indicating boldface. It is common to use 'boldface' as an adjective applicable to relativized classes and their theory.

Note that any relativized pointclass, other than the trivial class all of whose pointsets are empty, contains every type-0 pointset.[19]

(5.43) *Keeping in mind that type-0 pointsets are type-1 objects, and type-1 pointsets are type-2 objects, we see that relativized classes serve the useful purpose of isolating that part of type-2 complexity that is specific to type-2 by ignoring any distinction among type-1 objects: in effect, treating type-1 objects as structureless individuals, like real numbers in analysis or points in geometry.*

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[18] Note that if we similarly defined $\Gamma(z)$ for some $z \in U_0$, then $\Gamma(z) = \Gamma$. The same is true if $z \in U_1$ is recursive (as a subset of V_ω, or, equivalently, as a (total) function from ω to ω).

[19] Recall that a pointset X is of type-0 iff $X \subseteq U_{\mathfrak{s}}$, where $\mathfrak{s}$ is a 0-type, i.e., $\forall k < |\mathfrak{s}|\ \mathfrak{s}_k = 0$.

Based on (5.41.1) we have complexity pointclasses $\Sigma^i_n(z)$, $\Pi^i_n(z)$, and $\Delta^i_n(z)$, relative to a given type-1 object z. Essentially all of the preceding discussion may be carried out relative to a fixed type-1 object z. We may also relativize the various notions of recursiveness to an arbitrary object z by consideration of procedures using z as an oracle.

Based on (5.41.2) we have (fully) relativized—i.e., boldface—complexity pointclasses $\boldsymbol{\Sigma}^i_n$, $\boldsymbol{\Pi}^i_n$, and $\boldsymbol{\Delta}^i_n$, and most of the preceding discussion applies to them as well. As noted above, these are only of interest for type-1 pointsets: every type-0 pointset trivially belongs to all these classes.

5.2.9 Topological aspects

We alluded above[5.43] to the role played by relativized pointclasses in isolating structural features intrinsic to type-2. We will see that these fall under the rubric 'pointset topology': this is, in fact, the rationale for our use of 'pointset' in this discussion.

We begin by recalling that for any Σ^0_1 set $X \subseteq {}^\omega\omega$, there is an effective procedure $\mathcal{P}$ such that for any $f \in {}^\omega\omega$, $f \in X$ iff $\mathcal{P}f$ halts. Recall that the computation $\mathcal{P}f$ uses f as an oracle, and if it halts, it does so after only finitely many consultations of the oracle. For technical reasons it is convenient to suppose that $\mathcal{P}$ always asks for an initial segment of f, so at any stage in the computation it knows a finite initial segment of f. Then for some finite initial segment s of f, $\mathcal{P}$ concludes that $f \in X$ based on the fact that $s \subseteq f$. Thus, for every $f' \in {}^\omega\omega$, if $s \subseteq f'$ then $f' \in X$. Note that s itself is a suitable oracle for $\mathcal{P}$, inasmuch as $\mathcal{P}$, acting on the values supplied by s, halts without asking for any information not contained in s. For each $s \in {}^\omega\omega$, let $I_s = \{f \in {}^\omega\omega \mid s \subseteq f\}$. Let A be the set of $s \in {}^{<\omega}\omega$ such that $\mathcal{P}$ halts with s as an oracle, without asking for any information not in s. Then $X = \bigcup_{s\in A} I_s$. Obviously, A is Σ^0_1.[20]

Conversely, suppose A is a Σ^0_1 subset of ${}^{<\omega}\omega$. Let $\mathcal{P}$ be an effective procedure that, with input $f \in {}^\omega\omega$, dovetails an enumeration of A with an enumeration of f and halts if and when it finds an initial segment of f in A. Then $\mathcal{P}f$ halts iff $f \in \bigcup_{s\in A} I_s$, so $\bigcup_{s\in A} I_s$ is by definition a Σ^0_1 subset of ${}^\omega\omega$.

Thus, we have the following normal form for Σ^0_1 subsets of ${}^\omega\omega$.

(5.44) **Theorem** [ZF] *Suppose $X \subset {}^\omega\omega$. Then X is Σ^0_1 iff there exists a Σ^0_1 set $A \subseteq {}^{<\omega}\omega$ such that $X = \bigcup_{s\in A} I_s$.*

Proof The preceding remarks may be fashioned into a rigorous proof, but we may proceed more directly as follows. Suppose X is Σ^0_1. By definition[5.13.1] there exists a recursive (i.e., Δ^0_1) $A \subseteq {}^{<\omega}\omega$ such that for all $x \in {}^\omega\omega$, $x \in X$ iff $\exists a \in \omega\ x \restriction a \in A$. Then A is as desired.

Conversely, suppose $A \subseteq {}^{<\omega}\omega$ is Σ^0_1. Let $X = \bigcup_{s\in A} I_s$. Let $S \subseteq \omega \times {}^{<\omega}\omega$ be Δ^0_1 such that for all $s \in {}^{<\omega}\omega$, $s \in A \leftrightarrow \exists n \in \omega\ \langle n, s\rangle \in S$. Let $R = \{s \in {}^{<\omega}\omega \mid \exists m, n < |s|\ \langle n, s \restriction m\rangle \in S\}$. R is Δ^0_1, and for any $x \in {}^\omega\omega$

$$\begin{aligned} x \in X &\leftrightarrow \exists m \in \omega\ x \restriction m \in A \\ &\leftrightarrow \exists m, n \in \omega\ \langle n, x \restriction m\rangle \in S \\ &\leftrightarrow \exists a \in \omega\ \exists m, n < a\ \langle n, (x \restriction a) \restriction m\rangle \in S \\ &\leftrightarrow \exists a \in \omega\ x \restriction a \in R, \end{aligned}$$

[20] We make free use of birecursive bijections such as $\mathrm{Bin} : \omega \overset{\text{bij}}{\to} V_\omega$ to transfer complexity classifications to spaces that are not $U_{\mathfrak{s}}$ for some type $\mathfrak{s}$.

so R is as desired. $\square^{5.44}$

The preceding discussion may be carried out relative to any $z \in {}^{\omega}\omega$ to show that a set $X \subseteq {}^{\omega}\omega$ is $\Sigma^0_1(z)$ iff there exists a $\Sigma^0_1(z)$ set $A \subseteq {}^{<\omega}\omega$ such that $X = \bigcup_{s\in A} I_s$. And we may aggregate these for all $z \in {}^{\omega}\omega$ to show that a set $X \subseteq {}^{\omega}\omega$ is $\mathbf{\Sigma}^0_1$ iff there exists a $\mathbf{\Sigma}^0_1$ set $A \subseteq {}^{<\omega}\omega$ such that $X = \bigcup_{s\in A} I_s$.

(5.45) *We use the obvious extension of notions of definability and relative definability to subsets of V_ω (as opposed to subsets of $U_{\mathfrak{s}}$). The simplest way to carry the entire theory over is to use a birecursive bijection such as* $\mathrm{Bin} : \omega \overset{\text{bij}}{\to} V_\omega$. *Any infinite recursive subset of V_ω is interchangeable with ω for the present purpose.*

But every set $A \subseteq {}^{<\omega}\omega$ is $\mathbf{\Sigma}^0_1$ (since A is $\Sigma^0_1(A)$), so we have the following normal form for $\mathbf{\Sigma}^0_1$.

(5.46) **Theorem** [ZF] *Suppose $X \subseteq {}^{\omega}\omega$. Then X is $\mathbf{\Sigma}^0_1$ iff there exists a set $A \subseteq {}^{<\omega}\omega$ such that $X = \bigcup_{s\in A} I_s$.*

Proof Just given. $\square^{5.46}$

Note that in keeping with the comment 5.43 there is no restriction on A in (5.46).

This suggest that we adopt the following standard topologies on pointspaces.[21]

(5.47) **Definition** [ZF]

1. *Suppose A is a type-0 pointspace, broadly understood, e.g., ω or V_ω.*[5.45] *The* standard topology on A $\overset{\text{def}}{=}$ *the discrete topology, i.e., the topology in which every subset of A is open.*
2. *Suppose B is a type-1 pointspace of the form ${}^{C}A$—of which ${}^{\omega}\omega$ is our paradigm, but also broadly understood. The* standard topology on B $\overset{\text{def}}{=}$ *the product topology, where ${}^{C}A$ is regarded as the product of copies of A, one for each member of C, and A is given the discrete topology.*

By definition[3.197] the product topology on ${}^{C}A$, assuming the discrete topology on A, is generated by taking the sets $I_s \overset{\text{def}}{=} \{f \in {}^{C}A \mid s \subseteq f\}$ as a base, where s ranges over finite partial functions from C to A. That is, the open sets in the product topology are arbitrary unions of sets I_s. In the case of particular interest, viz., ${}^{\omega}\omega$, rather than allowing $s : \omega \rightharpoonup \omega$ to have any finite domain, we may and do specify that $\operatorname{dom} s$ be a finite ordinal, i.e., $s \in {}^{<\omega}\omega$.

We now have the following topological characterization of $\mathbf{\Sigma}^0_1$.

(5.48) **Theorem** [ZF] *Suppose $X \subseteq {}^{\omega}\omega$. Then X is $\mathbf{\Sigma}^0_1$ iff X is open (in the standard topology).*

Proof This is just a restatement of (5.46). $\square^{5.48}$

It is useful to adapt the procedural point of view for specific application to relativized pointsets. Recall that an effective procedure is a finitary thing in that it manipulates finitary objects according to a program that is itself finitary. We may also consider procedures $\mathcal{P}$ that manipulate finitary objects according to an

[21] Recall[3.185] that a topology on a set A is a set $\mathcal{T}$ of subsets of A—called the *open* sets—such that $0, A \in \mathcal{T}$, $\mathcal{T}$ is closed under finite intersections, and $\mathcal{T}$ is closed under arbitrary unions.

arbitrary program. Such a program is essentially a function P that operates on the state S_n of the computation at stage $n \in \omega$ to produce the state S_{n+1} of the computation at the next stage, taking into account the finitary input, which may be regarded as determining S_0. P is therefore representable as an arbitrary function from V_ω to V_ω—a type-1 object.

If we wish, we may model $\mathcal{P}$ as a fixed effective procedure $\mathcal{P}_0$, with access to P as an oracle. The oracle in this case is intrinsic to $\mathcal{P}$—*is* $\mathcal{P}$, if you will—but the procedure may still be thought of as effective relative to the oracle. We call this a *relative* procedure. As in the case of effective procedures, there are two sorts of relative procedures $\mathcal{P}$. A terminable procedure either halts after finitely many steps with a type-0 output, or never halts. An interminable procedure produces a sequence of type-0 objects as output, which may have any finite length, including 0, or length ω. Also as in the case of effective procedures, a relative procedure $\mathcal{P}$ of either sort acts on a type-1 input f by consulting f as an oracle.

Clearly, any function $F : \omega \to \omega$ and any $F : \omega \to {}^\omega\omega$ is computable by a relative procedure, because F is a type-1 object that may be used to define a procedure that computes it. So the only interesting case is that of relative procedures acting on type-1 inputs. The essential feature of such a procedure $\mathcal{P}$ is that its state at any stage is determined by a finite amount of information about the input f, which may be taken to be an initial segment of f in the canonical case of $f \in {}^\omega\omega$.

(5.49) *To avoid irrelevant complications, we will restrict our attention to relative procedures that compute total functions, i.e., to terminable procedures that always halt, and to interminable procedures that always produce an ω-sequence.*[4.58]

Let $F : {}^\omega\omega \to \omega$ be the function computed by a relative procedure $\mathcal{P}$ of the terminable sort. Suppose $Ff = n$. Then for some finite initial segment $s \subseteq f$, $\mathcal{P}$ halts with output n having obtained from f only the information contained in s. Thus, for any $f' \in {}^\omega\omega$, if $s \subseteq f'$ then $Ff' = n$, i.e., $I_s \subseteq F^{\leftarrow}\{n\}$. It follows that $F^{\leftarrow}\{n\} = \bigcup\{I_s \mid s \in {}^{<\omega}\omega \wedge I_s \subseteq F^{\leftarrow}\{n\}\}$, so $F^{\leftarrow}\{n\}$ is open. Since the union of open sets is open, $F^{\leftarrow}X$ is open for any $X \subseteq \omega$. F is therefore continuous.[22]

Conversely, suppose $F : {}^\omega\omega \to \omega$ is continuous. For each $n \in \omega$, let $A_n = \{s \in {}^{<\omega}\omega \mid I_s \subseteq F^{\leftarrow}\{n\}\}$. Since $\{n\}$ is open in the discrete topology, and F is continuous, $F^{\leftarrow}\{n\}$ is open, so $F^{\leftarrow}\{n\} = \bigcup A_n$. Note that $n \neq n' \to A_n \cap A_{n'} = 0$. Since F is total, for any $f \in {}^\omega\omega$ there exists $n \in \omega$ and a finite initial segment $s \subseteq f$ such that $s \in A_n$. Let $\mathcal{P}$ be the procedure that, given input $f \in {}^\omega\omega$, obtains successively longer finite initial segments $s \subseteq f$ until it finds one that is in A_n for some n, and then halts with output n. Clearly, for all $f \in {}^\omega\omega$, $\mathcal{P}f$ halts with output Ff.

(5.50) *Thus the continuous functions $F : {}^\omega\omega \to \omega$ are exactly those that are computable by a relative procedure.*

Now let $F : {}^\omega\omega \to {}^\omega\omega$ be the function computed by a relative procedure $\mathcal{P}$ of the interminable sort. Conceptually, $\mathcal{P}$, with input $f \in {}^\omega\omega$, acting finitarily, proceeds through an ω-sequence of steps, consulting f as an oracle and producing an ω-sequence Ff of numbers,[5.49] as its output. For each $t \in {}^{<\omega}\omega$, let A_t be the set of $s \in {}^{<\omega}\omega$ such that $\mathcal{P}$, with input s, computes t, i.e., $f \in I_s \to Ff \in I_t$. Then $f \in \bigcup\{I_s \mid s \in A_t\} \to Ff \in I_t$, and clearly also $Ff \in I_t \to f \in \bigcup\{I_s \mid s \in A_t\}$ (because at some point in the computation $\mathcal{P}f$, $\mathcal{P}$ has acquired enough information

[22] Recall that by definition, for any topological spaces A and B, a function $F : A \to B$ is continuous iff for every open $X \subseteq B$, $f^{\leftarrow}X$ is open.

from f to compute t). Thus $F^{\leftarrow}I_t = \bigcup\{I_s \mid s \in A_t\}$, which is open. Since every open set is a union of basic sets, it follows that $F^{\leftarrow}Y$ is open for all open $Y \subseteq {}^{\omega}\omega$. Hence F is continuous.

Conversely, suppose $F : {}^{\omega}\omega \to {}^{\omega}\omega$ is continuous. For each $t \in {}^{<\omega}\omega$, let $A_t = \{s \in {}^{<\omega}\omega \mid I_s \subseteq F^{\leftarrow}I_t\}$. Since F is continuous, $F^{\leftarrow}I_t$ is open, so $F^{\leftarrow}I_t = \bigcup\{I_s \mid s \in A_t\}$. Note that if $I_t \cap I_{t'} = 0$ then $A_t \cap A_{t'} = 0$. Since F is total, for any $f \in {}^{\omega}\omega$ there exists $g \in {}^{\omega}\omega$ such that $Ff = g$, so for any $k \in \omega$ there is a unique $t^k \in {}^{k}\omega$ (viz., $g \restriction k$) such that there exists a finite initial segment s of f such that $s \in A_{t^k}$. Let $\mathcal{P}$ be the procedure that—using $\{\langle s, t\rangle \mid s \in A_t\}$ as an internal oracle—given input $f \in {}^{\omega}\omega$, obtains successively longer finite initial segments s of f, successively identifying, for each $k \in \omega$, t^k as above. Since $t^k = g \restriction k$, $\mathcal{P}f$ computes g.

(5.51) *Thus the continuous functions $F : {}^{\omega}\omega \to {}^{\omega}\omega$ are exactly those that are computable by a relative procedure.*

Continuous functions on pointspaces are therefore[5.50, 5.51] the boldface (relativized) analog of recursive functions.

Topological continuity may also be characterized in terms of limits. In the case of first countable spaces such as pointspaces, it suffices to consider limits of ω-sequences. Recall that $\lim_{n\to\infty} x_n = x$ iff for any open G with $x \in G\ \exists N \in \omega\ \forall n > N\ x_n \in G$. Specializing to ${}^{\omega}\omega$ for simplicity, $\lim_{n\to\infty} f_n = f$ iff $\forall k \in \omega\ \exists N \in \omega\ \forall n > N\ f_n \restriction k = f \restriction k$.

It is natural in this setting to think of finite initial segments $s \in {}^{<\omega}\omega$ of $f \in {}^{\omega}\omega$ as *approximations* to f, and to think of $f, g \in {}^{\omega}\omega$ as close to each other if they have a long initial segment in common, and the standard topology on any pointspace may be obtained from a metric in the topological sense. In fact, virtually all of the boldface theory of pointspaces applies to arbitrary separable completely metrizable topological spaces.

To most mathematicians the most important separable completely metrizable topological space by far is that of the real numbers, and it was in this formulation that the subject was originally studied. In Section 5.3 we will take a topological, rather than an explicitly logical, approach to the description of pointsets. This is the framework in which the study of properties of pointsets classified according to the manner of their description was first undertaken, under the name *descriptive set theory*. One should keep in mind that this is strictly a "boldface" theory. It is often referred to as *classical* descriptive set theory by way of contrast with the theory based on logical complexity without relativization, as presented above, which may be called *effective* descriptive set theory.

5.2.10 Boldface universality and completeness

The essentially "lightface" notions[5.27] of recursive closure, 0-universality, and recursive completeness have the following "boldface" counterparts.

(5.52) **Definition** [ZF] *Suppose Γ is a pointclass.*

1. Γ *is* continuously closed $\overset{\text{def}}{\iff}$ Γ *is closed under continuous substitution, i.e., for any types $\mathfrak{s}$ and $\mathfrak{t}$, any $Y \in \Gamma$ of type t, and any continuous $F : U_{\mathfrak{s}} \to U_{\mathfrak{t}}$, $F^{\leftarrow}Y \in \Gamma$.*
2. *Suppose Γ is continuously closed, $\mathfrak{s}$ is a type, and $X \subseteq U_{\mathfrak{s}}$.*

1. X *is* 1-universal for $\Gamma \cap \mathcal{P} U_{\mathfrak{t}} \overset{\text{def}}{\iff} \mathfrak{s} = \langle 1 \rangle ^\frown \mathfrak{t}$ *and for every* $Y \in \Gamma \cap \mathcal{P} U_{\mathfrak{t}}$ *there exists* $x \in {}^{\omega}\omega$ *such that* $Y = \{y \in U_{\mathfrak{t}} \mid \langle x \rangle ^\frown y \in X\}$.
2. X *is* continuously Γ-complete $\overset{\text{def}}{\iff} X \in \Gamma$ *and for any type* $\mathfrak{t}$ *and* $Y \subseteq U_{\mathfrak{t}}$, *if* $Y \in \Gamma$ *then there is a continuous* $F : U_{\mathfrak{t}} \to U_{\mathfrak{s}}$ *such that* $Y = F^{\leftarrow} X$.

5.2.11 Tree representations

The use of ω-sequences as standard type-1 objects leads naturally to the use of trees in the description of pointsets. Recall[3.181] that a sequence tree $(T; <)$ on a set M is a set $T \subseteq {}^{<\omega}M$ of finite sequences from M, closed under initial segment, ordered by reverse inclusion: $s \leqslant t \leftrightarrow t \subseteq s$. Recall[3.183] that $[T]$ is the set of infinite branches of of a sequence tree T. The following definitions and theorem establish the relevance of sequence trees to the present topic.

Definition [ZF] *Suppose M is a set. The* sequence topology *or* standard topology *on ${}^{\omega}M$ is defined as follows.*[5.47.2] *For $s \in {}^{<\omega}M$, $I_s \overset{\text{def}}{=} \{x \in {}^{\omega}M \mid s \subseteq x\}$. These are the* basic open sets *or* intervals*. The open sets are unions of these.*

(5.53) **Definition** [ZF] *Suppose M is a set and ${}^{\omega}M$ is given the standard topology. Suppose $X \subseteq {}^{\omega}M$. The* tree from X $\overset{\text{def}}{=} T^X \overset{\text{def}}{=} \{x \restriction n \mid x \in X \wedge n \in \omega\}$.

(5.54) **Theorem** [ZF] *Suppose M is a set and ${}^{\omega}M$ is given the standard topology.*

1. *Suppose $X \subseteq {}^{\omega}M$. Then $\overline{X} = [T^X]$, i.e., the closure of X is the set of branches of the tree from X.*
2. *In particular, X is closed iff $X = [T^X]$.*

Proof 1 Let $G = \bigcup_{s \in ({}^{<\omega}M \setminus T^X)} I_s$. G is open by definition, and $I_s \subseteq G$ iff $s \notin T^X$ iff $X \cap I_s = 0$, so G is the largest open set disjoint from X, so $\overline{X} = {}^{\omega}M \setminus G$. Given $x \in {}^{\omega}M$,

$$x \in G \leftrightarrow \exists s \in ({}^{<\omega}M \setminus T^X)\ s \subseteq x \leftrightarrow x \notin [T^X],$$

so $[T^X] = \overline{X}$.

2 Immediate. □[5.54]

Note that, as far as pointspaces $U_{\mathfrak{s}}$ are concerned, Definition 5.53 applies directly only to $U_{\langle 1 \rangle}$. (Actually, not even directly here, as we have to interpolate the correspondence $x \leftrightarrow \langle x \rangle$, as we have agreed to do as necessary without explicit recognition.) To apply it more generally, we make use of pointwise operations on sequences in the following sense.

(5.55) **Definition** [ZF] *Suppose X, Y, R are sets and $u \in {}^{X}({}^{Y}R)$. Then $u^{\cdot} \overset{\text{def}}{=}$ the element of ${}^{Y}({}^{X}R)$ given by the condition that for all $x \in X$ and $y \in Y$,*

$$(u^{\cdot} y)x = (ux)y.$$

Note that $(u^{\cdot})^{\cdot} = u$. In the particular case of interest, $X = n \in \omega$ and $Y = \omega$, so that $u = \langle u_0, \dots, u_{n^-} \rangle$ with $u_m = \langle (u_m)_k \mid k \in \omega \rangle$, and

$$\langle u_0, \dots, u_{n^-} \rangle^{\cdot} = \big\langle \langle (u_0)_k, \dots, (u_{n^-})_k \rangle \,\big|\, k \in \omega \big\rangle. \tag{5.56}$$

Note that this is a recursive homeomorphism of ${}^n({}^\omega\omega)$ with ${}^\omega({}^n\omega)$, each endowed with the standard topology.

We use the same notation to indicate the corresponding correspondence between subsets of ${}^X({}^Y R)$ and ${}^Y({}^X R)$. Thus, given $A \subseteq {}^X({}^Y R)$,

$$A^{\cdot} = \{u^{\cdot} \mid u \in A\}.$$

The primordial application of trees in descriptive set theory is to Σ^1_1 and Π^1_1 pointsets. Suppose $X \subseteq {}^\omega\omega$ is Σ^1_1. By (5.35) there is a Π^0_1 set $S \subseteq {}^\omega\omega \times {}^\omega\omega$ such that for all $x \in {}^\omega\omega$, $x \in X \leftrightarrow \exists w \in {}^\omega\omega\ \langle x, w\rangle \in S$.[23] Thus, $x \in X \leftrightarrow \exists w \in {}^\omega\omega\ \langle x, w\rangle^{\cdot} \in S^{\cdot}$. Let[5.13.1] $R \subseteq {}^{<\omega}(\omega \times \omega)$ be recursive such that $S^{\cdot} = \{y \in {}^\omega(\omega \times \omega) \mid \forall^0 n\ y \restriction n \in R\}$. Let $T = \{s \in {}^{<\omega}(\omega \times \omega) \mid \forall n \leqslant |s| \mid s \restriction n \in R\}$. Then T is a recursive tree, and $S^{\cdot} = [T] = \{y \in {}^\omega(\omega \times \omega) \mid \forall^0 n\ y \restriction n \in T\}$, so

$$\begin{aligned} X &= \{x \in {}^\omega\omega \mid \exists^1 w\, \forall^0 n\ \langle x, w\rangle^{\cdot} \restriction n \in T\} \\ &= \{x \in {}^\omega\omega \mid \exists^1 w\ \langle x, w\rangle^{\cdot} \in [T]\}. \end{aligned}$$

Conversely, for any recursive tree $T \subseteq {}^{<\omega}(\omega \times \omega)$ the set

$$\{x \in {}^\omega\omega \mid \exists^1 w\ \langle x, w\rangle^{\cdot} \in [T]\}$$

is Σ^1_1.

Dually, for any $X \subseteq {}^\omega\omega$, X is Π^1_1 iff there is a recursive tree $T \subseteq {}^{<\omega}(\omega \times \omega)$ such that

$$X = \{x \in {}^\omega\omega \mid \forall^1 w\ \langle x, w\rangle^{\cdot} \notin [T]\}.$$

This relativizes immediately to any $z \in {}^\omega\omega$.

(5.57) **Theorem** [ZF] *Suppose $X \subseteq {}^\omega\omega$.*

1. *For any $z \in {}^\omega\omega$, X is $\Sigma^1_1(z)$ iff there is a $\Delta^0_1(z)$ tree T on $\omega \times \omega$ such that*

$$X = \{x \in {}^\omega\omega \mid \exists^1 w\ \langle x, w\rangle^{\cdot} \in [T]\}.$$

2. *X is $\boldsymbol{\Sigma}^1_1$ iff there is a tree T on $\omega \times \omega$ such that*

$$X = \{x \in {}^\omega\omega \mid \exists^1 w\ \langle x, w\rangle^{\cdot} \in [T]\}.$$

3. *Dually, X is $\Pi^1_1(z)$ iff there is a $\Delta^0_1(z)$ tree T on $\omega \times \omega$ such that*

$$X = \{x \in {}^\omega\omega \mid \forall^1 w\ \langle x, w\rangle^{\cdot} \notin [T]\}.$$

4. *X is $\boldsymbol{\Pi}^1_1$ iff there is a tree T on $\omega \times \omega$ such that*

$$X = \{x \in {}^\omega\omega \mid \forall^1 w\ \langle x, w\rangle^{\cdot} \notin [T]\}.$$

Note that in the boldface case there is no restriction on the definability of T, since it is a subset V_ω and is therefore recursive relative to some parameter, e.g., itself.

This normal form prompts the following definition.

(5.58) **Definition** [ZF] *Suppose T is a tree on $M \times N$.*

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[23] In this sort of situation it is conventional to quantify over the second argument, rather than the first, as in (5.35).

1. *For any* $s \in {}^{<\omega}M$,
   1. $T_{[s]} \overset{\text{def}}{=} \{t \in {}^{<\omega}N \mid |t| \leqslant |s| \wedge \langle s \restriction |t|, t\rangle^{\cdot} \in T\}$*; and*
   2. $T_s \overset{\text{def}}{=} \{t \in {}^{|s|}N \mid \langle s, t\rangle^{\cdot} \in T\}$.[24]
2. *For any* $x \in {}^{\omega}M$*, the* companion tree of x *(vis-à-vis* T*)* $\overset{\text{def}}{=} T_{[x]} \overset{\text{def}}{=} \{t \in {}^{<\omega}N \mid \langle x \restriction |t|, t\rangle^{\cdot} \in T\}$ $\big(= \bigcup_{n\in\omega} T_{[x \restriction n]}\big)$.
3. $\mathfrak{p}^{\cdot}[T] \overset{\text{def}}{=} \mathfrak{p}[T]^{\cdot} = \{x \in {}^{\omega}M \mid \exists y \in {}^{\omega}N\ \langle x, y\rangle^{\cdot} \in [T]\} = \{x \in {}^{\omega}M \mid [T_{[x]}] \neq 0\}$.

(5.59) *Recall*[3.182] *that if a tree* $T \subseteq {}^{<\omega}N$ *is wellfounded then* $[T] = 0$*; and conversely, assuming either* DC *or the wellorderability of* N*, if* $[T] = 0$ *then* T *is wellfounded. This is a crucial insight into the nature of sequence trees and may be regarded as the fundamental theorem of descriptive set theory.*

For the present application, $N = \omega$, so DC is not required. We therefore have the following very promising consequence of (5.57.3).

(5.60) **Theorem** [ZF] *Suppose* $X \subseteq {}^{\omega}\omega$ *and* $z \in {}^{\omega}\omega$*. Then* X *is* $\Pi^1_1(z)$ *iff there is a* $\Delta^0_1(z)$ *tree* $T \subseteq {}^{<\omega}(\omega \times \omega)$ *such that for all* $x \in {}^{\omega}\omega$*,* $x \in X$ *iff* $T_{[x]}$ *is wellfounded.*

This characterization of Π^1_1 prompts the following definition of an important Π^1_1 subset of ${}^{\omega}\omega$ that is recursively Π^1_1-complete.

(5.61) **Definition** [ZF]

1. *Given* $x \in {}^{\omega}\omega$*, let* R_x *for the nonce be* $\{\langle a, b\rangle \in {}^{2}(V_{\omega}) \mid x(\overleftarrow{B}\langle a, b\rangle) = 1\}$*. Thus,* R_x *is a binary relation on* V_{ω} *"coded by"* x*.*
2. LO $\overset{\text{def}}{=}$ *the set of* $x \in {}^{\omega}\omega$ *such that* R_x *is a reflexive linear order. For* $x \in$ LO*, we let* $\leqslant_x \overset{\text{def}}{=} R_x$*, and we let* $<_x$ *be the corresponding irreflexive order.*[25]
3. WO $\overset{\text{def}}{=}$ *the set of* $x \in$ LO *such that* $\leqslant_x$ *is a wellorder.*

In some applications, e.g., to sequence trees on ω, it would be more direct to work with (codes of) partial orders and wellfounded relations generally, instead of linear orders and wellorders specifically. There are, however, situations in which linearity is critical. The *Brouwer-Kleene ordering*[26] $\leqslant_{\text{BK}}$ of ${}^{<\omega}\omega$ allows us to substitute the latter for the former when dealing with sequence trees on ω.

(5.62) **Definition** [ZF] *Suppose* $s, t \in {}^{<\omega}\omega$*. Then* $s \leqslant_{\text{BK}} t \overset{\text{def}}{\iff}$ *either*

1. $\exists n \in \operatorname{dom} s \cap \operatorname{dom} t\, (s \restriction n = t \restriction n \wedge s(n) < t(n))$*; or*
2. $s \supseteq t$*.*

In other words, to compare $s, t \in {}^{<\omega}\omega$ in the $\leqslant_{\text{BK}}$ ordering we find the first n for which $s(n) \neq t(n)$, if there is any such n, and rank s and t according to their values at n; if there is no such n then one sequence is an initial segment of the other, and we rank the longer one as lower.

[24] Note that only in quite artificial examples is there any conflict between this definition and (3.183.2.1.1).

[25] The choice of reflexive, as opposed to irreflexive, order as the primary notion in the definition of LO is arbitrary.

[26] Also known as the *Lusin-Sierpiński* ordering.

(5.63) **Theorem** [ZF]

1. $\leqslant_{\mathrm{BK}}$ *linearly orders* ${}^{<\omega}\omega$.
2. *Suppose* $T \subseteq {}^{<\omega}\omega$ *is a sequence tree on* ω. *Then* $(T; \supseteq)$ *is wellfounded iff* $(T; \leqslant_{\mathrm{BK}})$ *is wellordered.*

Proof 1 Straightforward.

2 Suppose $(T; \supseteq)$ is not wellfounded. We may define a descending ω-sequence $s_0 \subsetneq s_1 \subsetneq \cdots$ in T by letting, for each $n \in \omega$, s_n be the first $s \in T$, according to the some fixed definable wellordering of ${}^{<\omega}\omega$, such that s properly extends all $s_m, m < n$, and $(T; \supseteq)$ is not wellfounded below s. By definition, $s_0 >_{\mathrm{BK}} s_1 >_{\mathrm{BK}} \cdots$, so $(T; \leqslant_{\mathrm{BK}})$ is not wellordered.

Conversely, suppose $(T; \leqslant_{\mathrm{BK}})$ is not wellordered. We may define a descending ω-sequence $s_0 >_{\mathrm{BK}} s_1 >_{\mathrm{BK}} \cdots$ in T as above. First note that for $n > 0$, $s_n \neq 0$, so $0 \in \operatorname{dom} s_n$, and $s_1(0) \geqslant s_2(0) \geqslant \cdots$. This nonincreasing sequence in ω is eventually constant. a_0 be the eventual value, and let n_0 be the least n such that $s_n(0) = a_0$. Since T is a tree, $\langle a_0 \rangle \in T$.

For every $n > n_0$, $s_n(0) = a_0$ and $1 \in \operatorname{dom} s_n$, so $\langle s_n(1) \mid n > n_0 \rangle$ is a nonincreasing sequence in ω, which is eventually constant. Let a_1 be the eventual value, and let n_1 be the least $n > n_0$ such that $s_n(1) = a_1$. Note that $\langle a_0, a_1 \rangle \in T$.

Proceeding in this fashion, we define $\langle a_m \mid m \in \omega \rangle$ such that $\forall n \in \omega\ \langle a_m \mid m < n \rangle \in T$; hence, $(T; \supseteq)$ is not wellfounded. $\square$[5.63]

(5.64) **Theorem** [ZF] WO *is recursively* Π^1_1*-complete and continuously* $\boldsymbol{\Pi}^1_1$*-complete.*

Proof It is easy to see that LO is arithmetical. WO is characterized within LO by the condition that for every $X \subseteq V_\omega$, if there is an element of X in the field of $\leqslant_x$, then there is a $\leqslant_x$-minimal such element. The universal quantification over $X \subseteq V_\omega$ makes this characterization Π^1_1. Thus, WO is Π^1_1.

Now suppose $X \subseteq {}^\omega\omega$ is Π^1_1. Let $T \subseteq {}^{<\omega}(\omega \times \omega)$ be a recursive tree such that for all $x \in {}^\omega\omega$, $x \in X$ iff $T_{[x]}$ is wellfounded.[5.60] Define $F : {}^\omega\omega \to {}^\omega 2$ by the condition that for every $x \in {}^\omega\omega$, for every $n \in \omega$, $(Fx)n = 1$ iff $\vec{B}\,n = \langle s, t \rangle$ for some $s, t \in T_{[x]}$ such that $s \leqslant_{\mathrm{BK}} t$. In the terminology of (5.61),

$$R_{Fx} = \leqslant_{\mathrm{BK}} \cap (T_{[x]} \times T_{[x]}),$$

so[5.63] $x \in X$ iff $T_{[x]}$ is wellfounded iff $Fx \in \mathrm{WO}$.

F is computed by the following procedure. Given $x \in {}^\omega\omega$, to obtain $(Fx)n$ for some $n \in \omega$, we first apply the recursive function $\vec{B}$ to n. If that turns out to be a 2-sequence $\langle s, t \rangle$, with $s, t \in {}^{<\omega}\omega$, we check whether $s \leqslant_{\mathrm{BK}} t$. If this is the case, we consult the oracle for x to obtain $x \restriction |s|$ and $x \restriction |t|$, and check whether $\langle x \restriction |s|, s \rangle^{\cdot}$ and $\langle x \restriction |t|, t \rangle^{\cdot}$ are in T (which is recursive by hypothesis). If they are then $(Fx)n = 1$; otherwise, $(Fx)n = 0$. Thus, F is recursive.

Since X was an arbitrary Π^1_1 set, WO is recursively Π^1_1-complete. The proof that WO is continuously $\boldsymbol{\Pi}^1_1$-complete is essentially the same. Given a $\boldsymbol{\Pi}^1_1$ set $X \subseteq {}^\omega\omega$. Let Let $T \subseteq {}^{<\omega}(\omega \times \omega)$ be a tree such that for all $x \in {}^\omega\omega$, $x \in X$ iff $T_{[x]}$ is wellfounded. Define $F : {}^\omega\omega \to {}^\omega 2$ as before. Since T is not typically recursive, R is not recursive, but it is recursive relative to T, and it is therefore continuous. $\square$[5.64]

For the present we are content to harvest one theorem to sustain us as we return to housekeeping chores. The proof provides a glimpse of the utility of the "fundamental theorem",[5.59, 5.60] of which we will see much more in the sequel.

(5.65) **Theorem** [ZF] *Suppose* $X, Y \subseteq {}^{\omega}\omega$ *are* $\Pi^1_1(z)$*. Then there exist* $\Pi^1_1(z)$ *sets* $X', Y' \subseteq {}^{\omega}\omega$ *such that*

1. $X' \subseteq X$*;*
2. $Y' \subseteq Y$*;*
3. $X' \cup Y' = X \cup Y$*; and*
4. $X' \cap Y' = 0$*.*

Proof Let S, T be $\Delta^0_1(z)$ sequence trees on $\omega \times \omega$ such that for all $x \in {}^{\omega}\omega$,

1. $x \in X$ iff $S_{[x]}$ is wellfounded; and
2. $x \in Y$ iff $T_{[x]}$ is wellfounded.

Let X' be the set of $x \in X$ such that either

1. $x \notin Y$; or
2. $x \in Y$ and $\operatorname{rk} S_{[x]} < \operatorname{rk} T_{[x]}$.

Let Y' be the set of $x \in Y$ such that either

1. $x \notin X$; or
2. $x \in X$ and $\operatorname{rk} T_{[x]} \leqslant \operatorname{rk} S_{[x]}$.

Clearly, X', Y' satisfy conditions (5.65).1–4. We must now show that they are $\Pi^1_1(z)$.

(5.66) **Claim** *Suppose* $x \in X$*. Then* $x \in X'$ *iff there does not exist an order-preserving* $f : T_{[x]} \to S_{[x]}$*.*

Proof Since $x \in X$, $S_{[x]}$ is wellfounded. Suppose $x \in X'$. Then by design either $T_{[x]}$ is not wellfounded or $T_{[x]}$ is wellfounded and $\operatorname{rk} S_{[x]} < \operatorname{rk} T_{[x]}$. By (3.184.3) in the former case and by (3.184.1) in the latter case, there does not exist an order-preserving $f : T_{[x]} \to S_{[x]}$.

Conversely, suppose there does not exist an order-preserving $f : T_{[x]} \to S_{[x]}$. Then either $T_{[x]}$ is not wellfounded or[3.184.1] $\operatorname{rk} T_{[x]} > \operatorname{rk} S_{[x]}$, so $x \in X'$. □[5.66]

(5.67) **Claim** *Suppose* $x \in Y$*. Then* $x \in Y'$ *iff there does not exist an order-preserving* $f : S_{[x]} \to (T_{[x]} \backslash \{0\})$*.*[27]

Proof Since $x \in Y$, $T_{[x]}$ is wellfounded. Suppose $x \in Y'$. Then either $S_{[x]}$ is not wellfounded or $S_{[x]}$ is wellfounded and $\operatorname{rk} T_{[x]} \leqslant \operatorname{rk} S_{[x]}$, in which case $\operatorname{rk}(T_{[x]} \backslash \{0\}) < \operatorname{rk} S_{[x]}$. By (3.184.3)[28] in the former case and by (3.184.1) in the latter case, there does not exist an order-preserving $f : S_{[x]} \to (T_{[x]} \backslash \{0\})$.

Conversely, suppose there does not exist an order-preserving $f : S_{[x]} \to (T_{[x]} \backslash \{0\})$. Then either $S_{[x]}$ is not wellfounded or[3.184.1] $\operatorname{rk} S_{[x]} > \operatorname{rk}(T_{[x]} \backslash \{0\})$, in which case $\operatorname{rk} S_{[x]} \geqslant \operatorname{rk} T_{[x]}$, so $x \in Y'$. □[5.67]

[27] The empty sequence 0 is the highest member of any sequence tree.

[28] We're using an obvious extension of Theorem 3.184 to $T_{[x]} \backslash \{0\}$.

Claims 5.66 and 5.67 provide $\Pi^1_1(z)$ descriptions of X' and Y', respectively.$\square$[5.65]

Theorem 5.65 asserts the *reduction* property for $\Pi^1_1(z)$ subsets of ${}^\omega\omega$, which may be extended by recursive substitution to Π^1_1 subsets of $U_\mathfrak{s}$ for any 1-type $\mathfrak{s}$. A similar argument applies to $\Pi^1_1(z)$ subsets of $U_\mathfrak{s}$ for 0-types $\mathfrak{s}$. Universally quantifying over the parameter z, we have the reduction property for $\boldsymbol{\Pi}^1_1$. (Of course, this is only of interest for 1-types.)

5.3 Classical descriptive set theory

> . . . usually he sat in a comfortable attitude, looking down, slightly stooped, with hands folded above his lap. He spoke quite freely, very clearly, simply and plainly: but when he wanted to emphasize a new viewpoint. . . then he lifted his head, turned to one of those sitting next to him, and gazed at him with his beautiful, penetrating blue eyes during the emphatic speech. . . . If he proceeded from an explanation of principles to the development of mathematical formulas, then he got up, and in a stately very upright posture he wrote on a blackboard beside him in his peculiarly beautiful handwriting: he always succeeded through economy and deliberate arrangement in making do with a rather small space. For numerical examples, on whose careful completion he placed special value, he brought along the requisite data on little slips of paper.
>
> Richard Dedekind describing his doctoral supervisor, Carl Friedrich Gauss

5.3.1 Introduction

As discussed above, the study of the Kleene pointclasses is often referred to as *effective* descriptive set theory, in contrast to the study of their fully relativized counterparts, which constitutes the *classical* theory. The classical theory actually has historical priority (hence the name) and it was developed originally as a theory of sets and functions of real numbers and pointset topology, arising naturally as an outgrowth of analysis. In this section we will develop the classical theory with only an occasional reference to the effective theory, tolerating a certain redundancy in the interest of a self-contained and historically representative (albeit not strictly chronologically accurate) presentation.

5.3.2 Real numbers

As noted above, there are multiple useful paradigms of the countably infinitary, ranging from $\mathcal{P}\omega$ to HC (hereditarily countable). The most familiar countably infinitary objects in mathematical practice, however, are geometrical points and their analytical counterparts: real numbers. Much of the early study of the foundations of the theory of the countably infinitary was undertaken in this context, and the relevant terminology has been adopted in foundational studies to describe corresponding notions in any appropriate setting.

As originally conceived, a geometrical point was regarded as a primitive structureless object, hardly a candidate for nomination as a paradigm of any notion of infinitarity. The usefulness of identifying points with numbers was made abundantly clear in the work of Descartes on analytic geometry, but numbers were themselves regarded as primitive and structureless. These numbers came to be called *real* to distinguish them from the *imaginary* and more generally *complex* numbers that were subsequently introduced to facilitate the discussion of polynomials and other functions. In this sense a real number is a special sort of number, but the term also came to be used with emphasis on its generality relative to such number types as natural numbers, integers, rational numbers, and algebraic numbers, for example. It was not until the nineteenth century that the requirements of mathematical rigor mandated that one provide a representation of these objects as part of an axiomatic theory.

A paradigm of the issues that stimulated this development is the problem of solutions of functional equations such as $\ulcorner f(x) = 0 \urcorner$, where f is a continuous function and x is a variable over the real numbers. For example, suppose $f(a) < 0$ and $f(b) > 0$. Obviously, for some x between a and b, $f(x) = 0$, but how to prove it?

A natural approach is to proceed as follows. In the interest of simplicity and without any loss of generality, suppose $a < b$. Let $a_0 = a$ and $b_0 = b$ and let c be the point midway between a_0 and b_0, i.e., $c = (a_0 + b_0)/2$. If $f(c) = 0$ we are finished. Otherwise, $f(c)$ is either < 0 or > 0. In the former case, let

$$\begin{aligned} a_1 &= c \\ b_1 &= b_0; \end{aligned}$$

in the latter case, let

$$\begin{aligned} a_1 &= a_0 \\ b_1 &= c. \end{aligned}$$

In either case $f(a_1) < 0$ and $f(b_1) > 0$. Now let $c = (a_1 + b_1)/2$. Again, if $f(c) = 0$ we are finished. Otherwise, we repeat the above construction, defining a_2 and b_2, and continue as long as necessary, generating sequences $\langle a_n \mid n \in N \rangle$ and $\langle b_n \mid n \in N \rangle$, where $0 < N \leqslant \omega$ and for every $n \in N$

1. letting $c = (a_{n^-} + b_{n^-})/2$,
   1. if $f(c) < 0$ then
   $$\begin{aligned} a_n &= c \\ b_n &= b_{n^-}; \end{aligned}$$
   2. if $f(c) > 0$ then
   $$\begin{aligned} a_n &= a_{n^-} \\ b_n &= c; \end{aligned}$$
2. $N = n + 1$ iff $f((a_n + b_n)/2) = 0$.

If $N = n + 1$ then $f((a_n + b_n)/2) = 0$, and we are finished. If $N = \omega$ we have infinite sequences $\langle a_n \mid n \in \omega \rangle$ and $\langle b_n \mid n \in \omega \rangle$ such that

(5.68)

1. $\forall n < \omega \; f(a_n) < 0$*;*
2. $\forall n < \omega \; f(b_n) > 0$*;*
3. $\forall n < \omega \; |a_n - b_n| = |a - b|/2^n$*; and*

4. $\forall m \leqslant n < \omega \; a_m \leqslant a_n < b_n \leqslant b_m$.

We now wish to claim that there exists a real number x such that

$$x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{5.69}$$

If there is, then since f is continuous,

$$f(x) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n).$$

Since $\forall n \; f(a_n) < 0$, $f(x) \leqslant 0$, and since $\forall n \; f(b_n) > 0$, $f(x) \geqslant 0$, so $f(x) = 0$.

Note that if both the limits in (5.69) exist then they are equal by virtue of (5.68.3). One approach to showing their existence is due to Cauchy and goes as follows. The sequence $\langle a_n \mid n \in \omega\rangle$ has the property that

$$\forall \varepsilon > 0 \, \exists n \, \forall m, m' > n \, |a_m - a_{m'}| < \varepsilon. \tag{5.70}$$

Such a sequence is called a *Cauchy sequence*, and Cauchy posited that any such sequence of real numbers has a limit.

Another approach, due to Dedekind, considers the set

$$L = \{a_n \mid n \in \omega\}.$$

Clearly L has an upper bound, e.g., b. Dedekind posited that any set of real numbers with an upper bound has a least upper bound. With this assumption, L has a least upper bound, say x. We claim that $x = \lim_{n\to\infty} a_n$, i.e., $\forall \varepsilon > 0 \, \exists n \, \forall m > n \; |a_m - x| < \varepsilon$. To show this, suppose $\varepsilon > 0$. Let n be such that[5.70]

$$\forall m, m' > n \; |a_m - a_{m'}| < \varepsilon/2.$$

We claim that $\forall m > n \, |a_m - x| < \varepsilon$. To show this, suppose toward a contradiction that for some $m > n$, $|a_m - x| \geqslant \varepsilon$. Then $a_m \leqslant x - \varepsilon$, so for all $m' > n$, $a_{m'} < a_m + \varepsilon/2 \leqslant x - \varepsilon/2$. Since $\langle a_x \mid k \in \omega\rangle$ is nondecreasing, $\forall k \in \omega \; a_k < x - \varepsilon/2$, so $x - \varepsilon/2$ is an upper bound for L that is less than x, contrary to the choice of x as the least upper bound for L.

Clearly Cauchy's approach is more direct than Dedekind's in this case, but they are equivalent. Both Cauchy's and Dedekind's principles are said to assert the *completeness* of the real numbers. Cauchy's approach generalizes readily to metric topological spaces, while Dedekind's generalizes to structures, like boolean algebras, that have an order relation.

We can implement Cauchy's or Dedekind's principle by simply declaring that the real numbers are the individuals of a structure $\mathbb{R} = (\mathbb{R}; 0, 1, +, \cdot, \leqslant)$,[29] which is an ordered field in the algebraic sense, with the additional property of completeness; but we still have to show that such a structure exists. The most direct way is to construct real numbers from some simpler precursor, and the rational number field $\mathbb{Q}$ is well suited to this purpose. $\mathbb{Q}$ is itself constructible from the ring $\mathbb{Z}$ of integers, which is in turn constructible from the monoid $\mathbb{N}$ of natural numbers, which Kronecker took as *gottgemacht*, but which von Neumann constructed as hereditarily finite sets.

Starting with $\mathbb{N} = (\omega; 0, 1, +.\cdot)$, we construct integers as pairs $\langle 0, n\rangle$ ($n \in \omega$) and $\langle 1, n\rangle$ ($n \in \omega\backslash\{0\}$), defining the operations of addition and multiplication so that

[29] it is customary to use the symbol $\ulcorner\mathbb{R}\urcorner$ for both the structure and for its set of individuals.

the $\langle 0, n\rangle$s are the non-negative integers, and the $\langle 1, n\rangle$s are the negative integers. We embed $\mathbb{N}$ in $\mathbb{Z}$ by the map $n \mapsto \langle 0, n\rangle$.

We then construct rational numbers as pairs $\langle a, b\rangle$ of integers with $b > 0$ and a, b having no factor in common other than 1, defining the operations of addition and multiplication so that $\langle a, b\rangle$ corresponds to a/b. We embed $\mathbb{Z}$ in $\mathbb{Q}$ by the map $a \mapsto \langle a, 1\rangle$.

So far the "numbers" we have constructed are hereditarily finite sets. To construct real numbers, however, we must leave HF.

(5.71) **Definition** *Using the Dedekind approach, we define a* Dedekind cut *in* $\mathbb{Q}$ *to be a nonempty* $x \subseteq \mathbb{Q}$ *that is bounded above, closed downward (or to the left), and has no greatest member:*

1. $\exists q \in \mathbb{Q}\, \forall q' \in x\ q' < q$;
2. $\forall q \in x\, \forall q' < q\ q' \in x$; *and*
3. $\forall q \in x\, \exists q' \in x\ q' > q$.

Note that a Dedekind cut is an infinite subset of HF, which is the simplest sort of infinitary object. We define $\mathbb{R}$ as the set of Dedekind cuts. For specificity, we will call these *Dedekind reals.* A rational number q is identified with the cut $\bar{q} \stackrel{\text{def}}{=} \{q' \in \mathbb{Q} \mid q' < q\}$. Before defining $\leqslant$, $+$, and $\cdot$ on $\mathbb{R}$, it is convenient to first define the negation and absolute value operations.

Definition

1. $-x \stackrel{\text{def}}{=} \{-q \mid q \in \mathbb{Q} \wedge \exists q' < q\ q' \notin x\}$.[30]
2. $|x| \stackrel{\text{def}}{=} \begin{cases} x & \textit{if } 0 \in x \\ -x & \textit{if } 0 \notin x. \end{cases}$
3. $x \leqslant y \stackrel{\text{def}}{\iff} x \subseteq y$.
4. $x + y \stackrel{\text{def}}{=} \{q + r \mid q \in x \wedge r \in y\}$.
5. *Multiplication is defined by cases.*
   1. *If* $x, y > 0$, $x \cdot y \stackrel{\text{def}}{=} \{q \cdot r \mid q \in x \wedge r \in y \wedge q, r \geqslant 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}$.
   2. *If either* $x = 0$ *or* $y = 0$ *then* $x \cdot y \stackrel{\text{def}}{=} 0$.
   3. *Otherwise*

$$x \cdot y = \begin{cases} |x| \cdot |y| & \textit{if } x, y < 0 \\ -(|x| \cdot |y|) & \textit{if } (x < 0 \wedge y > 0) \vee (x > 0 \wedge y < 0). \end{cases}$$

(These definitions may be made somewhat more elegant if we regard a Dedekind real as a pair consisting of a lower cut, which is a cut as we have defined it above, and an upper cut, which is the same, but closed upward, and which together contain all rationals except perhaps one.)

It is straightforward to show that with these definitions $\mathbb{R}$ is an ordered field. If X is a set of Dedekind reals with an upper bound then $\bigcup X$ is a Dedekind real that is clearly the least upper bound of X, so $\mathbb{R}$ is complete.

[30] Note that $-0 = 0$. We cannot define $-x$ as $\{-q \mid q \notin x\}$, because if x has a least upper bound q, then $-q$ would be the greatest member of $-x$, and we have defined Dedekind cuts as having no greatest member.

We have been careful to define Dedekind cuts in the rationals so that distinct cuts are distinct real numbers. We may also define real numbers via Cauchy sequences, but it is awkward (albeit feasible) to define a class of Cauchy sequences so that any given real number is represented by only one sequence in the class. The usual method is therefore to define a real number as an equivalence class of Cauchy sequences of rationals, where two sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are equivalent iff $a_0, b_0, a_1, b_1, \ldots$ is a Cauchy sequence. From our present standpoint, this is undesirable in that a real number is now an infinite set (in fact an uncountable set) of infinite subsets of HF: it is more infinitary than it should be.

5.3.3 Dedekind completion

The Dedekind construction is generally applicable to linear orders, and for future reference we pause here to make a few general remarks.

Recall[3.71.2] that $(X; <)$ is a linear (i.e., total) order iff $<$ is a binary relation on X and for all $x, y, z \in X$

1. $x < y \wedge y < z \rightarrow x < z$;
2. $x < y \vee x = y \vee y < x$; and
3. $x \not< x$.

We use '$\leqslant$' for the corresponding reflexive relation: $x \leqslant y \leftrightarrow x < y \vee x = y$, and we refer to $\leqslant$ also as a linear order.

Definition [ZF] *Suppose $(X; <)$ is a linear order. We may refer to $(X; <)$ as 'X'.*

1. *$x \in X$ is an* endpoint $\stackrel{\text{def}}{\iff}$ *x is the least or greatest member of X.*
2. *Suppose $x, y \in X$.*

$$\begin{aligned} (x, y) &\stackrel{\text{def}}{=} \{z \in X \mid x < z < y\} \\ [x, y] &\stackrel{\text{def}}{=} \{z \in X \mid x \leqslant z \leqslant y\} \\ (x, y] &\stackrel{\text{def}}{=} \{z \in X \mid x < z \leqslant y\} \\ [x, y) &\stackrel{\text{def}}{=} \{z \in X \mid x \leqslant z < y\}. \end{aligned}$$

3. *X is* dense $\stackrel{\text{def}}{\iff}$ *$\forall x, y \in X\, (x < y \rightarrow \exists z \in (x, y))$.*
4. *Suppose $Y \subseteq X$. Y is* dense in *X* $\stackrel{\text{def}}{\iff}$ *$\forall x, y \in X\, (x < y \rightarrow \exists z \in Y\ z \in (x, y))$.*
5. *X is* separable $\stackrel{\text{def}}{\iff}$ *X has a countable dense subset.*
6. *Suppose $Y \subseteq X$. $x \in X$ is an* upper (lower) bound *of Y* $\stackrel{\text{def}}{\iff}$ *for all $y \in Y$, $y \leqslant x$ ($y \geqslant x$).*
7. *X is* complete $\stackrel{\text{def}}{\iff}$ *for every nonempty $Y \subseteq X$, if Y has an upper bound then Y has a least upper bound.*

(5.72) **Theorem** [ZF] *Suppose $(X; <)$ is a linear order. X is complete iff for every nonempty $Y \subseteq X$, if Y has a lower bound then Y has a greatest lower bound.*

Proof Suppose $Y \subseteq X$ is nonempty. Clearly, if Y has a lower bound and y is the least upper bound of the set of lower bounds of Y, then y is the greatest lower bound of Y. Similarly, if Y has an upper bound and y is the greatest lower bound of the set of upper bounds of Y, then y is the least upper bound of Y. $\square^{5.72}$

To avoid irrelevant complications, the following discussion will be limited to dense linear orders without endpoints.

Definition [ZF] *Suppose $(X; <^X)$ is a dense linear order without endpoints.*

1. *A* Dedekind cut *in X $\overset{\text{def}}{=}$ a nonempty subset of X that is bounded above, closed downward, and has no greatest member.*[5.71]
2. *The* Dedekind completion of $(X; <^X)$ $\overset{\text{def}}{=}$ *the structure $(Y; <^Y)$, where*
    1. *Y is the set of Dedekind cuts in X; and*
    2. *$<^Y$ is the strict inclusion relation on Y, i.e., $Z <^Y Z' \leftrightarrow Z \subsetneq Z'$. Equivalently, $Z \leqslant^Y Z' \leftrightarrow Z \subseteq Z'$.*
3. *The* canonical embedding of $(X; <^X)$ in $(Y; <^Y)$ $\overset{\text{def}}{=} \iota^X \overset{\text{def}}{=}$ *the function $\iota : X \to Y$ defined by the condition that for any $x \in X$, $\iota^X x = \{y \in X \mid y <^X x\}$.*

ι^X is clearly an embedding of $(X; <^X)$ in its Dedekind completion $(Y; <^Y)$, i.e., an isomorphism of $(X; <^X)$ with a substructure of $(Y; <^Y)$. We will find it convenient to identify X with $\iota^{X \to} X$, and x with $\iota^X x$ for each $x \in X$.

(5.73) **Theorem** [ZF + $\mathsf{AC}_\omega(\mathbb{R})$]

1. *Suppose $(X; <^X)$ is a complete dense linear order without endpoints. Let $(Y; <^Y)$ be its Dedekind completion.*
    1. *$(Y; <^Y)$ is a complete dense linear order without endpoints.*
    2. *X (i.e., $\iota^{X \to} X$) is dense in $(Y; <^Y)$.*
2. *Suppose $(Y; <^Y)$ is a complete dense linear order without endpoints and $X \subseteq Y$ is dense. Let $<^X = <^Y \cap (X \times X)$.*
    1. *$(X; <^X)$ is a dense linear order without endpoints.*
    2. *$(Y; <^Y)$ is isomorphic to the Dedekind completion of $(X; <^X)$.*
    3. *$(Y; <^Y)$ is separable iff $(X; <^X)$ is separable.*

Proof 1 Straightforward. Note that for any $Z \subseteq Y$, if Z is nonempty and has an upper bound in $(Y; <^Y)$, then $\bigcup Z$ is the least upper bound of Z in $(Y; <^Y)$. $\square^{5.73.1}$

2 Also straightforward, given the following observations. $y \mapsto \{x \in X \mid x <^Y y\}$ is the *canonical embedding* of $(Y; <^Y)$ in the Dedekind completion of $(X; <^X)$. Any dense subset of $(X; <^X)$ is dense in $(Y; <^Y)$, so if $(X; <^X)$ is separable, so is $(Y; <^Y)$. On the other hand, suppose $Y' \subseteq Y$ is countable and dense. Note that any $y \in Y$ is uniquely determined by the set $\{y' \in Y' \mid y' < y\}$, and members of Y are in this way indexed by subsets of ω. Hence $\mathsf{AC}_\omega(\mathbb{R})$ (i.e., $\mathsf{AC}_\omega(\mathcal{P}\omega)$) implies the existence of a countable $X' \subseteq X$ such that $\forall y, y' \in Y' \, (y <^Y y' \to \exists x \in X' \; x \in (y, y'))$. X' is dense in X, so $(X; <^X)$ is separable. $\square^{5.73.2}$ $\square^{5.73}$

(5.74) **Definition** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *By virtue of (5.73) the Dedekind completion of a dense linear order $(X;<)$ without endpoints is—up to isomorphism—the minimum complete extension of $(X;<)$, and the unique complete extension in which X is dense, so we often refer to it as* the completion of $(X;<)$.

(5.75) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$]

1. $(\mathbb{Q};<)$ *is, up to isomorphism, the unique countable dense linear order without endpoints.*
2. *The Dedekind completion* $(\mathbb{R};<)$ *of* $(\mathbb{Q};<)$ *is, up to isomorphism, the unique complete separable dense linear order without endpoints.*

Proof 1 Suppose $(A;<^A)$ and $(B;<^B)$ are countable dense linear orders without endpoints. Let $\langle a_n \mid n \in \omega\rangle$ and $\langle b_n \mid n \in \omega\rangle$ be enumerations of A and B, respectively. Let $\langle \iota_n \mid n \in \omega\rangle$ be the sequence of partial isomorphisms of $(A;<^A)$ and $(B;<^B)$ defined by the following recursion.

1. Let $\iota_0 = 0$.
2. Suppose $n \in \omega$ is even. Let $k \in \omega$ be least such that $a_k \notin \operatorname{dom}\iota_n$, and let $l \in \omega$ be least such that for all $a' \in \operatorname{dom}\iota_n$, $a' <^A a_k \leftrightarrow \iota_n a' <^B b_l$. Let $\iota_{n+1} = \iota_n \cup \{(a_k, b_l)\}$.
3. Suppose $n \in \omega$ is odd. Let $l \in \omega$ be least such that $b_l \notin \operatorname{im}\iota_n$, and let $k \in \omega$ be least such that for all $b' \in \operatorname{im}\iota_n$, $b' <^B b_l \leftrightarrow \iota_n^{-1} b' <^A a_k$. Let $\iota_{n+1} = \iota_n \cup \{(a_k, b_l)\}$.

Since $(A;<^A)$ and $(B;<^B)$ are dense linear orders without endpoints, the extension steps are always feasible. Let $\iota = \bigcup_{n\in\omega}\iota_n$. Clearly, ι is an isomorphism of $(A;<^A)$ with $(B;<^B)$. $\square$[5.75.1]

2 It follows from (5.73.1) that $(\mathbb{R};<)$ is complete separable dense linear order without endpoints. Now suppose $(Y;<^Y)$ is a complete separable dense linear order without endpoints. Let $X \subseteq Y$ be countable and dense, and let $<^X = <^Y \cap (X \times X)$. Then $(X;<^X)$ is a countable dense linear order without endpoints, so it is isomorphic to $(\mathbb{Q};<)$.[5.75.1] By (5.73.2), $(Y;<^Y)$ is isomorphic to the Dedekind completion of $(X;<^X)$, which is $(\mathbb{R};<)$ by definition. $\square$[5.75.2] $\square$[5.75]

5.3.4 Topology of reals

In the preceding discussion we have seen how a close examination of the concept of real number leads to the realization that these objects are intrinsically infinitary, and the centrality of $\mathbb{R}$ as a mathematical concept makes it a natural introduction to infinitarity. From the standpoint of the set-theoretical study of the countably infinite, however, much of the structure of $\mathbb{R}$ is irrelevant. This includes addition, multiplication and the order relation. The most important feature of $\mathbb{R}$ for the study of the foundations of mathematics is its topology. This may be defined by taking the open intervals in $\mathbb{R}$, i.e., sets $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$, as a base, so an open set is a union of open intervals.

There are other topological spaces whose points are countably infinitary objects that are for foundational purposes essentially equivalent to $\mathbb{R}$ but are more easily managed. Two spaces are particularly important in this respect, viz., ${}^{\omega}2$ and ${}^{\omega}\omega$, with topologies defined as products of the discrete topologies on 2 and ω, respectively.

From the foundational standpoint the spaces $\mathbb{R}$, ${}^{\omega}2$, *and* ${}^{\omega}\omega$ *are essentially equivalent, and the members of any of these spaces are customarily referred to as* reals.

To define these topologies directly we proceed as follows. The definition is given for an arbitrary nonempty set A with the discrete topology.

(5.76) **Definition** [ZF] *Suppose* $\sigma \in {}^{<\omega}A$, *i.e.,* $\sigma : n \to A$, *where* $n < \omega$.

1. $I^A_\sigma \stackrel{\text{def}}{=} \{f \in {}^{\omega}A \mid \sigma \subseteq f\}$.[31]
2. *The sets* I^A_σ, $\sigma \in {}^{<\omega}A$ *are the* basic open intervals *of the* standard *topology on* ${}^{\omega}A$.
3. *Hence,* $X \subseteq {}^{\omega}A$ *is* open $\stackrel{\text{def}}{\iff}$ X *is a union of basic open intervals.*

If the set A *is known from the context, we may omit the superscript in* $\ulcorner I^A \urcorner$. *Unqualified references to the topology of* ${}^{\omega}A$ *refer to the standard topology.*

Definition [ZF]

1. ${}^{\omega}2$ *is the* Cantor space.
2. ${}^{\omega}\omega$ *is the* Baire space.

(5.77) **Theorem** [ZF]

1. ${}^{\omega}2$ *is homeomorphic to the* Cantor set *in* $\mathbb{R}$.[32]
2. ${}^{\omega}\omega$ *is homeomorphic to* $\mathbb{R}\backslash\mathbb{Q}$, *i.e., to the irrational real numbers with the relative topology.*

Proof See Note 10.15.

5.3.5 Metric spaces

$\mathbb{R}$, ${}^{\omega}2$, and ${}^{\omega}\omega$ are separable completely metrizable topological spaces as defined presently, and we will see that descriptive set theory is naturally regarded as the theory of definable subsets of such spaces.

The notion of metrizability derives from Cauchy's approach to the definition of completeness, mentioned above[5.70] in connection with $\mathbb{R}$. For reasons given there, we elected to use the alternative method of Dedekind to define the real number system $\mathbb{R}$ as the completion of $\mathbb{Q}$. We now pursue the notion of completeness in Cauchy's sense.

Definition [ZF] *A* metric *on a set* X *is a function* $d : X \times X \to \mathbb{R}$ *such that*

[31] Note that if $\sigma \in {}^{n}A$ then $\sigma = \{(m, \sigma m) \mid m \in n\}$, so the cardinality of $\sigma = |\sigma| = n$ and $\sigma = \{(m, \sigma m) \mid n \in |\sigma|\}$. Similarly, $f = \{(m, fm) \mid m \in \omega\}$, and $f \upharpoonright |\sigma| = \{(m, fm) \mid m \in |\sigma|\}$, so $\sigma \subseteq f \leftrightarrow \forall m \in |\sigma|\, fm = \sigma m$, i.e., considered as sequences, σ is the initial segment of f of length $|\sigma|$.

[32] The Cantor set is obtained as follows. Begin with $[0, 1]$. Remove the open middle third $(\frac{1}{3}, \frac{2}{3})$ to obtain $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then remove the open middle third from each of the remaining closed intervals. Continue *ad infinitum*. What is left is the Cantor set.

1. $d(x, y) \geqslant 0$.
2. $d(x, y) = 0$ *iff* $x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ [Triangle inequality].

The pair $\langle X, d \rangle$ *is a* metric space.

Definition [ZF] *Suppose* $\langle X, d \rangle$ *is a metric space.*

1. *For* $x \in X$ *and* $r > 0$*, the* open ball at x of radius r $\stackrel{\text{def}}{=} B(x, r) \stackrel{\text{def}}{=} \{y \in X \mid d(x, y) < r\}$.
2. *The* metric topology *on* X *(for the metric* d*)* $\stackrel{\text{def}}{=}$ *the topology generated by the open balls as a base.*

The ε-δ definition of continuity is meaningful and correct for metric spaces:

(5.78) **Theorem** [ZF] *Suppose* $\langle X_1, d_1 \rangle$ *and* $\langle X_2, d_2 \rangle$ *are metric spaces. Then a function* $f : X_1 \to X_2$ *is continuous at* $x \in X_1$ *iff for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $y \in X_1$,

$$d_1(x, y) < \delta \to d_2\big(fx, fy\big) < \varepsilon.$$

Proof Immediate from the definitions. □5.78

Two metric functions $d_1, d_2 : X \times X \to [0, \infty)$ generate the same topology iff for all $x \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for all $y \in X$, $d_1(x, y) < \delta \to d_2(x, y) < \varepsilon$ and $d_2(x, y) < \delta \to d_1(x, y) < \varepsilon$. We are usually concerned with the metric topology, not with a particular metric that produces it. Indeed, the mere fact that a topology is derived from a metric has consequences that may be stated without any reference to a metric. It is therefore useful to have the following definition:

Definition [ZF] *A topological space is* metrizable *iff its topology is generated by some metric function.*

Definition [ZF] *A sequence* $\langle x_n \mid n \in \omega \rangle$ *in a metric space* $\langle X, d(\cdot, \cdot) \rangle$ *is a* Cauchy sequence *iff for any* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$*, such that for all* $m, n > N$*,* $d(x_m, x_n) < \varepsilon$.

It is worth noting that cauchyness is not a purely topological notion. That is, one may have two metrics generating the same topology and a sequence that is Cauchy with respect to one metric but not the other.

5.3.6 Completeness

It is easy to see that every convergent sequence is Cauchy. The converse is not true in general. For example, the open interval $(0, 1) \subseteq \mathbb{R}$ with the usual topology is a metric space with the usual metric: $d(x, y) = |x - y|$. The sequence $\langle 1/n \mid n \in \omega \rangle$ is a Cauchy sequence, but it does not converge (because $0 \notin (0, 1)$). A more subtle example is the space of rational real numbers $\mathbb{Q}$ with the usual metric. If $\langle x_n \mid n \in \omega \rangle$ is any sequence in $\mathbb{Q}$ that converges in $\mathbb{R}$ to an irrational number, then $\langle x_n \mid n \in \omega \rangle$ is Cauchy, but it has no limit in $\mathbb{Q}$. $(0, 1)$ and $\mathbb{Q}$ are in a sense "incomplete". We codify this notion:

Definition [ZF]

1. *A metric space* $\langle X, d\rangle$ *is* complete $\overset{\text{def}}{\Longleftrightarrow}$ *every Cauchy sequence converges.*
2. *A topological space is* completely metrizable $\overset{\text{def}}{\Longleftrightarrow}$ *its topology is generated by a metric with respect to which it is complete.*

A metrizable space may be complete with respect to one metric and incomplete with respect to another. For example, the relative topology on the interval $(0, 1] \subseteq \mathbb{R}$ is generated by either of the metrics

1. $\rho_0(x, y) = |x - y|$;
2. $\rho_1(x, y) = |\frac{1}{x} - \frac{1}{y}|$.

$(0, 1]$ is incomplete with respect to ρ_0: $\langle \frac{1}{n+1} \mid n \in \omega\rangle$ is ρ_0-Cauchy and does not converge in $(0, 1]$. On the other hand, $(0, 1]$ with the metric ρ_1 is isomorphic to $[1, \infty)$ with the metric ρ_0 and is therefore clearly complete.

For this reason the adjective 'complete' applies only to metric spaces *per se*, not to purely topological spaces, even if they are completely metrizable. Accordingly, it would be inappropriate to refer to a completely metrizable space X as a complete metrizable space: it is not X itself that is complete, but rather X in conjunction with an appropriate metric.

(5.79) **Theorem** [ZF]

1. $\mathbb{R}$*, with the interval topology, is separable and completely metrizable.*
2. *Suppose* $A \neq 0$*.*
   1. ${}^{\omega}A$*, with the standard topology,*[5.76] *is completely metrizable.*
   2. *If* A *is countable,* ${}^{\omega}A$ *is separable.*

Proof 1 $x, y \mapsto |x - y|$ is clearly a metric on $\mathbb{R}$, and it clearly generates the interval topology. We will show that $\mathbb{R}$ is complete with respect to this metric. To this end suppose $\langle x_n \mid n \in \omega\rangle$ is a Cauchy sequence. Let $X = \{x \in \mathbb{R} \mid \exists N \in \omega\, \forall n > N\ x < x_n\}$. By virtue of the Cauchy property, $\{x_n \mid n \in \omega\}$ has an upper bound, so X has an upper bound. Let a be its least upper bound. X is clearly closed downward, so $X = (-\infty, a)$ or $X = (-\infty, a]$. Given $\varepsilon > 0$, let N be such that $\forall m, n > N\ |x_m - x_n| < \varepsilon/2$. We claim that $\forall n > N\ |x_n - a| < \varepsilon$. Suppose toward a contradiction that $n > N$ and $|x_n - a| \geqslant \varepsilon$. Then either $x_n \leqslant a - \varepsilon$ or $x_n \geqslant a + \varepsilon$. In the former case, $\forall m > N\ x_m < a - \varepsilon/2$, so $a - \varepsilon/2 \notin X$; contradiction. In the latter case, $\forall m > N\ x_m > a + \varepsilon/2$, so $a + \varepsilon/2 \in X$; also a contradiction. Hence, $\mathbb{R}$ is completely metrizable.

$\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so $\mathbb{R}$ is separable. $\square$[5.79.1]

2.1 For $f, g \in {}^{\omega}A$ with $f \neq g$, let $d(f, g) \overset{\text{def}}{=} 1/(n + 1)$, where n is least such that $fn \neq gn$. This generates the standard topology on ${}^{\omega}A$, and ${}^{\omega}A$ is clearly complete with respect to it.

2.2 Let a be a fixed member of A. For each $\sigma \in {}^{<\omega}A$, let $f^\sigma \in {}^{\omega}A$ be such that for all $n \in \omega$

$$f^\sigma(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ a & \text{otherwise.} \end{cases}$$

$\{f^\sigma \mid \sigma \in {}^{<\omega}A\}$ is a countable dense subset of ${}^{\omega}A$. $\square^{5.79.2}$ $\square^{5.79}$

Note that the natural metric on $\mathbb{Q}$ is $\mathbb{Q}$-valued, so we could have used the Cauchy machinery to define $\mathbb{R}$ from $\mathbb{Q}$. In general, however, a metric is necessarily $\mathbb{R}$-valued, so the definition of $\mathbb{R}$ must precede a full discussion of Cauchy's method, and this is another reason to use Dedekind's method for the initial definition of $\mathbb{R}$.

5.3.7 Polish spaces

As previously noted, for the purposes of descriptive set theory, $\mathbb{R}$, ${}^{\omega}2$, and ${}^{\omega}\omega$ are largely interchangeable; and, indeed, any separable completely metrizable space will serve as well. Spaces of this type were first extensively studied by Polish mathematicians and logicians, including notably Wacław Sierpiński, Kazimierz Kuratowski, and Alfred Tarski; hence:

(5.80) **Definition** [ZF] *A topological space X is* Polish $\overset{\text{def}}{\Longleftrightarrow}$ *X is separable and completely metrizable.*

The Baire space ${}^{\omega}\omega$ is universal among Polish spaces in the following sense.

(5.81) **Theorem** [ZF] *Suppose X is a Polish space. Then there exists a continuous $\iota : {}^{\omega}\omega \overset{\text{sur}}{\to} X$.*

Proof Let d be a metric that generates the topology of X and with respect to which it is complete. Let $\langle s_n \mid n \in \omega\rangle$ enumerate a dense subset S of X. For each $\sigma \in {}^{<\omega}\omega$ let $x_\sigma \in S$ and $B_\sigma \subseteq X$ be such that

(5.82)

1. $x_0 \in X$ *and* $B_0 = X$*;*
2. $|\sigma| > 0 \to B_\sigma = B(x_\sigma, 1/|\sigma|)$ *(the open ball at x_σ of radius $1/|\sigma|$);*
3. *for each $\sigma \in {}^{<\omega}\omega$, $\{x_{\sigma^\frown\langle n\rangle} \mid n \in \omega\} = S \cap \bigcap_{m \leqslant |\sigma|} B_{\sigma \restriction m}$.*[33]

By induction on length of sequences it is easy to show that this construction can be carried out by showing that

$$\forall \sigma \in {}^{<\omega}\omega \ x_\sigma \in S \cap \bigcap_{m<|\sigma|} B_{\sigma \restriction m}, \tag{5.83}$$

so B_σ has at least one point in common with $S \cap \bigcap_{m<|\sigma|} B_{\sigma \restriction m}$, viz., x_σ.

Suppose $f \in {}^{\omega}\omega$. For each $n \in \omega$, let $y_n = x_{f \restriction n}$. Suppose $n, n' > N > 0$. Then $y_n, y_{n'} \in B_{f \restriction N}$,[5.83] so $d(n, n') < 2/N$. Hence $\langle y_n \mid n \in \omega\rangle$ is a Cauchy sequence. Define ιf to be its limit.

It is straightforward to show that ι is continuous. To show that ι is surjective, suppose $x \in X$. We will define $f \in {}^{\omega}\omega$ so that

$$\forall n \in \omega \ x \in B_{f \restriction n}. \tag{5.84}$$

[33] Note that by using the fixed enumeration $\langle s_n \mid n \in \omega\rangle$ of S, there is no need to use a choice axiom to choose $x_{\sigma^\frown\langle n\rangle}$ for $n \in \omega$. The $x_{\sigma^\frown\langle n\rangle}$s need not be distinct.

Note that $x \in B_0 = X$. To define f recursively it therefore suffices to show that for any $\sigma \in {}^{<\omega}\omega$, if $x \in \bigcap_{m \leqslant |\sigma|} B_{\sigma \restriction m}$ then there exists $n \in \omega$ such that, $x \in B_{\sigma ^\frown \langle n \rangle}$. Suppose therefore that $x \in \bigcap_{m \leqslant |\sigma|} B_{\sigma \restriction m}$. Let $B = B(x, 1/(|\sigma| + 1))$. Then $B \cap \bigcap_{m \leqslant |\sigma|} B_{\sigma \restriction m}$ is open and nonempty, as it contains x. It therefore contains a point y in the dense set S.[34] Let[5.82.3] $n \in \omega$ be such that $x_{\sigma ^\frown \langle n \rangle} = y$. Let $\sigma' = \sigma ^\frown \langle n \rangle$. Then $d(x, x_{\sigma'}) < 1/(|\sigma| + 1) = 1/|\sigma'|$, so $x \in B_{\sigma'}$, as desired.

As noted above, this suffices to show that there exists $f \in {}^\omega\omega$ satisfying (5.84). Clearly, $\iota f = \lim_{n \to \infty} x_{f \restriction n} = x$. □[5.81]

(5.85) **Theorem** [ZF]

1. *A countable discrete topological space is Polish.*
2. [$\mathsf{AC}_\omega(\mathbb{R})$] *A nonempty closed subset of a Polish space is Polish with the relative topology.*
3. *A Polish space is second countable,*[3.193.2] *i.e., its topology has a countable base.*

Proof 1 Let $d(x, y) = 1$ if $x \neq y$. □[5.85.1]

2 Suppose X is Polish and $C \subseteq X$ is closed. Let d be a metric on X that generates its topology and with respect to which it is complete. Let $d' = d \restriction (C \times C)$. Then d' is a metric on C that generates the relative topology. Suppose $\langle c_n \mid n \in \omega \rangle$ is a d'-Cauchy sequence in C. Then $\langle c_n \mid n \in \omega \rangle$ is a d-Cauchy sequence in X and therefore has a limit, say c, in X. Since C is closed, $c \in C$. Hence, C is d'-complete.

To show that C is separable, suppose S is a countable dense set in X. Let $\iota : {}^\omega\omega \overset{\text{sur}}{\to} X$.[5.81] For each $s \in S$ and $n \in \omega \backslash \{0\}$, if $B(s, 1/n) \cap C \neq 0$, let $c_{s,n}$ be a member of $B(s, 1/n) \cap C$ (using $\mathsf{AC}_\omega(\mathbb{R})$ to choose $z_{s,n} \in {}^\omega\omega$ such that $\iota z_{s,n} \in B(s, 1/n) \cap C$ and letting $c_{s,n} = \iota z_{s,n}$). Let S' be the set of all these elements of C. S' is clearly countable. To see that it is dense, suppose $c \in C$ and $\varepsilon > 0$. We must show that there is a member of S' in $B(c, \varepsilon)$. Take $n > 1/\varepsilon$. Since S is dense in X, let $s \in S \cap B(c, 1/(2n))$. Note that $c \in B(s, 1/(2n))$, so $c_{s,2n}$ is defined, $c_{s,2n} \in B(s, 1/(2n))$, and $c_{s,2n} \in S'$. By the triangle inequality,

$$d(c, c_{s,2n}) \leqslant d(c, s) + d(s, c_{s,2n}) < 1/(2n) + 1/(2n) = 1/n < \varepsilon,$$

i.e., $c_{s,2n} \in B(c, \varepsilon)$, as desired. □[5.85.2]

3 Suppose X is a separable metric space. Let S be a countable dense subset X. Then $\{B(s, 1/n) \mid s \in S \wedge n \in \omega \backslash \{0\}\}$ is a countable base for the metric topology.
□[5.85.3] □[5.85]

The following theorem, which applies to topological spaces generally, is useful in manipulations of Polish function spaces.

(5.86) **Theorem** [ZF] *In the following, products of topological spaces are assumed to have product topologies. Suppose X, Y are topological spaces, and A, B are nonempty sets.*

1. ${}^A X \times {}^A Y$ *and* ${}^A(X \times Y)$ *are homeomorphic.*

[34] Again, using the fixed enumeration of S, we avoid an axiom of choice.

2. *Suppose A and B are equipollent. Then ${}^{A}X$ and ${}^{B}X$ are homeomorphic.*
3. *${}^{A}({}^{B}X)$ and ${}^{A\times B}X$ are homeomorphic.*

Proof 1 Let $j : {}^{A}X \times {}^{A}Y \to {}^{A}(X \times Y)$ be such that

$$(j\langle f, g\rangle)(a) = \langle f(a), g(a)\rangle,$$

for all $f \in {}^{A}X$, $g \in {}^{A}Y$, and $a \in A$. j is a homeomorphism.

2 Let $\pi : A \overset{\text{bij}}{\to} B$. Let $j : {}^{A}X \to {}^{B}X$ be such that

$$(jf)(\pi a) = f(a)$$

for all $f \in {}^{A}X$ and all $a \in A$. j is a homeomorphism.

3 Let $j : {}^{A}({}^{B}X) \to {}^{A\times B}X$ be such that

$$(jf)\langle a, b\rangle = (fa)(b)$$

for all $f \in {}^{A}({}^{B}X)$, $a \in A$, and $b \in B$. j is a homeomorphism. $\square^{5.86}$

As examples of the use of (5.86) we note that the following spaces are homeomorphic to ${}^{\omega}\omega$: ${}^{\omega}\omega \times {}^{\omega}2$,[35] ${}^{\omega}\omega \times {}^{\omega}\omega$, ${}^{\omega}({}^{\omega}\omega)$. These and similar equivalences are another illustration (in addition to (5.81)) of the universality of ${}^{\omega}\omega$ among Polish spaces.

The equivalence of ${}^{\omega}({}^{\omega}\omega)$ with ${}^{\omega}\omega$ is so frequently useful that it is worth positing a particular homeomorphism for future reference.

(5.87) *Let $p : {}^{2}\omega \overset{\text{bij}}{\to} \omega$ (a pairing function for ω). For each $n \in \omega$, let $j_n : {}^{\omega}\omega \to {}^{\omega}\omega$ be given by*

$$(j_n a)(m) = a(p\langle n, m\rangle).$$

For $a \in {}^{\omega}\omega$, let $ja = \langle j_n a \mid n \in \omega\rangle$. Note that $a \mapsto j_n a$ is continuous. j is a homeomorphism of ${}^{\omega}\omega$ with ${}^{\omega}({}^{\omega}\omega)$.

5.3.8 The Borel hierarchy

(5.88) **Definition** [ZF] *Suppose X is a topological space. For $0 < \alpha < \omega_1$ define $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ as follows.*

1. $\mathbf{\Sigma}^0_1 \overset{\text{def}}{=}$ *the set of open subsets of X.*
2. $\mathbf{\Pi}^0_1 \overset{\text{def}}{=}$ *the set of closed subsets of X.*
3. *For $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha \overset{\text{def}}{=}$ the set of countable unions $\bigcup_{n\in\omega} A_n$, where for each $n \in \omega$, $A_n \in \mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$.*
4. *For $\alpha > 1$, $\mathbf{\Pi}^0_\alpha \overset{\text{def}}{=}$ the set of countable intersections $\bigcap_{n\in\omega} A_n$, where for each $n \in \omega$, $A_n \in \mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$.*

Borel $\overset{\text{def}}{=} \bigcup_{0<\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$.

[35] This uses (5.86.1) together with the fact that $\omega \times 2$ and ω are equipollent, so as topological spaces with the discrete topology, they are homeomorphic.

For $n \in \omega\backslash\{0\}$ and $X = U_{\mathfrak{s}}$ a pointspace, $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$ are of course just the fully relativized Kleene arithmetical classes, e.g., $\boldsymbol{\Sigma}^0_n = \underset{\sim}{\Sigma}^0_n$. For $\alpha \geqslant \omega$, the Borel classes correspond to the Kleene hyperarithmetical classes, to which we alluded above. The complications surrounding the definition of *effective* countable union and intersection—corresponding to type-0 quantification in the effective ("light-face") theory and intrinsic to the definition of the hyperarithmetical hierarchy—do not arise in the "boldface" theory.

We are primarily interested in the case that X is a Polish space, but Definition 5.88 is appropriate for any topological space X. Note that we have omitted any reference to X in naming these classes, and we use these nonspecific names variously to refer to subsets of

1. *an arbitrary topological space;*
2. *a specific space under discussion; or*
3. *the canonical Polish space ${}^{\omega}\omega$.*

If necessary we indicate the relevant space with a superscript. As usual, we also use these class names as adjectives.

The following definition gives a useful way of coding Borel sets by reals.

(5.89) **Definition: Borel codes** [ZF]

1. *$S_\alpha, P_\alpha, C_\alpha \subseteq {}^{\omega}\omega$ are defined by recursion for $0 < \alpha < \omega_1$ as follows.*
   1. *$C_\alpha = \bigcup_{0<\beta<\alpha}(S_\beta \cup P_\beta)$.*
   2. *$S_1 = \{\langle n\rangle ^\frown x \mid 1 < n < \omega \wedge x \in {}^{\omega}\omega\}$.*
   3. *$P_\alpha = C_\alpha \cup \{\langle 0\rangle ^\frown x \mid x \in S_\alpha\}$.*
   4. *If $\alpha > 1$ then $S_\alpha = C_\alpha \cup \{\langle 1\rangle ^\frown x \mid x \in {}^{\omega}\omega \wedge \forall n \in \omega\ j_n x \in C_\alpha\}$.*[5.87]
2. $\mathrm{BC} \overset{\text{def}}{=} \bigcup_{0<\alpha<\omega_1} C_\alpha$.
3. *For codes $c \in \mathrm{BC}$ we define Borel sets B_c recursively as follows. We say that c* codes *B_c.*
   1. *If $c \in S_1$ and $c = \langle n\rangle ^\frown x$, then $B_c = \bigcup\{I_s \mid s \in {}^{<\omega}\omega \wedge x(\overleftarrow{B}\, s) = 1\}$.*
   2. *If $c \in P_\alpha$ and $c = \langle 0\rangle ^\frown x$, then $B_c = {}^{\omega}\omega\backslash B_x$.*
   3. *If $\alpha > 1$, $c \in S_\alpha$, and $c = \langle 1\rangle ^\frown x$, then $B_c = \bigcup_{n\in\omega} B_{j_n x}$.*

To justify the recursive definition of B_c we reason as follows. Suppose $c \in \mathrm{BC}$. It is easy to see that there exists $\alpha > 0$ such that $c \in C_{\alpha+1}\backslash C_\alpha$. c is therefore in $S_\alpha\backslash C_\alpha$ or $P_\alpha\backslash C_\alpha$, but not both. For the nonce, define the *rank of* c to be $\alpha\cdot 2$ if $c \in S_\alpha\backslash C_\alpha$ and $\alpha\cdot 2+1$ if $c \in P_\alpha\backslash C_\alpha$. Note that for any $\alpha > 1$, if $c \in C_\alpha$ then the rank of c is $< \alpha\cdot 2$. ($C_1 = 0$, $C_2 = S_1 \cup P_1$, etc.)

If c has the least rank, viz., 2, the $c \in S_1$, and B_c is defined directly by (5.89.3.1). If c has rank $\alpha\cdot 2+1$ for any α, then B_c is defined by (5.89.3.2) in terms of $B_{c'}$ for some c' of lower rank (viz., $\alpha\cdot 2$). If c has rank $\alpha\cdot 2$ for some $\alpha > 1$ then B_c is defined by (5.89.3.3) in terms of $B_{c'}$ for various c's which are in C_α and are therefore of lower rank.

(5.90) **Theorem** [ZF] *Suppose $0 < \alpha < \omega_1$.*

1. *Suppose $c \in S_\alpha$ (P_α). Then $B_c \in \Sigma^0_\alpha$ (Π^0_α).*
2. [$\mathsf{AC}_\omega(\mathbb{R})$] *Suppose $X \subseteq {}^{\omega}\omega$ is Σ^0_α (Π^0_α). Then there exists $c \in S_\alpha$ (P_α) such that $X = B_c$.*

Proof 1 Straightforward induction on α. $\square^{5.90.1}$

2 Also by induction on α. $\mathsf{AC}_\omega(\mathbb{R})$ is used to show that if for each $n \in \omega$ there exists $c \in \mathrm{BC}$ such that $X_n = B_c$ then there exists $\langle c_n \mid n \in \omega\rangle$ such that for each $n \in \omega$, $c_n \in \mathrm{BC}$ and $X_n = B_{c_n}$. $\square^{5.90.2}$ $\square^{5.90}$

(5.91) *The system (5.89) may be used to define codes for Borel subsets of any second-countable space X by redefining B_c for $c \in S_1$ so as to represent all the open subsets of X via an enumeration of countable base for the topology.*

One rather pedestrian application of Borel codes is to allow the use of $\mathsf{AC}_\omega(\mathbb{R})$ to assert the existence of choice functions $f : \omega \to \mathbf{Borel}$. Similar coding by reals is widely available in descriptive set theory.

(5.92) *For example, suppose X and Y are Polish spaces. Let $X_0 = Y_0 = 0$, and let $\langle X_n \mid 0 < n < \omega\rangle$ and $\langle Y_n \mid 0 < n < \omega\rangle$ be enumerations of bases for the X- and Y-topologies. Given a continuous function $f : X \to Y$, let $S_f = \{\langle m, n\rangle \in \omega \times \omega \mid f^{\to} X_m \subseteq Y_n\}$. Clearly, if $f, g : X \to Y$ are continuous then $f = g$ iff $S_f = S_g$, so $f \mapsto S_f$ is a coding of continuous functions by reals.*

(5.93) **Theorem** [ZF] *Suppose X and Y are Polish spaces.*

1. *If $0 < \alpha < \omega_1$ then $\mathbf{\Sigma}^0_\alpha = \{A \subseteq X \mid x \backslash A \in \mathbf{\Pi}^0_\alpha\}$.*
2. *$\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$.*
3. *If $0 < \alpha < \beta < \omega_1$ then*
   1. *$\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$;*
   2. *$\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$;*
   3. *$\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$;*
   4. *$\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$.*
4. *$\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ are continuously closed, i.e., if $f : X \to Y$ is continuous and $A \subseteq Y$ is in $\mathbf{\Sigma}^{0\,Y}_\alpha$ or $\mathbf{\Pi}^{0\,Y}_\alpha$ then $f^{\leftarrow} A$ is in $\mathbf{\Sigma}^{0\,X}_\alpha$ or $\mathbf{\Pi}^{0\,X}_\alpha$, respectively.*
5. *$\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ are closed under finite union and finite intersection.*
6. [$\mathsf{AC}_\omega(\mathbb{R})$] *$\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$) is closed under countable union (intersection).*
7. [$\mathsf{AC}_\omega(\mathbb{R})$] *$\mathbf{Borel}^X$ is the smallest class that contains all open sets and is closed under complementation (relative to X), and countable unions (equivalently, intersections). Equivalently, $\mathbf{Borel}^X$ is the smallest class that contains all open sets and all closed sets and is closed under countable unions and intersections.*

Proof 1 By induction on α, starting with the fact that closed sets are the complements of open sets. $\square^{5.93.1}$

2 Suppose a metric for X to have been chosen. Suppose $A \subseteq X$ is $\mathbf{\Sigma}^0_1$, i.e., open. For each $x \in A$, let $r_x = \sup\{r \in \mathbb{R} \mid r \leqslant 1 \wedge B(x, r) \subseteq A\}$, i.e. the supremum (least upper bound) of the set of $r \leqslant 1$ such that the open ball of radius r at x is included in A. Since A is open, $r_x > 0$.

Let $S \subseteq A$ be countable and dense in A. We claim that $A = \bigcup_{s \in S} C(s, r_s/2)$. Clearly, $\forall s \in S\ C(s, r_s/2) \subseteq A$, so $\bigcup_{s \in S} C(s, r_s/2) \subseteq A$. Suppose $x \in A$. Since S is dense in A, there exists $s \in S \cap B(x, r_x/3)$. It is easy to show that $r_s > 2r_x/3$ (triangle inequality), so $x \in C(s, r_s/2)$, so $x \in \bigcup_{s \in S} C(s, r_s/2)$.

Since each $C(s, r_s/2)$ is $\mathbf{\Pi}^0_1$, i.e., closed, A is $\mathbf{\Sigma}^0_2$. $\square^{5.93.2}$

3 Left to the reader. Use (5.93.2) to get things started. $\square^{5.93.3}$

4 Straightforward induction on α, given that the classes of open and closed sets are closed under continuous preimage. $\square^{5.93.4}$

5 Straightforward by induction on α, starting with the fact that the classes of open and closed sets are closed under finite union and intersection, and using the identity

$$\bigcup_{n \in \omega} A_n \cap \bigcup_{n \in \omega} B_n = \bigcup_{m,n \in \omega} (A_m \cap B_n)$$

and its dual. $\square^{5.93.5}$

6 Suppose for each $n \in \omega$, $X_n \in \Sigma^0_\alpha$. Thus, for each $n \in \omega$, there exists $\langle Y_m \mid m \in \omega \rangle$ such that $\forall m \in \omega\, \exists \beta < \alpha\ Y_m \in \Pi^0_\beta$ and $X_n = \bigcup_{m \in \omega} Y_m$. We wish to infer that there exists $\langle X^m_n \mid m, n \in \omega \rangle$ such that $\forall m, n \in \omega\, \exists \beta < \alpha\ X^m_n \in \Pi^0_\beta$ and $\forall n \in \omega\ X_n = \bigcup_{m \in \omega} X^m_n$, from which it follows that $X = \bigcup_{m,n \in \omega} X^m_n$ is in Σ^0_α (because $\omega \times \omega$ is equipollent with ω).

This inference is straightforward using AC_ω. To achieve it with $\mathsf{AC}_\omega(\mathbb{R})$, we use the fact that ${}^\omega({}^\omega\omega)$ is equipollent with ${}^\omega\omega$ to show that there exists $\langle c^m_n \mid m, n \in \omega \rangle$ such that $\forall m, n \in \omega\, \exists \beta < \alpha\ B_{c^m_n} \in \Pi^0_\beta$ and $\forall n \in \omega\ X_n = \bigcup_{m \in \omega} B_{c^m_n}$, where B_c is the Borel set coded by the Borel code c.[5.89] $\square^{5.93.6}$

7 Suppose $\mathcal{X} \subseteq \mathcal{P}X$ contains all open sets and is closed under complementation (relative to X), and countable unions. By induction on α, $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathcal{X}$, so $\mathbf{Borel}^X \subseteq \mathcal{X}$.

By (5.93.1) $\mathbf{Borel}^X$ is closed under complementation. To show it is closed under countable union, suppose A_n, $n \in \omega$, are Borel sets. Let α_n be least such that $A_n \in \mathbf{\Pi}^0_{\alpha_n}$. Use $\mathsf{AC}_\omega(\mathbb{R})$ to choose a wellordering of ω of order type α_n for each $n \in \omega$. Use these to define a composite wellordering of ω whose order type exceeds α_n for each $n \in \omega$. (In other words, $\mathsf{AC}_\omega(\mathbb{R})$ implies ω_1 is regular.) It follows that $\alpha = \sup_n \alpha_n < \omega_1$, so $\bigcup A_n \in \mathbf{\Sigma}^0_\alpha$, so it is Borel. $\square^{5.93.7}$ $\square^{5.93}$

Definition [ZF] *The following terminology is often applied to the lower levels of the Borel hierarchy:*

1. *'G', from the German* Gebiet *(region, domain) indicates open sets.*
2. *'F', from the French* fermé *(closed) indicates closed sets.*
3. *'δ', from the German* Durchschnitt *(intersection) indicates countable intersection.*

4. *'σ', from the French* somme *(sum, union) indicates countable union.*

Hence, $G = \mathbf{\Sigma}^0_1$, $F = \mathbf{\Pi}^0_1$, $G_\delta = \mathbf{\Pi}^0_2$, $F_\sigma = \mathbf{\Sigma}^0_2$, $G_{\delta\sigma} = \mathbf{\Sigma}^0_3$, *etc.*

In the context of boolean set algebras, 'σ' is used to indicate closure under countable union (and intersection); hence a σ-*algebra* in $\mathcal{P}X$ is a countably closed subalgebra of $\mathcal{P}X$. In the case of a topological space X, $\mathbf{Borel}^X$ is therefore the smallest σ-algebra containing all open sets.[5.93.7]

(5.94) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose X is a Polish space and $0 < \alpha < \omega_1$. There exists a $\mathbf{\Sigma}^0_\alpha$ set $U \subseteq {}^\omega\omega \times X$ that is* universal *in the sense that for any $\mathbf{\Sigma}^0_\alpha$ $A \subseteq X$ there exists $a \in {}^\omega\omega$ such that*

$$A = \{x \in X \mid \langle a, x\rangle \in U\}.$$

Remark This is just the notion of 1-universality[5.52.2.1] applied to an arbitrary Polish space. Note that $({}^\omega\omega \times X)\backslash U$ is correspondingly universal for $\mathbf{\Pi}^0_\alpha$.

Proof By induction on α. For $\alpha = 1$, i.e., for open sets, we proceed as follows. Let $G_0 = 0$, and let $G_1, G_2, \ldots$ be an enumeration of a countable base for the X-topology.[5.85.4] For each $n \in \omega$, let

$$U_n = \{\langle a, x\rangle \mid a \in {}^\omega\omega \wedge x \in G_{a(n)}\},$$

and let $U = \bigcup_{n\in\omega} U_n$. Clearly, each U_n is open,[36] so U is also open. Given an open $A \subseteq X$, there exists $a \in {}^\omega\omega$ such that $A = \bigcup_{n\in\omega} G_{a(n)}$. For any $x \in X$,

$$x \in A \leftrightarrow \exists n \in \omega\ x \in G_{a(n)} \leftrightarrow \langle a, x\rangle \in U.$$

Hence, U is universal for $\mathbf{\Sigma}^0_1$.

Now suppose U is universal for $\mathbf{\Sigma}^0_\alpha$. We will construct a universal $\mathbf{\Sigma}^0_{\alpha+1}$ set U'. Let p, j_n, j be as in (5.87), let

$$U'_n = \{\langle a, x\rangle \in {}^\omega\omega \times X \mid \langle j_n a, x\rangle \notin U\},$$

and let $U' = \bigcup_{n\in\omega} U'_n$. Since $\langle a, x\rangle \mapsto \langle j_n a, x\rangle$ is continuous, U'_n is $\mathbf{\Pi}^0_\alpha$, so U' is $\mathbf{\Sigma}^0_{\alpha+1}$. Given an arbitrary $\mathbf{\Sigma}^0_{\alpha+1}$ $A \subseteq X$, suppose $A = \bigcup_{n\in\omega} A_n$, where each A_n is $\mathbf{\Pi}^0_\alpha$, and for each $n \in \omega$ (using $\mathsf{AC}_\omega(\mathbb{R})$) let $a_n \in {}^\omega\omega$ be such that

$$A_n = \{x \in X \mid \langle a_n, x\rangle \notin U\}.$$

Let $a \in {}^\omega\omega$ be such that $\forall n, m \in \omega\ a(p(n, m)) = a_n(m)$, so $\forall n \in \omega\ j_n a = a_n$. For each $n \in \omega$

$$A_n = \{x \in X \mid \langle j_n a, x\rangle \notin U\} = \{x \in X \mid \langle a, x\rangle \in U'_n\},$$

so

$$A = \bigcup_{n\in\omega} A_n = \{x \in X \mid \langle a, x\rangle \in U'\}.$$

Hence, U' is universal for $\mathbf{\Sigma}^0_{\alpha+1}$.

Lastly, suppose $\alpha < \omega_1$ is a limit ordinal. Let $0 < \alpha_0 < \alpha_1 < \ldots$ be an ω-sequence cofinal in α, and for each $n \in \omega$, let U_n be universal for $\mathbf{\Sigma}^0_{\alpha_n}$ (using $\mathsf{AC}_\omega(\mathbb{R})$ to choose Borel codes for these). With p, j_n, j as above, let

$$U'_n = \{\langle a, x\rangle \in X \times {}^\omega\omega \mid \langle j_n a, x\rangle \notin U_n\},$$

[36] $\{I_s \times G_m \mid s \in {}^{<\omega}\omega \wedge m \in \omega\backslash\{0\}\}$ is a base for the (product) topology on ${}^\omega\omega \times X$.

and let $U' = \bigcup_{n\in\omega} U'_n$. The rest of the argument is essentially the same as before. $\square^{5.94}$

The proof of the following theorem is another example of a diagonal argument.

(5.95) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose $0 < \alpha < \omega_1$. Then there exists a $\mathbf{\Pi}^0_\alpha$ subset of ${}^\omega\omega$ that is not $\mathbf{\Sigma}^0_\alpha$.*

Remark Taking complements, it follows that there is a $\mathbf{\Sigma}^0_\alpha$ subset of ${}^\omega\omega$ that is not $\mathbf{\Pi}^0_\alpha$.

Proof Let $U \subseteq {}^\omega\omega \times {}^\omega\omega$ be $\mathbf{\Sigma}^0_\alpha$-universal. Let $A = \{x \in {}^\omega\omega \mid \langle x, x\rangle \notin U\}$. Since $x \mapsto \langle x, x\rangle$ is continuous, A is $\mathbf{\Pi}^0_\alpha$. If A is $\mathbf{\Sigma}^0_\alpha$ then there exists $a \in {}^\omega\omega$ such that for all $x \in X$, $x \in A \leftrightarrow \langle a, x\rangle \in U$, from which it follows that $\langle a, a\rangle \in U \leftrightarrow a \in A \leftrightarrow \langle a, a\rangle \notin U$. From this contradiction it follows that A is not $\mathbf{\Sigma}^0_\alpha$. $\square^{5.95}$

5.3.9 The projective hierarchy

Definition [ZF] *Suppose X is a Polish space and $A \subseteq X$. A is* analytic $\overset{\text{def}}{\iff}$ *there is a continuous $f : {}^\omega\omega \to X$ and a closed $C \subseteq {}^\omega\omega$ such that $f^{\to}C = A$. A is* coanalytic $\overset{\text{def}}{\iff}$ *$X \setminus A$ is analytic.* $\mathbf{Analytic}^X \overset{\text{def}}{=}$ *the set of analytic subsets of X.*[37]

'**Analytic**' without a superscript may refer to the analytic subsets of an arbitrary Polish space or specifically to $\mathbf{Analytic}^{{}^\omega\omega}$.

(5.96) **Definition** [ZF] *Suppose S is a set of 2-sequences. The* projection *of S* $\overset{\text{def}}{=} \mathfrak{p}\, S \overset{\text{def}}{=} \{x \mid \exists y\ \langle x, y\rangle \in S\}$.[38]

Recall[3.198] that if X and Y are topological spaces and $X \times Y$ is given the product topology, then the projection map $\langle x, y\rangle \mapsto x$ is continuous.

(5.97) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$]

1. *Suppose X is a Polish space. Then 0 and X are analytic (as subsets of X).*
2. **Analytic** *is closed under continuous image, i.e., if X and Y are Polish, $A \subseteq X$ is analytic, and $g : X \to Y$ is continuous, then $g^{\to}A$ is analytic.*[39]
3. *Suppose X is a Polish space and $A \subseteq X$. A is analytic iff A is the projection of a closed subset of $X \times {}^\omega\omega$.*
4. **Analytic** *is closed under continuous preimage, i.e., if X and Y are Polish, $A \subseteq X$ is analytic, and $g : Y \to X$ is continuous, then $g^{\leftarrow}A$ is analytic.*
5. **Analytic** *is closed under countable union and intersection.*
6. **Borel** $\subseteq$ **Analytic**.

Proof

[37] The adjective 'analytic' in classical descriptive set theory has an unfortunate similarity to the adjective 'analytical' in the effective theory. As we will see $\mathbf{Analytic} = \mathbf{\Sigma}^1_1$, which only adds to the confusion. Both terms are fixed in the literature, and one must simply observe the distinction scrupulously.

[38] We already have the notation '$\operatorname{dom} S$' for this notion, but '$\mathfrak{p}$' is more common in the current context.

[39] Note that $\operatorname{dom} g = X$. Given that g is continuous on X, $g \restriction Z$ is continuous in the relative topology on Z for any $Z \subseteq X$. It is not, however, true in general that any continuous function on Z can be extended to a continuous function on X, so we have to be specific as to domains.

1 Let[5.81] $f : {}^\omega\omega \overset{\text{sur}}{\to} X$ be continuous. Then $f^{\to}0 = 0$ and $f^{\to}({}^\omega\omega) = X$. $\square$[5.97.1]

2 Let $f : {}^\omega\omega \to X$ be continuous and $C \subseteq {}^\omega\omega$ be closed such that $A = f^{\to}C$. Then $g \circ f$ is continuous and $g^{\to}A = (g \circ f)^{\to}C$, so $g^{\to}A$ is analytic. $\square$[5.97.2]

3 Suppose X is a Polish space. Let $\pi : X \times {}^\omega\omega \to X$ be the projection map $\langle x, y\rangle \mapsto x$. Suppose $C \subseteq X \times {}^\omega\omega$ is closed. If $C = 0$ then $\mathfrak{p}\,C = 0$ and is therefore analytic.[5.97.1] If $C \neq 0$ then[5.85] C (with the relative topology) is a Polish space, and it is analytic (as a subset of itself).[5.97.1] $\pi \restriction C$ is continuous, so $\mathfrak{p}\,C = \pi^{\to}C$ is analytic.[5.97.2]

Conversely, suppose $A \subseteq X$ is analytic. Suppose $f : {}^\omega\omega \to X$ is continuous, $C \subseteq {}^\omega\omega$ is closed, and $A = f^{\to}C$. Let $B = \{\langle fx, x\rangle \mid x \in C\}$.

(5.98) **Claim** *B is closed.*

Proof Suppose $\langle\langle y_n, x_n\rangle \mid n \in \omega\rangle$ is a sequence in B that converges in $X \times {}^\omega\omega$. Then $\forall n \in \omega\ y_n = f(x_n)$, and $\langle x_n \mid n \in \omega\rangle$ is convergent. Let $x = \lim_{n\to\infty} x_n$, which is in C since C is closed. Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(x)$, so $\lim_{n\to\infty}\langle y_n, x_n\rangle = \langle fx, x\rangle$, which is in B. $\square$[5.98]

$A = \mathfrak{p}\,B$. $\square$[5.97.3]

4 Let $f : {}^\omega\omega \to X$ be continuous and $C \subseteq {}^\omega\omega$ be closed such that $A = f^{\to}C$, and let $Z = \{\langle b, c\rangle \in Y \times C \mid fc = gb\}$.

(5.99) **Claim** *Z is closed.*

Proof Suppose $\langle\langle b_n, c_n\rangle \mid n \in \omega\rangle$ is a sequence in Z that converges to $\langle b, c\rangle$. Then $b = \lim_{n\to\infty} b_n \in Y$, $c = \lim_{n\to\infty} c_n \in C$, and

$$fc = \lim_{n\to\infty} fc_n = \lim_{n\to\infty} gb_n = gb,$$

so $\langle b, c\rangle \in Z$. $\square$[5.99]

Since

$$\begin{aligned}\mathfrak{p}\,Z &= \{b \in Y \mid \exists c \in {}^\omega\omega\ fc = gb\} = \{b \in Y \mid \exists a \in A\ a = gb\}\\ &= g^{\leftarrow}A,\end{aligned}$$

$g^{\leftarrow}A$ is analytic.[5.97.3] $\square$[5.97.4]

5 Suppose X is a Polish space. Suppose for each $n \in \omega$, $A_n \subseteq X$ is analytic. Using $\mathsf{AC}_\omega(\mathbb{R})$ with a coding of continuous functions from ${}^\omega\omega$ to X by reals,[5.92] for each $n \in \omega$, let $f_n : {}^\omega\omega \to X$ be continuous such that $\operatorname{im} f_n = A_n$.

To show that $\bigcup_{n\in\omega} A_n$ is analytic, let $Z = \{\langle f_n x, \langle n\rangle^\frown x\rangle \mid n \in \omega \wedge x \in {}^\omega\omega\}$. Z is closed and

$$\mathfrak{p}\,Z = \bigcup_{n\in\omega} \operatorname{im} f_n = \bigcup_{n\in\omega} A_n,$$

so[5.97.3] $\bigcup_{n\in\omega} A_n$ is analytic.

To show that $\bigcap_{n\in\omega} A_n$ is analytic, let p, j_n, j be as in (5.87). For $n \in \omega$ let $Z_n = \{\langle f_n(j_n x), x\rangle \mid x \in {}^\omega\omega\}$. Since j_n and f_n are continuous, Z_n is closed, and clearly $\mathfrak{p}\, Z_n = A_n$. Let $Z = \bigcap_{n\in\omega} Z_n$. As an intersection of closed sets, Z is closed. Note that for $y \in X$ and $x \in {}^{\omega}\omega$,

$$\begin{aligned}\langle y, x\rangle \in Z &\leftrightarrow \forall n \in \omega\, \langle y, x\rangle \in Z_n \\ &\leftrightarrow \forall n \in \omega\; y = f_n(j_n x),\end{aligned}$$

and for $y \in X$,

$$\begin{aligned}y \in \bigcap_{n\in\omega} A_n &\leftrightarrow \forall n \in \omega\, \exists z \in {}^{\omega}\omega\; y = f_n(z) \\ &\leftrightarrow \exists h \in {}^{\omega}({}^{\omega}\omega)\, \forall n \in \omega\; y = f_n(hn) \\ &\leftrightarrow \exists x \in {}^{\omega}\omega\, \forall n \in \omega\; y = f_n(j_n x) \\ &\leftrightarrow y \in \mathfrak{p}\, Z,\end{aligned}$$

so[5.97.3] $\bigcap_{n\in\omega} A_n$ is analytic. □[5.97.5]

6 By (5.97.5) it is enough to show that all closed and all open sets are analytic.

If $A \subseteq X$ is closed, then either $A = 0$ or A with the relative topology is Polish, so[5.81] there exists a continuous map f from ${}^{\omega}\omega$ onto A. f is clearly also continuous from ${}^{\omega}\omega$ into X, and $\operatorname{im} f = A$, so A is analytic.

If $A \subseteq X$ is open, it is enough to show that A is a countable union of closed sets. To this end let $S \subseteq X$ be countable and dense. Then $S \cap A$ is countable and dense in A. It is easy to show that A is the union of the countable set $\{C(s, 1/n) \mid s \in S \cap A \wedge n > 0 \wedge C(s, 1/n) \subseteq A\}$ of closed balls. □[5.97.6] □[5.97]

Definition $[\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})]$ *Suppose X is a Polish space. For $n \in \omega\backslash\{0\}$, we define the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ recursively as follows. (A is an arbitrary subset of X, and $n > 0$.)*

1. $A \in \mathbf{\Sigma}^1_1 \overset{\text{def}}{\iff} A$ *is analytic.*
2. $A \in \mathbf{\Pi}^1_n \overset{\text{def}}{\iff} X\backslash A \in \mathbf{\Sigma}^1_n$.
3. $A \in \mathbf{\Sigma}^1_{n+1} \overset{\text{def}}{\iff}$ *for some* $\mathbf{\Pi}^1_n$ *set* $B \subseteq X \times {}^{\omega}\omega$, $A = \mathfrak{p}\, B$.
4. $A \in \mathbf{\Delta}^1_n \overset{\text{def}}{\iff} A \in \mathbf{\Sigma}^1_n$ *and* $A \in \mathbf{\Pi}^1_n$.

Projective $= \bigcup_{n=1}^{\infty} \mathbf{\Sigma}^1_n$.

As the notation implies, the projective pointclasses are the "boldface" versions of the Kleene analytical pointclasses.

(5.100) **Theorem** $[\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})]$ *Suppose $0 < n < \omega$.*

1. $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, *and* $\mathbf{\Delta}^1_n$ *are continuously closed, i.e., closed under continuous preimage.*
2. $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$.

Proof 1 $\mathbf{\Sigma}^1_1$ is continuously closed by (5.97.4).

If $\mathbf{\Sigma}^1_n$ is continuously closed, then so is $\mathbf{\Pi}^1_n$, since for any $f : X \to Y$ and $A \subseteq Y$, $f^{\leftarrow}(Y \backslash A) = X \backslash (f^{\leftarrow} A)$. If $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ are continuously closed, then so is $\mathbf{\Delta}^1_n$.

To complete the proof it suffices by induction to show that if $\mathbf{\Pi}^1_n$ is continuously closed then so is $\mathbf{\Sigma}^1_{n+1}$. To this end, assume $\mathbf{\Pi}^1_n$ is continuously closed, and suppose X and Y are Polish spaces, $A \subseteq Y$ is $\mathbf{\Sigma}^1_{n+1}$, and $f : X \to Y$ is continuous. Let $B \subseteq Y \times {}^{\omega}\omega$ be $\mathbf{\Pi}^1_n$ such that $A = \mathfrak{p}\, B$. Then for any $x \in X$

$$\begin{aligned} x \in f^{\leftarrow} A \leftrightarrow fx \in A \leftrightarrow \exists z \in {}^{\omega}\omega\ \langle fx, z \rangle \in B \\ \leftrightarrow \exists z \in {}^{\omega}\omega\ \langle x, z \rangle \in B', \end{aligned}$$

where $B' = \{\langle x, z \rangle \mid \langle fx, z \rangle \in B\}$. Since $\langle x, z \rangle \mapsto \langle fx, z \rangle$ is continuous, B' is $\mathbf{\Pi}^1_n$, so $f^{\leftarrow} A$ is $\mathbf{\Sigma}^1_{n+1}$. $\square$[5.100.1]

2 To show that $\mathbf{\Sigma}^1_1 \subseteq \mathbf{\Sigma}^1_2$, suppose X is Polish and $A \subseteq X$ is $\mathbf{\Sigma}^1_1$. Let $B \subseteq X \times {}^{\omega}\omega$ be closed such that $A = \mathfrak{p}\, B$. B is $\mathbf{\Pi}^1_1$,[5.97.6] so A is $\mathbf{\Sigma}^1_2$.

To show that $\mathbf{\Pi}^1_1 \subseteq \mathbf{\Sigma}^1_2$, suppose X is Polish and $A \subseteq X$ is $\mathbf{\Pi}^1_1$. Let $B = \{\langle x, z \rangle \in X \times {}^{\omega}\omega \mid x \in A\}$. $\langle x, z \rangle \mapsto x$ is continuous, so B is $\mathbf{\Pi}^1_1$.[5.100.1] $A = \mathfrak{p}\, B$, so A is $\mathbf{\Sigma}^1_2$.

Taking complements, it follows that $\mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1 \subseteq \mathbf{\Pi}^1_2$. By an easy induction, for all $n > 0$, $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Pi}^1_{n+1} \cap \mathbf{\Sigma}^1_{n+1} = \mathbf{\Delta}^1_{n+1}$. $\square$[5.100.2] $\square$[5.100]

(5.100.2) says that the projective hierarchy is cumulative. To show that it is a strict hierarchy, we first show that universal sets exist, as we did for the Borel hierarchy in (5.94).

(5.101) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_{\omega}(\mathbb{R})$] *Suppose X is a Polish space and $0 < n < \omega$. There exists a $\mathbf{\Sigma}^1_n$ set $U \subseteq {}^{\omega}\omega \times X$ that is* universal *in the sense that for any $\mathbf{\Sigma}^1_n$ $A \subseteq X$ there exists $a \in {}^{\omega}\omega$ such that*

$$A = \{x \in X \mid \langle a, x \rangle \in U\}.$$

Proof We prove this for all Polish spaces at once by induction on n. Given $n > 0$ and a Polish space X, let $X' = X \times {}^{\omega}\omega$, and let $V \subseteq {}^{\omega}\omega \times X'$ be a universal $\mathbf{\Sigma}^0_1(X')$[5.94] if $n = 1$, or a universal $\mathbf{\Sigma}^1_{n^-}(X')$, by induction hypothesis, if $n > 1$. Let $h : {}^2({}^{\omega}\omega) \to {}^{\omega}\omega$ be a homeomorphism,[5.86, 5.87] and let

$$\begin{aligned} U &= \{\langle a, x \rangle \in {}^{\omega}\omega \times X \mid \exists z \in {}^{\omega}\omega\ \langle a, \langle x, z \rangle \rangle \notin V\} \\ &= \{\langle a, x \rangle \in {}^{\omega}\omega \times X \mid \exists z \in {}^{\omega}\omega\ \langle \langle a, x \rangle, z \rangle \in V'\} \\ &= \mathfrak{p}\, V', \end{aligned}$$

where

$$V' = \{\langle \langle a, x \rangle, z \rangle \mid \langle a, \langle x, z \rangle \rangle \notin V\}.$$

Since $\langle \langle a, x \rangle, z \rangle \mapsto \langle a, \langle x, z \rangle \rangle$ is continuous, V' is $\mathbf{\Pi}^0_1$ (i.e., closed) if $n = 1$[5.93.4] and $\mathbf{\Pi}^1_{n^-}$ if $n > 1$,[5.100.1] so U is $\mathbf{\Sigma}^1_n$.

Now suppose $A \subseteq X$ is $\mathbf{\Sigma}^1_n$. Let $B \subseteq X' = X \times {}^{\omega}\omega$ be $\mathbf{\Pi}^0_1$ if $n = 1$, and $\mathbf{\Pi}^1_{n^-}$ if $n > 1$, such that $A = \mathfrak{p}\, B$. Let $a \in {}^{\omega}\omega$ be such that $B = \{\langle x, z \rangle \in X \times {}^{\omega}\omega \mid \langle a, \langle x, z \rangle \rangle \notin V\}$. For any $x \in X$

$$\begin{aligned} x \in A \leftrightarrow \exists z \in {}^{\omega}\omega\ \langle x, z \rangle \in B \leftrightarrow \exists z \in {}^{\omega}\omega\ \langle a, \langle x, z \rangle \rangle \notin V \\ \leftrightarrow \langle a, x \rangle \in U. \end{aligned}$$

So U is universal. $\square$[5.101]

(5.102) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose $0 < n < \omega$. There is $A \subseteq {}^\omega\omega$ such that A is $\mathbf{\Sigma}^1_n$ and not $\mathbf{\Pi}^1_n$ (and* vice versa*).*

Proof Let U be universal for $\mathbf{\Sigma}^1_n$ subsets of ${}^\omega\omega$, and let $A = \{x \in {}^\omega\omega \mid \langle x, x\rangle \in U\}$. $\square$[5.102]

Hence the projective hierarchy is strict.

5.4 Structural properties of pointclasses

We have developed effective descriptive set theory in Section 5.2 in the relatively narrow context of pointsets and pointclasses as defined in (5.11) and (5.12), whereas we have developed the classical theory in the relatively broad context of Polish spaces.[5.80] We have noted the universality of ${}^\omega\omega$ among Polish spaces,[5.81] and the homeomorphic equivalence of ${}^\omega\omega$ with various products via (5.86). Thus, for example, the spaces $\omega \times {}^\omega\omega$, ${}^\omega\omega \times {}^\omega 2$, $({}^\omega\omega \times {}^\omega\omega) \times {}^\omega\omega$, ${}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$, ${}^\omega({}^\omega\omega)$, etc., are all homeomorphic to ${}^\omega\omega$. With these considerations in mind, from now on we will—on the one hand—largely restrict our attention to homeomorphs of ${}^\omega\omega$, with occasional special consideration of ${}^\omega 2$ and related spaces. As always, ω is ubiquitous in multiple roles.

(5.103) *On the other hand, we will generalize our use of* pointspace*,* pointset *and* pointclass *to the broader context of spaces obtained from ${}^\omega\omega$, ${}^\omega 2$, and ω as above. When appropriate, we may state a definition or theorem for ${}^\omega\omega$ with the understanding that it is applicable* mutatis mutandis *to any pointspace in this sense.*

We begin this section with the celebrated 1917 theorem of Suslin[5.106] that $\mathbf{\Delta}^1_1 =$ **Borel**. This theorem alone justifies the effort we have put into the descriptive classification of pointsets, and it is an excellent illustration of the sort of understanding we may expect to gain of infinitarity from the study of descriptive set theory. We will actually present Suslin's theorem as a corollary of the later *analytic separation* theorem[5.104] of Lusin, which generalizes Suslin's result, and more clearly introduces some of the important structural properties of pointclasses that are the subject of this section.

(5.104) **Theorem: Analytic separation** [ZF] *Suppose X is a Polish space, and A_0 and A_1 are disjoint analytic subsets of X. Then there exists a Borel set B such that $A_0 \subseteq B$ and $B \cap A_1 = 0$.*

Remark We say that B *separates* A_0 and A_1, and A_0, A_1 are *Borel separable.*

Note that we have stated the result as a theorem of ZF, which it is. The proof we give, however, uses $\mathsf{AC}_\omega(\mathbb{R})$. We will presently provide a proof that does not rely on any choice principle.

Proof [$\mathsf{AC}_\omega(\mathbb{R})$] If either A_0 or A_1 is empty the theorem is trivial, so suppose neither is. There there exist continuous $f_0, f_1 : {}^\omega\omega \to X$ such that $\operatorname{im} f_0 = A_0$ and

$\operatorname{im} f_1 = A_1$.[40] For $\sigma \in {}^{<\omega}\omega$, let

$$A_0^\sigma = f_0{}^{\to} I_\sigma$$
$$A_1^\sigma = f_1{}^{\to} I_\sigma,$$

where $I_\sigma = \{z \in {}^\omega\omega \mid \sigma \subseteq z\}$.

(5.105) **Claim** *Suppose $A_0^{\sigma_0}$ and $A_1^{\sigma_1}$ are not Borel separable. Then for some $n_0, n_1 \in \omega$, $A_0^{\sigma_0{}^\frown\langle n_0\rangle}$ and $A_1^{\sigma_1{}^\frown\langle n_1\rangle}$ are not Borel separable.*

Proof Suppose toward a contradiction that for each $n_0, n_1 \in \omega$, $A_0^{\sigma_0{}^\frown\langle n_0\rangle}$ and $A_1^{\sigma_1{}^\frown\langle n_1\rangle}$ are Borel separable, and let B_{n_0,n_1} be Borel such that $A_0^{\sigma_0{}^\frown\langle n_0\rangle} \subseteq B_{n_0,n_1}$ and $B_{n_0,n_1} \cap A_1^{\sigma_1{}^\frown\langle n_1\rangle} = 0$ (using $\mathsf{AC}_\omega(\mathbb{R})$ to obtain suitable Borel codes). Note that

$$A_0^{\sigma_0} = \bigcup_{n_0\in\omega} A_0^{\sigma_0{}^\frown\langle n_0\rangle}$$
$$A_1^{\sigma_1} = \bigcup_{n_1\in\omega} A_1^{\sigma_1{}^\frown\langle n_1\rangle}.$$

Let

$$B = \bigcup_{n_0\in\omega}\bigcap_{n_1\in\omega} B_{n_0,n_1}.[41]$$

For each $n_0 \in \omega$, $A_0^{\sigma_0{}^\frown\langle n_0\rangle} \subseteq B_{n_0,n_1}$ for all $n_1 \in \omega$, so

$$A_0^{\sigma_0{}^\frown\langle n_0\rangle} \subseteq \bigcap_{n_1\in\omega} B_{n_0,n_1}.$$

Hence

$$A_0^{\sigma_0} \subseteq B.$$

For each $n_0 \in \omega$, $A_1^{\sigma_1{}^\frown\langle n_1\rangle} \cap B_{n_0,n_1} = 0$ for all $n_1 \in \omega$, so

$$A_1^{\sigma_1} \cap \bigcap_{n_1\in\omega} B_{n_0,n_1} = 0.$$

Hence

$$A_1^{\sigma_1} \cap B = 0.$$

Since B is Borel, this contradicts the assumed Borel inseparability of $A_0^{\sigma_0}$ and $A_1^{\sigma_1}$. $\square$[5.105]

Now suppose toward a contradiction that A_0 and A_1 are Borel inseparable. Note that $A_0 = A_0^0$ and $A_1 = A_1^0$, where 0 is the empty sequence. Use the claim to generate $z_0, z_1 \in {}^\omega\omega$ such that for all $n \in \omega$, $A_0^{z_0\restriction n}$ and $A_1^{z_1\restriction n}$ are Borel inseparable. Let $a_0 = f_0(z_0)$ and $a_1 = f_1(z_1)$. Then $a_0 \in A_0$ and $a_1 \in A_1$, so $a_0 \neq a_1$. Let D_0, D_1 be disjoint open subsets of X with $a_0 \in D_0$ and $a_1 \in D_1$.[42] Let $E_0 = f_0{}^{\leftarrow} D_0$ and

[40] A nonempty analytic set $A \subset X$ is $g^{\to}C$ for some continuous $g : {}^\omega\omega \to X$ and nonempty closed $C \subseteq {}^\omega\omega$. C (with the relative topology) is a Polish space, so there exists a continuous $h : {}^\omega\omega \overset{\text{sur}}{\to} C$. Let $f = g \circ h$. Then $A = \operatorname{im} f$.

[41] Alternatively, we could let $B = \bigcap_{n_1\in\omega}\bigcup_{n_0\in\omega} B_{n_0,n_1}$.

[42] Every metric space is Hausdorff. For example, let D_0 and D_1 be the respective open balls at a_0 and a_1 with radius $d(a_0, a_1)/2$.

$E_1 = f_1{}^{\leftarrow}D_1$. Then E_0, E_1 are open subsets of ${}^{\omega}\omega$ with $z_0 \in E_0$ and $z_1 \in E_1$. Let $n \in \omega$ be such that $I_{z_0 \restriction n} \subseteq E_0$ and $I_{z_1 \restriction n} \subseteq E_1$. Then

$$A_0^{z_0 \restriction n} = f_0{}^{\rightarrow} I_{z_0 \restriction n} \subseteq D_0$$
$$A_1^{z_1 \restriction n} = f_1{}^{\rightarrow} I_{z_1 \restriction n} \subseteq D_1.$$

Since D_0 and D_1 are disjoint, D_0 (likewise, $X \backslash D_1$) is a Borel set that separates $A_0^{z_0 \restriction n}$ and $A_1^{z_1 \restriction n}$; contradiction. $\square^{5.104}$

(5.106) **Theorem: Suslin's** $[\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})]$ *Suppose X is a Polish space and $A \subseteq X$. Then A is Borel iff A is $\boldsymbol{\Delta}^1_1$.*

Proof The $\rightarrow$ direction follows from (5.97.6). The $\leftarrow$ direction follows from (5.104), since A is the only set that separates A and $X \backslash A$. $\square^{5.106}$

5.4.1 Norms and the prewellordering property

Note that (5.104) has the corollary that disjoint $\boldsymbol{\Sigma}^1_1$ pointsets A_0 and A_1 are separable by a set B that is $\boldsymbol{\Delta}^1_1$, i.e., both B and $\neg B$ are $\boldsymbol{\Sigma}^1_1$. This is the important *separation* property of $\boldsymbol{\Sigma}^1_1$, which follows from the important *reduction* property of $\boldsymbol{\Pi}^1_1$, which in turn follows from the (yes, also important) *prewellordering* property of $\boldsymbol{\Pi}^1_1$.

The latter two of these properties will be familiar from Theorem 5.65 and its proof. We begin by defining norms and their associated prewellorderings

(5.107) **Definition** $[\mathsf{ZF}]$

1. *A* norm *on a set A* $\stackrel{\text{def}}{=}$ *a function $\varphi : A \rightarrow \mathrm{Ord}$.*
2. *A norm $\varphi : A \rightarrow \mathrm{Ord}$ is* regular $\stackrel{\text{def}}{\iff}$ *$\operatorname{im}\varphi$ is an ordinal.*
3. *Suppose $\varphi : A \rightarrow \mathrm{Ord}$ is a norm.*
    1. $\leqslant^{\varphi}$ $\stackrel{\text{def}}{=}$ *the prewellordering of A defined by the condition that for all $x, y \in A$*
    $$x \leqslant^{\varphi} y \stackrel{\text{def}}{\iff} \varphi x \leqslant \varphi y.$$
    $<^{\varphi}$ is of course the strict prewellordering associated with $\leqslant^{\varphi}$:
    $$x <^{\varphi} y \stackrel{\text{def}}{\iff} \varphi x < \varphi y.$$
    2. *Suppose $A \subseteq X$ for some pointspace X.*
        1. $\leqslant^*_{\varphi} \stackrel{\text{def}}{=} \{\langle x, y\rangle \in X \times X \mid x \in A \wedge (y \notin A \vee \varphi x \leqslant \varphi y)\}$.
        2. $<^*_{\varphi} \stackrel{\text{def}}{=} \{\langle x, y\rangle \in X \times X \mid x \in A \wedge (y \notin A \vee \varphi x < \varphi y)\}$.

Note that (unless $A = X$) $\leqslant^*_{\varphi}$ is not a (weak) preorder because it omits the component $\{\langle x, y\rangle \mid x, y \in X \backslash A\}$; thus the use of '$\leqslant$' to denote this relation is a little abusive of the notation. $<^*_{\varphi}$, on the other hand, is a bona fide (strict) preorder.

The starred relations associated with a norm φ may be also be described in terms of the associated function φ^* defined by the condition that for any $x \in X$,

$$\varphi^* x = \begin{cases} \varphi x & \text{if } x \in A \\ \infty & x \notin A, \end{cases}$$

where ∞ is an arbitrary non-ordinal understood to be greater than any ordinal for this purpose. Then

$$x \leqslant^*_\varphi y \leftrightarrow x \in A \wedge \varphi^* x \leqslant \varphi^* y$$
$$x <^*_\varphi y \leftrightarrow \varphi^* x < \varphi^* y.$$

Recall that $\breve{\Gamma}$ is $\neg\,\Gamma$.[5.18.2]

(5.108) **Definition** [ZF] *Suppose Γ is a recursively closed pointclass, X is a pointspace, $A \subseteq X$, and $\leqslant$ is a prewellordering of A. $\leqslant$ is a Γ-prewellordering $\stackrel{\text{def}}{\iff}$ there exist relations $\leqslant_\Gamma$ and $\leqslant_{\breve{\Gamma}}$ in Γ and $\breve{\Gamma}$ respectively such that for all $y \in X$*

$$y \in A \rightarrow \forall^1 x \big((x \in A \wedge x \leqslant y) \leftrightarrow x \leqslant_\Gamma y \leftrightarrow x \leqslant_{\breve{\Gamma}} y\big).$$

A norm φ on A is a Γ-norm $\stackrel{\text{def}}{\iff}$ the associated prewellorder $\leqslant^\varphi$ is a Γ-prewellordering.

Note that if $\leqslant^\varphi \in \Gamma \cap \breve{\Gamma}$ then φ is a Γ-norm, because we can let $\leqslant^\varphi_\Gamma = \leqslant^\varphi_{\breve{\Gamma}} = \leqslant^\varphi$. On the other hand, if φ is a Γ-norm, $A \in \Gamma$, and Γ is closed under $\wedge$, then $\leqslant^\varphi \in \Gamma$. As in this instance, it is often desirable that a pointclass have certain closure properties summarized by the notion of *adequacy*. As defined here, this is a stronger condition than is immediately necessary, but it is not so strong as to be unduly restrictive.

(5.109) **Definition** [ZF] *A pointclass Γ is* adequate $\stackrel{\text{def}}{\iff}$ *Γ includes Δ^0_1 and is closed under recursive substitution, $\vee$, $\wedge$, $\exists^<$ and $\forall^<$ (i.e., bounded quantification).*

(5.110) **Theorem** [ZF] *Suppose Γ is an adequate pointclass, $A \in \Gamma$, and φ is a norm on A. Then φ is a Γ-norm iff $\leqslant^*_\varphi$ and $<^*_\varphi$ are in Γ.*

Proof The proof is straightforward, but it is worth examining for the insight it provides into the relationships among the various orders involved. Suppose first that $\leqslant^*_\varphi$ and $<^*_\varphi$ are in Γ. Let

$$x \leqslant^\varphi_\Gamma y \leftrightarrow x \leqslant^*_\varphi y$$
$$\text{and } x \leqslant^\varphi_{\breve{\Gamma}} y \leftrightarrow \neg(y <^*_\varphi x).$$

These witness that φ is a Γ-norm.

Now suppose $\leqslant^\varphi_\Gamma \in \Gamma$ and $\leqslant^\varphi_{\breve{\Gamma}} \in \breve{\Gamma}$ witness that φ is a Γ-norm. Then

$$x \leqslant^*_\varphi y \leftrightarrow x \in A \wedge (x \leqslant^\varphi_\Gamma y \vee \neg y \leqslant^\varphi_{\breve{\Gamma}} x)$$
$$\text{and } x <^*_\varphi y \leftrightarrow x \in A \wedge \neg y \leqslant^\varphi_\Gamma x,$$

so $\leqslant^*_\varphi$ and $<^*_\varphi$ are in Γ. □[5.110]

(5.111) **Definition** [ZF] *Suppose Γ is a recursively closed pointclass. Let $\Delta = \Gamma \cap \breve{\Gamma}$. In the following definitions, unless otherwise stated, pointsets are presumed to be subsets of some fixed pointspace X, the specific identity of which is irrelevant.*

1. *Γ has the* separation *property $\stackrel{\text{def}}{\iff}$ for any pointsets $A, B \in \Gamma$, if $A \cap B = 0$ then there is a set $C \in \Delta$ that* separates *A and B, i.e., $A \subseteq C$ and $B \subseteq \neg C$.*
2. *Γ has the* reduction *property $\stackrel{\text{def}}{\iff}$ for any pointsets $A, B \in \Gamma$ there exist $A', B' \in \Gamma$ such that $\langle A', B' \rangle$* reduces *$\langle A, B \rangle$, i.e.,*

1. *$A' \subseteq A$ and $B' \subseteq B$;*
2. *$A' \cup B' = A \cup B$; and*
3. *$A' \cap B' = 0$.*

3. *Γ is* normed *or has the* prewellordering *property* $\overset{\text{def}}{\iff}$ *for any $A \in \Gamma$, A has a Γ-norm.*

4. *Γ has the* uniformization *property* $\overset{\text{def}}{\iff}$ *for every $A \subseteq {}^\omega\omega \times {}^\omega\omega$, if $A \in \Gamma$ then there exists $B \subseteq A$ such that $B \in \Gamma$ and for all $\langle x, y\rangle \in A$ there is a unique y' such that $\langle x, y'\rangle \in B$. We say that B* uniformizes *A.*

(5.112) **Theorem** [ZF] *Suppose Γ is an adequate pointclass.*

1. *If Γ has the prewellordering property then Γ has the reduction property.*
2. *If Γ has the reduction property then $\breve{\Gamma}$ has the separation property.*

Proof 1 Suppose Γ has the prewellordering property and $A, B \in \Gamma \cap U_{\mathfrak{s}}$ for some type $\mathfrak{s}$. Let

$$C = \{x ^\frown \langle 0\rangle \mid x \in A\} \cup \{x ^\frown \langle 1\rangle \mid x \in B\}.$$

Since Γ is adequate, $C \in \Gamma$. Let $\leqslant$ be a Γ-prewellordering of C, and let $\leqslant^*$ and $<^*$ be the corresponding relations on $U_{\mathfrak{s}^\frown\langle 0\rangle}$,[5.107.3.2] so that $\leqslant^*$ and $<^*$ are both in Γ.[5.112]

Let

1. $A' = \{x \in U_{\mathfrak{s}} \mid x ^\frown \langle 0\rangle \leqslant^* x ^\frown \langle 1\rangle\}$; and
2. $B' = \{x \in U_{\mathfrak{s}} \mid x ^\frown \langle 1\rangle <^* x ^\frown \langle 0\rangle\}$.

It is straightforward to show that $\langle A', B'\rangle$ reduces $\langle A, B\rangle$. $\square$[5.112.1]

2 Suppose Γ has the reduction property, $A, B \in \breve{\Gamma}$, and $A \cap B = 0$. Then $\neg A, \neg B \in \Gamma$ and $\neg A \cup \neg B = X$. Let $A', B' \in \Gamma$ reduce $\neg A, \neg B$. Thus,

1. *$A' \subseteq \neg A$;*
2. *$B' \subseteq \neg B$;*
3. *$A' \cup B' = \neg A \cup \neg B = X$; and*
4. *$A' \cap B' = 0$.*

Note that $A \subseteq \neg A'$, $B \subseteq \neg B'$, and $A' = \neg B'$. Let $C = B'$. Then C separates A, B, and $C \in \Delta$. $\square$[5.112.2] $\square$[5.112]

(5.113) **Theorem** [ZF] *For any $z \in {}^\omega\omega$, $\Pi^1_1(z)$ has the prewellordering property. Hence $\boldsymbol{\Pi}^1_1$ does too.*

Proof We will treat Π^1_1. The proof obviously relativizes to any $z \in {}^\omega\omega$. Suppose $A \subseteq {}^\omega\omega$ is Π^1_1. Then[5.60] there is a recursive tree $T \subseteq {}^{<\omega}(\omega \times \omega)$ such that for all $x \in {}^\omega\omega$, $x \in A$ iff $T_{[x]}$ is wellfounded. Let $\varphi : A \to \mathrm{Ord}$ be the norm on A given by the condition that $\varphi x = \operatorname{rk} T_{[x]}$.

For $x, y \in {}^\omega\omega$, let

1. $x \leqslant^{\varphi}_{\Pi^1_1} y$ iff there does not exist an order-preserving $f : T_{[y]} \to T_{[x]} \backslash \{0\}$; and

2. $x \leqslant^{\varphi}_{\Sigma^1_1} y$ iff there exists an order-preserving $f : T_{[x]} \to T_{[y]}$.

Then $\leqslant^{\varphi}_{\Pi^1_1}$ and $\leqslant^{\varphi}_{\Sigma^1_1}$ are respectively Π^1_1 and Σ^1_1 and satisfy[3.184] (5.108), thus witnessing that φ is a Π^1_1-norm on A. $\square$[5.113]

(5.114) **Theorem** [ZF] *Suppose* Γ *is an adequate pointclass.*

1. *Suppose* $A \in \Gamma \cap \mathcal{P} U_{\mathfrak{s}^\frown\langle 1\rangle}$ *admits a* Γ*-norm. Let* $B = \{x \in U_{\mathfrak{s}} \mid \exists^1 x' \; x^\frown\langle x'\rangle \in A\}$*. Then* B *admits an* $\exists^1\forall^1\Gamma$*-norm.*
2. *Thus, if* $\forall^1\Gamma \subseteq \Gamma$ *and* Γ *has the prewellordering property, then* $\exists^1\Gamma$ *has the prewellordering property.*

Proof The second assertion follows trivially from the first. Let φ be a Γ-norm on A, and for $x \in B$ let

$$\psi x = \inf\{\varphi\langle x, x'\rangle \mid \langle x, x'\rangle \in A\}.$$

Let $\leqslant^*_\psi$ and $<^*_\psi$ be the corresponding starred relations.[5.107.3.2] It is easily checked that

$$x \leqslant^*_\psi y \leftrightarrow \exists^1 x' \, \forall^1 y' \, (\langle x, x'\rangle \leqslant^*_\varphi \langle y, y'\rangle)$$
$$\text{and } x <^*_\psi y \leftrightarrow \exists^1 x' \, \forall^1 y' \, (\langle x, x'\rangle <^*_\varphi \langle y, y'\rangle),$$

so ψ is an $\exists^1\forall^1\Gamma$-norm. $\square$[5.114]

As an immediate corollary we have the following result.

(5.115) **Theorem** [ZF] $\Sigma^1_2(z)$ *and* $\boldsymbol{\Sigma}^1_2$ *have the prewellordering property.*

Applying (5.112.2, 3) with (5.113) and (5.115) we have the following.

(5.116) **Theorem** [ZF]

1. $\Pi^1_1(z)$*,* $\boldsymbol{\Pi}^1_1$*,* $\Sigma^1_2(z)$*, and* $\boldsymbol{\Sigma}^1_2$ *have the reduction property.*
2. $\Sigma^1_1(z)$*,* $\boldsymbol{\Sigma}^1_1$*,* $\Pi^1_2(z)$ *and* $\boldsymbol{\Pi}^1_2$ *have the separation property.*

5.4.2 Analytic boundedness

Recall[5.61] the set $\mathrm{WO} \subseteq {}^\omega\omega$ of (codes of) countable wellorders. The following theorem should come as no surprise.

(5.117) **Theorem** [ZF] *Let* $\varphi : \mathrm{WO} \to \mathrm{Ord}$ *be defined by the condition that for each* $x \in \mathrm{WO}$*,* φx *is the order type of the relation* R_x *as defined in (5.61.1) (which is a wellorder iff* $x \in \mathrm{WO}$*).* φ *is a* Π^1_1*-norm on* WO*.*

Proof Define $\leqslant_{\Pi^1_1}$ and $\leqslant_{\Sigma^1_1}$ as follows.

1. $x \leqslant_{\Pi^1_1} y$ iff $x, y \in \mathrm{WO}$ and there does not exist an order-preserving map from R_y into a proper initial segment of R_x.

2. $x \leqslant_{\Sigma^1_1} y$ iff $x, y \in \mathrm{LO}$ and there exists an order-preserving map from R_x into R_y.

Clearly, $\leqslant_{\Pi^1_1}$ and $\leqslant_{\Sigma^1_1}$ are respectively Π^1_1 and Σ^1_1 and satisfy (5.108). $\square$[5.117]

(5.118) **Theorem: $\mathbf{\Sigma}^1_1$ boundedness** [ZF] *Suppose $X \subseteq$ WO is $\mathbf{\Sigma}^1_1$. Then* $\sup\{\operatorname{ot} x \mid x \in X\} < \omega_1$.

Proof Suppose to the contrary that $\sup\{\operatorname{ot} x \mid x \in X\} = \omega_1$. Then for any $x \in {}^\omega\omega$,

$$x \in \mathrm{WO} \leftrightarrow \exists y \in X \ x \leqslant_{\Sigma^1_1} y,$$

where $\leqslant_{\Sigma^1_1}$ is the Σ^1_1 relation from the proof of (5.117). Thus, WO is $\mathbf{\Sigma}^1_1$. Since WO is (continuously) $\mathbf{\Pi}^1_1$-complete,[5.64] and $\mathbf{\Pi}^1_1 \nsubseteq \mathbf{\Sigma}^1_1$,[5.102] this is a contradiction. $\square$[5.118]

As noted above, the method used in this section to prove the reduction property for Π^1_1 is essentially the method used in our initial proof[5.65] of this result. The only new element is that we have isolated and defined the prewellordering property as a feature of independent interest. (A typical illustration of the value of this is the easy transfer to Σ^1_2.[5.116])

From the $\mathbf{\Pi}^1_1$-reduction property we directly obtain the $\mathbf{\Sigma}^1_1$-separation property,[5.112.2] as we could have done after (5.65). It is instructive to compare this with (5.104), which we have designated the *analytic separation theorem*. (5.104) states that disjoint $\mathbf{\Sigma}^1_1$ (i.e., analytic) sets are separable by a Borel set. Since **Borel** $\subseteq \mathbf{\Delta}^1_1$, this implies that $\mathbf{\Sigma}^1_1$ has the separation property,[5.111.1] but (5.104) supplies the additional information that $\mathbf{\Delta}^1_1 =$ **Borel**.[5.106]

Our proofs of Π^1_1-prewellordering and Π^1_1-reduction contain a suggestion of this stronger result inasmuch as we may imagine building a Π^1_1 set in a wellordered sequence of stages, and reducing a pair $\langle A, B\rangle$ of Π^1_1 sets by putting a point $x \in A \cup B$ in A' or B' according to whether and when it gets into A or B. Perhaps we can build a separating Borel set for disjoint $\mathbf{\Sigma}^1_1$ sets in a similar wellordered sequence of stages and thereby improve on the proof given of (5.104) by supplying a description of a separating set rather than simply deriving a contradiction from the hypothesis of its nonexistence.

Indeed, we can construct such a proof, and it has the advantage *vis-à-vis* (5.104) of not requiring a choice axiom. We present the proof also for the practice it gives in the use of trees, and as a way of lingering over the Suslin theorem, which is the gateway to descriptive set theory. In our defense, there is precedent for this sort of redundancy: more than 200 proofs have been published of the law of quadratic reciprocity, 8 by Gauss alone, who provided the first. Gauss also published four proofs of the fundamental theorem of algebra.

Actually, we will prove a more general result concerning κ-Suslin and κ-Borel sets, as defined in the next section.

5.4.3 κ-Borel and κ-Suslin

(5.119) **Definition** [ZF] *Suppose X is a topological space, and κ is an ordinal.*

1. *A set $\mathcal{A} \subseteq \mathcal{P}X$ is a κ-algebra $\overset{\text{def}}{\iff} 0 \in \mathcal{A}$ and $\mathcal{A}$ is closed under complementation and unions of length less than κ, i.e., if $\eta < \kappa$ and $\{A_\alpha \mid \alpha < \eta\} \subseteq \mathcal{A}$, then $\bigcup_{\alpha<\eta} A_\alpha \in \mathcal{A}$.*

2. *A set $A \subseteq X$ is κ-Borel (as a subset of X) $\overset{\text{def}}{\iff}$ it is in the smallest κ-algebra over X that contains all open sets.*

Obviously, the Borel sets in the original sense are just the κ-Borel sets for any $\omega < \kappa \leqslant \omega_1$. (Note that we have not required that κ be a cardinal.)

(5.120) **Definition** [ZF] *Suppose N is a set and κ is an ordinal. A set $A \subseteq {}^{\omega}N$ is κ-Suslin $\overset{\text{def}}{\iff}$ there is a tree T on $N \times \kappa$ such that $A = \mathfrak{p}^{\cdot}[T] = \{x \in {}^{\omega}N \mid \exists y \in {}^{\omega}\kappa\ \langle x, y\rangle^{\cdot} \in [T]\}$.*[5.58.3] *An arbitrary pointset is κ-Suslin $\overset{\text{def}}{\iff}$ it is the continuous preimage of a κ-Suslin subset of ${}^{\omega}\omega$.*[43]

The analytic ($\mathbf{\Sigma}^1_1$) pointsets are the ω-Suslin sets.

'κ-Suslin' is conventionally defined in terms of trees as we have just done, but it is a purely topological concept. Recall the definition[5.53] of the tree T^X from a set $X \subseteq {}^{\omega}M$ and the fact[5.54] that $[T^X] = \overline{X}$, i.e., the set of branches of T^X is the closure of X in the standard topology. Also recall[5.55] the operation $u \mapsto u^{\cdot}$ relating ${}^{X}({}^{Y}R)$ to ${}^{Y}({}^{X}R)$. Clearly, $A \subseteq {}^{\omega}N$ is κ-Suslin iff there exists a closed $C \subseteq {}^{\omega}N \times {}^{\omega}\kappa$ such that $A = \mathfrak{p}\,C$.

We will make repeated use of the following definition and simple fact.

Definition [ZF] *Suppose η is an ordinal and $(X; \prec)$ is a wellorder. The* lexicographic *order on ${}^{\eta}X$ is the binary relation $\prec^*$ defined by the following condition. Given $f, g \in {}^{\eta}X$, $f \prec^* g \overset{\text{def}}{\iff}$*

1. *$f \neq g$; and*
2. *letting α be the least $\alpha \in \eta$ such that $f\alpha \neq g\alpha$, $f\alpha \prec g\alpha$.*

It is easy to see that if η is finite, $\prec^*$ is a wellorder; whereas if η is infinite, it clearly is not. If $\eta = \omega$, however, any closed subset of ${}^{\eta}X$ has a $\prec^*$-least member.

(5.121) **Theorem** [ZF] *Suppose $(X; \prec)$ is a wellorder and $C \subseteq {}^{\omega}X$ is closed. Then C has a lexicographically least member. Equivalently, any sequence tree on X has a lexicographically least branch.*

Proof As always with closed subsets of spaces of the form ${}^{\omega}X$, it is convenient to think in terms of trees. Let $T = T^C$.[5.53] We will define a sequence $\langle t_n \mid n \in \omega\rangle$ of nodes of T by recursion on n so that for all $n \in \omega$,

1. $|t_n| = n$;
2. $t_n \subseteq t_{n+1}$; and
3. there exists $z \in [T]$ such that $t_n \subseteq z$.

Clearly, the first two conditions imply that $\{t_n \mid n \in \omega\}$ is a branch of T. The third condition is imposed to permit the recursion to continue. t_0 is necessarily 0. Given t_n satisfying the conditions, let $x \in X$ be $\prec$-least such that there exists $z \in [T]$ such that $t_n{}^\frown\langle x\rangle \subseteq z$, and let $t_{n+1} = t_n{}^\frown\langle x\rangle$. Clearly, $\bigcup_{n\in\omega} t_n$ is the lexicographically least branch of $[T]$; hence, the lexicographically least member of C. □[5.121]

Note that Definitions 5.119 and 5.120 are given for ordinals κ. (5.120) could reasonably be stated for an arbitrary set K in place of κ. In the presence of AC, of course, the K-Suslin sets are exactly the $|K|$-Suslin sets, but it is the wellordering of κ (as a set of ordinals) that gives the notion its significance. Trivially, any subset of ${}^{\omega}\omega$ is $({}^{\omega}\omega)$-Suslin.[44]

[43] This is an example of the convention stated above.[5.103] We will not take the trouble in every case to point out how to generalize a definition or theorem stated for one pointspace, usually ${}^{\omega}\omega$, to an arbitrary pointspace.

[44] Given $A \subseteq {}^{\omega}\omega$, let T be the set of sequences of the form $\langle\langle s_0, x\rangle, \langle s_1, x\rangle, \ldots, \langle s_{n^-}, x\rangle\rangle$, such that $x \in A$ and $\langle s_0, \ldots, s_{n^-}\rangle \subseteq x$. Clearly, $\mathfrak{p}^{\cdot}[T] = A$.

(5.122) **Theorem** [ZF]

1. *The $\mathbf{\Sigma}^1_1$ pointsets are exactly the ω-Suslin pointsets.*
2. *Every $\mathbf{\Sigma}^1_2$ pointset is ω_1-Suslin.*

Proof We proved the first assertion some time ago.[5.57.2] It is included here for completeness. To prove the second assertion, first suppose $A \subseteq {}^\omega\omega$ is $\mathbf{\Pi}^1_1$. Let $T \subseteq {}^{<\omega}(\omega \times \omega)$ be a tree such that $x \in A$ iff $T_{[x]}$ is wellfounded.

(5.123) *Let $\boldsymbol{s} = \langle \boldsymbol{s}_0, \boldsymbol{s}_1, \dots \rangle$ be a recursive enumeration of ${}^{<\omega}\omega$ such that*

$$\forall m, m' \in \omega\, (\boldsymbol{s}_m \subseteq \boldsymbol{s}_{m'} \rightarrow m \leqslant m'). \tag{5.124}$$

Note that $|\boldsymbol{s}_m| \leqslant m$, which is all we actually need for this application, but the stronger condition will be used later.

Let T' be the set of $\langle t, u\rangle^{\cdot}$ such that for some $n \in \omega$,

1. $t = \langle t_0, \dots, t_{n^{\cdot}} \rangle \in {}^n\omega$*;*
2. $u = \langle u_0, \dots, u_{n^{\cdot}} \rangle \in {}^n\omega_1$*;*
3. *for all $m, m' < n$, if $\boldsymbol{s}_m \subseteq \boldsymbol{s}_{m'} \in T_{[t]}$[5.58.1] then $u_m \geqslant u_{m'}$.*

In other words, the assignment of ordinals u_m to sequences $\boldsymbol{s}_m$ for $m < n$ and $\boldsymbol{s}_m \in T_{[t]}$ is order-preserving.[45]

T' is a tree.

Suppose $\langle x, y\rangle^{\cdot} \in [T']$. Then $\{(\boldsymbol{s}_m, y_m) \mid m \in \omega \wedge \boldsymbol{s}_m \in T_{[x]}\}$ is an order-preserving assignment of ordinals to $T_{[x]}$, from which it follows that $T_{[x]}$ is wellfounded. Conversely, if $T_{[x]}$ is wellfounded, let $f : T_{[x]} \rightarrow \omega_1$ be an order-preserving assignment of ordinals. (Since T is countable, countable ordinals suffice for this.) Let $y : \omega \rightarrow \omega_1$ be such that for all $m \in \omega$, if $\boldsymbol{s}_m \in T_{[x]}$ then $y_m = f\boldsymbol{s}_m$. (If $\boldsymbol{s}_m \notin T_{[x]}$ then y_m may have any value, say 0, to be definite.) Then $\langle x, y\rangle^{\cdot} \in [T']$. Thus $x \in A$ iff $T_{[x]}$ is wellfounded iff $x \in \mathfrak{p}^{\cdot}[T']$, so A is ω_1-Suslin.

Now suppose $B \subseteq {}^\omega\omega$ is $\mathbf{\Sigma}^1_2$. Then for some $\mathbf{\Pi}^1_1$ set $A \subseteq {}^\omega\omega \times {}^\omega\omega$, $B = \{x \in {}^\omega\omega \mid \exists x' \in {}^\omega\omega\, \langle x, x'\rangle \in A\}$. As we have just shown (essentially), there is a tree T on $\omega \times \omega \times \omega_1$ such that $A = \{\langle x, x'\rangle \in {}^\omega\omega \times {}^\omega\omega \mid \exists y \in {}^\omega\omega_1\, \langle x, x', y\rangle^{\cdot} \in [T]\}$. Thus $B = \{x \in {}^\omega\omega \mid \exists x' \in {}^\omega\omega\, \exists y \in {}^\omega\omega_1\, \langle x, x', y\rangle^{\cdot} \in [T]\}$. Using a definable bijection of $\omega \times \omega_1$ with ω_1, such as $\langle n, \alpha\rangle \mapsto \omega \cdot \alpha + n$, we may define a tree T' on $\omega \times \omega_1$ such that $B = \{x \in {}^\omega\omega \mid \exists y \in {}^\omega\omega_1\, \langle x, y\rangle^{\cdot} \in [T']\} = \mathfrak{p}^{\cdot}[T']$, so B is ω_1-Suslin. $\square$[5.122]

(5.125) **Theorem** [ZF] *Suppose κ is an infinite cardinal. Suppose A, A' are disjoint κ-Suslin pointsets. Then A, A' are separable by a $(\kappa+1)$-Borel set.*

Proof Let T, T' be sequence trees on $\omega \times \kappa$ such that $A = \mathfrak{p}^{\cdot}[T]$ and $A' = \mathfrak{p}^{\cdot}[T']$. Let S be the tree on $\omega \times \kappa \times \kappa$ defined by the condition that for each $n \in \omega$, $s \in {}^n\omega$ and $t, t' \in {}^n\kappa$,

$$\langle s, t, t'\rangle^{\cdot} \in S \leftrightarrow \langle s, t\rangle^{\cdot} \in T \wedge \langle s, t'\rangle^{\cdot} \in T'.$$

Note that if $\langle x, y, y'\rangle^{\cdot} \in [S]$ then $\langle x, y\rangle^{\cdot} \in [T]$ and $\langle x, y'\rangle^{\cdot} \in [T']$, so $x \in \mathfrak{p}^{\cdot}[T] = A$ and $x \in \mathfrak{p}^{\cdot}[T'] = A'$, which is impossible, as A and A' are assumed to be disjoint. Hence S is wellfounded. We will define by $\supseteq$-recursion on S a function $\langle s, t, t'\rangle^{\cdot} \mapsto$

[45] Remember that sequence trees grow downward, so that if $t \subseteq t'$ then t is *higher* than t'.

$B_{\langle s,t,t'\rangle}$ such that for each $\langle s,t,t'\rangle^{\cdot} \in S$, $B_{\langle s,t,t'\rangle}$ is $(\kappa+1)$-Borel and separates $\left\langle \mathfrak{p}^{\cdot}[T_{(\langle s,t\rangle^{\cdot})}], \mathfrak{p}^{\cdot}[T'_{(\langle s,t'\rangle^{\cdot})}]\right\rangle$.[46] This suffices, as $T_{(0)} = T$ and $T'_{(0)} = T'$, so $B_{\langle 0,0,0\rangle}$ is $(\kappa+1)$-Borel and separates $\langle A, A'\rangle$.

Suppose, therefore, that $\langle s,t,t'\rangle^{\cdot} \in S$, and suppose that for each immediate extension $\langle s_1,t_1,t'_1\rangle^{\cdot}$ of $\langle s,t,t'\rangle^{\cdot}$ in S, $B_{\langle s_1,t_1,t'_1\rangle}$ is $(\kappa+1)$-Borel and separates $\left\langle \mathfrak{p}^{\cdot}[T_{(\langle s_1,t_1\rangle^{\cdot})}], \mathfrak{p}^{\cdot}[T'_{(\langle s_1,t'_1\rangle^{\cdot})}]\right\rangle$. Let E and E' be the respective sets of immediate extensions of $\langle s,t\rangle$ and $\langle s,t'\rangle$. For $\langle s_1,t_1\rangle \in E$ and $\langle s'_1,t'_1\rangle \in E'$, let $C_{\langle s_1,t_1,s'_1,t'_1\rangle}$ be defined as follows.

1. Suppose $s_1 = s'_1$.
   1. Suppose $\langle s_1,t_1,t'_1\rangle^{\cdot} \in S$. Then
   $$C_{\langle s_1,t_1,s'_1,t'_1\rangle} = B_{\langle s_1,t_1,t'_1\rangle}.$$
   2. Suppose $\langle s_1,t_1,t'_1\rangle^{\cdot} \notin S$.
      1. Suppose $\langle s_1,t_1\rangle^{\cdot} \notin T$. Then
      $$C_{\langle s_1,t_1,s'_1,t'_1\rangle} = 0.$$
      2. Suppose $\langle s_1,t'_1\rangle^{\cdot} \notin T'$. Then
      $$C_{\langle s_1,t_1,s'_1,t'_1\rangle} = {}^{\omega}\omega.$$
2. Suppose $s_1 \neq s'_1$. Then
$$C_{\langle s_1,t_1,s'_1,t'_1\rangle} = \{x \in {}^{\omega}\omega \mid s_1 \subseteq x\}.$$

Note that in every case, $C_{\langle s_1,t_1,s'_1,t'_1\rangle}$ is $(\kappa+1)$-Borel and separates
$$\left\langle \mathfrak{p}^{\cdot}[T_{(\langle s_1,t_1\rangle^{\cdot})}], \mathfrak{p}^{\cdot}[T'_{(\langle s'_1,t'_1\rangle^{\cdot})}]\right\rangle.$$

Let
$$B_{\langle s,t,t'\rangle} = \bigcup_{\langle s_1,t_1\rangle \in E} \bigcap_{\langle s'_1,t'_1\rangle \in E'} C_{\langle s_1,t_1,s'_1,t'_1\rangle}.$$

Then $B_{\langle s,t,t'\rangle}$ is $(\kappa+1)$-Borel and separates $\left\langle \mathfrak{p}^{\cdot}[T_{(\langle s,t\rangle^{\cdot})}], \mathfrak{p}^{\cdot}[T'_{(\langle s,t'\rangle^{\cdot})}]\right\rangle$, as desired. $\square$[5.125]

The following theorem was proved independently by Kunen and Martin.

(5.126) **Theorem** [ZF] *Suppose κ is an infinite cardinal and $\prec$ is a κ-Suslin irreflexive wellfounded relation on $A \subseteq {}^{\omega}\omega$. Then $\mathrm{rk}(\prec) < \kappa^+$.*

Proof Let S be a sequence tree on $\omega \times \omega \times \kappa$ such that
$$\prec = \{\langle x,y\rangle \mid \exists z \in {}^{\omega}\kappa\ \langle x,y,z\rangle^{\cdot} \in [S]\}$$

T be the sequence tree on A consisting of all finite $\prec$-decreasing sequences in A. Clearly, for $n > 0$, the rank of $\langle a_0,\ldots,a_{n^-}\rangle \in T$ is just the rank of a_{n^-} in $\prec$, so $\mathrm{rk}(T \backslash \{0\}) = \mathrm{rk}(\prec)$.

Let U be the set of finite sequences
$$\big\langle \langle s_1, s_0, u_0\rangle, \ldots, \langle s_n, s_{n^-}, u_{n^-}\rangle \big\rangle$$
such that $n \in \omega$ and

[46] Recall[3.183] that for any sequence tree T on a set M and $s \in {}^{<\omega}M$, $T_{(s)} = \{t \in T \mid t \subseteq s \vee s \subseteq t\}$.

1. $\forall m \leqslant n\ s_m \in {}^n\omega$,
2. $\forall m < n\ u_m \in {}^n\kappa$; and
3. $\forall m < n\ \langle s_{m+1}, s_m, u_m\rangle^{\cdot} \in S$.

Let $\prec^*$ be the binary relation on U such that

$$\big\langle\langle s'_1, s'_0, u'_0\rangle, \ldots, \langle s'_{n'}, s'_{n'^-}, u'_{n'^-}\rangle\big\rangle \prec^* \big\langle\langle s_1, s_0, u_0\rangle, \ldots, \langle s_n, s_{n^-}, u_{n^-}\rangle\big\rangle$$
$$\leftrightarrow n' > n \wedge \forall m \leqslant n\ s'_m \supseteq s_m \wedge \forall m < n\ u'_m \supseteq u_m.$$

Suppose toward a contradiction that $\prec^*$ is not wellfounded. Since U is wellorderable, there is an infinite $\prec^*$-descending sequence $\langle w_k \mid k \in \omega\rangle$. Let

$$w_k = \big\langle\langle s^k_1, s^k_0, u^k_0\rangle, \ldots, \langle s^k_{n_k}, s^k_{n_k^-}, u^k_{n_k^-}\rangle\big\rangle.$$

Note that

1. $n_0 < n_1 < \cdots$;
2. $\forall m \leqslant n_k\ |s^k_m| = n_k$ and $\forall m < n_k\ |u^k_m| = n_k$; and
3. if $k \leqslant k'$ then $\forall m \leqslant n_k\ s^k_m \subseteq s^{k'}_m$ and $\forall m < n_k\ u^k_m \subseteq u^{k'}_m$.

Let $x_m = \bigcup_{n_k \geqslant m} s^k_m$ and $z_m = \bigcup_{n_k > m} u^k_m$. Then for each $m \in \omega$

1. $x_m \in {}^\omega\omega$;
2. $z_m \in {}^\omega\kappa$; and
3. $\langle x_m + 1, x_m, z_m\rangle^{\cdot} \in [S]$; so
4. $x_{m+1} \prec x_m$.

This contradicts the wellfoundedness of $\prec$. Hence $\prec^*$ is wellfounded.

Since $\prec^*$ is a wellfounded relation on a set of size κ, its rank is $< \kappa^+$, so it suffices to show that there is an order-preserving $\pi : (T\backslash\{0\}; \supsetneq)$ into $(U; \prec^*)$. For each $\langle x, y\rangle \in \prec$ let $z_{x,y}$ be the lexicographically least $z \in {}^\omega\kappa$ such that $\langle x, y, z\rangle^{\cdot} \in [S]$.[5.121] Given $w = \langle x_0, \ldots, x_n\rangle \in T\backslash\{0\}$,

1. if $n = 0$ then let $\pi w = 0$ (the empty sequence); and
2. if $n > 0$ then let

$$\pi w \quad = \quad \big\langle\langle x_1 \restriction n, x_0 \restriction n, z_{x_1,x_0} \restriction n\rangle, \ldots, \langle x_n \restriction n, x_{n^-} \restriction n, z_{x_n,x_{n^-}} \restriction n\rangle\big\rangle.$$

$\square$[5.126]

Using (5.126) together with (5.122.2) we have the following theorem first proved by Martin using a forcing argument.

Theorem [ZF] *Every $\mathbf{\Sigma}^1_2$ irreflexive wellfounded relation on a subset of ${}^\omega\omega$ has rank $< \omega_2$.*

Note that we also have another proof of (a somewhat more general statement of) (5.118) by applying (5.126) to (5.122.1).

5.4.4 Scales and uniformization

Another way to represent the sort of information about a pointset that is embodied in a Suslin representation is in terms of *semiscales* and *scales*.

(5.127) *We will specialize these definitions to subsets of* ${}^{\omega}\omega$*, but they are fundamentally topological and apply to pointsets generally.*

Definition [ZF]

1. *In general, if* f *is a prefunction*[47] *the* graph *of* $f \stackrel{\text{def}}{=} \mathfrak{gr}\, f \stackrel{\text{def}}{=} \{\langle x, y\rangle \mid (x, y) \in f\}$.
2. *For the purposes of this discussion, given a prefunction* $f \subseteq {}^{\omega}X \dot{\times} {}^{\omega}Y$*, the* pointwise graph *of* $f \stackrel{\text{def}}{=} \mathfrak{gr}^{\cdot}\, f \stackrel{\text{def}}{=} \{\langle x, y\rangle^{\cdot} \mid \langle x, y\rangle \in \mathfrak{gr}\, f\}$ $(= \{\langle x, y\rangle^{\cdot} \mid (x, y) \in f\})$*, where* $\langle x, y\rangle^{\cdot} = \big\langle\langle x(n), y(n)\rangle \,\big|\, n \in \omega\big\rangle$.[5.55]

Recall that for a binary relation R, $\mathfrak{p}\, R = \{x \mid \exists y\ \langle x, y\rangle \in R\}$.[48]

Definition [ZF] *Suppose* $A \subseteq {}^{\omega}\omega$ *and* $\kappa \in \mathrm{Ord}$*. A* semiscale on A to $\kappa \stackrel{\text{def}}{=}$ *a function* $\bar\varphi : A \to {}^{\omega}\kappa$ *such that* $\mathfrak{p}\, \overline{\mathfrak{gr}\, \bar\varphi} = A$*, where* $\overline{X}$ *is the topological closure of a set* X*.* $\bar\varphi$ *is a* semiscale $\stackrel{\text{def}}{\iff}$ *it is a semiscale to some ordinal.*

Since $\operatorname{dom} \bar\varphi = A$, $\mathfrak{p}\, \mathfrak{gr}\, \bar\varphi = A$, so $\mathfrak{p}\, \overline{\mathfrak{gr}\, \bar\varphi} \supseteq A$. Thus the essential characteristic of $\bar\varphi$ as a semiscale is that $\mathfrak{p}\, \overline{\mathfrak{gr}\, \bar\varphi} \subseteq A$.

It is customary to identify a semiscale $\bar\varphi$ on A to κ with the sequence $\langle \varphi_n \mid n \in \omega\rangle$ of norms on A to κ given by:

$$\varphi_n x = (\bar\varphi x) n.$$

(5.128) **Theorem** [ZF + $\mathsf{AC}_\omega(\mathbb{R})$] *Suppose* $\bar\varphi : A \to {}^{\omega}\kappa$*.* $\bar\varphi$ *is a semiscale iff*

(5.129) for any sequence $\langle x_n \mid n \in \omega\rangle$ of members of A, if

1. $\lim_{n\to\infty} x_n = x$, and
2. for all $m \in \omega$, $\lim_{n\to\infty} \varphi_m x_n$ exists,

then $x \in A$.

Proof $\to$ Suppose $\bar\varphi$ is a semiscale and Conditions 5.129 apply. Let $z \in {}^{\omega}\kappa$ be defined by the condition that $z(m) = \lim_{n\to\infty} \varphi_m x_n$. Then $z = \lim_{n\to\infty} \bar\varphi x_n$. Hence, $\langle x, z\rangle \in \overline{\mathfrak{gr}\, \bar\varphi}$, so $x \in A$.

$\leftarrow$ Suppose $\langle x, z\rangle \in \overline{\mathfrak{gr}\, \bar\varphi}$. Using $\mathsf{AC}_\omega(\mathbb{R})$, for each $n \in \omega$ let x_n be such that $x_n \restriction n = x \restriction n$ and $\bar\varphi(x_n) \restriction n = z \restriction n$. Then $\langle x_n \mid n \in \omega\rangle$ satisfies (5.129.1,2), so $x \in A$. $\square$[5.128]

The definition of *scale* imposes a key additional condition.

[47] Recall that a prefunction is just a class of ordered pairs. We seldom consider prefunctions that are not actually functions. The point of the $\mathfrak{gr}$ operation is simply to change ordered pairs to 2-sequences.

[48] '$\mathfrak{p}$' stands for 'projection', in this case projection along the first coordinate of a subset of the plane. We may also use '$\operatorname{dom} R$' to denote this, but we generally reserve dom for use with (pre)functions, which are classes of ordered pairs, rather than 2-sequences.

Definition [ZF] *Suppose* $A \subseteq {}^\omega\omega$, $\kappa \in \mathrm{Ord}$, *and* $\bar\varphi : A \to {}^\omega\kappa$ *is a semiscale.* $\bar\varphi$ *is a* scale on $A \subseteq {}^\omega\omega$ to κ $\overset{\text{def}}{\iff}$

$$\forall \langle x, z\rangle \in \overline{\mathfrak{gr}\,\bar\varphi}\ \forall m \in \omega\ (\bar\varphi x)(m) \leqslant z(m). \tag{5.130}$$

In other words, letting $z_0 = \bar\varphi x$, for each $m \in \omega$, $z_0(m)$ is the least member of the set $\{z(m) \mid \langle x, z\rangle \in \overline{\mathfrak{gr}\,\bar\varphi}\}$. We also say that z_0 is the *pointwise least* member of $\{z \mid \langle x, z\rangle \in \overline{\mathfrak{gr}\,\bar\varphi}\}$.

In terms of the norms φ_m ($m \in \omega$), as above, on the assumption of $\mathsf{AC}_\omega(\mathbb{R})$, the scale condition5.130 is

for any sequence $\langle x_n \mid n \in \omega\rangle$, *if*

1. $\lim_{n\to\infty} x_n = x$ *and*
2. *for each* $m \in \omega$, $\lim_{n\to\infty} \varphi_m x_n$ *exists*

then $\forall m \in \omega\ \varphi_m x \leqslant \lim_{n\to\infty} \varphi_m x_n$. *This is termed the* lower semicontinuity *property of scales.*

The following enhancements of the notion of (semi)scale are sometimes useful.

Definition [ZF] *Suppose* $\bar\varphi = \langle \varphi_m \mid m \in \omega\rangle$ *is a (semi)scale.*

1. $\bar\varphi$ *is* good $\overset{\text{def}}{\iff}$ *for every* $\langle x_n \mid n \in \omega\rangle \in {}^\omega A$, *if* $\langle \bar\varphi x_n \mid n \in \omega\rangle$ *converges then* $\langle x_n \mid n \in \omega\rangle$ *converges.*
2. $\bar\varphi$ *is* very good $\overset{\text{def}}{\iff}$ $\bar\varphi$ *is good and for all* $x, y \in A$ *and* $m \in \omega$, *if* $\varphi_m x \leqslant \varphi_m y$ *then* $\forall m' < m\ \varphi_{m'} x \leqslant \varphi_{m'} y$.
3. $\bar\varphi$ *is* excellent $\overset{\text{def}}{\iff}$ $\bar\varphi$ *is very good and for all* $x, y \in A$ *and* $m \in \omega$, *if* $\varphi_m x = \varphi_m y$ *then* $x \restriction m = y \restriction m$.[49]

As indicated at the beginning of this section, scales are closely related to Suslin trees. We have noted above that if $A \subseteq {}^\omega\omega$ is α-Suslin then A is $|\alpha|$-Suslin. Thus, for the most part, we may restrict our attention to cardinals, and only infinite cardinals are interesting. Suppose κ is an infinite cardinal. It is easy to use a κ-semiscale on A to define a κ-Suslin tree for A, and *vice versa.* It is also easy to use a κ-semiscale on A to define an excellent α-scale on A for some α such that $|\alpha| = \kappa$. With a little more effort, we can show that α may be taken to be κ. Thus we have the following theorem.

(5.131) **Theorem** [ZF] *Suppose* κ *is an uncountable cardinal and* $A \subseteq {}^\omega\omega$. *Then* A *is* κ*-Suslin iff* A *admits an excellent* κ*-scale.*

Proof $\leftarrow$ Since an excellent scale is *a fortiori* a semiscale, it suffices to define a κ-Suslin tree from a κ-semiscale $\bar\varphi$ on A. Let5.53

$$T = T^{\mathfrak{gr}^\cdot\,\bar\varphi} = \{\langle x, \bar\varphi x\rangle^\cdot \restriction m \mid x \in A \wedge m \in \omega\}.$$

Then $[T] = \overline{\mathfrak{gr}^\cdot\,\bar\varphi}$, so $[T]^\cdot = \overline{\mathfrak{gr}\,\bar\varphi}$. Since $\bar\varphi$ is a semiscale on A, $\mathfrak{p}^\cdot[T] = \mathfrak{p}[T]^\cdot = A$, so T is a Suslin tree for A.

[49] This notion obviously does not generalize immediately to an arbitrary pointspace in place of ${}^\omega\omega$.

$\rightarrow$ Suppose T is a sequence tree on $\omega \times \kappa$ such that $A = \mathfrak{p}^{\cdot}[T] = \mathfrak{p}[T]^{\cdot}$.[5.58.3] For each $x \in A$ let $f(x)$ be the lexicographically least branch of $T_{[x]}$.[5.121] It is easy to show that f is a semiscale on A to κ; indeed, for this it suffices that $f(x)$ be some branch of $T_{[x]}$, not necessarily the lexicographically least. (Since $[T]$ is closed (in ${}^{\omega}(\omega \times \kappa)$), $[T]^{\cdot}$ is closed (in ${}^{\omega}\omega \times {}^{\omega}\kappa$). Since $\mathfrak{gr}\, f \subseteq [T]^{\cdot}$, $\overline{\mathfrak{gr}\, f} \subseteq [T]^{\cdot}$, so f is a semiscale on A.)

To define an excellent scale on A we will make use of lexicographic orderings of products of products of ordinals, in particular, of sets of the form ${}^{m}(\kappa \times \omega)$. Thus, $\big\langle\langle\alpha_k, \beta_k\rangle \mid k < m\big\rangle < \big\langle\langle\alpha'_k, \beta'_k\rangle \mid k < m\big\rangle$ iff

1. $\big\langle\langle\alpha_k, \beta_k\rangle \mid k < m\big\rangle \neq \big\langle\langle\alpha'_k, \beta'_k\rangle \mid k < m\big\rangle$, and
2. letting k be least such that $\langle\alpha_k, \beta_k\rangle \neq \langle\alpha'_k, \beta'_k\rangle$, $\alpha_k < \alpha'_k$ or ($\alpha_k = \alpha'_k$ and $\beta_k < \beta'_k$).

The lexicographic ordering of finite products of wellordered sets is a wellorder. For each $m \in \omega$, let $T_m = T \cap {}^{m}(\kappa \times \omega)$, the mth level of T. Let $\lambda_m : T_m \overset{\text{bij}}{\rightarrow} \alpha_m$ be the isomorphism of T_m, lexicographically ordered, with the ordinal α_m which is the length of this order. Let $\lambda = \bigcup_{m\in\omega} \lambda_m$.

To define an excellent scale $\bar{\varphi} = \langle\varphi_m \mid m \in \omega\rangle$ on A, let

$$\varphi_m x = \lambda(\langle f(x), x\rangle^{\cdot} \restriction m).$$

Thus, $\bar{\varphi} = \lambda \circ \big\langle\langle f(x), x\rangle^{\cdot} \restriction m \,\big|\, m \in \omega\big\rangle$.

Suppose $\langle x, z\rangle \in \overline{\mathfrak{gr}\,\bar{\varphi}}$. Then there exists $w \in {}^{\omega}\kappa$ such that $z = \lambda \circ \big\langle\langle w, x\rangle^{\cdot} \restriction m \,\big|\, m \in \omega\big\rangle$ and $\langle x, w\rangle \in \overline{\mathfrak{gr}\, f}$. Since f is a semiscale, $x \in A$, so $\bar{\varphi}$ is a semiscale. Also, $w \in T_{[x]}$, so $f(x)$ is lexicographically $\leqslant w$, which implies that for each $m \in \omega$

$$\begin{aligned}\varphi_m x &= \lambda(\langle f(x), x\rangle^{\cdot} \restriction m) \leqslant \lambda(\langle w, x\rangle^{\cdot} \restriction m)\\ &= z(m),\end{aligned}$$

so $\bar{\varphi}$ is a scale.

To show that $\bar{\varphi}$ is a good scale, suppose $\langle x_n \mid n \in \omega\rangle \in {}^{\omega}A$, and suppose $\langle\bar{\varphi}x_n \mid n \in \omega\rangle$ converges. Then obviously $\langle x_n \mid n \in \omega\rangle$ converges, so $\bar{\varphi}$ is good. It is also trivial to show that $\bar{\varphi}$ is very good and excellent.

The only thing left to show is that T may be chosen so that $\operatorname{im}\lambda \subseteq \kappa$, for then $\bar{\varphi}$ is a κ-scale. Let S be a sequence tree on $\omega \times \kappa$ such that $A = \mathfrak{p}^{\cdot}[S]$, and suppose first that $\operatorname{cf}\kappa > \omega$. Let D be the set of $\langle x, w\rangle^{\cdot}$ such that $x \in A$ and $w = \langle\alpha\rangle^\frown z$, where $z \in S_{[x]}$ and $\forall m \in \omega\ z(m) < \alpha$. Let $T = T^D = \{\langle x, w\rangle^{\cdot} \restriction m \mid \langle x, w\rangle^{\cdot} \in D \wedge m \in \omega\}$ be the corresponding sequence tree on $\omega \times \kappa$. Since $\operatorname{cf}\kappa > \omega$, $A = \mathfrak{p}^{\cdot}[S] = \mathfrak{p}^{\cdot}[T]$. Since $\operatorname{cf}\kappa > \omega$ it is also true that for each $\alpha < \kappa$ and $m > 0$, $|\{\langle s, u\rangle^{\cdot} \in T_m \mid u(0) = \alpha\}| < \kappa$, so its order type is $< \kappa$, and the order type of T_m is κ. Thus, for each $m \in \omega$, $\lambda_m : T_m \rightarrow \kappa$, so $\bar{\varphi}$ is a κ-scale.

If $\operatorname{cf}\kappa = \omega$ a different trick must be used. Let $\langle\kappa_n \mid n \in \omega\rangle$ be an increasing sequence of infinite cardinals $< \kappa$ that is cofinal in κ. Given $z \in {}^{\omega}\kappa$, for each $m \in \omega$ let k^z_m be the least $k \in \omega$ such that $z(m) < \kappa_k$, and let $u^z_m = \langle k^z_m, 0, \ldots, 0, z(m)\rangle$ be the $(k_m + 2)$-sequence beginning with k^z_m and ending with $z(m)$, with zeros in between. Let $w^z = u^z_0{}^\frown u^z_1{}^\frown \cdots$. Now let $D = \{\langle x, w^z\rangle^{\cdot} \mid \langle x, z\rangle^{\cdot} \in [S]\}$, and let $T = T^D = \{\langle x, w\rangle^{\cdot} \restriction m \mid \langle x, w\rangle^{\cdot} \in D \wedge m \in \omega\}$. Clearly, for any $x \in {}^{\omega}\omega$, any $w \in T_{[x]}$ is w^z for some $z \in S_{[x]}$, so $A = \mathfrak{p}^{\cdot}[S] = \mathfrak{p}^{\cdot}[T]$. For any $m \in \omega$ the cardinality of T_m is $< \kappa_m$, so its order type is $< \kappa_m$. Thus, for each $m \in \omega$, $\lambda_m : T_m \rightarrow \kappa$, so $\bar{\varphi}$ is a κ-scale. $\square$[5.131]

The definability of scales is an important issue in descriptive set theory, closely related to the definability of prewellorderings.[5.108] We use the notation previously established for norms,[5.107.3] so that if $\varphi : A \to \mathrm{Ord}$ then $\leqslant^\varphi$ and $<^\varphi$ are the corresponding prewellordering relations on A. As above, we may dispense with norms in favor of their associated prewellorderings. Thus, by *scale* we may refer directly to a sequence $\bar{\leqslant} = \langle \leqslant_n \mid n \in \omega \rangle$ of prewellorders. (We could also refer to $\bar{<} = \langle <_n \mid n \in \omega \rangle$, but keep in mind that the definability classifications of the weak and strong orders typically differ.)

Definition [ZF] *Suppose Γ is a recursively closed pointclass, $A \subseteq {}^\omega\omega$, and $\bar{\leqslant} = \langle \leqslant_n \mid n \in \omega \rangle$ is a scale on A. $\bar{\leqslant}$ is a Γ-scale $\overset{\text{def}}{\iff}$ there exist S_Γ and $S_{\breve{\Gamma}} \subseteq \omega \times {}^\omega\omega \times {}^\omega\omega$ in Γ and $\breve{\Gamma}$, respectively, such that for all $x \in A$, $x' \in {}^\omega\omega$, and $m \in \omega$*

$$x' \leqslant_m x \leftrightarrow \langle m, x', x \rangle \in S_\Gamma \leftrightarrow \langle m, x', x \rangle \in S_{\breve{\Gamma}}.$$

Definition [ZF] *Suppose Γ is a recursively closed pointclass. Γ has the* scale property $\overset{\text{def}}{\iff}$ *for every $A \subseteq {}^\omega\omega$, if $A \in \Gamma$ then there exists a Γ-scale on A.*

(5.132) **Theorem** [ZF] *Suppose Γ is a recursively closed pointclass that is also closed under $\vee$, $\wedge$, $\exists^0$, $\forall^0$, and $\forall^1$. If Γ has the scale property then Γ has the uniformization property.*[5.111.4]

Proof Suppose $A \subseteq {}^\omega\omega \times {}^\omega\omega$ is in Γ and let $\bar{\varphi}$ be a Γ-scale on A.[5.127] Suppose $\bar{\varphi}$ is to κ, and for each $m \in \omega$ let λ_m be the order-preserving bijection between ${}^m(\kappa \times \omega)$ and some ordinal η_m. Let $\psi_m : A \to \eta_m$ be such that

$$\psi_m\langle x, y \rangle = \lambda_m \left(\left\langle \bar{\varphi}\langle x, y \rangle, y \right\rangle^{\cdot} \restriction m \right),$$

and let $\bar{\psi} = \langle \psi_m \mid m \in \omega \rangle$.

Since $\bar{\varphi}$ is a scale, $\bar{\psi}$ is a scale, with sufficient "goodness" for the present purpose. Let $\leqslant_m^\varphi$ and $\leqslant_m^\psi$ be the respective prewellorderings of A associated with φ_m and ψ_m, and let $\leqslant$ lexicographically order $\kappa \times \omega$. Then for any $\langle x, y \rangle \in A$ and any $\langle x', y' \rangle \in {}^\omega\omega \times {}^\omega\omega$,

$$\begin{aligned}
&\langle x', y' \rangle \leqslant_m^\psi \langle x, y \rangle \\
&\quad \leftrightarrow \forall i < m \left(\left\langle \bar{\varphi}\langle x', y' \rangle, y' \right\rangle^{\cdot} \restriction i = \left\langle \bar{\varphi}\langle x, y \rangle, y \right\rangle^{\cdot} \restriction i \right. \\
&\qquad \left. \to \left\langle \varphi_i\langle x', y' \rangle, y'(i) \right\rangle \leqslant \left\langle \varphi_i\langle x, y \rangle, y(i) \right\rangle \right) \\
&\quad \leftrightarrow \forall i < m \Big(\forall j < i \left(\langle x', y' \rangle \leqslant_j^\varphi \langle x, y \rangle \wedge \langle x, y \rangle \leqslant_j^\varphi \langle x', y' \rangle \wedge y'(j) = y(j) \right) \\
&\qquad \to \left(\langle x', y' \rangle \leqslant_i^\varphi \langle x, y \rangle \wedge \left(\langle x, y \rangle \leqslant_i^\varphi \langle x', y' \rangle \to y'(i) \leqslant y(i) \right) \right) \Big)
\end{aligned}$$

$\bar{\varphi}$ is a Γ-scale on A, so let S_Γ and $S_{\breve{\Gamma}}$ in Γ and $\breve{\Gamma}$, respectively, be such that for all $\langle x, y \rangle \in A$, $\langle x', y' \rangle \in {}^\omega\omega \times {}^\omega\omega$, and $m \in \omega$

$$\langle x', y' \rangle \leqslant_m^\varphi \langle x, y \rangle \leftrightarrow \langle m, x', y', x, y \rangle \in S_\Gamma \leftrightarrow \langle m, x', y', x, y \rangle \in S_{\breve{\Gamma}}.$$

Accordingly, let T_Γ be the set of $\langle m, x, y, x', y' \rangle$ such that

$$\begin{aligned}
\forall i < m \Big(&\forall j < i \left(\langle j, x', y', x, y \rangle \in S_{\breve{\Gamma}} \wedge \langle j, x, y, x', y' \rangle \in S_{\breve{\Gamma}} \wedge y'(j) = y(j) \right) \\
&\to \left(\langle i, x', y', x, y \rangle \in S_\Gamma \wedge \left(\langle i, x, y, x', y' \rangle \in S_{\breve{\Gamma}} \to y'(i) \leqslant y(i) \right) \right) \Big),
\end{aligned}$$

and let $T_{\check{\Gamma}}$ be the set of $\langle m, x, y, x', y'\rangle$ such that

$$\forall i < m \Big(\forall j < i \left(\langle j, x', y', x, y\rangle \in S_\Gamma \wedge \langle j, x, y, x', y'\rangle \in S_\Gamma \wedge y'(j) = y(j)\right) \\ \rightarrow \left(\langle i, x', y', x, y\rangle \in S_{\check{\Gamma}} \wedge (\langle i, x, y, x', y'\rangle \in S_\Gamma \rightarrow y'(i) \leqslant y(i))\right)\Big),$$

Then for all $\langle x, y\rangle \in A$, $\langle x', y'\rangle \in {}^\omega\omega \times {}^\omega\omega$, and $m \in \omega$

$$\langle x', y'\rangle \leqslant^\psi_m \langle x, y\rangle \leftrightarrow \langle m, x', y', x, y\rangle \in T_\Gamma \leftrightarrow \langle m, x', y', x, y\rangle \in T_{\check{\Gamma}}.$$

Since T_Γ and $T_{\check{\Gamma}}$ are in Γ and $\check{\Gamma}$, respectively, $\bar{\psi}$ is a Γ-scale.

Let $D = \{\langle x, y, \bar{\psi}\langle x, y\rangle\rangle^{\cdot} \mid \langle x, y\rangle \in A\}$, and let

$$U = T^D = \{\langle x, y, \bar{\psi}\langle x, y\rangle\rangle^{\cdot} \restriction m \mid \langle x, y\rangle \in A \wedge m \in \omega\}.$$

For $x \in \mathfrak{p}\, A$, let $C_x = \{\langle z, y\rangle^{\cdot} \mid \langle x, y, z\rangle^{\cdot} \in [U]\}$. Then C_x is closed. Let $\langle z^x, y^x\rangle^{\cdot}$ be its lexicographically least member. Then $\langle x, y^x, z^x\rangle^{\cdot} \in [U] = \overline{D}$. Since $\bar{\psi}$ is a scale, $\langle x, y^x\rangle \in A$ and $\forall m \in \omega\ \psi_m\langle x, y^x\rangle \leqslant z^x(m)$.

But z^x is also the lexicographically least element of $C_{x,y^x} = \{z \mid \langle x, y^x, z\rangle^{\cdot} \in [U]\}$. Since $\bar{\psi}\langle x, y^x\rangle$ is also in C_{x,y^x}, $z^x = \bar{\psi}\langle x, y^x\rangle$. Thus, $\langle\bar{\psi}\langle x, y^x\rangle, y^x\rangle^{\cdot}$ is the lexicographically least element of C_x. In particular, if $\langle x, y\rangle \in A$ then $\langle\bar{\psi}\langle x, y^x\rangle, y^x\rangle^{\cdot} \leqslant \langle\bar{\psi}\langle x, y\rangle, y\rangle^{\cdot}$ lexicographically, which by the design of $\bar{\psi}$ implies that $\forall m \in \omega\ \langle x, y^x\rangle \leqslant^\psi_m \langle x, y\rangle$. Also by the design of $\bar{\psi}$, this minimization property uniquely specifies y^x in $\{y \mid \langle x, y\rangle \in A\}$.

For each $m \in \omega$ let $\leqslant^*_m$ be derived from $\leqslant^\psi_m$ as in (5.107.3.2.1), i.e.,

$$\langle x, y\rangle \leqslant^*_m \langle x', y'\rangle \leftrightarrow \langle x, y\rangle \in A \wedge \left(\langle x', y'\rangle \notin A \vee \langle x, y\rangle \leqslant^\psi_m \langle x', y'\rangle\right).$$

Then[5.110]

(5.133)
$$\langle x, y\rangle \leqslant^*_m \langle x', y'\rangle \leftrightarrow \langle x, y\rangle \in A \wedge \left(\langle m, x, y, x', y'\rangle \in T_\Gamma \vee \langle m, x', y', x, y\rangle \notin T_{\check{\Gamma}}\right).$$

Let $B = \{\langle x, y\rangle \mid \forall y' \in {}^\omega\omega\ \forall m \in \omega\ \langle x, y\rangle \leqslant^*_m \langle x, y'\rangle\}$. Then

1. B is in Γ;[5.133]
2. $B \subseteq A$; and
3. for all $x \in \mathfrak{p}\, A$, y^x is the unique $y \in {}^\omega\omega$ such that $\langle x, y\rangle \in B$.

Thus B uniformizes A. □[5.132]

(5.134) **Theorem (Novikov-Kondo-Addison)** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *For any $z \in {}^\omega\omega$, $\Pi^1_1(z)$ has the scale property. Hence, $\Pi^1_1(z)$ has the uniformization property. It follows that $\boldsymbol{\Pi}^1_1$ has the scale and uniformization properties.*

Proof We will prove the theorem for Π^1_1. The proof generalizes immediately to $\Pi^1_1(z)$ for any $z \in {}^\omega\omega$, and accordingly to $\boldsymbol{\Pi}^1_1$. We have already seen how to define a Π^1_1-prewellordering for a Π^1_1 set $A \subseteq {}^\omega\omega$, using a recursive sequence tree T on $\omega \times \omega$ such that $A = \neg\, \mathfrak{p}^{\cdot}[T] = \{x \in {}^\omega\omega \mid T_{[x]}$ is wellfounded$\}$. For the nonce, for $x \in {}^\omega\omega$ and $s \in {}^{<\omega}\omega$, let[3.183.2.1.1]

$$T_{x,s} \stackrel{\text{def}}{=} (T_{[x]})_s = \{t \in {}^{<\omega}\omega \mid s^\frown t \in T_{[x]}\}.$$

Thus, $T_{x,s}$ is $T_{[x]}$ below s if $s \in T_{[x]}$; otherwise 0. If $x \in A$ then $T_{x,s}$ is a wellfounded tree—possibly empty, and certainly empty if $s \notin T_{[x]}$. If $s \in T_{[x]}$ then the rank $\operatorname{rk} T_{x,s}$ of $T_{x,s}$ is the rank $\operatorname{rk}^{T_{[x]}} s$ of s in $T_{[x]}$.

Recall[5.123] the fixed recursive enumeration $\boldsymbol{s}$ of ${}^{<\omega}\omega$. Given $x \in A$ and $m \in \omega$, let

$$\varphi_m x = \operatorname{rk} T_{x,\boldsymbol{s}(m)},$$

and let $\bar{\varphi} = \langle \varphi_m \mid m \in \omega \rangle$. For each $m \in \omega$, $\varphi_m : A \to \omega_1$. The proof of the prewellordering property of Π^1_1 shows that the norms φ_m are uniformly Π^1_1, so if $\bar{\varphi}$ is a scale it is a Π^1_1-scale.

Let $S = T^{\mathfrak{gr}^{\cdot}\,\bar{\varphi}} = \{\langle x, \bar{\varphi}x\rangle^{\cdot} \restriction m \mid x \in A \wedge m \in \omega\}$. Then $[S]^{\cdot} = \overline{\mathfrak{gr}\,\bar{\varphi}}$, the closure of the graph of $\bar{\varphi}$. Suppose $\langle x, z\rangle^{\cdot} \in [S]$. To show that $\bar{\varphi}$ is a scale we must show that

1. *$x \in A$; and*
2. *$\forall m \in \omega\ \varphi_m x \leqslant z(m)$.*

Let $f = z \circ \boldsymbol{s}^{-1} \restriction T_{[x]}$. Then $f : T_{[x]} \to \omega_1$, and for each $m \in \omega$

$$\boldsymbol{s}_m \in T_{[x]} \to f(\boldsymbol{s}_m) = z(m).$$

(5.135) **Claim** f *is order-preserving.*

Proof Suppose $\boldsymbol{s}_m \subseteq \boldsymbol{s}_{m'} \in T_{[x]}$. Then[5.124] $m \leqslant m'$. Let $n = |\boldsymbol{s}_{m'}| + 1$, and let $x' \in A$ be such that $x' \restriction n = x \restriction n$ and $(\bar{\varphi}x') \restriction n = z \restriction n$. Then

$$f(\boldsymbol{s}_m) = z(m) = \varphi_m x' = \operatorname{rk}^{T_{[x']}} \boldsymbol{s}_m \geqslant \operatorname{rk}^{T_{[x']}} \boldsymbol{s}_{m'} = \varphi_{m'} x' = z(m') = f(\boldsymbol{s}_{m'}).$$

□[5.135]

Since f is order-preserving,[5.135] $T_{[x]}$ is wellfounded, so $x \in A$. For any $s \in T_{[x]}$

$$f(s) \geqslant \operatorname{rk}^{T_{[x]}} s = \operatorname{rk} T_{x,s},$$

so for any $m \in \omega$

1. if $\boldsymbol{s}_m \in T_{[x]}$ then $z(m) = f(\boldsymbol{s}_m) \geqslant \operatorname{rk} T_{x,\boldsymbol{s}_m} = \varphi_m x$; and
2. if $\boldsymbol{s}_m \notin T_{[x]}$ then $z(m) \geqslant 0 = \operatorname{rk} T_{x,\boldsymbol{s}_m} = \varphi_m x$.

We have shown, therefore, that $\bar{\varphi}$ is a scale on A, and—as noted above—it is a Π^1_1-scale. Thus, Π^1_1 has the scale property. Since Π^1_1 has the closure properties required by (5.132), Π^1_1 has the uniformization property. □[5.134]

5.5 Perfect sets

Definition [ZF] *Suppose X is a topological space and $A \subseteq X$.*

1. *Suppose $a \in X$. a is an* isolated point *of A $\overset{\text{def}}{\iff}$ $a \in A$ and $a \notin \overline{(A\backslash\{a\})}$. In other words, $a \in A$ and there is a neighborhood N of a such that $N \cap A = \{a\}$.*
2. *A is* perfect $\overset{\text{def}}{\iff}$ *A is nonempty and closed and has no isolated points.*

(5.136) **Theorem** [ZF] *Suppose X is a Polish space and $A \subseteq X$ is perfect. Then there exists $B \subseteq A$ such that B is homeomorphic to ${}^{\omega}2$.*

Proof Let ρ be a metric for X. Let $\langle d_n \mid n \in \omega\rangle$ enumerate a dense set $D \subseteq X$. We will define $f : {}^{<\omega}2 \to D$ and $g : {}^{<\omega}2 \to (0, \infty)$ such that for all $s \in {}^{<\omega}2$,

1. $gs \leqslant 1/(|s| + 1)$;
2. $B(fs, gs) \cap A \neq 0$; and
3. $C(fs', gs'), C(fs'', gs'') \subseteq C(fs, gs)$ and $C(fs', gs') \cap C(fs'', gs'') = 0$, where $s' = s^\frown\langle 0\rangle$ and $s'' = s^\frown\langle 1\rangle$.

To begin, let $g0 = 1$. Let a be an arbitrary point in A. Let $n \in \omega$ be least such that $d_n \in B(a, 1) \cap D$, and let $f0 = d_n$. Now suppose fs and gs have been defined. Since A is perfect, for any $a \in A \cap B(fs, gs)$ there exists $a' \in A \cap B(fs, gs)\backslash\{a\}$. Thus, since $A \cap B(fs, gs) \neq 0$, there exist $a, a' \in A \cap B(fs, gs)$ with $a \neq a'$.

There exists $N \in \omega$ such that $N > 1/(gs)$, $N > 1/\rho(a, a')$, and $C(a, 1/N)$, $C(a', 1/N) \subseteq C(fs, gs)$. For any such N, there exist $d, d' \in D$ such that $d \in B(a, 1/4N)$ and $d' \in B(a', 1/4N)$. For any such d and d', $a \in B(d, 1/4N)$ and $a' \in B(d', 1/4N)$, so $B(d, 1/4N)$ and $B(d', 1/4N)$ both meet A, $C(d, 1/4N) \subseteq C(fs, gs)$, $C(d', 1/4N) \subseteq C(fs, gs)$, and $C(d, 1/4N) \cap C(d', 1/4N) = 0$.

Thus there exist $N, n, n' \in \omega$ such that

(5.137) *$N > 1/gs$; $B(d_n, 1/N)$ and $B(d_{n'}, 1/N)$ both meet A; $C(d_n, 1/N) \subseteq C(fs, gs)$; $C(d_{n'}, 1/N) \subseteq C(fs, gs)$; and $C(d_n, 1/N) \cap C(d_{n'}, 1/N) = 0$.*

Let $N \in \omega$ be least such that there exist $n, n' \in \omega$ such that (5.137) holds; then let $n \in \omega$ be least such that there exists $n' \in \omega$ such that (5.137) holds; and then let $n' \in \omega$ be least such that (5.137) holds. Let $f(s^\frown\langle 0\rangle) = d_n$, $f(s^\frown\langle 1\rangle) = d_{n'}$, and $g(s^\frown\langle 0\rangle) = g(s^\frown\langle 1\rangle) = 1/N$. This completes the definition of f and g.

Given $x \in {}^{\omega}2$, $\langle f(x \restriction n) \mid n \in \omega\rangle$ is a Cauchy sequence. Let $fx = \lim_{n\to\infty} f(x \restriction n)$. Note that $\bigcap_{n\in\omega} C(f(x \restriction n), g(x \restriction n)) = \{fx\}$. It is easy to see that $fx \in A$, for if it is not, then since A is closed, there is a neighborhood $\mathcal{N}$ of fx disjoint from A; however, for sufficiently large $n \in \omega$, $B(f(x \restriction n), g(x \restriction n)) \subseteq \mathcal{N}$, which contradicts the fact that $B(f(x \restriction n), g(x \restriction n))$ meets A.

f is clearly injective and continuous. Let $B = \operatorname{im} f$. It is easy to see that f^{-1} is continuous on B, so f is a homeomorphism of ${}^{\omega}2$ with B.[50] $\square$ 5.136

(5.138) **Theorem (Cantor-Bendixson)** [ZF] *Suppose X is a Polish space, and $F \subseteq X$ is closed and uncountable. Then $F = P \cup C$, where P is perfect and C is countable.*

Proof For any $A \subseteq X$, let A^i be the set of isolated points of A. For each $x \in A^i$ there exists $n \in \omega$ such that $B(x, 1/n) \cap A = \{x\}$. Let n_x be the least such n. Then $A\backslash A^i = A\backslash \bigcup_{x\in A^i} B(x, 1/n_x)$, so if A is closed then $A\backslash A^i$ is also closed. Define an ordinal-indexed sequence $\langle F_\alpha \mid \alpha \leqslant \eta\rangle$ as follows.

1. $F_0 = F$.
2. If $F^i_\alpha \neq 0$, let $F_{\alpha+1} = F_\alpha\backslash F^i_\alpha$; otherwise the construction terminates, and $\eta = \alpha$.

[50] Note that we have avoided any use of Choice.

3. If α is a limit ordinal, let $F_\alpha = \bigcap_{\beta\in\alpha} F_\beta$.

By induction on α, F_α is closed for all α. This is a strictly decreasing sequence of sets, so the construction must eventually terminate. This can only happen by virtue of Clause 2, at which point F_α is a closed set with no isolated points ($F_\alpha^i = 0$), and we have set $\eta = \alpha$. Let $P = F_\eta$ and let $C = \bigcup_{\alpha<\eta} F_\alpha^i$.

$F = P \cup C$, and F is assumed to be uncountable, so if C is countable, then P is nonempty, hence perfect, and we have the desired decomposition. To show that C is countable, let $D \subseteq X$ be a countable dense set. Let $\langle B_n \mid n \in \omega\rangle$ enumerate the (countable) set $\mathcal{B}$ of open balls $\{B(d, 1/m) \mid d \in D \wedge m \in \omega\backslash\{0\}\}$. Given a set $A \subseteq X$ and $x \in A^i$, there exists $m \in \omega$ such that $B(x, 1/m) \cap A = \{x\}$. Since D is dense, there exists $d \in D \cap B(x, 1/2m)$. Note that $B(d, 1/2m) \in \mathcal{B}$ and $B(d, 1/2m) \cap A = \{x\}$. For any set A and $x \in A^i$, let $B_x^A = B_n$, where n is least such that $B_n \cap A = \{x\}$. Note that if $x, x' \in A^i$ and $x \neq x'$ then $B_x^A \neq B_{x'}^A$.

We now define $f : C \to \mathcal{B}$ as follows. Given $x \in C$ there is a unique $\alpha < \eta$ such that $x \in F_\alpha^i$. Let $f(x) = B_x^{F_\alpha}$. It is straightforward to check that f is injective. Since $\mathcal{B}$ is countable, C is countable. $\square$[5.138]

If we are willing to use a little bit of Choice, a simpler proof of (5.138) may be formulated by defining C to be the set of $x \in F$ such that some neighborhood of x has countable intersection with F. Since X is second countable, C is a countable union of countable sets. Assuming AC_ω, C is countable; and $F\backslash C$ is now easily seen to be perfect.

By (5.136) and (5.81) a perfect subset of a Polish space has cardinality 2^ω, so Theorem 5.138 implies that a closed subset of a Polish space either is countable or has cardinality 2^ω. Thus, any counterexample to the continuum hypothesis, i.e., any subset of $\mathbb{R}$ with cardinality strictly intermediate between that of $\mathbb{N}$ and that of $\mathbb{R}$, cannot be a closed set. This result offered some encouragement that a proof along these lines might be found of the continuum hypothesis. The following theorem shows that this will not work.

(5.139) **Theorem** [ZFC] *Suppose X is an uncountable Polish space. There exists an uncountable $A \subseteq X$ such that A does not have a perfect subset.*

Proof By the preceding remarks, $|X| = 2^\omega$. Let $\mathcal{C}$ be the set of closed subsets of X. We will show that $|\mathcal{C}| = 2^\omega$. Since every singleton $\{x\}$ is closed, $|\mathcal{C}| \geqslant 2^\omega$, so it suffices to show that $|\mathcal{C}| \leqslant 2^\omega$. Let D be a countable dense subset of X. The set $\mathcal{B}$ of open balls $B(d, 1/n)$, with $d \in D$ and $n \in \omega\backslash\{0\}$, is countable, and is a base for the topology on X. The map that takes a closed set F to the set of members of $\mathcal{B}$ that are included in $X\backslash F$ (which is open) is an injection of $\mathcal{C}$ into $\mathcal{P}\mathcal{B}$, so $|\mathcal{C}| \leqslant 2^\omega$.

Let $\langle C_\alpha \mid \alpha \in 2^\omega\rangle$ enumerate $\mathcal{C}$. (Note that 2^ω is the cardinal, i.e., initial ordinal,[3.124.3] equipollent with ${}^\omega 2$.) Let $\prec$ be a wellordering of X with length 2^ω. Note that the set of $\prec$-predecessors of any $x \in X$ has cardinality less than 2^ω. For $\alpha < 2^\omega$, define x_α and y_α recursively as follows. Let x_α be the $\prec$-least $x \in X$ such that $\forall\beta < \alpha\,(x \neq x_\beta \wedge x \neq y_\beta)$. Let y_α be the $\prec$-least $y \in X$ such that if C_α is perfect then $y \in C_\alpha\backslash\{x_\beta \mid \beta \leqslant \alpha\}$. Since every perfect set has cardinality 2^ω, this construction can be carried out for 2^ω steps.

Let $A = \{x_\alpha \mid \alpha \in \eta\}$ and $B = \{y_\alpha \mid \alpha \in \eta\}$. Then $|A| = 2^\omega$ and $A \cap B = 0$, so A does not include any perfect set. $\square$[5.139]

Note that the wellorderings of X and the class $\mathcal{C}$ of closed sets in the proof of (5.139) are not *defined*; they are simply asserted to exist by AC, so the set A

is likewise not defined. The extent to which (5.138) holds for definable sets is an important topic in descriptive set theory.

In this context the following definition is convenient.

Definition [ZF] *Suppose X is a Polish space and $A \subseteq X$. A has the* perfect set property $\overset{\text{def}}{\iff}$ *A is countable or includes a perfect set.*

In Theorem 5.183 we show that every analytic set has the perfect set property.

5.6 Category

Definition [ZF] *Suppose X is a topological space and $A, B \subseteq X$. Recall that A is dense iff every nonempty open $G \subseteq X$ meets A.*

1. *A is* dense in B $\overset{\text{def}}{\iff}$ *$A \cap B$ is a dense subset of B in the sense of the relative topology on B, i.e., for every open set $G \subseteq X$, if $G \cap B \neq 0$ then $G \cap B \cap A \neq 0$.*
2. *A is* nowhere dense $\overset{\text{def}}{\iff}$ *it is not dense in any nonempty open set, i.e., every nonempty open G has a nonempty open subset G' such that $G' \cap A = 0$.*

We make frequent use of the obvious facts that if A is dense and $B \supseteq A$ then B is dense, and if A is nowhere dense and $B \subseteq A$, then B is nowhere dense.

(5.140) **Theorem** [ZF] *Suppose X is a topological space and $A \subseteq X$.*

1. *If A is nowhere dense then $\overline{A}$ is nowhere dense.*
2. *Suppose A is closed. Then A is nowhere dense iff $X \backslash A$ is dense.*
3. *A is nowhere dense iff $X \backslash A$ includes an open dense set.*

Proof 1 Suppose A is nowhere dense. To show that $\overline{A}$ is nowhere dense, suppose $G \subseteq X$ is nonempty and open. Then there exists a nonempty open $G' \subseteq G$ such that $G' \cap A = 0$, whence $G' \cap \overline{A} = 0$, since $\overline{A}$ is (by one version of its definition) the complement of the union of all open sets disjoint from A.

2 Straightforward.

3 Straightforward combining (1) and (2). $\square$5.140

Note that the complementary notion to 'nowhere dense' is 'includes an open dense', while the complementary notion to 'open dense' is 'closed and nowhere dense'.

A subset of X that is both open and dense comprises—in a sense—almost all of X. For example, the intersection of finitely many open dense sets is itself open and dense. The same cannot be said for arbitrary intersections of open dense sets. For suppose X is such that $A_x = X \backslash \{x\}$ is open and dense for any $x \in X$, which is true for any perfect Polish space (e.g., ${}^{\omega}\omega$ or ${}^{\omega}2$). Then for any $A \subseteq X$, $A = \bigcap_{x \notin A} A_x$, so A is an intersection of open dense sets.

Note that this typically represents A as an intersection of uncountably many open dense sets. The interesting question is therefore: What about countable intersections of open dense sets? For starters we observe that we should not expect such an intersection to be open. The class of open sets is required by definition to be closed under finite intersections, but it is not generally closed under countable intersections. The important question is: Must every countable intersection of open dense sets be dense? As it happens, the answer is affirmative for certain important classes of topological spaces, which include many of those with which we will have to do. In particular, we have the following theorem proved by René Baire in his doctoral thesis[1].

(5.141) **Baire category theorem (for Polish spaces)** [ZF] *Suppose X is Polish space. Then the intersection of a countable set of open dense sets is dense. Equivalently, the complement of the union of a countable set of nowhere dense sets is dense; or, a nonempty open set is not the union of a countable set of nowhere dense sets.*

Remark The Baire category theorem *per se* assumes that the space X is either a complete metric space (not necessarily separable) or a locally compact Hausdorff space. Its proof uses AC. In this section we are concerned primarily with Polish spaces, i.e., separable completely metrizable spaces. For these the Baire category theorem does not require any Choice, and $\mathsf{AC}_\omega(\mathbb{R})$ suffices for the rest of this section, with the notable exception of Vitali's theorem.[5.150]

Proof Suppose X is a separable complete metric space. Let

(5.142) $$\langle a_m \mid m \in \omega \rangle$$

enumerate a dense subset S of X. Suppose $\langle G_n \mid n \in \omega \rangle$ is sequence of open dense sets in X, and let G be a non-empty open set. We must show that $G \cap \bigcap_{n \in \omega} G_n \neq 0$. To this end we generate recursively sequences $\langle x_n \mid n \in \omega \rangle$ and $\langle N_n \mid n \in \omega \rangle$, such that

1. for each $n \in \omega$,
    1. $x_n \in S$;
    2. $n < N_n < \omega$; and
    3. $C(x_{n+1}, 1/N_{n+1}) \subseteq B(x_n, 1/N_n) \cap G_n$; and
2. $B(x_0, 1/N_0) \subseteq G$.

Since G is open and nonempty, and S is dense, we may let x_0 be the first member of $S \cap G$ in the given enumeration[5.142] of S, and let N_0 be the least positive integer N such that $B(x_0, 1/N) \subseteq G$.

Now suppose x_n and N_n have been generated. Since G_n is open and dense, $B(x_n, 1/N_n) \cap G_n$ is open and non-empty. Let x_{n+1} be the first member of $S \cap B(x_n, 1/N_n) \cap G_n$ in the given enumeration of S, and let N_{n+1} be the least integer $N > n + 1$ such that $C(x_{n+1}, 1/N) \subseteq B(x_n, 1/N_n) \cap G_n$.

By construction, if $m < n < \omega$ then $x_n \in B(x_m, 1/N_m)$, so $d(x_m, x_n) < 1/N_m < 1/(m + 1)$. $\langle x_n \mid n \in \omega \rangle$ is therefore a Cauchy sequence. Let $x = \lim_{n \to \infty} x_n$. Since the $C(x_n, 1/N_n)$'s are nested and closed, x is in all of them, so $x \in G \cap \bigcap_{n \in \omega} G_n$. $\square$[5.141]

Baire's theorem[5.141] suggests the following definition.

(5.143) **Definition** [ZF] *Suppose X is a topological space.*

1. X has the Baire property *or* is Baire $\overset{\text{def}}{\iff}$ *the intersection of any countable set of open dense subsets of X is dense, i.e., X satisfies Baire's theorem.*[5.141]
2. *Suppose $A \subseteq X$.*
   1. A *is* of first category *or* meager $\overset{\text{def}}{\iff}$ *A is a union of countable many nowhere dense sets; equivalently, $X \backslash A$ includes an intersection of countably many open dense sets.*
   2. A *is* of second category *or* nonmeager $\overset{\text{def}}{\iff}$ *A is not meager.*
   3. A *is* residual *or* comeager $\overset{\text{def}}{\iff}$ *$X \backslash A$ is meager, i.e., A includes an intersection of countably many open dense sets.*
   4. $\mathfrak{m}^X \overset{\text{def}}{=}$ *the set of meager subsets of X.*

The notion of category is of interest mainly for Baire spaces.

(5.144) **Theorem** [ZF] *Suppose X is a Baire space. $\mathfrak{m}^X$ is an ideal in the boolean algebra $\mathcal{P}X$ of subsets of X.*

Proof Clearly $0 \in \mathfrak{m}^X$, and for any $A, B \in \mathcal{P}X$,

1. $A \in \mathfrak{m}^X \wedge B \subseteq A \to B \in \mathfrak{m}^X$; and
2. $A, B \in \mathfrak{m}^X \to A \cup B \in \mathfrak{m}^X$.

Since X is Baire, a comeager set cannot be empty, so a comeager set cannot be meager; hence, $X \notin \mathfrak{m}^X$, so $\mathfrak{m}^X$ is an ideal in $\mathcal{P}X$. □[5.144]

The following terminology is useful.

Definition [ZF] *In the context of Baire category we use* almost all *or* almost every *to refer to the elements of a comeager set.*

Thus, for example, when we say that something is true for almost all $x \in X$, we mean that the set of x for which it is true is comeager. We may also say that the statement is true *almost everywhere*.

Definition [ZF] *Sets $A, B \subseteq X$ are* almost equal *iff for almost all x, $x \in A \iff x \in B$, i.e., the symmetric difference $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is meager. We also say that such sets are* categorically equivalent.

Since $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$, categorical equivalence is an equivalence relation.

(5.145) **Theorem** [ZF] *Suppose X is a topological space and $A \subseteq X$.*

1. *If A is open then $(\overline{A}) \backslash A$ is nowhere dense.*
2. *Complementarily, if A is closed then $A \backslash \operatorname{int} A$ is nowhere dense.*

Hence, any open (closed) set is categorically equivalent to its closure (interior).

Proof Straightforward. □[5.145]

It follows from (5.145) that any subset of X that is categorically equivalent to an open set is categorically equivalent to a closed set, and *vice versa*.

Definition [ZF] *Suppose X is a topological space and $A \subseteq X$. A* has the Baire property *or* is Baire $\overset{\text{def}}{\iff}$ *A is categorically equivalent to an open set.* $\mathbf{Baire}^X \overset{\text{def}}{=}$ *the set of Baire subsets of X.*

As stated in the remark following Theorem 5.141 we largely restrict our attention to Polish spaces in this chapter, in which setting we may usually restrict our use of Choice to $\mathsf{AC}_\omega(\mathbb{R})$. Some of the theorems that follow are provable for larger classes of spaces, with correspondingly stronger choice principles.

(5.146) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose X is a Polish space. A countable union of meager sets is meager.*

Proof Given meager sets $A_n \subseteq X$ $(n \in \omega)$, we use $\mathsf{AC}_\omega(\mathbb{R})$ with Borel codes[5.91] to choose for each $n \in \omega$ a countable set S_n of nowhere dense closed sets such that $A_n \subseteq \bigcup S_n$. Let $S = \bigcup\{S_n \mid n \in \omega\}$, and use $\mathsf{AC}_\omega(\mathbb{R})$ again to show that S is countable.[51] Since $\bigcup\{A_n \mid n \in \omega\} \subseteq \bigcup S$, $\bigcup\{A_n \mid n \in \omega\}$ is meager. $\square$[5.146]

(5.147) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose X is a Polish space.*

1. *X is Baire.*
2. $\mathbf{Baire}^X$ *contains all open and all closed subsets of X.*
3. $\mathbf{Baire}^X$ *is closed under complementation, countable union, and countable intersection.*
4. *Hence* $\mathbf{Borel}^X \subseteq \mathbf{Baire}^X$.

Proof 1 The Baire category theorem for Polish spaces.[5.141] $\square$[5.147.1]

2 By definition every open set is Baire. By (5.145) every closed set is Baire. $\square$[5.147.2]

3 Since categorical equivalence is an equivalence relation, it follows that if A is Baire then $X \backslash A$ is Baire.

Suppose for each $n \in \omega$, A_n is Baire. Using $\mathsf{AC}_\omega(\mathbb{R})$ with a coding of open sets by reals, for each $n \in \omega$, let G_n be open such that $A_n \triangle G_n$ is meager. Then $(\bigcup_{n\in\omega} A_n) \triangle (\bigcup_{n\in\omega} G_n) \subseteq \bigcup_{n\in\omega}(A_n \triangle G_n)$ is meager,[5.146] so $\bigcup_{n\in\omega} A_n$ is Baire.

Similarly, $\bigcap_{n\in\omega} A_n$ is Baire. We can prove this directly by using closed sets F_n in place of the open sets G_n above, or we can use the closure of the set of Baire sets under complementation. $\square$[5.147.3]

4 Immediate. $\square$[5.147.4] $\square$[5.147]

Like $\mathbf{Borel}^X$, therefore, $\mathbf{Baire}^X$ is a σ-algebra. Note that modulo the ideal of meager sets, **Baire** and **Borel** coincide, because every element of **Baire** is categorically equivalent to an open set.

[51] Borel codes are reals. Countable sets of closed sets are coded by reals using a bijection of ${}^\omega({}^\omega\omega)$ with ${}^\omega\omega$. The union of a countable set of countable sets S_n $(n \in \omega)$ of closed sets is shown to be countable by using $\mathsf{AC}_\omega(\mathbb{R})$ to choose an enumeration of each S_n (via codes) and combining these into an enumeration of $\bigcup\{S_n \mid n \in \omega\}$ using a bijection of ${}^2\omega$ with ω.

(5.148) **Theorem** [$\mathsf{ZF}{+}\mathsf{AC}_\omega(\mathbb{R})$] *Suppose X is a Polish space. Let* $\mathbf{Borel} = \mathbf{Borel}^X$ *and* $\mathfrak{m} = \mathfrak{m}^X \cap \mathbf{Borel}^X$*, the ideal of meager sets in* $\mathbf{Borel}$*.*

1. $\mathbf{Borel}/\mathfrak{m}$ *is countably complete.*
2. $\mathbf{Borel}/\mathfrak{m}$ *has a countable dense subset.*
3. $\mathbf{Borel}/\mathfrak{m}$ *is countably saturated.*
4. $\mathbf{Borel}/\mathfrak{m}$ *is complete.*

Proof For each $B \in \mathbf{Borel}$, $[B] \stackrel{\text{def}}{=} [B]_{\mathfrak{m}} \stackrel{\text{def}}{=} B + \mathfrak{m} \stackrel{\text{def}}{=}$ the $\mathfrak{m}$-equivalence class of B.

1 We must show that every countable subset of $\mathbf{Borel}/\mathfrak{m}$ has a least upper bound in $\mathbf{Borel}/\mathfrak{m}$. Suppose $\langle b_n \mid n \in \omega\rangle \in {}^{\omega}(\mathbf{Borel}/\mathfrak{m})$. Use $\mathsf{AC}_\omega(\mathbb{R})$ to infer the existence of a sequence $\langle c_n \mid n \in \omega\rangle$ of Borel codes[5.91] such that $\forall n \in \omega\ [B_{c_n}] = b_n$.[52] For each $n \in \omega$, let $B_n = B_{c_n}$. Let $B = \bigcup_{n\in\omega} B_n$, and let $b = [B]$. Clearly, $\forall n \in \omega\ b_n \leqslant b$. Suppose $b' \in \mathbf{Borel}/\mathfrak{m}$ is such that $\forall n \in \omega\ b_n \leqslant b'$. We must show that $b \leqslant b'$. To this end, let B' be Borel such that $[B'] = b'$. Then for each $n \in \omega$, $B_n \backslash B'$ is meager. Hence, $B\backslash B' = \left(\bigcup_{n\in\omega} B_n\right)\backslash B' = \bigcup_{n\in\omega}(B_n\backslash B')$ is meager,[5.146] so $b = [B] \leqslant [B'] = b'$. $\square$[5.148.1]

2 To obtain a countable dense subset of $\mathbf{Borel}/\mathfrak{m}$, let $\mathcal{G}$ be a countable base for the X-topology.[5.85.4] Let $D = \{[G] \mid G \in \mathcal{G}\}$. D is countable. By convention, $0 \notin \mathcal{G}$, so $0 \notin D$. Suppose $b \in \mathbf{Borel}/\mathfrak{m}$ and $b \neq 0$. Let B be an open set in b. Then $B \neq 0$, so for some $G \in \mathcal{G}$, $G \subseteq B$. Let $d = [G]$. Then $d \in D$ and $d \leqslant b$. So D is dense. $\square$[5.148.2]

3 Any boolean algebra $\mathfrak{B}$ with a countable dense set D is countably saturated. To prove this, note that any enumeration $\langle d_\alpha \mid \alpha < \kappa\rangle$ of D, where κ is either a finite ordinal or ω, yields a wellordering $\prec$ of D. Suppose $B \subseteq \mathfrak{B}$ is an antichain (of nonzero elements). For each $b \in B$ let d_b be the $\prec$-first $d \in D$ such that $d \leqslant b$. Note that $b \neq b' \rightarrow d_b \neq d_{b'}$. Hence $b \mapsto d_b$ is an injection of B into D. Hence B is countable. $\square$[5.148.3]

4 This follows from (5.148.1, 2). Suppose $\mathfrak{B}$ is a countably complete boolean algebra with a countable dense subset D. Suppose $B \subseteq \mathfrak{B}$. Let $D' = \{d \in D \mid \exists b \in B\ d \leqslant b\}$. Let $c = \bigvee D'$, which exists since D' is countable and $\mathfrak{B}$ is countably complete. It is straightforward to show that c is the supremum of B. $\square$[5.148]

5.6.1 Category in product spaces

Suppose X and Y are topological spaces. It is natural to inquire as to the relationship between category-theoretical properties of subsets of the product space $X \times Y$ and related subsets of X and Y. The following theorem is fundamental in this regard.

[52]Of course, we do not need the full machinery of Borel codes, since every element of $\mathbf{Borel}/\mathfrak{m}$ has an open (also a closed) representative.

(5.149) **Theorem: Kuratowski-Ulam (for Polish spaces)** [$\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$] *Suppose X and Y are Polish spaces. Let $X \times Y$ be given the product topology. Suppose $A \subseteq X \times Y$. For $x \in X$, let $A_x \overset{\text{def}}{=} \{y \in Y \mid \langle x, y\rangle \in A\}$.*

1. *If A is meager then A_x is meager for almost every $x \in X$.*
2. *If A has the Baire property then A_x has the Baire property for almost every $x \in X$.*
3. *If A has the Baire property and A_x is meager for almost every $x \in X$ then A is meager.*

Thus, if A has the Baire property then A_x has the Baire property for almost every $x \in X$ and A is meager iff A_x is meager for almost every $x \in X$.

Proof 1 First suppose $G \subseteq X \times Y$ is open and dense. Let $\mathcal{S}$ be a countable base for the topology on Y, and for each $D \in \mathcal{S}$, let $G^D = \{x \in X \mid \exists y \in D\ \langle x, y\rangle \in G\}$. Clearly, each G^D is open and dense. Let $H = \bigcap_{D \in \mathcal{S}} G^D$. Then H is comeager, and for any $x \in H$, G_x is an open dense subset of Y.

Now suppose $A \subseteq X \times Y$ is meager, and let $\langle G^n \mid n \in \omega\rangle$ be such that $G^n \subseteq X \times Y$ is open and dense for each $n \in \omega$, and $A \cap \bigcap_{n\in\omega} G^n = 0$. For each $n \in \omega$, let H^n be defined for G^n as H was defined for G in the preceding paragraph. Thus, each H^n is comeager, and for all $x \in H^n$, G^n_x is open dense. Let $H = \bigcap_{n\in\omega} H^n$. Then H is comeager,[53] and for any $x \in H$, $\left(\bigcap_{n\in\omega} G^n\right)_x$ is comeager, so A_x is meager. $\square^{5.149.1}$

2 Suppose $A \subseteq X \times Y$ has the Baire property, and let $G, D \subseteq X \times Y$ be respectively open and meager such that $A \triangle G = D$. For every $x \in X$, $A_x \triangle G_x = D_x$. For every $x \in X$, G_x is open, and for almost every $x \in X$, D_x is meager;[5.149.1] hence, for almost every $x \in X$, A_x has the Baire property. $\square^{5.149.2}$

3 Suppose $A \subseteq X \times Y$ has the Baire property and A_x is meager for almost every $x \in X$. Let $G, D \subseteq X \times Y$ be respectively open and meager such that $A \triangle G = D$. For any $x \in X$, $A_x \triangle G_x = D_x$ and G_x is open. By hypothesis, for almost every $x \in X$, A_x is meager, and by (5.149.1), for almost every $x \in X$, D_x is meager. Hence, for almost every $x \in X$, $G_x = 0$. Since the set of $x \in X$ such that $G_x \neq 0$ is open, it is empty, so G is empty, and A is therefore meager. $\square^{5.149}$

(5.150) **Theorem** [ZFC] *There is a set $V \subseteq \mathbb{R}$ that does not have the Baire property.*

Remark The construction in this proof is due to Vitali, and the set V constructed in this way is referred to as a (or, informally, the) *Vitali set*.

Since the set $\mathbb{Q}$ of rational numbers is countable, it is meager as a subset of $\mathbb{R}$. Hence, $V \setminus \mathbb{Q}$ is a subset of $\mathbb{R}\setminus\mathbb{Q}$ that is not Baire. Since ${}^\omega\omega$ is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$, there is a subset of ${}^\omega\omega$ that does not have the Baire property.

Proof Let $M \subseteq \mathbb{R}$ be maximal such that for every pair x, y of distinct members of M, $x - y \notin \mathbb{Q}$. (Use AC in the form of Zorn's lemma, for example.) For $a \in \mathbb{R}$, let $M_a = \{x + a \mid x \in M\}$, the *translate* of M by a. Note that if $a, a' \in \mathbb{Q}$ and $a \neq a'$

[53] Recall[5.146] that $\mathsf{AC}_\omega(\mathbb{R})$ implies that the set of comeager sets is closed under countable intersection.

then $M_a \cap M_{a'} = 0$, and $\mathbb{R} = \bigcup_{a \in \mathbb{Q}} M_a$. Since $\mathbb{Q}$ is countable, for some $a \in \mathbb{Q}$, M_a is not meager. Let $V = M_a$.

Suppose V has the Baire property. Let G be open such that $G \triangle V$ is meager. Then G is nonempty and $G \backslash V$ is meager. Let $b < c \in \mathbb{R}$ be such that $(b, c) \subseteq G$. Let $q < c - b$ be a positive rational. Let $G' = \{x + q \mid x \in G\}$, and $V' = \{x + q \mid x \in V\} = M_{a+q}$. As noted above, $V \cap V' = 0$. Since the translation map $x \mapsto x + q$ is a homeomorphism, $G' \backslash V'$ is meager. Note that $(b + q, c + q) \subseteq G'$, and $b < b + q < c < c + q$, so $I = (b + q, c)$ is a nonempty open interval and $I \subseteq G \cap G'$. Thus, $I \backslash V$ and $I \backslash V'$ are meager. But since $V \cap V' = 0$, $(I \backslash V) \cup (I \backslash V') = I$, which violates Baire's theorem. Hence V does not have the Baire property. □[5.150]

Like (5.139), Theorem 5.150 does not provide a definition of a non-Baire set, it simply shows that if the axiom of choice is true, then one exists. The extent to which definable sets have the Baire property is an important issue in descriptive set theory. In Theorem 5.181 we show that every analytic set has the Baire property.

5.7 Measure

As originally defined by Lebesgue, measure applied to subsets of $\mathbb{R}^n$, and this context is most important for our present purpose. For notational uniformity, we will deal with ${}^n\mathbb{R}$, the set of n-sequences of reals, rather than $\mathbb{R}^n$, the set of n-tuples of reals. We are also interested in measure theory on the Cantor space ${}^\omega 2$. Hence, we will give a treatment of measure that is sufficiently general to encompass both these examples, and we will use these as examples to illustrate the general theory.

We are not here interested in the use of measure to define a general notion of integration as in analysis. Indeed, for the purpose of descriptive set theory we are primarily interested in the mere fact of measurability of sets of reals as it relates to their definability, and that will be the focus of this discussion.

The reader is probably familiar with the informal use of '∞' to indicate a quasi-number that is greater than all ordinary numbers. For the purposes of measure theory, we use ∞ as an extension to the set of nonnegative real numbers. This leads to the familiar notation '$[0, \infty)$' for the set of nonnegative real numbers, and to the notation '$[0, \infty]$' for the extended system. For the purposes of measure theory, we are concerned with the topology, the order relation, and the addition operation on $[0, \infty]$. We are not concerned with multiplication. The rules relevant to ∞ in this context are as follows:

(5.151) *Suppose a is a nonnegative real number.*

1. $a < \infty$. *Hence*
    1. $(a, \infty) = \{b \in \mathbb{R} \mid a < b\}$;
    2. $[a, \infty) = \{b \in \mathbb{R} \mid a \leqslant b\}$;
    3. $(a, \infty] = (a, \infty) \cup \{\infty\}$; *and*
    4. $[a, \infty] = [a, \infty) \cup \{\infty\}$.
2. $a + \infty = \infty + a = \infty + \infty = \infty$.
3. $\{(a, \infty] \mid a \in [0, \infty)\}$ *is a neighborhood base for* ∞. *Hence, '*$\lim_{x \to \infty}$*' has the familiar meaning.*
4. *Consequently, if I is an arbitrary index set, $\forall i \in I \ x_i \in [0, \infty]$, and $y \in [0, \infty]$, then $\sum_{i \in I} x_i = y$ iff*

1. *for every finite* $I' \subseteq I$, $\sum_{i\in I'} x_i \leqslant y$, *and*
2. *for every* $y' < y$ *there exists a finite* $I' \subseteq I$ *such that* $\sum_{i\in I'} x_i > y'$.

In the context of (5.151.4) it is easy to see that if $x_i = \infty$ for some $i \in I$, or $\{i \in I \mid x_i > 0\}$ is uncountable, then $\sum_{i\in I} x_i = \infty$. Clearly, if $x_i = 0$ then it may be omitted without altering the sum. The interesting sums are therefore of the form $\sum_{i\in\omega} x_i$, where $x_i \in [0,\infty)$, and for these the usual rule applies: $\sum_{i\in\omega} x_i = \sup_{n\in\omega} \sum_{i=0}^{n} x_i$.

In the following discussion Ω is an arbitrary nonempty set, of which ${}^{n}\mathbb{R}$ and ${}^{\omega}2$ are our standard examples. We will be concerned with subsets of Ω, i.e., elements of the powerset $\mathcal{P}\,\Omega$. Note that $\mathcal{P}\,\Omega$ is a complete boolean algebra with the usual identifications:

1. $A \vee B = A \cup B$;
2. $A \wedge B = A \cap B$;
3. $\neg\, A = \Omega \backslash A$.

(5.152) **Definition** [ZF]

1. *Suppose* $A, B \in \mathcal{P}\,\Omega$, *and* $\mathcal{A} \subseteq \mathcal{P}\,\Omega$.
   1. $A \sqcup B \overset{\text{def}}{=} A \cup B$ *if* $A \cap B = 0$; *otherwise it is undefined.*
   2. *Similarly,* $\bigsqcup \mathcal{A} \overset{\text{def}}{=} \bigcup \mathcal{A}$ *if the members of* $\mathcal{A}$ *are pairwise disjoint; otherwise it is undefined.*

   We regard any assertion in which '$\sqcup$*' or '*$\bigsqcup$*' occurs as including the assertion of pairwise disjointness involved in its definition.*
2. $\mathfrak{S} \subseteq \mathcal{P}\,\Omega$ *is a* semiring *on* Ω $\overset{\text{def}}{\Longleftrightarrow}$
   1. $0 \in \mathfrak{S}$; *and*
   2. *for all* $A, B \in \mathfrak{S}$
      1. $A \cap B \in \mathfrak{S}$; *and*
      2. *there exists a finite* $\mathcal{A} \subseteq \mathfrak{S}$ *such that* $A \backslash B = \bigsqcup \mathcal{A}$.
3. $\mathfrak{R} \subseteq \mathcal{P}\,\Omega$ *is a* ring *on* Ω $\overset{\text{def}}{\Longleftrightarrow}$
   1. $0 \in \mathfrak{R}$; *and*
   2. *for all* $A, B \in \mathfrak{R}$
      1. $A \cup B \in \mathfrak{R}$; *and*
      2. $A \backslash B \in \mathfrak{R}$.

   Note that $A \cap B = A \backslash (A \backslash B)$, *so a ring is closed under intersection.*
4. $\mathfrak{A} \subseteq \mathcal{P}\,\Omega$ *is an* algebra *on* Ω *iff* $\mathfrak{A}$ *is a ring and* $\Omega \in \mathfrak{A}$, *so* $\mathfrak{A}$ *is closed under complementation (relative to* Ω*).*
5. *We use '*σ*' as a general indicator of countability. A* σ*-ring* $\overset{\text{def}}{=}$ *a ring closed under countable unions, and a* σ*-algebra* $\overset{\text{def}}{=}$ *an algebra closed under countable unions. Note that a* σ*-ring is also closed under countable intersections:* $\bigcap_{n\in\omega} A_n = A_0 \backslash \bigcup_{n\in\omega} (A_0 \backslash A_n)$.
6. *Suppose* $\mathcal{A} \subseteq \mathcal{P}\,\Omega$ *and* $0 \in \mathcal{A}$. μ *is a* measure on $\mathcal{A}$ $\overset{\text{def}}{\Longleftrightarrow}$

1. $\mu : A \to [0, \infty]$;
2. $\mu 0 = 0$; *and*
3. (Additivity) *If* A_n, $n \in \omega$ *are disjoint members of* $\mathcal{A}$ *and* $\bigcup_{n\in\omega} A_n \in \mathcal{A}$, *then* $\mu \bigcup_{n\in\omega} A_n = \sum_{n\in\omega} \mu A_n$.

7. μ *is an* outer measure *on* $\Omega \overset{\text{def}}{\iff}$
   1. $\mu 0 = 0$;
   2. (Monotonicity) *If* $A, B \in \mathcal{P}\Omega$ *and* $A \subseteq B$ *then* $\mu A \leqslant \mu B$;
   3. (Subadditivity) *If* $A_n \in \mathcal{P}\Omega$ *for each* $n \in \omega$ *then* $\mu \bigcup_{n\in\omega} A_n \leqslant \sum_{n\in\omega} \mu A_n$.

(5.153) **Examples**

1 The set of intervals $\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a_m \leqslant x < b_m\}$, where $\langle a_m \mid m \in n\rangle, \langle b_m \mid m \in n\rangle \in {}^n\mathbb{R}$, is a semiring on ${}^n\mathbb{R}$.[54]

Lebesgue measure is obtained by setting $\mu\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a_m \leqslant x < b_m\} = \prod_{m\in n}(b_m - a_m)$, assuming $\forall m \in n\ a_m \leqslant b_m$.

2 The set of intervals $I_s = \{x \in {}^\omega 2 \mid s \subseteq x\}$, where $s \in {}^{<\omega}2$, is a semiring on ${}^\omega 2$.[55] The *uniform measure* on ${}^\omega 2$ is obtained by setting $\mu I_s = 2^{-|s|}$.

5.7.1 Extending a measure to a σ-algebra

The next two theorems show how to extend a measure μ on a semiring $\mathfrak{S}$ to a measure on a σ-algebra that includes $\mathfrak{S}$. There are several ways to do this. In the approach given here we first define μ_2 as the unique extension of μ to the set $\mathfrak{C}$ of disjoint unions of members of $\mathfrak{S}$ by the prescription (5.152.6.3). Although $\mathfrak{C}$ is not a σ-algebra, we show that μ_2 has certain properties, viz., (5.154.5.1, 2, 3), that are common to all measures on σ-algebras. We then define the outer measure μ^* determined by μ, which may also be characterized in terms of μ_2, and the set $\mathfrak{M}$ of sets to which μ^* naturally assigns a measure (i.e., a measure *per se*, not just an outer measure). As a convenience, we make the assumption of σ-finiteness, which holds in all cases of interest to us, and this implies that $\mathfrak{M}$ is not empty, and in fact $\mathfrak{M} \supseteq \mathfrak{S}$. We conclude by using the properties (5.154.5.1, 2, 3) to show that $\mathfrak{M}$ is a σ-algebra.

(5.154) **Theorem** [ZF] *Suppose* $\mathfrak{S}$ *is a semiring on* Ω *and* $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$. *Let* $\mathfrak{R}$ *be the set of finite disjoint unions of members of* $\mathfrak{S}$. *Let* $\mathfrak{C}$ *be the set of countable disjoint unions of members of* $\mathfrak{S}$. *Suppose* μ *is a measure on* $\mathfrak{S}$.

1. $\mathfrak{R}$ *is the smallest ring that includes* $\mathfrak{S}$, *i.e., the ring* generated by $\mathfrak{S}$.
2. μ *extends uniquely to a measure* μ_1 *on* $\mathfrak{R}$.
3. $\mathfrak{C}$ *is the smallest set that includes* $\mathfrak{S}$ *and is closed under countable union.*

[54] The essential observation is that $[a, b)$ is the disjoint union of $[a, c)$ and $[c, b)$ for $a < c < b$, so if $a_m < c < b_m$, then $\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a_m \leqslant x < b_m\}$ is the disjoint union of $\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a_m \leqslant x < b'_m\}$ and $\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a'_m \leqslant x < b_m\}$, where for $m' \neq m$, $a'_{m'} = a_{m'}$ and $b'_{m'} = b_{m'}$, whereas $a'_m = b'_m = c$. The difference of two intervals may clearly be represented as the disjoint union of subintervals of the first interval obtained by iteration of the above dissection with appropriate intermediate points.

[55] The essential observation here is that I_s is the disjoint union of $I_{s^\frown\langle 0\rangle}$ and $I_{s^\frown\langle 1\rangle}$.

4. $\mathfrak{C}$ *is closed under (finite) intersection.*[56]
5. μ *extends uniquely to a measure* μ_2 *on* $\mathfrak{C}$*.* μ_2 *has the following properties.*

   1. (Monotonicity) *For any* $C, D \in \mathfrak{C}$*, if* $C \subseteq D$ *then* $\mu_2 C \leqslant \mu_2 D$*.*
   2. (Subadditivity) *For any* $C \in \mathfrak{C}$*, if* $C = \bigcup_{n\in\omega} C_n$ *with* $C_n \in \mathfrak{C}$ *for each* $n \in \omega$*, then* $\mu_2 C \leqslant \sum_{n\in\omega} \mu_2 C_n$*.*
   3. (Downward continuity) *Suppose* $C, C_0, C_1, \ldots$ *are members of* $\mathfrak{C}$*,* $\forall n \in \omega\ C_n \supseteq C_{n+1}$*,* $\mu_2 C_0 < \infty$*, and* $\bigcap_{n\in\omega} C_n \subseteq C$*. Then* $\lim_{n\to\infty} \mu_2 C_n \leqslant \mu_2 C$*.*

Proof See Note 10.16.

(5.155) **Definition** [ZF] *Suppose* $\mathfrak{S}$ *is a semiring on* Ω *and* μ *is a measure on* $\mathfrak{S}$*.*

1. *For any* $A \subseteq \Omega$*,* $\mu^* A \overset{\text{def}}{=}$ *the infimum of sums* $\sum_{n\in\omega} \mu S_n$*, where* $S_n \in \mathfrak{S}$ *for all* $n \in \omega$*, and* $A \subseteq \bigcup_{n\in\omega} S_n$*.*
2. $A \subseteq \Omega$ *is* μ-null $\overset{\text{def}}{\iff} \mu^* A = 0$*, i.e., for all* $\varepsilon > 0$ *there exist* $S_n \in \mathfrak{S}$*,* $n \in \omega$*, such that* $A \subseteq \bigcup_{n\in\omega} S_n$ *and* $\sum_{n\in\omega} \mu S_n < \varepsilon$*.*
3. $A \subseteq \Omega$ *is* μ-measurable $\overset{\text{def}}{\iff}$ *for all* $\varepsilon > 0$*, there exist* $S_n, T_n \in \mathfrak{S}$ $(n \in \omega)$ *such that*
   1. $A \subseteq \bigcup_{n\in\omega} S_n$*;*
   2. $\Omega \backslash A \subseteq \bigcup_{n\in\omega} T_n$*; and*
   3. $\sum_{m,n\in\omega} \mu(S_m \cap T_n) < \varepsilon$*.*
4. $\mathfrak{M}^\mu \overset{\text{def}}{=}$ *the set of* μ*-measurable sets.*
5. *For* $A \in \mathfrak{M}^\mu$*,* $\bar{\mu} A \overset{\text{def}}{=} \mu^* A$*.*
6. μ *is* σ-finite $\overset{\text{def}}{\iff}$ *there exist* $S_n \in \mathfrak{S}$ $(n \in \omega)$ *such that* $\forall n \in \omega\ \mu S_n < \infty$ *and* $\Omega = \bigcup_{n\in\omega} S_n$*.*

For any $X \subseteq [0, \infty]$, $\inf X$ is by definition the largest $x \in [0, \infty]$ such that $\forall x' \in X\ x \leqslant x'$. Hence, the infimum of the empty set exists and is ∞. $\mu^* A$ is therefore defined[5.155.1] for all $A \subseteq \Omega$, and μ^* is clearly an outer measure.[5.152.7] Note, however, that if μ is not σ-finite,[5.155.6] then no subset of Ω is μ-measurable.[5.155.3]

We will henceforth assume that measures are σ*-finite unless otherwise stated.*

(5.156) **Theorem** [ZF] *Suppose* $\mathfrak{S}$ *is a semiring on* Ω *and* $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$*. Suppose* μ *is a* σ*-finite measure on* $\mathfrak{S}$*. Then*

1. $\mathfrak{S} \subseteq \mathfrak{M}^\mu$*;*
2. $\mathfrak{M}^\mu$ *is a* σ*-algebra; and*
3. $\bar{\mu}$ *is the unique extension of* μ *to a measure on* $\mathfrak{M}^\mu$*.*

Proof See Note 10.17.

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[56] $\mathfrak{C}$ is not in general closed under difference or countable intersection.

5.7.2 Product measures

(5.157) **Definition** [ZF + AC_ω] *Suppose I is a finite set, and for each $i \in I$, μ_i is a σ-finite measure on a σ-algebra $\mathfrak{M}_i$ on a set Ω_i. Let $\Omega = \bigtimes_{i\in I} \Omega_i$, i.e., Ω is the set of functions $\langle x_i \mid i \in I\rangle$ such that $\forall i \in I\ x_i \in \Omega_i$. Let $\mathfrak{S}$ be the set of* rectangles *in Ω, i.e., product sets of the form $\bigtimes_{i\in I} A_i$, where $A_i \in \mathfrak{M}_i$ for all $i \in I$.*

1. *The* product of the measures $\mu_i \overset{\text{def}}{=} \bigtimes_{i\in I} \mu_i$ *is defined on $\mathfrak{S}$ by the condition that for all $A = \langle A_i \mid i \in I\rangle \in \mathfrak{S}$, $\mu A = \prod_{i\in I} \mu_i A_i$. Multiplication with an infinite factor is defined as follows.*[57]

$$\infty \cdot a = a \cdot \infty \overset{\text{def}}{=} \begin{cases} \infty & \text{if } a \in (0, \infty] \\ 0 & \text{if } a = 0. \end{cases}$$

2. *The* product of the measure algebras $\mathfrak{M}_i \overset{\text{def}}{=} \bigtimes_{i\in I} \mathfrak{M}_i \overset{\text{def}}{=}$ *the algebra $\mathfrak{M}^\mu$ of μ-measurable subsets of Ω.*

To show that this definition is legitimate we need only show that $\mathfrak{S}$ is a semiring on Ω and that μ as defined above is a measure on $\mathfrak{S}$. This is entirely straightforward. It is also clear that μ is σ-finite.

The main property of product measures for our purposes is Fubini's theorem for nullsets.[58]

(5.158) **Theorem** [ZF + AC_ω] *Suppose, for $i = 0, 1$, that μ_i is a σ-finite measure on a σ-algebra $\mathfrak{M}_i$ on a set Ω_i. Let $\mu = \mu_0 \times \mu_1$ be the product measure on the product algebra $\mathfrak{M} = \mathfrak{M}_0 \times \mathfrak{M}_1$ on $\Omega = \Omega_0 \times \Omega_1$.*[5.157] *Suppose $A \in \mathfrak{M}$. For $x \in \Omega_0$, let $A_x = \{y \in \Omega_1 \mid \langle x, y\rangle \in A\}$. Let $E = \{x \in \Omega_0 \mid \mu_1 A_x > 0\}$. Then $\mu A = 0$ iff $\mu_0 E = 0$.*

Proof See Note 10.18.

5.7.3 Measurability

Suppose $\Omega = {}^n\mathbb{R}$ and μ is Lebesgue measure.[5.153.1] Clearly any open interval is in $\mathfrak{M}^\mu$, so any open set—being a countable union of open intervals—is in $\mathfrak{M}^\mu$. Hence, any Borel set is in $\mathfrak{M}^\mu$, since $\mathfrak{M}^\mu$ is a σ-algebra. Note also that we may define the outer measure μ^* in terms of coverings by open intervals, since any interval $\{\langle x_0, \ldots, x_{n^-}\rangle \in {}^n\mathbb{R} \mid \forall m \in n\ a_m \leqslant x < b_m\}$ may be covered arbitrarily closely by an open interval. Let $\mathfrak{L} = \mathfrak{M}^\mu$, the algebra of Lebesgue measurable sets, and let $\lambda = \bar{\mu}$, Lebesgue measure.

Suppose $\Omega = {}^\omega 2$ and μ is the uniform measure.[5.153.2] The basic intervals I_s are open (as well as closed), so the remarks of the preceding paragraph apply.

Note that $\mathbb{R}$ may be mapped definably onto the set of ω-sequences of intervals in ${}^n\mathbb{R}$ and onto the set of ω-sequences of basic intervals in ${}^\omega 2$, so $\mathsf{AC}_\omega(\mathbb{R})$ is a sufficient choice principle for the development of the theory of Lebesgue measure and of the uniform measure on ${}^\omega 2$.

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[57] Note that this definition maintains the commutativity and associativity of multiplication, as well as distributivity over addition for infinite as well as finite sums. If we wished, we could obviate the consideration of infinite factors by letting $\mathfrak{S}$ be the set of rectangles that are products of factors with finite measure in their respective spaces.

[58] Fubini's theorem states the equality of integrals over product spaces to iterated integrals over the factor spaces. The theorem we state here is, in effect, the special case of functions that vanish almost everywhere.

(5.159) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose* $\Omega = {}^n\mathbb{R}$ *and* μ *is Lebesgue measure, or* $\Omega = {}^\omega 2$ *and* μ *is the uniform measure. The following are equivalent.*

1. *A is μ-null.*[5.155.2]
2. *For any $\varepsilon > 0$ there exists an open set G such that $A \subseteq G$ and $\mu G < \varepsilon$.*
3. *There exists a G_δ set G such that $A \subseteq G$ and $\mu G = 0$.*
4. *There exists a Borel set B such that $A \subseteq B$ and $\mu B = 0$.*

Proof Straightforward. For (2) let $G = \bigcap_{n\in\omega} G_n$, where G_n is open, $A \subseteq G_n$, and $\mu G_n < 1/(n+1)$. □[5.159]

(5.160) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose* $\Omega = {}^n\mathbb{R}$ *and* μ *is Lebesgue measure, or* $\Omega = {}^\omega 2$ *and* μ *is the uniform measure. The following are equivalent.*

1. *A is measurable.*
2. *For any $\varepsilon > 0$ there exist an open set G and a closed set F such that $F \subseteq A \subseteq G$ and $\mu(G\backslash F) < \varepsilon$.*
3. *There exist a G_δ set G and an F_σ set F such that $F \subseteq A \subseteq G$ and $\mu(G\backslash F) = 0$.*
4. *There exists a Borel set B such that $A \bigtriangleup B$ is μ-null.*

Proof Straightforward. □[5.160]

In measure theory the ideal $\mathfrak{n}$ of null sets plays an analogous role to the meager ideal $\mathfrak{m}$[5.143] in category theory, and the boolean algebra of Borel sets mod $\mathfrak{n}$ is a similarly natural construct, and we have the following analog of Theorem 5.148. We state the theorem for Lebesgue measure on $\mathbb{R}$, but it is obviously more general than that.

(5.161) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Let* **Borel** *be the boolean algebra of Borel subsets of $\mathbb{R}$, let μ be Lebesgue measure, and let $\mathfrak{n}$ be the ideal of null sets in* **Borel**. *Let* $\mathfrak{L} = \mathbf{Borel}/\mathfrak{n}$.

1. *$\mathfrak{L}$ is countably saturated.*
2. *$\mathfrak{L}$ is complete.*

Proof An element a of $\mathfrak{L}$ is an equivalence class of Borel sets that differ by a null set, so every $X \in a$ has the same μ-measure. We define $\mu(a)$ to be this common value. We can use $\mathsf{AC}_\omega(\mathbb{R})$ to choose Borel (codes of) representatives of countably many elements of $\mathfrak{L}$ at once, which allows us to show that μ is a countably additive measure on $\mathfrak{L}$. Given $m \in \mathbb{Z}$ (the integers—positive, negative, and zero), let $i_m = [(m, m+1)]$, i.e., the set of Borel sets μ-equivalent to the open interval $(m, m+1)$ in $\mathbb{R}$. Note that $\{i_m \mid m \in \mathbb{Z}\}$ is a maximal antichain in $\mathfrak{L}$, so for any $a \in |\mathfrak{L}|$, $a = \bigvee_{m\in\mathbb{Z}} a \cap i_m$. In particular, if $\mu(a) > 0$ then for some $m \in \mathbb{Z}$, $\mu(a \wedge i_m) > 0$.

1 Suppose A is an antichain in $\mathfrak{L}$, and suppose toward a contradiction that A is uncountable. For each $m \in \mathbb{Z}$ and each $n \in \omega\backslash\{0\}$, let $A_{m,n} = \{a \in A \mid \mu(a \wedge i_m) > 1/n\}$. Then $A = \bigcup_{m,n} A_{m,n}$, so for some m, n, $A_{m,n}$ is uncountable; otherwise, we can use $\mathsf{AC}_\omega(\mathbb{R})$ to obtain a choice function assigning to each $\langle m, n\rangle$ an enumeration of (Borel codes of representatives of members of) $A_{m,n}$, and we can dovetail these to obtain an enumeration of A. But by the additivity of μ, there can be at most n elements a of the antichain A such that $\mu(a \wedge i_m) > 1/n$.

2 Suppose $A \subseteq |\mathfrak{L}|$. We will show that A has a least upper bound in $\mathfrak{L}$. Let $B = \{b \in |\mathfrak{L}| \mid \forall a \in A\ b \geqslant a\}$ be the set of all upper bounds of A, and let $\mu_0 = \inf\{\mu b \mid b \in B\}$. If there exists $b \in B$ such that $\mu b = \mu_0$, we are finished, as b is the supremum of A in $\mathfrak{L}$. If not, use $\mathsf{AC}_\omega(\mathbb{R})$ (via Borel codes) to obtain a function $n \mapsto b_n$, for $n \in \omega$, such that $\mu(b_n) - \mu_0 < 1/(n+1)$. Let $b = \bigwedge_{n\in\omega} b_n$ (again using $\mathsf{AC}_\omega(\mathbb{R})$, and the fact that **Borel** is countably closed and $\mathfrak{n}$ is countably complete). Then $\mu b = \mu_0$, and $b \in B$, so $b = \bigvee A$. $\square$[5.161]

(5.162) **Theorem** [ZFC] *There exists $V \subseteq \mathbb{R}$ such that $V \notin \mathfrak{L}$, i.e., V is not Lebesgue-measurable.*

Proof We construct a Vitali set as in the proof of (5.150). Let $V \subseteq [0,1)$ be maximal such that for all $x, y \in V$, $x - y \notin \mathbb{Q}$. For $a \in [0,1) \cap \mathbb{Q}$, let $V_a = \{x \in [0,1) \mid x - a \in V \vee x - a + 1 \in V\}$.[59] Note that if $a, b \in [0,1) \cap \mathbb{Q}$ and $a \neq b$ then $V_a \cap V_b = 0$, because if $x - a = x - b \pm 1$ then $x - y \in \mathbb{Q}$. On the other hand, by the maximality of V, for any $x \in [0,1)$, there exists $y \in V$ such that $x - y \in \mathbb{Q}$. Let

$$a = \begin{cases} x - y & \text{if } x - y \geqslant 0 \\ x - y + 1 & \text{if } x - y < 0. \end{cases}$$

Then $a \in [0,1)$ and $x \in V_a$. So

$$[0,1) = \bigsqcup_{a \in [0,1)\cap\mathbb{Q}} V_a. \tag{5.163}$$

Suppose toward a contradiction that V is Lebesgue measurable. It is easy to show that Lebesgue measure is translation-invariant, because the measure of basic intervals is translation-invariant. Thus $\lambda V_a = \lambda V$ for all $a \in [0,1) \cap \mathbb{Q}$. Hence,[5.163] if $\lambda V = 0$ then $\lambda[0,1) = 0$, and if $\lambda V > 0$ then $\lambda[0,1) = \infty$, neither of which is consistent with the fact that $\lambda[0,1) = 1$. $\square$[5.162]

This proof is easily modified to show that there is a subset of ${}^{\omega}2$ that is not μ-measurable, where μ is the uniform measure. Of course, we assume the axiom of choice for this also.

Relying as it does on the axiom of choice, the proof of (5.162) is nonconstructive and does not give a definition of a nonmeasurable set. As in the case of the perfect set and Baire properties, the extent to which Lebesgue measurability obtains for definable sets is important question in descriptive set theory. In Theorem 5.181 we show that all analytic sets are Lebesgue measurable.

5.8 Determinacy

One of the greatest revelations concerning countably infinitary objects comes from a surprising source—the theory of games. Consider a two-person game of complete information in which the players alternate moves. A good example is the game of chess, although the games we will consider differ from chess in that there are no draws. Just for the sake of example, let us consider the game of chess modified so that a draw is counted as a win for, say, Black.

An important feature of chess is that there is a limit on the number of moves that can be played—if fifty moves have passed without a pawn being advanced or a

[59] We are in effect working with the quotient group $(\mathbb{R}, +)/\mathbb{Z}$, which is a circle group.

piece being captured then a draw is declared. Because of this it is easy to show that either White or Black possesses a strategy that will guarantee it a win regardless of its opponent's play. By a *strategy*, of course, we refer not to a fixed sequence of moves, but rather to a table of responses that prescribes a player's move in every conceivable situation. Thus a strategy for White would prescribe its first move, then would prescribe its second move as a function of Black's first move, then its third move as a function of Black's first and second moves taken together, etc.

Now suppose White does not have a winning strategy. We will describe a winning strategy for Black. Black's strategy is to play in such a way that White never has a winning strategy from that point on. (It is understood that a position in which White has won is considered to be a position from which point on White has a winning strategy.) Can Black do this? Certainly. Since by assumption White does not have a winning strategy at the outset, then it does not have a winning strategy after its first move. Now Black can find some move after which White still does not have a winning strategy, for if not, then by piecing together a winning strategy for the remainder of the game in response to each possible choice of Black, White could construct a winning strategy for the whole game. The same argument works throughout the game, and Black can always find a move after which White still does not have a winning strategy.

Clearly, White cannot win if Black plays by this strategy because the game has finite length. If the game ends with a win for White, then that is a position from which White has a winning strategy, which is just what Black's strategy always avoids. So this is a winning strategy for Black. To summarize, if White does not have a winning strategy, then Black does.

Clearly, White and Black cannot both have a winning strategy. Nobody knows which player in chess has a winning strategy (with the above proviso of draws being counted as wins for Black), but we may be sure that one of them does; and for our purposes that is the only important thing.[60] We say that a game in which one or the other player has a winning strategy is *determined*.

The essential character of the question does not change if we allow the set of possible moves to be infinite, although in general we must then use a choice principle to show that any game of finite length is determined.[61] If we consider games of infinite length, however, the situation is radically altered. The most general such game is defined by a set M of *moves*; an ordinal λ, which is the *length* of the game; and a subset A of ${}^{\lambda}M$, which is the *win set*.

We use 'Player I' and 'Player II', or simply 'I' and 'II', as labels for the hypothetical "players" of this game. Of course, these "players" do not exist, even in the weak sense attributed to mathematical entities; rather it is "their" plays and strategies that exist. We may avoid reference to players altogether by using 'I-play', 'II-play', 'I-strategy', and 'II-strategy' for sequences and functions of the appropriate types. Similarly, although we may refer to a "game" $\mathcal{G}_A$ corresponding to a set $A \subseteq {}^{\lambda}M$ to emphasize the context in which A is being considered, $\mathcal{G}_A$ is

[60] Although not important, it is perhaps interesting that the similar, but much simpler (with 5×10^{20} positions), game of checkers has been solved: either player can force a draw.

[61] The necessity of choice is easy to see. Suppose $\mathcal{S}$ is a set of nonempty sets. Consider the following game. The first player picks a member S of $\mathcal{S}$. The second player picks a member s of $\bigcup \mathcal{S}$. The second player wins iff $s \in S$. A strategy for the first player is essentially a member S of $\mathcal{S}$, and the second player can foil this by playing a member of S; so the first player does not have a winning strategy. If the game is determined, therefore, the second player has a winning strategy, and a winning strategy for the second player is a choice function for $\mathcal{S}$. Thus the statement that every game consisting of one move for each player is determined implies the axiom of choice.

not something different from A.

The most important class of games in this sense consists of those for which $\lambda = \omega$. The following definitions are specific to this class, but it is obvious how they should be modified for the general case.

(5.164) **Definition** [ZF]

1. *Composition/decomposition of plays*
   1. *Suppose $x \in {}^{\kappa}M$ and $y \in {}^{\kappa'}M$, where $\kappa, \kappa' \in \omega$ and $\kappa' \leqslant \kappa \leqslant \kappa' + 1$, or $\kappa = \kappa' = \omega$.*

      $$x * y \overset{\text{def}}{=} \{(2m, k) \mid (m, k) \in x\} \cup \{(2m+1, k) \mid (m, k) \in y\},$$

      as illustrated below:

      $$\begin{array}{rcccccc} x: & x_0 & & x_1 & & x_2 & \ldots \\ y: & & y_0 & & y_1 & & \ldots \\ x * y: & x_0 & y_0 & x_1 & y_1 & x_2 & \ldots \end{array}$$

   2. *Suppose $z \in {}^{\kappa}M$, where $\kappa \leqslant \omega$.*

      $$z^{\mathrm{I}} \overset{\text{def}}{=} \{(m, k) \mid (2m, k) \in z\}$$
      $$z^{\mathrm{II}} \overset{\text{def}}{=} \{(m, k) \mid (2m+1, k) \in z\},$$

      as illustrated below:

      $$\begin{array}{rcccccc} z: & z_0 & z_1 & z_2 & z_3 & z_4 & \ldots \\ z^{\mathrm{I}}: & z_0 & & z_2 & & z_4 & \ldots \\ z^{\mathrm{II}}: & & z_1 & & z_3 & & \ldots \end{array}$$

   Clearly, $z = z^{\mathrm{I}} * z^{\mathrm{II}}$.
2. *Strategies*
   1. *A* I*-strategy is a function $\sigma : {}^{<\omega}M \rightharpoonup M$ such that for each $n \in \omega$,* $\operatorname{dom}\sigma$ *contains every $z \in {}^{2n}M$ such that $\forall m < n\ z_{2m} = \sigma(z \restriction 2m)$.*
   2. *A* II*-strategy is a function $\tau : {}^{<\omega}M \rightharpoonup M$ such that for each $n \in \omega$,* $\operatorname{dom}\tau$ *contains every $z \in {}^{2n+1}M$ such that $\forall m < n\ z_{2m+1} = \tau(z \restriction (2m+1))$.*
3. *Composition of strategies and plays*
   1. *Suppose σ is a* I*-strategy and $y \in {}^{\kappa}M$ where $\kappa \leqslant \omega$.*
      1. *$\sigma * y \overset{\text{def}}{=} x * y$, where $x \in {}^{1+\kappa}M$ is given recursively by*

         $$x_m = \sigma\big((x \restriction m) * (y \restriction m)\big) = \sigma\langle x_0, y_0, x_1, y_1, \ldots, x_{m^-}, y_{m^-}\rangle,$$

         as illustrated below.

         $$\begin{array}{rccccccc} & x_0 & & x_1 & & x_2 & & \\ x: & \sigma\langle\rangle & & \sigma\langle x_0, y_0\rangle & & \sigma\langle x_0, y_0, x_1, y_1\rangle & \ldots \\ y: & & y_0 & & y_1 & & \ldots \\ \sigma * y: & \sigma\langle\rangle & y_0 & \sigma\langle x_0, y_0\rangle & y_1 & \sigma\langle x_0, y_0, x_1, y_1\rangle & \ldots \end{array}$$

      2. $\vec{\sigma} y \overset{\text{def}}{=} (\sigma * y)^{\mathrm{I}}$.[62]

[62]Thus, $\operatorname{dom}\vec{\sigma} = {}^{\leqslant\omega}M$ and $\vec{\sigma}^{\to}({}^{\kappa}M) \subseteq {}^{1+\kappa}M$. In the above definition of $\sigma * y$, $x = \vec{\sigma} y$.

2. *Suppose* τ *is a* II*-strategy and* $x \in {}^{\kappa}M$ *where* $\kappa \leqslant \omega$*.*

    1. $x * \tau \stackrel{\text{def}}{=} x * y$*, where* $y \in {}^{\kappa}M$ *is given recursively by*

    $$y_m = \tau\big((x \restriction (m+1)) * (y \restriction m)\big) = \tau\langle x_0, y_0, \ldots, x_{m^-}, y_{m^-}, x_m\rangle,$$

    as illustrated below.

    $$\begin{array}{rccccccc} x: & x_0 & & x_1 & & x_2 & & \ldots \\ & & y_0 & & y_1 & & y_2 & \\ y: & & \tau\langle x_0\rangle & & \tau\langle x_0, y_0, x_1\rangle & & \tau\langle x_0, y_0, x_1, y_1, x_2\rangle & \ldots \\ x * \tau: & x_0 & \tau\langle x_0\rangle & x_1 & \tau\langle x_0, y_0, x_1\rangle & x_2 & \tau\langle x_0, y_0, x_1, y_1, x_2\rangle & \ldots \end{array}$$

    2. $\vec{\tau}x \stackrel{\text{def}}{=} (x * \tau)^{\mathrm{II}}$.

4. *Composition of strategies: Suppose* σ *is a* I*-strategy and* τ *is a* II*-strategy. Then* $\sigma * \tau \stackrel{\text{def}}{=} x * y$*, where*

    1. $x, y \in {}^{\omega}M$*;*
    2. $x = \vec{\sigma}y$*; and*
    3. $y = \vec{\tau}x$*.*

The circularity of (5.164.4) is only apparent. x and y are defined recursively:

$$\begin{aligned} x_n &= \sigma\big((x \restriction n) * (y \restriction n)\big) \\ y_n &= \tau\big((x \restriction (n+1)) * (y \restriction n)\big). \end{aligned}$$

In the interest of clarity, we preferentially use 'σ' and 'τ' for I- and II-strategies, respectively.

The game $\mathcal{G}_A$ is defined by the stipulation that Player I wins iff $x * y \in A$, where x and y are the respective sequences in ${}^{\omega}M$ played by I and II.

(5.165) **Definition** [ZF] *Suppose* $A \subseteq {}^{\omega}M$*.*

1. *A* I*-strategy* σ *is* winning for A $\stackrel{\text{def}}{\Longleftrightarrow}$ $\forall y \in {}^{\omega}M \ \sigma * y \in A$*.*
2. *A* II*-strategy* τ *is* winning for A $\stackrel{\text{def}}{\Longleftrightarrow}$ $\forall x \in {}^{\omega}M \ x * \tau \notin A$*.*

A *is* determined *or* determinate $\stackrel{\text{def}}{\Longleftrightarrow}$ *there exists a winning* I*- or* II*-strategy for* A*.*

Note that there cannot exist both a winning I-strategy σ and a winning II-strategy τ, because in that case, letting $x = (\sigma * \tau)^{\mathrm{I}}$ and $y = (\sigma * \tau)^{\mathrm{II}}$, $\sigma * \tau = \sigma * y \in A$ and $\sigma * \tau = x * \tau \notin A$.

5.8.1 Borel determinacy

Unless otherwise stated, ${}^{\omega}M$ is given the standard topology, which is the product topology derived from the discrete topology on M, i.e., the topology for which the intervals $I_s = \{z \in {}^{\omega}M \mid s \subseteq z\}$ ($s \in {}^{<\omega}M$) constitute a base. If M is uncountable, this is not a Polish space, but the definition (5.88) of the Borel hierarchy is nonetheless appropriate and useful. The main result of this section is the theorem of ZFC that for any set M, any Borel subset of ${}^{\omega}M$ is determined; but equally important to our understanding of the countably infinitary are the metamathematical considerations surrounding this result, and we will mention some of these along the way.

For the purpose of discussing the determinacy of Borel subsets of a sequence space, it is convenient to use the following terminology. Recall that for any s, $\mathrm{Seq}(s)$ iff s is a finite sequence, i.e., s is a function and $\mathrm{dom}\, s$ is a finite ordinal. Remember that even in the context of a pure set theory such as ZF, we sometimes informally refer to predicate such as 'Seq' in terms of the class of objects satisfying it.

Recall[3.181] that a sequence tree T on a set M is a subset of ${}^{<\omega}M$ closed under initial segment, i.e., $\forall s \in T\, \forall m < |s|\ s \restriction m \in T$.[63] Recall[3.183] the following definitions for a sequence tree T.

1. Suppose $s \in T$.
    1. $T_s \overset{\text{def}}{=} \{t \in \mathrm{Seq} \mid s^\frown t \in T\}$.
    2. $T_{(s)} \overset{\text{def}}{=} \{t \in T \mid t \subseteq s \vee s \subseteq t\} = \{t \in T \mid t \subseteq s\} \cup (s^\frown T_s)$.
2. Suppose $n \in \omega$. $T|n \overset{\text{def}}{=} \{s \in T \mid |s| \leqslant n\}$. If $\mathcal{T}$ is a set of trees then $\mathcal{T}|n \overset{\text{def}}{=} \{T|n \mid T \in \mathcal{T}\}$.
3. $[T] \overset{\text{def}}{=}$ the set of infinite branches of T.

(5.166) **Definition** [ZF]

1. *A sequence tree T is* good $\overset{\text{def}}{\iff}$ *every $s \in T$ has a proper extension in T.*[64] *Note that every branch of a good tree is infinite.*
2. *Suppose T is a good tree.*
    1. *The* standard topology *on $[T]$ is the topology for which $\{[T_{(s)}] \mid s \in T\}$ is a base. Hence, for $X \subseteq [T]$,*
        1. *X is* open (relative to $[T]$) $\overset{\text{def}}{\iff}$ $X = \bigcup\{[T_{(s)}] \mid [T_{(s)}] \subseteq X\}$.
        2. *X is* closed (relative to $[T]$) $\overset{\text{def}}{\iff}$ $[T]\backslash X$ *is open.*
        3. *X is* clopen (relative to $[T]$) $\overset{\text{def}}{\iff}$ X *is open and closed.*
    2. *Suppose $X \subseteq [T]$ is closed in the standard topology. Then $T^X \overset{\text{def}}{=} \{s \in T \mid \exists x \in X\ s \subseteq x\}$. Note that T^X is a good tree, and $[T^X] = X$.*
3. *Suppose T, T' are trees and $T' \subseteq T$.*
    1. *T' is a* I-imposed subtree *of T* $\overset{\text{def}}{\iff}$ *for all $s \in T'$*
        1. *if s has a proper extension in T then s has a proper extension in T'.*
        2. *if $|s|$ is odd then every immediate extension of s in T is in T'.*
    2. *T' is a* II-imposed subtree *of T* $\overset{\text{def}}{\iff}$ *for all $s \in T'$*
        1. *if s has a proper extension in T then s has a proper extension in T'.*
        2. *if $|s|$ is even then every immediate extension of s in T is in T'.*
4. *Suppose T is a tree and σ is a subtree of T.*

[63] Note that 0 is in every nonempty sequence tree.

[64] Good trees are also called *pruned*, suggesting that all nodes without proper extensions have been lopped off. We feel this terminology is grammatically misleading in that in practice a good tree is not usually the result of such a pruning operation. Given that a tree with the property that every node has incompatible extensions is called *perfect*, it seems reasonable to use *good* in the sense defined here.

1. *σ is a* I-strategy $\overset{\text{def}}{\iff}$ *σ is a minimal* I*-imposed subtree of T, i.e., every $s \in \sigma$ of even length that has a proper extension in T has exactly one immediate extension in σ.*
2. *σ is a* II-strategy $\overset{\text{def}}{\iff}$ *σ is a minimal* II*-imposed subtree of T, i.e., every $s \in \sigma$ of odd length that has a proper extension in T has exactly one immediate extension in σ.*
3. *Suppose σ and τ are respectively a* I*- and a* II*-strategy in a nonempty good tree T. Clearly, $\sigma \cap \tau$ is a nonempty good tree with a single branch. $\sigma * \tau \overset{\text{def}}{=}$ the unique branch of $\sigma \cap \tau$. Note that $[\sigma] \cap [\tau] = \{\sigma * \tau\}$.*
4. *Suppose T is a sequence tree. Let* P *be* I *or* II.
   1. *$\mathcal{S}^T_{\mathrm{P}}$ is the set of* P*-strategies in T.*
   2. $\bar{\mathcal{S}}^T_{\mathrm{P}} \overset{\text{def}}{=} \bigcup_{n\in\omega} \mathcal{S}^{T|n}_{\mathrm{P}}$.
   3. $\mathcal{S}^T \overset{\text{def}}{=} \mathcal{S}^T_{\mathrm{I}} \cup \mathcal{S}^T_{\mathrm{II}}$ *and* $\bar{\mathcal{S}}^T \overset{\text{def}}{=} \bar{\mathcal{S}}^T_{\mathrm{I}} \cup \bar{\mathcal{S}}^T_{\mathrm{II}}$.
5. *A* game $\overset{\text{def}}{=}$ *a 2-sequence $\langle T, X\rangle$, where T is a nonempty good tree. Often $X \subseteq [T]$, or $X \subseteq {}^\omega M$ if T is a tree on M, but we do not require this.*
6. *σ is a* winning strategy *in a game $\langle T, X\rangle$* $\overset{\text{def}}{\iff}$
   1. *σ is a* I*-strategy in T and $[\sigma] \subseteq X$; or*
   2. *σ is a* II*-strategy in T and $[\sigma] \cap X = 0$.*

   $\langle T, X\rangle$ is a win *for* I *(*II*)* $\overset{\text{def}}{\iff}$ *there is a winning* I*- (*II*-)strategy in $\langle T, X\rangle$. Similarly, $\langle T, X\rangle$ is a* loss *for* I *(*II*)* $\overset{\text{def}}{\iff}$ *there is a winning* II*- (*I*-)strategy in $\langle T, X\rangle$.*
7. *Suppose T is a good tree. The* I- (II-)nonlosing subtree of T for X *is the set of $p \in T$ such that for every sequence $q \subseteq p$, $\langle T_{(q)}, X\rangle$ is not a loss for* I *(*II*).*

Suppose T is a good tree. Clearly, if σ is a winning I-strategy in $G = \langle T, X\rangle$ then for every II-strategy τ in T, $\sigma * \tau \in X$; similarly, if τ is a winning II-strategy in G then for every I-strategy σ in T, $\sigma * \tau \notin X$. Hence, there cannot exist both a winning I- and a winning II-strategy in G.

(5.167) **Theorem** [ZFC^-] *Suppose T is a good tree and $\langle T, X\rangle$ is not a loss for* I *(*II*). Let T' be the* I*- (*II*-)nonlosing subtree of T for X.*

1. *T' is a* I*- (*II*-)imposed subtree of T.*
2. *$\langle T', X\rangle$ is not a loss for* I *(*II*).*

Proof We will treat the case of I. The case of II is homologous.

1 Suppose $p \in T'$ and $|p|$ is odd (i.e., it is II's turn to move). We must show that for every immediate extension p' of p in T, $p' \in T'$. For this it suffices to show that $\langle T_{(p')}, X\rangle$ is not a loss for I. Suppose toward a contradiction that this is not the case for some such p'. Let σ be a winning II-strategy in $\langle T_{(p')}, X\rangle$. Then σ is a winning II-strategy in $T_{(p)}$; contradiction.

Now suppose $p \in T'$ and $|p|$ is even (I's turn to move). We must show that for some immediate extension p' of p in T, $p' \in T'$. Let P be the set of immediate extensions p in T, and suppose toward a contradiction that for each $p' \in P$ there is a winning II-strategy in $T_{(p')}$. By AC there is a function f such that for every such $p' \in P$, fp' is a winning II-strategy in $T_{(p')}$. Let $\sigma = \bigcup_{p' \in P} fp'$.[65] Since, by hypothesis, for each $p' \in P$, $[fp'] \cap X = 0$, $[\sigma] \cap X = 0$, i.e., σ is a winning II-strategy in $\langle T, X\rangle$; contradiction.

2 Suppose toward a contradiction that $\langle T', X\rangle$ is a loss for I. Let σ' be a winning II-strategy in $\langle T', X\rangle$. We will describe a winning II-strategy σ in $\langle T, X\rangle$, the existence of which contradicts the assumption that $\langle T, X\rangle$ is not a loss for I. Let II play as follows: II plays according to σ' unless and until it is confronted with a position $p \notin T'$. The first time this happens (if ever) II chooses a winning strategy in $\langle T_{(p)}, X\rangle$ and follows it for the rest of the game. Note that AC is used to show that an overall strategy σ as described here exists. It is easy to show that σ is a winning II-strategy in $\langle T, X\rangle$. $\square$[5.167]

Note that I has a winning strategy in $\langle T, X\rangle$ iff for some $p \in T$ of length 1, II has a winning strategy in $\langle T_p, [T_p]\backslash X_p\rangle$, where X_p is the set of ω-sequences x such that $p^\frown x \in X$.[5.166.6.1] Similarly, assuming AC_ω, II has a winning strategy in $\langle T, X\rangle$ iff for every $p \in T$ of length 1, I has a winning strategy in $\langle T_p, [T_p]\backslash X_p\rangle$.

(5.168) *Thus, if Γ is a class of subsets of ${}^\omega M$ closed under the operations $X \mapsto X_p$, $p \in {}^1M$—as is the case for any of the complexity classes we consider—and $\breve{\Gamma} = \{X \subseteq {}^\omega M \mid {}^\omega M\backslash X \in \Gamma\}$ is the dual class, then—assuming AC_ω—Γ-determinacy implies $\breve{\Gamma}$-determinacy.*

The following theorem was proven by Gale and Stewart in their seminal 1953 article introducing games of the sort under consideration here.

(5.169) **Theorem** [ZFC^-] *Suppose T is a good tree and $A \cap [T]$ is open or closed (in the standard topology on $[T]$). Then $\langle T, A\rangle$ is determined.*

Proof Suppose $A \cap [T]$ is open, and suppose $\langle T, A\rangle$ is not a win for I. Let T' be the II-nonlosing subtree of T for A. Let $\sigma \subseteq T'$ be any minimal II-imposed subtree of T'.[66] Note that σ is a II-strategy for T as well as for T'. We claim that σ is a winning strategy for II in $\langle T, A\rangle$. Suppose toward a contradiction that it is not. Let x be such that $z = x * \sigma \in A$. Then for some sequence $p \subseteq z$, $\forall y \in [T]\,(y \supseteq p \to y \in A)$. Hence, $\langle T_{(p)}, A\rangle$ is a win for I, so $p \notin T'$, which contradicts the fact that $p \in \sigma \subseteq T'$.

Thus, if $\langle T, A\rangle$ is not a win for I then it is a win for II, so it is determined. To handle the case that $S \cap [T]$ is closed we may reverse the roles of I and II, or we may invoke the general principle (5.168). $\square$[5.169]

The Gale-Stewart theorem was extended to $\mathbf{\Sigma}^0_2$ by Philip Wolfe in 1955.

(5.170) **Theorem** [ZFC^-] *Suppose T is a good tree and $A \cap [T]$ is $\mathbf{\Sigma}^0_2$ or $\mathbf{\Pi}^0_2$. Then $\langle T, A\rangle$ is determined.*

Proof It suffices to treat the case that A is $\mathbf{\Sigma}^0_2$.[5.168] Suppose $A = \bigcup_{n\in\omega} A_n$, where each $A_n \subseteq [T]$ is $\mathbf{\Pi}^0_1$, i.e., closed. Let $T^n = \{p \in T \mid \exists a \in A_n\ p \subseteq a\}$ be the (good) tree associated with A_n in the usual way, so that $A_n = [T^n]$.

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[65] Remember that we are regarding strategies as trees, and each fp' is a minimal II-imposed subtree of $T_{(p')}$. Since P contains every immediate extension of p, σ is a II-imposed subtree of $T_{(p)}$ and it is clearly minimal, so it is a II-strategy. '$\bigcup_{p' \in P} fp'$' is the set-theoretical formulation of the ludological description: II responds to the move of I creating the position p' by playing according to fp' for the rest of the game.

[66] Choice is required in general, although not if T is a tree on ω or any other naturally wellordered set.

Suppose $\langle T, A\rangle$ is not a win for I. We will describe a winning II-strategy in $\langle T, A\rangle$. Let T' be the II-nonlosing subtree of T for A.

(5.171) **Claim** *Suppose $p \in T'$ and $n \in \omega$. $\langle T'_p, A_n\rangle$ is a win for* II.

Proof By (5.169) it suffices to show that $\langle T'_p, A_n\rangle$ is not a win for I. Suppose toward a contradiction that σ' is a I-strategy in T'_p and $[\sigma'] \subseteq A_n$ (viewing a strategy as a tree). Then $\sigma' \subseteq T^n$. Let σ be the following I-strategy in T_p:

> I plays according to σ' as long as II plays on T'. If and when II plays to a position $q \notin T'$, I plays a winning strategy in $\langle T_q, A\rangle$, which exists because T' is the II-nonlosing subtree of T for A.[67]

Let $z \in [\sigma]$ be a result of I following this strategy. If II never played off T', then $z \in [\sigma']$, so $z \in A_n \subseteq A$. Otherwise, $z \in [\tau]$, where τ is the strategy followed by I after II strayed off T', so $z \in A$.

Thus, σ is a winning I-strategy in $\langle T_p, A\rangle$, contrary to the assumption that $p \in T'$. $\square$[5.171]

Using the claim, let σ be a II-strategy in T' with the following features:

> By hypothesis, $\langle T, A\rangle$ is not a win for I, so $0 \in T'$. Let $p_0 = 0$, let σ^0 be a winning II-strategy in $\langle T'_{(p_0)}, A_0\rangle$, and let II follow σ^0 for at least one move until a position $p_1 \in \sigma^0$ is reached such that $p_1 \notin T^0$, which must happen, as $[\sigma^0] \cap [T^0] = [\sigma^0] \cap A_0 = 0$. Now let σ^1 be a winning II-strategy in $\langle T'_{(p_1)}, A_1\rangle$ and let II follow σ^1 for at least one move until a position $p_2 \in \sigma^1$ is reached such that $p_2 \notin T^1$. Continue in this fashion *ad infinitum*.[68]

Clearly, σ is a winning II-strategy in $\langle T', A\rangle$. Since T' is a II-imposed subtree of T, σ is also a II-strategy in T and therefore a winning II-strategy in $\langle T, A\rangle$, as desired. $\square$[5.170]

In 1964, Morton Davis proved $\mathbf{\Sigma}^0_3$-determinacy. We will use the following lemma.

(5.172) **Theorem** [ZFC^-] *Suppose T is a good tree, $B \subseteq A \subseteq [T]$, B is $\mathbf{\Pi}^0_2$, and $\langle T, A\rangle$ is not a loss for* II. *Then there is a* II-*imposed $T^* \subseteq T$ such that*

1. $[T^*] \cap B = 0$; *and*
2. $\langle T^*, A\rangle$ *is not a loss for* II.

Proof Let T' be the II-nonlosing subtree of T for A.

(5.173)

1. *For $p \in T'$, say that p is* good *iff there is a* II-*imposed $T^* \subseteq T'_{(p)}$ such that*
   1. $[T^*] \cap B = 0$; *and*
   2. $\langle T^*, A\rangle$ *is not a loss for* II.
2. *Let G be the set of good positions in T'.*
3. *Let $\mathcal{T}$ be a function such that $\operatorname{dom}\mathcal{T} = G$ and for each $p \in G$, $\mathcal{T}p$ is a tree T^* witnessing that p is good according to (5.173.1).*

[67] Note that AC is used to show that such a σ exists.

[68] Note the implicit use of AC to provide the function $p_n \mapsto \sigma^n$.

Note that the the theorem is the statement that 0 is good.

Suppose $B = \bigcap_{n\in\omega} B_n$, where each B_n is open. For each $n \in \omega$ and $p \in T'$, let

$$C_n^p = \{z \in [T'] \mid \exists_{\mathrm{Seq}} q\,(p \subseteq q \subseteq z, [T'_{(q)}] \subseteq B_n, \text{ and } q \text{ is not good})\}. \tag{5.174}$$

(5.175) **Claim** *Suppose $p \in T'$, $n \in \omega$, and $\langle T'_{(p)}, A \cup C_n^p\rangle$ is not a win for* I. *Then p is good.*

Proof Let T'' be the II-nonlosing subtree of $T'_{(p)}$ for $A \cup C_n^p$. Let

$$S = \{q \in T'' \mid q \subseteq p \vee [T'_{(q)}] \nsubseteq B_n\}.$$

Note that S is a tree, not necessarily good. Let R be the set of $q \in T''\backslash S$ such that the immediate predecessor of q is in S. The members of R are pairwise incompatible, and for each $q \in R$, $[T'_{(q)}] \subseteq B_n$, so q is good (otherwise $\langle T'_{(q)}, C_n^p\rangle$ is already lost for II at q, so $q \notin T''$), and $\mathcal{T}q$ is a II-imposed subtree of $T'_{(q)}$ witnessing that q is good.[5.173.3] Let

$$T^* = S \cup \bigcup_{q\in R} \mathcal{T}q.$$

Since T'' is a II-imposed subtree of $T'_{(p)}$, and for every $q \in S$, every immediate successor of q in T'' is in $S \cup R$, and $\forall q \in R\ p \subseteq q$, T^* is a II-imposed subtree of $T'_{(p)}$.

We claim that T^* witnesses that p is good. To verify (5.173.1.1) suppose toward a contradiction that $z \in [T^*] \cap B$. Then $z \in B_n$. Since B_n is open, for some sequence $q \subseteq z$, $[T'_{(q)}] \subseteq B_n$. Let q_0 be the $\subseteq$-least $q \subseteq z$ such that q properly extends p and $[T'_{(q)}] \subseteq B_n$. Then $q_0 \in R$ and $z \in [\mathcal{T}q_0]$, so $z \notin B$.[5.173.3]

To verify (5.173.1.2) suppose toward a contradiction that σ is a winning I-strategy in $\langle T^*, A\rangle$. Note that for any $q \in \sigma$, $[T'_{(q)}] \nsubseteq B_n$, because otherwise there is a minimal $q \in \sigma$ such that q properly extends p and $[T'_{(q)}] \subseteq B_n$, and by definition $T^*_{(q)} = \mathcal{T}q$, so $\sigma_{(q)}$ is a I-strategy in $\mathcal{T}q$ that witnesses that $\langle \mathcal{T}q, A\rangle$ is a loss for II, since $[\sigma_{(q)}] \subseteq [\sigma] \subseteq A$, contradicting (5.173.3). $\square$[5.175]

Now suppose toward a contradiction that the conclusion of the theorem is false, i.e., 0 is not good. Let σ be a I-strategy in T' with the following features:

> Let $p_0 = 0$. By hypothesis, p_0 is not good. Let[5.175] σ^0 be a winning I-strategy in $\langle T'_{(p_0)}, A \cup C_0^{p_0}\rangle$. I follows σ^0 unless and until a position p_1 is reached such that $[T'_{(p_1)}] \subseteq B_0$ and p_1 is not good. If and when this first occurs, let σ^1 be a winning I-strategy in $\langle T'_{(p_1)}, A \cup C_1^{p_1}\rangle$. I now follows σ^1 unless and until a position p_2 is reached such that $[T'_{(p_2)}] \subseteq B_1$ and p_2 is not good. Continue *ad infinitum.*[69]

Note that if, for some $n \in \omega$, while I is following σ^n, a position p is never reached such that $[T'_{(p)}] \subseteq B_n$ and p is not good, then I follows σ^n for the rest of the game. The resulting play $z \in [\sigma]$ is in $A \cup C_n^{p_n}$, since σ^n is a winning I-strategy for this game, but $z \notin C_n^{p_n}$. Hence $z \in A$.

On the other hand, if, for every $n \in \omega$, while I is following σ^n, a position p *is* eventually reached such that $[T'_{(p)}] \subseteq B_n$ and p is not good, then the resulting play $z \in [\sigma]$ is in B_n for every n, so $z \in \bigcap_{n\in\omega} B_n = B \subseteq A$.

Hence, σ is a winning I-strategy in $\langle T', A\rangle$, contrary to hypothesis. $\square$[5.172]

[69] AC is of course used to provide a function $p_n \mapsto \sigma^n$.

(5.176) **Theorem** [ZFC$^-$] *Suppose T is a good tree and $A \cap [T]$ is $\mathbf{\Sigma}^0_3$ or $\mathbf{\Pi}^0_3$. Then $\langle T, A\rangle$ is determined.*

Proof Each case implies the other,[5.168] so suppose A is $\mathbf{\Sigma}^0_3$, say $A = \bigcup_{n\in\omega} A_n$, where each $A_n \subseteq [T]$ is $\mathbf{\Pi}^0_2$. Suppose $\langle T, A\rangle$ is not a win for I; hence, not a loss for II. Let $T \supseteq T^0 \supseteq T^1 \supseteq \cdots$ be a sequence of good trees with the following features:

Let[5.172] T^* be a II-imposed subtree of T such that

1. $[T^*] \cap A_0 = 0$; and
2. $\langle T^*, A\rangle$ is not a loss for II.

For each $p \in T^*$ of length 1 there exists an immediate extension p' of p such that $\langle T^*_{(p')}, A\rangle$ is not a loss for II. Let $S \subseteq T*$ be a set of positions of length 2 containing exactly one such immediate extension of each position in T^* of length 1. Let $T^0 = \bigcup_{p\in S} T^*_{(p)}$. Then T^0 is a II-imposed subtree of T^* that is minimal as regards its members of length 2, and is therefore a strategy for II's first move; and

$\langle T^0, A\rangle$ *is not a loss for* II.

Now let[5.172] T^* be a II-imposed subtree of T^0 such that

1. $[T^*] \cap A_1 = 0$; and
2. $\langle T^*, A\rangle$ is not a loss for II.

Let $S \subseteq T*$ be a set of positions of length 4 containing exactly one immediate extension of each position in T^* of length 3, such that for each $p \in S$, $\langle T^*_{(p)}, A\rangle$ is not a loss for II, and let $T^1 = \bigcup_{p\in S} T^*_{(p)}$. Continue *ad infinitum.*

Let $\sigma = \bigcap_{n\in\omega} T^n$. Then σ is a II-strategy in T, and $[\sigma] \subseteq [T^n]$ for each $n \in \omega$, so $[\sigma] \cap A_n = 0$ for each n, and $[\sigma] \cap A = [\sigma] \cap \bigcup_{n\in\omega} A_n = 0$, i.e., σ is a winning II-strategy in $\langle T, A\rangle$, as desired. □[5.176]

In 1972, Jeffrey Paris proved $\mathbf{\Sigma}^0_4$-determinacy. Paris's proof had a new element: it used the Power axiom. As noted above,[5.176] $\mathbf{\Sigma}^0_3$-determinacy is a theorem of ZFC$^-$, and it was natural to wonder whether $\mathbf{\Sigma}^0_4$-determinacy could be proved in ZFC$^-$. Harvey Friedman soon showed that the determinacy of certain simple combinations of $\mathbf{\Sigma}^0_5$ sets could not be proved in ZFC$^-$, and this was improved by D. A. (Tony) Martin to show that ZFC$^- \nvdash \ulcorner\mathbf{\Sigma}^0_4$-determinacy$\urcorner$.

This result was greeted with great enthusiasm by the set theory community, because it represented the first instance of a straightforward proposition about countably infinitary objects, viz., $\mathbf{\Sigma}^0_4$-determinacy, interesting in the ordinary mathematical way (not because of an overt logical connotation), provable in ZFC but not provable in ZFC$^-$.

The Friedman-Martin method also applied to show that $\mathbf{\Sigma}^0_5$-determinacy cannot be proved in ZFC$^-$+ $\ulcorner\mathcal{P}\omega$ exists$\urcorner$, $\mathbf{\Sigma}^0_6$-determinacy cannot be proved in ZFC$^-$+ $\ulcorner\mathcal{P}^2\omega$ exists$\urcorner$, etc. For each step up the Borel hierarchy, one more iteration of the powerset operation is needed to prove determinacy. This was shown to hold through all the levels of the Borel hierarchy. In 1975, Martin succeeded in proving Borel determinacy using ZFC$^-$ plus the assumption of the existence of $\mathcal{P}^\alpha\omega$ for all countable ordinals α. Within the proof, of course, each level of the hierarchy required one more iteration of $\mathcal{P}$.

(5.177) **Theorem** [ZFC] *Suppose T is a good tree and $X \subseteq [T]$ is Borel. Then $\langle T, X\rangle$ is determined.*

The proof of this theorem, even with Martin's subsequent simplification of the original version, is rather involved and is relegated to Note 10.19, so as not to impede the flow of the narrative.

The analysis of the use of the powerset operation in Martin's proof is given in Note 10.20, in terms of the operation $\mathcal{Q}$, defined in (10.135) so that

$$\mathcal{Q}M = \mathcal{F}M \cup \mathcal{F}\mathcal{P}\mathcal{F}M,$$

where $\mathcal{F}$ is the finitary closure operation (definable in ZF^-). The conclusion is as follows.

(5.178) **Theorem** [ZFC^-] *Suppose M is a set, $\rho < \omega_1$, $\mathcal{Q}^\rho M$ exists, and $X \subseteq {}^\omega M$ is $\mathbf{\Sigma}^0_{1+\rho+2}$. Then X is determined.*

The negative result of Friedman (with Martin's refinement) is given in Section 7.6.2.

(5.179) **Theorem** [ZFC] *There exists $A \subseteq {}^\omega\omega$ such that A is not determinate.*

Proof Using AC, let $\kappa = 2^\omega$, and let $\langle \sigma_\alpha \mid \alpha \in \kappa\rangle$ and $\langle \tau_\alpha \mid \alpha \in \kappa\rangle$ enumerate the I- and II-strategies, respectively. Let $\langle a^1_\alpha \mid \alpha \in \kappa\rangle$ and $\langle a^2_\alpha \mid \alpha \in \kappa\rangle$ be such that for each $\alpha \in \kappa$,

1. $a^1_\alpha = \sigma_\alpha * y$ for some $y \in {}^\omega\omega$ and $a^1_\alpha \neq \alpha^2_\beta$ for any $\beta < \alpha$.
2. $a^2_\alpha = x * \tau_\alpha$ for some $x \in {}^\omega\omega$ and $a^2_\alpha \neq \alpha^1_\beta$ for any $\beta \leqslant \alpha$.

Such sequences exist by virtue of AC and the fact that the maps $y \mapsto \sigma * y$ and $x \mapsto x * \tau$ are injective, so $\operatorname{Card}\{\sigma * y \mid y \in {}^\omega\omega\} = \operatorname{Card}\{x * \tau \mid x \in {}^\omega\omega\} = \kappa$. Let $A^1 = \{a^1_\alpha \mid \alpha \in \kappa\}$ and $A^2 = \{a^2_\alpha \mid \alpha \in \kappa\}$. Then $A^1 \cap A^2 = 0$. Let A be any subset of ${}^\omega\omega$ that includes A^2 and is disjoint from A^1, e.g., A^2. Suppose σ is a I-strategy. Then $\sigma = \sigma_\alpha$ for some $\alpha \in \kappa$, so there exists $y \in {}^\omega\omega$ such that $\sigma * y = \sigma_\alpha * y = a^1_\alpha \in A^1$. Hence $\sigma * y \notin A$, and σ is not a winning I-strategy for A. Similarly, there is no winning II-strategy for A. □5.179

5.9 Suslin's operation and analytic sets

Definition [ZF] *Suppose $A = \langle A_s \mid s \in {}^{<\omega}\omega\rangle$ is an ${}^{<\omega}\omega$-indexed family of sets.* operation A *or* Suslin's operation *applied to A* $\overset{\text{def}}{=}$

$$\mathcal{A}A \overset{\text{def}}{=} \bigcup_{x \in {}^\omega\omega} \bigcap_{n \in \omega} A_{x \restriction n}.$$

Note that if we let $B_s = \bigcap\{A_{s\restriction n} \mid n \in |s|\}$, then $\forall s, t \in {}^{<\omega}\omega\,(s \subseteq t \rightarrow B_s \supseteq B_t)$, and $\mathcal{A}\langle B_s \mid s \in {}^{<\omega}\omega\rangle = \mathcal{A}A$.

(5.180) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$] *Suppose X is a Polish space and $A \subseteq X$. The following are equivalent.*

1. *A is analytic.*
2. *$A = \mathcal{A}\langle B_s \mid s \in {}^{<\omega}\omega\rangle$, where each B_s is a closed subset of X.*
3. *$A = \mathcal{A}\langle B_s \mid s \in {}^{<\omega}\omega\rangle$, where each B_s is a Borel subset of X.*

Proof 1 → 2 Suppose $A \subseteq X$ is analytic. Let $f : {}^{\omega}\omega \to X$ be continuous such that $A = \operatorname{im} f$. Suppose $x \in {}^{\omega}\omega$, $b \in X$, and $fx \neq b$. Let M and N be disjoint open neighborhoods of fx and b, respectively. Then $f^{\leftarrow}M$ is open and contains x. Let s be an initial segment of x such that $I_s \subseteq f^{\leftarrow}M$. Then $f^{\rightarrow}I_s \cap N = 0$, so $b \notin \overline{(f^{\rightarrow}I_s)}$. It follows that for any $x \in {}^{\omega}\omega$

$$\bigcap_{n\in\omega} \overline{(f^{\rightarrow}I_{x\restriction n})} \subseteq \{fx\} \subseteq \bigcap_{n\in\omega} f^{\rightarrow}I_{x\restriction n} \subseteq \bigcap_{n\in\omega} \overline{(f^{\rightarrow}I_{x\restriction n})},$$

so $A = \mathcal{A}\langle\overline{(f^{\rightarrow}I_s)} \mid s \in {}^{<\omega}\omega\rangle$.

2 → 3 Trivial.

3 → 1 Suppose B_s is Borel for each $s \in {}^{<\omega}\omega$ and $A = \mathcal{A}\langle B_s \mid s \in {}^{<\omega}\omega\rangle$. Then

$$a \in A \leftrightarrow \exists x \in {}^{\omega}\omega\ a \in \bigcap_{n\in\omega} B_{x\restriction n} \leftrightarrow \exists x \in {}^{\omega}\omega\ \langle a, x\rangle \in B,$$

where $B = \{\langle a, x\rangle \mid a \in \bigcap_{n\in\omega} B_{x\restriction n}\}$. Clearly B is a Borel subset of $X \times {}^{\omega}\omega$, so A is analytic. $\square$5.180

(5.181) **Theorem** $[\mathsf{ZF} + \mathsf{AC}_{\omega}]$

1. *Suppose X is a separable Baire space (e.g., a Polish space). The set* $\mathbf{Baire}^X$ *of Baire subsets of X is closed under Suslin's operation $\mathcal{A}$.*
2. *Suppose μ is a σ-finite measure on a semiring $\mathfrak{S}$ in $\mathcal{P}\Omega$. Then the algebra $\mathfrak{M}^{\mu}$ of μ-measurable sets is closed under operation $\mathcal{A}$.*

Proof We will give the proofs both assertions simultaneously, placing text specific to assertion 1 and assertion 2 between round and square brackets respectively. Let ($\mathfrak{A} = \mathbf{Baire}^X$.) [$\mathfrak{A} = \mathfrak{M}^{\mu}$, and let $\bar{\mu}$ be the canonical extension of μ to $\mathfrak{A}$.]. Recall that $\mathfrak{A}$ is a σ-algebra. Suppose for each $s \in {}^{<\omega}\omega$, $A_s \in \mathfrak{A}$. Let $A = \mathcal{A}\langle A_s \mid s \in {}^{<\omega}\omega\rangle$. Without loss of generality, suppose that $s \subseteq t \to A_s \supseteq A_t$. Define by recursion A_s^{α} so that

$$\begin{aligned} A_s^0 &= A_s \\ A_s^{\alpha+1} &= \begin{cases} \bigcup_{n\in\omega} A^{\alpha}_{s^\frown\langle n\rangle} & \text{if } A_s^{\alpha}\backslash\bigcup_{n\in\omega} A^{\alpha}_{s^\frown\langle n\rangle} \text{ is (nonmeager) [non-null]} \\ A_s^{\alpha} & \text{otherwise} \end{cases} \\ A_s^{\alpha} &= \bigcap_{\beta\in\alpha} A_s^{\beta} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Note that for any $\alpha \in \mathrm{Ord}$

1. $A = \mathcal{A}\langle A_s^{\alpha} \mid s \in {}^{<\omega}\omega\rangle$;
2. for all $s, t \in {}^{<\omega}\omega$, $s \subseteq t \to A_s^{\alpha} \supseteq A_t^{\alpha}$; and
3. for all $s \in {}^{<\omega}\omega$
    1. $A_s^{\alpha+1} \subseteq A_s^{\alpha}$; and
    2. $A_s^{\alpha}\backslash A_s^{\alpha+1}$ is either empty or (nonmeager) [non-null].

4. if $\forall s \in {}^{<\omega}\omega\ A_s^\alpha \backslash A_s^{\alpha+1} = 0$ then $\forall \beta > \alpha\, \forall s \in {}^{<\omega}\omega\ A_s^\beta = A_s^\alpha$.

(5.182) **Claim** *For some $\alpha_0 < \omega_1$, for all $s \in {}^{<\omega}\omega$, $A_s^{\alpha_0} = A_s^{\alpha_0+1}$.*

Proof If not, then for some $s \in {}^{<\omega}\omega$, for uncountably many $\alpha < \omega_1$, $A_s^\alpha \backslash A_s^{\alpha+1}$ is (nonmeager) [non-null]. Since the sequence $\langle A_s^\alpha | \alpha \in \text{Ord}\rangle$ is monotone decreasing, this constitutes an uncountable set $\mathcal{N}$ of pairwise disjoint (nonmeager) [non-null] subsets of X. Since $\mathfrak{A}$ is a σ-algebra, $\mathcal{N} \subseteq \mathfrak{A}$.

(Suppose

1. $B, B' \in \mathcal{N}$;
2. $B \neq B'$;
3. $G \supseteq B$ and $G' \supseteq B'$ are open; and
4. $G \backslash B$ and $G' \backslash B'$ are meager.

Since $B \cap B' = 0$, $G \cap B'$ is meager, so $G \cap G'$ is meager. Since X is a Baire space, $G \cap G' = 0$ (as any comeager set is dense). Let S be a countable dense subset of X. Suppose $x \in S$. Then there is at most one $B \in \mathcal{N}$ such that for some open $G \supseteq B$, $G \backslash B$ is meager and $x \in G$. On the other hand, for any $B \in \mathcal{N}$ there exists an open $G \supseteq B$ such that $G \backslash B$ is meager, and since B is nonmeager, G is nonempty, so for some $x \in S$, $x \in G$. There is thus a function $f : S \overset{\text{sur}}{\rightharpoonup} \mathcal{N}$, and $\mathcal{N}$ is countable; contradiction.)

[For some $\epsilon > 0$, $\{B \in \mathcal{N} \mid \bar{\mu}B > \epsilon\}$ is uncountable. This is easily seen to contradict the σ-finiteness of μ.] $\square$[5.182]

Let[5.182] $\alpha_0 < \omega_1$ be such that $\forall s \in {}^{<\omega}\omega\ A_s^{\alpha_0} = A_s^{\alpha_0+1}$. By construction, for all $s \in {}^{<\omega}\omega$, $A_s^{\alpha_0} \backslash \bigcup_{n\in\omega} A_{s^\frown\langle n\rangle}^{\alpha_0}$ is (meager) [null]. Hence $M = \bigcup_{s\in{}^{<\omega}\omega} \left(A_s^{\alpha_0} \backslash \bigcup_{n\in\omega} A_{s^\frown\langle n\rangle}^{\alpha_0}\right)$ is (meager) [null].

Suppose $x \in A_0^{\alpha_0} \backslash M$. Define $f : \omega \to \omega$ as follows. Let $f0$ be the least m such that $x \in A_{\langle m\rangle}^{\alpha_0}$. Now let $f1$ be the least $m \in \omega$ such that $x \in A_{\langle f0,m\rangle}^{\alpha_0}$. In general, fn is the least m such that $x \in A_{f\restriction n^\frown\langle m\rangle}^{\alpha_0}$. Then $x \in \bigcap_{n\in\omega} A_{f\restriction n}^{\alpha_0}$, so $x \in \mathcal{A}\langle A_s^{\alpha_0} \mid s \in {}^{<\omega}\omega\rangle = \mathcal{A}\langle A_s \mid s \in {}^{<\omega}\omega\rangle = A$.

Conversely, if $x \in A$ then $x \in A_0^{\alpha_0}$. Thus, $A_0^{\alpha_0} \backslash M \subseteq A \subseteq A_0^{\alpha_0}$. Since α_0 is countable, $A_0^{\alpha_0} \in \mathfrak{A}$, so A differs from a member of $\mathfrak{A}$ by a (meager) [null] set, and A is therefore in $\mathfrak{A}$. $\square$[5.181]

It follows from (5.181) and (5.180) that every analytic set has the Baire property and is Lebesgue measurable (along with every coanalytic set and every set obtained from these by repeated applications of complementation, countable union, and operation $\mathcal{A}$). The following theorem shows that analytic sets have the perfect set property. Note that—unlike the Baire property and Lebesgue measurability—the fact that a set $A \subset X$ has the perfect set property does not necessarily imply that $X \backslash A$ has the perfect set property.

(5.183) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega$] *Suppose X is a Polish space and $A \subseteq X$ is analytic. A has the perfect set property.*

Proof The proof is similar to the proof of the Cantor-Bendixson theorem (5.138). Let $f : {}^\omega\omega \to X$ be continuous such that $\operatorname{im} f = A$. For $\alpha \in \text{Ord}$, let $T^\alpha \subseteq {}^\omega\omega$ be defined recursively as follows.

1. $T^0 = {}^{<\omega}\omega$.
2. $T^{\alpha+1} = \{s \in T^\alpha \mid |f^\rightarrow[T^\alpha_{(s)}]| > 1\}$.
3. If α is a limit ordinal then $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$.

Since ${}^{<\omega}\omega$ is countable, for some $\alpha_0 < \omega_1$, $T^{\alpha_0+1} = T^{\alpha_0}$. If $T^{\alpha_0} = 0$ then $A = f^{\rightarrow\omega}\omega = f^\rightarrow[T^0] \subseteq \bigcup_{\alpha<\alpha_0} \bigcup_{s\in T^\alpha\backslash T^{\alpha+1}} f^\rightarrow[T^\alpha_{(s)}]$, which is countable, since $\forall s \in (T^\alpha\backslash T^{\alpha+1})\ |f^\rightarrow[T_{(s)}]| \leqslant 1$.

Hence, if A is uncountable, T^{α_0} is nonempty. We will obtain a perfect subset of A. For each $s \in T^{\alpha_0}$, there exist $x_0, x_1 \in [T^{\alpha_0}_{(s)}]$ such that $fx_0 \neq fx_1$. Let N_0 and N_1 be disjoint neighborhoods of fx_0 and fx_1. Since f is continuous, there exist extensions s_0 and s_1 of s in T^{α_0} such that $f^\rightarrow T^{\alpha_0}_{(s_0)} \subseteq N_0$ and $f^\rightarrow T^{\alpha_0}_{(s_1)} \subseteq N_1$. Note that $f^\rightarrow[T^{\alpha_0}_{(s_0)}] \cap f^\rightarrow[T^{\alpha_0}_{(s_1)}] = 0$. For each $s \in T^{\alpha_0}$, let $\langle e_0 s, e_1 s\rangle$ be the first pair of extensions of s with this property in some fixed enumeration of ${}^{<\omega}\omega \times {}^{<\omega}\omega$.

Define $r : {}^{<\omega}2 \to T^{\alpha_0}$ so that

1. $r0 = 0$; and
2. for each $t \in {}^{<\omega}2$, $r(t^\frown\langle 0\rangle) = e_0(rt)$ and $r(t^\frown\langle 1\rangle) = e_1(rt)$.

Let $g : {}^{\omega}2 \to [T^{\alpha_0}]$ be such that $gx = \bigcup_{n\in\omega} r(x \upharpoonright n)$. Let $h = f \circ g$. h is a continuous injection of ${}^{\omega}2$ into A.

For convenience, let $C = {}^{<\omega}2$, regarded as a sequence tree, so $[C] = {}^{\omega}2$.

(5.184) **Claim** *$h^\rightarrow[C]$ is closed.*

Proof Suppose $y \in \overline{(h^\rightarrow[C])}$. Define $x \in {}^{\omega}2$ so that for each $n \in \omega$, if $y \in \overline{(h^\rightarrow[C_{(x\upharpoonright n^\frown\langle 0\rangle)}])}$ then $xn = 0$; otherwise $xn = 1$. In any topological space, if a point a is in the closure of a set D and $D = D' \cup D''$, then a is in the closure of D' or D''. It therefore follows by induction on $n \in \omega$ that $y \in \overline{(h^\rightarrow[C_{(x\upharpoonright n)}])}$ for all n. If $hx \neq y$ then there are disjoint neighborhoods N of hx and N' of y, and for some $n \in \omega$, $h^\rightarrow[C_{(x\upharpoonright n)}] \subseteq N$, which contradicts the fact that $y \in \overline{(h^\rightarrow[C_{(x\upharpoonright n)}])}$. Hence, $y = hx \in h^\rightarrow[C]$. $\square$[5.184]

To show that $h^\rightarrow[C]$ is perfect, suppose $x \in C$. For each $n \in \omega$, let $x_n \in {}^{\omega}2$ be given by

$$x_n(m) = \begin{cases} x(m) & \text{if } m \neq n \\ 1 - x(m) & \text{if } m = n. \end{cases}$$

Since h is injective, $hx_n \neq hx$, and since h is continuous, $hx = \lim_{n\to\infty} hx_n$, so hx is a limit point of $h^\rightarrow[C]\backslash\{hx\}$. $\square$[5.183]

5.10 Suslin's hypothesis

Although it is not strictly necessary, to avoid trivial complications we will restrict our discussion of linear orders in this section to dense linear orders without endpoints, as previously discussed in Section 5.3.3. Recall[5.75.2] the characterization of $(\mathbb{R}; <)$ as the unique (up to isomorphism) complete separable dense linear order without endpoints. Consider now the following proposal[5.186] of an alternative characterization.

Definition [ZFC] *Suppose $(X; <)$ is a dense linear order without endpoints.*

1. *An* antichain *in* $(X;<)$ *is set of disjoint nonempty open intervals.*
2. *Suppose* κ *is a cardinal.* $(X;<)$ *satisfies the* κ-chain condition $\overset{\text{def}}{\iff}$ $(X;<)$ *is* κ-cc $\overset{\text{def}}{\iff}$ *there is no antichain in* X *of cardinality* κ*.*
3. *The* countable chain condition *or* ccc *is the* ω_1*-cc.*

(5.185) **Theorem** [ZFC] *Suppose* $(X;<)$ *is a separable dense linear order without endpoints. Then* X *satisfies the countable chain condition.*

Proof Let $Y \subseteq X$ be countable and dense. Suppose $\mathcal{C}$ is a set of disjoint nonempty open intervals in X. For each $I \in \mathcal{C}$, let $y_I \in Y \cap I$. Then the y_Is are distinct members of Y, so $\mathcal{C}$ is countable. $\square$[5.185]

Note that the separability condition (i.e., the existence of a countable dense set) is critical to the unique characterization of the order type of $\mathbb{R}$ among complete dense linear orders without endpoints. It is natural to inquire whether the ostensibly weaker countable chain condition (ccc)[5.185] may be substituted for separability in (5.75.2).

(5.186) **Suslin's hypothesis:** $(\mathbb{R};<)$ *is, up to isomorphism, the unique complete ccc dense linear order without endpoints.* SH $\overset{\text{def}}{=}$ *Suslin's hypothesis.*

A counterexample to Suslin's hypothesis is called a *Suslin line*:

(5.187) **Definition** [ZFC] *A* Suslin line *is a complete ccc dense linear order without endpoints that is not separable (and is therefore not isomorphic to* $(\mathbb{R};<)$*).*

Thus Suslin's hypothesis is—somewhat inconveniently—that there is *no* Suslin line.

The definition of *Suslin line* is sometimes relaxed by omission of the condition of completeness. This is not a significant alteration.

(5.188) **Theorem** [ZFC] *Suppose* $(X;<^X)$ *is a dense linear order without endpoints. Let* $(Y;<^Y)$ *be its completion.*[5.74] $(X;<^X)$ *is ccc iff* $(Y;<^Y)$ *is ccc.*

Proof Clearly, an antichain in X is an antichain in Y. Conversely, suppose $\mathcal{C}$ is an antichain in Y. X is dense in Y.[5.74] Using AC, for each $(x,y) \in \mathcal{C}$, let $x' \in X$ be such that $x' \in (x,y)$, and let $y' \in X$ be such that $y' \in (x',y)$. Then $(x',y') \subseteq (x,y)$, and the set of all such intervals (x',y') is an antichain in X. $\square$[5.188]

Thus a Suslin line in the more general sense may be completed to a Suslin line in the restricted sense of (5.187).

5.11 Limitations of ZF

In this section we will review what we have deduced in ZF, with or without AC, about the countably infinitary, and we will pose some natural questions concerning the possibility of extending these results.

1. Regularity properties of pointsets
   1. Perfect set property
      1. ZF proves that $\mathbf{\Sigma}^1_1$ pointsets have the perfect set property.
      2. Does ZFC prove that Π^1_1 pointsets have the perfect set property?

2. Baire property and Lebesgue measurability
   1. ZF proves these for the closure of $\boldsymbol{\Sigma}^1_1$ under complementation and operation $\mathcal{A}$. Note that all these pointsets are in $\boldsymbol{\Delta}^1_2$.
   2. Does ZFC prove the Baire property or Lebesgue measurability for all Δ^1_2 pointsets?
3. Determinacy
   1. ZFC proves determinacy for $\boldsymbol{\Delta}^1_1$.
   2. Does ZFC prove determinacy for Σ^1_1?
4. For each of these regularity properties, ZFC proves the existence of pointset that does not have it, but the standard proofs do not *define* any such set. Does ZFC prove there is a definable such set? Does ZF (without AC) prove there exists such a set?

2. Structural properties of pointclasses
   1. ZF proves that $\Sigma^1_2(z)$ has the prewellordering property (indeed, the scale property). Hence $\Sigma^1_2(z)$ and $\boldsymbol{\Sigma}^1_2$ have the reduction property, and $\Pi^1_2(z)$ and $\boldsymbol{\Pi}^1_2$ have the separation property.
   2. Does ZFC prove the separation property for either Σ^1_3 or Π^1_3?

3. Other
   1. Does ZF prove the axiom of choice or its negation?
   2. Does ZFC prove the continuum hypothesis or its negation?
   3. Does ZFC prove Suslin's hypothesis[5.186] or its negation?

In general, what are the limits of the ability of ZF, with or without choice axioms, to explicate the countably infinitary? These questions inform much of the rest of our discussion of the foundations of mathematics.

5.12 Summary

Prior to this chapter, except for a few special topics that we could have omitted, we have not had occasion to invoke the axiom of Infinity. The essentially finitary theories S, C, and their congeners, have been sufficient to develop the general theory of structure, language, logic, definability, computability, and provability. We have not therefore supposed the existence of any element that is not an hereditarily finite set. The only necessarily infinite objects have been classes that are not required to be sets, and at no point have we supposed them to be members of any higher order object.

In this chapter we begin to investigate the implications of the admission of infinitary objects as elements. As we have come to see, such apparently simple objects as geometrical points and real numbers are intrinsically infinitary, so the study of infinitarity is essential to the foundations of mathematics.

In a sense, the subject matter of this chapter is that of the preceding chapter raised one level in the hierarchy of types as described here: type-0 objects being hereditarily finite, and type-$(n+1)$ objects being sets of type-n objects. The analysis of definability now takes separate account of quantification over each type. We refer to type-1 objects as *points* or *reals*, and focus our attention on type-2 objects, specifically pointsets, definable by formulas employing quantification over type 0 and type 1. From this standpoint, the present chapter is to type 1 as the preceding chapter is to type 0. There the focus was on definability of type-1 objects, here it is on definability of type-2 objects.

It is often useful to isolate that part of type-2 complexity that is specific to type 2 by allowing arbitrary type-1 objects as parameters in definitions. For example, a function $f : {}^\omega\omega \to {}^\omega\omega$ is recursive relative to a real iff it is continuous. In general, relativization in this sense transforms logical operations into topological operations, and the result is called *descriptive set theory*.

From this standpoint, the essential characteristic of standard type-1 pointspaces like ${}^\omega\omega$ is that they are Polish spaces: separable, completely metrizable topological spaces. Countable union and intersection correspond to type-0 existential and universal quantification, and the Borel hierarchy corresponds to the hierarchy of type-0 quantification, but it is easily extended into the transfinite, whereas the transfinite extension of type-0 quantification in the unrelativized ("effective") theory is relatively complicated. Projection corresponds to existential type-1 quantification, and alternation of this operation with complementation generates the projective hierarchy.

Our first indication of something intrinsically new and deeply significant happening at this level is the theorem of Suslin that $\boldsymbol{\Delta}^1_1 = \mathbf{Borel}$. This serves as an introduction to the topic of structural properties of pointclasses, including separation, reduction, prewellordering, and the scale and uniformization properties, and we carry these as high in the projective hierarchy as we can in ZFC (although we don't know that yet). Their extension to more complex pointsets is an important consideration in mathematics beyond ZFC.

Another motivation for the study of descriptive set theory is the investigation of the extent to which various regularity properties hold. We focus on four such properties: the perfect set property, the Baire property, Lebesgue measurability, and determinacy. Again, we carry the proofs of these properties as high as we can, and again we note that their extension (or not) to more complex pointsets is an important metamathematical issue.

We conclude with an outline of what we have done and some of the major questions left open by our development up to this point.

Chapter 6

The Metatheory of Membership

Contents

6.1 Introduction **415**
6.2 Satisfaction predicates **418**
6.3 Absoluteness **418**
6.3.1 Σ^1_1- and Σ^1_2-absoluteness 419
6.4 Reflection **421**
6.5 Ordinal-definability **425**
6.6 Consistency of the axiom of choice **426**
6.7 Consistency of ZFA + ¬AC **428**
6.7.1 Consistency of ZFA 429
6.7.2 Symmetric models 430
6.8 Summary **432**

6.1 Introduction

We have shown that the standard axiomatization of set theory, i.e., the theory ZF, implies that the set-theoretical universe is the union of the von Neumann hierarchy $\langle V_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ of sets, where

1. $V_0 = 0$;
2. for each $\alpha \in \mathrm{Ord}$, $V_{\alpha+1} = \mathcal{P}\, V_\alpha$; and
3. for each limit ordinal α, $V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}$.

Ord is the class of all ordinals. Additionally, ZF mandates that any criterion that may be formulated in the language of set theory, applied to the members of a given set, defines a set.

We have seen that this simple intuition is a sufficient foundation for most of mathematics, including all of mathematics as it applies to the physical world—and with the axiom of choice, nearly all the rest. As Cantor soon discovered, however, no sooner do we expose the axiomatic basis for our intuitive understanding of the membership relation than we confront fundamental questions about this relation which are palpably different from "ordinary mathematics", and which defy all attempts at resolution in the conventional sense. Chief among these are the axiom of choice, AC, and the continuum hypothesis, CH. The importance of these issues for the foundations of mathematics is reflected in their appearance as the first of "Hilbert's questions" for mathematicians of the twentieth century.

Like the parallel postulate *vis-à-vis* the other axioms of euclidean geometry, AC is not as "obviously true" of the set-theoretical universe as the axioms of ZF, but it is a powerful tool for proving theorems that assert the "orderliness" of that universe; in fact, without AC there are multiple formulations of the continuum hypothesis that are not provably equivalent.

In Chapters 7 and 8 we will show that AC cannot be proved or disproved in ZF, and that CH cannot be proved or disproved in $\mathsf{ZF}+\mathsf{AC}$. These results belong to the metatheory of membership, and like the metatheorems we have already seen, such as Gödel's incompleteness theorems, they are provable in basic set theory, S (without the axiom of infinity), or, equivalently, in Peano arithmetic, PA. A constituent of these finitary proofs is the demonstration that certain statements are provable in ZF. These statements are part of both the theory and the metatheory of membership. We encountered the same situation in the proofs of the incompleteness theorems, where we presented proofs in S partly in order to show that those proofs exist.

An important new element in the metatheory of ZF (which includes $\mathsf{Infinity}$) compared to that of S is that many of the sentences that have been shown to be unprovable in ZF are—like AC and CH—natural statements in the development of the theory for its own sake, unlike, say, $\ulcorner \mathrm{Con}\,\mathsf{S} \urcorner$, which is clearly a statement that is of greater interest for what it says about the *theory* of sets than for what it says about sets *per se*.[1]

In fact, the modern theory of membership is inextricably linked to its own metatheory. In large parts of the theory, most theorems are of the form $\mathsf{ZF}\cup\{\sigma\}\vdash\theta$, where σ is (provably in $\mathsf{S}+\ulcorner \mathrm{Con}\,\mathsf{ZF} \urcorner$) not provable in ZF. Moreover, proofs of statements without explicit metatheoretical content often contain metatheoretical arguments. To paraphrase Alexander Pope: The proper study of set theory is (to a great extent and unavoidably) set theory.

The principal theories for us are

1. S;[2]

2. C, essentially the same theory as S, but admitting proper classes;

3. $\mathsf{ZF}^- = \mathsf{S} + \mathsf{Infinity}$;

4. $\mathsf{GB}^- = \mathsf{C} + \mathsf{Infinity}$, essentially the same theory as ZF^-, but admitting proper classes;

5. $\mathsf{ZF} = \mathsf{ZF}^- + \mathsf{Power}$; and

6. $\mathsf{GB} = \mathsf{GB}^- + \mathsf{Power}$, essentially the same theory as ZF, but admitting proper classes.

[1] There are sentences in the theory of hereditarily finite sets with a less "logical" flavor than $\ulcorner \mathrm{Con}\,\mathsf{S} \urcorner$ or $\ulcorner$I am not provable in $\mathsf{S}\urcorner$ that have been shown to be true but not provable in S, but even these are not mainstream mathematics.

[2] S permits, but does not mandate, the existence of infinite sets. We use S usually to discuss hereditarily finite sets, and we could add an axiom of finiteness to restrict it to this setting, but this generally serves no useful purpose; on the contrary, it is sometimes convenient that S is a subtheory of ZF.

In this chapter we are primarily concerned with the metatheory of membership with the axiom of infinity (we have dealt with the metatheory of finitary set theory already), so ZF is the object theory of greatest interest.

We will be relatively informal in our dealings with variables, not always distinguishing between a variable v and the corresponding term $\bar{v}$, or between a metaname for variable and the variable itself in a typographic object language.

We will not necessarily distinguish notationally between theories and their extensions by definition; hence, 'S' refers not just to the original theory S, but to any extension of it by definition, for which we have previously used 'S^+'. We will, however, continue to be specific as to signatures. Recall that s is the signature of pure set theory with two binary predicate indices, for identity and membership. s^+ is an unspecified expansion that incorporates all the definitions made in the course of this work (and, if we wish, all definitions made by all sentient beings anywhere at any time past or future). Similarly, 'C' refers to any extension by definition of C, with signature c^+.

In a class theory such as C or GB, we may deal with structures $\mathfrak{S}$ that are proper classes. Recall the definitions of *valuation*,[1.58] *satisfactoriness*,[1.60] and *satisfaction*,[1.61] for structures in general, including proper classes, and of *satisfiability*.[1.71]

(6.1) *The following list of theorems will be a useful reference.*

1. (1.63) [C] *Every set structure is satisfactory.*
   (1.64) [ZF^-] *Every structure is satisfactory.*
2. (1.65) [C] *Given a formula ϕ, the $\{\phi\}$-satisfaction relation exists if the ψ-satisfaction relation exists for each immediate subformula of ϕ.*
   (1.67) [C] *The $\mathcal{E}_0^\rho$-valuation function exists, and for any $n \in \omega$, if the $\mathcal{E}_n^\rho$-valuation function exists then the $\mathcal{E}_{n+1}^\rho$-valuation function exists.*
3. (1.73) [C] *There does not exist a satisfaction relation for $(V_\omega; \in)$ that is definable over $(V_\omega; \in)$.*
4. (2.29) [C] *Suppose Θ is a consistent theory. Then Θ is satisfiable, i.e., there exists a satisfactory structure $\mathfrak{S}$ such that $\mathfrak{S} \models \Theta$.*
5. (2.174) [C] *Suppose $\mathfrak{S}$ is weakly satisfactory and $\mathfrak{S} \models \Theta$. If $\Theta \vdash \theta$ then $\mathfrak{S} \models \theta$; in particular, Θ is consistent.*
6. (2.176) [C] *Suppose $\vdash \sigma$. Then $\mathfrak{S} \models \sigma$.*
7. (2.183) [S] C *is a conservative extension of* S.
8. (3.217) [C] *Suppose $(V; \in)$ is weakly satisfactory. Then there is an* s*-sentence σ such that $(V; \in) \models \sigma$ and $\mathsf{S} \nvdash \sigma$.*
9. (4.99) [C] *Suppose* T *is a recursively enumerable ρ-theory in which* S *is interpretable. Let* $\mathrm{Con}\,\mathsf{T}$ *be a ρ-sentence that naturally expresses the consistency of* T. *If* T *is consistent then* $\mathsf{T} \nvdash \mathrm{Con}\,\mathsf{T}$.
10. (3.215) [C] $(V; \in) \models \mathsf{S}$. *Similarly:* [C] $(V_\omega; \in) \models \mathsf{S}$; [$\mathsf{GB}^-$] $(V; \in) \models \mathsf{ZF}^-$; [GB] $(V; \in) \models \mathsf{ZF}$.
11. [S] *Suppose* S *is consistent.*
    1. $\mathsf{C} \nvdash \ulcorner V_\omega$ *is weakly satisfactory*$\urcorner$.[3]
    2. *Similarly,* $\mathsf{GB} \nvdash \ulcorner V$ *is weakly satisfactory*$\urcorner$.

[3]Since[6.1.10] $\mathsf{C} \vdash \ulcorner(V_\omega; \in) \models \mathsf{S}\urcorner$, if $\mathsf{C} \vdash \ulcorner V_\omega$ is weakly satisfactory$\urcorner$, then[6.1.5] $\mathsf{C} \vdash \ulcorner\mathsf{S}$ is consistent$\urcorner$, contradicting Gödel's second incompleteness theorem.

6.2 Satisfaction predicates

It is in the nature of the membership relation that every class M is a set-theoretical world unto itself, viz., the structure $\mathfrak{M} = [\mathsf{s}, M, \pi]$, where $\pi_{[0]} = \{\langle x, x\rangle \mid x \in M\}$, and $\pi_{[1]} = \{\langle x, y\rangle \mid x, y \in M \wedge x \in y\}$. In customary shorthand fashion we use '$(M; \in)$', or simply 'M', to refer to $\mathfrak{M}$ in the appropriate context. The paradigm of these structures is of course $(V; \in)$.

The category of bounded quantification and the corresponding classification of formulas is particularly relevant to these structures. Suppose M is a class. For any ordinal α, let $M_\alpha = M \cap V_\alpha$. Let S^M_α be the Δ_0-satisfaction relation for $(M_\alpha; \in)$. Clearly, if $\alpha \leqslant \beta$, $(M_\alpha; \in) \preccurlyeq^{\Delta_0} (M_\beta; \in)$. Therefore[2.155] $\mathrm{Sat}^M_0 = \bigcup_{\alpha \in \mathrm{Ord}} S^M_\alpha$ is the Δ_0-satisfaction relation for $(M; \in)$. It is easily verified that Sat^M_0 is Δ_1, i.e., Sat^M_0 has both a Σ_1 and a Π_1 definition (with M as a parameter).

We have the following analog of (1.67).

(6.2) **Theorem** [GB] *For any class M and any $n \in \omega$, if there exists a Σ_n-satisfaction relation for $(M; \in)$ then there exists a Σ_{n+1}-satisfaction relation for $(M; \in)$. (Of course, we could also state this for Π instead of Σ.)*

Proof Straightforward. □[6.2]

Recall[4.54] that for any HF set x, $\hat{x}$ is the canonical s'-term that denotes x, where s' is the expansion of s by the addition of the zero and adjunction operations.

(6.3) **Theorem** [S] *Suppose $n \in \omega$. Then* $\mathsf{GB} \vdash$ ⌜*for any class M, the $\Sigma_{(\hat{n})}$-satisfaction relation exists for $(M; \in)$*⌝.[4]

Proof By induction on n, using the fact of the existence of a proof of (1.67) in GB. □[6.3]

These two theorems may be added to the list (6.1).

6.3 Absoluteness

(6.4) **Definition** [GB] *Suppose $\phi(v_0, \ldots, v_{n^-})$ is an s-formula with the free variables shown, and $M \subseteq N$.*

1. ϕ *is* absolute between M and N $\overset{\text{def}}{\iff}$

   (6.5) $$\forall x_0, \ldots, x_{n^-} \in M \left(M \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ x_0 & \cdots & x_{n^-}\end{smallmatrix}\right] \leftrightarrow N \models \phi\left[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ x_0 & \cdots & x_{n^-}\end{smallmatrix}\right]\right).$$

2. ϕ *is absolute* upward *or* downward $\overset{\text{def}}{\iff}$ *(6.5) holds with '$\leftrightarrow$' replaced by '$\rightarrow$' or '$\leftarrow$', respectively.*
3. ϕ *is* absolute for M $\overset{\text{def}}{\iff}$ *ϕ is absolute between M and V.*

(6.6) **Theorem** [GB] *Suppose ϕ is an s-formula.*

1. *If ϕ is Δ_0 then ϕ is absolute between transitive classes.*
2. *If ϕ is Σ_1 (Π_1) then ϕ is absolute upward (downward) between transitive classes.*

[4] It is not important that the canonical term be used; we just need a name for n.

3. *Suppose* ϕ^Σ *and* ϕ^Π *are respectively* Σ_1 *and* Π_1*. Then* ϕ *is absolute between transitive models of* $\phi \leftrightarrow \phi^\Sigma \leftrightarrow \phi^\Pi$*.*
4. *Suppose* T *is an* s*-theory and* ϕ *is* Δ_1^{T}*. Then* ϕ *is absolute between weakly satisfactory transitive models of* T*.*

Proof 1 Straightforward induction on the complexity of ϕ.

2 Immediate from (6.6.1).

3 Also immediate.

4 Use (6.1.5). $\square$[6.6]

(6.6.4) is not very useful, since we cannot even show (in GB) that V, for example, is weakly satisfactory. In practice, to deal with a specific formula ϕ, (6.6.3) may be used. We will not take the time now to formulate a general metatheorem to this effect; in Chapter 9 we will return to this topic in the context of an elementary embedding of the universe V into a transitive class.[9.25]

For sets M and N, Definition 6.4 makes sense in the context of ZF, but for the more important case of proper classes, we cannot define absoluteness for formulas in general in the context of a pure set theory. At best, we can define absoluteness for a class of formulas of bounded complexity, e.g., Σ_1.

Since we generally work in a class theory, we will not take the trouble to do this formally, but we note that for a single formula we may express absoluteness in purely set-theoretical terms using relativization[2.112] as follows.

Suppose ϕ *is an* s*-formula with free variables* $u_0, \ldots, u_{n^-}$*,* M *and* N *are unary predicates (typically in an extension-by-definition* s^+ *of* s*), and* $\forall x\, (M(x) \to N(x))$ *has been asserted. The assertion of absoluteness of* ϕ *between* M *and* N *is the* s^+*-sentence*

$$\forall u_0, \ldots, u_{n^-} \in M\, (\phi^M \leftrightarrow \phi^N).^{5}$$

This is generally adequate to deal with absoluteness in the context of a pure set theory. Obviously, it also serves in the context of a class theory, in which case we also have an equivalent formulation with M and N treated as classes.

6.3.1 Σ_1^1- and Σ_2^1-absoluteness

The following theorem is due to Mostowski (Part 1) and Shoenfield (Part 2).

(6.7) **Theorem** $[\mathsf{GB}^-]$ *There is a finite fragment* $\Theta \subseteq \mathsf{ZF}^-$ *such that for any transitive classes* $M \subseteq N$*, if* $M, N \models \Theta$ *then for any* s^1*-formula*[5.2] ϕ *with one free variable* u*, of type 1,*

1. *if* ϕ *is* Σ_1^1*, then for any* $x \in {}^\omega\omega \cap M$*,* $M \models \phi[x]$ *iff* $N \models \phi[x]$*; and*
2. *if* ϕ *is* Σ_2^1 *and* $\omega_1^N \subseteq M$*, then for any* $x \in {}^\omega\omega \cap M$*,* $M \models \phi[x]$ *iff* $N \models \phi[x]$*.*

[5] We are using the convenient notation by which '$u \in M$' means '$M(u)$', which is not to imply that M exists as a class.

Proof Recall that for any Σ^1_1 set $X \subseteq {}^\omega\omega$ there exists a recursive sequence tree T on $\omega \times \omega$, canonically defined from a formula for X, such that for all $x \in {}^\omega\omega$, $x \in X \leftrightarrow \exists y \in {}^\omega\omega\ \langle x, y\rangle^{\cdot} \in [T]$, the set of infinite branches of T. In other words $x \in X$ iff $T_{[x]}$ is nonwellfounded, where $T_{[x]}$ is $\{y \in {}^\omega\omega \mid \langle x, y\rangle^{\cdot} \in [T]\}$. Thus, for any Π^1_1 set $X \subseteq {}^\omega\omega$ there exists a tree T on $\omega \times \omega$, canonically defined from a formula for X, such that for all $x \in {}^\omega\omega$, $x \in X$ iff $T_{[x]}$ is wellfounded. In the proof of (5.122) we used this characterization of Π^1_1 to show that for any Σ^1_2 set $X \subseteq {}^\omega\omega$, there exists a tree T on $\omega \times \omega_1$, canonically defined from a formula for X, such that for all $x \in {}^\omega\omega$, $x \in X$ iff $T_{[x]}$ is nonwellfounded. Replacing ω_1 by an arbitrary ordinal α or by Ord in the definition of T, we obtain trees T^α (and a definition of a tree T^{Ord}, which is a proper class) such that $T = T^{\omega_1}$. In general,

(6.8)

1. *for any* $\alpha \leqslant \beta \leqslant \mathrm{Ord}$, $T^\alpha \subseteq T^\beta$*; and*
2. *for any* $\alpha \geqslant \omega_1$*, for all* $x \in {}^\omega\omega$, $x \in X$ *iff* $T_{[x]}$ *is not wellfounded.*

Let Θ be a finite fragment of ZF^- large enough that any transitive model of Θ contains every recursive subset of V_ω, and Θ establishes the equivalences just described along with the theorem that any wellfounded relation may be mapped order-preserving into the ordinals.[6] By virtue of the last condition, wellfoundedness is absolute for transitive models of Θ.[7]

Suppose ϕ is Σ^1_1 with one free variable u, of type 1. Let T be the canonical tree on $\omega \times \omega$ such that for all $x \in {}^\omega\omega$, $V \models \phi[x]$ iff $T_{[x]}$ is nonwellfounded. Suppose M is a transitive model of Θ. Then $T \in M$, and for any $x \in {}^\omega\omega \cap M$, $T_{[x]} \in M$ and $M \models \phi[x]$ iff $M \models \ulcorner [T_{[x]}]$ is wellfounded$\urcorner$. Since wellfoundedness is absolute for transitive models of Θ, $M \models \phi[x]$ iff $V \models \phi[x]$. The same is true of any other transitive model N of Θ. Thus, given $M \subseteq N$ as in the statement of the theorem, for any $x \in {}^\omega\omega \cap M$, $M \models \phi[x]$ iff $N \models \phi[x]$.

Now suppose ϕ is Σ^1_2. For each $\alpha \leqslant \omega_1$, let T^α be as described above,[6.8] so that for all $x \in {}^\omega\omega$, $V \models \phi[x]$ iff $T^{\omega_1}_{[x]}$ is nonwellfounded. Suppose M is a transitive model of Θ and $\omega_1 \subseteq M$. The definition of T^α is absolute between M and V. If $\omega_1 \in M$ then $T^{\omega_1} \in M$; otherwise, since $\omega_1 \subseteq M$, T^{ω_1} is a definable proper class in the sense of M. Since $M \models \Theta$, for any $x \in {}^\omega\omega \cap M$, $M \models \phi[x]$ iff $M \models \ulcorner T^{\omega_1}_{[[x]]}$ is nonwellfounded$\urcorner$ iff $M \models \ulcorner T^{[\omega_1]}_{[[x]]}$ is nonwellfounded$\urcorner$.[8] Since wellfoundedness is absolute, as is the formation of $T^\alpha_{[x]}$ from T^α and x, $M \models \phi[x]$ iff $T^{\omega_1}_{[x]}$ is nonwellfounded iff $V \models \phi[x]$.

Thus, for any $x \in {}^\omega\omega \cap M$, $M \models \phi[x]$ iff $V \models \phi[x]$. To obtain (6.7.2) as stated, we replace V by an arbitrary transitive $N \supseteq M$ such that $N \models \Theta$. $\square$[6.7]

[6] Since we have established the above equivalences in the theory ZF, we know at once that there exists such a $\Theta \subseteq \mathsf{ZF}$. A little reflection shows that Power is not required for this purpose: We can talk about the relevant trees on HF sets and countable ordinals without having to posit the existence of a set that contains all subsets of HF or all countable ordinals. Thus, there exists a finite $\Theta \subseteq \mathsf{ZF}^-$ that serves the purpose.

[7] Suppose M is transitive and $M \models \Theta$. Suppose $R \in M$ is a binary relation on $X \in M$, and $V \models \ulcorner [R]$ is wellfounded$\urcorner$, in the sense that every nonempty subset of X has an R-least member. Then clearly, $M \models \ulcorner [R]$ is wellfounded$\urcorner$ in the same sense. On the other hand, if $M \models \ulcorner [R]$ is wellfounded$\urcorner$ and $M \models \Theta$, then there exists $f \in M$ such that $M \models \ulcorner [f]$ is an order-preserving map of $[(X; R)]$ into the ordinals$\urcorner$. Therefore, $V \models \ulcorner [f]$ is an order-preserving map of $[(X, R)]$ into the ordinals$\urcorner$, so $V \models \ulcorner [R]$ is wellfounded$\urcorner$.

[8] If $\omega_1 \notin M$, so that $\omega_1 = \mathrm{Ord} \cap M$, then '$T^{[\omega_1]}_{[[x]]}$' is understood to mean '$T^{\mathrm{Ord}}_{[[x]]}$'.

6.4 Reflection

(6.9) **Theorem: Reflection principle** [GB$^-$] *Suppose $\langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is a monotone increasing continuous sequence of nonempty sets, i.e.*

1. $\forall\alpha < \beta \in \mathrm{Ord}$, $M_\alpha \subseteq M_\beta$; *and*
2. *for any limit ordinal* β, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$.

Let $M = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$. *Suppose* ϕ *is an* s*-formula. Then*

$$\forall_{\mathrm{Ord}}\alpha\, \exists_{\mathrm{Ord}}\beta \geqslant \alpha\ M_\beta \prec^{\{\phi\}} M,$$

i.e., M_β *is a* $\{\phi\}$*-elementary substructure of* M.

Proof If there does not exist a $\{\phi\}$-valuation function for M then[2.152] $M_\beta \prec^{\{\phi\}} M$ for all β, so suppose S is a $\{\phi\}$-valuation function for M.

(6.10) **Claim** *Suppose* $\alpha \in \mathrm{Ord}$. *There exists* $\beta > \alpha$ *such that for every* $\psi \in \overline{\{\phi\}}$ *and variable* u,

1. *if* $\exists u\, \psi \in \overline{\{\phi\}}$, *then for every* M_α*-assignment* A *for* $\exists u\, \psi$,

$$\exists x \in M\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1 \rightarrow \exists x \in M_\beta\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1; \textit{and}$$

2. *if* $\forall u\, \psi \in \overline{\{\phi\}}$ *then for every* M_α*-assignment* A *for* $\exists u\, \psi$,

$$\forall x \in M_\beta\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1 \rightarrow \forall x \in M\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1.$$

Proof This follows by Collection, since the class of objects $\langle\theta, A\rangle$ with $\theta \in \overline{\{\phi\}}$ and A an M_α-assignment for θ is a set. $\square$[6.10]

Let $\alpha \in \mathrm{Ord}$ be given. Let $\alpha_0 = \alpha$ and define α_n for $n > 0$ recursively by letting α_{n+1} be the least ordinal β witnessing Claim 6.10 with α_n for α. Let $\beta = \bigcup_{n\in\omega} \alpha_n$. Then $\beta \geqslant \alpha$ and every M_β-assignment is an M_{α_n}-assignment for some n, so for every $\psi \in \overline{\{\phi\}}$ and variable u,

1. if $\exists u\, \psi \in \overline{\{\phi\}}$, then for every M_β-assignment A for $\exists u\, \psi$,

$$\exists x \in M\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1 \rightarrow \exists x \in M_\beta\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1; \textit{and}$$

2. if $\forall u\, \psi \in \overline{\{\phi\}}$ then for every M_β-assignment A for $\exists u\, \psi$,

$$\forall x \in M_\beta\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1 \rightarrow \forall x \in M\ S\Big\langle \psi, A\Big\langle {u \atop x} \Big\rangle\Big\rangle = 1.$$

By the Tarski-Vaught criterion[2.153] $M_\beta \prec^{\{\phi\}} M$. $\square$[6.9]

In (6.9) we say that ϕ *reflects from* M *to* M_β, and (6.9) is the *reflection principle*. This term is also applied specifically to the case that $M_\alpha = V_\alpha$ for $\alpha > 0$, where the Power axiom is now presumed, so that we may infer that the V_αs are sets.

(6.11) **Theorem** [GB] *Suppose* ϕ *is an* s*-formula. Then*

$$\forall_{\mathrm{Ord}}\alpha\, \exists_{\mathrm{Ord}}\beta \geqslant \alpha\ V_\beta \prec^{\{\phi\}} V.$$

As a corollary, we have:

(6.12) **Theorem** [GB] *Suppose* ϕ *is an* s*-formula. Then* $\{\alpha \in \mathrm{Ord} \mid V_\alpha \prec^{\{\phi\}} V\}$ *is closed unbounded in* Ord.

Proof The unboundedness is (6.11). Closure follows from the fact that, given $\alpha < \beta$, if $V_\alpha \prec^{\{\phi\}} V$ and $V_\beta \prec^{\{\phi\}} V$ then $V_\alpha \prec^{\{\phi\}} V_\beta$. Thus, letting $C = \{\alpha \in \mathrm{Ord} \mid V_\alpha \prec^{\{\phi\}} V\}$, if γ is a limit point of C then V_γ is the union of a directed set of $\{\phi\}$-elementary substructures, viz., $\{V_\alpha \mid \alpha \in C \cap \gamma\}$, so for any $\alpha \in C \cap \gamma$, $V_\alpha \prec^{\{\phi\}} V_\gamma$. Hence $V_\gamma \prec^{\{\phi\}} V$, i.e., $\gamma \in C$. $\square$[6.12]

(6.13) **Theorem** [S] *If* ZF *is consistent, it is not finitely axiomatizable.*

Proof Suppose ZF is finitely axiomatizable. We can form a conjunction of any finite set of sentences, so for some s-sentence θ, $\mathsf{ZF} \vdash \theta$ and $\{\theta\} \vdash \sigma$ for all $\sigma \in \mathsf{ZF}$. Let $\hat{\theta}$ be the canonical s′-term for θ. Then

$$\mathsf{GB} \vdash \ulcorner V \models (\hat{\theta}) \urcorner,$$

so[6.11]

$$\mathsf{GB} \vdash \ulcorner \exists_{\mathrm{Ord}} \alpha \; V_\alpha \models (\hat{\theta}) \urcorner,$$

so[3.98]

$$\mathsf{GB} \vdash \ulcorner \exists_{\mathrm{Ord}} \alpha \; (\theta^{V_\alpha}) \urcorner.$$

Since GB is a conservative extension of ZF,

$$\mathsf{ZF} \vdash \ulcorner \exists_{\mathrm{Ord}} \alpha \; (\theta^{V_\alpha}) \urcorner,$$

so

$$\{\theta\} \vdash \ulcorner \exists_{\mathrm{Ord}} \alpha \; (\theta^{V_\alpha}) \urcorner,$$

so[3.98]

$$\{\theta\} \vdash \ulcorner \exists_{\mathrm{Ord}} \alpha \; V_\alpha \models (\hat{\theta}) \urcorner. \tag{6.14}$$

Note that for set structures like $(V_\alpha; \in)$, the existence of full valuation functions is demonstrable in S, so $V_\alpha \models \sigma$ iff $S\langle \sigma, 0 \rangle = 1$, where S is the full valuation function for V_α.

Working in ZF we may reason as follows:

Let α be the least ordinal such that $V_\alpha \models \theta$.[9] Since[6.14]

$$V_\alpha \models \ulcorner \exists_{\mathrm{Ord}} \alpha \; V_\alpha \models (\hat{\theta}) \urcorner,$$

for some $\alpha' < \alpha$,

$$V_\alpha \models \ulcorner V_{[\alpha']} \models (\hat{\theta}) \urcorner,$$

i.e. for some $S \in V_\alpha$, $V_\alpha \models \ulcorner [S]$ is the valuation function for $V_{[\alpha']}$, and $[S]\langle (\hat{\theta}), 0 \rangle = 1 \urcorner$. It is easy to see that S must indeed be the valuation function for $V_{\alpha'}$, and $S\langle \theta, 0 \rangle = 1$, so

$$V_{\alpha'} \models \theta.$$

But α is the least ordinal such that $V_\alpha \models \theta$.

[9]Since valuation functions for set structures are sets, the property of α that $V_\alpha \models \theta$ is expressible with only set quantification, so by Foundation, if there exists α such that $V_\alpha \models \theta$ then there exists a least such α.

ZF is therefore inconsistent. □[6.13]

On the other hand:

(6.15) **Theorem** [S] C *is finitely axiomatizable. Hence, any extension of* C *by finitely many axioms, e.g.,* GB^- *or* GB, *is finitely axiomatizable.*

Proof The only axiom schema in our formulation of C is Comprehension, and this may be replaced by a finite set of its instances. The following list is sufficient. As a convenience, we formulate the axioms in an extension C^+ by definition of the pair and ordered pair operations, whose definitions and basic properties do not depend on Comprehension. To obtain an axiomatization of C *per se*, we could substitute the definitions of these operations.

(6.16)

1. $\exists \mathrm{v}_0\, \forall \mathrm{v}_1\, \big(\mathrm{v}_1 \in \mathrm{v}_0 \leftrightarrow \exists \mathrm{v}_2, \mathrm{v}_3\, (\mathrm{v}_2 \in \mathrm{v}_3 \wedge \mathrm{v}_1 = (\mathrm{v}_2, \mathrm{v}_3))\big)$
2. $\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_1 \leftrightarrow \neg\, \mathrm{v}_2 \in \mathrm{v}_0)$
3. $\forall \mathrm{v}_0, \mathrm{v}_1\, \exists \mathrm{v}_2\, \forall \mathrm{v}_3\, (\mathrm{v}_3 \in \mathrm{v}_2 \leftrightarrow \mathrm{v}_3 \in \mathrm{v}_0 \wedge \mathrm{v}_3 \in \mathrm{v}_1)$
4. $\forall \mathrm{v}_0, \mathrm{v}_1\, \exists \mathrm{v}_2\, \forall \mathrm{v}_3\, \big(\mathrm{v}_3 \in \mathrm{v}_2 \leftrightarrow \exists \mathrm{v}_4, \mathrm{v}_5\, (\mathrm{v}_4 \in \mathrm{v}_0 \wedge \mathrm{v}_5 \in \mathrm{v}_1 \wedge \mathrm{v}_3 = (\mathrm{v}_4, \mathrm{v}_5))\big)$
5. $\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, (\mathrm{v}_2 \in \mathrm{v}_1 \leftrightarrow \exists \mathrm{v}_3\ (\mathrm{v}_2, \mathrm{v}_3) \in \mathrm{v}_0)$
6. $\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, \big(\mathrm{v}_2 \in \mathrm{v}_1 \leftrightarrow \exists \mathrm{v}_3, \mathrm{v}_4\, (\mathrm{v}_2 = (\mathrm{v}_4, \mathrm{v}_3) \wedge (\mathrm{v}_3, \mathrm{v}_4) \in \mathrm{v}_0)\big)$
7. $\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, \big(\mathrm{v}_2 \in \mathrm{v}_1 \leftrightarrow \exists \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_5\, (\mathrm{v}_2 = ((\mathrm{v}_3, \mathrm{v}_4), \mathrm{v}_5) \wedge (\mathrm{v}_3, (\mathrm{v}_4, \mathrm{v}_5)) \in \mathrm{v}_0)\big)$
8. $\forall \mathrm{v}_0\, \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, \big(\mathrm{v}_2 \in \mathrm{v}_1 \leftrightarrow \exists \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_5\, (\mathrm{v}_2 = ((\mathrm{v}_3, \mathrm{v}_5), \mathrm{v}_4) \wedge ((\mathrm{v}_3, \mathrm{v}_4), \mathrm{v}_5) \in \mathrm{v}_0)\big)$

Let Θ be the theory obtained from C by replacing Comprehension by these instances. Let Θ^+ be the extension by definition of the pair and ordered pair operations. We will use the ordered n-tuple notation.[3.58] Thus $\ulcorner(a, b, c)\urcorner$ means $\ulcorner((a, b), c)\urcorner$, etc.

(6.16.1–8) may be paraphrased as follows. Informally, we use lower case symbols for set variables. Universal closure is assumed. We also assume that the implicit quantification in the abstraction term is over all free variables unless otherwise stated.

(6.17)

1. $\{(x, y) \mid x \in y\}$ *exists*
2. $\{x \mid x \notin X\}$ *exists*
3. $\{z \mid z \in X \wedge z \in Y\}$ *exists*
4. $\{(x, y) \mid x \in X \wedge y \in Y\}$ *exists*
5. $\{y \mid \exists z\ (y, z) \in X\}$ *exists*
6. $\{(z, y) \mid (y, z) \in X\}$ *exists*
7. $\{((x, y), z) \mid (x, (y, z)) \in X\}$ *exists*
8. $\{((x, z), y) \mid ((x, y), z) \in X\}$ *exists*

(6.18) **Claim** *Suppose* $v_0, \ldots, v_{n^-}$ *are distinct variables and* ϕ *is a* c*-formula with* $\operatorname{Free}\phi \subseteq \{v_0, \ldots, v_{n^-}\}$. *Then*

$$\Theta^+ \vdash \exists u\, \forall w\, \Big(w \in u \leftrightarrow \exists v_0, \ldots, v_{n^-}\, \big(w = (v_0, \ldots, v_{n^-}) \wedge \phi\big)\Big),$$

i.e., $\Theta^+ \vdash\ \ulcorner(\{(v_0, \ldots, v_{n^-}) \mid \phi\})$ *exists*$\urcorner$.

Proof By induction on the complexity of ϕ. It suffices to work with formulas constructed using only $\in$, $\neg$, $\wedge$, and $\exists$, as any formula is Θ-equivalent to such a formula.

We first observe that Θ^+ proves the existence of 0 by Existence. Θ^+ proves the existence of V by applying (6.17.2) with 0 for v_0. For uniformity of notation we adopt the convention that an abstraction term $\ulcorner\{(\tau) \mid\}\urcorner$, with an empty condition, imposes no restriction on the variable in the term τ. Thus, for example, $\{(x, y) \mid \}$ is the class of all ordered pairs. Remember that the 1-tuple $(x) = x$ by definition.

(6.19) **Claim** Θ^+ *proves the existence of*

$$\begin{aligned} V &= \{(x_0) \mid \}, \\ V \dot{\times} V &= \{(x_0, x_1) \mid \}, \\ V \dot{\times} V \dot{\times} V &= \{(x_0, x_1, x_2) \mid \} \\ &= \{((x_0, x_1), x_2) \mid \} \\ &= (V \dot{\times} V) \dot{\times} V, \\ &\vdots \end{aligned}$$

Proof Repeated applications of (6.17.4). □6.19

(6.20) **Claim**

1. *For any number k, $\Theta^+ \vdash \ulcorner\{(x_0, \ldots, x_{k+1}) \mid x_k \in x_{k+1}\}$ exists$\urcorner$.*
2. *For any numbers $k < l$, $\Theta^+ \vdash \ulcorner\{(x_0, \ldots, x_l) \mid x_k \in x_l\}$ exists$\urcorner$.*
3. *For any numbers $k < l \leqslant m$, $\Theta^+ \vdash \ulcorner\{(x_0, \ldots, x_m) \mid x_k \in x_l\}$ exists$\urcorner$.*
4. *For any numbers $k, l < n$, $\Theta^+ \vdash \ulcorner\{(x_0, \ldots, x_{n^-}) \mid x_k \in x_l\}$ exists$\urcorner$.*

Proof 1 If $k = 0$, this follows from (6.17.1). If $k > 0$, we use the fact that $\Theta^+ \vdash \ulcorner\{(x_0, \ldots, x_{k^-}) \mid \}$ exists$\urcorner$ and $\Theta^+ \vdash \ulcorner\{(x_k, x_{k+1}) \mid x_k \in x_{k+1}\}$ exists$\urcorner$ to conclude, using (6.17.4), that $\Theta^+ \vdash \ulcorner\{((x_0, \ldots, x_{k^-}), (x_k, x_{k+1})) \mid x_k \in x_{k+1}$ exists$\urcorner$. Now use (6.17.7) to show that Θ^+ proves the existence of

$$\{(x_0, \ldots, x_{k+1}) \mid x_k \in x_{k+1}\} = \{(((x_0, \ldots, x_{k^-}), x_k), x_{k+1}) \mid x_k \in x_{k+1}\}.$$

2 By induction on $l > k$ for any fixed k. For $l = k+1$ this is (6.20.1). Now assume Θ^+ proves $\{(x_0, \ldots, x_l) \mid x_k \in x_l\}$ exists. Then Θ^+ proves $\{((x_0, \ldots, x_l), x_{l+1}) \mid x_k \in x_l\}$ exists by (6.17.4) with V for Y, i.e.,

$$\{(((x_0, \ldots, x_{l^-}), x_l), x_{l+1}) \mid x_k \in x_l\}$$

exists, so $\{(((x_0, \ldots, x_{l^-}), x_{l+1}), x_l) \mid x_k \in x_l\}$ exists by (6.17.8). By a change of variables, $\{(x_0, \ldots, x_{l+1}) \mid x_k \in x_{l+1}\} = \{(((x_0, \ldots, x_{l^-}), x_l), x_{l+1}) \mid x_k \in x_{l+1}\}$ exists.

3 By induction on $m \geqslant l$ for fixed k, l. For $m = l$ this is (6.20.2). Now assume Θ^+ proves $\{(x_0, \ldots, x_m) \mid x_k \in x_l\}$ exists. Use (6.17.4) to show that Θ^+ proves $\{(x_0, \ldots, x_{m+1}) \mid x_k \in x_l\} = \{((x_0, \ldots, x_m), x_{m+1}) \mid x_k \in x_l\}$ exists.

4 Θ^+ proves $\{(x_0, x_1) \mid x_1 \in x_0\}$ exists by virtue of (6.17.1) and (6.17.6). With this starting point, the preceding arguments show that for any numbers $l < k \leqslant m$, Θ^+ proves $\{(x_0, \dots, x_m) \mid x_k \in x_l\}$ exists. If $k \leqslant m$ then $\{(x_0, \dots, x_m) \mid x_k \in x_k\} = 0$, so it exists. Thus for any $k, l \leqslant m$, $\{(x_0, \dots, x_m) \mid x_k \in x_l\}$ exists. In the statement of the claim, we have just let $n = m + 1$. $\square^{6.20}$

Since we have limited our consideration to formulas ϕ using only the membership predicate, if ϕ is atomic it is $u \in v$ for some set variables u, v, or it is $u \in U$ for some set variable u and class variable U. (6.20) is Claim 6.18 for all atomic formulas ϕ of the first type. For atomic formulas of the second type, we observe that for numbers $k < n$, Θ^+ proves that $\{(x_0, \dots, x_{n\text{-}}) \mid x_k \in X\}$ exists by same argument as for (6.19) with one copy of V replaced by X.

Given that we have also restricted our attention to formulas generated by negation, conjunction, and existential quantification, (6.17.2), (6.17.3), and (6.17.5) suffice to prove (6.18) for all ϕ by induction on complexity. $\square^{6.18}$

It is fairly straightforward to show that any instance of the $\mathsf{Comprehension}$ schema of C is derivable from a suitable instance of (6.18). Details are left to the reader. $\square^{6.15}$

6.5 Ordinal-definability

(6.21) **Definition** [GB] *A set x is* ordinal-definable $\overset{\text{def}}{\iff}$ $\mathrm{OD}\, x$ $\overset{\text{def}}{\iff}$ *there exists an* s*-formula ϕ, a $\{\phi\}$-valuation function for V, distinct variables u, v_0, $\dots$, $v_{n\text{-}}$, and ordinals $\alpha_0, \dots, \alpha_{n\text{-}}$, such that x is the unique set y such that* $V \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]$.

Note that since this definition involves quantification over a class variable (a valuation function for V being a proper class), we cannot assert in GB that there is a *class* OD; nor can this definition be translated directly into the language $\mathcal{L}^{\mathsf{s}}$ of pure set theory. The reflection principle, however, allows us to prove its equivalence to the following definition, which is legitimate in ZF and which may be used in GB to define a class.

(6.22) **Definition** [ZF] *A set x is* ordinal-definable $\overset{\text{def}}{\iff}$ $\mathrm{OD}\, x$ $\overset{\text{def}}{\iff}$ *for some* s*-formula ϕ, distinct variables $u, v_0, \dots, v_{n\text{-}}$, and ordinals $\alpha_0, \dots, \alpha_n$, such that $\alpha_0, \dots, \alpha_{n\text{-}} \in \alpha_n$, and x is the unique $y \in V_{\alpha_n}$ such that* $V_{\alpha_n} \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]$.

(6.23) **Theorem** [GB] *A set is ordinal-definable in the sense of (6.21) iff it is ordinal-definable in the sense of (6.22).*

Proof Suppose there exists a $\{\phi\}$-valuation function for V, and x is the unique set y such that $V \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]$. Let[6.11] α_n be such that $\{x, \alpha_0, \dots, \alpha_{n\text{-}}\} \subseteq V_{\alpha_n}$, and $V_{\alpha_n} \prec^{\{\phi\}} V$. Then x is the unique set $y \in V_{\alpha_n}$ such that $V_{\alpha_n} \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]$.

Conversely, suppose x is the unique $y \in V_{\alpha_n}$ such that $V_{\alpha_n} \models \psi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]$. The s-formula $\ulcorner \alpha_0, \dots, \alpha_n$ are ordinals, $\{y, \alpha_0, \dots, \alpha_{n\text{-}}\} \subseteq V_{\alpha_n}$, and $V_{\alpha_n} \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]\urcorner$ is Δ_1^{ZF}. There are therefore a Σ_1 s-formula ϕ and distinct variables $u, v_0, \dots, v_n$, such that for all sets y,

$$\begin{aligned} V \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_n \\ y & \alpha_0 & \cdots & \alpha_n \end{smallmatrix}\right] &\leftrightarrow \left(y \in V_{\alpha_n} \wedge V_{\alpha_n} \models \psi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n\text{-}} \\ y & \alpha_0 & \cdots & \alpha_{n\text{-}} \end{smallmatrix}\right]\right) \\ &\leftrightarrow y = x. \end{aligned}$$

Since there exists a Σ_1-valuation function for V, there exists a $\{\phi\}$-valuation function for V. Hence x is ordinal-definable in the sense of (6.21). $\square^{6.23}$

When we use 'ordinal-definable' in the context of ZF we have (6.22) in mind. In the context of GB we may of course use the characterizations (6.21) and (6.22) interchangeably. As noted above, it is the existence of the latter form that allows us to conclude that the ordinal-definable sets form a class.

Definition [GB] $\mathrm{OD} \overset{\mathrm{def}}{=} \{x \in V \mid \mathrm{OD}\, x\}$.

6.6 Consistency of the axiom of choice

It is easy to define a wellordering of OD. Let us say that an OD-code is a sequence $\langle \beta, \phi, \alpha_0, \ldots, \alpha_{n^-}\rangle$, where β is an ordinal, n is a number, ϕ is an s-formula with $n+1$ free variables, and $\alpha_0, \ldots, \alpha_{n^-}$ are ordinals $< \beta$ (with the usual convention that if $n = 0$ then $\langle \alpha_0, \ldots, \alpha_{n^-}\rangle$ is the empty sequence 0). An OD-code $\langle \beta, \phi, \alpha_0, \ldots, \alpha_{n^-}\rangle$ corresponds to an OD set x iff x is the unique $y \in V_\beta$ such that $V_\beta \models \phi\left[\begin{smallmatrix} u & v_0 & \cdots & v_{n^-} \\ y & \alpha_0 & \cdots & \alpha_{n^-}\end{smallmatrix}\right]$, when $\langle u, v_0, \ldots, v_{n^-}\rangle$ is the enumeration of $\mathrm{Free}\,\phi$ in increasing order. The lexicographic ordering of OD codes is a wellorder.[10] Now we say that $x <^{\mathrm{OD}} y \overset{\mathrm{def}}{\iff}$ the first code for x strictly precedes the first code for y.

Definition [ZF] x *is* hereditarily ordinal-definable $\overset{\mathrm{def}}{\iff} \mathrm{HOD}\, x \overset{\mathrm{def}}{\iff}$ *every member of the transitive closure* $\mathrm{tc}\{x\}$ *is* OD.

Of course, in GB we define HOD as the class of HOD sets. By design, HOD is transitive, and all its members are OD.

(6.24) **Theorem (Gödel)** [GB] $\mathrm{HOD} \models \mathsf{ZFC}$.

Proof

(6.25) **Claim** *For any ordinal* α, $V_\alpha \cap \mathrm{HOD}$ *is* HOD.

Proof $V_\alpha \cap \mathrm{HOD}$ is OD according to (6.21), as it is definable over V from α by an s-formula ϕ obtained from the definitions we have just given of $\ulcorner\mathrm{OD}\urcorner$ and $\ulcorner\mathrm{HOD}\urcorner$. Of course, in the formulation of ϕ we have to use Definition 6.22 of $\ulcorner\mathrm{OD}\urcorner$, rather than Definition 6.21, to avoid class quantification. If we want to work completely in ZF, we may observe that ϕ works equally well to define $V_\alpha \cap \mathrm{HOD}$ from α in $V_{\alpha+\omega}$.

Since $V_\alpha \cap \mathrm{HOD}$ is transitive, $\mathrm{tc}\{V_\alpha \cap \mathrm{HOD}\} = V_\alpha \cap \mathrm{HOD}$, so $V_\alpha \cap \mathrm{HOD}$ is hereditarily OD. $\square^{6.25}$

It follows easily from the claim that HOD is almost universal. By (3.214) all that remains to be shown is that HOD models $\mathsf{Separation}$ and Choice. The argument for $\mathsf{Separation}$ parallels the corresponding argument showing that $V \models \mathsf{ZF}$.[3.215]

[10] Recall that the lexicographic ordering of sequences puts $s \leqslant t$ iff, letting i be least such that it is not the case that $i \in \mathrm{dom}\, s \wedge i \in \mathrm{dom}\, t \wedge s_i = t_i$, either

1. $i \notin \mathrm{dom}\, s$ (in which case $s \subseteq t$), or
2. $i \in \mathrm{dom}\, s \wedge i \in \mathrm{dom}\, t \wedge s_i < t_i$,

where some wellordering of the class from which the sequences are drawn is presumed. In the present case, formulas are ordered according to the canonical ω-ordering of V_ω, i.e., $\phi < \psi \leftrightarrow \overleftarrow{B}\,\phi < \overleftarrow{B}\,\psi$; and ordinals have the usual order.

Suppose

$$\sigma = \forall v_0, \ldots, v_{n-}\, \forall u\, \exists w\, \forall v\, (v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is an s-formula, and $u, v, w, v_0, \ldots, v_{n-}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{u, v, v_0, \ldots, v_{n-}\}$. Suppose S is a $\{\sigma\}$-valuation function for HOD.

We must show that

$$\forall y_0, \ldots, y_{n-} \in \mathrm{HOD}\ \forall x \in \mathrm{HOD}\ \exists z \in \mathrm{HOD}\ \forall y \in \mathrm{HOD}$$
$$\big(y \in z \leftrightarrow \big(y \in x \wedge S\big\langle \phi, \big\langle\begin{smallmatrix} u & v & v_0 & \cdots & v_{n-} \\ x & y & y_0 & \cdots & y_{n-}\end{smallmatrix}\big\rangle\big\rangle = 1\big)\big).$$

Suppose $x, y_0, \ldots, y_{n-} \in \mathrm{HOD}$, and let

$$z = \big\{ y \in \mathrm{HOD} \,\big|\, y \in x \wedge S\big\langle \phi, \big\langle\begin{smallmatrix} u & v & v_0 & \cdots & v_{n-} \\ x & y & y_0 & \cdots & y_{n-}\end{smallmatrix}\big\rangle\big\rangle = 1 \big\}.$$

Let F be a finite set of ordinals such that $x, y_0, \ldots, y_{n-}$ are definable from ordinals in F as in (6.22). By the reflection principle[6.9] there exists $\gamma \in \mathrm{Ord}$ such that $\{x, y_0, \ldots, y_{n-}\} \cup F \subseteq V_\gamma$ and $V_\gamma \cap \mathrm{HOD} \prec^{\{\phi\}} \mathrm{HOD}$. Let S_γ be the (full) valuation function for $V_\gamma \cap \mathrm{HOD}$. Then for any $y \in x$,

$$S\big\langle \phi, \big\langle\begin{smallmatrix} u & v & v_0 & \cdots & v_{n-} \\ x & y & y_0 & \cdots & y_{n-}\end{smallmatrix}\big\rangle\big\rangle = 1 \leftrightarrow S_\gamma\big\langle \phi, \big\langle\begin{smallmatrix} u & v & v_0 & \cdots & v_{n-} \\ x & y & y_0 & \cdots & y_{n-}\end{smallmatrix}\big\rangle\big\rangle = 1.$$

Note that S_γ has a uniform definition in terms of γ in any $V_{\gamma'}$ such that $\gamma + \omega \leqslant \gamma'$ (being generous). It follows that z is definable in $V_{\gamma+\omega}$ from ordinals in $F \cup \{\gamma\}$. Hence z is OD, so, since x is HOD and $z \subseteq x$, z is HOD. Hence,

$$\exists z \in \mathrm{HOD}\ \forall y \in \mathrm{HOD}\, \big(y \in z \leftrightarrow \big(y \in x \wedge S\big\langle \phi, \big\langle\begin{smallmatrix} u & v & v_0 & \cdots & v_{n-} \\ x & y & y_0 & \cdots & y_{n-}\end{smallmatrix}\big\rangle\big\rangle = 1\big)\big),$$

as claimed.

The definable wellordering $<^{\mathrm{OD}}$ of OD may be used to give a uniform definition of wellorderings $<^{\mathrm{OD}}_\alpha$ of each $V_\alpha \cap \mathrm{HOD}$, which are therefore ordinal-definable, indeed, hereditarily ordinal-definable. So $\mathrm{HOD} \models \ulcorner$for every ordinal α, there is a wellordering of $V_\alpha\urcorner$, which is a version of AC in the context of ZF. $\square^{6.24}$

(6.24) can be formulated without mention of proper classes only as a metatheorem, i.e., a theorem of S that states the ZF-provability of the axioms of ZFC relativized to the predicate HOD.

(6.26) **Theorem** [S] *Suppose σ is an axiom of* ZFC. *Then*

$$\mathsf{ZF} \vdash \sigma^{\mathrm{HOD}},$$

where σ^{HOD} is σ relativized to a unary predicate symbol defined as HOD.

Proof The short proof goes as follows. In the interest of clarity, let P be a unary predicate symbol in the extension s^+ of s defined as HOD; hence, σ^{HOD} in the statement of the theorem is σ^P. Let C be a constant symbol in the extension c^+ of c defined as the class HOD. Suppose σ is an axiom of ZFC. Let σ^P be the s^+-formula obtained by relativizing σ to the predicate P, and let σ^C be the c^+-formula obtained by relativizing σ to the class C. Clearly,

$$\mathsf{GB} \vdash \ulcorner(\sigma^P) \leftrightarrow (\sigma^C)\urcorner.$$

By (3.98)

$$\mathsf{GB} \vdash \ulcorner(\sigma^C) \leftrightarrow (C) \models (\sigma)\urcorner.$$

By (6.24)

$$\mathsf{GB} \vdash \ulcorner(C) \models (\sigma)\urcorner,$$

so

$$\mathsf{GB} \vdash \sigma^C,$$

so

$$\mathsf{GB} \vdash \sigma^P.$$

Since GB is a conservative extension of ZF,

$$\mathsf{ZF} \vdash \sigma^P,$$

i.e., $\mathsf{ZF} \vdash \sigma^{\mathrm{HOD}}$. $\square^{6.26}$

We may avoid all reference to (theories that allow) proper classes by refashioning the arguments up to and including the proof of (6.26) as metatheorems concerning ZF-provability; but this is—as indicated above—the long way 'round, and it's less appealing intuitively.

(6.27) **Theorem** [S] *If* ZF *is consistent then* ZFC *is consistent.*

Remark We will provide two proofs illustrating slightly different approaches.

Proof Method 1 Suppose ZFC is not consistent. Let π be a ZFC-proof of a contradiction $\sigma \wedge \neg\sigma$. Let Φ be the set of formulas occurring in π. Since Φ is finite, there exists a GB-proof π_1 of the existence of a Φ-valuation function for HOD.[11] Let π_2 be a GB-proof of (6.24). Let π_3 be a GB-proof that begins with π_1 to show the existence of a Φ-valuation function S for HOD; then uses π_2 to show that $S\langle\theta, 0\rangle = 1$ for each $\theta \in \Phi$ that is an axiom of ZFC used as a premise in π; then uses the sequence π of inferences to show that $S\langle\sigma \wedge \neg\sigma, 0\rangle = 1$, which contradicts the fact that, since $S\langle\sigma, 0\rangle = 1$ iff $S\langle\neg\sigma, 0\rangle = 0$, $S\langle\sigma \wedge \neg\sigma, 0\rangle = 0$. π_3 is therefore a GB-proof of a contradiction. Hence GB is inconsistent. Since GB is a conservative extension of ZF (provably in S), ZF is inconsistent.

Method 2 Suppose π is a ZFC-proof of a contradiction $\sigma \wedge \neg\sigma$. Let π^{HOD} be obtained by relativizing every formula in π to (the predicate) HOD. π^{HOD} is easily seen to be a $\mathsf{ZF}^{\mathrm{HOD}}$-proof of $\sigma^{\mathrm{HOD}} \wedge \neg\sigma^{\mathrm{HOD}}$. For each axiom θ of ZFC occurring as a premise of π, θ^{HOD} is the corresponding premise of π^{HOD}, and $\mathsf{ZF} \vdash \theta^{\mathrm{HOD}}$.[6.26] Hence, $\mathsf{ZF} \vdash \sigma^{\mathrm{HOD}} \wedge \neg\sigma^{\mathrm{HOD}}$, so ZF is inconsistent. $\square^{6.27}$

6.7 Consistency of $\mathsf{ZFA} + \neg\mathsf{AC}$

The inspiration for Cohen's proof of the independence of the axiom of choice came from previous work of Fraenkel and Mostowski showing the independence of the axiom of choice from the theory ZFA of set theory with atoms. In the simplest

[11] Of course, what π_1 proves is $\ulcorner$there exists a $(\hat{\Phi})$-valuation function for HOD$\urcorner$, where $\hat{x}$ is in general the canonical s'-term for an hereditarily finite set x. The proof that is asserted to exist could go as follows. First prove $\ulcorner$Every formula in $(\hat{\Phi})$ is in $\mathcal{E}^{\mathsf{s}}_{(\hat{n})}$.$\urcorner$ for some number n. Then prove (1.67). Conclude with $\ulcorner$Therefore there exists an $\mathcal{E}^{\mathsf{s}}_{\hat{0}}$-valuation function for HOD; hence, there exists an $\mathcal{E}^{\mathsf{s}}_{\hat{1}}$-valuation function for HOD; hence, there exists an $\mathcal{E}^{\mathsf{s}}_{\hat{2}}$-valuation function for HOD; ...; hence, there exists an $\mathcal{E}^{\mathsf{s}}_{\hat{n}}$-valuation function for HOD.$\urcorner$.

case there is a set A of atoms, or urelements, which are elements that are not sets and have no members. The universe is built from A by iteration of the powerset operation. In general, for any set X:

$$\begin{aligned} \mathcal{P}^0 X &\stackrel{\text{def}}{=} X \\ \mathcal{P}^{\alpha+1} X &\stackrel{\text{def}}{=} \mathcal{P}\mathcal{P}^\alpha X \\ \mathcal{P}^\alpha X &\stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} \mathcal{P}^\beta X \qquad \text{if } \alpha \text{ is limit} \\ V(X) \stackrel{\text{def}}{=} \mathcal{P}^\infty X &\stackrel{\text{def}}{=} \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{P}^\alpha X. \end{aligned}$$

6.7.1 Consistency of ZFA

The consistency of ZFA + ⌜A is nonempty⌝ relative to that of ZF is easy to prove. Working in GB, suppose A is a set. Define $\langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ recursively as follows:

1. $M_0 = \{\langle 0, a\rangle \mid a \in A\}$;
2. $M_{\alpha+1} = \{\langle 1, x\rangle \mid x \subseteq M_\alpha\}$;
3. if α is a limit ordinal, $M_\alpha = \bigcup_{\beta\in\alpha} M_\beta$.

Let $M = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$. Let $\mathfrak{M}$ be the structure with domain M; two unary predicate symbols, ⌜A⌝ for atoms and ⌜S⌝ for sets; and a binary predicate symbol ⌜E⌝ for membership:

1. $\mathsf{A}^{\mathfrak{M}} = \{\langle x\rangle \mid x \in M_0\}$;
2. $\mathsf{S}^{\mathfrak{M}} = \{\langle x\rangle \mid x \in M\backslash M_0\}$;
3. $\mathsf{E}^{\mathfrak{M}} = \{\langle x, \langle 1, y\rangle\rangle \mid x, y \in M \wedge x \in y\}$.

(6.28) **Theorem** [GB] *Given any structure $\mathfrak{M}$ defined from a set A as above, $\mathfrak{M} \models$ ZFA, i.e., for every axiom σ of ZFA, for every $\{\sigma\}$-valuation function S for $\mathfrak{M}$, $S\langle\sigma, 0\rangle = 1$.*

Proof The proof is entirely straightforward. □6.28

(6.29) *Using the above method we can obtain models of ZFA with various additional conditions on the set A of atoms, which can be used to derive consistency results from* Con ZF.

The general approach is as follows. Suppose Θ is a recursively enumerable theory, σ is an s-sentence, and C + σ proves that there is a structure $\mathfrak{M}$ that satisfies Θ.[12] We know[6.1] that this does not necessarily imply that C + σ proves that Θ is consistent, as we have the counterexample where Θ = ZF, σ is arbitrary, and $\mathfrak{M} = (V; \in)$. We do, however, have the following theorem.

(6.30) **Theorem** [S] *Under the conditions just stated, if ZF + σ is consistent then Θ is consistent.*

[12] ⌜$\mathfrak{M} \models \Theta$⌝ is formulated in terms of a Σ_1 definition of Θ.

Proof Suppose toward a contradiction that Θ is inconsistent, and let θ be a conjunction of a finite set of sentences in Θ such that $\neg\theta$ is a logical validity. Then $\mathsf{C}+\sigma$ proves that there is a structure $\mathfrak{M}$ such that $\mathfrak{M}\models\Theta$, and also that $\mathfrak{M}\models\neg\theta$. Each conjunct of θ can be shown in C to satisfy the postulated Σ_1 definition of Θ, so $\mathsf{C}+\sigma$ proves that $\mathfrak{M}\models\theta$. Thus $\mathsf{C}+\sigma$ is inconsistent, contrary to hypothesis. $\square$[6.30]

Note that this is the approach we have used in the proof of the relative consistency of ZFC using HOD. As in that case, we often simply recapitulate the argument in the proof of (6.30), rather than quoting the theorem.

In the particular case of interest, σ is the conjunction of the $\mathsf{Infinity}$ and Power axioms, so $\mathsf{C}+\sigma$ is GB. Working in GB, the set A used in the construction of the model $\mathfrak{M}$ as above for Theorem 6.28 may be taken to be infinite, in which case $\mathfrak{M}\models\mathsf{ZFA}+\ulcorner A\text{ is infinite}\urcorner$.

6.7.2 Symmetric models

Working in ZFA, if $\pi : A \overset{\text{bij}}{\to} A$ is a permutation of A, π naturally extends to an automorphism $\hat{\pi}$ of $V(A)$ defined by $\in$-recursion:

$$\hat{\pi}x = \begin{cases}\pi x & x\in A\\ \{\hat{\pi}y \mid y\in x\} & x\notin A.\end{cases}$$

Definition [ZF] *Suppose Γ is a group.*

1. $\mathcal{S}_\Gamma \overset{\text{def}}{=}$ *the set of subgroups of* Γ.
2. $\mathcal{F}$ *is a* filter on Γ, *or* Γ-filter $\overset{\text{def}}{\iff}$
   1. $\mathcal{F}$ *is a nonempty subset of* $\mathcal{S}_\Gamma$;
   2. *for any* $F, F'\in\mathcal{S}_\Gamma$, *if* $F\in\mathcal{F}$ *and* $F'\supseteq F$ *then* $F'\in\mathcal{F}$ *(so, in particular,* $\Gamma\in\mathcal{F}$*); and*
   3. *for any* $F, F'\in\mathcal{F}$, $F\cap F'\in\mathcal{F}$.

   A Γ*-filter* $\mathcal{F}$ *is* normal $\overset{\text{def}}{\iff} \forall F\in\mathcal{F}\,\forall\pi\in\Gamma\ \pi^{-1}F\pi\in\mathcal{F}$, *where* $\pi^{-1}F\pi = \{\pi^{-1}\rho\pi \mid \rho\in F\}$.

Definition [ZFA] *Suppose Γ is a group of permutations of A.*

1. *For* $x\in V(A)$, $\operatorname{sym}_\Gamma x \overset{\text{def}}{=} \{\pi\in\Gamma \mid \hat{\pi}x = x\}$. *Note that* $\operatorname{sym}_\Gamma x\in\mathcal{S}_\Gamma$.
2. *Suppose* $\mathcal{F}$ *is a normal filter on* Γ.
   1. *An element* $x\in V(A)$ *is* $\mathcal{F}$-symmetric $\overset{\text{def}}{\iff} \operatorname{sym}_\Gamma x\in\mathcal{F}$.
   2. *The class of* hereditarily $\mathcal{F}$-symmetric *elements of* $V(A) \overset{\text{def}}{=} V(A,\mathcal{F}) \overset{\text{def}}{=} \{x\in V(A) \mid \forall y\in\operatorname{tc}\{x\}\ \operatorname{sym}_\Gamma y\in\mathcal{F}\}$.

Note that for every $x\in V = V(0)$, the class of pure sets, $\operatorname{sym}_\Gamma x = \Gamma\in\mathcal{F}$, so $V\subseteq V(A,\mathcal{F})$.

Without loss of generality we suppose that all atoms are $\mathcal{F}$-symmetric (as we may ignore those that aren't), so all are in $V(A,\mathcal{F})$.

(6.31) **Theorem** [GBA] *Suppose $\mathcal{F}$ is a normal filter on a group Γ of permutations of the set A of atoms. For any $\pi\in\Gamma$, $\hat{\pi}\restriction V(A,\mathcal{F})$ is an automorphism of $V(A,\mathcal{F})$.*

Proof Straightforward. $\square^{6.31}$

(6.32) **Theorem** [GBA] *Suppose $\mathcal{F}$ is a normal filter on a group Γ of permutations of the set A of atoms. Then $V(A, \mathcal{F}) \models$* ZF*, i.e., for every axiom σ of* ZF *and every $\{\sigma\}$-valuation function S for $V(A, \mathcal{F})$, $S\langle\sigma, 0\rangle = 1$.*

Remark Alternatively, we have the metatheorem

Theorem [S] *Suppose σ is an axiom of* ZF*. Let M be a new unary predicate symbol, and let σ^M be σ relativized to M. Let u be a variable that does not occur except where explicitly indicated. Then* ZFA $\vdash$ ⌜*Suppose $\mathcal{F}$ is a normal filter on a group Γ of permutations of the set A of atoms, and suppose for all (u), $(\tilde{M}\langle u\rangle)$ iff (u) is hereditarily $\mathcal{F}$-symmetric. Then (σ^M).*⌝.

Proof For notational simplicity, let $M = V(A, \mathcal{F})$. By construction, M is transitive. Let $M_\alpha \overset{\text{def}}{=} \{y \in M \mid \operatorname{rk} y < \alpha\}$. Since automorphisms of M preserve rank, $\operatorname{sym}_\Gamma M_\alpha = \Gamma \in \mathcal{F}$, so M_α is $\mathcal{F}$-symmetric, and therefore $M_\alpha \in M$. For any set $x \subseteq M$, letting $\alpha = \operatorname{rk} x$, $x \subseteq M_\alpha \in M$, so M is almost universal. Hence,[3.214] M satisfies all axioms of ZF with the possible exception of Comprehension.

It therefore remains only to show that M satisfies Comprehension. Suppose

$$\sigma = \forall v_0, \ldots, v_{n^-} \,\forall u \,\exists w \,\forall v \,(v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is an s-formula, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{u, v, v_0, \ldots, v_{n^-}\}$. Suppose S is a $\{\sigma\}$-valuation function for M. Given $y_0, \ldots, y_{n^-}, x \in M$, we must show that there exists $z \in M$ such that for all $y \in M$, $y \in z \leftrightarrow y \in x \wedge S\big\langle \phi, \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} & u & v \\ y_0 & \cdots & y_{n^-} & x & y\end{smallmatrix}\big\rangle\big\rangle = 1$. In other words, letting

$$z = \big\{ y \in x \,\big|\, S\big\langle \phi, \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} & u & v \\ y_0 & \cdots & y_{n^-} & x & y\end{smallmatrix}\big\rangle\big\rangle = 1 \big\},$$

we must show that $z \in M$.

Since $z \subseteq M$, it is enough to show that z is $\mathcal{F}$-symmetric. Let $F = \operatorname{sym}_\Gamma x \cap \bigcap_{m \in n} \operatorname{sym}_\Gamma y_m$. Then for any $\pi \in F$, $\hat{\pi} x = x$ and for all $m \in n$, $\hat{\pi} y_m = y_m$. Since $\hat{\pi} \restriction M$ is an automorphism of M, for any $y \in x$,

$$S\big\langle \phi, \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} & u & v \\ y_0 & \cdots & y_{n^-} & x & y\end{smallmatrix}\big\rangle\big\rangle = 1 \leftrightarrow S\big\langle \phi, \big\langle\begin{smallmatrix} v_0 & \cdots & v_{n^-} & u & v \\ y_0 & \cdots & y_{n^-} & x & \hat{\pi} y\end{smallmatrix}\big\rangle\big\rangle = 1,$$

so $y \in z \leftrightarrow \hat{\pi} y \in z$, whence $\hat{\pi} z = z$. Thus, z is $\mathcal{F}$-symmetric. $\square^{6.32}$

A model of the form $V(A, \mathcal{F})$ is called a *symmetric inner model*. By a judicious choice of Γ and $\mathcal{F}$, we can arrange that such a model satisfies various sentences in addition to ZF.

(6.33) **Theorem (Fraenkel-Mostowski)** [S] *If* ZF *is consistent then so is* ZFA + ¬ AC.

Proof As discussed above, it suffices to work in GBA+⌜A is infinite⌝[6.29] to construct a symmetric inner model of ZFA + ¬ AC, which we now do.

Let Γ be the full permutation group of A. For each $s \in [A]^{<\omega}$ (the set of finite subsets of A) let $F_s = \{\pi \in \Gamma \mid \forall a \in s \;\pi a = a\}$. Let $\mathcal{F}$ be the filter generated by these groups, i.e., a subgroup F of Γ is in $\mathcal{F}$ iff for some $s \in [A]^{<\omega}$, $F_s \subseteq F$. Let $M = V(A, \mathcal{F})$. Note that for any permutation π of A, $\pi^{-1} F_s \pi = F_{\pi^{\leftarrow} s}$, so $\mathcal{F}$ is a normal filter.

By the definition[3.52] of 'infinite', A is not equipollent with a number, i.e., A is infinite in sense (3.133.3), and M also clearly believes this to be the case. We will show that as far as M is concerned, however, A is finite in sense (3.133.2) (and therefore also in sense (3.133.1)[3.134.2]). Since AC implies[3.149] that any set that is infinite in sense (3.133.3) is infinite in senses (3.133.1, 2) as well, $M \models \neg\,\mathsf{AC}$.

Suppose toward a contradiction that $f \in M$ and $f : B \overset{\text{sur}}{\to} A$ where $B \subsetneqq A$. By recursion on $n \in \omega$, define $A_0 = A$, and $A_{n+1} = f^{\leftarrow} A_n$. We will show by induction on $n \in \omega$ that $A_{n+1} \subsetneqq A_n$. For $n = 0$ this is true because $A_1 = B \subsetneqq A$. Suppose $A_{n+1} \subsetneqq A_n$. Then $A_{n+2} = f^{\leftarrow} A_{n+1} \subseteq f^{\leftarrow} A_n = A_{n+1}$, and $A_{n+1} \backslash A_{n+2} = f^{\leftarrow}(A_n \backslash A_{n+1})$. Since $A_n \backslash A_{n+1}$ is a nonempty subset of A and f is surjective to A, $A_{n+1} \backslash A_{n+2} \neq 0$.

Let $B_n = A_n \backslash A_{n+1}$. The B_ns are disjoint nonempty sets of atoms, and $f \restriction B_{n+1} : B_{n+1} \overset{\text{sur}}{\to} B_n$. Since $f \in M$, $\operatorname{sym}_\Gamma f \supseteq F_s$ for some finite $s \subseteq A$. Let $n > 0$ be such that $B_n \cap s = 0$. Let a be a member of B_{n+1}. Then $fa \in B_n$, and fa is therefore not in s. Let $\pi \in F_s$ be such that $\pi a = a$ and $\pi(fa) \neq fa$. Then $(\hat{\pi} f)a = (\hat{\pi} f)(\pi a) = \pi(fa) \neq fa$, so $\hat{\pi} f \neq f$, contradicting the fact that $\operatorname{sym}_\Gamma f \supseteq F_s$. □[6.33]

6.8 Summary

The principal foundational questions having to do with finitary objects have been dealt with in Chapters 1–4. In Chapter 5 we have begun the study of infinitary objects, where we have found a plethora of questions in search of answers. Given the fundamental position of the theory of membership in this regard, these questions essentially belong to the metatheory of membership.

If M is a transitive class then $(M; \in)$ is a model of Extension and Foundation, and the remaining axioms of ZF correspond to closure properties of M. On the other hand, every wellfounded setlike extensive relation is isomorphic to a transitive class. Hence, transitive class models loom large in the metatheory of membership. The paradigm of such a model is of course $(V; \in)$ itself.

The absoluteness of formulas between transitive models is a generally useful concept, and the absoluteness of Σ^1_1 and Σ^1_2 formulas between $(V; \in)$ and appropriate transitive models of a sufficient finite fragment of ZF^- is of particular importance for descriptive set theory.

The *reflection principle* is frequently useful in the setting of a hierarchy of sets wellordered by inclusion (such as $\langle V_\alpha \mid \alpha \in \mathrm{Ord} \rangle$). We use it to show (in S) that if ZF is consistent then it is not finitely axiomatizable. It follows that S is not finitely axiomatizable (on the same hypothesis, although it is known that Con S suffices) On the other hand, we show that C *is* finitely axiomatizable.

We introduce the important notion of ordinal definability and show (in GB) that the class HOD of hereditarily ordinal-definable sets is an inner model of ZFC. We state this also in the form of a metatheorem (of S) that for any axiom θ of ZFC, $\mathsf{ZF} \vdash \theta^{\mathrm{HOD}}$. Either way, we have derived the consistency of ZFC from that of ZF.

We finish with the symmetric inner model construction invented by Fraenkel and Mostowski to prove the consistency of the negation of AC in the context of the Zermelo-Fraenkel set theory with atoms, ZFA.

Chapter 7

Constructibility

Contents

7.1 The constructible universe **433**

7.1.1 Absoluteness of constructibility 437

7.2 A definable wellordering of L **438**

7.3 The condensation lemma **440**

7.4 The continuum hypothesis in L **443**

7.5 Relative constructibility **444**

7.6 $\mathbb{R}$ in L **445**

7.6.1 Suslin's hypothesis 447

7.6.2 Borel determinacy 452

7.7 Summary **464**

[Thomas Jech's *Set Theory*[12] is an excellent source for all aspects of the modern theory of membership, including constructibility.]

7.1 The constructible universe

As noted in the introduction to Chapter 6, there are fundamental questions about the membership relation that are not settled by the natural axiom system ZF, e.g., the axiom of choice and the continuum hypothesis. It was the critical insight of Kurt Gödel that the difficulty of bringing the axioms and intuition of ZF—as embodied in the concept of the von Neumann hierarchy—to bear on these questions springs from the indefinite nature of the successor step $V_\alpha \mapsto V_{\alpha+1}$, i.e., the formation of the powerset $\mathcal{P}\,V_\alpha$: there is nothing in this construction that suggests a means by which we might well-order the powerset $\mathcal{P}\,A$ of an arbitrary set A, nor is there any obvious way of defining (or otherwise demonstrating the existence of) a subset of $\mathcal{P}\,A$ of cardinality intermediate between that of A and $\mathcal{P}\,A$ (again for arbitrary A) or of showing that there is no such set.

Gödel asked whether one might achieve a resolution of these—and perhaps a good many other—questions by means of a more controlled process of admitting sets to V. In particular, suppose we admit only those sets that ZF requires us to admit. To this end we define a new hierarchy $\langle L_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ as follows. We will carry out the construction and the initial discussion in ZF^-, i.e., ZF without Power. This is not much extra work, and it will be important for certain applications.

(7.1) **Definition** [ZF^-]

1. $L_0 = 0$;
2. *for each ordinal* α, $L_{\alpha+1}$ *consists of all subsets of* L_α *definable over* $(L_\alpha; \in)$ *from parameters in* L_α;
3. *for each limit ordinal* α, $L_\alpha = \bigcup\{L_\beta \mid \beta < \alpha\}$.

Since we are working in ZF^-, we cannot justify (7.1.2) by defining $L_{\alpha+1}$ as a subset of $\mathcal{P}\,L_\alpha$. Instead, we proceed as follows. Let S be a valuation function for L_α. For each $n \in \omega$, each $n+1$-sequence $\langle v_0, \ldots, v_n\rangle$ of distinct variables, each s-formula ϕ with $\mathrm{Free}\,\phi \subseteq \{v_0, \ldots, v_n\}$, and each $(n-1)$-sequence $\langle x_1, \ldots, x_n\rangle$ from L_α, there is a unique set x such that

$$x = \left\{y \in L_\alpha \,\middle|\, S\left\langle \phi, \left\langle \begin{smallmatrix} v_0 & v_1 & \cdots & v_n \\ y & x_1 & \cdots & x_n \end{smallmatrix} \right\rangle\right\rangle = 1\right\}.$$

This defines a function f such that $\mathrm{dom}\, f$ consists of all suitable sequences $\big\langle\langle v_0, \ldots, v_n\rangle, \phi, \langle x_1, \ldots, x_n\rangle\big\rangle$, which is easily shown to be a set (without using Power). By $\mathsf{Replacement}$,[3.30] $\mathrm{im}\, f$ is a set.[1] We define $L_{\alpha+1}$ to be $\mathrm{im}\, f$.

(7.2) **Definition** [ZF^-]

1. x is constructible $\overset{\text{def}}{\iff} Lx \overset{\text{def}}{\iff} \exists_{\mathrm{Ord}}\alpha\; x \in L_\alpha$.
2. [GB^-] *The* constructible universe $\overset{\text{def}}{=} L \overset{\text{def}}{=} \{x \mid Lx\}$.

Note that (7.2.1) defines a unary predicate symbol $\ulcorner L \urcorner$ in ZF^-, while (7.2.2) defines the corresponding nulary operation symbol $\ulcorner L \urcorner$ in GB^-.[2]

(7.3) **Theorem** [GB^-]

1. *For every* $\alpha \in \omega + 1$, $L_\alpha = V_\alpha$.
2. *For every* $\alpha \in \mathrm{Ord}$
   1. L_α *is transitive;*
   2. $L_\alpha \cap \mathrm{Ord} = \alpha$;
   3. $L_\alpha \in L_{\alpha+1}$;
   4. *for every* $\beta \geqslant \alpha$, $L_\alpha \subseteq L_\beta$.
3. *Likewise*
   1. L *is transitive;*
   2. $L \cap \mathrm{Ord} = \mathrm{Ord}$, *i.e.*, $\mathrm{Ord} \subseteq L$;
   3. $\forall\alpha \in \mathrm{Ord}\;\; L_\alpha \in L$;
   4. $\forall\alpha \in \mathrm{Ord}\;\; L_\alpha \subseteq L$.

Proof Straightforward. $\square$[7.3]

For the remainder of this discussion we will generally state metatheorems in terms of ZF *and* GB, *keeping in mind that* ZF^- *and* GB^- *may be substituted whenever the power operation is not intrinsic to the matter at hand. We will occasionally emphasize this point.*

[1] As we have stated $\mathsf{Replacement}$, it deals with classes that are functions. In ZF^-, $\mathsf{Replacement}$ is a schema that states for each formula, that if it defines a functional relationship, its "image" on a set is a set.

[2] As usual, in the context of ZF^- we may use $\ulcorner \cdot \in L \urcorner$ informally to mean $\ulcorner L\cdot \urcorner$.

Definition [S] *The* axiom of constructibility *is the* s^+*-sentence* $\forall \mathrm{v}_0\ \boldsymbol{L}\bar{\mathrm{v}}_0$.[3] *In the context of* GB*, we may use '*$\boldsymbol{V}=\boldsymbol{L}$*'.*[4] *It is convenient and conventional to use '*$\boldsymbol{V}=\boldsymbol{L}$*' (or '*$\ulcorner V = L\urcorner$*') also in the context of* ZF*, with the understanding that it stands for* $\forall \mathrm{v}_0\ \boldsymbol{L}\bar{\mathrm{v}}_0$.

It is important to note that in calling this sentence an axiom we do not mean to imply that we add it to the canon of ZF, i.e., we do not *assert* that all sets are constructible—indeed, constructibility is widely viewed as an unjustifiable restriction on the notion of a set. As a hypothesis, however, its study is central to the metatheory of membership.

To say that ZF requires us to admit all sets in L is not quite accurate. It is conceivable that for some $\alpha \in \mathrm{Ord}$, $L_\alpha \models \mathsf{ZF}$, in which case of course, we are not obligated (by ZF) to add any more. As we will see, however, if M is a transitive set and $M \models \mathsf{ZF}$, then $L_\alpha \subseteq M$, where $\alpha = M \cap \mathrm{Ord}$. Moreover, $L_\alpha \models \mathsf{ZF}$ in this event.

We will show that

$$\begin{aligned} &\mathsf{GB} \vdash \ulcorner L \models \mathsf{ZF}\urcorner \\ &\mathsf{GB} \vdash \ulcorner L \models \mathsf{AC}\urcorner \\ &\mathsf{GB} \vdash \ulcorner L \models \mathsf{GCH}\urcorner. \end{aligned}$$

Hence,

$$\begin{aligned} \text{for any } \sigma \in \mathsf{ZF},\ &\mathsf{GB} \vdash \sigma^L \\ &\mathsf{GB} \vdash \mathsf{AC}^L \\ &\mathsf{GB} \vdash \mathsf{GCH}^L. \end{aligned}$$

With 'L' understood as the predicate[7.2.1] (as opposed to the class name[7.2.2]), these are s^+-sentences, and

$$\begin{aligned} \text{for any } \sigma \in \mathsf{ZF},\ &\mathsf{ZF} \vdash \sigma^L \\ &\mathsf{ZF} \vdash \mathsf{AC}^L \\ &\mathsf{ZF} \vdash \mathsf{GCH}^L. \end{aligned}$$

Thus, any proof of an s^+-sentence θ from $\mathsf{ZF}+\mathsf{AC}+\mathsf{GCH}$ can be converted to a proof of θ^L from ZF. Letting θ be, say, $\mathbf{0}=\mathbf{1}$, we see that if $\mathsf{ZF}+\mathsf{AC}+\mathsf{GCH}$ is inconsistent then $\mathsf{ZF} + \mathsf{AC} + \mathsf{GCH} \vdash \theta$, so $\mathsf{ZF} \vdash \theta^L$, so $\mathsf{ZF} \vdash \mathbf{0}=\mathbf{1}$, so ZF is inconsistent. We therefore will have proved (in S) that if ZF is consistent then so is $\mathsf{ZF}+\mathsf{AC}+\mathsf{GCH}$.

(7.4) **Theorem** [GB^-] $L \models \mathsf{ZF}^-$. *Assuming* Power*,* L *also satisfies* Power*, so* $L \models \mathsf{ZF}$.

Proof

(7.5) **Claim** L *is almost universal.*[3.101]

[3]We follow our convention of using bold symbols to represent expression-building operations, in this case '$\boldsymbol{L}$' for the unary formula-building operation corresponding to the constructibility predicate.

[4]Here '$\boldsymbol{L}$' denotes the c^+-term $\ulcorner L\urcorner$. Technically, we should write '$\boldsymbol{L}0$', '$\boldsymbol{L}$' denoting a nulary term-building operation.

Proof Suppose x is a subset of L. Since $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, by $\mathsf{Collection}$ there exists $\alpha \in \mathrm{Ord}$ such that $x \subseteq L_\alpha$. Since[7.3.3.3] $L_\alpha \in L$, the definition of almost universality is satisfied. □[7.5]

Thus[3.214] L satisfies all the axioms of ZF^- with the possible exception of $\mathsf{Comprehension}$, and, assuming Power, L also satisfies Power. The following claim therefore suffices to complete the proof.

(7.6) **Claim** $L \models \mathsf{Comprehension}$.

Proof Suppose

$$\sigma = \forall v_0, \ldots, v_{n\text{-}}\, \forall u\, \exists w\, \forall v\, (v \in w \leftrightarrow (v \in u \wedge \phi)),$$

where ϕ is an s-formula, and $u, v, w, v_0, \ldots, v_{n\text{-}}$ are distinct variables such that $\mathrm{Free}\,\phi \subseteq \{u, v, v_0, \ldots, v_{n\text{-}}\}$. Suppose S is a $\{\sigma\}$-valuation function for L.[5] We must show that

$$\text{(7.7)} \quad \forall y_0, \ldots, y_{n\text{-}} \in L\, \forall x \in L\, \exists z \in L\, \forall y \in L \\ \Big(y \in z \leftrightarrow \Big(y \in x \wedge S\Big\langle \phi, \Big\langle \begin{smallmatrix} u & v & v_0 & \cdots & v_{n\text{-}} \\ x & y & y_0 & \cdots & y_{n\text{-}} \end{smallmatrix} \Big\rangle \Big\rangle = 1\Big)\Big).$$

Let $x, y_0, \ldots, y_{n\text{-}} \in L$ be given, and let

$$z = \Big\{ y \in x \,\Big|\, S\Big\langle \phi, \Big\langle \begin{smallmatrix} u & v & v_0 & \cdots & v_{n\text{-}} \\ x & y & y_0 & \cdots & y_{n\text{-}} \end{smallmatrix} \Big\rangle \Big\rangle = 1 \Big\},$$

using (a single instance of) the class comprehension axiom together with the separation axiom of GB. Let $\beta \in \mathrm{Ord}$ be such that $x, y_0, \ldots, y_{n\text{-}} \in L_\beta$ and $L_\beta \prec^{\{\phi\}} L$.[6.9] Then

$$z = \Big\{ y \in x \,\Big|\, L_\beta \models \phi\Big[\begin{smallmatrix} u & v & v_0 & \cdots & v_{n\text{-}} \\ x & y & y_0 & \cdots & y_{n\text{-}} \end{smallmatrix}\Big] \Big\},$$

so $z \in L_{\beta+1}$. Hence, $z \in L$, and

$$\forall y \in L \Big(y \in z \leftrightarrow \Big(y \in x \wedge S\Big\langle \phi, \Big\langle \begin{smallmatrix} u & v & v_0 & \cdots & v_{n\text{-}} \\ x & y & y_0 & \cdots & y_{n\text{-}} \end{smallmatrix} \Big\rangle \Big\rangle = 1\Big)\Big),$$

as desired.[7.7] □[7.6] □[7.4]

(7.8) **Theorem** [S] *For every* $\sigma \in \mathsf{ZF}$, $\mathsf{ZF} \vdash \sigma^L$.

Remark The same holds for ZF^- in place of ZF.

Proof Suppose σ is an axiom of ZF. Then[7.4] $\mathsf{GB} \vdash \ulcorner L \models \ulcorner(\sigma)\urcorner\urcorner$. Therefore[3.98.2] $\mathsf{GB} \vdash \ulcorner(\sigma^L)\urcorner$, i.e., $\mathsf{GB} \vdash \sigma^L$. Since σ^L is an s^+-sentence, and GB is a conservative extension of ZF, $\mathsf{ZF} \vdash \sigma^L$, as claimed. □[7.8]

The proof just given of Theorem 7.8 has the virtue of brevity and makes use of several important general principles, but we do not have to be this sophisticated about it. We can prove the theorem directly (in ZF, without invoking GB and satisfaction for proper classes). We leave this as a (recommended) exercise for the reader.

[5] Note that we are not asserting that there is a $\{\sigma\}$-valuation function for L. If there is not, then the supposition is vacuous, and $L \models \sigma$ trivially.[1.61.1]

7.1.1 Absoluteness of constructibility

Assume ZF^-.

Suppose $\beta > \omega$ is a limit ordinal. Let $\alpha < \beta$ be given. It is easy to see that for every sufficiently large $\gamma < \beta$, L_γ contains every finite subset of L_α, every ordered pair (x, y) with $x, y \in L_\alpha$, every finite function $f : L_\alpha \rightharpoonup L_\alpha$ (including every finite sequence from L_α), and every $\langle \phi, A \rangle$, where ϕ is an s-formula and A is an L_α-assignment for ϕ.[6]

Let γ be as above. Then for any s-formula ϕ, $S^\phi_{L_\alpha} \subseteq L_\gamma$, where $S^\phi_{L_\alpha} = \{\langle \psi, A \rangle \mid \psi \in \overline{\{\phi\}} \wedge L_\alpha \models \psi[A]\}$ is the $\{\phi\}$-satisfaction relation for L_α. It is straightforward to show by induction on the complexity of ϕ that $S^\phi_{L_\alpha} \in L_{\gamma+1}$, using at each stage the fact that $S^\phi_{L_\alpha}$ is definable over L_γ from L_α and $S^\psi_{L_\alpha}$ for the immediate subformula(s) ψ of ϕ. It follows that the full satisfaction relation S_{L_α} for L_α is in $L_{\gamma+2}$ and therefore in L_β.

For a given limit ordinal $\beta > \omega$, this is true for every $\alpha < \beta$. It follows that the function $\{(L_\alpha, L_{\alpha+1}) \mid \alpha < \beta\}$ is definable over L_β. By induction we can show that for every $\gamma < \beta$, the sequence $\langle L_\alpha \mid \alpha < \gamma \rangle$ is in L_β, and the function $\{(\gamma, \langle L_\alpha \mid \alpha < \gamma \rangle) \mid \gamma < \beta\}$ is definable over L_β. Hence, $\langle L_\alpha \mid \alpha < \beta \rangle$ is definable over L_β.

Note that the same definition works for any L_β with $\beta > \omega$ a limit ordinal, and it also works for L. In fact, the definition is essentially the definition (7.1) of the L-hierarchy (the only difference being that we have specified that it be formulated in $\mathcal{L}^{\mathsf{s}}$, rather than in $\mathcal{L}^{\mathsf{s}^+}$, for technical reasons).

For later use we note that the definition may be given in Σ_1 form: $x = L_\alpha$ iff α is an ordinal and there exists a function $\langle \langle x_\gamma, S_\gamma \rangle \mid \gamma \leqslant \alpha \rangle$ such that

1. $x_0 = 0$;
2. for every limit $\gamma \in \alpha$, $x_\gamma = \bigcup_{\eta \in \gamma} x_\eta$;
3. for every $\gamma \in \alpha \backslash \{0\}$, S_γ is the satisfaction relation for x_γ; and
4. for every $\gamma \in \alpha$, $x_{\gamma+1}$ consists of exactly those subsets of x_γ definable over x_γ from parameters in x_γ;[7] and
5. $x = x_\alpha$.

(7.9) *Based on the preceding discussion, we let* Λ *be a fixed* Σ_1 *formula with* $\mathrm{Free}\,\Lambda = \{\mathrm{v}_0, \mathrm{v}_1\}$, *such that for any limit ordinal* $\beta > \omega$ *for any* $x, y \in L_\beta$,

$$L_\beta \models \Lambda\left[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 \\ x & y\end{smallmatrix}\right] \leftrightarrow \mathrm{Ord}\, y \wedge x = L_y.$$

(7.10) **Theorem** [S]

1. $\mathsf{GB}^- \vdash \ulcorner L \models \boldsymbol{V} = \boldsymbol{L} \urcorner$.
2. $\mathsf{ZF}^- \vdash (\boldsymbol{V} = \boldsymbol{L})^L$.

[6] Recall that s-expressions (including variables) are by definition in V_ω, hence in L_ω.[7.3.1]

[7] To keep this Σ_1, we cannot quantify over subsets of x_γ. Instead, we use quantification bounded by $\{x_{\gamma+1}, S_\gamma\}$ to say that each formula and x_γ-assignment for all but one of its free variables yields a member of $x_{\gamma+1}$ when interpreted via S_γ, and each member of $x_{\gamma+1}$ is obtainable in this way.

Proof These follow directly from the preceding discussion. Note that since $\boldsymbol{V} = \boldsymbol{L}$ is a single sentence, GB^- proves the existence of the $\{\boldsymbol{V} = \boldsymbol{L}\}$-satisfaction relation for L. $\square^{7.10}$

(7.11) **Theorem** [S] *Suppose* θ *is an* s*-sentence.*

$$\mathsf{ZF} \vdash \theta^L \leftrightarrow (\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}) \vdash \theta.$$

Remark Also with ZF^- for ZF.

Proof Suppose $\mathsf{ZF} \vdash \theta^L$. If $V = L$ the restriction of quantifiers to L is no restriction at all, i.e., $(\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}) \vdash \theta^L \leftrightarrow \theta$, so $(\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}) \vdash \theta$.

Conversely, suppose $(\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}) \vdash \theta$. Let π be a proof of θ from ZF together with $\boldsymbol{V} = \boldsymbol{L}$. Let π^L be the same proof with every quantifier restricted to L. Since $\mathsf{ZF} \vdash \sigma^L$ for every axiom σ of ZF,[7.8] and $\mathsf{ZF} \vdash (\boldsymbol{V} = \boldsymbol{L})^L$,[7.10.2] ZF proves all the hypotheses of π^L and therefore also the conclusion, viz., θ^L. $\square^{7.11}$

(7.12) **Theorem** [S] *If* $\mathrm{Con}\,\mathsf{ZF}$ *then* $\mathrm{Con}(\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L})$.

Proof Given a proof π of, say, $\mathbf{0} = \mathbf{1}$, in $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}$, let π^L be the same proof with every quantifier relativized to L. The premises of π^L are theorems of ZF, and the conclusion is $(\mathbf{0} = \mathbf{1})^L$, which is equivalent (over ZF) to $\mathbf{0} = \mathbf{1}$.[8] $\square^{7.12}$

7.2 A definable wellordering of L

The constructible hierarchy defines a prewellordering of L, with x preceding y if for some ordinal α, $x \in L_\alpha$ and $y \notin L_\alpha$. Note that the levels of this prewellordering, i.e., the levels of L, are the sets $L_{\alpha+1} \setminus L_\alpha$. We now wish to refine this to a wellordering $<^L$ of L by wellordering each of these levels. We do this for each α by comparing the s-formulas and parameters from L_α used in the definitions of the members of $L_{\alpha+1}$. The formulas—being members of V_ω—are ordered in type ω by the relation $\prec$,[3.211.3] which is Δ_1 over $V_\omega = L_\omega$. The parameters used in a definition form a finite sequence into L_α, and these sequences are wellordered lexicographically using the ordering $<^L$ restricted to L_α.

Definition [ZF^-] *We define the binary relation* $<^L_\alpha$ *for* $\alpha \in \mathrm{Ord}$ *by recursion on* α*, in such a way that for each ordinal* α*,* $<^L_\alpha$ *is a wellordering of* L_α*, and for each* $\beta < \alpha$*,* $<^L_\alpha$ *is an end extension of* $<^L_\beta$*. Recall that by definition a binary relation* R *is a class of 2-sequences with* $x\,R\,x' \leftrightarrow \langle x, x' \rangle \in R$*.*

1. $<^L_0 = 0$.
2. *If* α *is a limit ordinal then* $<^L_\alpha = \bigcup_{\beta<\alpha} <^L_\beta$.
3. *If* $\alpha = \beta + 1$ *then* $<^L_\alpha$ *is the set of 2-sequences* $\langle x, x' \rangle$ *such that* $x, x' \in L_\alpha$ *and either*
    1. $x, x' \in L_\beta$ *and* $x <^L_\beta x'$*; or*
    2. $x \in L_\beta$ *and* $x' \notin L_\beta$*; or*

[8] It might be thought that $(\mathbf{0} = \mathbf{1})^L$ is not just ZF-equivalent to $\mathbf{0} = \mathbf{1}$, but actually *is* $\mathbf{0} = \mathbf{1}$, since there are no quantifiers to relativize; however, $\mathbf{0} = \mathbf{1}$ is an s^+-sentence. When formulated as an s-sentence, it has quantifiers, but it is absolute for L.

3. $x, x' \notin L_\beta$ *and there exists an* s*-formula* ϕ *with free variables* $\mathrm{v}_0, \dots, \mathrm{v}_n$ *for some* $n \in \omega$ *and* $z = \langle z_0, \dots, z_{n^-} \rangle \in {}^n L_\beta$ *such that*[9]

$$x = \{y \in L_\beta \mid L_\beta \models \phi\big[\begin{smallmatrix} \mathrm{v}_n & \mathrm{v}_0 & \cdots & \mathrm{v}_{n^-} \\ y & z_0 & \cdots & z_{n^-} \end{smallmatrix}\big]\},$$

and for every s*-formula* ϕ' *with free variables* $\mathrm{v}_0, \dots, \mathrm{v}_{n'}$ *for some* $n' \in \omega$ *and* $z' = \langle z'_0, \dots, z'_{n'^-} \rangle \in {}^{n'} L_\beta$ *such that*

$$x' = \{y \in L_\beta \mid L_\beta \models \phi'\big[\begin{smallmatrix} \mathrm{v}_n & \mathrm{v}_0 & \cdots & \mathrm{v}_{n'^-} \\ y' & z'_0 & \cdots & z'_{n'^-} \end{smallmatrix}\big]\},$$

either

1. $z \subsetneq z'$*; or*
2. $z \nsubseteq z'$ *and* $z' \nsubseteq z$ *and, letting* m *be the least ordinal such that* $z_m \neq z'_m$, $z_m <^L_\beta z'_m$ *(lexicographic order); or*
3. $z = z'$ *and* $\phi \prec \phi'$*.*

Definition [ZF$^-$]

1. *For* $x, x' \in L$, $x <^L x' \overset{\text{def}}{\iff}$ *for any (equivalently, for all)* α *such that* $x, x' \in L_\alpha$, $x <^L_\alpha x'$*.*
2. [GB$^-$] $<^L \overset{\text{def}}{=} \bigcup_{\alpha \in \mathrm{Ord}} <^L_\alpha$.

It is straightforward to show that $<^L$ is a wellordering of L.

(7.13) **Theorem** [ZF$^-$] *If* $\alpha > \omega$ *is a limit ordinal,* L_α *satisfies all the axioms of* ZF *except possibly* Comprehension*,* Power*, and* Collection*.*[10]

Proof Straightforward. $\square^{7.13}$

(7.14) **Theorem** [ZF] *Suppose* α *is a limit ordinal. Then the definitions of* L_β *and* $<^L$ *are absolute for* L_α*, i.e., letting* $\ulcorner \cdot \in L_\cdot \urcorner$ *and* $\ulcorner \cdot <^L \cdot \urcorner$ *be the natural* s*-formulas defining these relations, for all* $x, y \in L_\alpha$ *and* $\beta \in \alpha$*,*

$$\begin{aligned} x \in L_\beta &\leftrightarrow \ulcorner [x] \in L_{[\beta]} \urcorner^{L_\alpha} \\ x <^L y &\leftrightarrow \ulcorner [x] <^L [y] \urcorner^{L_\alpha}. \end{aligned} \tag{7.15}$$

Proof The first of these equivalences was proved in Section 7.1.1, and the second is a straightforward exercise. $\square^{7.14}$

(7.15) also hold for $\alpha = \mathrm{Ord}$, i.e., for all $x, y \in L$ and $\beta \in \mathrm{Ord}$,

$$\begin{aligned} x \in L_\beta &\leftrightarrow (x \in L_\beta)^L \\ x <^L y &\leftrightarrow (x <^L y)^L. \end{aligned}$$

Thus,

(7.16) **Theorem** [ZF$^-$]

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[9] If $n = 0$ then $z = 0$.

[10] In particular, as discussed previously,§ 7.1.1 all pairs, ordered pairs, and finite sequences of members of L_α are in L_α.

1. *If $V = L$ then* $\ulcorner <^L \urcorner$ *wellorders the universe;* a fortiori*, every set is wellorderable, i.e.,* AC.
2. [GB⁻] $L \models \mathsf{AC}$.

(7.17) **Theorem** [S] *If* Con ZF *then* Con(ZF + AC).

Proof Use (7.12) and (7.16.1). $\square^{7.17}$

We have, of course, already proved (7.17) using the inner model HOD of hereditarily ordinal-definable sets. Historically, however, the above proof using L came earlier. It was proved by Gödel in 1949; HOD was defined (by Gödel) in 1965.

7.3 The condensation lemma

A key feature of the structure of L is its regularity, or repetitiveness, in the sense that there are many pairs $\beta < \alpha$ such that L_β is elementarily embeddable in L_α. The following theorem expresses this in a strong and useful way.

(7.18) **Theorem** [ZF⁻] *Suppose $B \prec^{\Sigma_1} L_\alpha$, where α is a limit ordinal. Let $\pi : B \overset{\text{sur}}{\to} M$ be the transitive collapse of B. Then for some $\beta \leqslant \alpha$, $M = L_\beta$.*

Remark An alternative proof of this theorem proceeds by formulating a Π_2 sentence Θ such that the transitive models of Θ are exactly the sets L_α, where α is a limit ordinal.[11] That this is possible is not surprising, since the key condition $\ulcorner V = L \urcorner$ may be taken to be $\forall \mathrm{v}_2\, \exists \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)$.[7.9] There is nevertheless some work to be done to show that a sufficient theory of language, satisfaction, etc., can be developed in a Π_2 fragment of ZF⁻.

The proof given below relieves us of this routine but tedious work, by using Λ itself, for which it is sufficient that it have the correct meaning when interpreted in structures L_α, α a limit ordinal. The proof has the additional advantages of bringing out some of the regularity alluded to above in the form of constructibility relationships preserved in Σ_1 substructures and their transitive collapses, and providing some practice in working with the constructible hierarchy.

Proof By Σ_1-elementarity, $(B, \in)$ satisfies the extension axiom, so π is injective. If $\alpha = \omega$ then $B = L_\omega$, so the result is trivial. Suppose therefore that $\alpha > \omega$. Let Λ be the Σ_1 formula referred to above[7.9] that says $\ulcorner (\mathrm{v}_0) = L_{(\mathrm{v}_1)} \urcorner$. Suppose $\beta \in B$ and $\operatorname{Ord} \beta$. Then $L_\alpha \models \exists \mathrm{v}_0\, \Lambda\bigl[{\mathrm{v}_1 \atop \beta}\bigr]$, so $B \models \exists \mathrm{v}_0\, \Lambda\bigl[{\mathrm{v}_1 \atop \beta}\bigr]$. Let $x \in B$ be such that $B \models \Lambda\bigl[{\mathrm{v}_0 \atop x}\,{\mathrm{v}_1 \atop \beta}\bigr]$. Then $L_\alpha \models \Lambda\bigl[{\mathrm{v}_0 \atop x}\,{\mathrm{v}_1 \atop \beta}\bigr]$, so $x = L_\beta$. Hence

$$\operatorname{Ord} \beta \wedge \beta \in B \to L_\beta \in B. \tag{7.19}$$

Now suppose $b \in B$. Then

$$L_\alpha \models \bigl(\exists \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\bigr)\bigl[{\mathrm{v}_2 \atop b}\bigr],$$

[11] One argues that $L_\alpha \models \Theta \to B \models \Theta \to M \models \Theta \to M = L_\beta$ for some (limit) β. The first implication follows from Σ_1-elementarity: Since Θ is Π_2, $\Theta = \forall u_0, \ldots, u_{n^-}\, \Theta'$ for some Σ_1 Θ'. Hence $B \models \Theta$ iff $\forall x_0, \ldots, x_{n^-} \in B$ $B \models \Theta'\bigl[{u_0 \atop x_0} \cdots {u_{n^-} \atop x_{n^-}}\bigr]$. But since $L_\alpha \models \Theta$, for any $x_0, \ldots, x_{n^-} \in B$, $L_\alpha \models \Theta'\bigl[{u_0 \atop x_0} \cdots {u_{n^-} \atop x_{n^-}}\bigr]$, so $B \models \Theta'\bigl[{u_0 \atop x_0} \cdots {u_{n^-} \atop x_{n^-}}\bigr]$ by Σ_1-elementarity.

so

$$B \models \big(\exists \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big)\big[\begin{smallmatrix}\mathrm{v}_2\\ b\end{smallmatrix}\big].$$

(7.20) *Let β be the least ordinal in B such that*

$$B \models \big(\exists \mathrm{v}_0\, (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big)\big[\begin{smallmatrix}\mathrm{v}_1 & \mathrm{v}_2\\ \beta & b\end{smallmatrix}\big].$$

Let $x \in B$ be such that

$$B \models (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1 & \mathrm{v}_2\\ x & \beta & b\end{smallmatrix}\big].$$

Then $b \in x$, and

$$L_\alpha \models \Lambda\big[\begin{smallmatrix}\mathrm{v}_0 & \mathrm{v}_1\\ x & \beta\end{smallmatrix}\big],$$

so $x = L_\beta$, $b \in L_\beta$ and $L_\beta \in B$.

(7.21) **Claim** *β is the least ordinal such that $b \in L_\beta$.*

Proof Suppose not. Then

$$L_\alpha \models \big(\exists \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_1 \in \mathrm{v}_3 \wedge \mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big)\big[\begin{smallmatrix}\mathrm{v}_2 & \mathrm{v}_3\\ b & \beta\end{smallmatrix}\big],$$

so

$$B \models \big(\exists \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_1 \in \mathrm{v}_3 \wedge \mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big)\big[\begin{smallmatrix}\mathrm{v}_2 & \mathrm{v}_3\\ b & \beta\end{smallmatrix}\big],$$

from which it follows as above that there exists $\beta' \in \beta \cap B$ such that

$$B \models \big(\exists \mathrm{v}_0\, (\mathrm{v}_2 \in \mathrm{v}_0 \wedge \Lambda)\big)\big[\begin{smallmatrix}\mathrm{v}_1 & \mathrm{v}_2\\ \beta' & b\end{smallmatrix}\big],$$

contradicting the minimality of β.[7.20] $\square$[7.21]

Hence,

$$b \in B \rightarrow \text{the least } \beta \text{ such that } b \in L_\beta \text{ is in } B.$$

It follows that for any ordinal γ,

$$\Big(\bigcup_{\beta \in \gamma} L_\beta\Big) \cap B = \Big(\bigcup_{\beta \in \gamma \cap B} L_\beta\Big) \cap B. \tag{7.22}$$

Claim *Suppose $\beta \in B$ is an ordinal. Then $\pi L_\beta = L_{\pi\beta}$.*[12]

Proof By induction on β. In other words, suppose the claim is false, and let β_0 be the least ordinal in B for which it fails. Clearly, $\beta_0 \neq 0$.

If β_0 is a limit ordinal then

$$L_{\beta_0} = \bigcup_{\beta \in \beta_0} L_\beta. \tag{7.23}$$

By (induction) hypothesis, for any $\beta \in \beta_0 \cap B$,

$$\pi L_\beta = L_{\pi\beta}. \tag{7.24}$$

Recall that by definition of the transitive collapse, for any $b \in B$,

$$\pi b = \pi^{\rightarrow} b. \tag{7.25}$$

[12] We know from (7.19) that $L_\beta \in B$.

Hence,

$$\begin{aligned}
\pi L_{\beta_0} = \pi^{\rightarrow} L_{\beta_0} = \pi^{\rightarrow} \bigcup_{\beta \in \beta_0} L_\beta &= \bigcup_{\beta \in \beta_0} \pi^{\rightarrow} L_\beta \\
&= \bigcup_{\beta \in \beta_0 \cap B} \pi^{\rightarrow} L_\beta = \bigcup_{\beta \in \beta_0 \cap B} \pi L_\beta \\
&= \bigcup_{\beta \in \beta_0 \cap B} L_{\pi\beta} = \bigcup_{\eta \in \pi\beta_0} L_\eta \\
&= L_{\pi\beta_0},
\end{aligned}$$

where, in addition to (7.23), (7.24), and (7.25), we have used (7.22) and the fact that since β_0 is a limit ordinal, $\beta_0 \cap B$ has limit order type,[13] so $\pi\beta_0$ is a limit ordinal.

Finally, suppose β_0 is a successor ordinal. Then its immediate predecessor is in B, i.e., $\beta_0 = \eta + 1$ and $\eta \in B$. Therefore[7.19] $L_\eta \in B$, and, by hypothesis, $\pi L_\eta = L_{\pi\eta}$.

Suppose $x \in L_{\eta+1} \cap B$. Then for some $n \in \omega$, some s-formula ϕ with $n+1$ free variables, say $v, w_0, \ldots, w_{n^-}$, and some $z_0, \ldots, z_{n^-} \in L_\eta$,

$$x = \left\{ y \in L_\eta \,\middle|\, L_\eta \models \phi\left[\begin{smallmatrix} v & w_0 & \cdots & w_{n^-} \\ y & z_0 & \cdots & z_{n^-} \end{smallmatrix}\right] \right\}.$$

Let d be a variable not in $\{v, w_0, \ldots, w_{n^-}\}$. Then ϕ^d is Δ_0, and for any transitive D and $b, c_0, \ldots, c_{n^-} \in D$,

$$\begin{aligned}
D \models \phi\left[\begin{smallmatrix} v & w_0 & \cdots & w_{n^-} \\ b & c_0 & \cdots & c_{n^-} \end{smallmatrix}\right] &\leftrightarrow \mathrm{Sat}_0\, \phi^d \left[\begin{smallmatrix} d & v & w_0 & \cdots & w_{n^-} \\ D & b & c_0 & \cdots & c_{n^-} \end{smallmatrix}\right] \\
&\leftrightarrow D' \models \phi^d \left[\begin{smallmatrix} d & v & w_0 & \cdots & w_{n^-} \\ D & b & c_0 & \cdots & c_{n^-} \end{smallmatrix}\right],
\end{aligned}$$

for any transitive D' containing D. In particular, letting u be a variable not in $\{d, v, w_0, \ldots, w_{n^-}\}$, and letting $\psi = \forall v \in d\,(v \in u \leftrightarrow \phi^d)$, ψ is Δ_0, and

$$L_\alpha \models \psi\left[\begin{smallmatrix} d & u & w_0 & \cdots & w_{n^-} \\ L_\eta & x & z_0 & \cdots & z_{n^-} \end{smallmatrix}\right].$$

It follows that

$$L_\alpha \models (\exists w_0, \ldots, w_{n^-} \in d\ \psi)\left[\begin{smallmatrix} d & u \\ L_\eta & x \end{smallmatrix}\right],$$

so

$$B \models (\exists w_0, \ldots, w_{n^-} \in d\ \psi)\left[\begin{smallmatrix} d & u \\ L_\eta & x \end{smallmatrix}\right].$$

Let $z'_0, \ldots, z'_{n^-} \in L_\eta \cap B$ be such that

$$B \models \psi\left[\begin{smallmatrix} d & u & w_0 & \cdots & w_{n^-} \\ L_\eta & x & z'_0 & \cdots & z'_{n^-} \end{smallmatrix}\right].$$

Then

$$M \models \psi\left[\begin{smallmatrix} d & u & w_0 & \cdots & w_{n^-} \\ \pi L_\eta & \pi x & \pi z'_0 & \cdots & \pi z'_{n^-} \end{smallmatrix}\right],$$

and $\pi L_\eta = L_{\pi\eta}$, so

$$\pi x = \left\{ y \in L_{\pi\eta} \,\middle|\, L_{\pi\eta} \models \phi\left[\begin{smallmatrix} v & w_0 & \cdots & w_{n^-} \\ y & \pi z'_0 & \cdots & \pi z'_{n^-} \end{smallmatrix}\right] \right\},$$

whence $\pi x \in L_{(\pi\eta)+1} = L_{\pi(\eta+1)} = L_{\pi\beta_0}$. Thus,

$$\pi^{\rightarrow} L_{\beta_0} \subseteq L_{\pi\beta_0}.$$

[13] By Σ_1-elementarity, B is closed under the successor operation on ordinals.

Conversely, if $\hat{x} \in L_{\pi\beta_0} = L_{\pi\eta+1}$ then for some $n \in \omega$, some s-formula ϕ with $n+1$ free variables, say $v, w_0, \ldots, w_{n^-}$, and some $\hat{z}_0, \ldots, \hat{z}_{n^-} \in L_{\pi\eta}$,

$$\hat{x} = \left\{ y \in L_{\pi\eta} \,\middle|\, L_{\pi\eta} \models \phi\left[\begin{smallmatrix} v & w_0 & \cdots & w_{n^-} \\ y & \hat{z}_0 & \cdots & \hat{z}_{n^-} \end{smallmatrix}\right] \right\}.$$

Since $L_{\pi\eta} = \pi L_\eta = \pi^{\rightarrow} L_\eta$, let $z_m = \pi^{-1} \hat{z}_m$ for $m \in n$, and let

$$x = \left\{ y \in L_\eta \,\middle|\, L_\eta \models \phi\left[\begin{smallmatrix} v & w_0 & \cdots & w_{n^-} \\ y & z_0 & \cdots & z_{n^-} \end{smallmatrix}\right] \right\}.$$

Then $x \in L_{\beta_0}$ and the argument just given shows that $\pi x = \hat{x}$. Thus

$$\pi^{\rightarrow} L_{\beta_0} \supseteq L_{\pi\beta_0}.$$

Hence $\pi L_{\beta_0} = \pi^{\rightarrow} L_{\beta_0} = L_{\pi\beta_0}$. □[7.18]

7.4 The continuum hypothesis in L

Definition [S] *The* generalized continuum hypothesis $\stackrel{\text{def}}{=}$ GCH $\stackrel{\text{def}}{=}$ ⌜*for every infinite cardinal* κ, $|\mathcal{P}\kappa| = \kappa^+$⌝.[14]

We will show that GCH holds in L. We begin with a simple cardinality computation.

(7.26) **Theorem** [ZFC] *Suppose α is an infinite ordinal. Then*

$$|L_\alpha| = |\alpha|.$$

Proof By induction on $\alpha \geqslant \omega$. Note that since $\alpha \subseteq L_\alpha$, $|L_\alpha| \geqslant |\alpha|$. As we have previously noted, $L_\omega = V_\omega$, which is countable, i.e., $|L_\omega| = \omega = |\omega|$.

Suppose $\alpha \geqslant \omega$ and $|L_\alpha| = |\alpha|$. We wish to show that $|L_{\alpha+1}| = |\alpha+1|$. Let $\kappa = |\alpha|$. Then $|\alpha+1| = \kappa$, so we wish to show that $|L_{\alpha+1}| = \kappa$. Each member of $L_{\alpha+1}$ is defined by a formula ϕ and an L_α-assignment for all but one of the free variables of ϕ. There are ω formulas and κ assignments, so there are κ such pairs, and therefore $|L_{\alpha+1}| \leqslant \kappa$. Since $|L_{\alpha+1}| \geqslant |\alpha|$, $|L_{\alpha+1}| = \kappa$.

Suppose now that α is a limit ordinal and $\forall \beta < \alpha\ |L_\beta| = |\beta|$. Then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$, so

$$|L_\alpha| \leqslant \sum_{\beta<\alpha} |\beta| \leqslant |\alpha| \cdot |\alpha| = |\alpha|.$$

Hence, $|L_\alpha| = |\alpha|$. □[7.26]

(7.27) **Theorem** [ZF] *Assume $V = L$. Then for every infinite cardinal κ, $\mathcal{P}\kappa \subseteq L_{\kappa^+}$. Hence,*[7.26] GCH.

Proof Assume $V = L$. Note that we therefore have AC. Suppose κ is an infinite cardinal, and suppose $X \subseteq \kappa$. Let $\alpha > \kappa$ be a limit ordinal such that $X \in L_\alpha$. Let M be an elementary substructure of L_α of cardinality κ that includes $\kappa+1 = \kappa \cup \{\kappa\}$ and contains X.[2.159.1] Let $\pi : M \to L_\beta$ be the transitive collapse.[7.18] Then $|L_\beta| = \kappa$,

[14] Note that as formulated here GCH implies the wellorderability of $\mathcal{P}\alpha$ for any ordinal α, and GCH is typically only considered in the context of AC. Obviously, GCH is also typically only considered in the context of Power, so the theories ZF^- and GB^- are not relevant.

so $\beta < \kappa^+$. Let $Y = \pi X$. Since $\kappa + 1$ is transitive, π is the identity on $\kappa + 1$ and $\pi\kappa = \kappa$. Since $L_\alpha \models [X] \subseteq [\kappa]$, $M \models [X] \subseteq [\kappa]$, so $L_\beta \models [Y] \subseteq [\kappa]$. For each $\gamma \in \kappa$,

$$\begin{aligned}\gamma \in X &\leftrightarrow L_\alpha \models [\gamma] \in [X] \leftrightarrow M \models [\gamma] \in [X]\\ &\leftrightarrow L_\beta \models [\pi\gamma] \in [\pi X] \leftrightarrow L_\beta \models [\gamma] \in [Y]\\ &\leftrightarrow \gamma \in Y.\end{aligned}$$

Hence $X = Y$, so $X \in L_\beta$. So every subset of κ is constructed before κ^+, as claimed. Since $|L_{\kappa^+}| = \kappa^+$, $|\mathcal{P}\kappa| = \kappa^+$. $\square$[7.27]

7.5 Relative constructibility

We have up to now considered constructibility in the absolute sense, but there are also relative notions of constructibility. These can be formulated in two principal ways. The first permits the use of an arbitrary set A as a predicate in structures analogous to $(L_\alpha; \in)$ in the fundamental definition (7.1). The second starts with $\mathrm{tc}\{A\}$ as the base structure (in place of 0, in effect) and proceeds with the definition (7.1) otherwise unaltered. In both cases the resulting class is an inner model of ZF. In the former case it is also a model of AC, whereas in the latter case it may not be.

The following definitions may also be formulated in fragments of ZF, such as ZF$^-$.

Definition [ZF] *Suppose A is a set.*

1. $L_0[A] = 0$*;*
2. *for each ordinal α, $L_{\alpha+1}[A]$ consists of all subsets of $L_\alpha[A]$ definable over $(L_\alpha[A]; \in, A \cap L_\alpha[A])$ from parameters in $L_\alpha[A]$; and*
3. *for each limit ordinal α, $L_\alpha[A] = \bigcup\{L_\beta[A] \mid \beta < \alpha\}$.*

In this definition, $(L_\alpha[A]; \in, A \cap L_\alpha[A])$ is a ρ-structure, where ρ is s expanded by the addition of a unary predicate whose denotation is $A \cap L_\alpha[A]$.

In the context of GB we define $L[A]$ as $\bigcup_{\alpha \in \mathrm{Ord}} L_\alpha[A]$. Note that $A \cap L_\alpha[A]$ is an element of $L_{\alpha+1}[A]$. Since A is a set, there exists an ordinal α such that for all $\beta > \alpha$, $A \cap L_\beta[A] = A \cap L_\alpha[A]$, from which it follows that $A \cap L[A] = A \cap L_\alpha[A]$. Hence, $A \cap L[A]$ is an element of $L[A]$, but A in general is not. In the context of ZF, $L[A]$ is definable from A as a parameter. $L[A]$ is a rather gentle modification of L, and the basic properties of L hold for $L[A]$ as well.

(7.28) **Theorem** [GBC] *Suppose A is a set.*

1. $L[A] \models$ ZFC.
2. *As noted above, $A \cap L[A] \in L[A]$. In general, suppose M is an inner model of* ZF *and $A \cap M \in M$. Let $A' = A \cap M$. Then $\ulcorner L[[A']]\urcorner^M = L[A'] = L[A]$.*
3. *In particular, letting $M = L[A]$, $\ulcorner L[[A']]\urcorner^{L[A]} = L[A]$, so $L[A] \models \ulcorner V = L[[A']]\urcorner$.*

In the context of ZF, in general, '$L[A] \models \sigma$' may be replaced by '$\sigma^{L[A]}$', and (7.28) may be formulated as a theorem schema.

Note that in some important cases, $A \in L[A]$. For example, if $A \subseteq \mathrm{Ord}$ then $A \cap L[A] = A$, so $A \in L[A]$. The following definition, on the other hand, guarantees that A is in the resulting class by putting it in at the beginning.

Definition [ZF] *Suppose A is a set.*

1. $L_0(A) = \mathrm{tc}\{A\}$*;*
2. *for each ordinal α, $L_{\alpha+1}(A)$ consists of all subsets of $L_\alpha(A)$ definable over $(L_\alpha(A); \in)$ from parameters in $L_\alpha(A)$; and*
3. *for each limit ordinal α, $L_\alpha(A) = \bigcup\{L_\beta(A) \mid \beta < \alpha\}$.*

As mentioned above, $L(A) \models \mathsf{ZF}$, but $L(A)$ may not model AC.

In the preceding discussion of constructibility we have made implicit use of formulas that describe key features of constructible hierarchies, often indicating them by means of the corner-bracket notation. For ease of future reference we now posit a choice of several specific such formulas, to which we assign names.

(7.29) [GB] *Let ρ be the expansion of s by the addition of a unary predicate. Let σ_0 and σ_1 be respectively an s and a ρ-sentence, and let ϕ_0 and ϕ_1 be respectively an s and a ρ-formula with free variables $\mathrm{v}_0, \mathrm{v}_1$, such that the following hold for any transitive class M and set A.*

1. $(M; \in) \models \sigma_0$ *iff either $M = L$ or $M = L_\alpha$ for some limit ordinal α.*
2. $(M; \in, A \cap M) \models \sigma_1$ *iff either $M = L[A]$ or $M = L_\alpha[A]$ for some limit ordinal α.*
3. *For any limit ordinal α, any $y \in L_\alpha$, and any x, $(L_\alpha; \in) \models \phi_0[x, y]$ iff $x <^L y$.*
4. *For any limit ordinal α, any $y \in L_\alpha[A]$, and any x, $(L_\alpha[A]; \in, A \cap L_\alpha[A]) \models \phi_1[x, y]$ iff $x <^{L[A]} y$.*

7.6 $\mathbb{R}$ in L

The axiom of constructibility provides much more information about the continuum than merely its size. In fact, there are very few interesting questions about $\mathbb{R}$ that are not settled by $\mathsf{ZF} + \boldsymbol{V = L}$. Many of these results follow from the existence of a definable, specifically a Δ^1_2, wellordering of $\mathbb{R}$ in L. As usual, we will address these issues using ${}^\omega\omega$, rather than $\mathbb{R}$ *per se*.

The wellordering we refer to is the canonical wellordering of L restricted to ${}^\omega\omega$. To compute its complexity we consider structures $\mathfrak{M} \subseteq V_\omega$ with signature l—which expands s by the addition of a unary operation symbol $\hat{L}$ intended to represent the constructible hierarchy. For this discussion, let $\mathfrak{L}_\alpha$ be the obvious structure with signature l such that $|\mathfrak{L}_\alpha| = L_\alpha$. As we know from the proof of (7.27), every member of ${}^\omega\omega$ is in L_α for some $\alpha < \omega_1$. Since L_α is countable in this case,[7.26] $\mathfrak{L}_\alpha$ is isomorphic to a structure $\mathfrak{M}$ as above.

Definition [ZF] $\mathcal{S} \overset{\text{def}}{=}$ *the set of l-structures $\mathfrak{M} \subseteq V_\omega$ such that $\mathfrak{M} \cong \mathfrak{L}_\alpha$ for some α.*

We will say an element x of ${}^\omega\omega$ is *represented in* $\mathfrak{M} \in \mathcal{S} \overset{\text{def}}{\iff}$ there exists $x' \in |\mathfrak{M}|$ such that $\pi x' = x$, where $\pi : |\mathfrak{M}| \to L_\alpha$ is the canonical isomorphism of $\mathfrak{M}$ with $\mathfrak{L}_\alpha$.

(7.30) **Theorem** [ZF] *$\mathcal{S}$ is Π^1_1.*

Proof In this discussion, 'structure' will be taken to mean 'l-structure included in V_ω'. The set S of structures $\mathfrak{M}$ such that $\in^{\mathfrak{M}}$ is extensional, is clearly arithmetical. The set S' of structures $\mathfrak{M} \in S$ such that $\in^{\mathfrak{M}}$ is wellfounded, is clearly Π^1_1, wellfoundedness being the property that *every* nonempty $X \subseteq |\mathfrak{M}|$ has an $\in$-minimal member. Since satisfaction relations exist and are unique, we have access to satisfaction in $\mathfrak{M}$ in a Π^1_1 way via constructions of the form $\ulcorner$for every satisfaction relation R for $\mathfrak{M}$ ...$\urcorner$. In this way we obtain a Π^1_1 characterization of the set S'' of extensional wellfounded structures $\mathfrak{M}$ such that $\hat{L}^{\mathfrak{M}}$ satisfies the definition of the constructible hierarchy.

Thus, S'' consists of the extensional wellfounded structures isomorphic to transitive sets N such that $L_{\mathrm{Ord}\,\cap N} \subseteq N$. Now it is easy to obtain a Π^1_1 characterization of $\mathcal{S}$. $\square$[7.30]

(7.31) **Theorem** $[\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}]$ *$<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ is a Δ^1_2 wellordering of ${}^\omega\omega$.*

Proof Using (7.30) we see that $<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ is Σ^1_2 because $x <^L y$ iff there exists a structure $\mathfrak{M}$ such that $\mathfrak{M} \in \mathcal{S}$, x, y are represented in $\mathfrak{M}$, say by x', y', and (for every satisfaction relation for $\mathfrak{M}$) $\mathfrak{M} \models [x'] <^L [y']$.

On the other hand, $<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ is Π^1_2 because $x <^L y$ iff for every structure $\mathfrak{M}$, if $\mathfrak{M} \in \mathcal{S}$ and x, y are represented in $\mathfrak{M}$, say by x', y', then (there exists a satisfaction relation for $\mathfrak{M}$ such that) $\mathfrak{M} \models [x'] <^L [y']$. $\square$[7.31]

(7.32) **Theorem** $[\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}]$ *There exists an uncountable Σ^1_2 set $A \subseteq {}^\omega\omega$ that has no uncountable $\boldsymbol{\Sigma}^1_1$ subset.* A fortiori *A has no nonempty perfect closed subset.*

Proof Let A be the set of $x \in \mathrm{WO}$[5.61] such that for all $y \in \mathrm{WO}$, if $\exists \alpha \in \omega_1\,(y \in L_\alpha \wedge x \notin L_\alpha)$ then $\mathrm{ot}\, y \neq \mathrm{ot}\, x$. Thus, if $x \in A$ then no member of WO constructed before x has the same rank, and for each $\alpha < \omega_1$ there are only countably many $x \in A$ such that $\mathrm{ot}\, x = \alpha$.

Note that $x \in A$ iff $x \in \mathrm{WO}$ and there exists $\mathfrak{M} \in \mathcal{S}$ such that x is represented in $\mathfrak{M}$, say by $x' \in |\mathfrak{M}|$, and for all $y' \in |\mathfrak{M}|$, if $\ulcorner \exists_{\mathrm{Ord}} \alpha\,([y'] \in \hat{L}_\alpha \wedge [x'] \notin \hat{L}_\alpha)\urcorner^{\mathfrak{M}}$ then $\mathrm{ot}\, y \neq \mathrm{ot}\, x$. Since $\mathcal{S}$ is Π^1_1,[7.30] A is Σ^1_2.

$|A| = \omega_1$, so A is uncountable. Suppose $B \subseteq A$ is an uncountable $\boldsymbol{\Sigma}^1_1$ set. Then since each "level" of A is countable, B is unbounded in A, violating (5.118). $\square$[7.32]

(7.33) **Theorem** $[\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}]$ *There exists an uncountable Π^1_1 set $A \subseteq {}^\omega\omega$ that has no uncountable $\boldsymbol{\Sigma}^1_1$ subset.* A fortiori *A has no nonempty perfect closed subset.*

Proof Let $B \subseteq {}^\omega\omega$ be an uncountable Σ^1_2 set that has no uncountable $\boldsymbol{\Sigma}^1_1$ subset.[7.32] Let $A' \subseteq {}^\omega\omega \times {}^\omega\omega$ be Π^1_1 such that $B = \{x \in {}^\omega\omega \mid \exists^1 y\, \langle x, y\rangle \in A'\}$. Let A'' be a Π^1_1 set that uniformizes A'.[5.134] Thus, for all $x \in {}^\omega\omega$,

1. $\exists^1 y\, \langle x, y\rangle \in A' \leftrightarrow \exists^1 y\, \langle x, y\rangle \in A''$; and
2. $\forall^1 y, y'\, (\langle x, y\rangle \in A'' \wedge \langle x, y'\rangle \in A'' \rightarrow y = y')$.

It follows that $B = \{x \in {}^\omega\omega \mid \exists^1 y\, \langle x, y\rangle \in A''\}$. Since B is uncountable, A'' is uncountable. We claim that A'' has no uncountable $\boldsymbol{\Sigma}^1_1$ subset. Suppose to the contrary that $X \subseteq A''$ is $\boldsymbol{\Sigma}^1_1$ and uncountable. Let $Y = \{x \in {}^\omega\omega \mid \exists^1 y\, \langle x, y\rangle \in X\}$.

Then Y is $\mathbf{\Sigma}^1_1$. Since the projection from A'' to B is injective, Y is uncountable, contradicting our choice of B as a set with no uncountable $\mathbf{\Sigma}^1_1$ subset.

Let $A = \{\langle x, y\rangle^{\cdot} \mid \langle x, y\rangle \in A''\}$.[5.56] Thus, $A \subseteq {}^\omega\omega$, and A is essentially equivalent to A''. In particular, A is an uncountable Π^1_1 set with no uncountable $\mathbf{\Sigma}^1_1$ subset. $\square$[7.33]

Using the definable wellordering $<^L$ of ${}^\omega\omega$, we can define a Vitali set A as the set of $x \in {}^\omega\omega$ such that for all $y \in {}^\omega\omega$, if $y <^L x$ then $x - y \notin \mathbb{Q}$. A does not have the Baire property and is not Lebesgue measurable.[5.150 5.162] A is easily seen to be Π^1_2. Hence, there exists a Π^1_2 subset of ${}^\omega\omega$ that is not Baire or Lebesgue measurable.

The following argument that shows that $<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ itself is not Baire or measurable, which is an improvement on the above result inasmuch as $<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ is not just Π^1_2 but Δ^1_2.

(7.34) **Theorem** [ZF + $\boldsymbol{V} = \boldsymbol{L}$] *$<^L \cap ({}^\omega\omega \times {}^\omega\omega)$ does not have the Baire property and is not Lebesgue measurable.*

Proof Given $C \subseteq {}^\omega\omega \times {}^\omega\omega$ and $z \in {}^\omega\omega$, let $C_z = \{w \in {}^\omega\omega \mid \langle z, w\rangle \in C\}$, and let $C^z = \{w \in {}^\omega\omega \mid \langle w, z\rangle \in C\}$. Let $A = <^L \cap ({}^\omega\omega \times {}^\omega\omega) = \{\langle x, y\rangle \in {}^\omega\omega \times {}^\omega\omega \mid x <^L y\}$ and let $B = ({}^\omega\omega \times {}^\omega\omega) \setminus A = \{\langle x, y\rangle \in {}^\omega\omega \times {}^\omega\omega \mid y \leqslant^L x\}$. Note that for every $z \in {}^\omega\omega$, A^z and B_z are countable and are therefore meager and Lebesgue-null.

Suppose A has the Baire property; then B does also. Since A^z is meager for all $z \in {}^\omega\omega$, A is meager;[5.149] and since B_z is meager for all $z \in {}^\omega\omega$, B is meager. Thus ${}^\omega\omega = A \cup B$ is meager—which it is not. So A does not have the Baire property.

Similarly, (5.158) shows that A is not Lebesgue measurable. $\square$[7.34]

7.6.1 Suslin's hypothesis

Recall Suslin's hypothesis,[5.186] which asserts that there does not exist a Suslin line.[5.187] In this section we will show that Suslin lines exist in L, i.e., (ZF+$\boldsymbol{V} = \boldsymbol{L}$) $\vdash \neg$ SH. It will be convenient to frame the question in terms of trees in the sense of (3.179).

(7.35) **Definition** [ZF] *A tree $(T, <)$ is a* Suslin tree $\overset{\text{def}}{\Longleftrightarrow}$

1. *the height of T is ω_1;*
2. *T has no uncountable antichain; and*
3. *T has no uncountable branch.*

We will show that there exists a Suslin line iff there exists a Suslin tree.

(7.36) **Definition** [ZF] *Suppose $(T; <)$ is a tree of height μ. $(T; <)$ is* normal $\overset{\text{def}}{\Longleftrightarrow}$

1. *for every $\alpha < \mu$, if α is not a successor ordinal then for every $x, y \in T$ of order α, if $\{z \in T \mid z < x\} = \{z \in T \mid z < y\}$ then $x = y$; and*
2. *for every $x \in T$, there are infinitely many elements of T above x at every level above $\mathrm{o}\, x$.*

Suppose T is normal. Specializing to $\alpha = 0$ in (7.36.1) we see that T has a unique element of order 0, which we call the *root* of T. Now suppose α is a limit ordinal and B is a branch of $T|\alpha \stackrel{\text{def}}{=} \{x \in T \mid \operatorname{o} x < \alpha\}$. Then B has at most one extension in T of length $\alpha + 1$.

Note that if $(T; <)$ is a tree and $T' \subseteq T$ is nonempty, then $(T'; <')$ is a tree, where $<' = < \cap (T' \times T')$ is the inherited order relation, and we may refer to subsets of T as subtrees of $(T; <)$ without explicit mention of this order relation.

(7.37) **Theorem** [ZFC] *If there exists a Suslin tree then there exists a normal Suslin tree.*

Proof Suppose $(T; <)$ is a Suslin tree. Since T has no uncountable antichain, each level of T is countable. Since $\operatorname{ht} T = \omega_1$, $|T| = \omega_1$. For any $x \in T$, let $T_x = \{y \in T \mid y \geqslant x\}$. Let $T' = \{x \in T \mid |T_x| = \omega_1\}$. Since each level of T is countable, at each level of T there is an element of T'. Hence, in particular, T' is nonempty. By the same token, for each $x \in T'$, at each level of T_x there is an element of T'. Hence, every element of T' has a successor in T' at every higher level. We now construct an extension of T' as follows. Let $\mathcal{B}$ be the set of $B \subseteq T'$ such that for some nonsuccessor ordinal α and distinct $x, y \in T'$ of order α, $B = \{z \in T' \mid z < x\} = \{z \in T' \mid z < y\}$. Let $B \mapsto x_B$ be an injection of $\mathcal{B}$ into $V \backslash T'$. Let $T'' = T' \cup \{x_B \mid B \in \mathcal{B}\}$, and expand the order relation on T' to T'' by adding the following:

1. $\langle x, x_B \rangle$, for all $B \in \mathcal{B}$ and $x \in B$;
2. $\langle x_B, y \rangle$, for all $B \in \mathcal{B}$ and $y \in T'$ such that $\forall x \in B\ x < y$; and
3. $\langle x_B, x_{B'} \rangle$, for all $B, B' \in \mathcal{B}$ such that $B \subsetneqq B'$.

In other words, we insert into T' a single new element immediately above each $B \in \mathcal{B}$. Note that T'' is a Suslin tree that satisfies (7.36.1), and each member of T'' has a successor at every higher level.

Note that every $x \in T''$ has incomparable successors in T''; otherwise the elements of T'' comparable with x constitute an uncountable branch of T'', and $x \cap T$ is an uncountable branch of T. Since T'' satisfies (7.36.1), every element of T'' has a successor with at least two immediate successors. Let T''' consist of those members of T'' that have at least two immediate successors in T''. T''' (with the inherited order) is a Suslin tree satisfying (7.36.1), and every member of T''' has at least two immediate successors.

Finally, let T'''' be the set of elements of T''' of nonsuccessor order. T'''' is a normal Suslin tree. □7.37

(7.38) **Theorem** [ZFC] *There exists a Suslin line iff there exists a Suslin tree.*

Proof → Suppose $(X; <)$ is a Suslin line, i.e., a complete dense linear order without endpoints that is ccc but not separable. Let C be the set of nondegenerate closed intervals in X, i.e., intervals $[x, y]$ such that $x < y$. Invoking AC we posit a fixed wellordering $\prec$ of C. Using this wellordering, we will define $\langle [x_\alpha, y_\alpha] \mid \alpha < \omega_1 \rangle \in {}^{\omega_1}C$ such that for each $\beta < \alpha < \omega_1$,

(7.39) *either* $[x_\alpha, y_\alpha] \subseteq (x_\beta, y_\beta)$ *or* $[x_\alpha, y_\alpha] \cap [x_\beta, y_\beta] = 0$.

To do this, let $[x_0, y_0]$ be the $\prec$-first member of C. Now suppose $0 < \gamma < \omega_1$, and $\langle [x_\alpha, y_\alpha] \mid \alpha < \gamma \rangle$ satisfies (7.39) for all $\beta < \alpha < \gamma$. $\bigcup_{\alpha < \gamma} \{x_\alpha, y_\alpha\}$ is a countable subset of X, so it is not dense in X, since X is not separable. Let (x, y) be a nonempty open interval in X that does not intersect it, and let $[x_\gamma, y_\gamma]$ be the $\prec$-first member of C included in (x, y). Clearly, (7.39) is satisfied for all $\beta < \alpha < \gamma+1$.

Let $\langle [x_\alpha, y_\alpha] \mid \alpha < \omega_1 \rangle$ be the sequence so constructed. Let $T = \{[x_\alpha, y_\alpha] \mid \alpha < \omega_1\}$, and let $<$ be the binary relation on T defined by the condition that $I < J$ iff $I \supsetneq J$. It is easy to check that $(T; <)$ is a tree.

(7.40) **Claim** *$(T; <)$ is a Suslin tree.*

Proof Clearly, an antichain $\mathcal{C}$ in T yields an antichain in X by replacing each $[x, y] \in \mathcal{C}$ by (x, y). Since X is ccc, T is ccc. In particular, each level of T is countable. Since $|T| = \omega_1$, $\operatorname{ht} T = \omega_1$.

Now suppose $B \subseteq T$ is a branch of length α. Then $B = \{I_\beta \mid \beta < \alpha\}$, where $\langle I_\beta \mid \beta < \alpha \rangle$ is a decreasing sequence of closed intervals such that for each $\beta < \alpha$, $I_{\beta+1}$ is included in the interior $\operatorname{int} I_\beta$ of I_β. Thus, $\{(\operatorname{int} I_{\beta+1}) \backslash I_\beta \mid \beta < \alpha\}$ is a set of disjoint nonempty open subsets of X (each one the union of two nonempty open intervals). Since X is ccc, $\alpha < \omega_1$. Hence, T has no uncountable branch. $\square^{7.40}$

$\leftarrow$ Conversely, suppose $(T; <)$ is a Suslin tree. By (7.37) we may assume T is normal. Note that the immediate successors of any $x \in T$ constitute a countably infinite set (else they would constitute an uncountable antichain). Use AC, for each $x \in T$, let $<_x$ be a fixed dense linear ordering without endpoints of this set—in other words, order it like $\mathbb{Q}$. Now order the set $\mathcal{B}$ of branches of T as follows. Suppose B and B' are distinct branches. Let α be the order type of their longest common initial segment. By construction, α cannot be a limit ordinal. Thus, $\alpha = \beta + 1$. Let x be the common member of B and B' at level β and let y and y' be their respective members at level $\beta + 1$. Let $B \prec B'$ iff $y <_x y'$. It is easy to show that $(\mathcal{B}; \prec)$ is a dense linear order without endpoints.

Note that for $B \prec B'$ as above, there exists z such that $y <_x z <_x y'$. Note that every branch C containing z lies in the interval (B, B'). Suppose toward a contradiction that $\mathcal{I}$ is an uncountable set of disjoint open intervals in $(\mathcal{B}; \prec)$. Using AC, for each $I \in \mathcal{I}$, let z_I be such that every branch containing z_I is in I. Then for distinct $I, I' \in \mathcal{I}$, z_I and $z_{I'}$ are incomparable, i.e., they form an uncountable antichain, which is impossible, since T is Suslin. Thus $(B; \prec)$ satisfies the countable chain condition.

Now suppose toward a contradiction that $\mathcal{C}$ is a countable dense set in $(\mathcal{B}; \prec)$. Let $\alpha < \omega_1$ be greater than the lengths of all $B \in \mathcal{C}$, and let x be any member of T at level α. Let y, y' be distinct immediate successors of x with $y <_x y'$, and let B, B' be branches containing y and y', respectively. Then (B, B') is an open interval in $(\mathcal{B}; \prec)$ that does not contain any $C \in \mathcal{C}$. Thus $(\mathcal{B}; \prec)$ is not separable.

The completion of $\prec$ is a therefore a Suslin line.[5.188] $\square^{7.38}$

We will now construct a Suslin tree in L, making use of the following combinatorial principle, which was isolated by Ronald Jensen in 1972 in the course of this construction, and is of great importance in its own right[13]. It is represented by the symbol '$\Diamond$' and is called the *diamond principle* for this reason.

(7.41) **Definition** [ZFC] $\Diamond \overset{\text{def}}{\iff}$ *there exists a sequence $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ such that*

1. $\forall \alpha \in \omega_1\ S_\alpha \subseteq \alpha$*; and*
2. *for all $X \subseteq \omega_1$, $\{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\}$ is stationary in ω_1.*

*Such a sequence $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ is called a $\Diamond$-*sequence *or* diamond-sequence.

(7.42) **Theorem** [ZF + $\boldsymbol{V} = \boldsymbol{L}$] $\Diamond$.

Proof Let $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ be defined by recursion as follows. Suppose $\alpha \in \omega_1$ and S_β has been defined for $\beta < \alpha$. If α is 0 or a successor ordinal, let $S_\alpha = 0$. Suppose α is a limit ordinal. To define S_α, we consider sets C, X such that

(7.43)

1. *$X \subseteq \alpha$;*
2. *$C \subseteq \alpha$ is closed and unbounded in α; and*
3. *$\forall \beta \in C \; X \cap \beta \neq S_\beta$.*

It is easy to see that there exist such pairs. Indeed, for any unbounded $C \subseteq \alpha$ of order type ω, there exists $X \subseteq \alpha$ so that $X \cap \beta \neq S_\beta$ for each $\beta \in C \backslash \{0\}$.

Let $\langle C, X \rangle$ be the $<^L$-first 2-sequence satisfying (7.43), and let $S_\alpha = X$.

Claim *$\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ is a $\Diamond$-sequence.*

Proof

(7.44) *Suppose toward a contradiction that $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ is not a $\Diamond$-sequence, and let $\langle C, X \rangle$ be the $<^L$-first 2-sequence such that*

1. *$X \subseteq \omega_1$;*
2. *$C \subseteq \omega_1$ is closed and unbounded in ω_1; and*
3. *$\forall \alpha \in C \; X \cap \alpha \neq S_\alpha$.*

Let M be a countable elementary substructure of L_{ω_2}. For every $\alpha \in M \cap \omega_1$, $L_{\omega_2} \models \ulcorner [\alpha]$ is countable$\urcorner$, so $M \models \ulcorner [\alpha]$ is countable$\urcorner$, so there exists $f \in M$ such that $M \models \ulcorner [f] : \omega \overset{\text{sur}}{\to} [\alpha] \urcorner$. It follows that $L_{\omega_2} \models \ulcorner [f] : \omega \overset{\text{sur}}{\to} [\alpha] \urcorner$, so $f : \omega \overset{\text{sur}}{\to} \alpha$; hence, $\alpha \subseteq M$. Thus, every ordinal in $M \cap \omega_1$ is included in M, so $M \cap \omega_1$ is an ordinal, say μ, which is countable, since M is countable.

By the condensation lemma, the transitive collapse of M is L_γ for some $\gamma \in \omega_1$. Let $\pi : M \to L_\gamma$ be the collapsing isomorphism. Note that ω_1, $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$, C, X, and $\langle C, X \rangle$ are in L_{ω_2} and are definable in L_{ω_2}, so they are in every $M \prec L_{\omega_2}$. Since $M \cap \omega_1 = \mu$, $\pi \omega_1 = \mu$, so $\mu = \ulcorner \omega_1 \urcorner^{L_\gamma}$. For each $\alpha < \mu$, $\pi S_\alpha = S_\alpha$, so $\pi \langle S_\alpha \mid \alpha < \omega_1 \rangle = \langle S_\alpha \mid \alpha < \mu \rangle$. Also, $\pi C = C \cap \mu$, $\pi X = X \cap \mu$, and $\pi \langle C, X \rangle = \langle C \cap \mu, X \cap \mu \rangle$.

Thus, $L_\gamma \models \ulcorner [\langle C \cap \mu, X \cap \mu \rangle]$ is the $<^L$-first 2-sequence $\langle C', X' \rangle$ such that C' is closed unbounded in ω_1, $X' \subseteq \omega_1$, and $\forall \alpha < \omega_1 \; X' \cap \alpha \neq S_\alpha \urcorner$. It is easy to check that $\ulcorner <^L \urcorner^{L_\gamma} = <^L \cap (L_\gamma \times L_\gamma)$, so $\langle C \cap \mu, X \cap \mu \rangle$ is the $<^L$-first 2-sequence $\langle C', X' \rangle$ such that C' is closed unbounded in μ, $X' \subseteq \mu$, and $\forall \alpha < \mu \; X' \cap \alpha \neq S_\alpha$.

S_μ is therefore by definition $X \cap \mu$. Since $L_\gamma \models \ulcorner [C \cap \mu]$ is unbounded in $[\mu] \urcorner$, $C \cap \mu$ is indeed unbounded in μ, so since C is closed, $\mu \in C$. This contradicts (7.44.3). $\square$[7.42]

(7.45) **Theorem** [ZFC] *If $\Diamond$ then there exists a Suslin tree, i.e., $\neg$ SH—Suslin's hypothesis is false. Thus,*[7.42] *if $V = L$ then $\neg$ SH.*

Proof We will construct a sequence $\langle T_\alpha \mid 0 < \alpha < \omega_1\rangle$ of trees $T_\alpha = (\tau_\alpha; \prec_\alpha)$, where $\langle \tau_\alpha \mid 0 < \alpha < \omega_1\rangle$ is a continuous increasing sequence of countable ordinals with $\tau_1 = 1$ and $\tau_{\alpha+1}\backslash\tau_\alpha$ $(= [\tau_\alpha, \tau_{\alpha+1}))$ is infinite for all $\alpha \in [1, \omega_1)$, e.g.

$$\tau_\alpha = \begin{cases} 1 & \text{if } \alpha = 1 \\ \omega \cdot \alpha & \text{if } \alpha > 1. \end{cases}$$

(We use ordinals as the domains of these trees to facilitate the use of a $\diamondsuit$-sequence in their construction.) The following conditions will be satisfied.

1. For $0 < \alpha < \omega_1$, T_α is a normal tree of height α.

2. For $0 < \beta < \alpha < \omega_1$, the T_β is the initial segment of T_α consisting of its elements of order $< \beta$.

Note that $\bigcup_{0<\alpha<\omega_1} \tau_\alpha = \omega_1$. Let $\prec = \bigcup_{0<\alpha<\omega_1} \prec_\alpha$, and let $T = (\omega_1; \prec)$. Since each T_α is a normal tree, T is a normal tree. The αth level of T is the interval $[\tau_\alpha, \tau_{\alpha+1})$ of ordinals.

Clearly, T has height ω_1. Since T is normal, if it has an uncountable branch B, it has an uncountable antichain, which may be obtained by choosing for each $x \in B$ an immediate successor c_x of x other than the one that is in B. Clearly, for $x, x' \in B$, if $x \neq x'$ then c_x and $c_{x'}$ are incomparable, so $\{c_x \mid x \in B\}$ is an uncountable antichain. Thus, in order that T be a Suslin tree, it is sufficient that it have no uncountable antichain. We will use a $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha \in \omega_1\rangle$ to arrange this.

Note that the above description requires $(\tau_1; \prec_1) = (1; 0)$, and if α is a limit ordinal, $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$ and $\prec_\alpha = \bigcup_{\beta<\alpha} \prec_\beta$. The construction is therefore determined by the following rules for generating $\prec_{\alpha+1}$ from $\prec_\alpha$. Note that $\tau_{\alpha+1}\backslash\tau_\alpha = [\tau_\alpha, \tau_{\alpha+1})$ is countably infinite.

If α is a successor ordinal, let $\prec_{\alpha+1}$ be such as to give each element of τ_α of order $\alpha - 1$ in T_α a (countably) infinite set of immediate successors in $T_{\alpha+1}$.

If α is a limit ordinal we must decide which branches of T_α to extend to level α. For each $x \in \tau_\alpha$ there is a branch B of T_α that contains x, as shown by the following argument. Given $x \in \tau_\alpha$, let $\langle \alpha_n \mid n \in \omega\rangle$ be a strictly increasing sequence of ordinals in α that is cofinal in α, with $\alpha_0 = \mathrm{o}\, x$. Let $\langle x_n \mid n \in \omega\rangle$ be an increasing sequence in T_α with $x_0 = x$, such that $\mathrm{o}\, x_n = \alpha_n$. This can be done because T_α is normal. Let $B = \{y \in \tau_\alpha \mid \exists n \in \omega\ y \prec x_n\}$.

If S_α is a maximal antichain in T_α, let $S = \{x \in \tau_\alpha \mid \exists x' \in S_\alpha\ x' \preccurlyeq_\alpha x\}$; otherwise, let $S = \tau_\alpha$. Note that S is necessarily infinite. For each $x \in S$ let B_x be a branch of T_α such that $x \in B_x$, and let $\mathcal{B} = \{B_x \mid x \in S\}$. Let $\langle y_B \mid B \in \mathcal{B}\rangle$ be a bijection of $\mathcal{B}$ with $[\tau_\alpha, \tau_{\alpha+1})$ (both of which are countably infinite), and let $\prec_{\alpha+1} = \prec_\alpha \cup \bigcup_{B\in\mathcal{B}} (B \times \{y_B\})$. In other words, each $B \in \mathcal{B}$ is extended uniquely to level α, and all other branches of T_α are terminated at this stage.

(7.46) **Claim** *For every $x \in \tau_\alpha$ there exists $z \in S$ such that $x \in B_z$.*

Proof If S_α is not a maximal antichain in T_α then $S = \tau_\alpha$ we may let $z = x$. If S_α is a maximal antichain in T_α, then $S = \{x \in \tau_\alpha \mid \exists x' \in S_\alpha\ x' \preccurlyeq_\alpha x\}$. Given $x \in \tau_\alpha$, since S_α is a maximal antichain there exists $w \in S_\alpha$ such that either $w \preccurlyeq_\alpha x$ or $x \preccurlyeq_\alpha w$. In the former case, $x \in S$, and we may let $z = x$. In the latter case, $x \in B_w$, so we may let $z = w$. $\square$[7.46]

It follows that $T_{\alpha+1}$ is normal. Our construction guarantees that if α is a limit ordinal and S_α is a maximal antichain in T_α, then every element of order α in T is above some member of S_α, hence every element of any order $\geqslant \alpha$ in T is above some member of S_α.

Hence, if $\operatorname{Lim}\alpha$ *and* S_α *is a maximal antichain in* T_α *then* S_α *is a maximal antichain in* T.

We now show that every antichain in T is countable. Note that any antichain may be extended to a maximal antichain, so we may suppose without loss of generality that $X \subseteq \omega_1$ is a maximal antichain in T. We will show that X is countable.

(7.47) *Let* $C \subseteq \omega_1$ *be the set of* $\alpha \in \omega_1$ *such that*

1. α *is a limit ordinal;*
2. $\tau_\alpha = \alpha$*; and*
3. $X \cap \alpha$ *is a maximal antichain in* T_α*.*

C is clearly closed. We now show that C is unbounded. Suppose $\alpha \in \omega_1$. Let $\langle \alpha_n \mid n \in \omega \rangle$ be a strictly increasing sequence in ω_1 such that

1. $\alpha_0 = \alpha$; and
2. for each $n \in \omega$,
    1. $\alpha_{n+1} \geqslant \tau_{\alpha_n}$; and
    2. every member of τ_{α_n} is comparable with some member of $X \cap \tau_{\alpha_{n+1}}$.

Then $\bigcup_{n \in \omega} \alpha_n$ is in C.

Since $\langle S_\alpha \mid \alpha \in \omega_1 \rangle$ is a $\diamondsuit$-sequence, $\{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\}$ is stationary, so there exists $\alpha \in C$ such that $X \cap \alpha = S_\alpha$. Then S_α is a maximal antichain in T_α,[7.47.3] which our construction has guaranteed is a maximal antichain in T, so $X = X \cap \alpha$, and X is therefore countable. $\square$[7.45]

7.6.2 Borel determinacy

In this section we prove the celebrated theorem of Harvey Friedman (as sharpened by D. A. Martin) stating that $\mathsf{ZF}^- \nvdash \Sigma^0_4$-determinacy, and, in general, for any countable ordinal ρ, $\mathsf{ZF}^- +$ ⌜the power operation may be applied ρ times in succession to ω⌝ $\nvdash \Sigma^0_{1+\rho+3}$-determinacy. Hence, the number of iterations of the power operation used in Martin's proof of Borel determinacy[5.178] is—level by level—the minimum number that suffices.

Specifically, Σ^0_4-determinacy fails in the minimum transitive model of ZF^-, i.e., in L_{β_0}, where $\beta_0 \overset{\text{def}}{=}$ the least ordinal β such that $L_\beta \models \mathsf{ZF}^-$. The general result uses a similarly minimum model that contains a ρ-sequence of successor cardinals.

The proof is rather long and relatively complicated, and it does not introduce any ideas needed elsewhere, so this is one that could reasonably be skipped at first reading. It does, however, demonstrate an ingenious linkage of disparate ideas that are individually fundamental, and it is well worth studying at one's leisure.

In the following discussion, recall that if $\alpha > \omega$ is a limit ordinal then L_α satisfies all the axioms of ZF^- except possibly $\mathsf{Comprehension}$ and $\mathsf{Collection}$,[7.13] and that L_α contains and correctly identifies pairs, ordered pairs, finite sequences, and satisfaction relations of objects in L_α.[7.14] We will make use of these and other absoluteness properties of L_α often without explicit mention.

Definition [ZF^-]

1. $\mathsf{c} : \mathrm{Ord} \overset{\mathrm{bij}}{\to} L \overset{\mathrm{def}}{=}$ *the* canonical enumeration of L, *defined by the condition that* $\alpha < \beta$ *iff* $\mathsf{c}\,\alpha <^L \mathsf{c}\,\beta$.
2. *Suppose* M *is a transitive set.* $\mathcal{D}\,M \overset{\mathrm{def}}{=}$ *the set of subsets of* M *definable over* $(M;\in)$ *from parameters in* M.

Thus, $L_{\alpha+1} = \mathcal{D}\,L_\alpha$. As noted in the discussion following (7.1), the existence of $\mathcal{D}\,M$ is demonstrable without the use of Power.

(7.48)

1. *It is clear from our definition of* $<^L$ *that for each ordinal* α, $\mathsf{c}^{\leftarrow} L_\alpha$ *is an initial segment of the ordinals, i.e., is an ordinal. In other words,* c *enumerates everything in* L_α *before anything in* $L \backslash L_\alpha$.
2. *Also clearly,* $\mathsf{c}^{\rightarrow}\alpha \subseteq L_\alpha$.

(7.49) **Theorem** [ZF^-] *Suppose* $\ulcorner[\alpha]$ *is regular*$\urcorner^{L_{\alpha+1}}$. *Then* $\mathsf{c}^{\rightarrow}\alpha = L_\alpha$.

Remark Since $\mathsf{c}^{\rightarrow}\alpha \subseteq L_\alpha$,[7.48.2] the import of the theorem is that $L_\alpha \subseteq \mathsf{c}^{\rightarrow}\alpha$.

Proof Assume the hypothesis, which is that α is a limit ordinal and for every $\beta < \alpha$ and $f \in L_{\alpha+1}$ such that $f : \beta \to \alpha$, $\sup \operatorname{im} f < \alpha$. If $\alpha = \omega$ then the conclusion is immediate, so suppose $\alpha > \omega$.

(7.50) **Claim** *Suppose* $f \in L_{\alpha+1}$ $(= \mathcal{D}\,L_\alpha)$.

1. *If* $f : \beta \rightharpoonup \alpha$ *for some* $\beta < \alpha$, *then* $\sup \operatorname{im} f < \alpha$.
2. *If* $f : V_\omega \rightharpoonup \alpha$, *then* $\sup \operatorname{im} f < \alpha$.
3. *If* $f : {}^n\beta \rightharpoonup \alpha$ *for some* $n \in \omega$ *and* $\beta < \alpha$, *then* $\sup \operatorname{im} f < \alpha$.
4. *If* $f : {}^{<\omega}\beta \rightharpoonup \alpha$ *for some* $\beta < \alpha$, *then* $\sup \operatorname{im} f < \alpha$.
5. *If* $f : V_\omega \times {}^{<\omega}\beta \rightharpoonup \alpha$ *for some* $\beta < \alpha$, *then* $\sup \operatorname{im} f < \alpha$.

Proof 1 Let $g : \beta \to \alpha$ be such that $g \supseteq f$ and $g\gamma = 0$ if $\gamma \in \beta \backslash \operatorname{dom} f$. Then $\operatorname{im} f \subseteq \operatorname{im} g$, and by hypothesis, $\sup \operatorname{im} g < \alpha$.

2 The canonical enumeration $\vec{B} : \omega \overset{\mathrm{bij}}{\to} V_\omega$ is in L_α, so $g = f \circ \vec{B} : \omega \rightharpoonup \alpha$ is in $L_{\alpha+1}$. $\sup \operatorname{im} g < \alpha$ by hypothesis, and $\operatorname{im} f = \operatorname{im} g$.

3 By induction on n. The case that $n = 0$ is trivial. Suppose the claim is true for n. Suppose $f : {}^{n+1}\beta \rightharpoonup \alpha$. Let $g : \beta \to \alpha$ be such that for every $\gamma \in \beta$, $g\gamma = \sup_{s \in {}^n\beta} f(s ^\frown \langle\gamma\rangle)$. By hypothesis, $\forall \gamma \in \beta\ g\gamma \in \alpha$. Clearly $\sup \operatorname{im} f \leqslant \sup \operatorname{im} g$, and since $\sup \operatorname{im} g < \alpha$, $\sup \operatorname{im} f < \alpha$.

4 Let $g : \omega \to \alpha$ be such that for each $n \in \omega$, $gn = \sup\{fs \mid s \in \operatorname{dom} f \wedge |s| = n\}$. Then $\forall n \in \omega\ gn \in \alpha$ and $\sup \operatorname{im} f = \sup \operatorname{im} g < \alpha$.

5 Similar to previous arguments. $\square$[7.50]

Now suppose toward a contradiction that $\mathsf{c}^{\rightarrow}\alpha \neq L_\alpha$. Then[7.48.2] $\mathsf{c}^{\rightarrow}\alpha \subsetneqq L_\alpha$. Since α is limit there exists $\beta < \alpha$ such that $\mathsf{c}^{\rightarrow}\alpha \subseteq L_\beta$.[7.48.1] Let β_0 be the least such β.

If β_0 is a limit ordinal then define $f : \beta_0 \to \alpha$ so that for each $\beta < \beta_0$, $f\beta$ is the least ordinal γ such that $\mathsf{c}_\gamma \notin L_\beta$. Then $\sup \operatorname{im} f < \alpha$ and $\mathsf{c}_{\sup \operatorname{im} f} \notin L_{\beta_0}$;[7.48.1] contradiction.

Hence β_0 is a successor ordinal. Let $\beta_1 = \beta_0 - 1$. Let $\gamma_1 = \mathsf{c}^{\leftarrow}L_{\beta_1}$. Note that $\gamma_1 \in \alpha$. Let S be the satisfaction relation for $(L_{\beta_1}; \in)$. Recall that $S \in L_\alpha$. Let $f : V_\omega \times {}^{<\omega}\gamma_1 \to \alpha$ be such that for any $\phi \in V_\omega$ and $s \in {}^{<\omega}L_{\beta_1}$, letting $n = |s|$,

1. if it is not the case that ϕ is an s-formula with $n+1$ free variables then $f\langle \phi, s\rangle = 0$; otherwise
2. letting $\langle v_0, \ldots, v_n\rangle$ be the enumeration of $\operatorname{Free}\phi$ in increasing canonical order, and letting
$$x = \big\{y \in L_{\beta_1} \,\big|\, \big\langle \phi, \big\langle \begin{smallmatrix} v_n & v_0 & \cdots & v_{n\text{-}} \\ y & \mathsf{c}_{s_0} & \cdots & \mathsf{c}_{s_{n\text{-}}} \end{smallmatrix} \big\rangle\big\rangle \in S\big\},$$

    1. if $\mathsf{c}^{-1}\, x < \alpha$ then $f\langle \phi, s\rangle = \mathsf{c}^{-1}\, x$; otherwise
    2. $f\langle \phi, s\rangle = 0$.

Note that $\operatorname{im} f = (\mathsf{c}^{\leftarrow}(\mathcal{D}\, L_{\beta_1})) \cap \alpha = (\mathsf{c}^{\leftarrow}L_{\beta_0}) \cap \alpha = \alpha$. But $f \in \mathcal{D}\, L_\alpha$, so[7.50.5] $\sup \operatorname{im} f < \alpha$; contradiction. $\square$[7.49]

(7.51) **Theorem** [ZF^-] *Suppose* $\ulcorner[\alpha]$ *is regular*$\urcorner^{L_{\alpha+1}}$. *Then for every* $x \in L_\alpha$ *and* $f \in L_{\alpha+1}$ *such that* $f : x \to \alpha$, $\sup \operatorname{im} f < \alpha$.

Proof Let[7.49] $\gamma < \alpha$ be such that $x \subseteq \mathsf{c}^{\rightarrow}\gamma$, and let $g = \{(\beta, f(\mathsf{c}_\beta)) \mid \mathsf{c}_\beta \in x\}$. Then $g \in L_{\alpha+1}$, $g : \gamma \rightharpoonup \alpha$, and $\operatorname{im} f = \operatorname{im} g$, so[7.50.1] $\sup \operatorname{im} f = \sup \operatorname{im} g < \alpha$. $\square$[7.51]

(7.52) **Theorem** [ZF^-] *Suppose* $\alpha > \omega$ *and* $\ulcorner[\alpha]$ *is regular*$\urcorner^{L_{\alpha+1}}$. *Then* $L_\alpha \models \mathsf{ZF}^-$.

Proof It follows from the assumption that $\alpha > \omega$ and α is a limit ordinal that L_α models all the axioms of ZF^- with the possible exception of $\mathsf{Collection}$ and $\mathsf{Comprehension}$.[7.13] We therefore have only to verify the latter two schemata.

We begin with $\mathsf{Collection}$. Suppose ϕ is an s-formula and a, v, v_0, …, $v_{n\text{-}}$ are distinct variables such that $\operatorname{Free}\phi \subseteq \{a, v, v_0, \ldots, v_{n\text{-}}\}$. Suppose y_0, …, $y_{n\text{-}}$, $x \in L_\alpha$ and
$$\forall y \in x\, \exists z \in L_\alpha\, \forall r \in L_\alpha\, \big(\big(L_\alpha \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} & v & a \\ y_0 & \cdots & y_{n\text{-}} & y & r \end{smallmatrix}\big]\big) \to r \in z\big).$$
Then
$$\forall y \in x\, \exists \beta < \alpha\, \forall r \in L_\alpha\, \big(\big(L_\alpha \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} & v & a \\ y_0 & \cdots & y_{n\text{-}} & y & r \end{smallmatrix}\big]\big) \to r \in L_\beta\big).$$
Let $f : x \to \alpha$ be such that for each $y \in x$, fy is the least β such that $\forall r \in L_\alpha\, \big(\big(L_\alpha \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} & v & a \\ y_0 & \cdots & y_{n\text{-}} & y & r \end{smallmatrix}\big]\big) \to r \in L_\beta\big)$. Then $f \in L_{\alpha+1}$, so $\sup \operatorname{im} f < \alpha$.[15] It follows that there exists $\beta < \alpha$ such that
$$\forall y \in x\, \forall r \in L_\alpha\, \big(\big(L_\alpha \models \phi\big[\begin{smallmatrix} v_0 & \cdots & v_{n\text{-}} & v & a \\ y_0 & \cdots & y_{n\text{-}} & y & r \end{smallmatrix}\big]\big) \to r \in L_\beta\big).$$

[15] The definition of f uses only the $\{\phi\}$-satisfaction relation for L_α, which is definable over L_α, unlike the full satisfaction relation for L_α.

Hence, Collection holds in L_α.

To prove Comprehension, we use a reflection argument as in the proof of (7.6).

(7.53) **Claim** *Suppose ϕ is an* s*-formula. The set $\{\beta < \alpha \mid L_\beta \prec^{\{\phi\}} L_\alpha\}$ is closed unbounded in α.*

Proof This is a variation on the downward Löwenheim-Skolem theorem.[2.159.1] Let S be the $\{\phi\}$-satisfaction relation for L_α. Note that S is definable over L_α. Using the Tarski-Vaught criterion[2.153], the canonical enumeration c, and bounding arguments similar to those used above, it is straightforward to show that $\{\beta < \alpha \mid L_\beta \prec^{\{\phi\}} L_\alpha\}$ is unbounded in α. That it is closed follows from (2.155), keeping in mind the general principle that if $\mathfrak{A} \prec^\Phi \mathfrak{C}$, $\mathfrak{B} \prec^\Phi \mathfrak{C}$, and $\mathfrak{A} \prec^0 \mathfrak{B}$, then $\mathfrak{A} \prec^\Phi \mathfrak{B}$, so $\{L_\beta \mid L_\beta \prec^{\{\phi\}} L_\alpha\}$ is a directed set under the relation $\prec^{\{\phi\}}$, and (2.155) applies. $\square$[7.53]

Suppose ϕ is an s-formula, and $u, v, v_0, \ldots, v_{n^-}$ are distinct variables such Free $\phi \subseteq \{u, v, v_0, \ldots, v_{n^-}\}$. Suppose $y_0, \ldots, y_{n^-}, x \in L_\alpha$. We must show that there exists $z \in L_\alpha$ such that

$$\forall y \in L_\alpha \big(y \in z \leftrightarrow \big(y \in x \wedge L_\alpha \models \phi\left[\begin{smallmatrix} u & v & v_0 & \cdots & v_{n^-} \\ x & y & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right]\big)\big). \tag{7.54}$$

To this end, let[7.53] $\beta < \alpha$ be such that $y_0, \ldots, y_{n^-}, x \in L_\beta$ and $L_\beta \prec^{\{\phi\}} L_\alpha$. Then there exists $z \in L_{\beta+1}$ such that

$$\forall y \in L_\beta \big(y \in z \leftrightarrow \big(y \in x \wedge L_\beta \models \phi\left[\begin{smallmatrix} u & v & v_0 & \cdots & v_{n^-} \\ x & y & y_0 & \cdots & y_{n^-} \end{smallmatrix}\right]\big)\big).$$

Since $L_\beta \prec^{\{\phi\}} L_\alpha$, (7.54) holds for z. $\square$[7.52]

(7.55) **Definition** [C]

1. r $\stackrel{\text{def}}{=}$ *the signature* s *with additional constants* $\ulcorner R \urcorner$ *and* $\ulcorner \rho \urcorner$.
2. $T^R \stackrel{\text{def}}{=}$ *the* r*-theory consisting of*
   1. ZF$^-$*;*
   2. $\ulcorner$*R is a wellorder of a subset of ω, and $\rho = \operatorname{ot} R$*$\urcorner$*;*[16]
   3. $\ulcorner V = L[R] \urcorner$*; and*
   4. $\ulcorner$*there exists an increasing ρ-sequence of infinite successor cardinals*$\urcorner$.
3. $T_0^R \stackrel{\text{def}}{=} T^R + \ulcorner \forall_{\mathrm{Ord}} \beta\ L_\beta[R] \not\models T^R \urcorner$.

(7.56) **Theorem** $[T^R]$

1. *ρ is a countable ordinal.*
2. *ω_ρ exists.*
3. *$V_{\omega+\rho}$ exists, i.e., the power operation may be applied ρ times in succession to V_ω.*

Proof 1 ρ is the order type of R, which is a wellorder of a countable set.

[16] We allow for the possibility that $R = 0$, so $\operatorname{ot} R = 0$; or $|R| = 1$, so $\operatorname{ot} R = 1$. The latter requires that R be a weak order relation. For order types > 1, we may use either weak or strong order relations.

2 If $\rho = 0$ then ω_ρ exists.

Suppose ρ is a limit ordinal. By (7.55.2.4) there is an increasing ρ-sequence of infinite cardinals, i.e., ω_α exists for all $\alpha < \rho$. Let $\lambda = \bigcup_{\alpha<\rho} \omega_\alpha$. λ is ω_ρ.

Suppose ρ is a successor ordinal. By (7.55.2.4) there is an increasing sequence $\langle \sigma_\alpha \mid \alpha < \rho \rangle$ of successor cardinals, and we may arrange that is an initial segment of the class of successor cardinals. Then clearly, for each $\alpha < \rho$, $\sigma_\alpha = \omega_{\alpha+1}$. Letting α be the immediate predecessor of ρ, $\sigma_\alpha = \omega_{\alpha+1} = \omega_\rho$, so ω_ρ exists.

3 In the proof of (7.27) we showed, in the context of $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}$, that every subset of an infinite cardinal κ is in L_{κ^+}, and that therefore $|\mathcal{P}\kappa| = |L_{\kappa^+}| = \kappa^+$. The same reasoning in the present context allows us to show by induction on $\alpha < \rho$ that

(7.57)

1. *$V_{\omega+\alpha}$ exists;*
2. *$|V_{\omega+\alpha}| = \omega_\alpha$;*
3. *every subset of $V_{\omega+\alpha}$ is in $L_{\omega_{\alpha+1}}[R]$; and therefore*
4. *$V_{\omega+\alpha+1}$ exists; and*
5. *$|V_{\omega+\alpha+1}| = \omega_{\alpha+1}$.*

If ρ is a successor ordinal, then (7.57.4) with $\alpha + 1 = \rho$ is the conclusion of the theorem. If ρ is a limit ordinal, we use the fact that $V_{\omega+\rho} = \bigcup_{\alpha<\rho} V_{\omega+\alpha}$. $\square^{7.56}$

(7.58) **Theorem** $[T_0^R]$

1. *ω_ρ is the largest cardinal, i.e., the order type of the class of infinite cardinals is $\rho + 1$.*
2. *Suppose $\zeta > \omega$. Let $M = L_\zeta[R]$. Suppose $\alpha < \zeta$. If ω_α^M exists then $\alpha \leqslant \rho$. In other words, the order type of the infinite M-cardinals is not greater than $\rho + 1$.*

Proof 1 To show that ω_ρ is the largest cardinal, suppose toward a contradiction that $\lambda = \omega_{\rho+1}$ exists. Since λ is a successor cardinal, it is regular, so $L_\lambda[R] \models \mathsf{ZF}^-$. (The proof is essentially that of (7.52) but simpler, since λ is actually regular, not just regular in $L_{\lambda+1}[R]$.) But for each $\alpha \leqslant \rho$, ω_ρ is a cardinal in $L_\lambda[R]$, so $L_\lambda[R] \models T^R$, contradicting the clause $\ulcorner \forall_{\mathrm{Ord}}\beta\ L_\beta[R] \not\models T^R \urcorner$ in (7.55.3).

2 Suppose toward a contradiction that $\alpha > \rho$. Let $\sigma = \omega_{\rho+1}^M$. Let $N = L_\sigma[R]$. Since $\sigma \in M$, $L_{\sigma+1}[R] \subseteq M$. Since σ is a regular cardinal in M, it is a regular cardinal in $L_{\sigma+1}[R]$, so $N \models \mathsf{ZF}^-$ (by (7.52) generalized to $L[R]$). Since every cardinal of M is a cardinal of N, $N \models T^R$. This violates Axiom 7.55.3 of our working theory T_0^R. $\square^{7.58}$

We will have use for a method of coding ordered pairs that does not increase the rank when applied to sets at least one of which is of infinite rank.

(7.59) **Theorem** $[\mathsf{ZF}^-]$ *Suppose X is a transitive set. There exists a unique function f such that $\operatorname{dom} f = X \times X$ and for all $x, y \in X$,*

1. *if $x, y \in V_\omega$ then $f\langle x, y\rangle = \langle x, y\rangle$ and*

2. *if either x or y is not in V_ω then*

$$f\langle x, y\rangle = \{f\langle 0, a\rangle \mid a \in x\} \cup \{f\langle 1, a\rangle \mid a \in y\}.$$

Proof Show that any two such functions agree on their common domain. Use this with Replacement to show that there exists a unique such function on $V_\omega \cup \operatorname{tc}\{x, y\}$ for any x, y. $\square^{7.59}$

(7.60) **Definition** [ZF$^-$] $P(x, y) \overset{\text{def}}{=}$ *the unique set z such that there exists f as in (7.59) with $X = \operatorname{tc}\{x, y\}$ such that $f\langle x, y\rangle = z$.*

(7.61) **Theorem** [ZF$^-$] *P is injective and does not increase rank above ω.*

Proof First we show that P is injective.

(7.62) **Claim** *Suppose α is an infinite ordinal and $\operatorname{rk} x, \operatorname{rk} y < \alpha$*

1. *If it is not the case that $x, y \in V_\omega$ then $P(x, y) \notin V_\omega$.*
2. *If $\operatorname{rk} x', \operatorname{rk} y' < \alpha$ and $x \neq x'$ or $y \neq y'$, then $P(x, y) \neq P(x', y')$.*

Proof By induction on $\alpha \geqslant \omega$. Suppose α is least where it fails, and let x, y exemplify this failure. Clearly, α is not a limit ordinal (including ω). Let $\alpha = \beta + 1$. Assume the result for β. To prove (1), note that either x or y has rank $\alpha \geqslant \omega$, so $P(x, y) = \{P(0, a) \mid a \in x\} \cup \{P(1, a) \mid a \in y\}$. If either x or y is infinite, then by induction hypothesis, $P(x, y)$ is infinite, hence not in V_ω. If x and y are both finite then one of them contains a set a that is not in V_ω. It follows that either $P(0, a)$ or $P(1, a)$ is in $P(x, y)$. Since $\operatorname{rk} a < \max\{\operatorname{rk} x, \operatorname{rk} y\} \leqslant \beta$, by induction hypothesis, neither $P(0, a)$ nor $P(1, a)$ is in V_ω, so $P(x, y) \notin V_\omega$.

To prove (2), suppose $P(x, y) = P(x', y')$, but either $x \neq x'$ or $y \neq y'$. If x and y are both in V_ω then clearly x' and y' are not both in V_ω, so by (1), which we have just proved for α, $P(x', y') \notin V_\omega$. Since $P(x, y) \in V_\omega$, $P(x, y) \neq P(x', y')$. Similarly, x' and y' are not both in V_ω. Hence

$$\begin{aligned} &\{P(0, a) \mid a \in x\} \cup \{P(1, a) \mid a \in y\} \\ &\qquad = P(x, y) = P(x', y') \\ &\qquad\qquad = \{P(0, a) \mid a \in x'\} \cup \{P(1, a) \mid a \in y'\} \end{aligned}$$

It follows by induction hypothesis that $x = x'$ and $y = y'$. $\square^{7.62}$

Now we show by induction on $\alpha \geqslant \omega$ that if $\operatorname{rk} x, \operatorname{rk} y < \alpha$ then $\operatorname{rk} P(x, y) < \alpha$. For $\alpha = \omega$ this is straightforward. Suppose α is least where it fails. Clearly, α is not a limit. Let $\alpha = \beta + 1$. Suppose $\operatorname{rk} x, \operatorname{rk} y < \alpha$. Since x and y are not both in V_ω, $P(x, y) = \{P(0, a) \mid a \in x\} \cup \{P(1, a) \mid a \in y\}$. For all $a \in x \cup y$, $\operatorname{rk} a < \beta$, so by induction hypothesis, $\operatorname{rk} P(0, a), \operatorname{rk} P(1, a) < \beta$. Hence, $\operatorname{rk} P(x, y) \leqslant \beta < \alpha$. $\square^{7.61}$

(7.63) **Definition** [ZF$^-$] *A set x* codes *a binary relation $s \overset{\text{def}}{\iff} s = \{\langle a, b\rangle \mid P(a, b) \in x\}$.*

We are chiefly interested in the use of this coding system to represent ordinals. Any ordinal in ω_1 is the order type of a binary relation on ω, which may be coded by a set in $V_{\omega+1}$. Any ordinal in ω_2 is the order type of a binary relation on ω_1, which is isomorphic to a binary relation on V_{ω_1} via the coding just mentioned, which is in turn coded by a set in $V_{\omega+2}$. This needs AC, but this will be available, as we will deal exclusively with models of $\boldsymbol{V} = \boldsymbol{L}$. In general, any ordinal in ω_α has a code in $V_{\omega+\alpha}$.

In the consideration of models M of the form $L_\zeta[R]$, it is often convenient to restrict one's attention to limit ordinals ζ. For this discussion we choose not to do this, and as a consequence we have to take care to define certain notions so as to be reasonable in this general context. For example, we might naïvely say that $\alpha < \zeta$ is a cardinal in M iff there is no function $f : \beta \overset{\text{sur}}{\to} \alpha$ in M with $\beta < \alpha$. Note, however, that if α is not a limit ordinal, say $\alpha = \theta + 1$, and $f : \beta \overset{\text{sur}}{\to} \alpha$, then f contains $(\gamma, \theta) = \{\{\gamma\}, \{\gamma, \theta\}\}$ for some γ, so $\operatorname{rk} f \geqslant \operatorname{rk} \theta + 3 = \operatorname{rk} \alpha + 2 = \alpha + 2$. Of course, $f \in M \to \operatorname{rk} f < \zeta$, so if $\alpha < \zeta$ but $\alpha + 2 \geqslant \zeta$ then there is no $f : \beta \overset{\text{sur}}{\to} \alpha$ in M. If ζ is a limit ordinal then this cannot happen, but if ζ is a successor ordinal, it does happen.

We will therefore say that $\alpha < \zeta$ is a cardinal in $L_\zeta[R]$ iff α is finite or α is a limit ordinal *and there is no function $f : \beta \overset{\text{sur}}{\to} \alpha$ in $L_\zeta[R]$ with $\beta < \alpha$.*[17]

(7.64) **Definition** $[T^R]$ *Suppose $\alpha \leqslant \rho$. As a notational convenience, let $U_\alpha \overset{\text{def}}{=} V_{\omega+\alpha}$.*

This definition is legitimate by virtue of (7.56). It is a notational convenience for two reasons. One is obviously that it allows us to use α where otherwise we would have to use $\omega + \alpha$. The other is that we never really have to use the fact that U_α exists, as we will not refer to its being a member of anything. We only refer to membership of sets in U_α, which we could state in terms of rank. That said, since they are available as sets, we will avail ourselves of their availability.

Definition $[T^R]$ *Define W_α by recursion on $\alpha \leqslant \rho$ as follows.*

1. $W_0 \overset{\text{def}}{=} V_\omega$.
2. *If $\alpha = \beta + 1$ then $W_\alpha \overset{\text{def}}{=}$ the set of w such that $w \in W_\beta$ or w codes*[7.63] *a wellorder of a subset of W_β.*
3. *If α is a limit ordinal then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.*

Note that since we are using the "flat" pairing operation P,[7.60]

$$\forall \alpha \leqslant \rho \; W_\alpha \subseteq U_\alpha \; (= V_{\omega+\alpha}),$$

so W_α is legitimately defined as a set. As just discussed in connection with (7.64) this is merely a convenience. At the expense of some circumlocution, we could define a predicate W so that $W(\alpha, x) \leftrightarrow x \in W_\alpha$.

(7.65) **Theorem** $[T_0^R]$ *Suppose η is an infinite ordinal. Then $|\eta| = \omega_\alpha$ for some $\alpha \leqslant \rho$,*[7.58.1] *and there exists w such that w codes a wellorder of a subset of W_α with $\operatorname{ot} w = \eta$.*

Proof Since $|\eta| = \omega_\alpha$, η is the order type of a binary relation S on a subset of ω_α. We now proceed by induction on α.

If $\alpha = 0$ then S is a relation on a subset of ω so it is coded by $w \subseteq V_\omega = W_0$.

Suppose $\alpha > 0$ and the theorem holds for all $\beta < \alpha$. For each $\xi \in \omega_\alpha$ let w_ξ be a representative of ξ in the following sense.

[17] Alternatively, we could solve this problem by using functional relations defined using the pairing function P, which does not increase rank.

1. If ξ is finite, let $w_\xi = \xi$.
2. If ξ is infinite, let β be such that $|\xi| = \omega_\beta$. Since $\xi < \omega_\alpha$, $\beta < \alpha$, so by induction hypothesis there exists w that codes a wellorder of a subset of W_β with $\operatorname{ot} w = \xi$. Let w_ξ be such a code. Note that $w_\xi \in W_{\beta+1} \subseteq W_\alpha$.

Since $\boldsymbol{V} = \boldsymbol{L}$ is in T^R, we have AC, so there exists a function $\xi \mapsto w_\xi$ as above. Let $S' = \{\langle w_\xi, w_{\xi'} \rangle \mid \langle \xi, \xi' \rangle \in S\}$. Then S' is a wellorder of a subset of W_α with order type η. Let w code S'. $\square^{7.65}$

(7.66) **Theorem** $[T_0^R]$ *Suppose ζ is an ordinal, and let $M = L_\zeta[R]$. Then for every $\eta < \zeta$ there exists $w \in M$ such that w codes a wellorder of a subset of W_ρ with* $\operatorname{ot} w = \eta$.

Proof Suppose $\delta < \zeta$ and δ is a successor cardinal in the sense of M. Let $M_\delta = L_\delta[R]$. Then $M_\delta \models \mathsf{ZF}^-$.[18] Let σ be the order type of the infinite successor cardinals in M_δ. Since $M_\delta \not\models T^R$,[7.55.3] $\sigma < \rho$.[7.55.2.4] M_δ is a model of T^R with clause (7.55.2.4) omitted. The proof of (7.65) is readily adapted to show that for every $\eta < \delta$ there exists $w \in M_\delta$ such that w codes a wellorder of a subset of W_σ with $\operatorname{ot} w = \eta$. Note that $w \in M$ and w codes a subset of W_ρ.

Let λ be the union of the successor cardinals in the sense of M. If $\eta < \lambda$ then there is an M-successor cardinal $\delta > \eta$, and we have just shown that in this case there exists $w \in M$ such that w codes a subset of W_σ for some $\sigma < \rho$, and $\operatorname{ot} w = \eta$. Note that by definition, $w \in W_\rho$.

Suppose, on the other hand, that $\eta \geqslant \lambda$. It is easy to see that λ is the largest M-cardinal, so there is a wellorder of λ of order type η. As in the proof of (7.65), this may be replaced by wellorder of codes for ordinals $< \lambda$, which we have just shown may be taken to be in W_ρ. Thus, in this case as well, there exists $w \in M$ such that w codes a wellorder of a subset of W_ρ with $\operatorname{ot} w = \eta$. $\square^{7.66}$

Definition [ZF] *Suppose r is a wellorder of a subset of ω. $\beta_r \overset{\text{def}}{=}$ the least β such that $L_\beta[r] \models T_0^R$.*

We will show that $L_{\beta_r}[r] \not\models \ulcorner\boldsymbol{\Sigma}^0_{1+\lceil\rho\rceil+3}\text{-determinacy}\urcorner$. The big idea behind Friedman's proof is to show that if $\mathfrak{M} = L_{\beta_r}[r]$ satisfies $\boldsymbol{\Sigma}^0_{1+\rho+3}$-determinacy, then the theory $\operatorname{Th}\mathfrak{M}$ of $\mathfrak{M}$ is definable over $\mathfrak{M}$. The proof then invokes the fact that $\operatorname{Th}\mathfrak{M}$ is not definable over $\mathfrak{M}$. We have previously proved results similar to this, but not this result exactly, so we supply a proof now.

(7.67) **Theorem** [C] *Suppose $\mathfrak{M}$ is any satisfactory s'-structure extending $(V_\omega; \in, \mathbf{0}, \backsim)$. Then $\operatorname{Th}\mathfrak{M}$ is not definable over $\mathfrak{M}$.*

Proof Suppose to the contrary that ϕ is an s-formula with one free variable v_0 such that for every s'-sentence θ,

$$\mathfrak{M} \models \phi\left[\begin{smallmatrix}\mathrm{v}_0\\ \theta\end{smallmatrix}\right] \leftrightarrow \mathfrak{M} \models \theta. \tag{7.68}$$

Let ψ be a formula with one free variable v_0 that says of its argument that it is an s'-formula with one free variable v_0 and that $\neg\,\phi$ holds for the sentence that results

[18] Remember[7.52] that it suffices that δ is regular in the sense of $L_{\delta+1}[R]$, which it is because it is a successor cardinal in the sense of $L_\zeta[R]$ and $\zeta > \delta$.

from substituting the canonical name of this formula for its free variable. Thus, for any s'-formula χ with one free variable v_0, letting $\tau = \mathrm{Nm}\,\chi$, the canonical s'-name for χ (as a member of V_ω), letting $\sigma = \chi\binom{\mathrm{v}_0}{\tau}$, and using (7.68),

$$\mathfrak{M} \models \psi\begin{bmatrix}\mathrm{v}_0\\ \chi\end{bmatrix} \leftrightarrow \mathfrak{M} \models \neg\,\phi\begin{bmatrix}\mathrm{v}_0\\ \sigma\end{bmatrix} \leftrightarrow \mathfrak{M} \not\models \sigma. \tag{7.69}$$

Now let $\chi = \psi$, so $\tau = \mathrm{Nm}\,\psi$ and $\sigma = \psi\binom{\mathrm{v}_0}{\tau}$. Then, since $\tau^{\mathfrak{M}} = \psi$, using (7.69),

$$\mathfrak{M} \models \sigma \leftrightarrow \mathfrak{M} \models \psi\begin{bmatrix}\mathrm{v}_0\\ \psi\end{bmatrix} \leftrightarrow \mathfrak{M} \not\models \sigma.$$

This contradiction establishes the theorem. $\square^{7.67}$

Definition $[T^R]$ *An* r*-structure* $\mathfrak{M}$ *is* good $\overset{\mathrm{def}}{\iff}$

1. $|\mathfrak{M}| \subseteq V_\omega$;
2. $\mathfrak{M} \models T_0^R$;
3. *the relation* $\{\langle a, b\rangle \mid \ulcorner [a] \in [b] \in \omega\urcorner^{\mathfrak{M}}\}$ *is isomorphic to* $\{\langle a, b\rangle \mid a \in b \in \omega\}$; *and*
4. $R^{\mathfrak{M}} \cong r$.

Suppose $\mathfrak{M}$ is good. Then the individuals of $\mathfrak{M}$ are hereditarily finite sets, and the order type of the finite ordinals in the sense of $\mathfrak{M}$ is ω. Although $\mathfrak{M}$ is not in general wellfounded, it resembles L in certain important respects. First note that since $\omega^{\mathfrak{M}}$ has order type ω, $V_\omega^{\mathfrak{M}}$ is isomorphic to V_ω and $\mathfrak{M}$ correctly identifies the language $\mathcal{L}^{\mathsf{r}}$. (Statements of this sort, here and below, are of course to be understood modulo isomorphic equivalence.) $\mathfrak{M}$ also satisfies all of the theory of constructibility that does not depend on the Power axiom. As it is generally easy to supply appropriate absoluteness arguments *en passant*, we will often not explicitly state them.

(7.70) **Definition** $[T^R]$ *Suppose* $\mathfrak{M}$ *is a good structure.*

1. $\Omega^{\mathfrak{M}} \overset{\mathrm{def}}{=}$ *the set* $\{w \mid \ulcorner \mathrm{Ord}\,[w]\urcorner^{\mathfrak{M}}\}$ *of* $\mathfrak{M}$*-ordinals, with the corresponding order. Note that* $\Omega^{\mathfrak{M}}$ *is linearly ordered, but is not necessarily wellordered.*
2. $\bar{\Omega}^{\mathfrak{M}} \overset{\mathrm{def}}{=}$ *the maximum wellordered initial segment of* $\Omega^{\mathfrak{M}}$ *(which is the union of wellordered initial segments of* $\Omega^{\mathfrak{M}}$*).*
3. $\bar{V}^{\mathfrak{M}} \overset{\mathrm{def}}{=} \bigcup_{a \in \bar{\Omega}^{\mathfrak{M}}} \ulcorner V_{[a]}\urcorner^{\mathfrak{M}}$*, which is the maximum wellfounded initial segment of* $\mathfrak{M}$*.*
4. $\pi_{\mathfrak{M}} \overset{\mathrm{def}}{=}$ *the transitive collapse of* $\bar{V}^{\mathfrak{M}}$*, i.e., the (unique) isomorphism of* $(\bar{V}^{\mathfrak{M}}; \in^{\mathfrak{M}})$ *with a transitive set.* $\hat{c} \overset{\mathrm{def}}{=} \pi_{\mathfrak{M}} c$*, when the identity of the structure* $\mathfrak{M}$ *may be inferred from the context.*

Note that $\ulcorner \mathrm{Ord}\,\rho\urcorner^{\mathfrak{M}}$, so there is a corresponding $a \in \Omega^{\mathfrak{M}}$, in the sense that the set of $\mathfrak{M}$-ordinals below a has order type ρ. Hence, $\bar{\Omega}^{\mathfrak{M}}$ has order type at least $\rho + 1$. It is easy to see that the order type of $\bar{\Omega}^{\mathfrak{M}}$ must be a limit ordinal, so it is at least $\rho + \omega$.[19]

Note that for any $a \in \bar{\Omega}^{\mathfrak{M}}$, $\pi_{\mathfrak{M}}{\to}\ulcorner V_{[a]}\urcorner^{\mathfrak{M}} \subseteq V_\alpha$, where $\alpha = \pi_{\mathfrak{M}}\,a$; and $\pi_{\mathfrak{M}} \restriction \ulcorner V_\omega\urcorner^{\mathfrak{M}}$ is an isomorphism of $\ulcorner V_\omega\urcorner^{\mathfrak{M}}$ with V_ω. Note that if $\mathfrak{N}$ is a transitive substructure of $\mathfrak{M}$, i.e., if $\forall x \in |\mathfrak{N}|\, \forall y \in |\mathfrak{M}|\, (\ulcorner [y] \in [x]\urcorner^{\mathfrak{M}} \to y \in |\mathfrak{N}|)$, then $\pi_{\mathfrak{N}} \subseteq \pi_{\mathfrak{M}}$.

[19]It is also easy to see that it must be at least $\rho + \rho$, $\rho \cdot \rho$, etc.

Definition $[T^R]$ *Suppose r is a wellorder of a subset of ω, and $\mathfrak{M}$ is good.*

1. *Suppose $a \in \Omega^{\mathfrak{M}}$. $\mathfrak{M}_a \stackrel{\text{def}}{=}$ the substructure of $\mathfrak{M}$ with domain $|L_a[r]^{\mathfrak{M}}|$.*
2. *Suppose $A \subseteq \Omega^{\mathfrak{M}}$ is an initial segment. $\mathfrak{M}_A \stackrel{\text{def}}{=}$ the substructure of $\mathfrak{M}$ with domain $\bigcup_{b \in A} |\mathfrak{M}_b|$.*
3. *An* initial segment of $\mathfrak{M}$ $\stackrel{\text{def}}{=}$ *a substructure of $\mathfrak{M}$ of the form $\mathfrak{M}_A$, where A is an initial segment of $\Omega^{\mathfrak{M}}$.*
4. *$\mathfrak{M}_A$ is of* limit *type* $\stackrel{\text{def}}{\iff}$ *A has no greatest element; otherwise $\mathfrak{M}_A$ is of* successor *type.*

It is important to recognize that $\mathfrak{M}_A = \mathfrak{M}_a$ iff a is the least $\mathfrak{M}$-ordinal not in A. If $\mathfrak{M}$ is illfounded, there may not be such an element a for a given proper initial segment $A \subseteq \Omega^{\mathfrak{M}}$.

Definition $[T^R]$

1. *Suppose S is a satisfaction relation. $\mathfrak{S}^S \stackrel{\text{def}}{=}$ the corresponding structure.*
2. *$\mathcal{S} \stackrel{\text{def}}{=}$ the set of $S \subseteq V_\omega$ such that S is an r-satisfaction relation.*
3. *$S \in \mathcal{S}$ is* good $\stackrel{\text{def}}{\iff}$ *$\mathfrak{S}^S$ is good.*

As a notational convenience we may refer to a satisfaction relation when the corresponding structure is meant, e.g., C_0^M is $C_0^{\mathfrak{S}^M}$.

Note that $\mathcal{S}$ is Π^0_2, the most complex parts of the description being the statement that for every existential formula $\phi = \exists v\, \psi$ and every $\mathfrak{S}^S$-assignment A for ϕ, $\langle \phi, A\rangle \in S \leftrightarrow \exists a \in |\mathfrak{S}^S|\ \langle \psi, A\langle {v \atop a}\rangle\rangle \in S$; and the corresponding statement for universal formulas.

Note that if ϕ is a Σ^{s}_n (Π^{s}_n) formula then the set of $\langle \mathfrak{S}, A\rangle$ such that $\mathfrak{S} \subseteq V_\omega$ is an s-structure, A is an $\mathfrak{S}$-assignment for ϕ, and $\mathfrak{S} \models \phi[A]$ is Σ^0_n (Π^0_n), but it is not generally in any smaller complexity class. On the other hand, the set of $\langle S, A\rangle$ such that $S \subseteq V_\omega$ is an s-satisfaction relation, A is an $\mathfrak{S}^S$-assignment for ϕ, and $\mathfrak{S}^S \models \phi[A]$ is Π^0_2. Moreover, if we condition on $S \in \mathcal{S}$ then it is Δ^0_1, i.e., it is the intersection of a Δ^0_1 set with $\mathcal{S}$. For these reasons we work with satisfaction relations, rather than with structures *per se*, in the following discussion.

Definition $[T^R]$ *Suppose M is a good satisfaction relation, $a \in |M|$, and $\eta \leqslant \rho$.*

1. *$M^a \stackrel{\text{def}}{=}$ the set of $b \in |M|$ such that $\ulcorner [b]$ is constructed at an earlier stage than $[a] \urcorner^M$.*
2. *$M_\eta \stackrel{\text{def}}{=}$ the set of $b \in |M|$ such that b is in the wellfounded part of M and $\hat{b} \subseteq U_\eta$.*
3. *Naturally, $M^a_\eta \stackrel{\text{def}}{=} M^a \cap M_\eta$.*

(7.71) **Definition** $[T^R]$ *Suppose M^1, M^2 are good satisfaction relations and $a \in |M^1|$. Then $M^1 \leqslant^a M^2$* $\stackrel{\text{def}}{\iff}$

1. *for all* $b^1 \in M^{1a}_\rho$ *there exists* $b^2 \in M^2_\rho$ *such that* $\hat{b}^1 = \hat{b}^2$, *and if* $\ulcorner[b^1]$ *codes a wellorder*$\urcorner^{M^1}$ *then* $\ulcorner[b^2]$ *codes a wellorder*$\urcorner^{M^2}$*; and*
2. *for all* $b^1 \in M^{1a}_\rho$, $b^2 \in M^2_\rho$, *and* $c^2 \in M^{2(b^2+1)}_\rho$, *if* $\hat{b}^1 = \hat{b}^2$ *then there exists* $c^1 \in M^{1a}_\rho$ *such that* $\hat{c}^1 = \hat{c}^2$.[20]

(7.72) **Theorem** $[T^R]$ *The set of* $\langle M^1, M^2, a\rangle$ *such that* M^1, M^2 *are good satisfaction relations,* $a \in |M^1|$, *and* $M^1 \preccurlyeq^a M^2$, *is* $\mathbf{\Pi}^0_{1+\rho+2}$.

Proof As noted above, the set $\mathcal{S}$ of r-satisfaction relations $S \in V_{\omega+1}$ such that $S \models T^R_0$ is Π^0_2. The additional condition that $R^S = R$ is $\Pi^0_1(R)$, hence $\mathbf{\Pi}^0_1$. Hence, the class of good satisfaction relations is $\mathbf{\Pi}^0_2$. Let $F \subseteq V_{\omega+1}$ be this class.

Suppose M^1, M^2 are good satisfaction relations, $a^1 \in M^1_0$ and $a^2 \in M^2_0$. Then $\hat{a}^1 = \hat{a}^2$ iff for all canonical terms τ for a member of V^ω, $\ulcorner(\tau) \in [a^1]\urcorner^{M^1}$ iff $\ulcorner(\tau) \in [a^2]\urcorner^{M^2}$. Thus there is a Π^0_1 relation $E_0 \subseteq V_{\omega+1} \times V_{\omega+1} \times V_\omega \times V_\omega$, such that for good satisfaction relations M^1, M^2, $a^1 \in M^1_0$, and $a^2 \in M^2_0$,

$$\hat{a}^1 = \hat{a}^2 \leftrightarrow E_0(M^1, M^2, a^1, a^2),$$

the universal quantification (equivalently, countable conjunction or intersection) over terms being the dominant feature, once the restriction to good satisfaction relations has been imposed.

For each $\eta \leqslant \rho$, there is a Δ^0_1 relation $I_\eta \subseteq V_{\omega+1} \times V_\omega \times V_\omega$ such that for any good satisfaction relation M, $a \in |M|$, and $b \in V_\omega$,[21]

$$b \in M^a_\eta \leftrightarrow I_\eta(M, a, b).$$

Define $E_\eta \subseteq V_{\omega+1} \times V_{\omega+1} \times V_\omega \times V_\omega$ by recursion on $\eta \leqslant \rho$ so that

(7.73)
$$\begin{aligned} &E_\eta(M^1, M^2, a^1, a^2) \\ &\quad\leftrightarrow \bigwedge\nolimits_{\alpha<\eta} \bigwedge\nolimits_{b^1, b^2 \in V_\omega} \big(I_\alpha(M^1, a^1, b^1) \wedge I_\alpha(M^2, a^2, b^2) \wedge E_\alpha(M^1, M^2, b^1, b^2) \\ &\qquad \to \big(\ulcorner[b^1] \in [a^1]\urcorner^{M^1} \leftrightarrow \ulcorner[b^2] \in [a^2]\urcorner^{M^2}\big)\big). \end{aligned}$$

Now suppose $0 < \eta \leqslant \rho$, M^1, M^2 are good satisfaction relations, $a^1 \in M^1_\eta$, and $a^2 \in M^2_\eta$. Then $\hat{a}^1 = \hat{a}^2$ iff for all $\alpha < \eta$, $b^1 \in M^{1a^1}_\alpha$ and $b^2 \in M^{2a^2}_\alpha$, if $\hat{b}^1 = \hat{b}^2$ then $\hat{b}^1 \in \hat{a}^1 \leftrightarrow \hat{b}^2 \in \hat{a}^2$. Hence, by induction on η, for all $\eta \leqslant \rho$, good satisfaction relations M^1, M^2, $a^1 \in M^1_\eta$, and $a^2 \in M^2_\eta$,

$$\hat{a}^1 = \hat{a}^2 \leftrightarrow E_\eta(M^1, M^2, a^1, a^2).$$

Since η is countable in (7.73), if E_α is $\mathbf{\Pi}^0_{1+\alpha}$ for all $\alpha < \eta$, then E_η is $\mathbf{\Pi}^0_{1+\eta}$. Hence, by induction, E_η is $\mathbf{\Pi}^0_{1+\eta}$ for all $\eta \leqslant \rho$.

Using these complexity bounds in (7.71) together with the existence of Δ^0_1 relations expressing that $a \in |M|$, $\ulcorner[a]$ codes a wellorder$\urcorner^M$, $a \in M_\eta$ and $b \in M^{\{a\}}_\eta$ for good satisfaction relations M, we obtain the conclusion of the theorem. $\square^{7.72}$

[20] We interpret $\ulcorner b^2 + 1\urcorner$ literally as $b^2 \cup \{b^2\}$, which is constructed at the next stage after b^2. Thus $c^2 \in M^{2(b^2+1)}_\rho$ iff c_2 is constructed at the same stage as b^2 or earlier. We could use $\ulcorner\{b^2\}\urcorner$, but $\ulcorner b^2 + 1\urcorner$ is preferable for its mnemonic value.

[21] What makes this relation specific to η is a reference to the element of ω that corresponds to η in the wellorder r.

(7.74) **Theorem** $[T_0^R]$ *Suppose, M^1 and M^2 are good satisfaction relations, $a \in |M^1|$, $M^1 \preccurlyeq^a M^2$, and M^2 is wellfounded. Then M^{1a} is wellfounded and $\operatorname{ot}\Omega^{M^{1a}} \leqslant \operatorname{ot}\Omega^{M^2}$. Hence, M^{1a} is isomorphic to an initial segment of M^2.*

Proof $M^1 \models T_0^R$, so (7.66) holds in M^1 and for any $b \in \Omega^{M^{1a}}$ there exists $c^1 \in M_\rho^{1a}$ such that $\ulcorner [c^1]$ codes a wellorder of order type $[b]\urcorner^{M^1}$. As noted following Definition 7.70, the wellfounded initial segment of M^1 includes considerably more than M_ρ^1. In particular it contains the element $d^1 \in |M^1|$ such that $\ulcorner [d^1]$ is the binary relation coded by $[c^1]\urcorner^{M^1}$,[7.63] which has only finitely greater rank (to put it mildly). Note that $\hat{d}^1$ is the binary relation coded by $\hat{c}^1$. Since $M^1 \preccurlyeq^a M^2$, there exists $c^2 \in M_\rho^2$ such that $\hat{c}^1 = \hat{c}^2$ and $\ulcorner [c^2]$ codes a wellorder$\urcorner^{M^2}$. Let $d^2 \in |M^2|$ be the element such that $\ulcorner [d^2]$ is the binary relation coded by $[c^2]\urcorner^{M^2}$. Then $\hat{d}^2$ is the binary relation coded by $\hat{c}^2$, so $\hat{d}^1 = \hat{d}^2$. Since M^2 is wellfounded, the relation $E^2 = \{\langle e_0^2, e_1^2\rangle \mid \ulcorner\langle [e_0^2], [e_1^2]\rangle \in [d^2]\urcorner^{M^2}\}$ is a wellorder with order type $< \operatorname{ot}\Omega^{M^2}$.

Hence $\hat{d}^2$, which is $\{\langle \hat{e}_0^2, \hat{e}_1^2\rangle \mid \langle e_0^2, e_1^2\rangle \in E^2\}$, is a wellorder. Since $\hat{d}^1 = \hat{d}^2$, the relation $E^1 = \{\langle e_0^1, e_1^1\rangle \mid \langle \hat{e}_0^1, \hat{e}_1^1\rangle \in \hat{d}^1\}$ is isomorphic with E^2. In M^1 there is an isomorphism of d^1 with $\Omega^{M^{1b}}$, which is therefore also isomorphic to E^2.

Every proper initial segment of $\Omega^{M^{1a}}$ is included in $\Omega^{M^{1b}}$ for some $b \in \Omega^{M^{1a}}$, so $\Omega^{M^{1a}}$ is wellordered, and $\operatorname{ot}\Omega^{M^{1a}} \leqslant \operatorname{ot}\Omega^{M^2}$. By induction on $\Omega^{M^{1a}}$ it is easily shown that $M^{1a} \cong L_\alpha[R]$, where $\alpha = \operatorname{ot}\Omega^{M^{1a}}$. Similarly, M^2 is isomorphic to an initial segment of $L[R]$, so M^{1a} is isomorphic to an initial segment of M^2. $\square$[7.74]

(7.75) **Theorem** [ZF] *Suppose ρ is a countable ordinal and $r \subseteq \omega \times \omega$ is a wellorder of order type ρ. Then $L_{\beta_r}[r] \not\models \ulcorner \mathbf{\Sigma}^0_{1+[\rho]+3}$-determinacy$\urcorner$.*

Proof For each sentence θ in the language of set theory, let $X_\theta \subseteq {}^{<\omega}\omega$ be defined as follows. Note that we regard players I and II as coding subsets of V_ω, viz., $\vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{I}})$ and $\vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{II}})$, which are to be satisfaction relations M^1 and M^2 as we have been discussing above.

(7.76) *$z \in X_\theta$ iff, letting $M^1 = \vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{I}})$ and $M^2 = \vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{II}})$,*

1. *M^1 is a good satisfaction relation such that $M^1 \models \theta$; and*
2. *if M^2 is a good satisfaction relation then either*
    1. *$M^1 \preccurlyeq M^2$, i.e., for all $a \in |M^1|$, $M^1 \preccurlyeq^a M^2$; or*
    2. *there is a least $a \in \Omega^{M^1}$ such that $M^1 \not\preccurlyeq^a M^2$.*

(7.77) **Claim** *If $L_{\beta_r}[r] \models \ulcorner$II has a winning strategy in $\langle {}^{<\omega}\omega, X_{[\theta]}\rangle\urcorner$ then $L_{\beta_r}[r] \models \neg\,\theta$.*

Proof Suppose $\sigma \in L_{\beta_r}[r]$ is such that $L_{\beta_r}[r] \models \ulcorner [\sigma]$ is a winning II-strategy in $\langle {}^{<\omega}\omega, X_{[\theta]}\rangle\urcorner$, i.e., $L_{\beta_r}[r] \models \ulcorner \forall x \in {}^{\omega}\omega\ x * [\sigma] \notin X_{[\theta]}\urcorner$. Since $L_{\beta_r}[r]$ is a transitive model of ZF^-, by Π_1^1-absoluteness,[6.7] $\forall x \in {}^{\omega}\omega\ x * \sigma \notin X_\theta$.

Suppose toward a contradiction that $L_{\beta_r}[r] \models \theta$, and let $x \in {}^{\omega}\omega$ be such that $\vec{B}^{\rightarrow}(\operatorname{im} x)$ is a satisfaction relation M^1 for a structure isomorphic to $L_{\beta_r}[r]$. Let $z = x * \sigma$, and let $M^2 = \vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{II}})$. Since $L_{\beta_r}[r] \models T_0^R + \theta$, (7.76.1) is satisfied. Since $z \notin X_\theta$, M^2 must be a good satisfaction relation, and both (7.76.2.1) and (7.76.2.2) must fail. But if (7.76.2.1) fails then, since Ω^{M^1} is wellordered, (7.76.2.2) holds. $\square$[7.77]

(7.78) **Claim** *If* $L_{\beta_r}[r] \models \ulcorner$I *has a winning strategy in* $\langle {}^{<\omega}\omega, X_{\lceil\theta\rceil}\rangle\urcorner$ *then* $L_{\beta_r}[r] \models \theta$.

Proof Suppose $\sigma \in L_{\beta_r}[r]$ is such that $L_{\beta_r}[r] \models \ulcorner\lceil\sigma\rceil$ is a winning I-strategy in $\langle {}^{<\omega}\omega, X_{\lceil\theta\rceil}\rangle\urcorner$, i.e., $L_{\beta_r}[r] \models \ulcorner\forall y \in {}^{\omega}\omega\ \lceil\sigma\rceil * y \in X_{\lceil\theta\rceil}\urcorner$. Again by Π^1_1-absoluteness, $\forall y \in {}^{\omega}\omega\ \sigma * y \in X_\theta$.

Suppose toward a contradiction that $L_{\beta_r}[r] \models \neg\,\theta$, and let $y \in {}^{\omega}\omega$ be such that $\vec{B}^{\rightarrow}(\operatorname{im} y)$ is a satisfaction relation M^2 for a structure isomorphic to $L_{\beta_r}[r]$. Let $z = \sigma * y$, and let $M^1 = \vec{B}^{\rightarrow}(\operatorname{im} z^{\mathrm{I}})$. Since $z \in X_\theta$, M^1 is a good satisfaction relation and $M^1 \models \theta$. Since $L_{\beta_r}[r] \not\models \theta$, $M^1 \not\cong L_{\beta_r}[r]$. Since $L_{\beta_r}[r]$ is the minimum transitive model M of T^R such that $R^M = r$, M^1 is not isomorphic to any initial segment of M^2, so[7.74] it is not the case that $M^1 \preccurlyeq M^2$. Hence, (7.76.2.1) is not satisfied, so (7.76.2.2) must hold.

(7.79) *Let* $a \in \Omega^{M^1}$ *be least such that* $M^1 \not\preccurlyeq^a M^2$.

Clearly, a is not a limit element of Ω^{M^1}. Let a^1 be the immediate predecessor of a in Ω^{M^1}. Then $M^1 \preccurlyeq^{a^1} M^2$, so M^{1a^1} is isomorphic to an initial segment of M^2. Since M^2 is wellfounded, this is M^{2a^2} for some $a^2 \in \Omega^{M^2}$. Let ι be the isomorphism (which is unique, as the structures are wellfounded). Note that a is $a^1 + 1$ in the sense of M^1, and $M^{1(a^1+1)}$ is $\mathcal{D}\, M^1_{a^1}$ as computed in M^1. Similarly, $M^{2(a^2+1)}$ is $\mathcal{D}\, M^{2a^2}$ as computed in M^2.

This allows us to extend ι to an isomorphism of $M^{1(a^1+1)}$ (i.e., M^{1a}) with $M^{2(a^2+1)}$ as follows. Recall that $\ulcorner V_\omega\urcorner^{M^1}$ is isomorphic to V_ω, so the language $\mathcal{L}^{\mathsf{r}}$ as construed in M^1 is isomorphic to $\mathcal{L}^{\mathsf{r}}$. Hence, if $x \in |M^{1(a^1+1)}|$ then there exist an r-formula ϕ, variables $v_0, \ldots, v_n$, and $z^1_1, \ldots, z^1_n \in |M^{1a^1}|$ such that for all $y \in |M^{1a^1}|$, $\ulcorner\lceil y\rceil \in \lceil x\rceil\urcorner^{M^1}$ iff $M^{1a^1} \models \phi\left[\begin{smallmatrix} v_0 & v_1 & \cdots & v_n \\ y & z_1 & \cdots & z_n\end{smallmatrix}\right]$. Let $\iota\, x$ be the corresponding element of $|M^2|$.

From this it follows that $M^1 \preccurlyeq^a M^2$, contradicting (7.79). $\square$[7.78]

Now suppose toward a contradiction that $L_{\beta_r}[r] \models \ulcorner\mathbf{\Sigma}^0_{1+\lceil\rho\rceil+3}$-determinacy$\urcorner$. It follows from (7.77) and (7.78) that $\operatorname{Th} L_{\beta_r}[r]$ is definable over $L_{\beta_r}[r]$. This is impossible by (7.67). Hence $L_{\beta_r}[r] \not\models \ulcorner\mathbf{\Sigma}^0_{1+\lceil\rho\rceil+3}$-determinacy$\urcorner$. $\square$[7.75]

7.7 Summary

The constructible universe is a conservative's utopia, admission to which is granted only to those sets whose existence is mandated by ZF, and every individual of which is assigned a unique identifying (ordinal) number. The power of the axiom is exploited by meticulous examination of the sequence in which sets are entered into the roster of L.

The almost universality of L is essentially immediate, from which it follows that L satisfies all of ZF with the possible exception of Comprehension, but this follows rather easily from the reflection principle and the generative principle of L, which is specifically designed to yield Comprehension, so $L \models$ ZF. The definition of the sequence $\langle L_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is sufficiently local that it is absolute[7.9] for structures $(L_\beta; \in)$ for limit ordinals β, from which it follows that $L \models \ulcorner V = L\urcorner$.

The admission of sets into L is sufficiently methodical that it is easy to produce a wellordering of L that has a definition that is absolute for L. Hence, $L \models$ AC.

The condensation lemma is the simplest example of the structural theorems that make the axiom of constructibility so powerful. We use it to show that $L \models \mathsf{GCH}$, the generalized continuum hypothesis.

Not surprisingly, $\ulcorner V = L \urcorner$ answers many additional questions left open by ZF. In particular, in the sense of L there is a Δ^1_2 wellordering of $\mathbb{R}$. This can be used to define an uncountable Π^1_1 set without a perfect subset, and a Δ^1_2 set lacking the Baire property and Lebesgue measurability. In fact, many issues in descriptive set theory are settled by $\ulcorner V = L \urcorner$ via the definable wellordering of $\mathbb{R}$.

In another and far more fruitful direction we have the refutation of Suslin's hypothesis in L. This construction of Jensen exploits the repetitive nature of the construction of L to define the powerful combinatorial principle $\diamondsuit$, which can be used to construct a Suslin tree.

We conclude with the proof that Σ^0_4-determinacy does not hold in the minimum model of ZF^- (and the generalization of this proof to all Borel sets), which is a good example of the sort of analysis to which the constructible hierarchy lends itself.

Chapter 8

Possible Worlds

Contents

8.1 Introduction . . . **468**
- 8.1.1 A universe of names . . . 469
- 8.1.2 Category applied . . . 471
- 8.1.3 Genericity . . . 473
- 8.1.4 Forcing . . . 474

8.2 Genericity and forcing . . . **475**
- 8.2.1 Partial orders and filters . . . 476
- 8.2.2 Generic extensions . . . 478
- 8.2.3 Forcing . . . 484
- 8.2.4 Definability of forcing . . . 487
- 8.2.5 Generic extensions again . . . 489
- 8.2.6 "Arguing with generic extensions" . . . 490

8.3 Boolean-valued structures . . . **492**
- 8.3.1 Definability of boolean valuation . . . 500
- 8.3.2 Forcing *vs* boolean valuation . . . 501

8.4 Logical considerations . . . **503**

8.5 The theory of a generic extension . . . **505**
- 8.5.1 Generic extension preserves ZF . . . 507
- 8.5.2 "Arguing in a generic extension" . . . 513
- 8.5.3 Generic extension preserves AC . . . 517
- 8.5.4 Arguing in a boolean-valued universe . . . 521

8.6 Relative consistency proofs . . . **521**
- 8.6.1 $\mathrm{Con}(\mathsf{ZF} + V \neq L)$ . . . 523

8.7 Chain conditions and saturation . . . **524**

8.8 Closure and distributivity . . . **527**

8.9 Independence of the continuum hypothesis . . . **529**

8.10 Independence of the axiom of choice . . . **532**

8.11 Product forcing . . . **534**
- 8.11.1 Easton forcing . . . 537

8.12 Class forcing . . . **540**
- 8.12.1 The necessity of pretameness . . . 543
- 8.12.2 Pretame forcing . . . 544

8.12.3 Pretame generic extension preserves ZF^- 545
8.12.4 Easton's theorem 547
8.12.5 Coding the universe by a real 549
8.13 Iterated forcing 549
8.13.1 Generic is not so special 558
8.13.2 Transfinite iteration 559
8.13.3 Martin's axiom 561
8.14 Some forcing constructions 565
8.14.1 A generic Suslin tree 565
8.14.2 A generic $\Diamond$-sequence 567
8.14.3 Silver's singular cardinals theorem 568
8.14.4 A model in which every set of reals has the Baire property, is Lebesgue-measurable, and has the perfect set property 570
8.15 Summary 575

Cogitor, ergo sum.[1]

$V[G]$

[Thomas Jech's *Set Theory*[12] is an excellent source for all aspects of the modern theory of membership. Kenneth Kunen's *Set Theory*[16] is also an excellent source, with particular emphasis on the topic of this chapter.]

8.1 Introduction

Following Gödel's proof that $\boldsymbol{V} = \boldsymbol{L}$, and therefore also its consequences, including AC and GCH, are consistent relative to ZF, the question remained: do AC, GCH, etc., follow from ZF? It was strongly suspected that they do not, i.e., that their negations are consistent with ZF. Starting from Con ZF, the method of inner models cannot prove $\mathrm{Con}(\mathsf{ZF} + \neg\,\mathsf{AC})$ or $\mathrm{Con}(\mathsf{ZF} + \neg\,\mathsf{CH})$. The reason is that each of these latter statements implies $\mathrm{Con}(\mathsf{ZF} + \boldsymbol{V} \neq \boldsymbol{L})$, and this cannot be proved by the method of inner models. For suppose one could prove that any model of ZF has an inner model of $\mathsf{ZF} + \boldsymbol{V} \neq \boldsymbol{L}$. Then in particular, any model of $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}$ has an inner model of $\mathsf{ZF} + \boldsymbol{V} \neq \boldsymbol{L}$. But this is impossible as L has no proper inner models: it is the minimum model of ZF that includes Ord. Even if we consider shorter, as well as thinner, models, we cannot falsify $\boldsymbol{V} = \boldsymbol{L}$, because any initial segment of L satisfies $\boldsymbol{V} = \boldsymbol{L}$.

Thus, after Gödel had invented (discovered?) L, $\mathrm{Con}(\mathsf{ZF} + \boldsymbol{V} \neq \boldsymbol{L})$ became the Holy Grail of set theorists. It would be necessary to prove at least that much if the independence of the AC and CH were to be proved, and the method of inner models would not work. The way was found in 1963 by Paul Cohen, a mathematician who was not primarily a set theorist. His idea was rather simple, but very deep. It set off an explosion of activity in the field of set theory that has continued unabated to the present.

This introduction is devoted to arriving at Cohen's method by a process of discovery, and it is hoped that it is a useful motivation, but it may be omitted without loss of continuity, as all the ideas are presented formally beginning in Section 8.2.

[1] I am thought, therefore I am.

Suppose M is a transitive set and $M \models \mathsf{ZF}$. We make no assumption about the size of M at the present time. Later we may require that it be countable, or we may permit it to be a proper class, even the whole universe: $M = V$. For now, however, we simply suppose M is a set.

We will make reference from time to time to "people living in M" as opposed to people (like ourselves, perhaps?) living in the "real world". Let us say a little more about these people. They are actually a most admirable race. They have complete knowledge of M in the sense that if you specify any s-formula ϕ, with free variables $u_0, \ldots, u_{n^-}$, and any $a_0, \ldots, a_{n^-} \in M$ then the M-people can tell you whether $M \models \phi\left[\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ a_0 & \cdots & a_{n^-} \end{smallmatrix}\right]$. In other words, they know the satisfaction relation of M, which is a lot more than we know about "our" world. (We wouldn't even qualify as M-people if M were taken to be the class HF of hereditarily finite sets.) We also refer to what M "knows" or "thinks" in the same vein. Any such reference is to the satisfaction relation for M.

One thing that the M-people cannot do, of course, is see any set outside of M. For example, if M is countable, they will not realize it. Any function that witnesses the countability of M, i.e., any $f : \omega \overset{\text{sur}}{\to} M$, will not be in M—since $M \models \mathsf{ZF}$—and so will not be accessible to them. Nevertheless, they can *imagine* objects outside of M, and, as we will see, they may be able to say quite a bit about them.

Let A be an infinite set in M. The possibility exists that there is a subset of A that is not in M. In fact, if M is countable, there will certainly be a subset of A that is not in M, because A has uncountably many subsets. Let us consider *extending* M by the adjunction of an arbitrary non-empty subset G of A. This is only interesting if G is not in M, but for the time being, we will only assume that G is not empty (and that only for technical reasons). We want the extension to be a model of ZF, but we want it to be the smallest model of ZF that includes M and contains G. We wish, therefore, to define a transitive set $M[G]$ with the following properties:

(8.1)

1. $M \subseteq M[G]$,
2. $G \in M[G]$,
3. $M[G] \models \mathsf{ZF}$,
4. *if N is any other transitive model of* ZF *with* $M \cup \{G\} \subseteq N$ *then* $M[G] \subseteq N$.

8.1.1 A universe of names

Much of this discussion will express the point of view of the people of M, and it will be important to be able to talk about $M[G]$ in M. For this we will need names for everything in $M[G]$, and these names must be in M. Since G is not in M and is, in fact, somewhat arbitrary, the denotations of the names cannot be known to (the people of) M. Only when a definite G is given can the names be said to denote definite sets.

The denotation convention is such as to make $M[G]$ a transitive set containing the same ordinals as M. Conditions 8.1.1 and 8.1.2 will also be satisfied.

For any name τ, let τ^G be the denotation (yet to be defined) of τ in $M[G]$. The mapping $\tau \mapsto \tau^G$ will be definable over $M[G]$. The purpose of τ^G is to name something in $M[G]$, and to motivate its definition we note that if $M[G]$ satisfies (8.1) and $\sigma \in M$, then by virtue of the definability of $\tau \mapsto \tau^G$ over $M[G]$, the set

$\{\tau^G \mid \exists g \in G\ \langle \tau, g\rangle \in \sigma\}$ is in $M[G]$. Indeed, we may regard $\sigma \in M$ as a name for this set, and define σ^G to be $\{\tau^G \mid \exists g \in G\ \langle \tau, g\rangle \in \sigma\}$. So we have the following definition:

(8.2) *For any $\sigma \in M$ and any $G \subseteq A$,*

$$\sigma^G = \{\tau^G \mid \exists g \in G\ \langle \tau, g\rangle \in \sigma\}.$$

For a slightly different emphasis, we may use the equivalent statement:

$$\sigma^G = \{\tau^G \mid \sigma^{\rightharpoonup}\{\tau\} \,\mathrm{meets}\, G\}.$$[2]

This definition makes sense if it is interpreted as defining $\tau \mapsto \tau^G$ by recursion on the rank of τ, because $\tau \in \operatorname{dom}\sigma \to \operatorname{rk}\tau < \operatorname{rk}\sigma$.

Now we define $M[G]$ by means of the system of denotations $\tau \mapsto \tau^G$.

(8.3) *If M is a transitive model of ZF, A is an element of M, and G is a non-empty subset of A, we define*

$$M[G] = \{\tau^G \mid \tau \in M\}.$$

Observe that (8.2) defines each element of $M[G]$ as a subset of $M[G]$, so $M[G]$ is transitive.

Let's see whether $M[G]$, so defined, satisfies (8.1). Begin with (8.1.1). By recursion on rank define the map $x \mapsto \check{x}$ so that

$$\check{x} = \{\langle \check{y}, a\rangle \mid y \in x \wedge a \in A\}. \tag{8.4}$$

Then for any nonempty $G \subseteq A$,

$$\check{x}^G = \{\check{y}^G \mid y \in x\},$$

so by induction we know that $\check{x}^G = x$ for all $x \in M$. Hence, $M \subseteq M[G]$.

Next, let's deal with (8.1.2). We need to define a name for G. This is quite easy. Let

$$\Gamma = \{\langle \check{a}, a\rangle \mid a \in A\},$$

where $x \mapsto \check{x}$ is the system just defined of standard names for members of M.

$$\begin{aligned}
\Gamma^G &= \{\gamma^G \mid \Gamma^{\rightharpoonup}\{\gamma\} \,\mathrm{meets}\, G\} \\
&\qquad \text{(by definition of } \tau \mapsto \tau^G\text{)} \\
&= \{\check{a}^G \mid a \in G\} \\
&\qquad \text{(by definition of } \Gamma\text{)} \\
&= \{a \mid a \in G\} \\
&\qquad \text{(because } \check{a}^G = a\text{)} \\
&= G \\
&\qquad \text{(just because).}
\end{aligned}$$

Hence, $G \in M[G]$.

Now we come to (8.1.3). How do we know that $M[G]$ is a model of ZF? Truth be told, it sometimes isn't. But—amazingly enough—it *almost always* is. What do we mean here by 'almost always'? Essentially what is meant in Section 5.6: the

[2]Recall that X meets Y iff $X \cap Y \neq 0$.)

set of G for which $M[G]$ is a model of ZF is comeager. There we dealt with reals, which are essentially subsets of ω. Here we have to do with subsets of an arbitrary set A. But everything we did there can be carried out in the more general case.

Some of the axioms of ZF hold in $M[G]$ no matter what G we choose. An example is Pair. If τ and σ are names in M for sets t and s, respectively, in $M[G]$, then $\bigcup_{a\in A}\{\langle\tau,a\rangle,\langle\sigma,a\rangle\}$ is a name for $\{t,s\}$. But it is clear that not all the axioms of ZF hold in $M[G]$ for every G. To provide a simple example, first observe that $M[G]$ has the same ordinals as M. Suppose M is countable and $A=\omega$. Let $G\subseteq A$ encode a countable well-ordering of length greater than any ordinal in M. G cannot be in any transitive model of ZF whose ordinals are the same as those of M, because it is a theorem of ZF that every well-ordering is isomorphic to an ordinal.

This property of coding a large ordinal is obviously a special property of G, and—employing the intuition gained in Section 5.6—you may be willing to accept (pending proof, of course) that the set of Gs with this property is meager. But what about all the other ways that some axiom of ZF could be violated in $M[G]$? Well, if each violation only occurs for a meager set of Gs, we might hope to be able to avoid all of them. Reasoning now very loosely, if M is countable, then there are only countably many ways in which ZF can fail to be satisfied by $M[G]$. If each type of failure occurs for only a meager set of G's, then (since the union of a countable collection of meager sets is meager) the set of G's for which $M[G]\not\models\mathsf{ZF}$ is meager. Hence, $M[G]\models\mathsf{ZF}$ for almost every G.

Continuing to reason in the same vein, it occurs to us that everything about $M[G]$ might be a good deal more regular (and accessible to people in M) if we restrict our attention to only those G's that have no special properties, i.e., to those that are *generic* in some sense. This was Cohen's insight, and—*mirabile dictu*—it works.

8.1.2 Category applied

We will illustrate Cohen's method in the special case that we are adding a real number to M. In the process all the remaining features of the general method will arise naturally, and we will indicate as we go along what the general method is. Some familiarity is presumed with the theory of Baire category as discussed in Section 5.6, but as noted above, one may skip this introduction altogether and proceed straight to Section 8.2, which does not use the concept of category explicitly.

To fit the pattern we have developed so far, we should represent a real number as a subset of some set A. We have already seen that there are various ways to do this. The most convenient for our present purpose, since notions of category are going to play a role, is to represent a real as the set of rational open intervals that contain it. For the nonce, 'rational open interval' will be defined as follows:

(8.5) *A* rational open interval *is a 2-sequence $\langle a,b\rangle$ with $a,b\in\mathbb{Q}$ and $0\leqslant a<b\leqslant 1$. Let P be the set of rational open intervals.*

Notice that we have not defined a rational open interval to be an actual interval in the real number line $\mathbb{R}$, but rather the pair of rationals that constitute the endpoints. This is because we are going to be adding reals to M, and we do not want the identity of a member of P to change during this process. As defined in (8.5), P consists of finitary objects. There is therefore a one-to-one correspondence between P and ω.

For convenience we restrict attention to the reals strictly between 0 and 1, i.e., to the open unit interval $(0,1)$, which we denote by '$\mathbf{\Omega}$'. For the rest of this section, we will use the lightface 'Ω' to denote $\langle 0,1\rangle$, which is the element of P corresponding to $\mathbf{\Omega}$.

There is a natural way to order the elements of P. Just say that $\langle a',b'\rangle \leqslant \langle a,b\rangle$ if and only if $a \leqslant a'$ and $b' \leqslant b$. Let $\mathbb{P} = (P;\leqslant)$. $\mathbb{P}$ is a partial order, and most of what we have to say about it applies to any partial order.

The *raison d'être* of $\mathbb{P}$ is that real numbers may be identified with subsets of P. This is done as follows. For any real x, define F_x to be the set of rational open intervals that contain x. Distinct reals determine different subsets of P in this way. Of course, not every subset of P is F_x for some real x. Let us now see how the F_xs can be characterized in terms of $\mathbb{P}$, without reference to the existence of reals that generate them. This is all part of the program of learning how to say as much as possible about $M[G]$ while still being comprehensible to the people in M.

The F_xs are examples of *filters* on $\mathbb{P}$.

Definition [S] *Suppose $\mathbb{P} = (P;\leqslant)$ is a partial order. F is a* filter on $\mathbb{P}$ $\overset{\text{def}}{\iff}$

1. $F \subseteq P$;
2. $F \neq 0$; *and*
3. *for all* $p,q \in P$
    1. *if $p \leqslant q$ and $p \in F$, then $q \in F$,*
    2. *if $p \in F$ and $q \in F$, then there is $r \in F$ such that $r \leqslant p$ and $r \leqslant q$.*

It is clear that for any $x \in \mathbb{R}$, F_x is a filter. x can be recovered from F_x by taking the intersection of all the members of F_x, regarded as actual intervals in $\mathbb{R}$. There are filters on $\mathbb{P}$ other than the F_x's: for example, $\{\langle a,b\rangle \in P \mid a < \frac{1}{3} \wedge b > \frac{2}{3}\}$. The intersection of all of these intervals is the interval $[\frac{1}{3},\frac{2}{3}]$, not a single real. Other filters have empty intersection, for example, $\{\langle 0,a\rangle \mid a \in (0,1)\}$, so they don't define reals either.

In the spirit of "almostness" the natural question is: For a filter on $\mathbb{P}$, is it the rule or the exception that its intersection consists of a single real? In other words, is the set of F_xs comeager or meager in the space of all filters on $\mathbb{P}$?

To properly formulate and answer this question, we must define an appropriate topology on $\mathcal{F}$, the set of all filters on $\mathbb{P}$. This can be done for the set $\mathcal{F}$ of filters on any partial order $\mathbb{P}$. In the particular case we are discussing here, the topology closely parallels the usual topology on $\mathbb{R}$.

The *standard topology* on $\mathcal{F}$ is defined by the basic open sets O_p, where for any $p \in P$, O_p is the set of filters that contain p. An arbitrary open set is then any union of basic open sets. Note that a set $\mathcal{X}$ of filters is open iff there is a set $X \subseteq P$ such that $\mathcal{X}$ is the set of filters that meet X, i.e., for any filter F, $F \in \mathcal{X} \leftrightarrow F \cap X \neq 0$.

Notice that for any real x, F_x is in O_p just in case x is in p (interpreting p as an interval in $\mathbb{R}$). So the standard topology defined here for $\mathcal{F}$, when restricted to the set of F_x's, corresponds exactly to the usual topology on $\mathbb{R}$.

The whole theory of category can be developed for the standard topology on the set of filters on any partial order. The key concept, as you will recall, is that of an open dense set. Let O be an open subset of $\mathcal{F}$. O is completely determined by the basic open sets that it includes. To say that O is dense means that any open set intersects O. This is the same as saying that any basic open set includes one of the basic open sets included in O. This prompts the following definition.

Suppose $X \subseteq P$.

1. $\lceil X \rceil \stackrel{\text{def}}{=} \{q \in P \mid \exists p \in X\, q \leqslant p\}$.
2. X *is* dense $\stackrel{\text{def}}{\iff} \forall p \in P\, \exists q \leqslant p\ q \in X$.
3. X *is* predense $\stackrel{\text{def}}{\iff} \lceil X \rceil$ *is dense, i.e.,* $\forall p \in P\, \exists q \leqslant p\, \exists r \in X\ q \leqslant r$.

Note that a filter meets X iff it meets $\lceil X \rceil$. One sees that a set $X \subseteq P$ is predense iff the set of filters that meet X is dense in the standard topology. Also, a dense set is *a fortiori* predense, so if X is dense then the set of filters that meet X is dense.

Now let us answer the question posed above: Is the set of F_x's comeager or meager in the space of all filters on $\mathbb{P}$? In fact, it is comeager. To show this, we must show that it includes the intersection of a countable family of open dense sets.

Thus, we must exhibit a countable family of dense subsets of P such that if a filter meets all of them then it is an F_x. For each rational number $a \in \mathbf{\Omega}$, let D_a be the set of $p \in P$ such that there exists $q = \langle q_0, q_1 \rangle \in P$ such that $q_0 < a < q_1$ and $q \perp p$, where $q \perp p \stackrel{\text{def}}{\iff} q$ and p are *incompatible* $\stackrel{\text{def}}{\iff} \neg \exists r\, (r \leqslant q \wedge r \leqslant p)$ (the intervals defined by q and p do not overlap). Each D_a is dense, and there are countably many of them. The interested reader may wish to complete the proof that if a filter meets every D_a then its intersection consists of a single real.

8.1.3 Genericity

The stage is now set to implement the program suggested above. As before, consider extending M by the adjunction of a subset G of P, but now require that G be a filter. Let $\mathcal{F}$ be the set of filters on $\mathbb{P}$. Consider the set $\{G \in \mathcal{F} \mid \exists x\ G = F_x\}$. We have just seen that this set is comeager. This can be framed in terms of $M[G]$. We have

$\{G \in \mathcal{F} \mid M[G] \models \ulcorner \exists x\ \lfloor G \rfloor = F_x \urcorner\}$ *is comeager.*

Now let ϕ, with free variables $u_0, \dots, u_{n^-}$, be an arbitrary s-formula, and let $\tau_0, \dots, \tau_{n^-}$ be names (in M) for elements of $M[G]$. Consider $\big\{G \in \mathcal{F} \,\big|\, M[G] \models \phi\big[\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau_0^G & \cdots & \tau_{n^-}^G \end{smallmatrix}\big]\big\}$. This is a particular set of filters.

(8.6) *If M is countable this type of set is Borel.*

Proof The proof of this is not difficult, but it involves a welter of detail, so we will only give the briefest sketch of it. One begins with the observation that if $\tau, \sigma \in M$, $p \in P$, and $\langle \tau, p \rangle \in \sigma$, then the set $\{G \in \mathcal{F} \mid \tau^G \in \sigma^G\}$ includes the basic open set O_p. One is tempted to say that $\{G \in \mathcal{F} \mid \tau^G \in \sigma^G\} = \bigcup_{p \in \sigma^{\rightarrow}\{\tau\}} O_p$, which is a union of basic open sets and hence is open; however, τ^G might be in σ^G by virtue of being equal to ρ^G for some $\rho \in M$ such that $\sigma^{\rightarrow}\{\rho\}$ meets G. Including all these cases involves a countable union of sets of the form $\{G \in \mathcal{F} \mid \tau^G = \rho^G \wedge \sigma^{\rightarrow}\{\rho\}\ \text{meets}\ G\}$. If all these sets are Borel, then so is $\{G \mid \tau^G \in \sigma^G\}$. The key point is that the formulas $\tau = \rho$ that we have to consider are all "simpler" than $\tau \in \sigma$ because ρ has lower rank than σ (else ρ could not be in $\operatorname{dom} \sigma$).

Similarly, any set of the form $\{G \mid \tau^G = \sigma^G\}$ can be written in terms of countable unions and intersections and complements of sets of the form $\{G \mid \rho^G \in \tau^G\}$ and $\{G \mid \rho^G \in \sigma^G\}$ (basically because $\tau^G = \sigma^G$ if and only if $\forall \rho \in M\, (\rho^G \in \tau^G \leftrightarrow \rho^G \in \sigma^G)$), where, as before, ρ may be restricted to have lower rank than τ or σ, respectively. By a careful ordering of the primitive sentences $\tau \in \sigma$ and $\tau = \sigma$ by the ranks of their constituent names one can show by transfinite induction that all the sets $\{G \mid \tau^G \in \sigma^G\}$ and $\{G \mid \tau^G = \sigma^G\}$ are Borel.

Once the primitive sentences have been taken care of, the rest is easy. An arbitrary sentence is built up from primitive sentences by the use of negation, conjunction, disjunction, universal quantification, and existential quantification. In terms of the sets $\left\{G \,\middle|\, M[G] \models \phi\left[\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau_0^G & \cdots & \tau_{n^-}^G \end{smallmatrix}\right]\right\}$ these logical operations correspond to the set theoretic operations of complementation, intersection, union, countable intersection, and countable union, respectively (assuming, as we have been, that M is countable). Hence they are all Borel. □[8.6]

8.1.4 Forcing

It's high time that we explicitly introduce the language that the people of M use to talk about $M[G]$, which we have been using implicitly in the previous section. It is obtained from the language of set theory by adding a term for each name. As we have defined names above, every element of M is a name, but we could, and often do, define a restricted class of names by requiring that a name τ be a binary relation such that $\operatorname{im}\tau \subseteq |\mathbb{P}|$ and $\operatorname{dom}\tau$ consists entirely of names. Clearly, for any $\tau \in M$ there exists a name σ in this restricted sense, such that $\sigma^G = \tau^G$. We construct a language within M from these terms in the usual way. If ϕ is a sentence of this language, the interpretation of ϕ in $M[G]$ has already been defined. For reasons that will become clear momentarily, this language is called a *forcing language*, specifically the $\mathbb{P}$-*forcing language of* M if we have used a class of names restricted for use with $\mathbb{P}$. We will not provide further detail at this time.

We saw in the previous section that for any sentence ϕ of the forcing language, $\{G \mid M[G] \models \phi\}$ is Borel, assuming M is countable. Recall[5.147] that every Borel set has the Baire property, i.e., it is almost equal to an open set. So

For any sentence ϕ of the forcing language, there is an open set $O_\phi \subseteq \mathcal{F}$ such that for almost every filter G, $G \in O_\phi \leftrightarrow M[G] \models \phi$.

O_ϕ is in a sense the *truth value* of ϕ.

Note that $\{G \mid M[G] \models \phi\} \cup \{G \mid M[G] \models \neg\,\phi\} = \mathcal{F}$. Hence, almost every filter G is in $O_\phi \cup O_{\neg\,\phi}$. Given $p \in P$, since O_p is not meager, it meets either O_ϕ or $O_{\neg\,\phi}$. (In fact, $O_\phi \cup O_{\neg\,\phi}$ is dense; hence, since O_p is nonempty, it meets either O_ϕ or $O_{\neg\,\phi}$.)

Thus, for any formula ϕ of the forcing language, for any $p \in P$, there is a $q \leqslant p$ such that either *for almost every G containing q, $M[G] \models \phi$,* or *for almost every G containing q, $M[G] \models \neg\,\phi$.*

This suggests the following definition.

If $p \in P$ and ϕ is a sentence of the forcing language, p forces *ϕ $\overset{\text{def}}{\iff} p \Vdash \phi \overset{\text{def}}{\iff}$ for almost every G containing p, $M[G] \models \phi$.*

It may seem a bit strong to say that p forces ϕ in this case because there is still a meager set of filters that may disobey the general rule. But why worry about them? There are only countably many sentences in the forcing language if M is countable; hence, we can throw out all the uncooperative filters and still have a comeager set of them left. While we're at it, let's also throw out that meager set of filters G such that G does not correspond to a real number. Call the remaining filters *generic*.

(8.7) *Let g_G be the real number determined by a generic filter G, and call such a real* generic. *Note that g_G uniquely determines G, and we use the inverse notation as well: If x is a generic real, then G_x is that generic filter F such that $g_F = x$.*

Then we have the following characterization of forcing.

For any rational open interval p, $p \Vdash \phi$ if and only if for every generic real x in the interval p, $M[G_x] \models \phi$.

Can it be that the people in M are aware of which elements p of P force which sentences ϕ of the forcing language? This hardly seems possible, since we have defined forcing in terms of *all* filters on $\mathbb{P}$, and hardly any of them are in M. In fact, *none* of the generic ones are in M. Moreover, we have assumed M to be countable, which is very difficult for the people in M to imagine. Nevertheless,

The forcing relation is definable over M.

This remarkable fact is the very heart of the matter, and with it we will be able to show that for every generic G, $M[G] \models \mathsf{ZF}$.

It is straightforward to show that if G is generic then $G \notin M$, so $M[G]$ is a nontrivial extension of M. Since it has the same ordinals as M, the constructible universe as defined in $M[G]$ is the same as in M, so $M[G] \models \ulcorner [G] \notin L \urcorner$, whence, $M[G] \models \ulcorner V \neq L \urcorner$.

By varying the partial order $\mathbb{P}$, one can similarly obtain models of ZF satisfying a wide variety of sentences, including $\neg\,\mathsf{AC}$ and $\neg\,\mathsf{CH}$. This was Cohen's epiphany. In the following sections we supply the details and develop additional machinery that facilitates its implementation.

8.2 Genericity and forcing

From this point forward, the development will be detailed and rigorous. There are three interlocking ideas:

1. a *partial order* $\mathbb{P}$ in a transitive model M of set theory, whose elements are regarded as *conditions* on an *M-generic filter* G on $\mathbb{P}$;

2. a *generic extension* $M[G]$, including M and containing G along with all the other sets named by terms of the appropriate *forcing language* $\mathcal{L}^{M,\mathbb{P}}$; and

3. the *forcing relation* $\Vdash^{M,\mathbb{P}}$, *defined over* M in a suitable sense, with the derived property that if for every $p \in |\mathbb{P}|$ there is an M-generic filter G with $p \in G$, then for any $p \in |\mathbb{P}|$ and sentence $\phi \in \mathcal{L}^{M,\mathbb{P}}$, $p \Vdash^{M,\mathbb{P}} \phi$ (i.e., p *forces* ϕ) iff for every M-generic filter G on $\mathbb{P}$, $M[G] \models \phi$.

Additionally, we have the notions of the *regular algebra* $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$ of $\mathbb{P}$, and the *$\mathfrak{A}$-valued universe* $M^{\mathfrak{A}}$, which is another—very intuitive and productive—way of formulating these ideas.

These themes form an organic whole, and, like the best ideas in mathematics, they are really a way of thinking. In this chapter we spend more time on their development than is strictly necessary before proceeding to applications, with the intention that all the above points of view be fully aired and integrated and available at the touch of a wand. The proofs in this section are not difficult, but they are given in detail—either in the main text or in the notes—in the hope that the repetition of basic themes will help you to reach more quickly that level of mastery at which you can say: 'Yes, the proofs in this section are not difficult'.

While this scheme of presentation has distinct pedagogic advantages and is quite efficient in its way, it may exceed the reader's tolerance for delay of gratification, and it would not be out of order to glance ahead from time to time when the going gets tedious—to Sections 8.6 and 8.9, for example—to see this machinery at work.

8.2.1 Partial orders and filters

(8.8) **Definition** [GB] *Suppose* $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ *is a partial order,*[3.71] *i.e.,* $\leqslant$ *is transitive, reflexive, and antisymmetric. Let* $<$ *be the corresponding strong order relation:* $q < p \overset{\text{def}}{\iff} (q \leqslant p \wedge p \neq q)$.

1. *Suppose* $X \subseteq |\mathbb{P}|$.

$$\lceil X \rceil \overset{\text{def}}{=} \{q \in |\mathbb{P}| \mid \exists p \in X\, q \leqslant p\}$$
$$\lfloor X \rfloor \overset{\text{def}}{=} \{q \in |\mathbb{P}| \mid \exists p \in X\, q \geqslant p\}.$$

2. *p and q are* compatible $\overset{\text{def}}{\iff} p \parallel q \overset{\text{def}}{\iff}$ *they have a common extension, i.e.,* $\exists r\, (r \leqslant p \wedge r \leqslant q)$. *Otherwise, they are* incompatible, $p \perp q$.
3. *Suppose* $D \subseteq |\mathbb{P}|$.
   1. D *is* dense $\overset{\text{def}}{\iff} \forall p \in |\mathbb{P}|\, \exists q \in D\; q \leqslant p$.
   2. D *is* dense below p $\overset{\text{def}}{\iff} \forall q \leqslant p\, \exists r \in D\; r \leqslant q$.
   3. D *is* predense $\overset{\text{def}}{\iff} \lceil D \rceil$ *is dense.*
   4. D *is* predense below p $\overset{\text{def}}{\iff} \lceil D \rceil$ *is dense below* p.
4. *A* filter on $\mathbb{P}$ *is a nonempty set* $F \subseteq |\mathbb{P}|$ *such that*
   1. $\forall p \in F\, \forall q \geqslant p\; q \in F$ *(i.e.,* F *is "closed upward",* $\lfloor F \rfloor = F$*).*
   2. $\forall p, q \in F\, \exists r \in F\, (r \leqslant p \wedge r \leqslant q)$.[3]
5. *Suppose* S *is a class. A filter* $F \subseteq |\mathbb{P}|$ *is* S-generic $\overset{\text{def}}{\iff}$ *for every* $D \in S$*, if* D *is a dense subset of* $|\mathbb{P}|$*, then* F *meets* D*, i.e.,* $F \cap D \neq 0$.[4]

(8.9) **Definition** [GB] *Suppose* $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ *is a partial order,* $p, q \in |\mathbb{P}|$*, and* $X \subseteq |\mathbb{P}|$.

1. X *is* open $\overset{\text{def}}{\iff} \forall p \in X\, \forall q \leqslant p\; q \in X$ *(i.e.,* X *is "closed downward",* $\lceil X \rceil = X$*).*
2. *The* complement of X $\overset{\text{def}}{=} X^{\perp} \overset{\text{def}}{=} \left\{p \in |\mathbb{P}| \;\middle|\; \forall q \in X\; p \perp q\right\}$.
3. *The* completion of X $\overset{\text{def}}{=} \overline{X} \overset{\text{def}}{=} X^{\perp\perp}$.
4. X *is* regular $\overset{\text{def}}{\iff} X = \overline{X}$.

[3] Note that for any $p \in |\mathbb{P}|$, $\lfloor\{p\}\rfloor$ is a filter; these are the *principal* filters.

[4] The common situation is that $(S; \in)$ is a model of ZF, and F is used to extend S as described above.[§ 8.1]

(8.10) **Theorem** [GB] *Suppose $\mathbb{P}$ is a partial order and $X \subseteq |\mathbb{P}|$.*

1. *$X^\perp$ is open, $X \cap X^\perp = 0$, and $X \subseteq \overline{X}$.*
2. *If X is regular then X is open.*
3. *$\overline{X} = \{p \in |\mathbb{P}| \mid X$ is predense below $p\}$.*
4. *$\overline{X^\perp} = X^{\perp\perp\perp} = X^\perp$. Hence, $X^\perp$ is regular.*
5. *$\lceil X \rceil \cup X^\perp$ is dense.*
6. *If a filter F is $\{\lceil X \rceil \cup X^\perp\}$-generic then F meets X iff F meets $\overline{X}$.*
7. *In particular, if F is $\{\lceil X \rceil \cup X^\perp\}$-generic and X is predense below p and $p \in F$, then F meets X.*

Proof 1, 2 Straightforward.

3 Clearly

$$\begin{aligned}\{p \in |\mathbb{P}| \mid X \text{ is predense below } p\} &= \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, \exists r \in X\, q \parallel r\} \\ &= \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, q \notin X^\perp\}.\end{aligned}$$

Suppose $p \in \overline{X} = X^{\perp\perp}$. Then for any $q \leqslant p$, since $q \parallel p$, $q \notin X^\perp$. Inversely, suppose $p \notin X^{\perp\perp}$. Then p is compatible with something in $X^\perp$, so for some $q \leqslant p$, q extends (i.e., is $\leqslant$) something in $X^\perp$, so $q \in X^\perp$, since $X^\perp$ is open.[8.9.2]

4 Clearly, for any $X \subseteq Y \subseteq |\mathbb{P}|$, $X^\perp \supseteq Y^\perp$. Clearly, $X \subseteq X^{\perp\perp}$, so $X^\perp \supseteq (X^{\perp\perp})^\perp = X^{\perp\perp\perp}$. But also, $X^\perp \subseteq (X^\perp)^{\perp\perp} = X^{\perp\perp\perp}$.

5 Suppose $p \in |\mathbb{P}|$. Then either $p \in X^\perp$ or p is compatible with something in X, i.e., for some $q \leqslant p$, q extends something in X, i.e., $q \in \lceil X \rceil$.

6 In the nontrivial direction, suppose F is $\{\lceil X \rceil \cup X^\perp\}$-generic and F meets $\overline{X}$. Since F is a filter, any two members of F are compatible, so F does not meet $\overline{X}^\perp = X^\perp$. By the genericity hypothesis, F therefore meets $\lceil X \rceil$, and since filters are closed upward, F meets X.

7 If X is predense below p then $p \in \overline{X}$,[8.10.3] so if $p \in F$ then F meets $\overline{X}$, so[8.10.6] F meets X. □[8.10]

(8.11) **Theorem** [GB] *Suppose $\mathbb{P}$ is a partial order, $|\mathbb{P}|$ is wellorderable, and S is countable. Then there exists an S-generic filter on $\mathbb{P}$.*

Proof Let $\prec$ wellorder $|\mathbb{P}|$, and let $\langle X_n \mid n \in \omega \rangle$ be an enumeration of S. Without loss of generality, suppose each X_n is a dense subset of $|\mathbb{P}|$. Let $\langle p_n \mid n \in \omega \rangle$ be the sequence in $|\mathbb{P}|$ such that

1. p_0 is the $\prec$-first member of X_0; and
2. for each $n > 0$, p_n is the $\prec$-first $p \in X_n$ such that $p \leqslant p_{n^-}$.

Let $F = \{p \in |\mathbb{P}| \mid \exists n \in \omega\ p_n \leqslant p\}$. Clearly, F is an S-generic filter on $\mathbb{P}$. □[8.11]

8.2.2 Generic extensions

As sketched in the introduction,§ 8.1 the concept of forcing arises from that of generic extension.

(8.12) *Specifically, if M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, and ϕ is a sentence in the* forcing language $\mathcal{L}^{M,\mathbb{P}}$*, then—under appropriate circumstances—we say that p* forces *ϕ iff for every M-generic filter G on $\mathbb{P}$, if $p \in G$ then $M[G] \models \phi$. We will refer to this as the* extrinsic *definition of forcing.*

Note that we have specified that $\mathbb{P}$ is a member of M. Since M is transitive, if $\mathbb{P} \in M$ then $\mathbb{P} \subseteq M$. One may also consider generic extensions of M using a partial order $\mathbb{P}$ that is included in M but is not necessarily a member of M. We will do this in Section 8.12 under the rubrics *class-generic* and *class forcing*. For the present, we will restrict our attention to the case that $\mathbb{P} \in M$, which we may refer to using the terms *set-generic* and *set forcing* by way of specification.

(8.13) *For technical reasons, we will generally suppose that a partial order $\mathbb{P}$ used for forcing has a greatest element, which we denote by '$\mathbf{1}$', or more specifically by '$\mathbf{1}^{\mathbb{P}}$'. This is not significantly restrictive: if we wish to use a partial order that does not naturally have a greatest element, we simply add an element that is greater than all the rest. In many cases, 0 is naturally the maximum element of $\mathbb{P}$, and we could adopt the convention that this is always the case, without any adverse consequence.*

We now define the forcing language $\mathcal{L}^{M,\mathbb{P}}$, and the structure that we have informally indicated by '$M[G]$', which interprets it. First we define the *forcing terms*, which are the constant symbols of $\mathcal{L}^{M,\mathbb{P}}$. We may, as we did in our introductory remarks, regard any member of M as a forcing term, but it is conventional and convenient to restrict our attention to a particular subclass $M^{\mathbb{P}} \subseteq M$. It will be clear that these are sufficient to name every element of $M[G]$.

(8.14) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P} = (|\mathbb{P}|; \leqslant) \in M$ is a partial order. We define $\langle M^{\mathbb{P}}_\alpha \mid \alpha \in \mathrm{Ord} \cap M\rangle$ by recursion on ordinals α as follows.*

1. $M^{\mathbb{P}}_0 = 0$;
2. $M^{\mathbb{P}}_\alpha = \bigcup_{\beta\in\alpha} M^{\mathbb{P}}_\beta$ *for limit α;*
3. $M^{\mathbb{P}}_{\alpha+1} = M \cap \mathcal{P}(M^{\mathbb{P}}_\alpha \times |\mathbb{P}|)$*, i.e., the powerset of $M^{\mathbb{P}}_\alpha \times |\mathbb{P}|$ in the sense of M.*

$M^{\mathbb{P}} \overset{\text{def}}{=} \bigcup_{\alpha\in M} M^{\mathbb{P}}_\alpha$.

(8.15) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order with maximum $\mathbf{1}$. We define a mapping $x \mapsto \check{x}$ from M into $M^{\mathbb{P}}$ by $\in$-recursion:*

$$\check{x} \overset{\text{def}}{=} \{\langle \check{y}, \mathbf{1}\rangle \mid y \in x\}.^{5}$$

We define

$$\mathsf{G} \overset{\text{def}}{=} \{\langle \check{p}, p\rangle \mid p \in |\mathbb{P}|\}.$$

[5] This definition differs from (8.4) in a way that will soon be seen to be inessential. The reason for the present definition is that it is also applicable when $\mathbb{P}$ is not in M, but is only included in M. The opportunity to define $\check{x}$ this way is one reason we require that $\mathbb{P}$ have a maximum element.

Note that $\mathsf{G} \in M^{\mathbb{P}}$.

(8.16) **Definition** [S] $\mathsf{s}^{\mathsf{V}} \overset{\text{def}}{=}$ *the signature obtained from* s *by the addition of a unary predicate symbol* V. *As for other commonly used predicate symbols, such as '$\in$' and '$=$', we use a bold version of 'V', viz., '$\boldsymbol{\mathsf{V}}$', to denote the corresponding formula-building operation.*

(8.17) **Definition** [GB] *The forcing language $\mathcal{L}^{M,\mathbb{P}}$, or just $\mathcal{L}^{\mathbb{P}}$ when M is understood, is constructed from the signature $\mathsf{s}^{M,\mathbb{P}}$, which is s^{V} extended by the addition of the members of $M^{\mathbb{P}}$ as constants.*

Both M and $\mathbb{P}$ must be specified to define the forcing language, as above, and the forcing relation, to be defined presently. If M is unspecified, it is presumed that $M = V$, the class of all sets. Note that from the standpoint of M, V is M, so setting $M = V$ is equivalent to "working in M".

(8.18) **Definition** [GB] *Suppose M is a transitive model of* ZF *and G is a filter on $\mathbb{P} \in M$.*

1. *We define τ^G for $\tau \in M^{\mathbb{P}}$ by recursion:*[8.2]

$$\tau^G \overset{\text{def}}{=} \{\tau'^G \mid \exists p \in G\ \langle \tau', p\rangle \in \tau\}.$$

2. $M[G] \overset{\text{def}}{=} \{\tau^G \mid \tau \in M^{\mathbb{P}}\}$.[8.3]
3. $\mathfrak{M}[G] \overset{\text{def}}{=}$ *the* $\mathsf{s}^{M,\mathbb{P}}$*-structure* $(M[G]; \in, M, \tau^G)_{\tau \in M^{\mathbb{P}}}$, *where* $\mathsf{V}^{\mathfrak{M}[G]} = M$ *and* $\tau^{\mathfrak{M}[G]} = \tau^G$ *for each* $\tau \in M^{\mathbb{P}}$.

We may use '$M[G]$' informally to denote $\mathfrak{M}[G]$.

The following theorem explains the purpose of $\check{x}$ and G.

(8.19) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and G is a filter on $\mathbb{P} \in M$.*

1. $\forall x \in M\ \check{x}^G = x$.
2. $\mathsf{G}^G = G$.

Proof 1 By induction. Since $\mathbf{1}$ is an element of any filter, $\check{x}^G = \{\check{y}^G \mid \langle y, \mathbf{1}\rangle \in \check{x}\} = \{\check{y}^G \mid y \in x\} = \{y \mid y \in x\} = x$.

2 $\mathsf{G}^G = \{\check{p}^G \mid \langle \check{p}, p\rangle \in \mathsf{G}\} = \{p \mid p \in G\} = G$. □[8.19]

We may now implement the extrinsic definition of forcing.[8.12] Note that we state the definition in ZF, so we are necessarily considering only the case that M is a set. Consequently, the existence of satisfaction relations for M and $M[G]$ is demonstrable.

(8.20) **Definition** [ZF] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order. Suppose $p \in |\mathbb{P}|$ and ϕ is a sentence of $\mathcal{L}^{M,\mathbb{P}}$. Then $p \Vdash^{*} \phi \overset{\text{def}}{\iff}$ for every M-generic filter G on $\mathbb{P}$, $p \in G \rightarrow M[G] \models \phi$.*

This definition is most useful *per se* when for every $p \in |\mathbb{P}|$ there exists an M-generic filter G on $\mathbb{P}$ with $p \in G$. This condition is met if M is countable,[8.11] and we will have occasion to make use of this fact; but the extrinsic definition of forcing is useful in another way, viz., as motivation for an intrinsic definition that is actually more fundamental.

We arrive at the intrinsic definition by observing that when the extrinsic definition is applicable, it implies that $\Vdash^*$ has attributes that amount to a recursive definition that yields, for each s^{V}-formula ϕ, with n free variables, an $\mathsf{s}^{M,\mathbb{P}}$-formula $\phi^{\Vdash}$ with $n+1$ free variables, such that for $p \in |\mathbb{P}|$ and $\tau_0, \ldots, \tau_{n^-} \in M^{\mathbb{P}}$, if we define $p \Vdash \phi(\tau_0, \ldots, \tau_{n^-}) \overset{\text{def}}{\iff} (M; \in, \mathbb{P}) \models \phi^{\Vdash}[p, \tau_0, \ldots, \tau_{n^-}]$, then $\Vdash^* = \Vdash$. The recursive definition of $\Vdash$ thus arrived at is meaningful quite independently of the existence of generic filters, and it therefore serves as an intrinsic definition of forcing.

(8.21) **Theorem** [ZF] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order. Suppose for every $p \in |\mathbb{P}|$ there exists an M-generic filter G on $\mathbb{P}$,*[6] *and define $\Vdash^*$ as above.*[8.20] *Suppose ϕ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence.*

1. *Suppose $p \in |\mathbb{P}|$. Then*
   1. *if $\phi = \tau \in \tau'$ then*
   $$p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \; \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash^* \tau_0 = \tau);$$
   2. *if $\phi = \tau = \tau'$ then*
   $$\begin{aligned} p \Vdash^* \phi \leftrightarrow & \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau') \\ & \wedge \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau); \end{aligned}$$
   3. *if $\phi = \mathsf{V}(\tau)$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \, \exists x \in M \, (r \Vdash^* \tau = \check{x})$;*
   4. *if $\phi = \neg \psi$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \; q \nVdash^* \psi$;*
   5. *if $\phi = \psi \wedge \psi'$ then $p \Vdash^* \phi \leftrightarrow (p \Vdash^* \psi \wedge p \Vdash^* \psi')$;*
   6. *if $\phi = \psi \vee \psi'$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \, (r \Vdash^* \psi \vee r \Vdash^* \psi')$;*
   7. *if $\phi = \psi \rightarrow \psi'$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, (q \Vdash^* \psi \rightarrow q \Vdash^* \psi')$;*
   8. *if $\phi = \psi \leftrightarrow \psi'$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, (q \Vdash^* \psi \leftrightarrow q \Vdash^* \psi')$;*
   9. *if $\phi = \forall v \, \psi$ then $p \Vdash^* \phi \leftrightarrow \forall \tau \in M^{\mathbb{P}} \; p \Vdash^* \psi\binom{v}{\tau}$; and*
   10. *if $\phi = \exists v \, \psi$ then $p \Vdash^* \phi \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \, \exists \tau \in M^{\mathbb{P}} \; r \Vdash^* \psi\binom{v}{\tau}$.*
2. *Suppose G is an M-generic filter on $\mathbb{P}$. Then*
$$M[G] \models \phi \leftrightarrow \exists p \in G \; p \Vdash^* \phi.$$

Proof Since filters are closed upward, the following fact is obvious, but is stated here for future reference.

Suppose $p \Vdash^* \phi$ and $q \leqslant p$. Then $q \Vdash^* \phi$.

We prove (8.21.1) and (8.21.2) simultaneously by induction, attending first to sentences ϕ of the form $\tau \in \tau'$ and $\tau = \tau'$. We order these by associating to $\tau \in \tau'$ and $\tau = \tau'$ the pair $\langle \operatorname{rk} \tau, \operatorname{rk} \tau' \rangle$ and ordering the ordinal pairs by $\preccurlyeq$, which is defined as follows

[6] As discussed above, this supposition is just a way of getting to the intrinsic definition of forcing; it is not a condition for its use.

(8.22) *Given a* 2*-sequence* s *of ordinals, let* $\bar{s}$ *be the (unique)* 2*-sequence of the form* $s \circ \pi$*, where* $\pi : 2 \overset{\text{bij}}{\to} 2$ *is a permutation of* $\{0, 1\}$*, and* $\bar{s}_0 \geqslant \bar{s}_1$*.* $s \preccurlyeq s' \overset{\text{def}}{\iff} \bar{s}$ *precedes* $\bar{s}'$ *lexicographically.*

Note that $\preccurlyeq$ is a prewellorder, sequences related by a permutation of their domain (viz., 2) are at the same level of $\preccurlyeq$, and if s and s' differ at one coordinate, then whichever is lower at that coordinate is lower in $\preccurlyeq$.

The remaining sentences are ordered by logical complexity.

The induction hypothesis for a given sentence ϕ is therefore that (8.21.1) and (8.21.2) hold for all $\phi' \prec \phi$. When working with this hypothesis, we make use of the fact that (8.21.1) provides a definition over M of $\Vdash^*$ by recursion. (In fact, this recursion is precisely the intrinsic definition of forcing that is the endpoint of this analysis.) Thus, if X is a bounded subset of M (i.e., $X \subseteq Y \in M$) defined by reference to $\Vdash^*$ applied to sentences ϕ' that precede ϕ, then, since $M \models \mathsf{Comprehension}$, $X \in M$. The typical use of this is to show that if G is M-generic on $\mathbb{P}$, $p \in G$, and X is a subset of $|\mathbb{P}|$ that is dense below p, then G meets X.

(8.23) *We use this in particular to show that our induction hypothesis at stage* ϕ *implies that for each* $\phi' \prec \phi$*,* $\{p \in |\mathbb{P}| \mid p \Vdash^* \phi'\}$ *is regular.*

Proof Let $S = \{p \in |\mathbb{P}| \mid p \Vdash^* \phi'\}$, and suppose $p \in |\mathbb{P}|$ is such that S is dense below p. Then, since $S \in M$, if G is M-generic and $p \in G$ then G meets S, i.e, for some $r \in G$, $r \Vdash^* \phi'$; hence, $M[G] \models \phi'$. Thus, $p \Vdash^* \phi'$, i.e., $p \in S$. $\square$[8.23]

Note that the $\leftarrow$ direction of (8.21.2) follows immediately from the definition of $\Vdash^*$, so we only have to prove the $\to$ direction. We now justify with the induction step clause by clause.

$(\phi = \tau \in \tau')$ Let

$$S = \{p \mid \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \to q \nVdash^* \tau_0 = \tau)\}.$$

Note that S is open.

$$S^\perp = \{p \mid \forall q \leqslant p \, \exists r \leqslant q \, \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash^* \tau_0 = \tau)\}.$$

Note that

1. $S \cap S^\perp = 0$;[8.10.1] and
2. since S is open, $S \cup S^\perp$ is dense.[8.10.5]

By induction hypothesis, $S \in M$, so every M-generic filter meets S or $S^\perp$, but not both.

(8.24) **Claim** *Suppose* G *is* M*-generic on* $\mathbb{P}$*. Then* $M[G] \models \tau \in \tau'$ *iff* G *meets* $S^\perp$*. Equivalently,* $M[G] \models \tau \notin \tau'$ *iff* G *meets* S*.*

Proof Suppose $p \in G \cap S^\perp$. Then $\{r \mid \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash^* \tau_0 = \tau)\}$ is in M and is dense below p, so there exists $r \in G$ and $\langle \tau_0, r' \rangle \in \tau'$ such that $r \leqslant r'$ and $r \Vdash^* \tau_0 = \tau$. It follows that $r' \in G$, so $\tau_0^G \in \tau'^G$ and, by induction hypothesis, $\tau_0^G = \tau^G$, so $M[G] \models \tau \in \tau'$.

Conversely, suppose $M[G] \models \tau \in \tau'$. Then there exists $\langle \tau_0, r' \rangle \in \tau'$ such that $r' \in G$ and $\tau_0^G = \tau^G$. By induction hypothesis, there exists $s \in G$ such that

$s \Vdash^* \tau_0 = \tau$. Now let p be an arbitrary member of G. Since G is a filter, there exists $q \in G$ such that $q \leqslant p, r', s$. But then $q \leqslant p$, $q \leqslant r'$, and $q \Vdash^* \tau_0 = \tau$, so $p \notin S$. Thus, G does not meet S, so G meets $S^\perp$. $\square^{8.24}$

It follows immediately from the claim that if $p \in S^\perp$ then $p \Vdash^* \tau \in \tau'$. Inversely, if $p \notin S^\perp$, then since $S^\perp \cup S$ is dense and $S^\perp$ is open, there exists $q \leqslant p$ such that $q \in S$. Let G be M-generic such that $q \in G$. Then $M[G] \models \tau \notin \tau'$ and $p \in G$, so $p \nVdash^* \tau \in \tau'$. This completes the proof of (8.21.1.1).

To prove (8.21.2) for this case, suppose $M[G] \models \tau \in \tau'$. By the claim, G meets $S^\perp$, say at p. Then $p \Vdash^* \tau \in \tau'$. $\square^{\phi = \tau \in \tau'}$

($\phi = \tau = \tau'$) Let

$$\begin{aligned} S_0 &= \{s \in |\mathbb{P}| \mid \exists \langle \tau_0, r' \rangle \in \tau\, (s \leqslant r' \wedge \forall t \leqslant s\ t \nVdash^* \tau_0 \in \tau')\} \\ S_1 &= \{s \in |\mathbb{P}| \mid \exists \langle \tau_0, r' \rangle \in \tau'\, (s \leqslant r' \wedge \forall t \leqslant s\ t \nVdash^* \tau_0 \in \tau)\} \\ S &= S_0 \cup S_1. \end{aligned}$$

Note that S_0, S_1, and S are open.

By definition,

$$S_0^\perp = \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (q \leqslant r' \rightarrow \exists r \leqslant q\ r \Vdash^* \tau_0 \in \tau')\}.$$

Let

$$S_0' = \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau')\}.$$

We claim that $S_0^\perp = S_0'$. Clearly, $S_0' \subseteq S_0^\perp$. Conversely, suppose $p \in S_0^\perp$. Suppose $q \leqslant p$, $\langle \tau_0, r' \rangle \in \tau$, and $q \leqslant r'$. Then for all $q' \leqslant q$, since $q' \leqslant p$ and $q' \leqslant r'$, $\exists r \leqslant q'\ r \Vdash^* \tau_0 \in \tau'$. Thus, $\{r \mid r \Vdash^* \tau_0 \in \tau'\}$ is dense below q. Using the induction hypothesis, we know that $\{r \mid r \Vdash^* \tau_0 \in \tau'\}$ is regular;[8.23] hence, $q \Vdash^* \tau_0 \in \tau'$, which proves the claim.

Similarly,

$$S_1^\perp = \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau)\}.$$

Thus,

$$\begin{aligned} p \in S^\perp \leftrightarrow & \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau') \\ & \wedge \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash^* \tau_0 \in \tau). \end{aligned} \tag{8.25}$$

By induction hypothesis, $S \in M$, so, as before, any M-generic filter G meets S or $S^\perp$, but not both.

(8.26) **Claim** *Suppose G is M-generic on $\mathbb{P}$. Then $M[G] \models \tau \neq \tau'$ iff G meets S. Equivalently, $M[G] \models \tau = \tau'$ iff G meets $S^\perp$.*

Proof Suppose G meets S, i.e., there exists $s \in S \cap G$. Suppose $s \in S_0$; the case that $s \in S_1$ is handled analogously. Let $\langle \tau_0, r' \rangle \in \tau$ be such that $s \leqslant r'$ and $\forall t \leqslant s\ t \nVdash^* \tau_0 \in \tau'$. Then $r' \in G$, so $\tau_0^G \in \tau^G$. But $\tau_0^G \notin \tau'^G$; otherwise, by induction hypothesis, for some $s' \in G$, $s' \Vdash^* \tau_0 \in \tau'$, and letting t be any common extension of s and s', $t \leqslant s$ and $t \Vdash^* \tau_0 \in \tau'$, contrary to assumption. Thus, $\tau_0^G \in \tau^G$ and $\tau_0^G \notin \tau'^G$, so $\tau^G \neq \tau'^G$.

Now suppose $\tau^G \neq \tau'^G$. Then either there exists $x \in \tau^G$ such that $x \notin \tau'^G$ or *vice versa*. Assume the former; the latter case is treated analogously. Let $\langle \tau_0, r' \rangle \in \tau$

be such that $r' \in G$ and $\tau_0^G = x$, so $\tau_0^G \notin \tau'^G$. We wish to show that G meets S, so suppose toward a contradiction that G meets $S^\perp$. Then G meets the larger set $S_0^\perp$, and since $r' \in G$, G meets $S_0^\perp$ below r', i.e., there exists $p \leqslant r'$ such that $p \in G$ and $\forall q \leqslant p\ q \notin S_0$. Since $p \leqslant r'$, $\forall q \leqslant p\, \exists t \leqslant q\ t \Vdash^* \tau_0 \in \tau'$. Since G is M-generic and $p \in G$, there exists $t \in G$ such that $t \Vdash^* \tau_0 \in \tau'$,[7] so $\tau_0^G \in \tau'^G$. This contradiction establishes that G meets S, as claimed. $\square^{8.26}$

As before, it follows immediately from the claim that if $p \in S^\perp$ then $p \Vdash^* \tau = \tau'$. Inversely, if $p \notin S^\perp$, then p is compatible with a member of S; since S is open, there exists $q \leqslant p$ such that $q \in S$. Let G be M-generic such that $q \in G$. Then[8.26] $M[G] \models \tau \neq \tau'$ and $p \in G$, so $p \nVdash^* \tau = \tau'$. In view of (8.25), this completes the proof of (8.21.1.2).

To prove (8.21.2) for this case, suppose $M[G] \models \tau = \tau'$. By the claim, G meets $S^\perp$, say at p. Then $p \Vdash^* \tau = \tau'$. $\square^{\phi = \tau = \tau'}$

($\phi = \mathsf{V}(\tau)$) Let $S = \{s \in |\mathbb{P}| \mid \exists x \in M\ s \Vdash^* \tau = \check{x}\}$. By induction hypothesis, $S \in M$, and S is open, so $S \cup S^\perp$ is dense. It is straightforward to show, using the induction hypothesis, that

1. for any M-generic filter G, G meets S iff $\tau^G \in M$; and

2. for any $p \in |\mathbb{P}|$, $p \in S^{\perp\perp}$ iff every M-generic filter G containing p meets S.

Since $p \in S^{\perp\perp} \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists x \in M\ (r \Vdash^* \tau = \check{x})$, these considerations suffice to prove (8.21) for $\phi = \mathsf{V}(\tau)$. $\square^{\phi = \mathsf{V}(\tau)}$

($\phi = \neg\, \psi$) Let $S = \{s \in |\mathbb{P}| \mid s \Vdash^* \psi\}$. The induction hypothesis implies that

1. for any M-generic filter G, G meets $S^\perp$ iff $M[G] \not\models \psi$ iff $M[G] \models \neg\, \psi$; and

2. for any $p \in |\mathbb{P}|$, $p \in S^\perp$ iff every M-generic filter G containing p meets $S^\perp$.

$p \in S^\perp \leftrightarrow \forall q \leqslant p\ q \nVdash^* \psi$, so the theorem holds for $\phi = \neg\, \psi$. $\square^{\phi = \neg\, \psi}$

($\phi = \psi \wedge \psi'$) (8.21.1.5) is immediate. (8.21.2) follows from the fact that if $p, p' \in G$ are such that $p \Vdash^* \psi$ and $p' \Vdash^* \psi'$, there there exists $q \in G$ extending both p and p', and $q \Vdash^* \psi \wedge \psi'$. $\square^{\phi = \psi \wedge \psi'}$

($\phi = \psi \vee \psi'$) (8.21.1.6) follows from the fact that $\psi \vee \psi'$ is equivalent to $\neg(\neg\, \psi \wedge \neg\, \psi')$. A direct argument from the definition of $\Vdash^*$ is also easy.

To prove (8.21.2) suppose $M[G] \models \psi \vee \psi'$. Then $M[G] \models \psi$ or $M[G] \models \psi'$. In the former instance, $\exists p \in G\ p \Vdash^* \psi$, whereas in the latter, $\exists p \in G\ p \Vdash^* \psi'$. In either event $p \Vdash^* \psi \vee \psi'$ by virtue of (8.21.1.6). $\square^{\phi = \psi \vee \psi'}$

[7] Or we could use (8.23) to conclude that $p \Vdash^* \tau_0 \in \tau'$.

$(\phi = \psi \to \psi')$ The fact that $\psi \to \psi'$ is equivalent to $\neg \psi \vee \psi'$ yields directly that

$$p \Vdash^* \phi \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, (r \Vdash^* \neg \psi \vee r \Vdash^* \psi').$$

Suppose $\forall q \leqslant p\, \exists r \leqslant q\, (r \Vdash^* \neg \psi \vee r \Vdash^* \psi')$. Suppose $q \leqslant p$ and $q \Vdash^* \psi$. Then $\forall q' \leqslant q\, \exists r \leqslant q'\; r \Vdash^* \psi'$, so $q \Vdash^* \psi'$, which proves the forward direction.[8.23]

Conversely, suppose $\forall q \leqslant p\, (q \Vdash^* \psi \to q \Vdash^* \psi')$. We will show that $\forall q \leqslant p\, \exists r \leqslant q\, (r \Vdash^* \neg \psi \vee r \Vdash^* \psi')$. To this end, suppose $q \leqslant p$ and $\forall r \leqslant q\; r \nVdash^* \neg \psi$. It suffices to show that $\exists r \leqslant q\; r \Vdash^* \psi'$. In fact, we will show that $q \Vdash^* \psi'$. As we have shown above, $r \Vdash^* \neg \psi$ iff $\forall s \leqslant r\; s \nVdash^* \psi$, so $\forall r \leqslant q\, \exists s \leqslant r\; s \Vdash^* \psi$. Hence[8.23] $q \Vdash^* \psi$, so $q \Vdash^* \psi'$.

To prove (8.21.2) suppose $M[G] \models \psi \to \psi'$. Then either $M[G] \models \neg \psi$ or $M[G] \models \psi'$. In the former instance there exists $p \in G$ such that $p \Vdash^* \neg \psi$, so $\forall q \leqslant p\; q \nVdash^* \psi$; in the latter, there exists $p \in G$ such that $p \Vdash^* \psi'$, so $\forall q \leqslant p\; q \Vdash^* \psi'$. In either case for all $q \leqslant p$, if $q \Vdash^* \psi$ then $q \Vdash^* \psi'$, so $p \Vdash^* \psi \to \psi'$ by (8.21.1.6). $\square^{\phi = \psi \to \psi'}$

$(\phi = \psi \leftrightarrow \psi')$ Straightforward. $\square^{\phi = \psi \leftrightarrow \psi'}$

$(\phi = \forall v\, \psi)$ (8.21.1.9) is immediate using the fact that every element of $M[G]$ is τ^G for some $\tau \in M^{\mathbb{P}}$. To prove (8.21.2) we proceed as follows.[8] Suppose G is M-generic and $M[G] \models \forall v\, \psi$. Let $S = \left\{p \in |\mathbb{P}| \,\middle|\, \exists \tau \in M^{\mathbb{P}}\, \forall q \leqslant p\; q \nVdash^* \psi\binom{v}{\tau}\right\}$. Then $S \cup S^{\perp}$ is in M, by induction hypothesis, and is dense, so there exists $p \in G$ such that $p \in S$ or $p \in S^{\perp}$. Suppose $p \in S$. Let $\tau \in M^{\mathbb{P}}$ be such that $\forall q \leqslant p\; q \nVdash^* \psi\binom{v}{\tau}$. By induction hypothesis, there exists $p' \in G$ such that $p' \Vdash^* \psi\binom{v}{\tau}$. Then p and p' are both in the filter G, but they can have no common extension.

Thus, $p \notin S$; hence, $p \in S^{\perp}$, so $\forall \tau \in M^{\mathbb{P}}\, \forall q \leqslant p\, \exists r \leqslant q\; r \Vdash^* \psi\binom{v}{\tau}$. Hence,[8.23] $\forall \tau \in M^{\mathbb{P}}\; p \Vdash^* \psi\binom{v}{\tau}$, so $p \Vdash^* \forall v\; \psi$. $\square^{\phi = \forall v\, \psi}$

$(\exists v\, \psi)$ $\exists v\, \psi$ is equivalent to $\neg \forall v\; \neg \psi$. $\square^{8.21}$

8.2.3 Forcing

It is important to note that (8.21.1) asserts that the relation $\Vdash^*$, defined extrinsically in terms of M-generic filters, satisfies the stated equivalences. Clearly, these have the pattern of a recursive definition, and we used this in the proof to show that the set $\{\langle p, \phi\rangle \mid \phi \in \Phi \wedge p \Vdash^* \phi\}$, where Φ is a set of $\mathsf{s}^{M,\mathbb{P}}$-sentences of bounded complexity, is definable over M and is therefore in M (since $|\mathbb{P}| \in M$ and $M \models \mathsf{Comprehension}$). As explained above, the definition is useful even when generic filters do not exist, and from this standpoint the primary purpose of Theorem 8.21 is to discover the appropriate definition of forcing. In this section we will investigate the ramifications of this definition.

Given a transitive model M of ZF and a partial order $\mathbb{P} \in M$, we let $\Vdash^{M,\mathbb{P}}$ be the forcing relation as defined intrinsically, i.e., defined over M by recursion using

[8] $\forall$ is a sort of generalized $\wedge$, but the straightforward adaptation of the proof of (8.21.2) for the latter operation does not work for the former since we cannot conclude, from the fact that for each $\tau \in M^{\mathbb{P}}$ there exists $p_\tau \in G$ such that $p_\tau \Vdash^* \psi\binom{v}{\tau}$, that there exists $p \in G$ that extends every p_τ; plus, we don't have the axiom of choice to pick p_τs.

the clauses (8.21.1). Of course, to the extent that the extrinsically defined relation $\Vdash^*$ exists for $M, \mathbb{P}$, it is identical to $\Vdash^{M,\mathbb{P}}$ (as shown by induction using (8.21.1). For notational simplicity, we omit one or both superscripts in '$\Vdash^{M,\mathbb{P}}$' when the relevant class M or partial order $\mathbb{P}$ is clear from the context.

(8.27) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order. We define $\Vdash \; = \; \Vdash^{M,\mathbb{P}}$ for sentences of the form $\tau \in \tau'$ and $\tau = \tau'$ for $\tau, \tau' \in M^{\mathbb{P}}$ by recursion as follows, with sentences ordered*[8.22] *as in the proof of (8.21).*

1. $$p \Vdash \tau \in \tau' \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \; \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash \tau_0 = \tau).$$

2. $$\begin{aligned} p \Vdash \tau = \tau' \leftrightarrow & \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau') \\ & \wedge \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau). \end{aligned}$$

As for the remainder of the definition, when M is a proper class the same limitations apply relating to the presence of unbounded quantification in (8.21.1.9, 10) as in the definition of satisfaction, and we operate within these constraints in the same way: by consideration of partial forcing relations.[9]

(8.28) **Definition** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and Φ is a class of $\mathsf{s}^{M,\mathbb{P}}$-formulas. Each $\phi \in \Phi$ is derived from an s^{V}-formula ϕ' by substitution of forcing terms for (some or all of its) free variables. Let Φ' be the class of all such s^{V}-formulas, together with all atomic s^{V}-formulas: $u \in u'$, $u = u'$, $\mathsf{V}u$. $\Phi^{M,\mathbb{P}} \overset{\text{def}}{=}$ the class of $M^{\mathbb{P}}$-sentences $\phi\binom{u_0 \cdots u_{n^-}}{\tau_0 \cdots \tau_{n^-}}$, where $\phi \in \overline{\Phi'}$,*[10] *$\langle u_0, \ldots, u_{n^-} \rangle$ enumerates* $\mathrm{Free}\,\phi$, *and $\tau_0, \ldots, \tau_{n^-} \in M^{\mathbb{P}}$.*

Recall that the definition of semantic valuation (and satisfaction) of an expression ϵ in a structure $\mathfrak{S}$ requires that all the free variables of ϵ be assigned values in $|\mathfrak{S}|$. For an individual $t \in |\mathfrak{S}|$ that is the denotation of a term τ, i.e., $\tau^{\mathfrak{S}} = t$, the substitution of τ for a variable v has the same effect as the assignment of t to v. In the case that for every $t \in |\mathfrak{S}|$ there exists τ such that $\tau^{\mathfrak{S}} = t$, assignment may be dispensed with altogether in favor of substitution. This applies in the case of forcing, since $M^{\mathbb{P}}$ is a complete class of terms denoting members of $M[G]$. Thus, we have no need here for the notion of *assignment*—indeed, it would be inappropriate to use this notion, as until G has been specified, it is not known what the individuals of $M[G]$ will be.

(8.29) **Definition** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and Φ is a class of $\mathsf{s}^{M,\mathbb{P}}$-formulas. F is a $\Phi^{M,\mathbb{P}}$-forcing relation $\overset{\text{def}}{\Longleftrightarrow}$ F is a binary relation such that $p \, F \, \sigma \rightarrow p \in |\mathbb{P}| \wedge \sigma \in \Phi^{M,\mathbb{P}}$, and for all $p \in |\mathbb{P}|$ and $\phi \in \Phi^{M,\mathbb{P}}$,*

1. *if $\phi = \tau \in \tau'$ then $p \, F \, \phi \leftrightarrow p \Vdash^{M,\mathbb{P}} \tau \in \tau'$;*[11]
2. *if $\phi = \tau = \tau'$ then $p \, F \, \phi \leftrightarrow p \Vdash^{M,\mathbb{P}} \tau = \tau'$;*
3. *if $\phi = \mathsf{V}(\tau)$ then $p \, F \, \phi \leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \, \exists x \in M \; r \, F \, (\tau = \check{x})$.*

[9] Recall that we avoided this issue in (8.21), since in the context of ZF M is necessarily a set.

[10] Recall that $\overline{\Phi'}$ is the class of subformulas of members of Φ'.

[11] Note that we use the forcing relation already defined[8.27] for sentences $\tau \in \tau'$ and $\tau = \tau'$.

4. *if* $\phi = \neg\,\psi$ *then* $p\,F\,\phi \leftrightarrow \forall q \leqslant p\ \ \neg\ q\,F\,\psi$*;*
5. *if* $\phi = \psi \wedge \psi'$ *then* $p\,F\,\phi \leftrightarrow (p\,F\,\psi \wedge p\,F\,\psi')$*.*
6. *if* $\phi = \psi \vee \psi'$ *then* $p\,F\,\phi \leftrightarrow \forall q \leqslant p\,\exists r \leqslant q\,(r\,F\,\psi \vee r\,F\,\psi')$*.*
7. *if* $\phi = \psi \rightarrow \psi'$ *then* $p\,F\,\phi \leftrightarrow \forall q \leqslant p\,(q\,F\,\psi \rightarrow q\,F\,\psi')$*.*
8. *if* $\phi = \psi \leftrightarrow \psi'$ *then* $p\,F\,\phi \leftrightarrow \forall q \leqslant p\,(q\,F\,\psi \leftrightarrow q\,F\,\psi')$*.*
9. *if* $\phi = \forall v\,\psi$ *then* $p\,F\,\phi \leftrightarrow \forall \tau \in M^{\mathbb{P}}\ p\,F\,\psi\binom{v}{\tau}$*.*
10. *if* $\phi = \exists v\,\psi$ *then* $p\,F\,\phi \leftrightarrow \forall q \leqslant p\,\exists r \leqslant q\,\exists \tau \in M^{\mathbb{P}}\ r\,F\,\psi\binom{v}{\tau}$*.*

F *is a* partial $M^{\mathbb{P}}$-forcing relation $\overset{\text{def}}{\iff}$ *it is a* $\Phi^{M,\mathbb{P}}$*-forcing relation for some* Φ*.*

(8.30) **Theorem** [GB] *Suppose* M *is a transitive model of* ZF*,* $\mathbb{P} \in M$ *is a partial order,* Φ_0 *and* Φ_1 *are classes of* $\mathsf{s}^{M,\mathbb{P}}$*-formulas, and* F_0 *and* F_1 *are respectively a* $\Phi_0^{M,\mathbb{P}}$*- and a* $\Phi_1^{M,\mathbb{P}}$*-forcing relation. Then*

$$\forall p \in |\mathbb{P}|\,\forall \phi \in \left(\Phi_0^{M,\mathbb{P}} \cap \Phi_1^{M,\mathbb{P}}\right)(p\,F_0\,\phi \leftrightarrow p\,F_1\,\phi).$$

Roughly speaking, partial forcing relations agree on their "common domain".

Proof By induction on rank[8.22] of terms followed by logical complexity, as in the proof of (8.21). □[8.30]

(8.31) **Theorem** [ZF] *Suppose* M *is a transitive model of* ZF *and* $\mathbb{P} \in M$ *is a partial order. Then there is a unique* $(\mathcal{F}^{\mathsf{s}^{\mathsf{V}}})^{M,\mathbb{P}}$*-forcing relation, where* $\mathcal{F}^{\mathsf{s}^{\mathsf{V}}}$ *is the set of all* s^{V}*-formulas.*

Proof Since we are working in ZF, M is a set, so there is no impediment to the standard proof of the existence of relations defined by recursion. □[8.31]

When M is a proper class, we must make do with this analog of (1.65).

(8.32) **Theorem** [GB] *Suppose* M *is a transitive model of* ZF *and* $\mathbb{P} \in M$ *is a partial order.*

1. *Suppose* ϕ *is an atomic* $\mathsf{s}^{M,\mathbb{P}}$*-formula. Then the* $\{\phi\}^{M,\mathbb{P}}$*-forcing relation exists.*
2. *Suppose* ψ_0, ψ_1, ϕ *are* $\mathsf{s}^{M,\mathbb{P}}$*-formulas,* v *is a variable, and the* $\{\psi_0\}^{M,\mathbb{P}}$*- and* $\{\psi_1\}^{M,\mathbb{P}}$*-forcing relations exist. Then the* $\{\phi\}^{M,\mathbb{P}}$*-forcing relation exists if*
   1. ϕ *is a subformula of* ψ_0*;*
   2. $\phi = \psi_0(T)$*, where* T *is a substitution for (some or all) free variables of* ψ_0*;*
   3. $\phi = \neg\,\psi_0,\ \psi_0 \vee \psi_1,\ \psi_0 \wedge \psi_1,\ \psi_0 \rightarrow \psi_1,\ \psi_0 \leftrightarrow \psi_1,\ \exists v\,\psi_0,$ *or* $\forall v\,\phi_0$*.*

Proof Straightforward. □[8.32]

As in the case of satisfaction relations, (8.32) may be used to show that GB proves the existence of the $\{\phi\}^{M,\mathbb{P}}$-forcing relation for any specific formula ϕ, and we have the following analog of (1.67). Recall that for any expression ϵ in an HF signature, $\hat{\epsilon}$ is the standard name of ϵ.

(8.33) **Theorem** [S] *Suppose* ϕ *is an* s^{V}*-formula. Then* GB $\vdash$ ⌜*for any transitive model* M *of* ZF *and partial order* $\mathbb{P} \in M$*, there exists a* $\{(\hat{\phi})\}^{M,\mathbb{P}}$*-forcing relation*⌝*.*

Proof By induction on the grammatical complexity of ϕ. $\square^{8.33}$

(8.34) **Definition** [GB] *Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, and $p \in |\mathbb{P}|$.*

1. *Suppose ϕ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence. Then $p \Vdash^{M,\mathbb{P}} \phi \overset{\text{def}}{\iff}$ for every $\{\phi\}^{M,\mathbb{P}}$-forcing relation F, $p\,F\,\phi$.*
2. *Suppose Φ is a class of $\mathsf{s}^{M,\mathbb{P}}$-sentences. Then $p \Vdash^{M,\mathbb{P}} \Phi \overset{\text{def}}{\iff}$*

$$\forall \phi \in \Phi \; p \Vdash^{M,\mathbb{P}} \phi.\text{[12]}$$

We omit one or both of the superscripts on '$\Vdash$' as circumstances permit.

(8.35) *In addition to our use of '$\Vdash$' to represent the predicate just defined, we may also use it to denote an assumed partial forcing relation. This serves a useful mnemonic purpose and conforms to conventional usage, which largely ignores the potential nonexistence of full forcing relations for proper classes. For specificity we may use '$\Vdash^F$' with reference to a partial forcing relation F, analogously to our use of '$\models^S$' to denote a partial satisfaction relation S.*

(8.36) **Theorem** [GB] *For any $\mathsf{s}^{M,\mathbb{P}}$-sentence σ, $\{p \in |\mathbb{P}| \mid p \Vdash \sigma\}$ is regular.*

Proof See Note 10.21.

(8.37) **Definition** [GB] *Suppose σ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence and $p \in |\mathbb{P}|$. p* decides *σ $\overset{\text{def}}{\iff} p \mid \sigma \overset{\text{def}}{\iff} p \Vdash \sigma \vee p \Vdash \neg\sigma$.*

(8.38) **Theorem** [GB] *For any $\mathsf{s}^{M,\mathbb{P}}$-sentence σ, $\{p \in |\mathbb{P}| \mid p \mid \sigma\}$ is dense.*

Proof If there is no $\{\neg\sigma\}^{M,\mathbb{P}}$-forcing relation then $\{p \in |\mathbb{P}| \mid p \Vdash \sigma\} = \{p \in |\mathbb{P}| \mid p \Vdash \neg\sigma\} = |\mathbb{P}|$, which is dense. So suppose $\Vdash$ is a $\neg\sigma$-forcing relation.[8.35] The theorem is now immediate from the definition.[8.29] $\square^{8.38}$

8.2.4 Definability of forcing

Since $\Vdash^{M,\mathbb{P}}$ subsumes the satisfaction relation for M (via relativization of s-formulas to V), the full forcing relation—like the full satisfaction relation for M—is clearly not definable over M, and (letting $M = V$) $\Vdash^{\mathbb{P}}$ is not definable in the context of ZF. Nevertheless, as we have discussed above, there is a local sense in which $\Vdash^{M,\mathbb{P}}$ is definable in M and $\Vdash^{\mathbb{P}}$ is definable in the context of ZF, which we now characterize.

We will define, for each s^{V}-formula ϕ with n free variables, an s-formula $\phi^{\Vdash}$ with $n+2$ free variables, such that for any transitive model M of ZF, partial order $\mathbb{P} \in M$, and $p, x_0, \ldots, x_{n^-} \in M$,

$$M \models \phi^{\Vdash}[\mathbb{P}, p, x_0, \ldots, x_{n^-}] \\ \leftrightarrow p \in |\mathbb{P}| \wedge x_0, \ldots, x_{n^-} \in M^{\mathbb{P}} \wedge p \Vdash^{M,\mathbb{P}} \phi(x_0, \ldots, x_{n^-}).$$

We first observe that although (8.27) is formulated in GB, if we specialize to the case that $M = V$, we may regard it as a definition by recursion, formulated in

[12] Note that this definition does not refer to $\Phi^{M,\mathbb{P}}$-forcing relations, but rather to $\{\phi\}^{M,\mathbb{P}}$-forcing relations for $\phi \in \Phi$.

ZF, of $\ulcorner p \Vdash^{\mathbb{P}} \tau \in \tau' \urcorner$ and $\ulcorner p \Vdash^{\mathbb{P}} \tau = \tau' \urcorner$. Like any predicate defined in ZF, it may be expressed by an s-formula, and we let $\phi^{\Vdash}$ be a fixed such formula, for ϕ of either of these forms. In short:

$$(8.39)\qquad \begin{aligned} (u_0 \in u_1)^{\Vdash}(\mathbb{P}, p, \tau_0, \tau_1) &\overset{\text{def}}{=} \ulcorner p \Vdash^{\mathbb{P}} \tau_0 \in \tau_1 \urcorner \\ (u_0 = u_1)^{\Vdash}(\mathbb{P}, p, \tau_0, \tau_1) &\overset{\text{def}}{=} \ulcorner p \Vdash^{\mathbb{P}} \tau_0 = \tau_1 \urcorner. \end{aligned}$$

Now we define

$$(8.40)\qquad (\mathsf{V}\, u)^{\Vdash}(\mathbb{P}, p, \tau) \overset{\text{def}}{=} \ulcorner \forall q \leqslant^{\mathbb{P}} p \, \exists r \leqslant^{\mathbb{P}} q \, \exists x \, (r \Vdash^{\mathbb{P}} \tau = \check{x}) \urcorner.$$

We complete the definition by recursion on the logical complexity of ϕ.

Definition [S] *Suppose ϕ is an s^{V}-formula.*

1. *If ϕ is atomic, $\phi^{\Vdash}$ is defined by (8.39) or (8.40).*
2. *Suppose ϕ is a complex formula.*
   1. *Suppose $\phi = \neg\, \psi$. Then*

      $$\phi^{\Vdash}(\mathbb{P}, p, \dots) \overset{\text{def}}{=} \ulcorner \forall q \leqslant^{\mathbb{P}} p \;\; \neg\, (\psi^{\Vdash})(\mathbb{P}, q, \dots) \urcorner.$$

      Analogous definitions apply for $\phi = \psi \vee \psi'$, $\psi \wedge \psi'$, $\psi \rightarrow \psi'$, or $\psi \leftrightarrow \psi'$.
   2. *Suppose $\phi(\dots) = \forall v\, \psi(\dots, v, \dots)$. Then*

      $$\phi^{\Vdash}(\mathbb{P}, p, \dots) \overset{\text{def}}{=} \ulcorner \forall \tau \in V^{\mathbb{P}} \, (\psi^{\Vdash})(\mathbb{P}, p, \dots, \tau, \dots) \urcorner,$$

      where $\ulcorner \in V^{\mathbb{P}} \urcorner$ is of course shorthand for an s*-expression. An analogous definition applies for $\phi = \exists v\, \psi$.*

(8.41) **Theorem** [S] *Suppose ϕ is an s^{V}-formula with n free variables. Then*

GB $\vdash$ $\ulcorner$*Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, and p, x_0, $\dots$, $x_{n^-} \in M$. Then*

$$M \models (\hat{\phi^{\Vdash}})[\mathbb{P}, p, x_0, \dots, x_{n^-}] \\ \leftrightarrow p \in |\mathbb{P}| \wedge x_0, \dots, x_{n^-} \in M^{\mathbb{P}} \wedge p \Vdash^{M,\mathbb{P}} (\hat{\phi})(x_0, \dots, x_{n^-}).\urcorner,$$

where $\hat{\phi}$ and $\hat{\phi^{\Vdash}}$ are the standard names of ϕ and $\phi^{\Vdash}$.

Proof Straightforward. $\square$[8.41]

Note that the definition of the mapping $\phi \mapsto \phi^{\Vdash}$ is formulated in S, as is the above theorem. They constitute, if you will, a metadefinition and a metatheorem. This "meta" aspect of the forcing predicate is implicit in the traditional development of forcing in the context of ZF, although it is not always explicitly recognized.[13]

In GB we can assert the definability of the forcing relation directly, predicated on the hypothesis of the existence of the appropriate partial forcing relation, of course.

[13] Alternatively, we may extend ZF by the addition of a new predicate symbol $\ulcorner \Vdash \urcorner$, with axioms that correspond to the usual recursive definition of the forcing relation. Note that these axioms allow us to generate a definition of $\ulcorner p \Vdash^{\mathbb{P}} \phi(\dots) \urcorner$ for any given ϕ, but this definition has quantifier depth that increases with that of ϕ, and the axioms do not yield a definition of $\Vdash^{\mathbb{P}}$ in its entirety. Note also that, since $\ulcorner \Vdash \urcorner$ is not introduced by definition, we must explicitly extend the axiom schemas of ZF to formulas that incorporate the new symbol. This theory is a conservative extension of ZF, so it is largely immaterial which approach we use to the description of forcing over V in ZF.

(8.42) **Theorem** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, ϕ is an* s^{V}*-formula with n free variables, and the $\{\phi\}^{M,\mathbb{P}}$-forcing relation exists. Then the $\{\phi^{\Vdash}\}$-satisfaction relation for M exists, and for any $p \in |\mathbb{P}|$ and $x_0, \dots, x_{n^-} \in M^{\mathbb{P}}$,*

$$M \models \phi^{\Vdash}[\mathbb{P}, p, x_0, \dots, x_{n^-}] \leftrightarrow p \Vdash^{M,\mathbb{P}} \phi(x_0, \dots, x_{n^-}).$$

Proof We derive the conclusion for all subformulas of ϕ (including ϕ itself) by induction on the logical complexity of the subformulas. $\square$[8.42]

Whenever we speak of forcing in a purely set-theoretical context—for example, within a transitive model $(M, \in)$—forcing relationships are to be understood in this way, i.e., $\ulcorner [p] \Vdash^{[\mathbb{P}]} (\phi)(\dots)\urcorner$ is understood as $\phi^{\Vdash}(\mathbb{P}, p, \dots)$. Thus, if M is a transitive model of ZF, then

$$\begin{aligned} M \models \ulcorner [p] \Vdash^{[\mathbb{P}]} (\phi)([x_0], \dots, [x_{n^-}])\urcorner &\leftrightarrow M \models \phi^{\Vdash}[\mathbb{P}, p, x_0, \dots, x_{n^-}] \\ &\leftrightarrow p \Vdash^{M,\mathbb{P}} \phi(x_0, \dots, x_{n^-}) \end{aligned}$$

(as long as the $\{\phi\}^{M,\mathbb{P}}$-forcing relation exists).

(8.43) *A typical application of (8.42) is to use* Comprehension^M *to show that a subset of M that is defined in terms of $\Vdash^{M,\mathbb{P}}$ is a member of M. Similarly, we use* Collection^M *to show that a function from a member of M into M that is defined in terms of $\Vdash^{M,\mathbb{P}}$ is bounded in M.*

8.2.5 Generic extensions again

In Section 8.2.2 we defined[8.20] the relation $\Vdash^{*}$ in terms of generic extensions of a transitive model M of ZF using a partial order $\mathbb{P} \in M$, under the restriction that M be a set, and the assumption that for every $p \in |\mathbb{P}|$ there is an M-generic filter G on $\mathbb{P}$ with $p \in G$. We obtained an intrinsic characterization[8.21.1] of $\Vdash^{*}$ in the form of a recursive definition. In Section 8.2.3 we applied this definition to an arbitrary transitive model M of ZF, possibly a proper class, without any assumption as to the existence of M-generic filters.

We now close the loop by showing that to the extent to which M-generic filters exist, the corresponding extensions of M behave as expected.

Definition [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and G is an M-generic filter on $\mathbb{P}$. $M[G] \stackrel{\text{def}}{=} \{\tau^G \mid \tau \in M^{\mathbb{P}}\}$. $\mathfrak{M}[G] \stackrel{\text{def}}{=}$ the $\mathsf{s}^{M,\mathbb{P}}$-structure with domain $M[G]$ in which*

1. *each constant term $\tau \in M^{\mathbb{P}}$ is interpreted as τ^G, and*
2. V *is interpreted as M.*

$\in$ and $=$ have their usual meaning. As is customary with transitive classes considered as structures, we may refer to $\mathfrak{M}[G]$ as '$M[G]$'.

The following theorem is an initial segment of (8.45), as it were, but it can be stated without reference to satisfaction relations and is sufficiently distinctive that it may stand on its own.

(8.44) **Theorem** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and G is an M-generic filter on $\mathbb{P}$. For all $\tau, \tau' \in M^{\mathbb{P}}$,*

1. $\tau^G \in \tau'^G \leftrightarrow \exists p \in G\ p \Vdash \tau \in \tau'$; *and*
2. $\tau^G = \tau'^G \leftrightarrow \exists p \in G\ p \Vdash \tau = \tau'$.

Proof See Note 10.22.

(8.45) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, G is an M-generic filter on $\mathbb{P}$, and σ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence. Suppose there is a $\{\sigma\}^{M,\mathbb{P}}$-forcing relation. Then there is a $\{\sigma\}^{M[G]}$-satisfaction relation, and*

$$M[G] \models \sigma \leftrightarrow \exists p \in G\, p \Vdash \sigma.$$

Proof See Note 10.23.

8.2.6 "Arguing with generic extensions"

Much of the theory of forcing consists of theorems of the form

(8.46) ⌜if M is a transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \dots, \tau_{n^-} \in M^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$⌝.

Formulated in GB, this sort of theorem applies to arbitrary classes M. Formulated in ZF, it applies strictly only to sets M, but it is easily reformulated to apply to any definable class M (such as $M = V$ or L), or to an arbitrary class referred to by means of a unary predicate symbol. In this case, ZF is extended by enlarging the axiom schemas to include all formulas in the expanded signature.

Not surprisingly, proofs of theorems of this sort may be simplified—often considerably—by the additional assumption that M is countable, because in this case we may use the theorem that for any $p \in |\mathbb{P}|$ there is an M-generic filter G on $\mathbb{P}$ such that $p \in G$, so that $\Vdash^{M,\mathbb{P}}$ is equal to the extrinsically defined relation $\Vdash^{*M,\mathbb{P}}$, i.e.,

(8.47) ZF $\vdash$ ⌜if M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, and $\tau_0, \dots, \tau_{n^-} \in M^{\mathbb{P}}$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$ iff for every M-generic filter G on $\mathbb{P}$, $p \in G \rightarrow M[G] \models (\phi)[\tau_0^G, \dots, \tau_{n^-}^G]$⌝,

in conjunction with the fact that

ZF $\vdash$ ⌜if M is a transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $\tau_0, \dots, \tau_{n^-} \in M^{\mathbb{P}}$, and G is an M-generic filter G on $\mathbb{P}$, then $M[G] \models (\phi)[\tau_0^G, \dots, \tau_{n^-}^G]$ iff $\exists p \in G\ (\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$⌝.

Wouldn't it be nice if we could use the existence of a ZF-proof of

⌜if M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \dots, \tau_{n^-} \in M^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$⌝,

in the course of which we could use the lemma (8.47), to prove (8.46), which makes no assumption regarding the countability of M, and indeed applies in the "absolute" case that $M = V$?

In other words, wouldn't it be nice if we could derive (8.46) from the hypothesis

(8.48) ZF $\vdash$ ⌜if M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \dots, \tau_{n^-} \in M^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \dots, \tau_{n^-})^M$⌝?

Yes, it would be; in fact, it is, if we slightly strengthen (8.48) as follows. Let s^{M} be the expansion of s by the addition of a constant symbol M. Consider the s^{M}-theory

$$\Theta = \mathsf{ZF} \cup \{\ulcorner(\mathsf{M}) \text{ is a countable transitive set}\urcorner\} \cup \{\theta^{\mathsf{M}} \mid \theta \in \mathsf{ZF}\} \tag{8.49}$$

and the hypothesis

(8.50) $\Theta \vdash \ulcorner$if $\mathbb{P} \in \mathsf{M}$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \ldots, \tau_{n\text{-}} \in \mathsf{M}^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^{\mathsf{M}}$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^{\mathsf{M}}\urcorner$.

Note that we have replaced the single sentence

$$\ulcorner\mathsf{M} \models \mathsf{ZF}\urcorner \tag{8.51}$$

by the schema

$$\{\theta^{\mathsf{M}} \mid \theta \in \mathsf{ZF}\} \tag{8.52}$$

consisting of every axiom of ZF relativized to M. Since (8.51) implies (8.52), but not *vice versa*, (8.50) is, at least ostensibly, a stronger hypothesis than (8.48).

In practice (8.52) may indeed replace (8.51) in arguments concerning generic extensions of a transitive set M, but of course the substitution entails a good bit of circumlocution if it is observed rigorously, just the sort of thing we're trying to avoid in this book, and we will subsequently see how to obviate this inconvenience, but for now we will proceed on the basis of (8.50), rather than (8.48).

Thus, suppose (8.50). Let σ be the conjunction of the ZF-axioms θ for which θ^{M} occurs as a premise in the proof that is there asserted to exist. Then

(8.53) $\mathsf{ZF} \vdash \ulcorner$if M is a countable transitive set such that $(\sigma)^M$, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \ldots, \tau_{n\text{-}} \in M^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^M$, then $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^M\urcorner$,

We may therefore proceed as follows.[14] Suppose π is a ZF-proof whose existence (8.53) asserts. We have the following GB-proof:

> $\ulcorner$Suppose M is a transitive model of ZF, $\mathbb{P} \in M$ is a partial order, $p \in |\mathbb{P}|$, $\tau_0, \ldots, \tau_{n\text{-}} \in M^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^M$. Let Φ be a finite set of s-formulas that contains (σ), $(\phi^{\Vdash})$ and (ψ), and any other odds and ends that may be needed, such as $\ulcorner\cdot$ is a partial order$\urcorner$. Let S be a countable Φ-elementary substructure of $(M; \in)$ that contains $\mathbb{P}$, p, $\tau_0, \ldots, \tau_{n\text{-}}$. Let $\iota : S \to M'$ be the collapse of S to a transitive set M'. Let $\mathbb{P}' = \iota\mathbb{P}$, $p' = \iota p$, etc. Since $M \models \mathsf{ZF}$, $(\sigma)^M$, so $(\sigma)^{M'}$. Also, $(\psi)(\mathbb{P}', p', \tau'_0, \ldots, \tau'_{n\text{-}})^{M'}.\urcorner$ We now insert the proof π and continue. $\ulcorner$Thus, since M' is countable, $(\sigma)^{M'}$, $\mathbb{P}' \in M'$ is a partial order, $p' \in |\mathbb{P}'|$, $\tau'_0, \ldots, \tau'_{n\text{-}} \in M'^{\mathbb{P}'}$, and $(\psi)(\mathbb{P}', p', \tau'_0, \ldots, \tau'_{n\text{-}})^{M'}$, it follows that $(\phi^{\Vdash})(\mathbb{P}', p', \tau'_0, \ldots, \tau'_{n\text{-}})^{M'}$. By elementarity, $(\phi^{\Vdash})(\mathbb{P}, p, \tau_0, \ldots, \tau_{n\text{-}})^M.\urcorner$

Thus, we have the following metatheorem.

(8.54) **Theorem** [S] *Suppose ϕ and ψ are s-formulas with n and $n+2$ free variables, respectively, and suppose (8.50). Then* $\mathsf{GB} \vdash$ (8.46)*. With an appropriate treatment of proper classes as discussed above,* $\mathsf{ZF} \vdash$ (8.46).

This metatheorem may be used to justify the common practice of using generic filters in proofs of forcing theorems even in circumstances in which they do not provably exist. We will return to this topic in Section 8.5.2 under the rubric "arguing in a generic extension" (as opposed to "arguing with generic extensions") where we prove a version of (8.54) in which we stipulate that M is an inner model of ZF and V is a $\mathbb{P}$-generic extension of M.

[14] We'll formulate the argument in GB so as to facilitate reference to proper classes M, but it is easily adapted to ZF for definable classes M, or for arbitrary classes in the extension of ZF to an expanded signature as discussed above.

8.3 Boolean-valued structures

We postpone further analysis of the forcing relation and generic extensions for the moment in order to present a description of these structures in boolean algebraic terms. Deriving from the description just given and entirely equivalent to it, the boolean algebraic description offers a natural and valuable perspective that deepens our understanding of the basic principles while providing useful analytical tools. Some of the basic definitions and properties of boolean algebras are presented in Section 3.10.2.

Suppose ρ is a relational signature, and $\mathfrak{A}$ is a complete boolean algebra. In an $\mathfrak{A}$-valued ρ-structure $\mathfrak{S}$, an n-ary predicate index P is interpreted as a function $P^{\mathfrak{S}} : {}^n|\mathfrak{S}| \to |\mathfrak{A}|$. One may think of $P^{\mathfrak{S}}\langle a_0, \ldots, a_{n^-}\rangle$ as the truth value of P at $\langle a_0, \ldots, a_{n^-}\rangle$, where truth values are elements of $|\mathfrak{A}|$. Note that if $\mathfrak{A}$ is the 2-element boolean algebra, this reduces to the usual notion of a structure, except that we have represented each relation $R \subseteq {}^n|\mathfrak{S}|$ by its characteristic function $R' : {}^n|\mathfrak{S}| \to 2$ given by

$$R'\langle u_0, \ldots, u_{n^-}\rangle = \begin{cases} 1 & \text{if } \langle u_0, \ldots, u_{n^-}\rangle \in R \\ 0 & \text{if } \langle u_0, \ldots, u_{n^-}\rangle \notin R. \end{cases}$$

$[\![\phi[A]]\!] \stackrel{\text{def}}{=}$ the $\mathfrak{A}$-value of a ρ-formula ϕ at an $\mathfrak{S}$-assignment A of its free variables, where the $\mathfrak{A}$-values of complex formulas are determined by interpreting logical operations by the corresponding algebraic operations, just as Boole had in mind when he formulated the notion of the algebras that bear his name. Thus,

$$\begin{aligned}
[\![\neg\,\phi[A]]\!] &= \neg\,[\![\phi[A]]\!] \\
[\![(\phi \vee \psi)[A]]\!] &= [\![\phi[A]]\!] \vee [\![\psi[A]]\!] \\
[\![(\phi \wedge \psi)[A]]\!] &= [\![\phi[A]]\!] \wedge [\![\psi[A]]\!] \\
[\![(\phi \to \psi)[A]]\!] &= [\![\phi[A]]\!] \to [\![\psi[A]]\!] \\
[\![(\phi \leftrightarrow \psi)[A]]\!] &= [\![\phi[A]]\!] \leftrightarrow [\![\psi[A]]\!] \\
[\![\exists u\ \phi[A]]\!] &= \bigvee_{x \in |\mathfrak{S}|} [\![\phi[A\langle {}^u_x \rangle]]\!] \\
[\![\forall u\ \phi[A]]\!] &= \bigwedge_{x \in |\mathfrak{S}|} [\![\phi[A\langle {}^u_x \rangle]]\!].
\end{aligned}$$

Note that a product $\prod_{x \in X} \mathfrak{S}_x$ of ρ-structures $\mathfrak{S}_x$ is an $\mathfrak{A}$-valued structure, where $\mathfrak{A}$ is the subset algebra of the index set X.

(8.55) **Definition** [GB] *Suppose $\mathfrak{S}$ is an $\mathfrak{A}$-valued ρ-structure. $\mathfrak{S}$ is* full $\stackrel{\text{def}}{\iff}$ *for every ρ-formula ϕ and $\mathfrak{S}$-assignment A for all the free variables of ϕ except u, there exists $x \in |\mathfrak{S}|$ such that*

$$[\![\phi[A\langle {}^u_x \rangle]]\!] = [\![\exists u\ \phi[A]]\!].$$

In general, Choice is required to prove the fullness of a naturally defined boolean-valued structure. We have already seen this in the proof of Łoś' theorem (2.164). Note that if U is an ultrafilter on $\mathfrak{A}$ then we may define the quotient structure $\mathfrak{S}/U$ as in the special case of product structures. If ρ is without identity, $|\mathfrak{S}/U| = |\mathfrak{S}|$. If ρ is with identity the individuals of $\mathfrak{S}/U$ are equivalence classes $\hat{x}$, where

$\hat{x} \overset{\text{def}}{=} \{y \in |\mathfrak{S}| \mid [\![=[x,y]]\!] \in U$.[15] We state the appropriate generalization of (2.164) as a ZF-theorem to avoid complications of proper class structures. Note that once we have assumed $\mathfrak{S}$ is full, we do not need Choice.

(8.56) **Theorem** [ZF] *Suppose $\mathfrak{S}$ is a full $\mathfrak{A}$-valued ρ-structure and U is an ultrafilter on $\mathfrak{A}$. Then for any ρ-formula ϕ and $\mathfrak{S}$-assignment for ϕ,*

$$\mathfrak{S}/U \models \phi\big[\hat{x}_0, \dots, \hat{x}_{n^{\cdot}}\big] \leftrightarrow [\![\phi[x_0, \dots, x_{n^{\cdot}}]]\!] \in U.$$

Proof The proof is by induction on logical complexity, with the quantification step(s) requiring fullness. $\square^{8.56}$

We now show how the idea of forcing leads naturally to the consideration of boolean-valued structures.

Throughout this section we restrict our attention to partial orders and boolean algebras that are sets, as opposed to proper classes.

(8.57) **Definition** [GB] *Suppose $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ is a partial order.*

1. *For $p, q \in |\mathbb{P}|$, $p \leqslant q \overset{\text{def}}{\iff} \forall r \leqslant p \; r \parallel q$.*[8.8.2]
2. *For $p, q \in |\mathbb{P}|$, $p \approx q \overset{\text{def}}{\iff} (p \leqslant q \wedge q \leqslant p)$.*
3. *$\mathbb{P}$ is* separative $\overset{\text{def}}{\iff} \forall p, q \in |\mathbb{P}| \, (p \approx q \rightarrow p = q)$.

The following equivalences are easily derived:[8.9.3]

$$(8.58) \qquad \begin{aligned} &p \leqslant q \leftrightarrow p \in \overline{\{q\}} \leftrightarrow \overline{\{p\}} \subseteq \overline{\{q\}} \\ &p \approx q \leftrightarrow \overline{\{p\}} = \overline{\{q\}}. \end{aligned}$$

The separativity property gets its name from the equivalence

$$p \not\leqslant q \leftrightarrow \exists r \leqslant p \; r \perp q,$$

i.e., r "separates" p from q.

Clearly, $\leqslant$, which is simply $\subseteq$ for the sets $\overline{\{p\}}$,[8.58] is transitive and reflexive. Hence $\approx$ (which is $=$ for the sets $\overline{\{p\}}$) is an equivalence relation. It is clear from (8.57.3) that $\mathbb{P}/\approx$ is a separative partial order, and from (8.58) that it is isomorphic to the structure $\overline{\mathbb{P}}$ defined by:

(8.59) **Definition** [GB] $\overline{\mathbb{P}} = (|\overline{\mathbb{P}}|; \leqslant)$, *where* $|\overline{\mathbb{P}}| = \{\overline{\{p\}} \mid p \in |\mathbb{P}|\}$, *and* $\leqslant = \subseteq$.

(8.60) **Theorem** [GB] *$\overline{\mathbb{P}}$ is—up to isomorphism—the unique separative partial order $\mathbb{Q}$ such that there exists a homomorphism $h : \mathbb{P} \overset{\text{sur}}{\rightarrow} \mathbb{Q}$ such that $\forall p, q \in |\mathbb{P}| \big(p \parallel q \leftrightarrow h(p) \parallel h(q)\big)$.*

Proof Straightforward. $\square^{8.60}$

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[15] We denote the equivalence class of x by '$\hat{x}$' instead of the usual '$[x]$' to avoid confusion with the notation for assignment; and we place the identity predicate symbol before, rather than between, its arguments, to permit the indication of assignment in the usual way, as '$=[x,y]$' instead of '$[x]=[y]$'.

Definition [GB] *The* regular algebra of $\mathbb{P} \stackrel{\text{def}}{=} \mathfrak{R}\,\mathbb{P} \stackrel{\text{def}}{=}$ *the structure with domain* $\{X \subseteq |\mathbb{P}| \mid X$ *is regular*$\}$ *and the following relation and operations, where* $X, Y \in |\mathfrak{R}\,\mathbb{P}|$.[16]

1. $X \leqslant Y \overset{\text{def}}{\Longleftrightarrow} X \subseteq Y$;
2. $\mathbf{1} \stackrel{\text{def}}{=} |\mathbb{P}|$ *and* $\mathbf{0} \stackrel{\text{def}}{=} 0$;
3. $\neg X \stackrel{\text{def}}{=} X^{\perp}$;
4. $X \wedge Y \stackrel{\text{def}}{=} X \cap Y$;
5. $X \vee Y \stackrel{\text{def}}{=} \neg(\neg X \wedge \neg Y) = \overline{X \cup Y}$.

More generally, for $\mathcal{X} \subseteq |\mathfrak{R}\,\mathbb{P}|$,

6. $\bigwedge \mathcal{X} \stackrel{\text{def}}{=} \bigcap \mathcal{X}$;
7. $\bigvee \mathcal{X} \stackrel{\text{def}}{=} \neg(\bigwedge_{X \in \mathcal{X}} \neg X) = \overline{\bigcup \mathcal{X}}$.

We will also find it useful to have the following operations, which can be defined in any boolean algebra.

8. $X \rightarrow Y \stackrel{\text{def}}{=} \neg X \vee Y$;
9. $X \leftrightarrow Y \stackrel{\text{def}}{=} (X \rightarrow Y) \wedge (Y \rightarrow X)$.

It is a routine exercise to show that these operations form a boolean algebra. Note that $\bigwedge \mathcal{X}$ and $\bigvee \mathcal{X}$ exist for all $\mathcal{X} \subseteq |\mathfrak{R}\,\mathbb{P}|$, so

Theorem [GB] $\mathfrak{R}\,\mathbb{P}$ *is a complete boolean algebra.*

Definition [C] *Suppose* $\mathfrak{A}$ *is a boolean algebra.* $\mathfrak{A}^{+} \stackrel{\text{def}}{=}$ *the partial order with domain* $|\mathfrak{A}| \setminus \{\mathbf{0}\}$ *and the order relation of* $\mathfrak{A}$.

The partial order $\overline{\mathbb{P}}$ is clearly a substructure of the partial order $\mathfrak{R}\,\mathbb{P}^{+}$; moreover, $|\overline{\mathbb{P}}|$ is dense in $\mathfrak{R}\,\mathbb{P}^{+}$, and we have the following theorem.

(8.61) **Theorem** [GB] $\mathfrak{R}\,\mathbb{P}$ *is—up to isomorphism—the unique extension of* $\overline{\mathbb{P}}$ *to a complete boolean algebra in which* $|\overline{\mathbb{P}}|$ *is dense.*

Proof Straightforward. $\square$[8.61]

Definition [GB] *Suppose* $\mathfrak{A}$ *is a boolean algebra. Then* $\mathfrak{A}^{+}$ *is clearly a separative partial order. We define* $\mathfrak{R}\,\mathfrak{A}$ *to be* $\mathfrak{R}\,(\mathfrak{A}^{+})$, *and—identifying* $P \in \mathfrak{A}$ *with* $[P] \in \mathfrak{R}\,\mathfrak{A}$*—we call this the* regular completion *of* $\mathfrak{A}$.

$\mathfrak{R}\,\mathfrak{A}$ is—up to isomorphism—the unique complete boolean algebra in which $\mathfrak{A}$ is a dense subalgebra.[8.61]

It follows from (8.61) that if $\mathfrak{A}$ and $\mathfrak{A}'$ are complete boolean algebras with dense subsets P and P' that are isomorphic as partial orders, then $\mathfrak{A} \cong \mathfrak{A}'$. As a simple application of this we have

[16] We use bold symbols here for the one and zero elements of $\mathfrak{R}\,\mathbb{P}$ to distinguish them from the sets $\{0\}$ and 0, but we will not maintain this distinction scrupulously.

(8.62) **Theorem** [ZF + DC] *There is, up to isomorphic equivalence equivalence, just one atomless complete boolean algebra with a countable dense subset.*

Proof Suppose $\mathfrak{A}$ is an atomless complete boolean algebra and $\{x_n \mid n \in \omega\}$ is dense in $\mathfrak{A}$. We will construct a dense embedding $p \mapsto a_p$ of $\mathbb{P}$ in $\mathfrak{A}$, where $\mathbb{P}$ is the partial order such that $|\mathbb{P}| = {}^{<\omega}2$, and $q \leqslant^{\mathbb{P}} p \leftrightarrow p \subseteq q$. Thus, $\mathfrak{A} \cong \mathfrak{R}\,\mathbb{P}$.

We begin by letting $a_0 = \mathbf{1}$. We now proceed in ω stages. At stage $n \in \omega$, (nonzero) a_p will have been chosen for all $p \in {}^{n}2$; and we will choose nonzero $a_{p^\frown\langle 0\rangle}$ and $a_{p^\frown\langle 1\rangle}$ in such a way that

$$a_{p^\frown\langle 0\rangle} \wedge a_{p^\frown\langle 1\rangle} = \mathbf{0}$$
$$\text{and } a_{p^\frown\langle 0\rangle} \vee a_{p^\frown\langle 1\rangle} = a_p.$$

Given that we have proceeded in this fashion up to this point, $\bigvee_{p\in {}^{n}2} a_p = \mathbf{1}$, so there exists $q \in {}^{n}2$ such that $a_q \wedge x_n \neq \mathbf{0}$. Since $\mathfrak{A}$ is atomless, we may choose nonzero $a_{q^\frown\langle 0\rangle} < a_q \wedge x_n$ and let $a_{q^\frown\langle 1\rangle} = a_q \wedge \neg\, a_{q^\frown\langle 0\rangle}$. For $p \in {}^{n}2$ other than q, we choose nonzero $a_{p^\frown\langle 0\rangle}$ and $a_{p^\frown\langle 1\rangle}$ arbitrarily.

Since $\{x_n \mid n \in \omega\}$ is dense in $\mathfrak{A}$, $\{a_p \mid p \in {}^{<\omega}2\}$ is dense in $\mathfrak{A}$. $\square$[8.62]

Note that the partial order of nonempty open intervals with rational endpoints, used in the introduction to this chapter, is countable and atomless, so its regular algebra is a paradigm of atomless boolean algebras with a countable dense set.

Suppose M is a transitive model of ZF and $\mathbb{P} \in M$ is a partial order. Let $\mathfrak{A}$ be $\mathfrak{R}\,\mathbb{P}$ in the sense of M, i.e., it consists of the regular subsets of $|\mathbb{P}|$ that are in M. Since $\overline{\mathbb{P}}$ is dense in $\mathfrak{A}$, filters on $\mathbb{P}$ correspond to filters on $\mathfrak{A}$, and generic filters correspond to generic filters, via the map $F \mapsto \{a \in |\mathfrak{A}| \mid \exists p \in F\ \overline{\{p\}} \subseteq a\} = \{a \in |\mathfrak{A}| \mid \underline{a \cap F \notin 0}\}$ taking a filter on $\mathbb{P}$ to a filter on $\mathfrak{A}$, and the inverse map $F \mapsto \{p \in |\mathbb{P}| \mid \overline{\{p\}} \in F\}$.

Suppose ϕ is a sentence of the forcing language $\mathcal{L}^{M,\mathbb{P}}$. The set $\{p \in |\mathbb{P}| \mid p \Vdash^{M,\mathbb{P}} \phi\}$ is a regular subset of $|\mathbb{P}|$,[8.36] and by virtue of the definability of forcing, it is in $\mathfrak{A}$. It is natural to regard this as the *truth value* of ϕ in a sense. Indeed, the forcing terms themselves may be regarded as the individuals of a $\mathfrak{R}\,\mathbb{P}$*-valued structure*, which interprets a $\mathfrak{R}\,\mathbb{P}$-valued language and logic. If we recall that boolean algebra was originally conceived by George Boole as a representation of the logic of propositions, the inevitability of this point of view becomes obvious.

The following identities, which are useful in their own right, reinforce the analogy of forcing with boolean logic when compared with Definition 8.29.

(8.63) **Theorem** [GB] *Suppose $\mathbb{P}$ is a partial order; $p, q, r \in |\mathbb{P}|$; $X, Y \in |\,\mathfrak{R}\,\mathbb{P}|$; and $\mathcal{X} \subseteq |\,\mathfrak{R}\,\mathbb{P}|$.*

1. $p \in \neg\, X \leftrightarrow \forall q \leqslant p\ q \notin X$.
2. $p \in X \wedge Y \leftrightarrow (p \in X \wedge p \in Y)$.
3. $p \in X \vee Y \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, (r \in X \vee r \in Y)$.
4. $p \in X \rightarrow Y \leftrightarrow \forall q \leqslant p\, (q \in X \rightarrow q \in Y)$.
5. $p \in X \leftrightarrow Y \leftrightarrow p \in (X \rightarrow Y) \wedge p \in (Y \rightarrow X)$.
6. $p \in \bigwedge \mathcal{X} \leftrightarrow \forall X \in \mathcal{X}\ p \in X$.
7. $p \in \bigvee \mathcal{X} \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists X \in \mathcal{X}\ p \in X$.

Proof 1 Because X is open.

2, 3 By definition.

4 Using the fact that X and Y are regular,

$$\begin{aligned} p \in X \to Y &\leftrightarrow p \in (\neg X \vee Y) \\ &\leftrightarrow \forall q \leqslant p \, \exists r \leqslant q \, (r \in \neg X \vee r \in Y) \\ &\leftrightarrow \forall q \leqslant p \, (\exists r \leqslant q \; r \in \neg X \vee \exists r \leqslant q \; r \in Y) \\ &\leftrightarrow \forall q \leqslant p \, (q \notin X \vee \exists r \leqslant q \; r \in Y) \\ &\leftrightarrow \forall q \leqslant p \, (q \in X \to \exists r \leqslant q \; r \in Y) \\ &\leftrightarrow \forall q \leqslant p \, (q \in X \to \forall s \leqslant q \, \exists r \leqslant s \; r \in Y) \\ &\leftrightarrow \forall q \leqslant p \, (q \in X \to q \in Y). \end{aligned}$$

5 By definition.

6, 7 Generalizing (2, 3). $\square$[8.63]

Definition [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A}$ is a boolean algebra in M. We define $M^{\mathfrak{A}}_{\alpha}$ by recursion on ordinals α as follows.*

1. $M^{\mathfrak{A}}_0 = 0$.
2. $M^{\mathfrak{A}}_{\alpha} = \bigcup_{\beta \in \alpha} M^{\mathfrak{A}}_{\beta}$ *for limit $\beta \in M$.*
3. $M^{\mathfrak{A}}_{\alpha+1}$ *is the set of functions $f \in M$ such that $f : M^{\mathfrak{A}}_{\alpha} \to |\mathfrak{A}|$.*

$M^{\mathfrak{A}} = \bigcup_{\alpha \in M} M^{\mathfrak{A}}_{\alpha}$.
We define a mapping $x \mapsto \check{x}$ from M into $M^{\mathfrak{A}}$ by $\in$-recursion:

$$\check{x} \stackrel{\text{def}}{=} \{(\check{y}, \mathbf{1}) \mid y \in x\}.$$

We define

$$\mathsf{G} \stackrel{\text{def}}{=} \{(\check{a}, a) \mid a \in |\mathfrak{A}|\}.$$

$\mathfrak{M}^{\mathfrak{A}}$ is the corresponding $\mathfrak{A}$-valued structure. The universe of $\mathfrak{M}^{\mathfrak{A}}$ is $M^{\mathfrak{A}}$, and the language appropriate to $\mathfrak{M}^{\mathfrak{A}}$ is $\mathcal{L}^{M,\mathfrak{A}} = \mathcal{L}^{\mathsf{s}^{M,\mathfrak{A}}}$, which is the language $\mathcal{L}^{\mathsf{s}}$ of pure set theory extended by the addition of the members of $M^{\mathfrak{A}}$ as constant terms,[8.17] together with a unary predicate V[8.16] intended to be interpreted as membership in M.

We define $[\![\sigma]\!] = [\![\sigma]\!]^{\mathfrak{A}}$ for atomic sentences $x = y$ and $x \in y$, for $x, y \in M^{\mathfrak{A}}_{\alpha}$, by recursion on α.

(8.64) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra, which is to say M is a complete boolean algebra in the sense of M, i.e., for any $X \subseteq |\mathfrak{A}|$, if $X \in M$ then $\bigvee X$ and $\bigwedge X$ exist in $\mathfrak{A}$. We define $[\![x = y]\!]$ and $[\![x \in y]\!]$ for $x, y \in M^{\mathfrak{A}}$ by recursion with sentences ordered as in the proof of (8.21).*

1. $$[\![x \in y]\!] = \bigvee_{z \in \operatorname{dom} y} \big(y(z) \wedge [\![z = x]\!]\big).$$

2. $$[\![x = y]\!] = \bigwedge_{z \in \operatorname{dom} x} \big(x(z) \to [\![z \in y]\!]\big) \wedge \bigwedge_{z \in \operatorname{dom} y} \big(y(z) \to [\![z \in x]\!]\big).$$

(8.65) **Theorem** [GB] *Suppose $x, y \in M^{\mathfrak{A}}$.*

1. $[\![x = y]\!] = [\![y = x]\!]$.
2. $[\![x = x]\!] = 1$.
3. $y \in \operatorname{dom} x \to x(y) \leqslant [\![y \in x]\!]$.

Proof 1 This is immediate from Definition 8.64.2 by symmetry.

2 By induction on the rank of x. $[\![x = x]\!] = \bigwedge_{z \in \operatorname{dom} x} \big(x(z) \to [\![z \in x]\!]\big)$,[8.64.2] so we must show that for each $z \in \operatorname{dom} x$, $x(z) \leqslant [\![z \in x]\!]$. Since[8.64.1]

$$[\![z \in x]\!] = \bigvee_{w \in \operatorname{dom} x} \big(x(w) \wedge [\![w = z]\!]\big),$$

$[\![z \in x]\!] \geqslant (x(z) \wedge [\![z = z]\!]) = x(z)$, given that $[\![z = z]\!] = 1$ by induction hypothesis.

3 Suppose $y \in \operatorname{dom} x$. Then[8.64.1]

$$x(y) = x(y) \wedge [\![y = y]\!] \leqslant [\![y \in x]\!],$$

since $[\![y = y]\!] = \mathbf{1}$.[8.65.2] □[8.65]

In the following theorem, all variables range over $M^{\mathfrak{A}}$.

(8.66) **Theorem** [GB]

1. $[\![x = y]\!] \wedge [\![y = z]\!] \leqslant [\![x = z]\!]$.
2. $[\![x \in y]\!] \wedge [\![x = z]\!] \leqslant [\![z \in y]\!]$.
3. $[\![y \in x]\!] \wedge [\![x = z]\!] \leqslant [\![y \in z]\!]$.

Proof See Note 10.24.

Note that (8.65.1–2) and (8.66.1–3) say that the axioms of identity are $\mathfrak{A}$-validities. For example, (8.66.2) is equivalent to

$$[\![x \in y \wedge x = z \to z \in y]\!] = \mathbf{1}\,.$$

Next we define $[\![\phi]\!]$ for all $\mathsf{s}^{M,\mathfrak{A}}$-sentences ϕ by recursion on the logical complexity of ϕ. It is hoped that by now the reader is well enough acquainted with the ontologic limitations of the theory of classes to recognize that if M is a proper class, $[\![\cdot]\!]$ cannot be defined by recursion at one stroke, as this would entail positing the existence of a class defined by quantification over proper classes, viz., partial valuation functions. Thus, we proceed as we did for satisfaction and forcing relations.

Definition [GB] *Suppose M is a transitive model of* ZF*, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and Φ is a class of $\mathsf{s}^{M,\mathfrak{A}}$-formulas. We define $\Phi^{M,\mathfrak{A}}$ analogously to $\Phi^{M,\mathbb{P}}$.*

(8.67) **Definition** [GB] *Suppose M is a transitive model of* ZF*, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and Φ is a class of* s*-formulas. F is a $\Phi^{M,\mathfrak{A}}$-valuation function $\overset{\text{def}}{\iff} F : \Phi^{M,\mathfrak{A}} \to |\mathfrak{A}|$, and for all $\phi, \psi \in \mathcal{L}^{M,\mathfrak{A}}$,*

1. $F(x \in y) = [\![x \in y]\!]$;[17]
2. $F(x = y) = [\![x = y]\!]$;
3. *if* $\phi = \mathsf{V}(x)$ *then* $F(\phi) = \bigvee_{x' \in M} [\![x = \check{x}']\!]$;
4. *if* $\phi = \neg\,\psi$ *then* $F(\phi) = \neg\, F(\psi)$;
5. *if* $\phi = \psi \wedge \psi'$ *then* $F(\phi) = F(\psi) \wedge F(\psi')$;
6. *if* $\phi = \psi \vee \psi'$ *then* $F(\phi) = F(\psi) \vee F(\psi')$.
7. *if* $\phi = \psi \rightarrow \psi'$ *then* $F(\phi) = F(\psi) \rightarrow F(\psi')$.
8. *if* $\phi = \psi \leftrightarrow \psi'$ *then* $F(\phi) = F(\psi) \leftrightarrow F(\psi')$.
9. *if* $\phi = \forall v\, \psi$ *then* $F(\phi) = \bigwedge_{x \in M^{\mathfrak{A}}} F\big(\psi(x)\big)$.
10. *if* $\phi = \exists v\, \psi$ *then* $F(\phi) = \bigvee_{x \in M^{\mathfrak{A}}} F\big(\psi(x)\big)$.

F is a partial $M^{\mathfrak{A}}$-valuation function $\overset{\text{def}}{\iff}$ *it is a* $\Phi^{M,\mathfrak{A}}$*-valuation function for some* Φ.

We now recapitulate Theorems 8.30, 8.31, and 8.32.

Theorem [GB] *Partial* $M^{\mathfrak{A}}$*-valuation functions agree on their common domain.*

Theorem [ZF] *The full* $M^{\mathfrak{A}}$*-valuation function exists. (M is necessarily a set in the context of* ZF*.)*

(8.68) **Theorem** [GB]

1. *If* ϕ *is an atomic* $\mathsf{s}^{M,\mathfrak{A}}$*-formula then the* $\{\phi\}^{M,\mathfrak{A}}$*-valuation function exists.*
2. *If* ψ_0, ψ_1, ϕ *are* $\mathsf{s}^{M,\mathfrak{A}}$*-formulas, v is a variable, and the* $\{\psi_0\}^{M,\mathfrak{A}}$*- and* $\{\psi_1\}^{M,\mathfrak{A}}$*-valuation functions exist, then the* $\{\phi\}^{M,\mathfrak{A}}$*-valuation function exists if*
    1. ϕ *is a subformula of* ψ_0;
    2. $\phi = \psi_0(T)$, *where T is a substitution for (some or all) free variables of* ψ_0;
    3. $\phi = \neg\,\psi_0,\ \psi_0 \vee \psi_1,\ \psi_0 \wedge \psi_1,\ \psi_0 \rightarrow \psi_1,\ \psi_0 \leftrightarrow \psi_1,\ \exists v\, \psi_0,$ *or* $\forall v\, \phi_0$.

As in the case of satisfaction relations and forcing relations, (8.68) may be used to show that GB proves the existence of the $\{\phi\}^{M,\mathfrak{A}}$-valuation function for any specific formula ϕ:

(8.69) **Theorem** [S] *Suppose* ϕ *is an* s^{V}*-formula. Then* GB $\vdash$ ⌜*for any transitive model M of* ZF *and M-complete boolean algebra* $\mathfrak{A} \in M$*, there exists a* $\{(\hat{\phi})\}^{M,\mathfrak{A}}$*-valuation function*⌝.

Proof By induction on the grammatical complexity of ϕ. □8.69

Recall that if $\mathbb{P}$ is a partial order in a transitive model M of ZF, and σ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence, then $p \Vdash^{M,\mathbb{P}} \sigma$ iff for every $\{\sigma\}^{M,\mathbb{P}}$-forcing relation F, $p\, F\, \sigma$.[8.34.1] Thus, if there is no $\{\sigma\}^{M,\mathbb{P}}$-forcing relation, then $p \Vdash^{M,\mathbb{P}} \sigma$ for all $p \in |\mathbb{P}|$. Clearly, the existence of $M^{\mathfrak{R}\,\mathbb{P}}$-valuation functions parallels that of $M^{\mathbb{P}}$-forcing relations, so it is appropriate to define the boolean valuation operation to have the value $\mathbf{1}$ (i.e., $|\mathbb{P}|$, in the case of $\mathfrak{R}\,\mathbb{P}$) if a partial valuation function does not exist.[18] The following is therefore our formal definition of the boolean valuation operation in GB.

[17] Note that we use the valuation operation already defined for sentences $x \in x'$ and $x = x'$.

[18] As an operation symbol ⌜$[\![\cdot]\!]^{\cdot,\cdot}$⌝ must have a defined value for any arguments.

(8.70) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra.*

1. *Suppose ϕ is an $\mathsf{s}^{M,\mathfrak{A}}$-sentence. If there exists a $\{\phi\}^{M,\mathfrak{A}}$-valuation function F then $[\![\phi]\!]^{M,\mathfrak{A}} \stackrel{\text{def}}{=} F\phi$; otherwise, $[\![\phi]\!]^{M,\mathfrak{A}} \stackrel{\text{def}}{=} \mathbf{1}$.*
2. *Suppose Φ is a class of $\mathsf{s}^{M,\mathfrak{A}}$-sentences. Then $[\![\Phi]\!]^{M,\mathfrak{A}} \stackrel{\text{def}}{=} \bigwedge_{\phi\in\Phi} [\![\phi]\!]^{M,\mathfrak{A}}$.*[19]

We omit one or both of the superscripts on '$[\![\,]\!]$' as circumstances permit.

The following definitions and theorems place (8.18), (8.19), (8.44), and (8.45) in the framework of boolean valuation.

(8.71) **Definition** [GB] *Suppose M is a transitive model of* ZF*, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and G is an M-generic filter on $\mathfrak{A}$. We define x^G for $x \in M^{\mathfrak{A}}$ by $\in$-recursion (equivalently, for $x \in M^{\mathfrak{A}}_\alpha$ by recursion on α):*

$$x^G \stackrel{\text{def}}{=} \{x'^G \mid x' \in \operatorname{dom} x \wedge x(x') \in G\}.$$

(8.72) **Theorem** [GB] *Suppose M, $\mathfrak{A}$, and G are as in (8.71). For all $x, x' \in M^{\mathfrak{A}}$,*

1. $x^G \in x'^G \leftrightarrow [\![x \in x']\!] \in G$;
2. $x^G = x'^G \leftrightarrow [\![x = x']\!] \in G$;
3. $\forall x \in M\ \check{x}^G = x$; *and*
4. $\mathsf{G}^G = G$.

Proof Straightforward as a translation of (8.44) into the language of boolean valuation. □8.72

Definition [GB] *Suppose M, $\mathfrak{A}$, and G are as in (8.71). $M[G] \stackrel{\text{def}}{=} \{x^G \mid x \in M^{\mathfrak{A}}\}$. $\mathfrak{M}[G] \stackrel{\text{def}}{=}$ the $\mathsf{s}^{M,\mathfrak{A}}$-structure with domain $M[G]$ in which*

1. *each constant term $x \in M^{\mathfrak{A}}$ is interpreted as x^G,*
2. V *is interpreted as M.*

$\in$ and $=$ have their usual meaning.

(8.73) **Theorem** [GB] *Suppose M, $\mathfrak{A}$, and G are as in (8.71), and $\sigma \in \mathcal{L}^{M,\mathfrak{A}}$. Suppose there is a $\{\sigma\}^{M,\mathfrak{A}}$-valuation function. Then there is a $\{\sigma\}^{\mathfrak{M}[G]}$-satisfaction relation, and*

$$\mathfrak{M}[G] \models \sigma \leftrightarrow [\![\sigma]\!] \in G.$$

Proof Straightforward as a translation of (8.45) into the language of boolean valuation. □8.73

(8.74) **Theorem** [GB] *If ϕ is a formula with the single free variable u, then*

1. $[\![x = y]\!] \wedge [\![\phi(x)]\!] \leqslant [\![\phi(y)]\!]$.[20]

[19]Note that this definition does not refer to $\Phi^{M,\mathfrak{A}}$-valuation functions, but rather to $\{\phi\}^{M,\mathfrak{A}}$-valuation functions for $\phi \in \Phi$.

[20]Note that (8.66) is a special case.

2. $[\![\exists u \in x\, \phi]\!] = \bigvee_{y \in \operatorname{dom} x} \left(x(y) \wedge [\![\phi(y)]\!]\right)$.
3. $[\![\forall u \in x\, \phi]\!] = \bigwedge_{y \in \operatorname{dom} x} \left(x(y) \rightarrow [\![\phi(y)]\!]\right)$.

Proof See Note 10.25.

We have the following analog of (8.37).

Definition [GB] *Suppose σ is an $\mathsf{s}^{M,\mathfrak{A}}$-sentence and $a \in |\mathfrak{A}|$. a* decides *$\sigma \overset{\text{def}}{\Longleftrightarrow} a \mid \sigma \overset{\text{def}}{\Longleftrightarrow} a \leqslant [\![\sigma]\!]$ or $a \leqslant \neg\, [\![\sigma]\!]$ $(= [\![\neg\, \sigma]\!])$.*

The analog of (8.38) states that $\{a \in |\mathfrak{A}| \mid a \leqslant [\![\sigma]\!] \vee a \leqslant \neg\, [\![\sigma]\!]\}$ is dense in $\mathfrak{A}^+$. This is obviously equivalent to the statement for any $b \in |\mathfrak{A}^+|$, $\{a \in |\mathfrak{A}| \mid a \leqslant b \vee a \leqslant \neg\, b\}$ is dense, i.e., that $\{b, \neg\, b\}$ is predense. This is a special case of the following theorem.

(8.75) **Theorem** [ZF] *Suppose $\mathfrak{A}$ is a complete boolean algebra and $S \subseteq |\mathfrak{A}|$. Then S is predense below $\bigvee S$.*

Proof Straightforward. $\square$[8.75]

Note in particular that if M is a transitive model of ZF and $\mathfrak{A} \in M$ is an M-complete boolean algebra then any M-generic filter G on $\mathfrak{A}$ is an ultrafilter, i.e., $\forall a \in |\mathfrak{A}|\, (a \in G \vee \neg\, a \in G)$.

8.3.1 Definability of boolean valuation

Our discussion of the definability of forcing[§ 8.2.4] applies *mutatis mutandis* to boolean valuation. In particular, we may define for each s^{V}-formula ϕ with n free variables, an s-formula $\phi^{[\![\,]\!]}$ with $n+2$ free variables such that '$\phi^{[\![\,]\!]}(\mathfrak{A}, a, \dots)$' says that $a = [\![\phi(\dots)]\!]^{\mathfrak{A}}$:

(8.76) **Definition** [S] *Suppose ϕ is an s^{V}-formula.*

1. *If $\phi = u_0 \in u_1$ then $\phi^{[\![\,]\!]}(\mathfrak{A}, a, x_0, x_1) \overset{\text{def}}{=} \ulcorner a = [\![x_0 \in x_1]\!]^{\mathfrak{A}} \urcorner$.*
2. *If $\phi = u_0 = u_1$ then $\phi^{[\![\,]\!]}(\mathfrak{A}, a, x_0, x_1) \overset{\text{def}}{=} \ulcorner a = [\![x_0 = x_1]\!]^{\mathfrak{A}} \urcorner$.*
3. *If $\phi = \mathsf{V} u$ then $\phi^{[\![\,]\!]}(\mathfrak{A}, a, x) \overset{\text{def}}{=} \ulcorner a = \bigvee \left\{ [\![x = \check{y}]\!]^{\mathfrak{A}} \,\middle|\, y \in V \right\} \urcorner$.*
4. *If $\phi = \neg\, \psi$ then $\phi^{[\![\,]\!]}(\mathfrak{A}, a, \dots) \overset{\text{def}}{=} \ulcorner (\psi^{[\![\,]\!]})(\mathfrak{A}, \neg\, a, \dots) \urcorner$. Analogous definitions apply for $\phi = \psi \vee \psi'$, $\psi \wedge \psi'$, $\psi \rightarrow \psi'$, or $\psi \leftrightarrow \psi'$.*
5. *If $\phi(\dots) = \forall v\, \psi(\dots, v, \dots)$ then $\phi^{[\![\,]\!]}(\mathfrak{A}, a, \dots) \overset{\text{def}}{=} \ulcorner a = \bigwedge \left\{ b \in |\mathfrak{A}| \,\middle|\, \exists y \in V^{\mathfrak{A}}\, (\psi^{[\![\,]\!]})(\mathfrak{A}, b, \dots, y, \dots) \right\} \urcorner$. An analogous definition applies for $\phi = \exists v\, \psi$.*

We have the following analog of (8.41).

Theorem [S] *Suppose ϕ is an s^{V}-formula with n free variables. Then*

$\mathsf{GB} \vdash \ulcorner$*Suppose M is a transitive model of ZF, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $a, x_0, \dots, x_{n^\cdot} \in M$. Then*

$$M \models (\hat{\phi^{[\![\,]\!]}})[\mathfrak{A}, a, x_0, \dots, x_{n^\cdot}] \\ \leftrightarrow x_0, \dots, x_{n^\cdot} \in M^{\mathfrak{A}} \wedge a = [\![(\hat{\phi})(x_0, \dots, x_{n^\cdot})]\!]^{M,\mathfrak{A}} . \urcorner,$$

where $\hat{\phi}$ and $\phi^{\hat{[\![\,]\!]}}$ are the standard names of ϕ and $\phi^{[\![\,]\!]}$.

8.3.2 Forcing *vs* boolean valuation

By regarding the elements of $M^{\mathfrak{A}}$ as individuals of the $\mathfrak{A}$-valued structure $\mathfrak{M}^{\mathfrak{A}}$ we have to a degree untethered them from their significance as terms denoting individuals in a generic extension of M. We have recognized this point of view by using nonspecific symbols such as 'x', rather than more "term-specific" symbols like 'τ' that we have typically used for the elements of $M^{\mathbb{P}}$, where $\mathbb{P}$ is a partial order. We now establish the exact correspondence of $M^{\mathbb{P}}$ with $M^{\mathfrak{R}\mathbb{P}}$, and in recognition of this correspondence we will relax the latter notational convention.

Suppose M is a transitive model of ZF and $\mathbb{P} \in M$ is a partial order. We will define a subclass $\bar{M}^{\mathbb{P}}$ of $M^{\mathbb{P}}$ as follows.

(8.77) **Definition** [GB] *For $x \in M^{\mathbb{P}}$, the* regularization of $x \overset{\text{def}}{=} \bar{x}$ *is defined by recursion so that*

$$\bar{x} = \{\langle \bar{y}, p\rangle \mid y \in \operatorname{dom} x \wedge p \in \overline{x^{\rightarrow}\{y\}}\}.$$

The regularization of $M^{\mathbb{P}} \overset{\text{def}}{=} \bar{M}^{\mathbb{P}} \overset{\text{def}}{=} \{\bar{x} \mid x \in M^{\mathbb{P}}\}$*. The members of $\bar{M}^{\mathbb{P}}$ are the* regular *terms.*

Note that $\operatorname{dom} \bar{x} = \{\bar{y} \mid y \in \operatorname{dom} x\}$. It is easily shown by induction that $\bar{\bar{x}} = \bar{x}$ for any $x \in M^{\mathbb{P}}$.

Note that for $x \in \bar{M}^{\mathbb{P}}$, $y \in \operatorname{dom} x$, and $p \in |\mathbb{P}|$, $x^{\rightarrow}\{y\}$ is a regular subset of $|\mathbb{P}|$, i.e., an element of $\mathfrak{R}\,\mathbb{P}$; hence the following definition.

(8.78) **Definition** [GB] *For $x \in M^{\mathbb{P}}$, we define $\hat{x} \in M^{\mathfrak{R}\,\mathbb{P}}$ by recursion so that*

$$\hat{x} = \{(\hat{y}, a) \mid \bar{y} \in \operatorname{dom} \bar{x} \wedge a = \bar{x}^{\rightarrow}\{\bar{y}\}\}.$$

$x \mapsto \hat{x}$ is clearly a bijection of $\bar{M}^{\mathbb{P}}$ with $M^{\mathfrak{A}}$.

(8.79) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order. Let $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$. Suppose $p \in |\mathbb{P}|$ and $x, y \in M^{\mathbb{P}}$.*

1. $p \Vdash^{M,\mathbb{P}} x \in y$ *iff* $p \Vdash^{M,\mathbb{P}} \bar{x} \in \bar{y}$ *iff* $p \in [\![\hat{x} \in \hat{y}]\!]^{M,\mathfrak{A}}$.
2. $p \Vdash^{M,\mathbb{P}} x = y$ *iff* $p \Vdash^{M,\mathbb{P}} \bar{x} = \bar{y}$ *iff* $p \in [\![\hat{x} = \hat{y}]\!]^{M,\mathfrak{A}}$.

Proof We proceed by induction on rank to prove (1) and (2) simultaneously.

1 For the first equivalence:

$$\begin{aligned}
p \Vdash \bar{x} \in \bar{y} &\leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists z \in \operatorname{dom} \bar{y}\, \exists r' \in \bar{y}^{\rightarrow}\{z\}\, (r \leqslant r' \wedge r \Vdash z = \bar{x}) \\
&\leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists \bar{z} \in \operatorname{dom} \bar{y}\, \exists r' \in \overline{y^{\rightarrow}\{z\}}\, (r \leqslant r' \wedge r \Vdash \bar{z} = \bar{x}) \\
&\leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists z \in \operatorname{dom} y\, (r \in \overline{y^{\rightarrow}\{z\}} \wedge r \Vdash z = x) \\
&\leftrightarrow \forall q \leqslant p\, \exists z \in \operatorname{dom} y\, \exists r \leqslant q\, (r \in \lceil y^{\rightarrow}\{z\}\rceil \wedge r \Vdash z = x) \\
&\leftrightarrow \forall q \leqslant p\, \exists z \in \operatorname{dom} y\, \exists r' \in y^{\rightarrow}\{z\}\, \exists r \leqslant q\, (r \leqslant r' \wedge r \Vdash z = x) \\
&\leftrightarrow \forall q \leqslant p\, \exists \langle z, r'\rangle \in y\, \exists r \leqslant q\, (r \leqslant r' \wedge r \Vdash z = x) \\
&\leftrightarrow p \Vdash x \in y.
\end{aligned}$$

The second equivalence is also straightforward.

2 For the first equivalence we use the fact that

$$\begin{aligned}&\forall q \leqslant p\, \forall z \in \operatorname{dom} x\, \forall r' \in \overline{x^{\rightarrow}\{z\}}\, (q \leqslant r' \rightarrow q \Vdash z \in y)\\ &\leftrightarrow \forall q \leqslant p\, \forall z \in \operatorname{dom} x\, (q \in \overline{x^{\rightarrow}\{z\}} \rightarrow q \Vdash z \in y)\\ &\leftrightarrow \forall q \leqslant p\, \forall \langle z, r'\rangle \in x\, (q \leqslant r' \rightarrow q \Vdash z \in y),\end{aligned}$$

where the last equivalence follows from the fact that $\{q \Vdash z \in y\}$ is regular, and the corresponding fact with x and y exchanged. Again, the second equivalence is straightforward. $\square^{8.79}$

Analogously, in terms of M-generic filters G on $\mathbb{P}$, we will show by induction that for any $x \in M^{\mathbb{P}}$, $\bar{x}^G = x^G$. To this end, suppose $\bar{y}^G = y^G$ for all $y \in \operatorname{dom} x$. Then

$$\begin{aligned}\bar{x}^G &= \{\bar{y}^G \mid \bar{y} \in \operatorname{dom} \bar{x} \wedge \exists p \in G\, \langle \bar{y}, p\rangle \in \bar{x}\}\\ &= \{y^G \mid \bar{y} \in \operatorname{dom} \bar{x} \wedge \exists p \in G\, \langle \bar{y}, p\rangle \in \bar{x}\}\\ &= \{y^G \mid y \in \operatorname{dom} x \wedge \exists p \in G\; p \in \overline{x^{\rightarrow}\{y\}}\}\\ &= \{y^G \mid y \in \operatorname{dom} x \wedge \exists p \in G\; p \in x^{\rightarrow}\{y\}\}\\ &= \{y^G \mid y \in \operatorname{dom} x \wedge \exists p \in G\, \langle y, p\rangle \in x\}\\ &= x^G,\end{aligned}$$

where we have used the fact that an M-generic filter G meets the regular completion $\overline{X}$ of a subset X of $|\mathbb{P}|$ in M iff G meets X.

If we let $\hat{G}$ be the filter on $\mathfrak{R}\,\mathbb{P}$ corresponding to G under the map $x \mapsto \hat{x}$, then $\hat{G}$ is also M-generic, and it is easy to show that for any $x \in M^{\mathbb{P}}$, $x^G = \hat{x}^{\hat{G}}$.

(8.80) *We extend the correspondence $x \mapsto \hat{x}$ to formulas of the languages $\mathcal{L}^{M,\mathbb{P}}$ and $\mathcal{L}^{M,\mathfrak{R}\,\mathbb{P}}$. Given the above equivalences, we will not always trouble to distinguish corresponding entities in the two points of view.*

(8.81) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order and $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$. For any sentence $\sigma \in \mathcal{L}^{M,\mathbb{P}}$,*

$$[\![\hat{\sigma}]\!]^{M,\mathfrak{A}} = \{p \in |\mathbb{P}| \,|\, p \Vdash^{M,\mathbb{P}} \sigma\}.$$

Proof Straightforward comparison of the definitions of $\Vdash^{M,\mathbb{P}}$ and $[\![\,]\!]^{\mathfrak{A}}$. Note that the case that there is no $\{\sigma\}^{M,\mathbb{P}}$-forcing relation (equivalently, no $\{\hat{\sigma}\}^{M,\mathfrak{R}\,\mathbb{P}}$-valuation function) is properly handled by Definition 8.70 (cf., the discussion preceding the definition). $\square^{8.81}$

Going the other way, suppose M is a transitive model of ZF*, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $\mathbb{P} = \mathfrak{A}^+$, the partial order of nonzero elements of $\mathfrak{A}$. Then $p \Vdash^{M,\mathbb{P}} \sigma$ iff $p \in [\![\sigma]\!]^{M,\mathfrak{A}}$.*

The technique described in Section 8.2.6 for deriving forcing results by "arguing with generic extensions" applies equally to boolean valued structures. In this connection Theorems 8.73 and 8.75 are useful—the latter because it means that for any generic filter G, $[\![\bigvee S]\!] \in G \leftrightarrow \exists a \in S\; [\![a]\!] \in G$.

As an illustration of the method, consider Theorem 8.74. Suppose M is a countable transitive model of ZF and $\mathfrak{A} \in M$ is an M-complete boolean algebra. For

(8.74.1) we must show that any M-generic filter that contains $[\![x = y]\!] \wedge [\![\phi(x)]\!]$ contains $[\![\phi(y)]\!]$. To this end, suppose G is M-generic and contains $[\![x = y]\!] \wedge [\![\phi(x)]\!]$. Then $M[G] \models x = y \wedge \phi(x)$.[21] By ordinary logic, $M[G] \models \phi(y)$. Hence, $[\![\phi(y)]\!] \in G$.

Now consider (8.74.2). Suppose G is M-generic. Then

$$\begin{aligned}
[\![\exists u \in x\ \phi]\!] \in G &\leftrightarrow M[G] \models \exists u \in x\, \phi \\
&\leftrightarrow \exists c \in x^G\ M[G] \models \phi[c] \\
&\leftrightarrow \exists y \in \operatorname{dom} x\ \big(x(y) \in G \wedge M[G] \models \phi[y^G]\big) \\
&\leftrightarrow \exists y \in \operatorname{dom} x\ \big(x(y) \in G \wedge [\![\phi(y)]\!] \in G\big) \\
&\leftrightarrow \bigvee_{y \in \operatorname{dom} x} (x(y) \wedge [\![\phi(y)]\!]) \in G.
\end{aligned}$$

8.4 Logical considerations

Throughout this section we continue to restrict our attention to partial orders and boolean algebras that are sets, as opposed to proper classes.

The correspondences (8.45), (8.73), and (8.81), form the sides of a triangle whose vertices are the notions of forcing, boolean valuation, and satisfaction in a generic extension. Depending on the situation, any of these three points of view may be superior to the others, and it is standard practice to use them interchangeably.

In this connection two quite different existence questions must be addressed. The first is that of forcing relations, boolean valuations, and satisfaction relations for proper class structures. We have already thoroughly discussed this issue and shown that in practice it may essentially be ignored (as it typically is in the literature).

The second question is that of the existence of generic filters, which we will address in detail beginning in Section 8.5.2. For the present, suffice it to say at that for many purposes this issue may be finessed by regarding a statement about satisfaction of a sentence σ in $M[G]$ as a *façon de parler*, the meaning of which is the corresponding statement about $\{p \in |\mathbb{P}| \mid p \Vdash \sigma\}$ or $[\![\sigma]\!]$.

The final element to be incorporated into this complex is logic.

Definition [GB] *Suppose M is a transitive model of* ZF.

1. *Suppose $\mathfrak{A} \in M$ is an M-complete boolean algebra. An $\mathsf{s}^{M,\mathfrak{A}}$-sentence σ is* $M^{\mathfrak{A}}$-valid $\overset{\text{def}}{\iff} [\![\sigma]\!]^{M,\mathfrak{A}} = 1$.

2. *Equivalently, if $\mathbb{P} \in M$ is a partial order, an $\mathsf{s}^{M,\mathbb{P}}$-sentence is* $M^{\mathbb{P}}$-valid $\overset{\text{def}}{\iff} \Vdash^{M,\mathbb{P}} \sigma$ $\overset{\text{def}}{\iff} \forall p \in |\mathbb{P}|\ p \Vdash^{M,\mathbb{P}} \sigma$ *(iff $\mathbf{1} \Vdash^{M,\mathbb{P}} \sigma$, where $\mathbf{1}$ is as usual the maximum element of $\mathbb{P}$).*

3. $\mathsf{s}^{M,\mathfrak{A}}$- *($\mathsf{s}^{M,\mathbb{P}}$-)formulas ϕ and ψ are $M^{\mathfrak{A}}$- ($M^{\mathbb{P}}$-)*equivalent $\overset{\text{def}}{\iff} \bar{\forall}(\sigma \leftrightarrow \psi)$ *is an $M^{\mathfrak{A}}$- ($M^{\mathbb{P}}$-)validity.*[22]

[21] Remember that x and y are terms of $\mathcal{L}^{M,\mathfrak{A}}$. We could also write that $M[G] \models \ulcorner [x^G] = [y^G] \wedge (\phi)[x^G] \urcorner$.

[22] Recall that $\bar{\forall}$ is the operation of universal closure, i.e., universal quantification of all the free variables in a formula.

We may omit to mention M, $\mathbb{P}$, or $\mathfrak{A}$ if we may do so without confusion, and if $M = V$ we regularly omit to mention it.

The considerations of Section 2.8 relating satisfactoriness and logic apply *mutatis mutandis* to forcing and logic. The following theorem corresponds to Theorems 2.175 and 2.176. Recall that these have the same conclusion and differ only in the premise Infinity, which is present in the statement of the former but not the latter. The following theorem resembles (2.175) in that it is stated in GB, but its proof resembles that of (2.176) in that it exploits the subformula property of the deductive system $\mathbf{LK}^-$.

(8.82) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra.*

1. *Suppose an $\mathsf{s}^{M,\mathfrak{A}}$-sentence σ is a logical validity, i.e., $\vdash \sigma$, then $[\![\sigma]\!] = \mathbf{1}$.*
2. *Hence, if $\mathsf{s}^{M,\mathfrak{A}}$-formulas σ and θ are logically equivalent, i.e., σ and θ have the same free variables and $\vdash \sigma \leftrightarrow \theta$, then for any substitution T of forcing terms for the free variables, $[\![\sigma(T)]\!] = [\![\theta(T)]\!]$.*

Proof See Note 10.26.

For the sake of this discussion we make the following definition.

Definition [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P} \in M$ is a partial order.*

1. *$M^{\mathbb{P}}$ is* weakly forceful $\overset{\text{def}}{\Longleftrightarrow}$ *for every finite set Φ of $\mathsf{s}^{M,\mathbb{P}}$-formulas there is a $\Phi^{M,\mathbb{P}}$-forcing relation.*
2. *$M^{\mathbb{P}}$ is* strongly forceful *or simply* forceful $\overset{\text{def}}{\Longleftrightarrow}$ *there is a full $M, \mathbb{P}$-forcing relation.*

We define forceful *and* weakly forceful *analogously for an M-complete boolean algebra $\mathfrak{A} \in M$.*

(8.83) **Theorem** [GB] *Suppose M is a set, M is a transitive model of* ZF*, and $\mathbb{P} \in M$ is a partial order ($\mathfrak{A} \in M$ is an M-complete boolean algebra). Then $M^{\mathbb{P}}$ ($M^{\mathfrak{A}}$) is forceful, i.e., the full forcing relation (boolean valuation) exists.*

Proof This is essentially (8.31), stated in the terminology of forcefulness, and the proof is entirely straightforward. $\square^{8.83}$

The following theorem corresponds to (2.174).

(8.84) **Theorem** [GB] *Suppose $M^{\mathfrak{A}}$ is weakly forceful, Θ is an $\mathsf{s}^{M,\mathfrak{A}}$-theory, and for every $\theta \in \Theta$, $[\![\theta]\!]^{M,\mathfrak{A}} = \mathbf{1}$.*

1. *Θ is consistent.*
2. *If θ is an $\mathsf{s}^{M,\mathfrak{A}}$-sentence and $\Theta \vdash \theta$ then $[\![\theta]\!]^{M,\mathfrak{A}} = \mathbf{1}$.*

Proof Straightforward. $\square^{8.84}$

We have the following "meta" version of (8.84).

(8.85) **Theorem** [S] *Suppose σ is an s^{V}-sentence and $\vdash \sigma$.*

1. *Let $\hat{\sigma}$ be the standard name of σ.* $\mathsf{GB} \vdash \ulcorner$*Suppose M is transitive, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $M \models \mathsf{ZF}$. Then $[\![(\hat{\sigma})]\!]^{M,\mathfrak{A}} = \mathbf{1}$. In particular (letting $M = V$), if $\mathfrak{A}$ is a complete boolean algebra then $[\![(\hat{\sigma})]\!]^{\mathfrak{A}} = \mathbf{1}$.*$\urcorner$.
2. $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}} \vdash \ulcorner$*Suppose $\mathfrak{A}$ is a complete boolean algebra. Then $(\sigma^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$.*$\urcorner$.

Proof Suppose π is a proof of σ in any of the deductive systems we have considered.

1 A GB-proof of the indicated sentence begins with a proof of the existence of a $\Phi^{M,\mathfrak{A}}$-partial valuation $[\![\cdot]\!]$ for the set Φ of formulas occurring in π.[8.69] This is followed by the sequential explicit demonstration that the sequents in π are $[\![\cdot]\!]$-validities, as in the proof of (8.82.1). There is no need to suppose that π has the subformula property.

2 Here, rather than showing that a sufficient partial valuation exists—a notion that cannot be formulated in ZF—we put together a $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$-proof by showing that the corresponding valuation definitions $\phi^{[\![\,]\!]}$ behave in such a way as to permit the transformation of π into $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$-proof of $\ulcorner$Suppose $\mathfrak{A}$ is a complete boolean algebra. Then $(\sigma^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$.$\urcorner$. □[8.85]

8.5 The theory of a generic extension

The following technical results are important to an understanding of forcing and are frequently useful. Our first use of them will be in proving that forcing preserves the axioms of ZF.

(8.86) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathfrak{A} \in M$ is an M-complete boolean algebra, $S \in M$ and $S \subseteq M^{\mathfrak{A}}$. Then $\exists x \in M^{\mathfrak{A}}\, \forall y \in S\ [\![y \in x]\!] = \mathbf{1}$.*

Proof Let $x = \{(y, \mathbf{1}) \mid y \in S\}$. □[8.86]

(8.87) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathfrak{A}$ is an M-complete boolean algebra, $S \in M$ is a set of pairwise incompatible elements of $|\mathfrak{A}|$, and $\langle x_s \mid s \in S\rangle \in M$ is a function from S to $M^{\mathfrak{A}}$. Then there exists $x \in M^{\mathfrak{A}}$ such that*

$$\forall s \in S\ s \leqslant [\![x = x_s]\!].$$

Remark It may be helpful to think of this theorem and its proof in terms of generic extensions. We are looking for a term $x \in M^{\mathfrak{A}}$ such that for any M-generic filter G on $\mathfrak{A}$, if $s \in G$ then $x^G = x_s^G$. Since the members of S are pairwise incompatible, only one of them can be in any given filter G, so this project seems feasible. We will define x in such a way as to guarantee that for each $s \in S$, if $s \in G$ then $y^G \in x^G \leftrightarrow y^G \in x_s^G$ for any y in $\operatorname{dom} x_s$ and $\operatorname{dom} x$.[8.88] This is enough to show that for any $s \in S$, if $s \in G$ then $x^G = x_s^G$, as desired.[8.89]

Proof Let $\alpha \in \mathrm{Ord} \cap M$ be such that $\forall s \in S\ x_s \in M_\alpha^{\mathfrak{A}}$. For $s \in S$, let $x_s' \in M_{\alpha+1}^{\mathfrak{A}}$ be such that $\operatorname{dom} x_s' = M_\alpha^{\mathfrak{A}}$ and for all $y \in M_\alpha^{\mathfrak{A}}$

$$x_s'(y) = \begin{cases} x_s(y) & \text{if } y \in \operatorname{dom} x_s \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let $x \in M^{\mathfrak{A}}_{\alpha+1}$ with domain $M^{\mathfrak{A}}_{\alpha}$ be given by

$$x(y) = \bigvee_{s \in S} \big(s \wedge x'_s(y)\big).$$

For any $y \in M^{\mathfrak{A}}_{\alpha}$ and $s \in S$,

$$s \wedge x(y) = s \wedge x'_s(y),$$

and therefore

$$\begin{aligned} s \wedge [\![y \in x]\!] &= s \wedge \bigvee_{z \in \operatorname{dom} x} \big(x(z) \wedge [\![z = y]\!]\big) = \bigvee_{z \in M^{\mathfrak{A}}_{\alpha}} \big(s \wedge x(z) \wedge [\![z = y]\!]\big) \\ &= \bigvee_{z \in M^{\mathfrak{A}}_{\alpha}} \big(s \wedge x'_s(z) \wedge [\![z = y]\!]\big) = \bigvee_{z \in \operatorname{dom} x_s} \big(s \wedge x_s(z) \wedge [\![z = y]\!]\big) \\ &= s \wedge \bigvee_{z \in \operatorname{dom} x_s} \big(x_s(z) \wedge [\![z = y]\!]\big) \\ &= s \wedge [\![y \in x_s]\!]. \end{aligned} \tag{8.88}$$

Also, for any $y \in \operatorname{dom} x$,

$$s \wedge \neg\, x(y) = s \wedge \neg\, x'_s(y).$$

Hence,

$$\begin{aligned} & s \wedge [\![x = x_s]\!] \\ &\quad = s \wedge \bigwedge_{y \in \operatorname{dom} x} \big(x(y) \rightarrow [\![y \in x_s]\!]\big) \wedge \bigwedge_{y \in \operatorname{dom} x_s} \big(x_s(y) \rightarrow [\![y \in x]\!]\big) \\ &\quad = \bigwedge_{y \in \operatorname{dom} x} \big(s \wedge (\neg\, x(y) \vee [\![y \in x_s]\!])\big) \wedge \bigwedge_{y \in M^{\mathfrak{A}}_{\alpha}} \big(s \wedge (\neg\, x'_s(y) \vee [\![y \in x]\!])\big) \\ &\quad = \bigwedge_{y \in \operatorname{dom} x} \big(s \wedge (\neg\, x(y) \vee [\![y \in x]\!])\big) \wedge \bigwedge_{y \in \operatorname{dom} x} \big(s \wedge (\neg\, x(y) \vee [\![y \in x]\!])\big) \\ &\quad = s, \end{aligned} \tag{8.89}$$

since $\forall y \in \operatorname{dom} x\; x(y) \leqslant [\![y \in x]\!]$. Hence, $s \leqslant [\![x = x_s]\!]$, as claimed. $\square$[8.87]

We have frequent occasion to consider expressions in forcing languages that are conveniently described in informal mathematical language. We may create a name for such an expression in the usual way with corner quotation marks. To denote the boolean value of such a quoted expression we sometimes omit the quotation marks—in effect incorporating them into the double brackets of the boolean value notation. Substitution of terms for (often implicit) variables in such expressions is indicated as usual by light round brackets, and assignment of values to variables is indicated as usual by light square brackets. Thus, for example,

$$[\![(x) \text{ is a subset of } (y)]\!]$$

is understood as

$$[\![\ulcorner (x) \text{ is a subset of } (y) \urcorner]\!],$$

i.e.,

$$[\![\forall u\, (u \in x \rightarrow u \in y)]\!],$$

or some equivalent expression. Similarly for arguments of the forcing or satisfaction relations, as in

$$p \Vdash \ulcorner (\tau) \text{ is a subset of } (\tau') \urcorner,$$

or

$$\mathfrak{M} \models \ulcorner (\tau) \text{ is a subset of } [y] \urcorner,$$

where in the latter example τ is a term of the language appropriate to $\mathfrak{M}$ and y is an element of $|\mathfrak{M}|$, and we have substituted (a constant term) τ for one variable and assigned y to another variable (neither variable being shown explicitly). Note that since forcing languages have terms for all the elements of the structures in question, assignment is seldom called for. We will not be scrupulous in the use of these conventions, and if an expression does not make sense as written, the reader should not hesitate to supply the proper indicators, or simply ignore the issue (the usual attitude). For example,

$$[\![x \subseteq y]\!]$$

should be understood as

$$[\![\ulcorner (x) \subseteq (y) \urcorner]\!].$$

It is useful to have the notions of $\mathfrak{A}$-pairs and $\mathfrak{A}$-ordered pairs.

(8.90) **Definition** [GB] *Given $x, y \in M^{\mathfrak{A}}$,*

1. $\{x, y\}^{\mathfrak{A}} \overset{\text{def}}{=} \{(x, \mathbf{1}), (y, \mathbf{1})\}$. $\{x\}^{\mathfrak{A}} \overset{\text{def}}{=} \{x, x\}^{\mathfrak{A}} = \{(x, \mathbf{1})\}$.
2. $(x, y)^{\mathfrak{A}} \overset{\text{def}}{=} \{\{x\}^{\mathfrak{A}}, \{x, y\}^{\mathfrak{A}}\}^{\mathfrak{A}}$.

It is straightforward to show that the following are $\mathfrak{A}$-valid for any $x, y \in M^{\mathfrak{A}}$:

1. $\ulcorner (\{x, y\}^{\mathfrak{A}}) = \{(x), (y)\} \urcorner$, i.e., e.g., $\forall u \left(u \in \{x, y\}^{\mathfrak{A}} \leftrightarrow (u = x \vee u = y)\right)$.
2. $\ulcorner ((x, y)^{\mathfrak{A}}) = ((x), (y)) \urcorner$.

8.5.1 Generic extension preserves ZF

(8.91) **Theorem** [GB] *Suppose M is transitive, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $M \models \mathsf{ZF}$. Then every axiom of $\mathsf{ZF}^{\mathsf{s}^{\vee}}$ is an $M^{\mathfrak{A}}$-validity, where $\mathsf{ZF}^{\mathsf{s}^{\vee}}$ is ZF with the axiom schemas extended to all $\mathsf{s}^{\vee}$-formulas.*

Remark As a convenience, we will work with the theory $\mathsf{ZF}^{\mathsf{s}^{M,\mathfrak{A}}}$ in the signature $\mathsf{s}^{M,\mathfrak{A}}$, which is trivially included in the deductive closure of $\mathsf{ZF}^{\mathsf{s}^{\vee}}$ in $\mathcal{L}^{\mathsf{s}^{M,\mathfrak{A}}}$ and so is essentially equivalent.

Proof Remember that for any $\mathsf{s}^{M,\mathfrak{A}}$-sentence σ, if the $\{\sigma\}^{M,\mathfrak{A}}$-valuation function does not exist then $[\![\sigma]\!] = 1$,[23] so we will assume throughout this proof that the valuation function exists for the axiom in question. There is no need to invoke the fact that there is a GB-proof of the existence of the $\{\sigma\}$-valuation function for any specific sentence σ, and this would not work in any event, as there are infinitely many axioms in ZF, and we cannot insert infinitely many proofs into a single proof.

[23] We recognize that this is true by fiat, but the choice was not arbitrary.

Extension The axiom is equivalent to

$$\forall u\,\forall v\,\big((\forall w\in u\ w\in v\wedge\forall w\in v\ w\in u)\to u=v\big),$$

so by (8.82) it is sufficient to show that this is a validity. Since

$$\begin{aligned}&[\![\forall u\,\forall v\,((\forall w\in u\ w\in v\wedge\forall w\in v\ w\in u)\to u=v)]\!]\\&\qquad=\bigwedge_{x,y\in M^{\mathfrak{A}}}\big(([\![\forall w\in x\ w\in y]\!]\wedge[\![\forall w\in y\ w\in x]\!])\to[\![x=y]\!]\big),\end{aligned}$$

we must show that for any $x,y\in M^{\mathfrak{A}}$,

$$\big(([\![\forall w\in x\ w\in y]\!]\wedge[\![\forall w\in y\ w\in x]\!])\to[\![x=y]\!]\big)=\mathbf{1},$$

i.e.,

$$[\![\forall w\in x\ w\in y]\!]\wedge[\![\forall w\in y\ w\in x]\!]\leqslant[\![x=y]\!].$$

This follows from (8.74.3) and Definition 8.64.2.

The proof in terms of generic extensions simply amounts to observing that $M[G]$ is a transitive class and therefore satisfies Extension.

Comprehension We must show that for any $\mathsf{s}^{M,\mathfrak{A}}$-formula ϕ and distinct variables $u,v,w,v_0,\ldots,v_{n^-}$, with all free variables of ϕ in the set $\{v,v_0,\ldots,v_{n^-}\}$,

$$[\![\forall v_0\cdots\forall v_{n^-}\,\forall u\,\exists w\,\forall v\,(v\in w\leftrightarrow(v\in u\wedge\phi))]\!]=\mathbf{1}.$$

We must therefore show that for any $y_0,\ldots,y_{n^-},x\in M^{\mathfrak{A}}$, letting $\phi'=\phi\binom{v_0\ \cdots\ v_{n^-}}{y_0\ \cdots\ y_{n^-}}$ and $\phi'(y)=\phi'\binom{v}{y}$,

$$\bigvee_{z\in M^{\mathfrak{A}}}\bigwedge_{y\in M^{\mathfrak{A}}}[\![y\in z\leftrightarrow(y\in x\wedge\phi'(y))]\!]=\mathbf{1}.$$

We will actually show that for some $z\in M^{\mathfrak{A}}$,

$$\bigwedge_{y\in M^{\mathfrak{A}}}[\![y\in z\leftrightarrow(y\in x\wedge\phi'(y))]\!]=\mathbf{1},$$

i.e., for all $y\in M^{\mathfrak{A}}$,

$$[\![y\in z]\!]=[\![y\in x\wedge\phi'(y)]\!].$$

Let

$$z=\langle x(y)\wedge[\![\phi'(y)]\!]\mid y\in\operatorname{dom}x\rangle.$$

Then for any $y\in M^{\mathfrak{A}}$, using (8.64.1) and (8.74.2) and, as usual, the fact that logical equivalents have equal boolean value,

$$\begin{aligned}[\![y\in z]\!]&=\bigvee_{y'\in\operatorname{dom}z}(z(y')\wedge[\![y'=y]\!])\\&=\bigvee_{y'\in\operatorname{dom}x}(x(y')\wedge[\![\phi'(y')]\!]\wedge[\![y'=y]\!])\\&=[\![\exists v'\in x(\phi'(v')\wedge v'=y)]\!]\\&=[\![y\in x\wedge\phi'(y)]\!].\end{aligned}$$

Existence Since the join of the empty set in a boolean algebra is $\mathbf{0}$, it follows from the definition[8.64.1] that for any $y \in M^{\mathfrak{A}}$,

$$[\![y \in 0]\!] = \mathbf{0}.\text{[24]}$$

Hence, $[\![\forall y\ y \notin 0]\!] = \mathbf{1}$, and $[\![\exists x\, \forall y\, y \notin x]\!] = \mathbf{1}$.[25]

Pair Given $x, y \in M^{\mathfrak{A}}$, let $z = \{(x, \mathbf{1}), (y, \mathbf{1})\}$. Then $[\![x \in z \wedge y \in z]\!] = \mathbf{1}$.

Collection We must show that for any $x \in M^{\mathfrak{A}}$, distinct variables a, v, w, and $\mathsf{s}^{M,\mathfrak{A}}$-formula ϕ with free variables in $\{a, v\}$,[26]

$$[\![\forall v \in x\, \exists w\, \forall a\, (\phi \to a \in w) \to \exists w\, \forall v \in x\, \forall a\, (\phi \to a \in w)]\!] = \mathbf{1},$$

i.e.,

$$[\![\forall v \in x\, \exists w\, \forall a\, (\phi \to a \in w)]\!] \leqslant [\![\exists w\, \forall v \in x\, \forall a\, (\phi \to a \in w)]\!]. \tag{8.92}$$

For any $y \in \operatorname{dom} x$, let α_y be the least ordinal α such that

$$\bigvee_{z \in M^{\mathfrak{A}}_{\alpha}} [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!] = \bigvee_{z \in M^{\mathfrak{A}}} [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!].$$

Using the fact that $M \models \mathsf{ZF}$, in particular $M \models \mathsf{Collection}$, and $\mathfrak{A}$ is in M, not just included in M, one can show that $\alpha_y \in M$. Let $\alpha = \sup_{y \in \operatorname{dom} x} \alpha_y$. The same sort of argument shows that $\alpha \in M$. For any $y \in \operatorname{dom} x$,

$$\bigvee_{z \in M^{\mathfrak{A}}_{\alpha}} [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!] = \bigvee_{z \in M^{\mathfrak{A}}} [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!].$$

Let $z_0 = \{(a, \mathbf{1}) \mid a \in M^{\mathfrak{A}}_{\alpha}\}$. Then for any $z \in M^{\mathfrak{A}}_{\alpha}$, $[\![z \subseteq z_0]\!] = \mathbf{1}$, so $\forall a \in M^{\mathfrak{A}}\, ([\![a \in z]\!] \leqslant [\![a \in z_0]\!])$. Thus,

$$\begin{aligned}
[\![\forall v \in x\, \exists w\, \forall a\, (\phi \to a \in w)]\!] &= \bigwedge_{y \in \operatorname{dom} x} \Big(x(y) \to \bigvee_{z \in M^{\mathfrak{A}}_{\alpha}} [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!] \Big) \\
&\leqslant \bigwedge_{y \in \operatorname{dom} x} \big(x(y) \to [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z_0)]\!] \big) \\
&\leqslant \bigvee_{z \in M^{\mathfrak{A}}} \bigwedge_{y \in \operatorname{dom} x} \big(x(y) \to [\![\forall a\, (\phi\tbinom{v}{y} \to a \in z)]\!] \big) \\
&= [\![\exists w\, \forall v \in x\, \forall a\, (\phi \to a \in w)]\!],
\end{aligned}$$

as claimed.[8.92]

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[24] Note that 0 is the empty set, regarded as an element of $M^{\mathfrak{A}}$, while $\mathbf{0}$ is the zero element of $\mathfrak{A}$ (which happens to be 0 if $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$ for some $\mathbb{P}$).

[25] Any $x \in M^{\mathfrak{A}}$ such that $\operatorname{im} x \subseteq \{\mathbf{0}\}$ also represents the empty set with boolen value $\mathbf{1}$.

[26] We have skipped the superfluous step of instantiating parameters $v_0, \ldots, v_{n^-}$ as we did in the proof of Comprehension. In other words, the role of ϕ' in that proof is played by ϕ itself in this proof.

Infinity $\exists u\,\big(\exists v\,(v \in u \wedge \forall w\; w \notin v) \wedge \forall v \in u\, \exists w \in u\; v \in w\big)$ is a convenient form of Infinity for this purpose, and it suffices to show that

$$[\![\exists u\,(\exists v\,(v \in u \wedge \forall w\; w \notin v) \wedge \forall v \in u\, \exists w \in u\; v \in w)]\!] = \mathbf{1}.$$

To this end, let $x_0 = 0$, and for each $n \in \omega$, let $x_{n+1} = \{(x_n, \mathbf{1})\}$. Let $S = \{x_n \mid n \in \omega\}$. Let $x = \{(x_n, \mathbf{1}) \mid n \in \omega\}$. Since $\mathsf{Infinity}^M$, $x \in M^{\mathfrak{A}}$. Then

$$\begin{aligned}
&[\![\exists u\,(\exists v\,(v \in u \wedge \forall w\; w \notin v) \wedge \forall v \in u\, \exists w \in u\; v \in w)]\!]\\
&\qquad \geqslant [\![\exists v\,(v \in x \wedge \forall w\; w \notin v) \wedge \forall v \in x\, \exists w \in x\; v \in w]\!]\\
&\qquad \geqslant [\![(0 \in x \wedge \forall w\; w \notin 0) \wedge \forall v \in x\, \exists w \in x\; v \in w]\!]\\
&\qquad = [\![\forall v \in x\, \exists w \in x\; v \in w]\!]\\
&\qquad = \bigwedge_{y \in \operatorname{dom} x} \Big(x(y) \to \bigvee_{z \in \operatorname{dom} x} (x(z) \to [\![y \in z]\!])\Big)\\
&\qquad = \mathbf{1},
\end{aligned}$$

as claimed.

Foundation Given Infinity we can show the existence of transitive closures without the use of Foundation, and this allows us to infer any instance of the Foundation schema from the single instance that states that every nonempty set has a $\in$-minimal member. We must therefore show that

$$[\![\forall u\,(\exists v\; v \in u \to \exists v \in u\, \forall w \in v\; w \notin u)]\!] = \mathbf{1},$$

i.e., for all $x \in M^{\mathfrak{A}}$,

$$[\![\exists v\; v \in x \to \exists v \in x\, \forall w \in v\; w \notin x)]\!] = \mathbf{1}.$$

Suppose toward a contradiction that this is not true, i.e.,

$$[\![\exists v\; v \in x \wedge \forall v \in x\, \exists w \in v\; w \in x)]\!] = a > \mathbf{0}.$$

Then $[\![\exists v\; v \in x]\!] \geqslant a$, so $[\![\exists v\; v \in x]\!] \wedge a = a < \mathbf{0}$, so there exists $y \in M^{\mathfrak{A}}$ such that $[\![y \in x]\!] \wedge a > \mathbf{0}$. Let y be an $\in$-minimal example. Since

$$\begin{aligned}
[\![y \in x]\!] \wedge a &\leqslant [\![y \in x]\!] \wedge [\![\forall v \in x\, \exists w \in v\; w \in x)]\!]\\
&= [\![y \in x]\!] \wedge \bigwedge_{y' \in M^{\mathfrak{A}}} [\![y' \in x \to \exists w \in y'\; w \in x)]\!]\\
&\leqslant [\![y \in x]\!] \wedge [\![y \in x \to \exists w \in y\; w \in x)]\!]\\
&= [\![y \in x]\!] \wedge [\![\exists w \in y\; w \in x)]\!]\\
&\leqslant [\![\exists w \in y\; w \in x]\!] = \bigvee_{z \in \operatorname{dom} y} \big(y(z) \wedge [\![z \in x]\!]\big)\\
&\leqslant \bigvee_{z \in \operatorname{dom} y} [\![z \in x]\!].
\end{aligned}$$

Thus,

$$\begin{aligned}
0 < [\![y \in x]\!] \wedge a = [\![y \in x]\!] \wedge a \wedge a \leqslant \bigvee_{z \in \operatorname{dom} y} [\![z \in x]\!] \wedge a = \bigvee_{z \in \operatorname{dom} y} ([\![z \in x]\!] \wedge a)\\
= 0,
\end{aligned}$$

by virtue of the $\in$-minimality of y; contradiction.

Power Given $x \in M^{\mathfrak{A}}$, we will show that there is $y \in M^{\mathfrak{A}}$ such that for any $z \in M^{\mathfrak{A}}$

$$[\![z \subseteq x \rightarrow z \in y]\!] = \mathbf{1}.$$

For each $z \in M^{\mathfrak{A}}$, let $w_z \in M^{\mathfrak{A}}$ be such that $\operatorname{dom} w_z = \operatorname{dom} x$ and

$$\forall u \in \operatorname{dom} x \ w_z(u) = x(u) \wedge [\![u \in z]\!].$$

Let

$$S = \{w_z \mid z \in M^{\mathfrak{A}}\}.$$

Note that for any $z \in M^{\mathfrak{A}}$, $w_z : \operatorname{dom} x \to |\mathfrak{A}|$, so $S \in M$. Note that[8.74.3]

$$\begin{aligned} [\![w_z \subseteq z]\!] &= \bigwedge_{u \in \operatorname{dom} w_z} (w_z(u) \rightarrow [\![u \in z]\!]) \\ &= \bigwedge_{u \in \operatorname{dom} x} (x(u) \wedge [\![u \in z]\!] \rightarrow [\![u \in z]\!]) \\ &= \mathbf{1}. \end{aligned} \tag{8.93}$$

(8.94) **Claim** $[\![z \subseteq x]\!] \leqslant [\![z = w_z]\!]$.

Proof Since $[\![z = w_z]\!] = [\![z \subseteq w_z]\!] \wedge [\![w_z \subseteq z]\!]$ and $[\![w_z \subseteq z]\!] = \mathbf{1}$,[8.93] it suffices to show that

$$[\![z \subseteq x]\!] \leqslant [\![z \subseteq w_z]\!].$$

Since

$$[\![z \subseteq x]\!] = \bigwedge_{u \in \operatorname{dom} z} (z(u) \rightarrow [\![u \in x]\!]),$$

and

$$[\![z \subseteq w_z]\!] = \bigwedge_{u \in \operatorname{dom} z} (z(u) \rightarrow [\![u \in w_z]\!])$$

it suffices to show that for any $u \in \operatorname{dom} z$

$$z(u) \rightarrow [\![u \in x]\!] \leqslant z(u) \rightarrow [\![u \in w_z]\!],$$

or, equivalently,

$$z(u) \wedge [\![u \in x]\!] \leqslant z(u) \wedge [\![u \in w_z]\!],$$

i.e.,

$$\begin{aligned} & z(u) \wedge \bigvee_{v \in \operatorname{dom} x} (x(v) \wedge [\![v = u]\!]) \\ & \qquad \leqslant z(u) \wedge \bigvee_{v \in \operatorname{dom} x} (x(v) \wedge [\![v \in z]\!] \wedge [\![v = u]\!]). \end{aligned}$$

For this it suffices to show for any $v \in \operatorname{dom} x$ that

$$z(u) \wedge x(v) \wedge [\![v = u]\!] \leqslant z(u) \wedge x(v) \wedge [\![v \in z]\!] \wedge [\![v = u]\!].$$

Since $z(u) \leqslant [\![u \in z]\!]$, it suffices to show that

$$[\![u \in z]\!] \wedge [\![v = u]\!] \leqslant [\![u \in z]\!] \wedge [\![v \in z]\!] \wedge [\![v = u]\!].$$

This follows from the fact[8.66.2] that

$$[\![u \in z]\!] \wedge [\![v = u]\!] \leqslant [\![v \in z]\!].$$

$\square$[8.94]

Let[8.86] $y \in M^{\mathfrak{A}}$ be such that $\forall w \in S\ [\![w \in y]\!] = 1$. Then for any $z \in M^{\mathfrak{A}}$

$$[\![z = w_z]\!] = [\![z = w_z]\!] \wedge [\![w_z \in y]\!] \leqslant [\![z \in y]\!],$$

so[8.94]

$$[\![z \subseteq x]\!] \leqslant [\![z = w_z]\!] \leqslant [\![z \in y]\!],$$

so y is as desired. $\square$[8.91]

We have the following "meta" version of (8.91).

(8.95) **Theorem** [S] *Suppose $\theta \in \mathsf{ZF}^{\mathsf{s}^\vee}$, i.e., θ is an axiom of $\mathsf{ZF}^{\mathsf{s}^\vee}$.*

1. *As usual, let $\hat{\theta}$ be the standard name for θ. $\mathsf{GB} \vdash$ ⌜Suppose M is transitive, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $M \models \mathsf{ZF}$. Then $[\![(\hat{\theta})]\!]^{M,\mathfrak{A}} = \mathbf{1}$. In particular (letting $M = V$), if $\mathfrak{A}$ is a complete boolean algebra then $[\![(\hat{\theta})]\!]^{\mathfrak{A}} = \mathbf{1}$.⌝.*
2. *$\mathsf{ZF} \vdash$ ⌜Suppose $\mathfrak{A}$ is a complete boolean algebra. Then $(\theta^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$.⌝.*

Proof 1 Since $\mathsf{GB} \vdash$ ⌜$(\hat{\theta})$ is an axiom of $\mathsf{ZF}^{\mathsf{s}^\vee}$⌝(as GB proves every true Σ^0_1 sentence), this follows directly from (8.91).

2 The proof of (8.91) is easily modified to a $\mathsf{ZF}^{\mathsf{s}^\vee}$-proof of ⌜Suppose $\mathfrak{A}$ is a complete boolean algebra. Then $(\theta^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$.⌝ for any given $\theta \in \mathsf{ZF}^{\mathsf{s}^\vee}$. $\square$[8.95]

Note that Theorem 8.91 only says that every *axiom* of $\mathsf{ZF}^{\mathsf{s}^\vee}$ is $M^{\mathfrak{A}}$-valid, not that every *theorem* of $\mathsf{ZF}^{\mathsf{s}^\vee}$—or even of ZF—is $M^{\mathfrak{A}}$-valid. On the contrary:

(8.96) **Theorem** [S] *If ZF is consistent then $\mathsf{GB} \nvdash$ ⌜for some complete boolean algebra $\mathfrak{A}$, every theorem of ZF is a $V^{\mathfrak{A}}$-validity⌝.*

Proof $\mathsf{GB} \vdash$ ⌜for every complete boolean algebra $\mathfrak{A}$, $[\![\,\mathbf{0} = \mathbf{1}\,]\!]^{\mathfrak{A}} = \mathbf{0}$⌝. Thus, if $\mathsf{GB} \vdash$ ⌜for some complete boolean algebra $\mathfrak{A}$, every theorem of ZF is a $V^{\mathfrak{A}}$-validity⌝then $\mathsf{GB} \vdash$ ⌜$\mathbf{0} = \mathbf{1}$ is not a theorem of ZF⌝, i.e., $\mathsf{GB} \vdash$ ⌜ZF is consistent⌝, which it does not, if ZF is consistent. $\square$[8.96]

On the other hand, we have the following metatheorem that strengthens (8.95) to say that every theorem of $\mathsf{ZF}^{\mathsf{s}^\vee}$ is provably valid.

(8.97) **Theorem** [S] *Suppose $\mathsf{ZF}^{\mathsf{s}^\vee} \vdash \theta$.*

1. *Let $\hat{\theta}$ be the standard name for θ. $\mathsf{GB} \vdash$ ⌜Suppose M is transitive, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $M \models \mathsf{ZF}$. Then $[\![(\hat{\theta})]\!]^{M,\mathfrak{A}} = \mathbf{1}$. In particular (letting $M = V$), if $\mathfrak{A}$ is a complete boolean algebra then $[\![(\hat{\theta})]\!]^{\mathfrak{A}} = \mathbf{1}$.⌝.*
2. *$\mathsf{ZF} \vdash$ ⌜Suppose $\mathfrak{A}$ is a complete boolean algebra. Then $(\theta^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$.⌝.*

Proof This follows directly from (8.95) and (8.85). $\square^{8.97}$

In practice, metatheorems such as (8.85), (8.95), and (8.97), which state the GB- or ZF-provability of propositions in the theory of forcing, may be regarded as descriptions of classes of lemmas. In the course of proving a theorem in GB or ZF we may invoke one of these lemmas, even though we have not explicitly stated or proved it. The proof of the metatheorem is accepted as sufficient evidence that a proof of the lemma exists. From the standpoint of logical rigor, this is perfectly acceptable—indeed, it is more reliable than the common practice of "leaving the details to the reader", which often means "I haven't really bothered to check all the details, but I'm confident that a complete formal proof exists". We have previously remarked that in practice what passes for a proof is almost always a proof sketch, which is in effect a proof in S that a proof (in whatever theory we are dealing with, e.g., GB or ZF) exists. Note that (8.85), etc., are S-theorems, so their use is epistemologically as sound as anything we do.[27]

8.5.2 "Arguing in a generic extension"

We now reconsider the method of "arguing with generic extensions", presented in Section 8.2.6, from the point of view suggested by (8.91). We call the present method "arguing in a generic extension". Suppose M is a transitive model of ZF. If $\mathbb{P} \in M$ is a partial order and G is M-generic on $\mathbb{P}$, then, since $\mathfrak{M}[G] \models \mathsf{ZF}$, the theory of partial orders and forcing holds in $\mathfrak{M}[G]$. This is true in particular for forcing with $\mathbb{P}$ over $M = \mathsf{V}^{\mathfrak{M}[G]}$. For any $S \in M$, S is dense in $\mathbb{P}$ iff it is dense in the sense of $\mathfrak{M}[G]$, so

$$\mathfrak{M}[G] \models \ulcorner \lceil G \rceil \text{ is a } \mathsf{V}\text{-generic filter on } \lceil \mathbb{P} \rceil \urcorner.$$

Similarly, if $\mathfrak{A} \in M$ is an M-complete boolean algebra then

$$\mathfrak{M}[G] \models \ulcorner \lceil G \rceil \text{ is a } \mathsf{V}\text{-generic filter on } \lceil \mathfrak{A} \rceil \urcorner.$$

The following theorem expresses these facts without the hypothesis of the existence of an M-generic filter. In fact, they hold with $M = V$, the entire universe of sets.

(8.98) **Theorem** [GB] *Suppose M is a transitive model of ZF and $\mathfrak{A} \in M$ is an M-complete boolean algebra. The following are $\mathfrak{A}$-valid.*

1. $\ulcorner \mathsf{V}$ *is transitive*$\urcorner$, *i.e.*, $\forall u \forall v \left(\mathsf{V}(u) \wedge v \in u \to \mathsf{V}(v)\right)$.[28]
2. $\ulcorner \forall x \,(\mathrm{Ord}\, x \to \mathsf{V}(x)) \urcorner$.
3. $\ulcorner (\mathsf{G})$ *is a* V*-generic filter on* $(\check{\mathfrak{A}}) \urcorner$.

[27] Actually, primitive recursive arithmetic may replace S as a metatheory for most purposes, and this is important for a finer analysis of provability, but for our purposes, S is a satisfactory base theory.

[28] One of the functions of the corner-quote convention in the context of a pure set theory is to indicate that a statement apparently referring to a proper class, such as V, is to be translated into purely set-theoretical terms, as we have done here.

Proof 1 Recall that for $x \in M$, $\check{x} = \{(\check{y}, \mathbf{1}) \mid y \in x\}$, and that $[\![\mathsf{V}(x)]\!] = \bigvee_{x' \in M} [\![x = \check{x}']\!]$.8.67.3 Note also that for any $x \in M$ and $y \in M^{\mathfrak{A}}$,

$$\begin{aligned}[\![y \in \check{x}]\!] &= \bigvee_{z \in \operatorname{dom} \check{x}} \big(\check{x}(z) \wedge [\![z = y]\!]\big) \\ &= \bigvee_{z \in x} [\![\check{z} = y]\!]\end{aligned}$$

We need to show that for every $x, y \in M^{\mathfrak{A}}$,

$$\bigvee_{x' \in M} [\![x = \check{x}']\!] \wedge [\![y \in x]\!] \leqslant \bigvee_{y' \in M} [\![y = \check{y}']\!].$$

Suppose $x' \in M$. Then

$$\begin{aligned}[\![x = \check{x}']\!] \wedge [\![y \in x]\!] &\leqslant [\![y \in \check{x}']\!] \\ &= \bigvee_{y' \in x'} [\![y = \check{y}']\!] \\ &\leqslant \bigvee_{y' \in M} [\![y = \check{y}']\!].\end{aligned}$$

2 Suppose toward a contradiction that $\ulcorner \forall x\, (\operatorname{Ord} x \to \mathsf{V}(x)) \urcorner$ is not $\mathfrak{A}$-valid, i.e.,

$$[\![\ulcorner \forall x\, (\operatorname{Ord} x \to \mathsf{V}(x)) \urcorner]\!] < \mathbf{1}\,. \tag{8.99}$$

Then[29]

$$[\![\ulcorner \text{there exists } x \text{ such that every ordinal in } \mathsf{V} \text{ is in } x \urcorner]\!] > \mathbf{0}\,. \tag{8.100}$$

Hence, there exists $x \in M^{\mathfrak{A}}$ such that

$$[\![\ulcorner \text{every ordinal in } \mathsf{V} \text{ is in } x \urcorner]\!] > \mathbf{0}\,.$$

It is easy to show that for every ordinal $\alpha \in M$, $[\![\,\mathbf{Ord}(\check{\alpha})]\!] = \mathbf{1}$. Thus, for every $\alpha \in \operatorname{Ord}^M$, $\exists y \in \operatorname{dom} x\ [\![y = \check{\alpha}]\!] > \mathbf{0}$. Since $M \models \mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$, x and $|\mathfrak{A}|$ are in M, and $\operatorname{Ord}^M \notin M$, and since the relevant forcing relations are definable over M, there exist $y \in \operatorname{dom} x$ and distinct $\alpha, \beta \in \operatorname{Ord}^M$ such that

$$[\![y = \check{\alpha}]\!] = [\![y = \check{\beta}]\!] > \mathbf{0}\,.$$

It follows that $[\![y = \check{\alpha}]\!] \wedge [\![y = \check{\beta}]\!] > \mathbf{0}$, so $[\![\check{\alpha} = \check{\beta}]\!] > \mathbf{0}$; however, it is easy to show that if $\alpha \neq \beta$ then $[\![\check{\alpha} = \check{\beta}]\!] = \mathbf{0}$.

3 The essential point is that for any $a \in |\mathfrak{A}|$, $[\![\check{a} \in \mathsf{G}]\!] = a$. Suppose $s \in M$. Let $A = |\mathfrak{A}|$. We must show that

$$[\![\ulcorner \text{if } (\check{s}) \cap (\check{A}) \text{ is dense then } (\mathsf{G}) \text{ meets } (\check{s}) \urcorner]\!] = \mathbf{1}.$$

[29] We are sketching a GB-proof. To convince ourselves that (8.100) follows from (8.99) we note that $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}} \vdash \ulcorner$if there is an ordinal not in V then there exists x such that every ordinal in V is in $x\urcorner$; and we then use (8.97). From now on, we will typically use this sort of inference without specific recognition.

$\llbracket \ulcorner(\check{s}) \cap (\check{A})$ is dense$\urcorner \rrbracket$ is $\mathbf{1}$ or $\mathbf{0}$ according as $s \cap A$ is or is not dense. Thus, we must show that if $s \cap A$ is dense then

$$\llbracket \exists u \in \mathsf{G}\ u \in \check{s} \rrbracket = \mathbf{1}\,.$$

This is true, since

$$\begin{aligned}\llbracket \exists u \in \mathsf{G}\ u \in \check{s} \rrbracket &= \bigvee_{a \in A} (a \wedge \llbracket \check{a} \in \check{s} \rrbracket) = \bigvee_{a \in A \cap s} a \\ &= \mathbf{1},\end{aligned}$$

since $A \cap s$ is dense. □[8.98]

Theorem 8.98 suggests a quite useful method for proving forcing relations. In practice we typically frame the argument as follows.

To show that $p \Vdash^{\mathbb{P}} \phi$, we show that if we enlarge the universe V to $V[G]$, where G is a V-generic filter on $\mathbb{P}$ and $p \in G$, then $\mathfrak{V}[G] \models \phi$.

Informally, suppose we have reached a certain point in a discussion of forcing, and we wish to prove $\ulcorner p \Vdash^{\mathbb{P}} \phi \urcorner$ for some partial order $\mathbb{P}$. Let $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$.[30] We temporarily transfer the discussion to $V^{\mathfrak{A}}$ and we regard everything we have said up to this point as referring to $\{\check{x} \mid x \in V\}$, which is an image of V in $V^{\mathfrak{A}}$ (which is itself included in V). So translated, every sentence of the preceding discussion has $\mathfrak{A}$-value $\mathbf{1}$ or $\mathbf{0}$, and ordinary logic has applied. Every theorem we have proved has of course had $\mathfrak{A}$-value $\mathbf{1}$. By Theorem 8.98, $\ulcorner(\mathsf{G})$ is a (V)-generic filter on $(\check{\mathfrak{A}})\urcorner$ also has $\mathfrak{A}$-value $\mathbf{1}$, so we may continue to reason in the ordinary way with this new hypothesis, and every conclusion we draw will have $\mathfrak{A}$-value $\mathbf{1}$. Suppose we draw the conclusion that $p \in G \to \phi$. Then

$$\llbracket \check{p} \in \mathsf{G} \to \phi \rrbracket = \mathbf{1},$$

so

$$\llbracket \check{p} \in \mathsf{G} \rrbracket \leqslant \llbracket \phi \rrbracket.$$

Since $\llbracket \check{p} \in \mathsf{G} \rrbracket = p$, $p \leqslant \llbracket \phi \rrbracket$, i.e.,

$$p \Vdash \phi.$$

Formally, we proceed as follows. Let s^* be a signature that extends the signature s of set theory by the addition of a unary predicate symbol V and constant symbols P and G.

(8.101) *Let Θ be the s^*-theory consisting of*

1. $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$*;*
2. $\ulcorner\mathsf{V}$ *is transitive and contains every ordinal*$\urcorner$*;*
3. ZF^{V}*, i.e., all axioms of* ZF *relativized to* V*;*
4. $\ulcorner\mathsf{V}(\mathsf{P})$ *and* P *is a partial order*$\urcorner$*;*
5. $\ulcorner\mathsf{G}$ *is a* V*-generic filter on* $\mathsf{P}\urcorner$*;*
6. $\ulcorner$*every set is* x^{G} *for some* $x \in \mathsf{V}^{\mathsf{P}}\urcorner$*.*

[30] In this discussion and elsewhere we will move freely between partial orders and their regular algebras via the correspondence $\hat{\ }$ without necessarily mentioning the transformation.[8.80]

The following theorem is often useful in framing propositions about generic extensions of inner models of ZF in terms of countable models of finite fragments of ZF.

(8.102) **Theorem** [S] *For any finite subset T of Θ*[8.101] *there is a finite subset F of* ZF *such that* $\mathsf{ZF} \vdash$ $\ulcorner$*for any transitive (set) model M of $(\hat{F})$, partial order $\mathbb{P} \in M$, and M-generic filter G on $\mathbb{P}$, $M[G] \models (\hat{T})$ with $\mathsf{V}, \mathsf{P}, \mathsf{G}$ interpreted respectively as M, $\mathbb{P}$, and G*$\urcorner$*, where $\hat{T}$ and $\hat{F}$ are, as usual, the standard names for T and F.*

Proof Let F_0 be a finite fragment of ZF that is sufficient to develop the theory of forcing within any transitive model M of F_0 (e.g., the entire set of axioms we have specifically used in this book so far to prove ZF-theorems). For each $\theta \in T$, let F_θ be an additional finite fragment of ZF such that if $M \models (F_0 \cup F_\theta)$ then $M[G] \models \theta$. Let $F = F_0 \cup \bigcup_{\theta \in T} F_\theta$. □[8.102]

The following theorem formally defines and establishes the method of "arguing in a generic extension". The similarity to "arguing with generic extensions"[§ 8.2.6] is evident, and the proof of applicability is similar. The difference is largely a matter of point of view: extrinsic when arguing with generic extensions, and intrinsic when arguing in a generic extension.

(8.103) **Theorem** [S] *Suppose ψ and ϕ are* s*-formulas with $n+2$ and n free variables, respectively. Suppose* $\Theta \vdash$

(8.104) $\ulcorner$*for all $p \in |\mathsf{P}|$ and $x_0, \ldots, x_{n^-} \in \mathsf{V}^{\mathsf{P}}$, if $(\psi^{\mathsf{V}})(\mathsf{P}, p, x_0, \ldots, x_{n^-})$ then $p \in \mathsf{G} \rightarrow (\phi)(x_0^{\mathsf{G}}, \ldots, x_{n^-}^{\mathsf{G}})$*$\urcorner$.

Then $\mathsf{ZF} \vdash$

(8.105) $\ulcorner$*if $\mathbb{P}$ is a partial order, $p \in |\mathbb{P}|$, $x_0, \ldots, x_{n^-} \in V^{\mathbb{P}}$, and $(\psi)(\mathbb{P}, p, x_0, \ldots, x_{n^-})$, then $(\phi^{\Vdash})(\mathbb{P}, p, x_0, \ldots, x_{n^-})$*$\urcorner$.

Proof Let ψ and ϕ be given, and let T be a finite subset of Θ such that T proves (8.104). Let F be a finite subset of ZF as in (8.102). We now show how to construct a proof of (8.105) in ZF (without being too fussy about use vs. mention). We begin by supposing toward a contradiction that it is not the case. We use the reflection principle,[6.9] appropriately formulated in ZF, to show that there is an ordinal α such that $V_\alpha \models F$ and there exist $\mathbb{P}', p', x'_0, \ldots, x'_{n^-} \in V_\alpha^{\mathbb{P}}$ such that (8.105) fails. Take a countable elementary substructure of V_α containing these elements and collapse it to a transitive set M. Let $\mathbb{P}$, etc., be the images of $\mathbb{P}'$, etc., under the collapsing map. Then M is a countable transitive model of F, with a partial order $\mathbb{P} \in M$, $p \in |\mathbb{P}|$, and $x_0, \ldots, x_{n^-} \in M^{\mathbb{P}}$, such that $M \models \psi[\mathbb{P}, p, x_0, \ldots, x_{n^-}]$, and $p \nVdash^{M,\mathbb{P}} \phi(x_0, \ldots, x_{n^-})$. We let G be an M-generic filter on $\mathbb{P}$ such that $p \in G$ and $M[G] \not\models \phi[x_0^G, \ldots, x_{n^-}^G]$. Since $M \models F$, $M[G] \models T$. It follows that $M[G] \models \phi[x_0^G, \ldots, x_{n^-}^G]$, a contradiction.[31] □[8.105]

As in the method of "arguing with generic extensions", Θ implements the hypothesis that V is a model of ZF by positing each axiom of ZF relativized to V.[8.49] In a pure set theory we have no other option, as ZF is not finitely axiomatizable, and proper classes do not exist. In GB, however, we have generally implemented the hypothesis that M is a model of ZF as the single sentence $\ulcorner M \models \mathsf{ZF}\urcorner$, which leads to the following formulation of the above method.

[31] In effect, we justify the method of arguing in a hypothetical generic extension of V by arguing in an actual generic extension of a countable transitive model of a finite fragment of ZF.

Let c^* be the signature c with additional constants V, P, and G; we also treat V as a unary predicate in the usual way.

(8.106) *Let Θ' be the c^*-theory which is Θ with the following changes:*

1′. GB.

3′. $\ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$.

When "arguing in a generic extension", we will naturally reason from Θ' rather than Θ. The question is whether Theorem 8.103 applies with Θ' in place of Θ.

In the absence of an affirmative answer to this question, the usefulness of $\ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$ as a premise in forcing arguments undertaken in GB is much diminished, as one must maintain a parallel development from $\ulcorner \mathsf{ZF}^{\mathsf{V}} \urcorner$ to use when deriving forcing relations by "arguing in a generic extension". Thus, the following theorem is a great convenience.[32]

(8.107) **Theorem** [S] *Suppose ψ and ϕ are s-formulas with $n+2$ and n free variables, respectively. Suppose* $\Theta' \vdash$

$\ulcorner$*for all* $p \in |\mathsf{P}|$ *and* $x_0, \ldots, x_{n^-} \in \mathsf{V}^{\mathsf{P}}$, *if* $(\psi^{\mathsf{V}})(\mathsf{P}, p, x_0, \ldots, x_{n^-})$ *then* $p \in \mathsf{G} \rightarrow (\phi)(x_0^{\mathsf{G}}, \ldots, x_{n^-}^{\mathsf{G}})\urcorner$.

Then $\mathsf{ZF} \vdash$

$\ulcorner$*if* $\mathbb{P}$ *is a partial order,* $p \in |\mathbb{P}|$, $x_0, \ldots, x_{n^-} \in V^{\mathbb{P}}$, *and* $(\psi)(\mathbb{P}, p, x_0, \ldots, x_{n^-})$, *then* $(\phi^{\Vdash})(\mathbb{P}, p, x_0, \ldots, x_{n^-})\urcorner$.

Proof This follows immediately from Theorem 8.103 and (8.108). $\square^{8.107}$

(8.108) **Theorem** [S] *Θ' is a conservative extension of Θ in the sense that for any s^*-sentence σ, if $\Theta' \vdash \sigma$ then $\Theta \vdash \sigma$.*

Proof See Note 10.27.

8.5.3 Generic extension preserves AC

(8.109) **Theorem** [GB] *Suppose M is a transitive model of ZFC and $\mathfrak{A}$ is an M-complete boolean algebra. Then $M^{\mathfrak{A}}$ is full,*[8.55] *i.e., for each $\mathcal{L}^{M,\mathfrak{A}}$-formula ϕ with the single free variable u, there exists $x \in M^{\mathfrak{A}}$ such that*

$$[\![\exists u\, \phi]\!] = [\![\phi(x)]\!].$$

Remark Note that for the first time we assume $M \models \mathsf{AC}$.

As in the case of (8.87) it is useful to consider this in terms of generic extensions. We are looking for a witness for $\exists u\, \phi$, i.e., $x \in M^{\mathfrak{A}}$ such that for any generic G, if $M \models \exists u\, \phi$ then $M[G] \models \phi(x)$. The first step is to obtain a sufficient set of local witnesses $\{x_\alpha \mid \alpha < \eta\}$, so that $[\![\exists u\, \phi]\!] = \bigvee_{\alpha<\eta} [\![\phi(x_\alpha)]\!]$. We then obtain a sequence $\langle a_\alpha \mid \alpha < \eta \rangle$[8.110] of disjoint elements of $|\mathfrak{A}|$ such that $a_\alpha \leqslant [\![\phi(x_\alpha)]\!]$ and $\bigvee_{\alpha<\eta} a_\alpha = \bigvee_{\alpha<\eta} [\![\phi(x_\alpha)]\!]$. Thus, if $a_\alpha \in G$ then $M[G] \models \phi(x_\alpha)$, and we can use (8.87) to obtain $x \in M^{\mathfrak{A}}$ such that if $a_\alpha \in G$ then $M[G] \models x = x_\alpha$, so $M[G] \models \phi(x)$. Since G is generic, if $\left(\bigvee_{\alpha<\eta} a_\alpha\right) \in G$ then for some $\alpha < \eta$, $a_\alpha \in G$. Since

[32] Like the notion of satisfaction for proper classes used in this book and the other theorems specific to it, Theorems 8.107 and 8.108 are due to the author.

$[\![\exists u\,\phi]\!] = \bigvee_{\alpha<\eta}[\![\phi(x_\alpha)]\!] = \bigvee_{\alpha<\eta} a_\alpha$, if $[\![\exists u\,\phi]\!] \in G$ then $a_\alpha \in G$ for some $\alpha < \eta$, so $M[G] \models \phi(x)$. In terms of boolean valuation, $[\![\exists u\,\phi]\!] \leqslant [\![\phi(x)]\!]$. Conversely (and this is true for any x) if $M[G] \models \phi(x)$ then $M[G] \models \exists u\,\phi$, so $[\![\phi(x)]\!] \leqslant [\![\exists u\,\phi]\!]$. Hence $[\![\exists u\,\phi]\!] = [\![\phi(x)]\!]$. All we do in the following proof is eliminate any reference to generic extensions.

Proof Using $\mathsf{Collection}$ in M, let $\kappa \in \mathrm{Ord} \cap M$ be such that

$$\forall a \in |\mathfrak{A}|\, (\exists x \in M^{\mathfrak{A}}\ [\![\phi(x)]\!] = a \rightarrow \exists x \in M^{\mathfrak{A}}_\kappa\ [\![\phi(x)]\!] = a).$$

Since $M \models \mathsf{AC}$, there is a wellordering of $M^{\mathfrak{A}}_\kappa$ in M. Use such a well ordering to construct in M a maximal (i.e., not extendible) sequence $\langle x_\alpha \mid \alpha < \eta\rangle$ of members of $M^{\mathfrak{A}}$ such that for all $\beta \in \eta$

$$[\![\phi(x_\beta)]\!] \nleqslant \bigvee_{\alpha<\beta}[\![\phi(x_\alpha)]\!].$$

For $\beta < \eta$ let

(8.110) $$a_\beta = [\![\phi(x_\beta)]\!] - \bigvee_{\alpha<\beta}[\![\phi(x_\alpha)]\!].$$

Then

$\langle a_\beta \mid \beta < \eta\rangle$ *is a sequence of nonzero incompatible elements of* $|\mathfrak{A}|$.

(8.111) **Claim** $\bigvee_{\alpha<\eta}[\![\phi(x_\alpha)]\!] = [\![\exists u\,\phi]\!]$.

Proof Since $\forall x \in M^{\mathfrak{A}}\ [\![\phi(x)]\!] \leqslant [\![\exists u\,\phi]\!]$, if $\bigvee_{\alpha<\eta}[\![\phi(x_\alpha)]\!] \neq [\![\exists u\,\phi]\!]$ then $\bigvee_{\alpha<\eta}[\![\phi(x_\alpha)]\!] < [\![\exists u\,\phi]\!]$; hence,[8.67.10] for some $x \in M^{\mathfrak{A}}_\kappa$, $[\![\phi(x)]\!] \nleqslant \bigvee_{\alpha<\eta}[\![\phi(x_\alpha)]\!]$, contradicting the maximality of $\langle x_\alpha \mid \alpha < \eta\rangle$. □[8.111]

Using Theorem 8.87 let $x \in M^{\mathfrak{A}}$ be such that for all $\beta < \eta$

$$a_\beta \leqslant [\![x_\beta = x]\!].$$

Since $a_\beta \leqslant [\![\phi(x_\beta)]\!]$,[8.110] for every $\beta < \eta$,[8.74.1]

$$a_\beta \leqslant [\![x_\beta = x]\!] \wedge [\![\phi(x_\beta)]\!] \leqslant [\![\phi(x)]\!].$$

Hence,[8.111, 8.110]

$$[\![\exists u\,\phi]\!] = \bigvee_{\beta<\eta}[\![\phi(x_\beta)]\!] = \bigvee_{\beta<\eta} a_\beta \leqslant [\![\phi(x)]\!] \leqslant [\![\exists u\,\phi]\!],$$

so $[\![\exists u\,\phi]\!] = [\![\phi(x)]\!]$ as desired. □[8.109]

We give two proofs of the following theorem: the first applying the definitions of forcing/boolean value directly in the manner of the arguments given so far, which we may call "working in M", and the second "working in $M[G]$".[§ 8.5.2] The latter is much shorter than the former.[33]

(8.112) **Theorem** [GB] *Suppose M is transitive, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and $M \models \mathsf{ZF}$. If M also models AC then AC is an $M^{\mathfrak{A}}$-validity.*

Proof

[33] When "working in $M[G]$" we imagine ourselves in $M[G]$ and construct a proof in the theory Θ'. Formally, this proof must be set in the framework of the method of "arguing in a generic extension" in order to draw a conclusion about the forcing relation.

Working in M We will use the form of AC that states that for any set of non-empty sets there is a choice function. Suppose $x \in M^{\mathfrak{A}}$. We must show that

$$[\![\forall v \in x\, \exists w\; w \in v \to \exists f\, (\mathbf{Fcn}\, f \wedge \forall v \in x\, f(v) \in v)]\!] = \mathbf{1},$$

i.e.,

$$[\![\forall v \in x\, \exists w\; w \in v]\!] \leqslant [\![\exists f\, (\mathbf{Fcn}\, f \wedge \forall v \in x\, f(v) \in v)]\!].$$

We will show that there exists $F \in M^{\mathfrak{A}}$ such that

$$(8.113) \qquad [\![\forall v \in x\, \exists w\; w \in v]\!] \leqslant [\![\, \mathbf{Fcn}\, F \wedge \forall v \in x\, \exists w\, (w \in v \wedge (v, w)^{\mathfrak{A}} \in F)]\!].$$ [34]

The strategy is to use AC^M first to wellorder $\operatorname{dom} x$ so that we may recursively redefine $x(y)$ for $y \in \operatorname{dom} x$ so as to ensure that each member of x is (in an $\mathfrak{A}$-valued sense) listed only once. Then we use AC^M to show (via (8.109)) that for each $y \in \operatorname{dom} x$ there exists z whose $\mathfrak{A}$-value to be in y is $[\![\exists v\; v \in y]\!]$, and we use AC^M to pick such a z for each $y \in \operatorname{dom} x$. We use these objects to define F.

Using AC in M, let $\langle y_\alpha \mid \alpha < \eta\rangle \in M$ enumerate $\operatorname{dom} x$. Define recursively

$$(8.114) \qquad a_\alpha = x(y_\alpha) - \bigvee_{\beta<\alpha} (a_\beta \wedge [\![y_\alpha = y_\beta]\!]).$$

Let $x' \in M^{\mathfrak{A}}$ be such that $\operatorname{dom} x' = \operatorname{dom} x$ and for all $\alpha < \eta$

$$x'(y_\alpha) = a_\alpha.$$

Since $x'(y_\alpha) = a_\alpha \leqslant x(y_\alpha)$ for all $\alpha < \eta$, $[\![x' \subseteq x]\!] = \mathbf{1}$. Also,

$$\begin{aligned}
[\![x \subseteq x']\!] &= \bigwedge_{\alpha<\eta} (x(y_\alpha) \to [\![y_\alpha \in x']\!]) \\
&= \bigwedge_{\alpha<\eta} \Big(x(y_\alpha) \to \bigvee_{\beta<\eta} (x'(y_\beta) \wedge [\![y_\beta = y_\alpha]\!])\Big) \\
&= \mathbf{1},
\end{aligned}$$

because for any $\alpha < \eta$

$$\begin{aligned}
\bigvee_{\beta\leqslant\alpha} (x'(y_\beta) \wedge [\![y_\beta = y_\alpha]\!]) &= a_\alpha \vee \bigvee_{\beta<\alpha} (a_\beta \wedge [\![y_\beta = y_\alpha]\!]) \\
&\geqslant x(y_\alpha).
\end{aligned}$$

Thus,

$$(8.115) \qquad [\![x = x']\!] = \mathbf{1}.$$

Using (8.109) and AC^M, we infer that there exists $\langle z_\alpha \mid \alpha < \eta\rangle \in M$ such that for each $\alpha < \eta$,

$$[\![z_\alpha \in y_\alpha]\!] = [\![\exists w\; w \in y_\alpha]\!].$$

Let

$$F = \{((y_\alpha, z_\alpha)^{\mathfrak{A}}, a_\alpha) \mid \alpha < \eta\}.$$

[34] Note that by Theorem 8.109 the preceding formula implies the existence of such an F.

(8.116) **Claim** $[\![\,\mathbf{Fcn}\,F]\!] = \mathbf{1}$.

Proof We first observe that for any $s \in M^{\mathfrak{A}}$

$$\begin{aligned}[\![s \in F]\!] &= \bigvee_{\alpha<\eta} \big(a_\alpha \wedge [\![s = (y_\alpha, z_\alpha)^{\mathfrak{A}}]\!]\big)\\ &\leqslant \bigvee_{\alpha<\eta} \big(a_\alpha \wedge [\![\ulcorner (s) \text{ is an ordered pair}\urcorner]\!]\big)\\ &\leqslant [\![\ulcorner (s) \text{ is an ordered pair}\urcorner]\!],\end{aligned}$$

so $[\![\ulcorner\text{every member of } (F) \text{ is an ordered pair}\urcorner]\!] = \mathbf{1}$. Moreover, for any $s, t \in M^{\mathfrak{A}}$,

$$\begin{aligned}&[\![\ulcorner s \text{ and } t \text{ are in } (F) \text{ and have the same first item}\urcorner]\!]\\ &\quad= \bigvee_{\alpha,\beta<\eta} \big(a_\alpha \wedge a_\beta \wedge [\![s = (y_\alpha, z_\alpha)^{\mathfrak{A}}]\!] \wedge [\![t = (y_\beta, z_\beta)^{\mathfrak{A}}]\!] \wedge [\![y_\alpha = y_\beta]\!]\big)\\ &\quad= \bigvee_{\alpha<\eta} \big(a_\alpha \wedge [\![s = (y_\alpha, z_\alpha)^{\mathfrak{A}}]\!] \wedge [\![t = (y_\alpha, z_\alpha)^{\mathfrak{A}}]\!]\big)\\ &\quad\leqslant \bigvee_{\alpha<\eta} \big(a_\alpha \wedge [\![s = t]\!]\big)\\ &\quad\leqslant [\![s = t]\!],\end{aligned}$$

since[8.114] $\beta < \alpha \rightarrow a_\alpha \wedge a_\beta \wedge [\![y_\alpha = y_\beta]\!] = \mathbf{0}$. It follows that $[\![\ulcorner$every member of (F) is an ordered pair, and any members of (F) with the same first item are identical$\urcorner]\!] = \mathbf{1}$, i.e., $[\![\,\mathbf{Fcn}\,F]\!] = \mathbf{1}$. □[8.116]

Since $[\![x = x']\!] = \mathbf{1}$,[8.115] it suffices to show that

$$[\![\forall v \in x' \, \exists w \ w \in v]\!] \leqslant [\![\forall v \in x' \, \exists w \, (w \in v \wedge (v, w)^{\mathfrak{A}} \in F)]\!],$$ [8.113]

which the following computation does:

$$\begin{aligned}[\![\forall v \in x' \, \exists w \, (w \in v \wedge (v, w)^{\mathfrak{A}} \in F)]\!] &= \bigwedge_{\alpha<\eta} \Big(x'(y_\alpha) \rightarrow \bigvee_{z \in M^{\mathfrak{A}}} [\![z \in y_\alpha \wedge (y_\alpha, z)^{\mathfrak{A}} \in F)]\!]\Big)\\ &\geqslant \bigwedge_{\alpha<\eta} \big(a_\alpha \rightarrow [\![z_\alpha \in y_\alpha \wedge (y_\alpha, z_\alpha)^{\mathfrak{A}} \in F]\!]\big)\\ &\geqslant \bigwedge_{\alpha<\eta} \big(a_\alpha \rightarrow ([\![\exists w \ w \in y_\alpha]\!] \wedge a_\alpha)\big)\\ &= \bigwedge_{\alpha<\eta} \big(a_\alpha \rightarrow [\![\exists w \ w \in y_\alpha]\!]\big)\\ &= [\![\forall v \in x' \, \exists w \ w \in v]\!].\end{aligned}$$

Working in $M[G]$ For the sake of variety, we will present the argument in terms of forcing, as opposed to boolean value. Since we know that $M[G] \models \mathsf{ZF}$, we may use any of the many equivalent formulations of AC over ZF. We will use $\ulcorner$every set is the surjective image of an ordinal$\urcorner$.[35]

Suppose $x \in M[G]$. Let $\rho \in M^{\mathbb{P}}$ be such that $x = \rho^G$. Using AC in M, let $\langle \sigma_\alpha \mid \alpha < \eta \rangle \in M$ enumerate $\operatorname{dom} \rho$. Let $f(\alpha) = \sigma_\alpha^G$ for $\alpha < \eta$. Then $f : \eta \overset{\text{sur}}{\rightarrow} x$.

[35] If $f : \eta \overset{\text{sur}}{\rightarrow} x$, we may use the fact that η is wellordered to define a bijection of a subset of η with x, and us this to define a wellordering of x.

To apply Theorem 8.103 explicitly, we observe that the preceding argument amounts to a proof of AC in the theory Θ with the added assumption of AC^{V}. Thus it is a theorem of ZFC that for any partial order $\mathbb{P}$ and any $p \in |\mathbb{P}|$, $p \Vdash^{\mathbb{P}} \mathsf{AC}$, i.e., AC is a $\mathbb{P}$-validity. $\square$[8.112]

8.5.4 Arguing in a boolean-valued universe

The preceding discussion has illustrated the legitimacy and usefulness of reasoning as though a generic extension of V exists, even though it doesn't. We may achieve the same advantage somewhat differently by working in the corresponding boolean-valued universe, which does exist. Of course, nothing is free, and when we "argue in $V^{\mathfrak{A}}$", where $\mathfrak{A}$ is a (V-)complete boolean algebra, we must use $\mathfrak{A}$-valued logic. For $\mathfrak{A}$-validities and their negations, this reduces to ordinary 2-valued logic, and when $V^{\mathfrak{A}}$ is full[8.55] we also have witnesses in $V^{\mathfrak{A}}$ for valid existential sentences, so the correspondence with ordinary logic is even closer. Recall[8.109] that if $V \models \mathsf{AC}$ then $V^{\mathfrak{A}}$ is full. Clearly, working in $V^{\mathfrak{A}}$ is really the same as working in V with the $\mathfrak{A}$-value operation,[36] which is the same as working in V with the $\mathfrak{A}$-forcing relation, but the point of view is sometimes advantageous.

It is standard in discussions of forcing to use whichever of the above techniques is best suited to the task at hand, and with practice one can become quite nimble at jumping from one possible world to another.

8.6 Relative consistency proofs

Suppose σ is an s-sentence and we wish to use forcing to prove $\mathrm{Con}(\mathsf{ZF}+\sigma)$, which is the statement that $\mathsf{ZF}+\sigma$ is consistent. We must assume at least the consistency of ZF, which, according to Gödel's second incompleteness theorem, is not provable in ZF (assuming, of course, that ZF is consistent). We therefore typically formulate the result in terms of *relative consistency*, i.e., 'if $\mathrm{Con}\,\mathsf{ZF}$ then $\mathrm{Con}(\mathsf{ZF}+\sigma)$', which is a finitary statement, for which we wish to give a finitary proof. More generally, we wish to prove results of the form 'if $\mathrm{Con}(\mathsf{ZF}+\theta)$ then $\mathrm{Con}(\mathsf{ZF}+\sigma)$'. It may be that $\mathrm{Con}(\mathsf{ZF}+\theta)$ follows from $\mathrm{Con}\,\mathsf{ZF}$—this is the case, for example, if θ is $\ulcorner V = L \urcorner$—but this is exceptional, and in practice most relative consistency results are between theories that are not provably consistent relative to ZF.

We will describe several methods of establishing such a result, which are really just different descriptions of a single method. It is worth noting before proceeding that while relative consistency proofs were the initial motivation for the concept of forcing, its applications in set theory go far beyond that. In this regard it is in a similar position to that of the concept of constructibility.

The starting point is always the choice of a suitable partial order or, equivalently, complete boolean algebra. The first method described deals directly with the forcing relation/boolean valuation, while the second and third methods accomplish the same goal via the techniques described above under the rubrics "arguing with generic extensions" and "arguing in a generic extension".

Method 1 The argument can be framed in terms of partial orders or boolean algebras. We will use boolean algebras here. Suppose $\mathsf{ZF}+\theta$ is consistent and $\mathsf{ZF}+\theta \vdash$ $\ulcorner$there exists a complete boolean algebra $\mathfrak{A}$ such that $(\sigma)^{[\![\,]\!]}(\mathfrak{A}, \mathbf{1})\urcorner$. We

[36] In our first proof of (8.112) we were more or less working in $M^{\mathfrak{A}}$.

will show that $\mathsf{ZF} + \sigma$ is consistent. Suppose toward a contradiction that $\mathsf{ZF} + \sigma$ is inconsistent, i.e., $\mathsf{ZF} \vdash \neg\,\sigma$. Then[8.97.2] $\mathsf{ZF} \vdash \ulcorner$for every complete boolean algebra $\mathfrak{A}$, $((\neg\,\sigma)^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})\urcorner$. By definition,[8.76.4] $(\neg\,\sigma)^{[\![\,]\!]}(\mathfrak{A}, a) \overset{\text{def}}{=} \ulcorner(\sigma^{[\![\,]\!]})(\mathfrak{A}, \neg\, a)\urcorner$. Hence, $\mathsf{ZF} + \theta \vdash \ulcorner$there exists a complete boolean algebra $\mathfrak{A}$ such that $(\sigma^{[\![\,]\!]})(\mathfrak{A}, \mathbf{1})$ and $(\sigma^{[\![\,]\!]})(\mathfrak{A}, \mathbf{0})\urcorner$, which implies $\mathsf{ZF} + \theta$ is inconsistent, since $\mathsf{ZF} \vdash \ulcorner$for any complete boolean algebra $\mathfrak{A}$ there exists a unique $a \in |\mathfrak{A}|$ such that $(\sigma^{[\![\,]\!]})(\mathfrak{A}, a)\urcorner$.

Method 2: "Arguing with generic extensions" Suppose $\mathsf{ZF} + \theta \vdash \ulcorner$there exists a partial order $\mathbb{P}$ such that $\Vdash^{\mathbb{P}} (\sigma)\urcorner$(by which we mean, of course, $\ulcorner(\sigma^{\Vdash})(\mathbb{P}, \mathbf{1})\urcorner$, where we have assumed for convenience that $\mathbb{P}$ has maximum element $\mathbf{1}$). Suppose there is a transitive set model of $\mathsf{ZF} + \theta$. Then there is a countable transitive model M of $\mathsf{ZF} + \theta$. Suppose $\mathbb{P} \in M$ is a partial order such that $\Vdash^{\mathbb{P}} \sigma$. Since M is countable, there exists an M-generic filter G on $\mathbb{P}$, and $\mathfrak{M}[G] \models (\mathsf{ZF} + \sigma)$. Hence there is a model of $\mathsf{ZF} + \sigma$, so $\mathsf{ZF} + \sigma$ is consistent.

This approach is less than satisfactory, in that it assumes not just that $\mathsf{ZF} + \theta$ has a set model, which by the completeness theorem follows from the hypothesis of consistency of $\mathsf{ZF} + \theta$, but that it has a transitive model, which does not follow (in $\mathsf{ZF} + \theta$) from the hypothesis of consistency of $\mathsf{ZF} + \theta$, unless $\mathsf{ZF} + \theta$ is inconsistent, as can easily be shown. The following improvement uses only the consistency of $\mathsf{ZF} + \theta$.

Suppose $\mathsf{ZF} + \sigma$ is inconsistent. Then by the compactness theorem, there is a finite set F of axioms of ZF such that $F + \sigma$ is inconsistent. Let ζ be the conjunction of F. Let F' be a finite subset of $\mathsf{ZF} + \theta$ such that $F' \vdash \ulcorner$there exists a partial order $\mathbb{P}$ such that $\Vdash^{\mathbb{P}} (\sigma \wedge \zeta)\urcorner$.

By the reflection principle, $\mathsf{ZF} + \theta$ proves that there is a countable transitive model of F'. The following is therefore a proof in $\mathsf{ZF} + \theta$: $\ulcorner$Let M be a countable transitive model of F'. Let $\mathbb{P} \in M$ be such that $\Vdash^{\mathbb{P}} \sigma$, and let G be an M-generic filter on $\mathbb{P}$. Then $M[G] \models (F + \sigma)$. But $F + \sigma$ is inconsistent, so $M[G] \not\models (F + \sigma)$.$\urcorner$

Hence $\mathsf{ZF} + \theta$ is inconsistent.

Method 3: "Arguing in a generic extension" We will use the terminology of Section 8.5.2. V, P, and G are respectively a unary predicate index and two constant indices. We define extensions of the signature s of pure set theory by adding V to obtain s^{V}, adding P to obtain s^{P}, and adding V, P and G to obtain s^*. Suppose θ is an s^{P}-sentence and $\mathsf{ZF} + \theta \vdash \ulcorner\mathsf{P}$ is a partial order$\urcorner$.[37] It is clear from the discussion in Section 8.5.2—Theorem 8.98 in particular—that every axiom of the theory Θ[8.101] is a P-validity provably in $\mathsf{ZF} + \theta$. $\mathsf{ZF} + \theta$ also proves θ^{V}. Thus, if $\mathsf{ZF} + \theta$ is consistent then $\Theta + \theta^{\mathsf{V}}$ is consistent. It follows from (8.108) that $\Theta' + \theta^{\mathsf{V}}$ is consistent. Suppose $\Theta' + \theta^{\mathsf{V}} \vdash \sigma$. Then $\Theta' + \sigma$ is consistent. Since $\mathsf{GB} \subseteq \Theta'$, $\mathsf{GB} + \sigma$ is consistent, so $\mathsf{ZF} + \sigma$ is consistent.

To summarize this method, working in GB from the supposition that the universe is $\mathsf{V}[\mathsf{G}]$, where V is an inner model of $\mathsf{ZF} + \theta$, and G is a V-generic filter on P, we prove σ. It then follows that if $\mathsf{ZF} + \theta$ is consistent then $\mathsf{ZF} + \sigma$ is consistent.[38]

[37] In general, θ says more about P than merely that it is a partial order; often it uniquely defines P. θ may also say things that have nothing to do with P directly. For example, θ might say that $V = L$ or that an inaccessible cardinal exists.

[38] We have, of course, shown that $\mathsf{GB} + \sigma$ is consistent, but this is not any stronger, since GB is a conservative extension of ZF.

8.6.1 Con(ZF + $V \neq L$)

For our first relative consistency result using forcing we will settle the question with which we began this chapter by showing that the axiom of constructibility is not a theorem of ZF. Suppose M is a transitive model of ZF and $\mathbb{P} \in M$ is a partial order with the property that

$$\forall p \in |\mathbb{P}|\, \exists q, q' \leqslant p\, (q \perp q'), \tag{8.117}$$

i.e., any element of $|\mathbb{P}|$ has incompatible extensions.

Claim *Suppose G is M-generic on $\mathbb{P}$. Then $G \notin M$.*

In other words, if $F \in M$ is a filter on $\mathbb{P}$ then F is not M-generic. To prove this, we observe that by virtue of (8.117), any $p \in |\mathbb{P}|$ has an extension that is not in F (since members of a filter are compatible by definition). Hence $D = |\mathbb{P}| \backslash F$ is dense. Since $F \in M$ and $|\mathbb{P}| \in M$, $D \in M$, and F does not meet D, so F is not M-generic. Since L is included in any model of ZF that contains all the ordinals, $L^{\mathfrak{M}[G]} \subseteq M$, so $\mathfrak{M}[G] \models \ulcorner \mathsf{G} \notin L \urcorner$.

For the sake of definiteness, we may use the following partial order $\mathbb{P} = (|\mathbb{P}|; \leqslant)$:

(8.118)

1. $p \in |\mathbb{P}|$ *iff*
    1. p *is finite;*
    2. p *is a function;*
    3. $\operatorname{dom} p \subseteq \omega$*;*
    4. $\operatorname{im} p \subseteq 2$ $(= \{0,1\})$.
2. *For* $p, q \in |\mathbb{P}|$
$$q \leqslant p \leftrightarrow q \supseteq p.$$

Using any of the methods outlined above,[§ 8.6] we may construct a finitary proof of the following theorem.

(8.119) **Theorem** [S] *If* ZF *is consistent then so is* ZF + $\ulcorner V \neq L \urcorner$.

Note that if G is a filter on $\mathbb{P}$ and $p, p' \in G$, then p and p' are compatible, so $p \cup p' \in |\mathbb{P}|$. If fact, since $p \cup p'$ is the weakest (i.e., highest, i.e., smallest) condition that extends both p and p', $p \cup p' \in G$. Let $x^G = \bigcup G$. Then x^G is a function, $\operatorname{dom} x^G \subseteq \omega$, and $\operatorname{im} x^G \subseteq 2$. For any $n \in \omega$ the set $\{p \in |\mathbb{P}| \mid n \in \operatorname{dom} p\}$ is dense in $\mathbb{P}$, so $n \in \operatorname{dom} x^G$. Thus $\operatorname{dom} x^G = \omega$. There is a simple bijection between sets $S \subseteq \omega$ and functions $\chi^S : \omega \to 2$:

$$\chi^S(n) = 1 \leftrightarrow n \in S.$$

So G is equivalent to a subset of ω.

Because of their inaugural role in the presentation of Cohen's method of forcing, $\mathbb{P}^{8.118}$ *and its regular algebra* $\mathfrak{R}\,\mathbb{P}$ *are often referred to as the* Cohen order *and* Cohen algebra*. Note that by virtue of (8.62) any atomless complete boolean algebra with a countable dense set is isomorphic to the Cohen algebra.*

8.7 Chain conditions and saturation

Definition [GBC] *Suppose* $\mathbb{P}$ *is a partial order. An* antichain *in* $\mathbb{P}$ *is a set of pairwise incompatible elements of* $|\mathbb{P}|$*. Suppose* κ *is a cardinal.* $\mathbb{P}$ *satisfies (or has) the* κ-chain condition $\overset{\text{def}}{\Longleftrightarrow}$ *any antichain in* $\mathbb{P}$ *has size* $< \kappa$*. We also say that* $\mathbb{P}$ *is* κ-cc*. The* countable chain condition (ccc) *is the* ω_1*-chain condition, and we also say that a partial order satisfying this condition is* ccc*.*

For boolean algebras $\mathfrak{A}$, these notions are defined with respect to the partial order $\mathfrak{A}^+$ of nonzero elements of $\mathfrak{A}$.[39]

(8.120) **Theorem** [ZFC] *Suppose* $\mathbb{P}$ *is a separative partial order, and* $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$*. Then for any cardinal* κ*,* $\mathbb{P}$ *is* κ*-cc iff* $\mathfrak{A}$ *is* κ*-cc.*

Proof $\mathbb{P}$ is naturally isomorphic to a dense set in $\mathfrak{A}$, so an antichain in $\mathbb{P}$ corresponds to an antichain in $\mathfrak{A}$, and, given an antichain in $\mathfrak{A}$, using AC, there exists an antichain of the same size in $\mathbb{P}$. $\square$[8.120]

Chain conditions in boolean algebras are often stated in terms of the equivalent concept of *saturation.*

(8.121) **Definition** [GBC] *Suppose* $\mathfrak{B}$ *is a boolean algebra and* κ *is a cardinal.*

1. $\mathfrak{A}$ *is* κ-saturated $\overset{\text{def}}{\Longleftrightarrow}$ $\mathfrak{A}$ *has the* κ*-chain condition.*
2. $\operatorname{sat}\mathfrak{A} \overset{\text{def}}{=}$ *the least cardinal* κ *such that* $\mathfrak{A}$ *is* κ*-saturated.*

The following theorem shows that for most purposes it suffices to consider κ-chain conditions for uncountable regular cardinals κ.

(8.122) **Theorem** [ZFC] *Suppose* $\mathfrak{A}$ *is an infinite complete boolean algebra. Then* $\operatorname{sat}\mathfrak{A}$ *is a regular uncountable cardinal.*

Proof Let $\kappa = \operatorname{sat}\mathfrak{A}$. We will first show that κ is uncountable by using a well-ordering $\prec$ of $|\mathfrak{A}|$ to define a strictly descending ω-sequence $\langle a_n \mid n \in \omega\rangle$ of elements of $\mathfrak{A}$. The construction will be such that for each $n \in \omega$, there are infinitely many elements of $|\mathfrak{A}|$ below a_n. Let $a_0 = \mathbf{1}$. Given a_n with infinitely many elements below it, let a be the $\prec$-first $a \in |\mathfrak{A}|$ such that $\mathbf{0} < a < a_n$. If a has infinitely elements below it, let $a_{n+1} = a$; otherwise, $a_n - a$ necessarily has infinitely many elements below it, and we let $a_{n+1} = a_n - a$. $\{a_n - a_{n+1} \mid n \in \omega\}$ is an infinite antichain in $\mathfrak{A}$.

To show that κ is regular, suppose toward a contradiction that it is singular. For $a \in |\mathfrak{A}^+|$, let $\mathfrak{A}_a$ be the boolean algebra such that $|\mathfrak{A}_a| = \{b \in |\mathfrak{A}| \mid b \leqslant a\}$, with the operations and order relation inherited from $\mathfrak{A}$. Let $s(a) = \operatorname{sat}\mathfrak{A}_a$. Note that if $b \in |\mathfrak{A}_a^+|$ then $s(b) \leqslant s(a)$. We will say that a is *stable* $\overset{\text{def}}{\Longleftrightarrow}$ $\forall b \in |\mathfrak{A}_a^+|\ s(b) = s(a)$. Given $a \in |\mathfrak{A}^+|$, there exists $b \in |\mathfrak{A}_a^+|$ such that $s(b)$ is least, and any such b is stable. Hence the set S of stable elements of $\mathfrak{A}$ is dense. Let $A \subseteq S$ be a maximal antichain

[39] Note that a *chain condition* is defined in terms of antichains. In the case of a complete boolean algebra $\mathfrak{A}$, antichains are closely related to chains, as follows. Suppose $\langle a_\alpha \mid \alpha \in \lambda\rangle$ is a λ-sequence of pairwise incompatible (disjoint) elements of $|\mathfrak{A}|$. Let $b_\alpha = \bigcup_{\beta<\alpha} a_\beta$. Then $b_0 < b_1 < \cdots < b_\alpha < \cdots$, so $\langle b_\alpha \mid \alpha \in \lambda\rangle$ is a chain. Conversely, given a chain in $\mathfrak{A}$ in which every element has an immediate successor, the set of differences between consecutive elements is an antichain.

in S. Since S is dense, A is a maximal antichain in $\mathfrak{A}$, so $\bigvee A = \mathbf{1}$. We say that A is a *partition of* $\mathfrak{A}$.

By hypothesis, $|A| < \kappa$. Since all successor cardinals are regular, κ is a limit cardinal, and for all cardinals $\lambda < \kappa$ there exists an antichain in $\mathfrak{A}$ of size λ. Suppose λ is a cardinal and $|A| \leqslant \lambda < \kappa$. Then $\lambda^+ < \kappa$. Let B be an antichain in $\mathfrak{A}$ of size λ^+. For each $a \in A$ let $B_a = \{b \wedge a \mid b \in B \text{ and } b \wedge a \neq \mathbf{0}\}$. Let $B' = \bigcup_{a \in A} B_a$. Since A and B are antichains, B' is an antichain. Since $\bigvee A = \mathbf{1}$, for each $b \in B$, for some $a \in A$, $b \wedge a \neq \mathbf{0}$. Thus, there exists an injection of B into B', and therefore $|B'| \geqslant \lambda^+$. It follows that for some $a \in A$, $|B_a| \geqslant \lambda^+$. Since B_a is an antichain in $\mathfrak{A}_a$, $s(a) > \lambda^+$.

Thus $\{s(a) \mid a \in A\}$ is unbounded in κ. Let $\langle \lambda_\alpha \mid \alpha < \operatorname{cf} \kappa \rangle$ be an increasing sequence of cardinals $< \kappa$ with limit κ. For each $\alpha < \operatorname{cf} \kappa$ let $a_\alpha \in A$ be such that $s(a_\alpha) \geqslant \lambda_\alpha^+$ and $\forall \beta < \alpha \; s(a_\alpha) > s(a_\beta)$. Note that the second condition guarantees that $\langle a_\alpha \mid \alpha < \operatorname{cf} \kappa \rangle$ is injective. For each $\alpha < \operatorname{cf} \kappa$ let X_α be an antichain in $\mathfrak{A}_{a_\alpha}$ of size λ_α. Then $\bigcup_{\alpha < \operatorname{cf} \kappa} X_\alpha$ is an antichain in $\mathfrak{A}$ of size κ; contradiction. $\square$[8.122]

The following theorem gives one of the most important consequences of a chain condition for the theory of forcing.

(8.123) **Theorem** [GB] *Suppose*

1. *M is a transitive model of* ZFC*;*
2. *$\kappa, \lambda, \mathbb{P} \in M$;*
3. *$M \models \ulcorner [\kappa]$ and $[\lambda]$ are cardinals, $[\kappa]$ is regular, and $[\mathbb{P}]$ is a partial order with the $[\kappa]$-chain condition$\urcorner$;*
4. *G is an M-generic filter on $\mathbb{P}$;*
5. *$f \in M[G]$; and*
6. *$f : \lambda \to M$.*

Then there exists $X \in M$ such that $\operatorname{im} f \subseteq X$ *and*

1. *if $\lambda < \kappa$ then $|X|^M < \kappa$, and*
2. *if $\lambda \geqslant \kappa$ then $|X|^M \leqslant \lambda$,*

where $|X|^M$ is the cardinality of X in the sense of M, i.e., $\ulcorner |[X]| \urcorner^M$.

Proof Let $\dot{f} \in M^{\mathbb{P}}$ be such that $f = \dot{f}^G$, and let $p \in G$ be such that $p \Vdash \ulcorner \dot{f}$ is a function from $\check{\lambda}$ into $\mathsf{V}\urcorner$.

Working in M, it is straightforward to show that

(8.124) *for each $\alpha < \lambda$, the set of conditions $r \leqslant p$ such that for some $x \in M$, $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$, is dense below p.*

Suppose $\alpha < \lambda$, and consider sets A such that

(8.125)

1. *A is a binary relation;*
2. *for each $\langle r, x \rangle \in A$, $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$;*
3. *$\langle r, x \rangle \in A \wedge \langle r', x' \rangle \in A \to r \perp r'$.*

Note that by the κ-chain condition for $\mathbb{P}$, (8.126) *if A satisfies (8.125) then $|A| < \kappa$.*

If $\mathcal{A}$ is a chain[40] of sets A satisfying (8.125) then clearly $\bigcup \mathcal{A}$ satisfies (8.125). It follows by Zorn's lemma that there is a maximal such set.

Suppose A is maximal satisfying (8.125). Then $\operatorname{dom} A$, i.e., $\{r \mid \exists x\, \langle r, x\rangle \in A\}$, is predense below p. To show this, suppose suppose toward a contradiction that $\operatorname{dom} A$ is not predense below p. Let $q \leqslant p$ be incompatible with everything in $\operatorname{dom} A$. Let[8.124] $r \leqslant q$ and x be such that $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$ (still working in M). Then $\langle r, x\rangle \notin A$ and $A \cup \{\langle r, x\rangle\}$ satisfies (8.125), a contradiction.

Still supposing A is maximal satisfying (8.125), and stepping into $M[G]$ momentarily, let $x = f(\alpha) \in M$. Let $q \leqslant p$, $q \in G$, $q \Vdash \dot{f}(\check{\alpha}) = \check{x}$. Since $\operatorname{dom} A$ is predense below p, and $p \in G$, there exists $q' \in G \cap \operatorname{dom} A$. Let x' be such that $\langle q', x'\rangle \in A$. Let $r \in G$ extend both q and q'. Then $r \Vdash \dot{f}(\check{\alpha}) = \check{x}$ and $r \Vdash \dot{f}(\check{\alpha}) = \check{x}'$, so $x = x' \in \operatorname{im} A$. We have therefore shown that

$$f(\alpha) \in \operatorname{im} A.$$

Back in M, we now use the axiom of choice again to conclude that there exists $\langle A_\alpha \mid \alpha < \lambda\rangle$ such that for each $\alpha < \lambda$, A_α is maximal satisfying (8.125) for α. Let $X = \bigcup_{\alpha<\lambda} \operatorname{im} A_\alpha = \{x \mid \exists \alpha < \lambda\, \exists r\, \langle r, x\rangle \in A_\alpha\}$.

Since[8.126] $\forall \alpha < \lambda\ |A_\alpha| < \kappa$,

1. if $\lambda < \kappa$, since κ is regular, $|X| < \kappa$; and
2. if $\lambda \geqslant \kappa$, $|X| \leqslant \lambda$.

$\square$[8.123]

As a straightforward corollary, we have:

(8.127) **Theorem** [GB] *Suppose M is a transitive model of* ZFC*, $M \models \ulcorner [\kappa]$ is a regular cardinal and $[\mathbb{P}]$ is a partial order with the $[\kappa]$-chain condition$\urcorner$, and G is an M-generic filter on $\mathbb{P}$. Then*

1. *$M[G] \models \ulcorner [\kappa]$ is a regular cardinal$\urcorner$.*
2. *If $\lambda > \kappa$ and $M \models \ulcorner [\lambda]$ is a cardinal$\urcorner$, then $M[G] \models \ulcorner [\lambda]$ is a cardinal$\urcorner$.*

In particular, if $M \models \ulcorner [\mathbb{P}]$ has the countable (i.e., ω_1-)chain condition$\urcorner$*, then every cardinal in the sense of M is a cardinal in the sense of $M[G]$.*

Proof 1 Suppose $\eta < \kappa$ is a cardinal in $M[G]$, $f \in M[G]$, and $f : \eta \to \kappa$. By Theorem 8.123 there exists $X \in M$ such that $|X|^M < \kappa$ and $\operatorname{im} f \subseteq X$. Since $\ulcorner [\kappa]$ is regular$\urcorner^M$, $X \cap \kappa$ is not cofinal in κ, so $\operatorname{im} f$ is not cofinal in κ.

2 Suppose $\eta < \lambda$ is a cardinal in $M[G]$, $f \in M[G]$, and $f : \eta \to \lambda$. Then[8.123] there exists $X \in M$ such that $|X|^M \leqslant \max\{\eta, \kappa\} < \lambda$ and $\operatorname{im} f \subseteq X$. Since λ is a cardinal in M, $\lambda \nsubseteq X$, so f is not a surjection. $\square$[8.127]

The following combinatorial theorem of Shanin is useful.

(8.128) **Theorem** [ZFC] *Suppose X is an uncountable set of finite sets. There exists an uncountable $X' \subseteq X$ and a set d such that for all distinct $x, y \in X'$, $x \cap y = d$.*

[40] Here we use 'chain' to mean a set of sets linearly ordered by inclusion, as in Zorn's lemma.

Remark A set of finite sets with the property asserted for X' is called a Δ-*system*, and the present theorem is sometimes referred to as the Δ-*lemma*.

Proof It is enough to show that for any $n \in \omega$, the theorem holds for any uncountable set X of sets of size n, because for any uncountable set X of finite sets, for some $n \in \omega$, $\{x \in X \mid |x| = n\}$ is uncountable.

The proof is by induction on n. The result is trivial for $n = 1$ (in which case $d = 0$). Suppose it holds for n and suppose X is an uncountable set of sets of size $n + 1$.

Suppose first that there exists a such that $Y = \{x \in X \mid a \in x\}$ is uncountable. Let $Y' = \{x \backslash \{a\} \mid x \in Y\}$. Then Y' is an uncountable set of sets of size n, so by induction hypothesis there is an uncountable $Y' \subseteq Y$ and d' such that for all distinct $x, y \in Y'$, $x \cap y = d'$. Let $X' = \{y \cup \{a\} \mid y \in Y'\}$ and let $d = d' \cup \{a\}$. Then X' is an uncountable subset of X, and for all distinct $x, y \in X'$, $x \cap y = d$.

If there exists no such element a, we construct a sequence $\langle x_\alpha \mid \alpha \in \omega_1 \rangle$ of pairwise disjoint members of X, and an increasing sequence $\langle Y_\alpha \mid \alpha \in \omega_1 \rangle$ of countable subsets of X, such that

(8.129) *for all* $\alpha \in \omega_1$

1. $x_\alpha \notin Y_\alpha$; *and*
2. $\forall y \in X\, (y \cap x_\alpha \neq 0 \rightarrow y \in Y_{\alpha+1})$.

We do this recursively. At stage α, we first define Y_α and then let x_α be the first member of $X \backslash Y_\alpha$ according to some fixed wellordering of X.

1. If $\alpha = 0$ let $Y_\alpha = 0$. Thus, x_0 is simply a member of $X \backslash 0 = X$.
2. If $\operatorname{Lim} \alpha$ let $Y_\alpha = \bigcup_{\beta < \alpha} Y_\beta$.
3. If $\alpha = \beta + 1$ let $Y_\alpha = Y_\beta \cup \{y \in X \mid y \cap x_\beta \neq 0\}$.

Note that countability of Y_α is preserved at limit stages, and at successor stages, since x_β is finite and no member of x_β is in uncountably many members of X. Since X is uncountable, $X \backslash Y_\alpha$ is nonempty, so x_α may be chosen for each α.

Now suppose $\beta < \alpha < \alpha_1$. Then $x_\alpha \cap x_\beta = 0$; otherwise, $x_\alpha \in Y_{\beta+1} \subseteq Y_\alpha$,[8.129.2] which it is not.[8.129.1] $\{x_\alpha \mid \alpha \in \omega_1\}$ is therefore a Δ-system (with common intersection 0). $\square$[8.128]

8.8 Closure and distributivity

(8.130) **Definition** [ZFC] *Suppose* κ *is a cardinal.*

1. *A partial order* $\mathbb{P}$ *is* κ-closed $\overset{\text{def}}{\iff}$ *for every* $\lambda \leqslant \kappa$, *for every descending* λ-*chain* $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_\alpha \geqslant \cdots$ $(\alpha < \lambda)$, *there exists* $p \in |\mathbb{P}|$ *such that* $p \leqslant p_\alpha$ *for all* $\alpha < \lambda$.
2. *A partial order* $\mathbb{P}$ *is* κ-distributive $\overset{\text{def}}{\iff}$ *for every set* S *of open dense subsets of* $|\mathbb{P}|$, *with* $|S| \leqslant \kappa$, $\bigcap S \neq 0$. *Note that if* $\mathbb{P}$ *is* κ-*distributive then for every set* S *of open dense subsets of* $|\mathbb{P}|$, *with* $|S| \leqslant \kappa$, $\bigcap S$ *is actually dense (and of course open).*

3. *If $\mathfrak{A}$ is a complete boolean algebra, these definitions are made with reference to the partial order $\mathfrak{A}^+$.*

4. *A partial order is $<\kappa$-*closed (-distributive) $\overset{\text{def}}{\iff}$ *it is λ-closed (-distributive) for all $\lambda < \kappa$.*

Note 10.28 explains the use of the term 'distributivity' for this property in the context of a more general definition of distributivity for boolean algebras.

(8.131) **Theorem** [ZFC] *Suppose $\mathbb{P}$ is a partial order and κ is a cardinal. If $\mathbb{P}$ is κ-closed then $\mathbb{P}$ is κ-distributive.*

Proof Suppose $S = \{s_\alpha \mid \alpha < \kappa\}$ is a set of open dense subsets of $|\mathbb{P}|$. Using an appropriate choice function, we will define recursively $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_\alpha \geqslant \cdots (\alpha < \kappa)$, such that $\forall\alpha < \kappa\; p_\alpha \in s_\alpha$. For each $\alpha < \kappa$, supposing p_β has been chosen for every $\beta < \alpha$, since $\mathbb{P}$ is κ-closed, there exists p extending every p_β, $\beta < \alpha$. Since s_α is dense, there exists $p' \leqslant p$ such that $p' \in s_\alpha$, and we choose p_α to be some such p'.

Now let p extend p_α for all $\alpha < \kappa$. Since each s_α is open, $p \in s_\alpha$ for every $\alpha < \kappa$. Hence, $\mathbb{P}$ is κ-distributive. $\square$[8.131]

(8.132) **Theorem** [GB] *Suppose M is a transitive model of* ZFC*, $M \models \ulcorner [\kappa]$ is an infinite cardinal and $[\mathbb{P}]$ is a $[\kappa]$-distributive partial order$\urcorner$, and G is M-generic on $\mathbb{P}$. Suppose $f \in M[G]$ is a function from κ into M. Then $f \in M$.*

Proof Let $\dot{f} \in M^{\mathbb{P}}$ be such that $\dot{f}^G = f$. Let $p_0 \in G$ be such that

$$p_0 \Vdash \dot{f} : \check{\kappa} \to \mathsf{V}. \tag{8.133}$$

For each $\alpha < \kappa$ let

$$B_\alpha = \{p \leqslant p_0 \mid \exists x\; p \Vdash \dot{f}\check{\alpha} = \check{x}\}.$$

Then[8.133] each B_α is open and dense below p_0, and $\langle B_\alpha \mid \alpha < \kappa\rangle \in M$, so $B = \bigcap_{\alpha<\kappa} B_\alpha$ is open and dense below p_0, and $B \in M$.

Hence, there exists $p \in G \cap B$. Let $g : \kappa \to M$ be defined by the condition that $\forall\alpha < \kappa\; p \Vdash \dot{f}\check{\alpha} = \check{g\alpha}$. Then g is definable over M, so $g \in M$, and $\forall\alpha < \kappa\; p \Vdash \dot{f}\check{\alpha} = \check{g}\check{\alpha}$, so $p \Vdash \dot{f} = \check{g}$. Hence, $f = \dot{f}^G = \check{g}^G = g \in M$. $\square$[8.132]

(8.134) **Theorem** [GB] *Suppose M is a transitive model of* ZFC*, $M \models \ulcorner [\kappa]$ is an infinite cardinal and $[\mathbb{P}]$ is a $<[\kappa]$-distributive partial order$\urcorner$, and G is M-generic on $\mathbb{P}$. Then for all $\lambda \leqslant \kappa$, if λ is a cardinal in M then λ is a cardinal in $M[G]$ with the same cofinality.*

Proof Straightforward corollary of (8.132). $\square$[8.134]

Finally we present a combinatorial theorem useful in proving chain conditions.

(8.135) **Theorem** [ZFC] *Suppose κ is a regular cardinal such that $2^{<\kappa} = \kappa$. Let P consist of functions into κ of size less than κ such that*

$$\forall p, p' \in P\; \big(p \neq p' \to \exists x\, (x \in \operatorname{dom} p \cap \operatorname{dom} p' \wedge p(x) \neq p'(x))\big). \tag{8.136}$$

Then $|P| \leqslant \kappa$.

Proof Let $\bar{P} = \{p \mid \exists p' \in P\ p \subseteq p'\}$. We will construct a κ-sequence $0 = P_0 \subseteq P_1 \subseteq \cdots$ of subsets of P such that for all $\alpha < \kappa$, letting $B_\alpha = \bigcup_{p \in P_\alpha} \operatorname{dom} p$,

1. $|P_\alpha| \leqslant \kappa$;
2. $\forall p \in \bar{P}\left(\operatorname{dom} p \subseteq B_\alpha \rightarrow \left(\exists q \in P\,(p = q \upharpoonright B_\alpha) \rightarrow \exists q \in P_{\alpha+1}\,(p = q \upharpoonright B_\alpha)\right)\right)$;
3. if α is a limit ordinal then $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$;

and $P = \bigcup_{\alpha<\kappa} P_\alpha$.

Clearly Property 1 is maintained at limit stages. We will show that it can be maintained at successor stages. Suppose $|P_\alpha| \leqslant \kappa$. Then $|B_\alpha| \leqslant \kappa$. For any set S, let $[S]^{<\kappa} = \{X \subseteq S \mid |X| < \kappa\}$. Since κ is regular, any member of $[\kappa]^{<\kappa}$ is a subset of some $\lambda < \kappa$, so $[\kappa]^{<\kappa} = \bigcup_{\lambda<\kappa} \mathcal{P}\lambda$. By hypothesis, $\lambda < \kappa \rightarrow 2^\lambda \leqslant \kappa$, so $\left|[\kappa]^{<\kappa}\right| \leqslant \sum_{\lambda<\kappa} 2^\lambda \leqslant \kappa \cdot \kappa = \kappa$. Since $|B_\alpha| \leqslant \kappa$, $[B_\alpha]^{<\kappa} \leqslant \kappa$. For any $B \in [B_\alpha]^{<\kappa}$ there are $\kappa^{|B|}$ functions from B to κ. Since κ is regular, any such function f maps B into λ for some $\lambda < \kappa$, and $f \subseteq \{(d, \alpha) \mid d \in B \wedge \alpha \in \lambda\}$, so there are at most $2^{|B| \cdot \lambda} \leqslant \kappa$ such functions. It follows that $|\{p \in \bar{P} \mid \operatorname{dom} p \subseteq B_\alpha\}| \leqslant \kappa$, so no more than κ elements have to be added to P_α to form $P_{\alpha+1}$ so as to satisfy Property 2.

It remains to show that $P = \bigcup_{\alpha<\kappa} P_\alpha$. Suppose $p \in P$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. Since $|\operatorname{dom} p| < \kappa$ and κ is regular, for some $\alpha < \kappa$, $\operatorname{dom} p \cap B_\alpha = \operatorname{dom} p \cap B$. Let $p' = p \upharpoonright B_\alpha$. Then $p' \in \bar{P}$, so there exists $q \in P_{\alpha+1}$ such that $q \upharpoonright B_\alpha = p' = p \upharpoonright B_\alpha = p \upharpoonright B$. Since $\operatorname{dom} q \subseteq B_{\alpha+1} \subseteq B$, p and q agree on their common domain, so[8.136] $q = p$.

Hence $|P| = \sum_{\alpha<\kappa} |P_\alpha| \leqslant \kappa \cdot \kappa = \kappa$. □[8.135]

8.9 Independence of the continuum hypothesis

This section is devoted to questions involving the sizes of powersets, in particular the size of $\mathcal{P}\omega$. Recall that $|\mathcal{P}\omega| = |\mathbb{R}|$, the set of real numbers, i.e., the continuum. The *continuum hypothesis*, CH is the assertion that $2^\omega = \omega_1$, the first uncountable cardinal. Recall that $L \models \mathsf{CH}$. Throughout this section we will be working in ZFC.

Suppose M is a transitive model of ZFC. As noted above, an M-generic filter for the partial order (8.118) produces a subset of ω not in M. To violate CH we want a partial order that produces a large number of new subsets of ω. Working in M, suppose κ is an infinite cardinal. Let $\mathbb{P}_\kappa$ be defined as follows.

(8.137)

1. $p \in |\mathbb{P}_\kappa|$ *iff*
   1. p *is finite;*
   2. p *is a function;*
   3. $\operatorname{dom} p \subseteq \omega \times \kappa$*;*
   4. $\operatorname{im} p \subseteq 2$.
2. *For* $p, q \in |\mathbb{P}_\kappa|$

$$q \leqslant p \leftrightarrow q \supseteq p.$$

For $p \in |\mathbb{P}_\kappa|$ and $\alpha \in \kappa$, let

$$p_\alpha = \{(n, i) \mid (\langle n, \alpha\rangle, i) \in p\}.$$

As before,[8.118] if G is M-generic on $\mathbb{P}_\kappa$, then, letting $g = \bigcup G$, $g : \omega \times \kappa \to 2$. For each $\alpha \in \kappa$, let $g_\alpha : \omega \to 2$ be defined by

$$g_\alpha n = g\langle n, \alpha\rangle.$$

A straightforward dense set argument shows that $\alpha \neq \alpha' \to g_\alpha \neq g_{\alpha'}$, so in $M[G]$, $\{g_\alpha \mid \alpha \in \kappa\}$ is a set of κ distinct functions from ω to 2.

If we begin by taking κ to be, say, ω_2^M, and if ω_1^M and ω_2^M remain cardinals in $M[G]$, then $M[G]$ says that there are at least ω_2 subsets of ω, so CH fails. In fact, all cardinals of M remain cardinals in $M[G]$, as we will now show.

(8.138) **Theorem** [ZFC] $\mathbb{P}_\kappa$ *has the countable chain condition.*

Proof Suppose toward a contradiction that there exists an uncountable set of pairwise incompatible conditions in $\mathbb{P}_\kappa$. Since conditions are finite, by (8.128) there exists $d \subseteq \omega \times \kappa$ and an uncountable $X' \subseteq X$ such that for any distinct $p, p' \in X'$, $\operatorname{dom} p \cap \operatorname{dom} p' = d$. There are only finitely many functions from d into 2 ($2^{|d|}$, to be precise), so for some $p'' : d \to 2$, there is an uncountable $X'' \subseteq X'$ such that $p \restriction d = p''$ for every $p \in X''$. Let p and p' be distinct elements of X''. Then since $p, p' \in X'$, $\operatorname{dom} p \cap \operatorname{dom} p' = d$, and $p \restriction d = p'' = p' \restriction d$, so $p \cup p'$ is a function, hence a member of P_κ, which is a common extension of p and p'; contradiction. $\square$[8.138]

(8.139) **Theorem** [S] *If* ZFC *is consistent then* $\mathsf{ZFC} + \neg\,\mathsf{CH}$ *is consistent.*

Proof Suppose M is a transitive model of ZFC. Let $\mathbb{P} = \ulcorner \mathbb{P}_{\omega_2} \urcorner^M$. Let G be an M-generic filter on $\mathbb{P}$. Since $\ulcorner (\mathbb{P})$ has the countable chain condition$\urcorner^M$,[8.138] forcing with $\mathbb{P}$ preserves cardinals,[8.127] so ω_1^M and ω_2^M are cardinals in $M[G]$. Let $\kappa = \omega_2^M$. Then $\kappa = \omega_2^{M[G]}$. As shown above, letting $g = \bigcup G$, and letting $g_\alpha(n) = g\langle n, \alpha\rangle$ for each $\alpha < \kappa$, $\alpha \mapsto g_\alpha$ is an injection of κ into ${}^\omega 2$ in $M[G]$. $\square$[8.139]

The following theorem generalizes this to an arbitrary regular cardinal κ in place of ω and also shows how to make 2^κ exactly λ with appropriate assumptions concerning cardinal exponentiation in the ground model:

(8.140) **Theorem** [ZFC] *Suppose* κ, λ *are cardinals,* κ *is regular,* $2^{<\kappa} = \kappa$, *and* $\lambda^\kappa = \lambda$. *Then there is a partial order* $\mathbb{P}$ *such that*

1. *forcing with* $\mathbb{P}$ *preserves cardinals, i.e., for any cardinal* η, $\Vdash^{\mathbb{P}} \ulcorner (\check{\eta})$ *is a cardinal*$\urcorner$; *and*
2. $\Vdash^{\mathbb{P}} \ulcorner 2^{(\check{\kappa})} = (\check{\lambda})\urcorner$.

Remark If we assume GCH then the conditions of the theorem are satisfied if κ is regular and $\operatorname{cf} \lambda > \kappa$.

We could state an equivalent theorem in GB in terms of generic extensions of a transitive model M of ZF, as we have done above, in (8.127) for example. As we have stated the theorem, we have, in effect, placed ourselves in M, with ZFC as our theory. Nevertheless, we will informally refer to V, V-generic filters G on $\mathbb{P}$, and generic extensions $V[G]$ as things that exist. This practice is quite common in discussions of forcing and should be understood in the context of "arguing in a generic extension" as discussed above.[§ 8.5.2]

Proof Let $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ be as follows:

1. $|\mathbb{P}|$ is the set of functions p such that

    1. $\operatorname{dom} p \subseteq \kappa \times \lambda$,
    2. $\operatorname{im} p \subseteq 2$, and
    3. $|p| < \kappa$.

2. $q \leqslant p \leftrightarrow p \subseteq q$.

$\mathbb{P}$ is obviously $<\kappa$-closed, hence $<\kappa$-distributive, so[8.134] all cardinals $\leqslant \kappa$ remain cardinals in $V[G]$ for any V-generic G on $\mathbb{P}$.

It is also easy to show that $\mathbb{P}$ has the κ^+-chain condition. For suppose $P \subseteq |\mathbb{P}|$ is an antichain. Then κ and P satisfy the conditions of Theorem 8.135 and therefore $|P| \leqslant \kappa < \kappa^+$. Hence[8.127] every cardinal $> \kappa$ remains a cardinal in $V[G]$.

Working in $V[G]$ we now calculate $2^\kappa = \left|{}^\kappa 2\right|$. Let $g = \bigcup G$. Since G is a filter on $\mathbb{P}$, any two members of G are compatible, so g is a function—the members of G are, in effect, approximations to g. For any $\pi \in \kappa \times \lambda$, the set of conditions p such that $\pi \in \operatorname{dom} p$ is dense, so $\operatorname{dom} g = \kappa \times \lambda$. For each $\alpha \in \lambda$, let $g_\alpha \in {}^\kappa 2$ be given by

$$g_\alpha \beta = g\langle \beta, \alpha \rangle,$$

for all $\beta \in \kappa$. For any $\alpha, \alpha' \in \lambda$, the set of conditions p such that $\exists \beta \in \kappa \ \big(\langle \beta, \alpha \rangle, \langle \beta, \alpha' \rangle \in \operatorname{dom} p \wedge p\langle \beta, \alpha \rangle \neq p\langle \beta, \alpha' \rangle\big)$ is dense, so $g_\alpha \neq g_{\alpha'}$. It follows that $\left|{}^\kappa 2\right| \geqslant \lambda$.

Still working in $V[G]$, we conclude by showing that $\left|{}^\kappa 2\right| \leqslant \lambda$. Suppose $f : \kappa \to 2$. Let $\tau \in V^{\mathbb{P}}$ be any name for f, i.e., $f = \tau^G$. Define $\hat{\tau} : \kappa \to |\mathfrak{R}\,\mathbb{P}|$ by

$$\hat{\tau}(\beta) = [\![\tau(\check{\beta}) = 0]\!] = \{p \in |\mathbb{P}| \mid p \Vdash \tau(\check{\beta}) = 0\}.$$

Note that for any $\beta \in \kappa$,

$$f(\beta) = 0 \leftrightarrow G \cap \hat{\tau}(\beta) \neq 0.$$

Now suppose $f' : \kappa \to 2$ and let τ' and $\hat{\tau}'$ be to f' as τ and $\hat{\tau}$ are to f, so for any $\beta \in \kappa$,

$$f'(\beta) = 0 \leftrightarrow G \cap \hat{\tau}'(\beta) \neq 0.$$

So if $f \neq f'$ then $\hat{\tau} \neq \hat{\tau}'$.

It is therefore enough to show that $\big||\mathfrak{R}\,\mathbb{P}|\big|^\kappa \leqslant \lambda$.

We first observe that for any $a \in \mathfrak{R}\,\mathbb{P}$, i.e., any regular open subset of $\mathbb{P}$, if X is a maximal set of incompatible elements of a, then $a = \overline{X} = X^{\perp\perp}$.[41] Since $\mathbb{P}$ satisfies the κ^+-chain condition, there are at most $|\mathbb{P}|^\kappa$ such sets X, so $\big||\mathfrak{R}\,\mathbb{P}|\big| \leqslant \big||\mathbb{P}|\big|^\kappa$.

$|(\kappa \times \lambda) \times 2| = \lambda$, so $\big||\mathbb{P}|\big| \leqslant \lambda^{<\kappa} \leqslant \lambda^\kappa = \lambda$, so $\big||\mathfrak{R}\,\mathbb{P}|\big| \leqslant \lambda^\kappa = \lambda$, so $\big||\mathfrak{R}\,\mathbb{P}|\big|^\kappa \leqslant \lambda^\kappa = \lambda$, as desired.

Therefore, in $V[G]$, $2^\kappa = \lambda$. □[8.140]

[41] Otherwise some $p \in a$ is compatible with something in $X^\perp$, so some $q \leqslant p$ is incompatible with everything in X. But since a is open, $q \in a$, so X is not a maximal set of incompatible elements of a.

8.10 Independence of the axiom of choice

The inspiration for Cohen's proof of the unprovability of AC from ZF came from previous work of Fraenkel and Mostowski showing the unprovability of AC from the theory ZFA of sets with atoms.[§ 6.7] The essential concept is that of a *symmetric extension*, which in this case is a submodel of a generic extension.

Assume GBC. Let $\mathfrak{A}$ be a complete boolean algebra, and let $V^{\mathfrak{A}}$ be the $\mathfrak{A}$-valued universe. Suppose π is an automorphism of $\mathfrak{A}$. Define $\hat{\pi} : V^{\mathfrak{A}} \to V^{\mathfrak{A}}$ by recursion on rank so that

$$\hat{\pi}x = \{(\hat{\pi}y, \pi a) \mid (y, a) \in x\}.$$

By induction on α one shows that $\hat{\pi} : V^{\mathfrak{A}}_\alpha \xrightarrow{\text{bij}} V^{\mathfrak{A}}_\alpha$, so $\hat{\pi} : V^{\mathfrak{A}} \xrightarrow{\text{bij}} V^{\mathfrak{A}}$. Clearly, $\hat{\pi}\check{x} = \check{x}$ for all x.

(8.141) **Theorem** [GBC] *Suppose $\mathfrak{A}$ is a complete boolean algebra, π is an automorphism of $\mathfrak{A}$, and $\hat{\pi}$ is as above. Suppose ϕ is an* s*-formula with n free variables and $x_0, \ldots, x_{n^-} \in V^{\mathfrak{A}}$. Then*

$$[\![\phi(\hat{\pi}x_0, \ldots, \hat{\pi}x_{n^-})]\!] = \pi[\![\phi(x_0, \ldots, x_{n^-})]\!].$$

Proof Straightforward. □[8.141]

Definition [GBC] *As in Section 6.7 suppose Γ is a group of automorphisms of $\mathfrak{A}$. For each $x \in V^{\mathfrak{A}}$ let*

$$\operatorname{sym}_\Gamma x \stackrel{\text{def}}{=} \{\pi \in \Gamma \mid \hat{\pi}x = x\}.$$

*Suppose $\mathcal{F}$ is a normal filter of subgroups of Γ. Then $x \in V^{\mathfrak{A}}$ is $\mathcal{F}$-*symmetric $\stackrel{\text{def}}{\iff} \operatorname{sym}_\Gamma x \in \mathcal{F}$. *$V^{\mathfrak{A}}(\mathcal{F}) \stackrel{\text{def}}{=}$ the class of hereditarily $\mathcal{F}$-symmetric members of $V^{\mathfrak{A}}$, i.e., the smallest $M \subseteq V^{\mathfrak{A}}$ such that for all $x \in V^{\mathfrak{A}}$, if x is $\mathcal{F}$-symmetric and* $\operatorname{dom} x \subseteq M$, *then $x \in M$.*

Definition [GBC] *Suppose M is a model of* ZFC *and the above construction has been carried out relative to M. Suppose G is an M-generic $\mathfrak{A}$-filter. Let $M[G, \mathcal{F}] \stackrel{\text{def}}{=} \{x^G \mid x \in M^{\mathfrak{A}}(\mathcal{F})\}$.*

Note that $M \subseteq M[G, \mathcal{F}] \subseteq M[G]$. We call $M[G, \mathcal{F}]$ a *symmetric extension* of M.

(8.142) **Theorem** [GBC] *Suppose M is a model of* ZFC, *$\mathfrak{A}$ is an M-complete boolean algebra in M, $\Gamma \in M$ is a group of automorphisms of $\mathfrak{A}$, $\mathcal{F} \in M$ is a normal filter on Γ, and G is an M-generic filter on $\mathfrak{A}$. Then $M[G, \mathcal{F}] \models$* ZF.

Proof The proof closely parallels that of the corresponding Theorem 6.32 in the context of ZFA, set theory with atoms. We will "work in $M[G]$". Let $N = M[G, \mathcal{F}]$. N is a proper class definable from M, $\mathfrak{A}$, G, and $\mathcal{F}$. It is transitive by design. For $\alpha \in \mathrm{Ord}$, let $N_\alpha = N \cap M_\alpha[G]$. Since for any $\pi \in \Gamma$, $\hat{\pi} : M_\alpha[G] \to M_\alpha[G]$, it is easy to see that $\operatorname{sym}_\Gamma N_\alpha = \Gamma$, so $N_\alpha \in N$. It follows that N is almost universal.

Hence,[3.214] N models ZF with the possible exception of Comprehension. The proof that N satisfies each instance of the Comprehension schema is also essentially the same as for ZFA. □[8.142]

(8.143) **Theorem** [S] *Suppose* ZF *is consistent. Then* ZF + ¬ AC *is consistent.*

Proof We will show in particular that it is consistent with ZF that there exist a set of reals that is infinite, but is finite in the sense (3.133.2). The construction parallels that of the Fraenkel-Mostowski model used in the proof of Theorem 6.33. Reals generic over a ground model M play the role of atoms.

It suffices to show in GB that for any transitive model M of ZF there is an M-complete boolean algebra $\mathfrak{A} \in M$, a group $\Gamma \in M$ of automorphisms of $\mathfrak{A}$, and a normal filter $\mathcal{F} \in M$ on Γ, such that for any M-generic filter G on $\mathfrak{A}$, $M[G, \mathcal{F}] \models \neg\,\mathsf{AC}$. The following argument is to be understood in this way.

Working in M, let $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ be the following partial order for adding an ω-sequence of Cohen reals.

1. $p \in |\mathbb{P}|$ iff
   1. p is a finite function;
   2. $\operatorname{dom} p \subseteq \omega \times \omega$; and
   3. $\operatorname{im} p \subseteq 2$.
2. For $p, q \in |\mathbb{P}|$
$$q \leqslant p \leftrightarrow q \supseteq p.$$

Let $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$. Identify $\mathbb{P}$ in the usual way with its canonical embedding in $\mathfrak{A}$.

Suppose G is M-generic over $\mathbb{P}$, and work in $M[G]$. Let $g = \bigcup G$. For $n \in \omega$, let $g_n : \omega \to 2$ be given by
$$g_n m = g\langle n, m\rangle.$$
Let $A = \{g_n \mid \omega\}$.

For $n, m \in \omega$ and $k \in 2$, let
$$\begin{aligned} a_{n,m,k} &= [\![(\langle n, m\rangle, k)\check{} \in \mathsf{G}]\!] \\ &= \{p \in |\mathbb{P}| \mid (\langle n, m\rangle, k) \in p\}. \end{aligned}$$

For $n \in \omega$, let $\dot{g}_n$ be the function with domain $\big\{(m,k)\check{} \,\big|\, (m,k) \in \omega \dot{\times} 2\big\}$ such that for each $(m,k) \in \omega \dot{\times} 2$
$$\dot{g}_n\big((m,k)\check{}\big) = a_{n,m,k},$$
so $\dot{g}_n^G = g_n$. Let
$$\dot{A} = \{(\dot{g}_n, \mathbf{1}) \mid n \in \omega\},$$
so $\dot{A}^G = A$.

Suppose ρ is a permutation of ω. Let π_ρ^P be the automorphism of $\mathbb{P}$ given by
$$\pi_\rho^P p = \{(\langle \rho n, m\rangle, k) \mid (\langle n, m\rangle, k) \in p\},$$
and let π_ρ be the corresponding automorphism of $\mathfrak{A}$:
$$\pi_\rho a = \{\pi_\rho^P p \mid p \in a\}.$$

Let Γ be the group of automorphisms of $\mathfrak{A}$ of the form π_ρ, ρ a permutation of ω, and for each $\pi \in \Gamma$, let $\hat{\pi}$ be the corresponding operation on $M^{\mathfrak{A}}$. Clearly, for $n \in \omega$, $\hat{\pi}_\rho \dot{g}_n = \dot{g}_{\rho n}$, and $\hat{\pi}\dot{A} = \dot{A}$.

For $s \in [\omega]^{<\omega}$, let R_s be the set of permutations ρ of ω such that $\rho n = n$ for all $n \in s$, and let $F_s = \{\pi_\rho \mid \rho \in R_s\}$. Let $\mathcal{F}$ be the filter of subgroups of Γ generated by the F_ss. $\mathcal{F}$ is normal.

Clearly $\dot{g}_n \in M^{\mathfrak{A}}(\mathcal{F})$ for each $n \in \omega$, and $\dot{A} \in M^{\mathfrak{A}}(\mathcal{F})$. Let $N = M[G, \mathcal{F}]$. Then $N \models \mathsf{ZF}$,[8.142] $g_n \in N$ for each $n \in \omega$, and $A \in N$. Clearly, $N \models \ulcorner [A]$ is infinite$\urcorner$, i.e., not equipollent with any finite ordinal.

(8.144) **Claim** $N \models {}^{\ulcorner}[A]$ *is finite in the sense (3.133.2).*

Proof Suppose toward a contradiction that there exists $f \in N$ such that $\operatorname{dom} f \subsetneq A$ and $\operatorname{im} f = A$. Let $\dot{f} \in M^{\mathfrak{A}}(\mathcal{F})$ be such that $\dot{f}^G = f$. Let $s \in [\omega]^{<\omega}$ be such that $\operatorname{sym}_\Gamma f \supseteq F_s$. Let $S = \{g_n \mid n \in s\}$. Since S is a finite subset of N, $S \in N$.

Let $A_0 = A$, and for each $n \in \omega$, let $A_{n+1} = f^{\leftarrow} A_n$. Then, as in the proof of (6.33), $A_{n+1} \subsetneq A_n$. Let $B_n = A_n \backslash A_{n+1}$. Then $f \restriction B_{n+1} : B_{n+1} \overset{\text{sur}}{\to} B_n$. Let $n_0 > 0$ be such that $B_{n_0} \cap S = 0$, and let g be a member of B_{n_0+1}. Then $f(g) \in B_{n_0}$, so $f(g) \neq g$ and $f(g) \notin S$. Let n, n' be such that $g = g_n$ and $f(g) = g_{n'}$. Note that $n' \neq n$ and $n' \notin s$. Let $p \in G$ be such that

$$p \Vdash \mathbf{Fcn}\, \dot{f} \wedge (\dot{g}_n, \dot{g}_{n'})^{\mathfrak{A}} \in \dot{f}.$$

$\operatorname{dom} p$ is finite, so there exists $n'' \in \omega$ be such that $n'' \notin s \cup \{n, n'\}$ and $\forall m \in \omega\ \langle n'', m\rangle \notin \operatorname{dom} p$. Let $\rho = \{(n', n''), (n'', n')\}$, and let $\pi = \pi_\rho$. Then $rho \in R_s$, so $\pi \in F_s$; hence, $\hat{\pi}\dot{f} = \dot{f}$. Also, $\hat{\pi}\dot{g}_n = \dot{g}_n$ and $\hat{\pi}\dot{g}_{n'} = \dot{g}_{n''}$. Hence

$$\hat{\pi} p \Vdash (\dot{g}_n, \dot{g}_{n''})^{\mathfrak{A}} \in \dot{f}.$$

By design, $\hat{\pi} p \parallel p$. Let q be a common extension of p and $\hat{\pi} p$. Then

$$q \Vdash \mathbf{Fcn}\, \dot{f} \wedge (\dot{g}_n, \dot{g}_{n'})^{\mathfrak{A}} \in \dot{f} \wedge (\dot{g}_n, \dot{g}_{n''})^{\mathfrak{A}} \in \dot{f}.$$

Since $\Vdash g_{n'} \neq g_{n''}$, this is a contradiction. $\square_{8.144}$ $\square_{8.143}$

8.11 Product forcing

Definition [ZF] *Suppose $\mathbb{P}_0, \mathbb{P}_1$ are partial orders.*

1. $\mathbb{P}_0 \times \mathbb{P}_1 \overset{\text{def}}{=}$ *the partial order with domain $|\mathbb{P}_0| \times |\mathbb{P}_1|$ and order relation $\leqslant$ given by*
$$\langle p_0, p_1\rangle \leqslant \langle p_0', p_1'\rangle \leftrightarrow p_0 \leqslant^{\mathbb{P}_0} p_0' \wedge p_1 \leqslant^{\mathbb{P}_1} p_1'.$$
2. *Suppose G is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$.*
$$G^0 \overset{\text{def}}{=} \{p_0 \in |\mathbb{P}_0| \mid \exists p_1\ \langle p_0, p_1\rangle \in G\}$$
$$G^1 \overset{\text{def}}{=} \{p_1 \in |\mathbb{P}_1| \mid \exists p_0\ \langle p_0, p_1\rangle \in G\}.$$

(8.145) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathbb{P}_0, \mathbb{P}_1 \in M$ are partial orders.*

1. *Suppose G_0, G_1 are filters on $\mathbb{P}_0, \mathbb{P}_1$, respectively. Let $G = G_0 \times G_1$. Then G is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$, $G_0 = G^0$, and $G_1 = G^1$.*
2. *Suppose G is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$. Then G^0, G^1 are filters on $\mathbb{P}_0, \mathbb{P}_1$, respectively; and $G = G^0 \times G^1$.*
3. *Suppose G_0, G_1 are filters on $\mathbb{P}_0, \mathbb{P}_1$, respectively. The following are equivalent:*
   1. *$G_0 \times G_1$ is M-generic.*
   2. *G_0 is M-generic and G_1 is $M[G_0]$-generic.*
   3. *G_1 is M-generic and G_0 is $M[G_1]$-generic.*
4. *Suppose G_0, G_1 are filters on $\mathbb{P}_0, \mathbb{P}_1$, respectively, and $G_0 \times G_1$ is M-generic. Then $M[G] = M[G_0][G_1] = M[G_1][G_0]$.*

Proof 1 Straightforward. $\square$[8.145.1]

2 Suppose G is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$. To show that G^0 is a filter, suppose first that $p_0 \in G^0$ and $q_0 \geqslant p_0$. We must show that $q_0 \in G^0$. Let p_1 be such that $\langle p_0, p_1 \rangle \in G$. Then $\langle q_0, p_1 \rangle \in G$, so $q_0 \in G^0$. Next suppose $p_0, p_0' \in G^0$. We must show that p_0, p_0' have a common extension in G^0. Let p_1, p_1' be such that $\langle p_0, p_1 \rangle, \langle p_0', p_1' \rangle \in G$. Let $\langle p_0'', p_1'' \rangle \in G$ extend both $\langle p_0, p_1 \rangle$ and $\langle p_0', p_1' \rangle$. Then $p_0'' \in G^0$ and extends both p_0 and p_0'. Thus, G^0 is a filter.

Similarly, G^1 is a filter.

Obviously, $G \subseteq G^0 \times G^1$. Conversely, suppose $p_0 \in G^0$ and $p_1 \in G^1$. We must show that $\langle p_0, p_1 \rangle \in G$. Let p_1', p_0' be such that $\langle p_0, p_1' \rangle, \langle p_0', p_1 \rangle \in G$, and let $\langle p_0'', p_1'' \rangle \in G$ be a common extension. Then $\langle p_0'', p_1'' \rangle$ extends $\langle p_0, p_1 \rangle$, so $\langle p_0, p_1 \rangle \in G$. $\square$[8.145.2]

3 (3.2) and (3.3) have essentially the same content, so it suffices to show that (3.1) is equivalent to (3.2)

3.1→3.2 Suppose $G = G_0 \times G_1$ is M-generic. Recall[8.145.1] that $G^0 = G_0$ and $G^1 = G_1$. To show that G_0 is M-generic, suppose $X \in M$ is dense in $\mathbb{P}_0$. Then $X \times |\mathbb{P}_1|$ is dense in $\mathbb{P}_0 \times \mathbb{P}_1$, so there exists $\langle p_0, p_1 \rangle \in G \cap (X \times |\mathbb{P}_1|)$. Thus, $p_0 \in X$ and $p_0 \in G^0 = G_0$.

To show that G_1 is $M[G_0]$-generic, suppose $X_1 \in M[G_0]$ is dense in $\mathbb{P}_1$. Let $\dot{X}_1 \in M^{\mathbb{P}_0}$ be such that $\dot{X}_1^{G_0} = X_1$. Note that $M[G_0] \models \ulcorner [X_1] \text{ is dense in } [\mathbb{P}_1] \urcorner$. Let $p_0 \in G_0$ be such that $p_0 \Vdash^{\mathbb{P}_0} \ulcorner (\dot{X}_1) \text{ is dense in } (\check{\mathbb{P}}_1) \urcorner$. Let X be the set of $\langle q_0, q_1 \rangle \in |\mathbb{P}_0| \times |\mathbb{P}_1|$ such that

1. $q_0 \perp p_0$; or
2. $q_0 \leqslant p_0$ and $q_0 \Vdash^{\mathbb{P}_0} \check{q}_1 \in \dot{X}_1$.

Note that $X \in M$. We claim that X is dense in $\mathbb{P}_0 \times \mathbb{P}_1$. To show this, suppose $\langle r_0, r_1 \rangle \in |\mathbb{P}_0| \times |\mathbb{P}_1|$. We must show that $\langle r_0, r_1 \rangle$ has an extension in X. If $r_0 \perp p_0$ then $\langle r_0, r_1 \rangle \in X$, so suppose $r_0 \parallel p_0$, and let $s_0 \leqslant r_0, p_0$. Then $s_0 \Vdash^{\mathbb{P}_0} \ulcorner (\dot{X}_1) \text{ is dense in } (\check{\mathbb{P}}_1) \urcorner$, so for some $q_0 \leqslant s_0$ and $q_1 \leqslant r_1$, $q_0 \Vdash^{\mathbb{P}_0} \check{q}_1 \in \dot{X}_1$. Thus, $\langle q_0, q_1 \rangle \leqslant \langle r_0, r_1 \rangle$ and $\langle q_0, q_1 \rangle \in X$.

Thus, $X \in M$ is dense in $|\mathbb{P}_0| \times |\mathbb{P}_1|$. Since G is assumed to be M-generic, there exists $\langle q_0, q_1 \rangle \in G \cap X$. Note that $q_0 \in G^0 = G_0$ and $q_1 \in G^1 = G_1$. Since $p_0 \in G^0$, q_0 is compatible with p_0, so $q_0 \leqslant p_0$ and $q_0 \Vdash^{\mathbb{P}_0} \check{q}_1 \in \dot{X}_1$. Since $q_0 \in G_0$, $M[G_0] \models [q_1] \in [X_1]$, so $q_1 \in X_1$. Thus G_1 meets X_1. Since X_1 is an arbitrary set in $M[G_0]$ that is dense in $\mathbb{P}_1$, G_1 is $M[G_0]$-generic.

3.2→3.1 Suppose G_0 is M-generic and G_1 is $M[G_0]$-generic, and suppose $X \in M$ is a dense subset of $|\mathbb{P}_0| \times |\mathbb{P}_1|$. We must show that G meets X.

Let $X^1 = \{p_1 \mid \exists p_0 \in G_0 \; \langle p_0, p_1 \rangle \in X\}$. Note that $X^1 \in M[G_0]$. We claim that X^1 is dense in $\mathbb{P}_1$. To show this, suppose $q_1 \in |\mathbb{P}_1|$. Let X^0 be the set of $p_0 \in |\mathbb{P}_0|$ such that there exists $p_1 \leqslant q_1$ such that $\langle p_0, p_1 \rangle \in X$. X^0 is dense in $\mathbb{P}_0$ because for any $q_0 \in |\mathbb{P}_0|$ there exists $\langle p_0, p_1 \rangle \in X$ that extends $\langle q_0, q_1 \rangle$. Hence G_0 meets X^0, i.e., there exist $p_0 \in G_0$ and $p_1 \leqslant q_1$ such that $\langle p_0, p_1 \rangle \in X$, which is to say, there exists $p_1 \leqslant q_1$ such that $p_1 \in X^1$. Since q_1 is arbitrary, X^1 is dense.

Since G_1 is $M[G_0]$-generic there exists $p_1 \in G_1$ such that $p_1 \in X^1$, i.e., for some $p_0 \in G_0$, $\langle p_0, p_1 \rangle \in X$. Since $\langle p_0, p_1 \rangle \in G$, G meets X. Since X is an arbitrary dense set in M, G is M-generic. $\square^{8.145.3}$

4 Straightforward. $\square^{8.145.4}$ $\square^{8.145}$

Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}$ be the regular algebras of partial orders $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$. Let the maps π_0 and π_1 from $|\mathbb{P}_0| \times |\mathbb{P}_1|$ to $|\mathbb{P}_0|$ and $|\mathbb{P}_1|$, respectively, be the projection maps:

$$\pi_0\langle p_0, p_1\rangle = p_0$$
$$\pi_1\langle p_0, p_1\rangle = p_1.$$

Then the inverse image maps $\pi_0{}^{\leftarrow}$ and $\pi_1{}^{\leftarrow}$ are the dual embeddings of $\mathfrak{A}_0$ and $\mathfrak{A}_1$, respectively, in $\mathfrak{A}$:

$$\pi_0{}^{\leftarrow} X_0 = X_0 \times |\mathbb{P}_1|$$
$$\pi_1{}^{\leftarrow} X_1 = |\mathbb{P}_0| \times X_1,$$

where X_0 and X_1 are regular subsets of $|\mathbb{P}_0|$ and $|\mathbb{P}_1|$, respectively.

(8.146) **Theorem** [GB] *$\pi_0{}^{\leftarrow}$ and $\pi_1{}^{\leftarrow}$ are complete embeddings of $\mathfrak{A}_0$ and $\mathfrak{A}_1$ in $\mathfrak{A}$.*[42]

Proof Straightforward. $\square^{8.146}$

The following generalization of (8.132) illustrates the interplay of chain conditions and closure properties in product forcing and is used in the next section.

(8.147) **Theorem** [GB] *Suppose M is a transitive model of ZFC; $\mathbb{P}_0, \mathbb{P}_1 \in M$ are partial orders; κ is a cardinal in M; $M \models \ulcorner[\mathbb{P}_0]$ satisfies the $[\kappa]^+$-chain condition and $[\mathbb{P}_1]$ is $[\kappa]$-closed$\urcorner$; G_0, G_1 are filters on $\mathbb{P}_0, \mathbb{P}_1$, respectively; $G = G_0 \times G_1$ is an M-generic filter on $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$; $f \in M[G]$; and $f : \kappa \to M$. Then $f \in M[G_0]$.*

Proof Since $M[G] \models \mathsf{ZF}$, there exists $A \in M$ such that $\operatorname{im} f \subseteq A$.[43] Let $\dot{f} \in M^{\mathbb{P}}$ be such that $\dot{f}^G = f$, and let $p = \langle p_0, p_1 \rangle \in G$ be such that $p \Vdash^{M,\mathbb{P}} \dot{f} : \check{\kappa} \to \check{A}$.

Arguing in M: For each $\alpha < \kappa$, let B_α be the set of q_1 extending p_1 in $\mathbb{P}_1$ such that there exists a maximal antichain X below p_0 in $\mathbb{P}_0$ such that for every $q_0 \in X$ there exists $a \in A$ such that $\langle q_0, q_1 \rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}$.

(8.148) **Claim** *In M: For each $\alpha < \kappa$, B_α is open and dense below p_1.*

Proof In M: B_α is clearly open. Suppose $q_1 \leqslant p_1$. We will show that there exists $q_1' \leqslant q_1$ such that $q_1' \in B_\alpha$. Using a fixed wellordering of $A \cup |\mathbb{P}_0| \cup |\mathbb{P}_1|$, we will define by recursion sequences $\langle q_1^\beta \mid \beta < \eta \rangle$ and $\langle q_0^\beta \mid \beta < \eta \rangle$ such that for all $\beta, \beta' < \eta$,

1. $q_1^\beta \leqslant q_1$ and $q_0^\beta \leqslant p_0$;

---

[42] $\mathfrak{A}$ is the *sum* or *coproduct* $\mathfrak{A}_0 \oplus \mathfrak{A}_1$ of $\mathfrak{A}_0$ and $\mathfrak{A}_1$ in the category of complete boolean algebras, as $\mathbb{P}_0 \times \mathbb{P}_1$ is the *product* $\mathbb{P}_0 \otimes \mathbb{P}_1$ of $\mathbb{P}_0$ and $\mathbb{P}_1$ in category-theoretic terminology. (Of course, to be formally correct, the morphisms must also be specified in these definitions.)

[43] For example, let $\alpha \in M \cap \mathrm{Ord}$ be such that $M[G] \models \ulcorner[f] : [\kappa] \to V_{[\alpha]}$, and let $A = V_\alpha^M$.

2. for some a, $\langle q_0^\beta, q_1^\beta\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}$; and

3. if $\beta < \beta'$ then $q_1^\beta \geqslant q_1^{\beta'}$ and $q_0^\beta \perp q_0^{\beta'}$.

Suppose $\langle q_1^\beta \mid \beta < \gamma\rangle$ and $\langle q_0^\beta \mid \beta < \gamma\rangle$ are as specified. Since $\mathbb{P}_0$ satisfies the κ^+-chain condition, $\gamma < \kappa^+$. If $\{q_0^\beta \mid \beta < \gamma\}$ is a maximal incompatible set of conditions below p_0, the recursion is finished, and we let $\eta = \gamma$. Otherwise, proceed as follows.

1. Let r_1 be the first (in the sense of the fixed wellordering posited at the outset) member of $|\mathbb{P}_1|$ that extends q_1^β for all $\beta < \gamma$, which is possible because $\mathbb{P}_1$ is κ-closed.

2. Let r_0 be the first member of $|\mathbb{P}_0|$ below p_0 and incompatible with q_0^β for all $\beta < \gamma$.

3. Since $\langle r_0, r_1\rangle \Vdash^{\mathbb{P}} \dot{f} : \check{\kappa} \to \check{A}$, there exists $\langle s_0, s_1\rangle \leqslant \langle r_0, r_1\rangle$ such that for some $a \in A$, $\langle s_0, s_1\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}$. Let $\langle q_0^\gamma, q_1^\gamma\rangle$ be the first such pair $\langle s_0, s_1\rangle$ in lexicographic order (using the fixed wellorderings of $|\mathbb{P}_0|$ and $|\mathbb{P}_1|$).

As previously noted, γ is always $< \kappa^+$, so at some point the construction terminates with $\eta < \kappa^+$. Let $X = \{q_0^\beta \mid \beta < \eta\}$. X is a maximal antichain in $\mathbb{P}_0$ below p_0. Let q_1' extend q_1^β for all $\beta < \eta$. For each $\beta < \eta$ there exists $a \in A$ such that $\langle q_0^\beta, q_1^\beta\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}$, and $\langle q_0^\beta, q_1'\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}$, as well. Hence $q_1' \in B_\alpha$, as desired. $\square$[8.148]

Still in M: Using AC again, and using the κ-closure of $\mathbb{P}_1$, we will show that $B = \bigcap_{\alpha<\kappa} B_\alpha$ is dense below p_1. To this end, suppose $q_1^0 \leqslant p_1$ is given, and construct a decreasing κ-sequence $\langle q_1^\alpha \mid \alpha < \kappa\rangle$ such that for each $\alpha < \kappa$, $q_1^\alpha \in B_\alpha$. Then let q_1' be a common extension of every q_1^α. q_1' is an extension of p_1 in B.

Let $\langle X_\alpha^{q_1} \mid q_1 \in B \wedge \alpha \in \kappa\rangle$ and $\langle a_\alpha^{q_0,q_1} \mid q_0 \in X_\alpha^{q_1} \wedge q_1 \in B \wedge \alpha \in \kappa\rangle$ be such that for each $q_1 \in B$ and $\alpha \in \kappa$, $X_\alpha^{q_1}$ is a maximal antichain below p_0, and for every $q_0 \in X_\alpha^{q_1}$, $\langle q_0, q_1\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}_\alpha^{q_0,q_1}$.

In $M[G]$: Since G_1 is M-generic over $\mathbb{P}_1$ and $p_1 \in G_1$, there exists $q_1 \in G_1 \cap B$.

In $M[G_0]$: Define $g : \kappa \to A$ as follows. Given $\alpha < \kappa$, since $p_0 \in G_0$ and $X_\alpha^{q_1}$ is a maximal antichain below p_0, G_0 contains exactly one condition in $X_\alpha^{q_1}$, say q_0^α. Let $g(\alpha) = a_\alpha^{q_0^\alpha,q_1}$. Note that $g \in M[G_0]$, since that's our whole universe for this part of the discussion. Note also that $\langle q_0^\alpha, q_1\rangle \Vdash^{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{a}_\alpha^{q_0^\alpha,q_1}$.

In $M[G]$: For each $\alpha < \kappa$, $q_0^\alpha \in G_0$, so $\langle q_0^\alpha, q_1\rangle \in G$, and $f(\alpha) = a_\alpha^{q_0^\alpha,q_1} = g(\alpha)$. I.e., $f = g$; hence, $f \in M[G_0]$. $\square$[8.147]

8.11.1 Easton forcing

We will use the notion of product forcing to prove the following theorem, which shows that the only constraints imposed by ZFC on the function $\kappa \mapsto 2^\kappa$, for regular cardinals κ, are the obvious condition of monotonicity and the condition that $\kappa < \operatorname{cf} 2^\kappa$.[3.157.1] Theorem 8.149 shows how to adjust the sizes of powersets of the members of a set of regular cardinals and is a limited version of the celebrated theorem of Easton, which applies the technique to the proper class of all regular cardinals. Easton's theorem generalizes a previous result of Solovay dealing with finite sets of regular cardinals.

(8.149) **Theorem** [GBC] *Suppose M is a transitive model of* ZFC+GCH *and $F \in M$ is a function whose domain consists of regular cardinals of M and whose values are cardinals of M, such that $M \models \ulcorner$for all $\kappa, \lambda \in \operatorname{dom} [F]$,*

1. *if $\kappa \leqslant \lambda$ then $[F]\kappa \leqslant [F]\lambda$; and*
2. *$\kappa < \operatorname{cf}[F]\kappa\urcorner$.*

Then there exists a partial order $\mathbb{P} \in M$ such that

1. *for every κ, λ, if $M \models \ulcorner[\kappa]$ is a cardinal with cofinality $[\lambda]$, then $\Vdash^{M,\mathbb{P}} \ulcorner(\check{\kappa})$ is a cardinal with cofinality $(\check{\lambda})\urcorner$; and*
2. *$\Vdash^{M,\mathbb{P}} \ulcorner\forall\kappa \in \operatorname{dom}(\check{F})\ 2^\kappa = (\check{F})\kappa\urcorner$.*

Remark We could state the conclusion of the theorem informally in terms of generic extensions thusly:

> There exists a generic extension $M[G]$ such that $M[G]$ has the same cardinals and cofinalities as M and for all $\kappa \in \operatorname{dom} F$, $M[G] \models 2^{[\kappa]} = [F]\kappa$.

Proof Let $D = \{\langle\kappa, \beta, \alpha\rangle \mid \kappa \in \operatorname{dom} F \wedge \beta < \kappa \wedge \alpha < F\kappa\}$, and for any regular cardinal ν, let

1. $D^{\leqslant\nu} = \{\langle\kappa, \beta, \alpha\rangle \in D \mid \kappa \leqslant \nu\}$; and
2. $D^{>\nu} = \{\langle\kappa, \beta, \alpha\rangle \in D \mid \kappa > \nu\}$.

Let $\mathbb{P} = (P; \leqslant)$, where $\leqslant = \supseteq$ and P is the set of functions $p : D \rightharpoonup 2$ such that

(8.150) *for every regular cardinal ν, $|\operatorname{dom} p \cap D^{\leqslant\nu}| < \nu$.*

Given a regular cardinal λ, for each $p \in P$, let

1. $p^{\leqslant\lambda} = p \upharpoonright D^{\leqslant\lambda}$; and
2. $p^{>\lambda} = p \upharpoonright D^{>\lambda}$.

Let

1. $P^{\leqslant\lambda} = \{p^{\leqslant\lambda} \mid p \in P\}$; and
2. $P^{>\lambda} = \{p^{>\lambda} \mid p \in P\}$.

For $\kappa \in \operatorname{dom} F$, let

1. $p_\kappa = \{(\langle\beta, \alpha\rangle, i) \mid (\langle\kappa, \beta, \alpha\rangle, i) \in p\}$;
2. $P_\kappa = \{p_\kappa \mid p \in P\}$; and
3. $\mathbb{P}_\kappa = (P_\kappa; \leqslant)$, where $\leqslant = \supseteq$.

Note that the cardinality condition (8.150) implies that $|p_\kappa| < \kappa$, so $\mathbb{P}_\kappa$ is the partial order used in the proof of Theorem 8.140 with $\lambda = F\kappa$, and the analysis used there is applicable.

Suppose G is an M-generic filter on P. Let $g = \bigcup G$. Then $g : D \to 2$ ($\operatorname{dom} g = D$). For each $\kappa \in \operatorname{dom} F$ and $\alpha < F\kappa$ we let $g^\kappa_\alpha \in {}^\kappa 2$ be given by

$$g^\kappa_\alpha \beta = g\langle\kappa, \beta, \alpha\rangle.$$

As before, for each $\kappa \in \operatorname{dom} F$ and $\alpha, \alpha' \in F\kappa$, if $\alpha \neq \alpha'$ then $g^\kappa_\alpha \neq g^\kappa_{\alpha'}$, so

(8.151) $$M[G] \models \ulcorner |^{[\kappa]}2| \geqslant |[F\kappa]| \urcorner.$$

It remains to be shown that

$$M[G] \models \ulcorner |^{[\kappa]}2| \leqslant |[F\kappa]| \urcorner$$

and that the cardinals and cofinalities of $M[G]$ are are exactly those of M. Toward this end we note that $p \mapsto \langle p^{\leqslant\lambda}, p^{>\lambda}\rangle$ is an isomorphism of P with $P^{\leqslant\lambda} \times P^{>\lambda}$ for any regular cardinal λ. Thus, for any regular cardinal λ in the sense of M, $M[G] = M[G^{\leqslant\lambda}][G^{>\lambda}]$.

(8.152) **Claim** *In the sense of M: For any regular λ, $P^{\leqslant\lambda}$ satisfies the λ^+-chain condition.*

Proof Arguing in M: Note that if $p, p' \in P^{\leqslant\lambda}$ agree on their common domain, then $p \cup p'$ is a common extension of p, p'. Thus, if $X \subseteq P^{\leqslant\lambda}$ is an antichain then for any $p, p' \in X$, if $p \neq p'$ then $\exists x \in \operatorname{dom} p \cap \operatorname{dom} p'\ p(x) \neq p'(x)$. Since GCH holds, $2^{<\lambda} = \lambda$. The members of X are functions into 2 of size less than λ. Thus, (8.135) applies with λ for κ, and $|X| \leqslant \lambda$. $\square$[8.152]

(8.153) **Claim** *In the sense of M: For any regular λ, $P^{>\lambda}$ is λ-closed.*

Proof Arguing in M: Suppose X is a set of pairwise compatible elements of $P^{>\lambda}$ and $|X| \leqslant \lambda$. Let $p = \bigcup X$. Then $p : D^{>\lambda} \rightharpoonup 2$. Suppose ν is a regular cardinal. If $\nu \leqslant \lambda$ then $\operatorname{dom} p \cap D^{\leqslant\nu} = 0$, and if $\nu > \lambda$ then $|\operatorname{dom} p \cap D^{\leqslant\nu}| \leqslant \sum_{q\in X} |\operatorname{dom} q \cap D^{\leqslant\nu}| < \nu$, because $|\operatorname{dom} q \cap D^{\leqslant\nu}| < \nu$ for each $q \in X$,[8.150] $|X| \leqslant \lambda$, $\lambda < \nu$, and ν is regular. Hence, (8.150) is satisfied, so $p \in P^{>\lambda}$, and $p \leqslant q$ for all $q \in X$. $\square$[8.153]

To show that cardinals and cofinalities are preserved in the extension to $M[G]$ it suffices to show that every regular cardinal in M is regular in $M[G]$, i.e., if $M \models \ulcorner [\kappa]$ is a regular cardinal$\urcorner$ then $M[G] \models \ulcorner \operatorname{cf}[\kappa] = [\kappa]\urcorner$. Suppose toward a contradiction that κ is regular in M, $\lambda < \kappa$, $f : \lambda \to \kappa$, $f \in M[G]$, and $\operatorname{im} f$ is cofinal in κ. We apply (8.147) with $\mathbb{P}_0 = \mathbb{P}^{\leqslant\lambda}$ and $\mathbb{P}_1 = \mathbb{P}^{>\lambda}$ to conclude that $f \in M[G^{\leqslant\lambda}]$. But since $\mathbb{P}^{\leqslant\lambda}$ satisfies the λ^+-chain condition, it satisfies the κ-chain condition, so by (8.127) κ is regular in $M[G^{\leqslant\lambda}]$. Hence $\operatorname{im} f$ is not cofinal in κ; contradiction.

As noted above,[8.151] for each $\kappa \in \operatorname{dom} F$, $M[G] \models \ulcorner |^{[\kappa]}2| \geqslant |[F\kappa]|\urcorner$. By virtue of the preservation of cardinals, it follows that $M[G] \models \ulcorner 2^{[\kappa]} \geqslant [F\kappa]\urcorner$. We now show that $M[G] \models \ulcorner 2^{[\kappa]} \leqslant [F\kappa]\urcorner$. Again we use (8.147), this time with $\mathbb{P}_0 = \mathbb{P}^{\leqslant\kappa}$ and $\mathbb{P}_1 = \mathbb{P}^{>\kappa}$, to conclude that ${}^\kappa 2 \cap M[G] = {}^\kappa 2 \cap M[G_0]$, where $G_0 = G^{\leqslant\kappa}$. We will show that $|{}^\kappa 2|^{M[G_0]} \leqslant F\kappa$.

Suppose $f \in {}^\kappa 2 \cap M[G_0]$, and suppose $\dot f \in M^{\mathbb{P}_0}$ is such that $\dot f^{G_0} = f$. Let $f^0 \in M$ be the function on κ defined by the condition that

$$f^0(\alpha) = \{p \in |\mathbb{P}_0| \mid p \Vdash^{\mathbb{P}_0} (\dot f : \check\kappa \to \check 2 \wedge \dot f(\check\alpha) = \check 0)\}.$$

Then for any $\alpha \in \kappa$, $f(\alpha) = 0$ iff G_0 meets $f^0(\alpha)$. Thus if $f, g \in {}^\kappa 2 \cap M[G_0]$ and $f \neq g$ then $f^0 \neq g^0$. Note that each $f^0(\alpha)$ is a regular open subset of $\mathbb{P}_0$, i.e., a member of the regular algebra $\mathfrak{R}\,\mathbb{P}_0$. It follows that $|{}^\kappa 2|^{M[G_0]} \leqslant |{}^\kappa \mathfrak{R}\,\mathbb{P}_0|^M$, so we must show that

(8.154) $$|{}^\kappa \mathfrak{R}\,\mathbb{P}_0|^M \leqslant F\kappa.$$

In M: Let $\lambda = F\kappa$. Let $P_0 = |\mathbb{P}_0|$. We first compute $|P_0|$. Let $D_0 = D^{\leqslant\kappa}$. Clearly, $|D_0| = \lambda$. For size computations, we will refer to a fixed ordering of D_0 of length λ, and we will identify subsets of D_0 with subsets of λ in this way. In general, given a set X and a cardinal ν, we let $[X]^\nu \stackrel{\text{def}}{=}$ the set of subsets of X of size ν, $[X]^{<\nu} \stackrel{\text{def}}{=}$ the set of subsets of X of size $< \nu$, etc.

Suppose $p \in P_0$. Then $p : X \to 2$ for some $X \in [D_0]^{<\kappa}$. For a given X, the number of conditions p with domain X is no greater than $2^{|X|} \leqslant 2^\kappa \leqslant \lambda$, by GCH. Thus $|P_0| \leqslant |[D_0]^{<\kappa}| \cdot \lambda = |[\lambda]^{<\kappa}| \cdot \lambda = |[\lambda]^{<\kappa}|$.

Since $\operatorname{cf}\lambda > \kappa$, any $X \in [\lambda]^{<\kappa}$ is bounded below λ, and is therefore in $\mathcal{P}\nu$ for some $\nu < \lambda$. By GCH, $2^\nu \leqslant \lambda$ for each $\nu < \lambda$, so $|[\lambda]^{<\kappa}| = \lambda$.

Thus, $|P_0| = \lambda$. Next we compute $|\mathfrak{R}\,P_0|$. Suppose $a \in \mathfrak{R}\,\mathbb{P}_0$, and suppose $X \subseteq a$ is a maximal antichain in a. Then $a = \overline{X}$. Since $\mathbb{P}_0$ satisfies the κ^+-chain condition, $|X| \leqslant \kappa$. Thus, $X \mapsto \overline{X}$ maps $[P_0]^{\leqslant\kappa}$ onto $\mathfrak{R}\,\mathbb{P}_0$, so $|\mathfrak{R}\,\mathbb{P}_0| \leqslant |[\lambda]^{\leqslant\kappa}|$. Since $\operatorname{cf}\lambda > \kappa$, any $X \in [\lambda]^{\leqslant\kappa}$ is bounded below λ, so by a similar computation to the preceding, $|[\lambda]^{\leqslant\kappa}| = \lambda$. Thus, $|\mathfrak{R}\,P_0| = \lambda$. Similar arguments show that $|{}^\kappa\mathfrak{R}\,\mathbb{P}_0| = \lambda$.

The foregoing argument in M justifies (8.154) and this completes the proof. $\square$[8.149]

8.12 Class forcing

As mentioned above, Easton's theorem *per se* adjusts the powersets of all regular cardinals at once. To accomplish this by forcing requires the use of a partial order $\mathbb{P}$ which is a proper class, to which the theory of forcing we have presented does not immediately apply.

(8.155) *An indication of the issues that must be addressed in a general theory of forcing with proper classes is given by consideration of the following two partial orders.*

1. *Let $\mathbb{P}$ consist of all finite functions $p : \omega \rightharpoonup \mathrm{Ord}$. It is easy to see that forcing with $\mathbb{P}$ adds a function $g : \omega \stackrel{\text{sur}}{\to} \mathrm{Ord}$. Thus, if G is M-generic on $\mathbb{P}$ then $M[G]$ violates an instance of the* $\mathsf{Collection}$ *schema.*[44]
2. *Recall the partial order $\mathbb{P}_\kappa$,*[8.137] *forcing with which adds distinct functions $g_\alpha : \omega \to 2$ for $\alpha \in \kappa$. Let $\mathbb{P}_{\mathrm{Ord}}$ be defined analogously as the class of finite functions $p : \omega \times \mathrm{Ord} \rightharpoonup 2$. Forcing with $\mathbb{P}_{\mathrm{Ord}}$ adds distinct functions $g_\alpha : \omega \to 2$ for $\alpha \in \mathrm{Ord}$. If G is M-generic on $\mathbb{P}_{\mathrm{Ord}}$ then $M[G]$ does not satisfy the* Power *axiom, as $\mathcal{P}\omega$ is a proper class in $M[G]$.*

If $\mathbb{P}$ is merely included in M but is not in M, it is not sufficient to consider the structure $(M; \in)$ as a ground model; rather we must deal with the structure $(M; \in, \mathbb{P})$, with $\mathbb{P}$ treated as a unary predicate.

Definition [GB] s^{P} *is the signature* s *of basic set theory expanded by the addition of the new unary predicate symbol, and* ZF^{P} *is the extension of* ZF *in which the axiom schemas are stated for all* s^{P}*-formulas, not just for* s*-formulas, and* P *is stated to be a partial order with a maximum element (which we typically denote by '**1**').*[45]

[44] Like set forcing, proper class forcing does not add new ordinals.

[45] The requirement of a maximum element is a convenience. The utility of this will first be apparent in the definition of $\check{x}$ in (8.157). (See (8.13) and the footnote to (8.15).) If we wish to use a partial order that does not naturally have a maximum element, we simply add one.

Thus we suppose a transitive class M and partial order $\mathbb{P} \subseteq M$ such that $(M;\in,\mathbb{P}) \models \mathsf{ZF}^{\mathsf{P}}$. Note that in this case (since $(M;\in,\mathbb{P}) \models \mathsf{Comprehension}^{\mathsf{P}}$) for every $x \in M$, $x \cap \mathbb{P} \in M$. Thus, if $\mathbb{P} \notin M$ then $\mathbb{P} \nsubseteq x$ for any $x \in M$. In particular, $\mathbb{P}$ is not included in any V_α in the sense of M. Thus, $\mathbb{P}$ is a proper class in the sense of M for the "usual" reason: it is "bigger than any set", or "goes all the way to the top of the universe" in the sense of M.

Note that if $\mathbb{P}$ is an element of M then ZF^{P} is not a significant extension of ZF, inasmuch as $\mathsf{ZF} + \exists u\,\forall v\,(v \in u \leftrightarrow \mathsf{P}v) \vdash \mathsf{ZF}^{\mathsf{P}}$, so we may work directly with ZF in this case. On the other hand, it should be understood that when $\mathbb{P} \notin M$, the statement that M is a model of ZF should generally be taken to mean that $(M;\in,\mathbb{P}) \models \mathsf{ZF}^{\mathsf{P}}$.

Independent of whether $\mathbb{P}$ is a set or a proper class in the sense of M, M itself may be a set or a proper class (in the sense of V, as it were). For some purposes, e.g., if we wish to be able to prove that M-generic filters exist, it is convenient to suppose that M is a countable set. Obviously, we are "outside M" when we do this. On the other hand, the demonstration that the forcing relation and related objects are definable over M essentially requires their construction "within M", i.e., as though $M = V$.

(8.156) **Theorem** [GB] *There exists a Σ_1 s^{P}-formula θ with two free variables such that for any transitive model $(M;\in,\mathbb{P})$ of ZF^{P}, letting*

$$X_\alpha = \{x \in M \mid \theta^{(M;\in,\mathbb{P})}(\alpha, x)\}$$

for each $\alpha \in \mathrm{Ord} \cap M$,

1. $X_0 = 0$*;*
2. $X_\alpha = \bigcup_{\beta\in\alpha} X_\beta$ *for limit α; and*
3. $X_{\alpha+1} = M \cap \mathcal{P}(X_\alpha \times |\mathbb{P}|)$*, i.e., the class of subsets of $X_\alpha \times |\mathbb{P}|$ in M.*

Proof If $\mathbb{P}$ is a set in M then (8.156.1–3) describes a recursion of the conventional type, and $\{\langle\alpha, x\rangle \mid x \in X_\alpha\}$ is definable over $(M;\in,\mathbb{P})$ in the usual way, i.e., we take θ to be $\ulcorner$there exists a sequence $\langle X_\beta \mid \beta \leqslant (\alpha)\rangle$ satisfying (8.156.1–3) with P for $\mathbb{P}$ and $(x) \in X_{(\alpha)}\urcorner$.[46] The reason this works is that for any $\alpha \in M$, $\langle X_\beta \mid \beta \leqslant \alpha\rangle \in M$. If $\mathbb{P} \notin M$, the recursion must be handled differently:

1. If M is itself a proper class then each X_α for $\alpha > 0$ is a proper class, and a different treatment is required simply to show that the indicated sequence exists.

2. If M is a set, we still have to show that the sequence is definable over M, which is essentially the same problem.

We therefore take θ to be $\ulcorner$there exists $m \in M$ and a sequence $\langle X_\beta \mid \beta \leqslant (\alpha)\rangle$ satisfying (8.156.1–3) with $|\mathsf{P}| \cap m$ for $|\mathbb{P}|$, and $(x) \in X_{(\alpha)}\urcorner$. Suppose $m, m' \in M$ and $m \subseteq m'$. Let X^m_α and $X^{m'}_\alpha$ be the sets defined by (8.156.1–3) with m and m' respectively for M. It is easy to show by induction that for every $\alpha \in M$, $X^m_\alpha \subseteq X^{m'}_\alpha$, and that the classes $X_\alpha = \bigcup_{m\in M} X^m_\alpha = \{x \in M \mid \theta^{(M;\in,\mathbb{P})}(\alpha, x)\}$ satisfy (8.156.1–3). $\square_{8.156}$

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[46] In this case there is also a Π_1 definition of the sequence, viz., $\ulcorner$for every sequence $\langle X_\beta \mid \beta \leqslant (\alpha)\rangle$ satisfying (8.156.1–3), $(x) \in X_{(\alpha)}\urcorner$.

The following definition is to be understood in light of (8.156), which is important, not just because it demonstrates the existence of an indexed family $\mathcal{M}$ such that, letting $M^{\mathbb{P}}_{\alpha} = \mathcal{M}_{[\alpha]}$, (8.156.1–3) are satisfied, but also because it shows that $\mathcal{M}$, which is obviously uniquely specified by (8.156.1–3), is definable over $(M; \in, \mathbb{P})$.

Definition [GB] *Suppose $(M; \in, \mathbb{P})$ is a transitive model of ZF^{P}. For each $\alpha \in \mathrm{Ord} \cap M$, let*

$$M^{\mathbb{P}}_{\alpha} = \{x \in M \mid \theta^{(M; \in, \mathbb{P})}(\alpha, x)\},$$

where θ is the formula referred to in (8.156). Thus,

1. $M^{\mathbb{P}}_{0} = 0$;
2. $M^{\mathbb{P}}_{\alpha} = \bigcup_{\beta \in \alpha} M^{\mathbb{P}}_{\beta}$ *for limit α;*
3. $M^{\mathbb{P}}_{\alpha+1} = M \cap \mathcal{P}(M^{\mathbb{P}}_{\alpha} \times |\mathbb{P}|)$.

$M^{\mathbb{P}} \stackrel{\text{def}}{=} \bigcup_{\alpha \in M} M^{\mathbb{P}}_{\alpha}$.

(8.157) **Definition** [GB] *Suppose $(M; \in, \mathbb{P})$ is a transitive model of ZF^{P}. We define a mapping $x \mapsto \check{x}$ from M into $M^{\mathbb{P}}$ by $\in$-recursion:*

$$\check{x} \stackrel{\text{def}}{=} \{\langle \check{y}, \mathbf{1} \rangle \mid y \in x\}.^{47}$$

Note that if $\mathbb{P}$ is a proper class for M then $\{\langle \check{p}, p \rangle \mid p \in |\mathbb{P}|\}$—which we would ordinarily use as a forcing term denoting the generic filter—is also a proper class and is therefore not a term.

Recall[8.16] the definition of s^{V} as the signature obtained from s by the addition of a unary predicate symbol V. The following definition adds the new symbol P and a new symbol G to the language $\mathcal{L}^{M,\mathbb{P}}$ defined previously.[8.17]

Definition [GB] *The forcing language $\mathcal{L}^{M,\mathbb{P}}$ is constructed from the signature $\mathsf{s}^{M,\mathbb{P}}$, which is s^{V} extended by the addition of the members of $M^{\mathbb{P}}$ as constants, and P and G as predicate symbols.*

If $\mathfrak{M} = (M; \in, \mathbb{P})$ is a transitive model of ZF^{P} and G is a filter on $\mathbb{P}$ we define τ^{G} for $\tau \in M^{\mathbb{P}}$, $M[G]$, and $\mathfrak{M}[G]$ as before,[8.18] with $\mathsf{P}^{\mathfrak{M}[G]} = \mathbb{P}$ and $\mathsf{G}^{\mathfrak{M}[G]} = G$.

A theory of forcing over a transitive model $\mathfrak{M} = (M; \in, \mathbb{P})$ of ZF^{P} requires a relation $\Vdash^{M,\mathbb{P}}$ satisfying (8.29) with the requisite definability over $\mathfrak{M}$. Recall that if $\mathbb{P} \in M$ we are able to define[8.27] $\Vdash^{M,\mathbb{P}}$ for sentences $\tau \in \tau'$ and $\tau = \tau'$ by recursion on rank within M. Thus this relation is Δ_1 over M. If $\mathbb{P}$ is a proper class in the sense of M this straightforward approach does not work, because to ascertain whether, say, $p \Vdash \tau \in \tau'$, we must know whether $r \Vdash \tau_0 = \tau$ for all $r \leqslant q \leqslant p$, which is generally a proper class of conditions.

We must impose a condition that guarantees, in effect, that the recursion can be effected by looking at sets of extensions, rather than all extensions. The following condition is sufficient, and we will see that it is also necessary if generic extensions are to satisfy $\mathsf{Collection}$.

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[47] If $\mathbb{P}$ is a set we could let $\check{x} \stackrel{\text{def}}{=} \{\langle \check{y}, p \rangle \mid y \in x \wedge p \in |\mathbb{P}|\}$, but this will clearly not do if $\mathbb{P}$ is a proper class, so it is useful that $\mathbb{P}$ has a greatest element.

Definition [GB] *Suppose* $(M;\in,\mathbb{P})$ *is a transitive model of* ZF^{P}. $\mathbb{P}$ *is* pretame *(vis-à-vis* M*)* $\overset{\text{def}}{\iff}$ *for every* $m \in M$*,* m*-indexed family* $D = [D_{[i]} \mid i \in m] \subseteq M$ *definable over* $(M;\in,\mathbb{P})$*, and* $p \in |\mathbb{P}|$*, if* $D_{[i]}$ *is predense below* p *for every* $i \in m$*, then there exist* $q \leqslant p$ *and an* m*-indexed family* $d = [d_{[i]} \mid i \in m] \in M$*, such that* $\forall i \in m\,(d_{[i]} \subseteq D_{[i]})$ *and* $d_{[i]}$ *is predense below* q.

In other words, in the sense of $(M;\in,\mathbb{P})$, every definable set-indexed family of subclasses of $|\mathbb{P}|$ predense below p may be refined to an identically indexed family of sub*sets* of $|\mathbb{P}|$, predense below some fixed extension of p.

Recall that $A \subseteq |\mathfrak{S}|$ is definable over a structure $\mathfrak{S}$ iff there is a formula ϕ in the signature of $\mathfrak{S}$ with free variables $u, u_0, \ldots u_{n^-}$, and a $\{\phi\}$-satisfaction relation S for $\mathfrak{S}$, such that for some $a_0, \ldots, a_{n^-} \in |\mathfrak{S}|$,

$$A = \left\{a \in |\mathfrak{S}| \,\middle|\, \models^S \phi\begin{bmatrix} u & u_0 & \cdots & u_{n^-} \\ a & a_0 & \cdots & a_{n^-}\end{bmatrix}\right\}.$$

It is therefore equivalent to say that $\mathbb{P}$ *is pretame* vis-à-vis M *iff* $(M;\in,\mathbb{P}) \models \mathsf{ZFP}$, *which is* ZF^{P} *extended by the following axiom schema:*

Pretameness

⌜Suppose $p \in |\mathsf{P}|$ and $\forall i \in m\,\forall q \leqslant p\,\exists r\,((\!(\phi)\!)(i, r, a_0, \ldots, a_{n^-}) \wedge q \parallel r)$. Then there exist $q \leqslant p$ and $d = \langle d_i \mid i \in m\rangle$ such that for all $i \in m$, $\forall r \in d_i\ (\!(\phi)\!)(i, r, a_0, \ldots, a_{n^-})$ and $\forall q' \leqslant q\,\exists r \in d_i\ q' \parallel r$.⌝, where ϕ is any s^{P}-formula with $n+2$ free variables.

Note that any transitive model M of ZF with a partial order $\mathbb{P} \in M$ is naturally a model of ZFP, because the pretameness condition for $\mathbb{P}$ is trivially satisfied if $\mathbb{P}$ is a set, so the present formulation is not specific to class forcing, but subsumes the more usual sort of forcing, which we may refer to for specificity as *set forcing*.

8.12.1 The necessity of pretameness

We have introduced pretameness as a condition on a partial order $\mathbb{P} \subseteq M$ that renders the recursive definition of $\Vdash^{M,\mathbb{P}}$ definable over $(M;\in,\mathbb{P})$. We will see that it also guarantees that the first of the two potential defects of proper class forcing mentioned at the beginning of this section[8.155.1] does not occur. In fact, if $(M;\in,\mathbb{P}) \models \mathsf{ZFP}$ then ZF^- is $\mathbb{P}$-valid, where ZF^- consists of all the axioms of ZF other than Power.[48]

As noted above, pretameness is necessary as well as sufficient for this. For suppose $\mathfrak{M} = (M;\in,\mathbb{P})$ is a countable transitive model of ZFP and for all $\mathfrak{M}$-generic filters G on $\mathbb{P}$, $M[G] \models \mathsf{ZF}^-$. Suppose toward a contradiction that $m \in M$, $D = [D_{[i]} \mid i \in m] \subseteq M$ is definable over $\mathfrak{M}$, $p \in |\mathbb{P}|$, and $D_{[i]}$ is predense below p for every $i \in m$, but there do not exist $q \leqslant p$ and an m-indexed family d in M such that for all $i \in m$, $d_{[i]}$ is included in $D_{[i]}$ and is predense below q. Then in particular for any ordinal $\alpha \in M$, for any $q \leqslant p$, there exists $i \in m$ such that $D_{[i]} \cap M_\alpha$ is not predense below q, i.e. there exists $r \leqslant q$ such that $r \perp (D_{[i]} \cap M_\alpha)$. Thus $\{r \in |\mathbb{P}| \mid \exists i \in m\,(r \perp (D_{[i]} \cap M_\alpha))\}$ is dense below p for each $\alpha \in M$, and it is clearly $\mathfrak{M}$-definable.

Suppose G is $\mathfrak{M}$-generic on $\mathbb{P}$ and $p \in G$. Working in $M[G]$, which satisfies ZF^- by assumption, let $f : m \to \mathrm{Ord}$ be defined by the condition that for each $i \in m$, $f(i)$ is the least ordinal α such that $G \cap D_{[i]} \cap M_\alpha \neq 0$. Since each $D_{[i]}$ is predense below p, G meets each of these, so f is total. By virtue of $\mathsf{Collection}$, f is bounded,

[48] We will call a partial order $\mathbb{P}$ *tame* iff it is pretame and in addition $\Vdash^{\mathbb{P}} \mathsf{Power}$.

so there exists $\alpha \in \mathrm{Ord}$ such that for all $i \in m$, $G \cap D_{[i]} \cap M_\alpha \neq 0$. On the other hand, by the argument in the preceding paragraph, there exists $r \in G$ and $i \in m$ such that $r \perp (D_{[i]} \cap M_\alpha)$, so $G \cap D_{[i]} \cap M_\alpha = 0$.

8.12.2 Pretame forcing

(8.158) **Definition** [GB] *Suppose $(M; \in, \mathbb{P})$ is a transitive model of* ZFP. $A^{M,\mathbb{P}} \overset{\text{def}}{=}$ *the class of sentences of $\mathcal{L}^{M,\mathbb{P}}$ of the form $\tau \in \tau'$ or $\tau = \tau'$, where $\tau, \tau' \in M^{\mathbb{P}}$. In other words, $A^{M,\mathbb{P}}$ consists of the atomic sentences of $\mathcal{L}^{M\mathbb{P}}$ other than those of the form* $\mathsf{V}(\tau)$.

(8.159) **Theorem** [GB] *There exists a Π_1 s^{P}-formula θ with two free variables such that for any transitive model $(M; \in, \mathbb{P})$ of* ZFP, *if we let*

$$p \Vdash^{M,\mathbb{P}} \phi \overset{\text{def}}{\iff} \theta^{(M;\in,\mathbb{P})}(p, \phi)$$

for any $p \in |\mathbb{P}|$ and $\phi \in A^{M,\mathbb{P}}$, then for all $\tau, \tau' \in M^{\mathbb{P}}$

1. $$p \Vdash^{M,\mathbb{P}} \tau \in \tau' \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash^{M,\mathbb{P}} \tau_0 = \tau),$$

and

2. $$\begin{aligned} p \Vdash^{M,\mathbb{P}} \tau = \tau' \leftrightarrow & \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \to q \Vdash^{M,\mathbb{P}} \tau_0 \in \tau') \\ & \wedge \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \to q \Vdash^{M,\mathbb{P}} \tau_0 \in \tau). \end{aligned}$$

Proof See Note 10.29.

(8.160) *Theorem 8.159 provides the definition of the forcing relation $\Vdash^{M,\mathbb{P}}$ for sentences in $A^{M,\mathbb{P}}$ over an arbitrary transitive model $(M; \in, \mathbb{P})$ of* ZFP. *We complete the definition of $\Vdash^{M,\mathbb{P}}$ as for set forcing,*[8.29] *with the added stipulation that*

$$p \Vdash^{M,\mathbb{P}} \mathsf{G}(\tau) \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists r' \in |\mathbb{P}| \, (r \leqslant r' \wedge r \Vdash^{M,\mathbb{P}} \tau = \check{r}').$$

(8.32), and (8.34) go through essentially unchanged except for this addition. Likewise, we have (8.36) and (8.38).

Definability considerations are also the same as before, with the inconsequential exception that the forcing relation for sentences in $A^{M,\mathbb{P}}$ is now only asserted to be Π_1 over $\mathfrak{M}$, not Δ_1 as in the case of set forcing.

Theorems 8.44 and 8.45 on generic extensions are also valid in the general case.

At this point in the discussion of forcing above we introduced the boolean-valuation approach and developed the basic theory mostly in that framework. That approach does not work directly for class forcing, as the "elements" of the regular algebra $\mathfrak{R}\,\mathbb{P}$, when $\mathbb{P}$ is a proper class, are themselves proper classes, and thus are not elements *per se*, but it is easy to translate it into order-theoretic terms, and we will use the results of this analysis to the extent that it is applicable when $\mathbb{P}$ is a proper class in M and $(M; \in, \mathbb{P})$ models ZFP.

For this purpose, $[\![\phi]\!]$ should be translated as $\overline{\{p \mid p \Vdash \phi\}}$, and $x(y)$ (for x, y in the imagined structure $M^{\mathfrak{R}\,P}$) should be translated as $\overline{x \to \{y\}} = \{p \in |\mathbb{P}| \mid \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle y, r' \rangle \in x \, (r \leqslant r')\}$. Thus, for example, (8.65.3) becomes

$$y \in \operatorname{dom} x \to \{p \mid \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle y, r' \rangle \in x \;\, r \leqslant r'\} \subseteq \{p \mid p \Vdash y \in x\},$$

i.e.,

$$y \in \operatorname{dom} x \wedge \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle y, r' \rangle \in x\; r \leqslant r' \rightarrow p \Vdash y \in x,$$

but it does not have to be this complicated. It is neater simply to show that for all $y \in M^{\mathbb{P}}$ and $p \in |\mathbb{P}|$,

$$\langle y, p \rangle \in x \rightarrow p \Vdash y \in x.$$

For suppose this is true and suppose $y \in \operatorname{dom} x$ and $\forall q \leqslant p\, \exists r \leqslant q\, \exists \langle y, r' \rangle \in x\; r \leqslant r'$. Then $\forall q \leqslant p\, \exists r \leqslant q\, \exists r' \, (r \leqslant r' \wedge r' \Vdash y \in x)$. Thus, the class of conditions r such that $r \Vdash y \in x$ is dense below p, so[8.36] $p \Vdash y \in x$.

The following corresponds to (8.65) and (8.74).

(8.161) **Theorem** [GB] *Suppose* $(M; \in, \mathbb{P})$ *is a transitive model of* ZFP, $x, y, z \in M^{\mathbb{P}}$, *and* $p \in |\mathbb{P}|$.

1. $p \Vdash x = y \leftrightarrow p \Vdash y = x$.
2. $\Vdash x = x$.
3. $\langle y, p \rangle \in x \rightarrow p \Vdash y \in x$.
4. *If* ϕ *is a formula with the single free variable* u, *then*
   1. $p \Vdash \big(x = y \wedge \phi(x)\big) \rightarrow p \Vdash \phi(y)$.
   2. $p \Vdash \exists u \in x\, \phi \leftrightarrow \forall q \leqslant p\, \exists \langle y, r' \rangle \in x\, \exists r \leqslant q\, (r \leqslant r' \wedge r \Vdash \phi(y))$.
   3. $p \Vdash \forall u \in x\, \phi \leftrightarrow \forall q \leqslant p\, \forall \langle y, r' \rangle \in x\, (q \leqslant r' \rightarrow q \Vdash \phi(y))$.

8.12.3 Pretame generic extension preserves ZF$^-$

The following is the analog of (8.90).

Definition [GB] *Given* $x, y \in M^{\mathbb{P}}$,

1. $\{x, y\}^{\mathbb{P}} \stackrel{\text{def}}{=} \{\langle x, \mathbf{1} \rangle, \langle y, \mathbf{1} \rangle\}$. $\{x\}^{\mathbb{P}} \stackrel{\text{def}}{=} \{x, x\}^{\mathbb{P}} = \{\langle x, \mathbf{1} \rangle\}$.
2. $(x, y)^{\mathbb{P}} \stackrel{\text{def}}{=} \{\{x\}^{\mathbb{P}}, \{x, y\}^{\mathbb{P}}\}^{\mathbb{P}}$.

(8.162) **Theorem** [GB] *Suppose* $(M; \in, \mathbb{P})$ *is a transitive model of* ZFP. *Then every axiom of* $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$ *with the possible exception of* Power *is a* $M^{\mathbb{P}}$*-validity, where* $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$ *is* ZF *with the axiom schemas extended to all* s^{V}*-formulas.*

Proof See Note 10.30. □

We have now seen that the assumption of pretameness of a partial order $\mathbb{P}$ in the context of a structure $\mathfrak{M} = (M; \in, \mathbb{P})$ allows us to define $\Vdash^{M,\mathbb{P}}$ within $\mathfrak{M}$, and it allows us to show that $\mathfrak{M} \models \mathsf{ZF}^-$. The Power axiom is not valid for every pretame class forcing, so to obtain $\mathfrak{M} \models \mathsf{ZF}$ we must explicitly assume it.

Definition [GB] *Suppose* $\mathfrak{M} = (M; \in, \mathbb{P})$ *is a transitive model of* ZFP. *Then* $\mathfrak{M}$ *is* tame $\stackrel{\text{def}}{\Longleftrightarrow} \Vdash^{M,\mathbb{P}}$ Power.

If M is countable then there are only countable many subsets of M definable over $\mathfrak{M} = (M; \in, \mathbb{P})$, so for every $p \in |\mathbb{P}|$ there exists an $\mathfrak{M}$-generic filter on $\mathbb{P}$, and we may "argue with generic extensions" over countable models. Likewise, the method of "arguing in a generic extension"[8.5.2] is applicable in class forcing as in set forcing.

(8.163) **Theorem** [GB] *Suppose $(M;\in,\mathbb{P})$ is a transitive model of $\mathsf{ZFP}+\mathsf{AC}$. Then $\Vdash^{M,\mathbb{P}} \mathsf{ZFC}^-$, where it is to be understood that the axiom schemas of ZFC^- are extended to all s^{V}-formulas.*

Proof The proof given of (8.112) by arguing in a generic extension works. $\square^{8.163}$

As a simple and useful application of class forcing we will show how to generically add a wellordering of the universe to any transitive model of ZFC without adding any new sets.

(8.164) **Theorem** [GBC] *Suppose $(M;\in)$ is a transitive model of ZFC. Then there is a partial order $\mathbb{P}$ definable over $(M;\in)$ such that $\mathfrak{M}=(M;\in,\mathbb{P})$ is tame, and for any $\mathfrak{M}$-generic filter G on $\mathbb{P}$, there is a wellordering of $M[G]$ definable over $\mathfrak{M}[G]$. Specifically (and more formally), $\Vdash^{M,\mathbb{P}}$ $\ulcorner V=\mathsf{V}$ and $\bigcup \mathsf{G}:\mathrm{Ord}\overset{\mathrm{bij}}{\to} V\urcorner$.*

Proof Let $\mathbb{P}$ consist of all injections $p\in M$ such that $\operatorname{dom} p$ is an ordinal, ordered by reverse inclusion. To show that $\mathbb{P}$ is pretame, suppose $p\in|\mathbb{P}|$, κ is a regular cardinal in M, and $[D_{[\gamma]}\mid\gamma\in\kappa]$ is a κ-indexed family of subclasses of $|\mathbb{P}|$, each predense below p, definable over M.

Let $f:\kappa\to\mathrm{Ord}^M$ be defined as follows over M.

1. $f0$ is the least $\delta\in\mathrm{Ord}$ such that $p\in V_\delta$.
2. If $\operatorname{Lim}\alpha$ then $f\alpha=\bigcup_{\beta<\alpha}f\beta$.
3. Given $f\alpha$, $f(\alpha+1)$ is $\delta+1$, where δ is the least ordinal greater than $f\alpha$ such that for all $q\in V_{f\alpha}\cap|\mathbb{P}|$ and all $\gamma<\kappa$, if there exists $r\leqslant q$ and $s\in D_{[\gamma]}$ such that $r\leqslant s$, then there exist $r,s\in V_\delta$ such that $r\leqslant q$, $s\in D_{[\gamma]}$, and $r\leqslant s$.

Note that f is strictly increasing. Let $\delta_0=\sup_{\alpha<\kappa}f\alpha$, and let $X=V_{\delta_0}\cap|\mathbb{P}|$. By assumption, for each $\gamma<\kappa$, $D_{[\gamma]}$ is predense below p.

Hence, for any $q\leqslant p$, if $q\in X$ then there exist $r,s\in X$ such that $r\leqslant q$, $s\in D_{[\gamma]}$, and $r\leqslant s$.

Since $M\models\mathsf{AC}$ we may posit some fixed wellordering of X, which we use to define sequences $\langle p_\alpha\mid\alpha<\kappa\rangle$ and $\langle s_\alpha\mid\alpha<\kappa\rangle$ in X such that

1. $\langle p_\alpha\mid\alpha<\kappa\rangle$ is a decreasing sequence below p.
2. For each $\alpha\in\kappa$, $s_\alpha\in D_{[\alpha]}$ and $p_\alpha\leqslant s_\alpha$.

Suppose $\alpha<\kappa$ and we have defined p_β for all $\beta<\alpha$.

1. If $\alpha=0$ let $q=p_0$.
2. If $\alpha=\beta+1$ let $q=p_\beta$.
3. If $\operatorname{Lim}\alpha$, let $q=\bigcup_{\beta<\alpha}p_\beta$. Since κ is regular in M and f is strictly increasing, there exists $\delta<\delta_0$ such that $\{p_\beta\mid\beta<\alpha\}\subseteq V_\delta$. It follows that $\bigcup_{\beta<\alpha}p_\beta\in V_{\delta+1}\subseteq V_{\delta_0}$, so $q\in X$.

In each case, let p_α be the first $r\in X$ such that $r\leqslant q\wedge\exists s\in X\,(s\in D_{[\alpha]}\wedge r\leqslant s)$, and let s_α be the first $s\in X$ with this property.

Now let $q=\bigcup_{\alpha<\kappa}p_\alpha$, and let d be the κ-indexed family defined by the condition that $d_{[\gamma]}=\{s_\gamma\}$. Note that $d_{[\gamma]}\subseteq D_{[\gamma]}$. By construction, for each $\gamma\in\kappa$, $p_\gamma\leqslant s_\gamma$, so $q\leqslant s_\gamma$, and $d_{[\gamma]}$ is therefore trivially predense below q.

Since $\mathfrak{M} = (M; \in, \mathbb{P})$ is pretame, $\Vdash^{M,\mathbb{P}}$ is well defined, and $\Vdash^{M,\mathbb{P}} \mathsf{ZFC}^-$. Since $\mathbb{P}$ is α-closed for every $\alpha \in \mathrm{Ord}^M$, forcing with $\mathbb{P}$ adds no new functions from α into M. Since $\Vdash^{M,\mathbb{P}} \mathsf{AC}$, forcing with $\mathbb{P}$ adds no new subsets of M, and therefore adds no new sets whatsoever, because any new set of minimal rank would be a new subset of M. Thus, $\Vdash^{M,\mathbb{P}} \mathsf{Power}$, so $\Vdash^{M,\mathbb{P}} \mathsf{ZFC}$ ($\mathbb{P}$ is tame).

For any $\alpha \in \mathrm{Ord}^M$ and for any $x \in M$, $\{p \in |\mathbb{P}| \mid \alpha \in \operatorname{dom} p\}$ and $\{p \in |\mathbb{P}| \mid x \in \operatorname{im} p\}$ are dense in $\mathbb{P}$. Hence, $\Vdash^{M,\mathbb{P}} \ulcorner \bigcup \mathsf{G} : \mathrm{Ord} \overset{\text{bij}}{\to} \mathsf{V} \urcorner$. $\square$8.164

8.12.4 Easton's theorem

We are now in a position to state and prove Easton's theorem on adjusting the size of 2^κ for all regular cardinals κ.

(8.165) **Theorem** [GBC] *Suppose $(M; \in)$ is a transitive model of $\mathsf{ZFC} + \mathsf{GCH}$, and there is a wellordering of M definable over M. Suppose $F \subseteq M$ is a function definable over $(M; \in)$, and let F be a defined operation symbol for F, so that $\mathsf{F}^{(M;\in)} = F$. Suppose $\operatorname{dom} F$ consists of regular cardinals of $(M; \in)$ and $\operatorname{im} F$ consists of cardinals of $(M; \in)$, such that $(M; \in) \models \ulcorner$for all $\kappa, \lambda \in \operatorname{dom} \mathsf{F}$,*

1. *if $\kappa \leqslant \lambda$ then $\mathsf{F}\kappa \leqslant \mathsf{F}\lambda$; and*
2. *$\kappa < \operatorname{cf} \mathsf{F}\kappa \urcorner$.*

Then there exists a partial order $\mathbb{P} \subseteq M$ definable over M such that

1. *$\mathfrak{M} = (M; \in, \mathbb{P})$ is tame;*
2. *for every κ, λ, if $\mathfrak{M} \models \ulcorner [\kappa]$ is a cardinal with cofinality $[\lambda] \urcorner$, then $\Vdash^{M,\mathbb{P}} \ulcorner (\check{\kappa})$ is a cardinal with cofinality $(\check{\lambda}) \urcorner$; and*
3. *$\Vdash^{M,\mathbb{P}} \ulcorner \forall \kappa \in \operatorname{dom} \mathsf{F}\ 2^\kappa = \mathsf{F}\kappa \urcorner$.*

Proof Working in M we define the following classes. As usual, any reference to a defined proper class is to be interpreted by means of the definition as a statement about sets. Let $D = \{\langle \kappa, \beta, \alpha \rangle \mid \kappa \in \operatorname{dom} F \wedge \beta < \kappa \wedge \alpha < F\kappa\}$, and for any cardinal ν, let

1. $D^{<\nu} = \{\langle \kappa, \beta, \alpha \rangle \in D \mid \kappa < \nu\}$;
2. $D^{\leqslant\nu} = \{\langle \kappa, \beta, \alpha \rangle \in D \mid \kappa \leqslant \nu\}$;
3. $D^{\geqslant\nu} = \{\langle \kappa, \beta, \alpha \rangle \in D \mid \kappa \geqslant \nu\}$; and
4. $D^{>\nu} = \{\langle \kappa, \beta, \alpha \rangle \in D \mid \kappa > \nu\}$.

Since $\operatorname{dom} \mathsf{F}$ consists of regular cardinals, if ν is singular then $D^{\leqslant\nu} = D^{<\nu}$ and $D^{\geqslant\nu} = D^{>\nu}$.

Let $\mathbb{P} = (P; \leqslant)$, where $\leqslant = \supseteq$ and P is the set of functions $p : D \rightharpoonup 2$ such that

for every regular cardinal ν, $|\operatorname{dom} p \cap D^{\leqslant\nu}| < \nu$.

Given a cardinal λ, for each $p \in P$, let $p^{<\lambda} = p \restriction D^{<\lambda}$, etc. Let $P^{<\lambda} = \{p^{<\lambda} \mid p \in P\}$, etc. As before, $p \mapsto \langle p^{\leqslant\lambda}, p^{>\lambda} \rangle$ and $p \mapsto \langle p^{<\lambda}, p^{\geqslant\lambda} \rangle$ are isomorphisms of P with $P^{\leqslant\lambda} \times P^{>\lambda}$ and $P^{<\lambda} \times P^{\geqslant\lambda}$ for any cardinal λ.

(8.166) **Claim** $\mathfrak{M} = (M; \in, \mathbb{P})$ *is pretame.*

Proof Suppose $p \in |\mathbb{P}|$ and $[D_{[\gamma]} \mid \gamma \in \kappa]$ is a κ-indexed family of subclasses of $|\mathbb{P}|$ predense below p, definable over $\mathfrak{M}$ (i.e., over M, since $\mathbb{P}$ is definable over M).

Let κ_0 be a regular cardinal in M such that $\kappa \leqslant \kappa_0$ and $p \in P^{\leqslant \kappa_0}$. For $n \in \omega$, define κ_n by recursion so that κ_{n+1} is the least regular cardinal in M such that $F(\kappa_n) \leqslant \kappa_{n+1}$. Note that $\kappa_0 < \kappa_1 < \cdots$. Let $\lambda = \sup_{n \in \omega} \kappa_n$. Note that $\operatorname{cf} \lambda = \omega$.

Let $P_0 = P^{<\lambda}$. Note that P_0 is also $P^{<\lambda'}$, where $\lambda' = \ulcorner \lambda^+ \urcorner^M$. Let $P_1 = P^{\geqslant \lambda}$, and note that P_1 is also $P^{\geqslant \lambda'}$. Since $M \models \mathsf{GCH}$, $|P_0| = \lambda$, and we let $\big\langle \langle p_\alpha, i_\alpha \rangle \,\big|\, \alpha < \lambda \big\rangle$ be an enumeration in order type λ of the set of pairs $\langle p', i \rangle$ such that $p' \in P_0$, $p' \leqslant p$, and $i \in \kappa$.

P_1 is λ-closed, and we define a decreasing sequence $\langle s_\alpha \mid \alpha \in \lambda \rangle$ in P_1, and a sequence $\langle r_\alpha \mid \alpha \in \lambda \rangle$ in P, such that for all $\alpha \in \lambda$,

1. if $\alpha = 0$ then $s_\alpha = 0$;

2. $s_{\alpha+1}$ is the first (with respect to the definable wellordering of M posited in the statement of the theorem) $s \leqslant s_\alpha$ in P_1 such that there exists $r \in D_{[i_\alpha]}$ and $t \leqslant p_\alpha$ in P_0 such that $t \cup s \leqslant r$, and let r_α be the first such r (for s);

3. if $\operatorname{Lim} \alpha$ then $s_\alpha = \bigcup_{\beta < \alpha} s_\beta$.

Let $s = \bigcup_{\alpha \in \lambda} s_\alpha$, and let $R = \{r_\alpha \mid \alpha \in \lambda\}$.

Let $q = p \cup s$. For each $i \in \kappa$, let $d_{[i]} = R \cap D_{[i]}$. We will show that $d_{[i]}$ is predense below q. To this end, suppose $q' \leqslant q$. Then $q' = p' \cup s'$, with $p' \in P_0$ and $s' \in P_1$, $p' \leqslant p$ and $s' \leqslant s$. Let $\alpha \in \lambda$ be such that $p_\alpha = p'$ and $i_\alpha = i$. Then $r_\alpha \in d_{[i]}$, and there exists $t \leqslant p'$ in P_0 such that $t \cup s_{\alpha+1} \leqslant r_\alpha$. Thus, $t \cup s' \leqslant r_\alpha$, and $t \cup s' \leqslant p' \cup s' = q'$. Thus, $d_{[i]}$ is predense below q, as claimed. $\square^{8.166}$

The proof may now be completed as for (8.149). The computation of the sizes of powersets also shows that $\Vdash^{M,\mathbb{P}} \mathsf{Power}$. $\square^{8.165}$

Note that we have assumed in Theorem 8.165 that there is a wellordering of M definable over M. This is the case, for example, if $M \models \ulcorner V = L \urcorner$. We have also assumed that the function F is definable over M. We obtain a slightly more general result if we work with transitive models $\mathfrak{M} = (M; \in, A)$ in an expanded signature with an additional unary predicate represented by $A \subseteq M$, where $\mathfrak{M}$ models ZF extended so that the axiom schemas are stated for all formulas in the expanded signature. Note that any collection of such predicates of any arity may be encoded by a single unary predicate A. Note also that we have essentially subsumed this level of generality in our consideration of models $(M; \in, \mathbb{P})$, where $\mathbb{P} \subseteq M$ is a partial order, since we may always obtain an isomorphic partial order $\mathbb{P}'$ such that $|\mathbb{P}'| = A$. The function F and the supposed wellordering of M mentioned in the statement of (8.165) may now be regarded as encoded in A.

It is also worth noting that we do not have to assume the existence of a definable wellordering of M, as we may use $\mathsf{Collection}$ as in the proof of (8.164) to obtain a subset of P that is large enough to permit the construction to go through, and we may use AC to infer that this set is wellordered. We may also use (8.164) itself to obtain a generic extension of M with a definable wellordering of its universe, and then use the construction of (8.165) to adjust the sizes of powersets. This is an example of *iterated forcing*, which we discuss in Section 8.13.

8.12.5 Coding the universe by a real

As another example of proper class forcing we mention the following result of Jensen. The proof is beyond the scope of this book

(8.167) **Theorem** [GBC] *Suppose* $(M;\in,A)$ *is a transitive model of* ZFC. *Then there is an* $(M;\in,A)$*-definable partial order* $\mathbb{P}\subseteq M$*, such that*

1. $A=|\mathbb{P}|$ *(just a convenience);*
2. $\mathfrak{M}=(M;\in,\mathbb{P})$ *is tame;*
3. $\Vdash^{M,\mathbb{P}}$ ⌜*there exists* $r\subseteq\omega$ *such that* $V=L[r]$ *and* $\mathbb{P},\mathsf{G}$ *are definable from* r⌝.

Suppose we start with a transitive model of ⌜$V=L$⌝, extend it by Easton forcing to a model in which $2^\kappa=\kappa^{++}$ for every regular cardinal κ, and then extend this to a model of ⌜$V=L[r]$⌝. Note that, in general, forcing over a transitive model M of ZFC with a partial order of cardinality κ (in the sense of M) cannot increase the size of 2^λ for any cardinal $\lambda\geqslant\kappa$. Thus, set forcing over L cannot increase the powersets of an unbounded class of cardinals; hence, r is not in any set-generic extension of L.

It follows (as a theorem of S) that if ZF is consistent then so is ZFC+ ⌜there exists a real that is not set-generic over L⌝. Prior to Jensen's result, the only known examples of reals r such that $L[r]$ differs so significantly from L at all levels, were obtained from powerful hypotheses with consistency strength well beyond that of ZF.

8.13 Iterated forcing

We have previously mentioned the possibility of iterating forcing constructions. In the simplest case, we have a partial order $\mathbb{P}$ and a term $\dot{\mathbb{Q}}\in M^{\mathbb{P}}$ such that $\Vdash^{\mathbb{P}}$ ⌜$(\dot{\mathbb{Q}})$ is a partial order⌝. Suppose G is an M-generic filter on $\mathbb{P}$. Then $\dot{\mathbb{Q}}^G$ is a partial order in $M[G]$, and we may consider a generic extension $M[G][H]$ of $M[G]$, where H is a $M[G]$-generic filter on $\dot{\mathbb{Q}}^G$. We will now show how to accomplish this extension in one step.

Recall that we have supposed for technical reasons—which are particularly compelling in the case of iterated forcing—that all partial orders under consideration have a maximum element, denoted by '$\mathbf{1}$'.

(8.168) *We will suppose that* $\dot{\mathbf{1}}=\mathbf{1}^{\dot{\mathbb{Q}}}\in M^{\mathbb{P}}$ *is such that* $\Vdash^{\mathbb{P}}$ ⌜$(\dot{\mathbf{1}})$ *is the maximum element of* $(\dot{\mathbb{Q}})$⌝.

(8.169) **Definition** [GB] *Suppose* M *is a transitive model of* ZF, $\mathbb{P}\in M$ *is a partial order,* $p\in|\mathbb{P}|$, *and* $x\in M^{\mathbb{P}}$. *Let* $\alpha\in\mathrm{Ord}^M$ *be least such that* $\exists y\in M^{\mathbb{P}}_\alpha\; p\Vdash^{\mathbb{P}} y=x$. $[x]_p\overset{\mathrm{def}}{=}\{y\in M^{\mathbb{P}}_\alpha\mid p\Vdash^{\mathbb{P}} y=x\}$. *In this context* $[x]\overset{\mathrm{def}}{=}[x]_{\mathbf{1}^{\mathbb{P}}}$.

(8.170) **Definition** [ZF] *Suppose* M *is a transitive model of* ZF, $\mathbb{P}\in M$ *is a partial order, and* $\dot{\mathbb{Q}}\in M^{\mathbb{P}}$ *is such that* $\Vdash^{\mathbb{P}}$ ⌜$(\dot{\mathbb{Q}})$ *is a partial order*⌝. *We define the partial order* $\mathbb{P}*\dot{\mathbb{Q}}$ *as follows:*

1. $|\mathbb{P}*\dot{\mathbb{Q}}|=\{\langle p,[\dot q]_p\rangle\mid p\in|\mathbb{P}|\wedge\Vdash^{\mathbb{P}}\dot q\in|\dot{\mathbb{Q}}|\}$;

2. $\langle p_1, [\dot{q}_1]_{p_1}\rangle \leqslant^{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, [\dot{q}_0]_{p_0}\rangle \leftrightarrow p_1 \leqslant^{\mathbb{P}} p_0 \wedge p_1 \Vdash^{\mathbb{P}} \dot{q}_1 \leqslant^{\dot{\mathbb{Q}}} \dot{q}_0$.

To show that this definition is well made we observe that if $[\dot{q}_0']_{p_0} = [\dot{q}_0]_{p_0}$ *and* $[\dot{q}_1']_{p_1} = [\dot{q}_1]_{p_1}$ *then* $p_0 \Vdash \dot{q}_0' = \dot{q}_0$ *and* $p_1 \Vdash \dot{q}_1' = \dot{q}_1$, *so*

$$p_1 \leqslant^{\mathbb{P}} p_0 \wedge p_1 \Vdash^{\mathbb{P}} \dot{q}_1' \leqslant^{\dot{\mathbb{Q}}} \dot{q}_0' \leftrightarrow p_1 \leqslant^{\mathbb{P}} p_0 \wedge p_1 \Vdash^{\mathbb{P}} \dot{q}_1 \leqslant^{\dot{\mathbb{Q}}} \dot{q}_0.$$

In the following discussion we will omit the specifying superscripts on '$\leqslant$' when the context makes it clear which order is intended, and we will do the same for '$\Vdash$' as the occasion arises.

Note that $\mathbb{P} * \dot{\mathbb{Q}}$ is indeed a partial order, since it is antisymmetric. For if $\langle p_1, [\dot{q}_1]_{p_1}\rangle \leqslant \langle p_0, [\dot{q}_0]_{p_0}\rangle$ and $\langle p_0, [\dot{q}_0]_{p_0}\rangle \leqslant \langle p_1, [\dot{q}_1]_{p_1}\rangle$ then, letting $p = p_0 = p_1$, $p \Vdash \dot{q}_0 = \dot{q}_1$, so $[\dot{q}_0]_{p_0} = [\dot{q}_0]_p = [\dot{q}_1]_p = [\dot{q}_1]_{p_1}$; hence, $\langle p_0, [\dot{q}_0]_{p_0}\rangle = \langle p_1, [\dot{q}_1]_{p_1}\rangle$. This is the purpose of defining $\mathbb{P} * \dot{\mathbb{Q}}$ in terms of the equivalence classes $[\dot{q}]_p$, rather than directly in terms of $\dot{q}$, which would define $\mathbb{P} * \dot{\mathbb{Q}}$ as a preorder. It should be noted that preorders serve most forcing purposes just as well as partial orders, and some authors indeed develop the entire theory of forcing in terms of preorders—also referred to as *quasi-orders*. In this case one must nevertheless from time to time pay attention to the equivalent partial orders, e.g., in cardinality computations when their size is important. Note also that it would be inadvisable to define $\mathbb{P} * \dot{\mathbb{Q}}$ as $\{\langle p, \dot{q}\rangle \mid p \in |\mathbb{P}| \wedge \dot{q} \in M^{\mathbb{P}} \wedge \Vdash^{\mathbb{P}} \dot{q} \in |\dot{\mathbb{Q}}|\}$, since this is a proper class over M; thus, some restriction on $\dot{q}$, such as to minimum rank, is necessary in any event.

(8.171) **Definition** [GB] *Under the conditions of (8.170) suppose G is an M-generic filter on $\mathbb{P}$, $\mathbb{Q} = \dot{\mathbb{Q}}^G$, and H is an $M[G]$-generic filter on $\mathbb{Q}$.*

$$G * H \stackrel{\text{def}}{=} \{\langle p, [\dot{q}]_p\rangle \in |\mathbb{P} * \dot{\mathbb{Q}}| \mid p \in G \wedge \dot{q}^G \in H\}.$$

The following theorem corresponds to the comment justifying (8.170.2).

(8.172) **Theorem** [GB] *Under the conditions of (8.171)*

1. *for any $p, \dot{q}$ such that $\langle p, [\dot{q}]_p\rangle \in |\mathbb{P} * \dot{\mathbb{Q}}|$,*

$$\langle p, [\dot{q}]_p\rangle \in G * H \leftrightarrow p \in G \wedge \dot{q}^G \in H;$$

and

2. *$G * H$ is a filter on $\mathbb{P} * \dot{\mathbb{Q}}$.*

Proof The first assertion follows directly from the fact that if $[\dot{q}']_p = [\dot{q}]_p$ and $p \in G$ then $\dot{q}^G = \dot{q}'^G$.

For the second assertion, we first observe that $G*H$ is nonempty. Next, suppose $\langle p_1, [\dot{q}_1]_{p_1}\rangle \in G * H$ and $\langle p_1, [\dot{q}_1]_{p_1}\rangle \leqslant \langle p_0, [\dot{q}_0]_{p_0}\rangle$. Then by virtue of (8.172.1), $p_1 \in G$ and $\dot{q}_1^G \in H$. Since $p_1 \Vdash \dot{q}_1 \leqslant \dot{q}_0$, $\dot{q}_0^G \in H$, so $\langle p_0, [\dot{q}_0]_{p_0}\rangle \in G * H$. A similar argument shows that any two elements of $G*H$ have a common extension in $G*H$. $\square_{8.172}$

(8.173) **Theorem** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, $\dot{\mathbb{Q}} \in M^{\mathbb{P}}$ is such that $\Vdash^{\mathbb{P}}$ ⌜$(\dot{\mathbb{Q}})$ is a partial order⌝, and G is an M-generic filter on $\mathbb{P}$. Then $|\dot{\mathbb{Q}}^G| = \{\dot{q}^G \mid \Vdash^{\mathbb{P}} \dot{q} \in |\dot{\mathbb{Q}}|\}$.*

Remark In other words, every member of $|\dot{\mathbb{Q}}^G|$ has a name that denotes a member of $\dot{\mathbb{Q}}$ in every $\mathbb{P}$-generic extension of M.

Proof Suppose $q \in |\dot{\mathbb{Q}}^G|$. Let $\dot{q}$ be such that $\dot{q}^G = q$. Replacing $\dot{\mathbf{1}}$ and $\dot{q}$ by their regularizations[8.77] if necessary, we may assume that they are regular terms, i.e., members of $\bar{M}^{\mathbb{P}}$. Let $p_0 \in G$ be such that $p_0 \Vdash \dot{q} \in |\dot{\mathbb{Q}}|$. Let

$$\dot{q}' = \{\langle x, p\rangle \in \dot{q} \mid p \in \overline{\{p_0\}}\} \cup \{\langle x, p\rangle \in \dot{\mathbf{1}} \mid p \in \{p_0\}^{\perp}\}.$$

Note that $\overline{\{p_0\}}$ and $\{p_0\}^{\perp}$ are regular sets that are complementary elements of $\mathfrak{R}\,\mathbb{P}$. Note also that for any $x \in M^{\mathbb{P}}$, $\dot{q}^{\rightarrow}\{x\}$ and $\dot{\mathbf{1}}^{\rightarrow}\{x\}$ are in $\mathfrak{R}\,\mathbb{P}$, and

$$\begin{aligned}\dot{q}'^{\rightarrow}\{x\} &= \big((\dot{q}^{\rightarrow}\{x\}) \cap \overline{\{p_0\}}\big) \cup \big((\dot{\mathbf{1}}^{\rightarrow}\{x\}) \cap \{p_0\}^{\perp}\big)\\ &= \big((\dot{q}^{\rightarrow}\{x\}) \wedge \overline{\{p_0\}}\big) \cup \big((\dot{\mathbf{1}}^{\rightarrow}\{x\}) \wedge \{p_0\}^{\perp}\big),\end{aligned}$$

using algebraic notation to the extent applicable. $\dot{q}'^{\rightarrow}\{x\}$ is not necessarily in $\mathfrak{R}\,\mathbb{P}$, but this does not matter.

Thus, for any x, if G meets $\dot{q}'^{\rightarrow}\{x\}$ then G meets $\dot{q}^{\rightarrow}\{x\}$, since G does not meet $\{p_0\}^{\perp}$. Conversely, if G meets $\dot{q}^{\rightarrow}\{x\}$ then, since $X = \overline{\{p_0\}} \cup \{p_0\}^{\perp}$ is dense and $\dot{q}^{\rightarrow}\{x\}$ is open, G meets $(\dot{q}^{\rightarrow}\{x\}) \cap X$. So G meets $(\dot{q}^{\rightarrow}\{x\}) \cap \overline{\{p_0\}}$, so G meets $\dot{q}'^{\rightarrow}\{x\}$. Hence,

$$\begin{aligned}\dot{q}'^G &= \{x^G \mid G \text{ meets } \dot{q}'^{\rightarrow}\{x\}\}\\ &= \{x^G \mid G \text{ meets } \dot{q}^{\rightarrow}\{x\}\}\\ &= \dot{q}^G.\end{aligned}$$

It suffices therefore to show that $\Vdash \dot{q}' \in |\dot{\mathbb{Q}}|$. This can be shown directly, but it is easier and more intuitive to "argue in a generic extension". Suppose G' is any M-generic filter on $\mathbb{P}$. Since $\overline{\{p_0\}}$ and $\{p_0\}^{\perp}$ are complementary elements of $\mathfrak{R}\,\mathbb{P}$, G' meets one or the other, but not both. If G' meets $\overline{\{p_0\}}$ then $\dot{q}'^{G'} = \dot{q}^{G'}$, as we have just shown, so $M[G'] \models \dot{q}' \in |\dot{\mathbb{Q}}|$. Likewise, if G' meets $\{p_0\}^{\perp}$ then by the corresponding argument, $\dot{q}'^{G'} = \dot{\mathbf{1}}^{G'}$, so $M[G'] \models \dot{q}' \in |\dot{\mathbb{Q}}|$. Thus, $\Vdash \dot{q}' \in |\dot{\mathbb{Q}}|$, as claimed. $\square$[8.173]

(8.174) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, and $\dot{\mathbb{Q}} \in M^{\mathbb{P}}$ is such that $\Vdash^{\mathbb{P}} \ulcorner(\dot{\mathbb{Q}})$ is a partial order$\urcorner$.*

1. *Suppose G is an M-generic filter on $\mathbb{P}$, $\mathbb{Q} = \dot{\mathbb{Q}}^G$, and H is an $M[G]$-generic filter on $\mathbb{Q}$. Then*
   1. *$G * H$ is an M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$; and*
   2. *$M[G * H] = M[G][H]$.*
2. *Suppose I is an M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$. Let*

   $$\begin{aligned}G &= \{p \in |\mathbb{P}| \mid \langle p, [\dot{\mathbf{1}}]_p\rangle \in I\}\\ H &= \{\dot{q}^G \mid \exists p\, \langle p, [\dot{q}]_p\rangle \in I\}.\end{aligned}$$

   Then
   1. *G is an M-generic filter on $\mathbb{P}$;*
   2. *H is an $M[G]$-generic filter on $\mathbb{Q}$; and*
   3. *$I = G * H$.*

Proof 1.1 We've already shown that $G * H$ is a filter.[8.172.2] Suppose $Z \in M$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. We will show that $G * H$ meets Z. Working in $M[G]$, let $Y = \{\dot{q}^G \mid \exists p \in G\, \langle p, [\dot{q}]_p \rangle \in Z\}$.

We will show that Y is dense in $\mathbb{Q}$. To this end, suppose $q_0 \in |\mathbb{Q}|$. Let $\dot{q}_0$ be such that $\dot{q}_0^G = q_0$ and $\Vdash \dot{q}_0 \in |\dot{\mathbb{Q}}|$.[8.173] Working in M, let $X = \{p \in |\mathbb{P}| \mid \exists \dot{q}_1\, (p \Vdash \dot{q}_1 \leqslant \dot{q}_0 \wedge \langle p, [\dot{q}_1]_p \rangle \in Z)\}$.

We will show that X is dense in $\mathbb{P}$. To this end, suppose $p_0 \in |\mathbb{P}|$. Then $\langle p_0, [\dot{q}_0]_p \rangle \in |\mathbb{P} * \dot{\mathbb{Q}}|$. Let $\langle p_1, [\dot{q}_1]_{p_1} \rangle \leqslant \langle p_0, [\dot{q}_0]_{p_0} \rangle$ be such that $\langle p_1, [\dot{q}_1]_{p_1} \rangle \in Z$. Then $p_1 \Vdash \dot{q}_1 \leqslant \dot{q}_0$,[8.170.2] so $p_1 \in X$ and $p_1 \leqslant p_0$.

Thus, X is dense in $\mathbb{P}$, and since G is an M-generic filter on $\mathbb{P}$, there exists $p_1 \in G \cap X$. Let $\dot{q}_1$ be such that $p_1 \Vdash \dot{q}_1 \leqslant \dot{q}_0$ and $\langle p_1, [\dot{q}_1]_{p_1} \rangle \in Z$. Let $q_1 = \dot{q}_1^G$. Then $q_1 \leqslant q_0$, and there exists $p \in G$ such that $\langle p, [\dot{q}_1]_p \rangle \in Z$, viz., p_1, so $q_1 \in Y$.

Thus, Y is dense in $\mathbb{Q}$, and since H is an $M[G]$-generic filter on $\mathbb{Q}$, there exists $q \in H \cap Y$. Let $\dot{q}$ and $p \in G$ be such that $\dot{q}^G = q$ and $\langle p, [\dot{q}]_p \rangle \in Z$. Then $\langle p, [\dot{q}]_p \rangle \in G * H$,[8.171] so $G * H$ meets Z.

1.2 Since $M[G * H]$ and $M[G][H]$ are models of ZF, it is enough to show that $G, H \in M[G * H]$ and $G * H \in M[G][H]$, which is straightforward.

2.1 Note that if, for some $\dot{q}$, $\langle p, [\dot{q}]_p \rangle \in I$, then $\langle p, [\dot{\mathbf{1}}]_p \rangle \in I$, so $p \in G$. It is easy to show that G is a filter. Suppose $X \in M$ is dense in $\mathbb{P}$. Then $Y = \{\langle p, [\dot{q}]_p \rangle \mid p \in X\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Hence, there exists $\langle p, [\dot{q}]_p \rangle \in I \cap Y$. $\langle p, [\dot{\mathbf{1}}]_p \rangle \in I$, so $p \in G$. Hence, G is M-generic.

2.2 We first show that H is a filter. It is clearly nonempty. Suppose $\dot{q}_0^G, \dot{q}_0'^G \in H$. Let p_0, p_0' be such that $\langle p_0, [\dot{q}_0]_{p_0} \rangle, \langle p_0', [\dot{q}_0']_{p_0'} \rangle \in I$. Let $\langle p_1, [\dot{q}_1]_{p_1} \rangle$ be a common extension of $\langle p_0, [\dot{q}_0]_{p_0} \rangle$ and $\langle p_0', [\dot{q}_0']_{p_0'} \rangle$ in I. Then $\dot{q}_1^G \in H$. Since $p_1 \in G$, $\dot{q}_1^G \leqslant \dot{q}_0^G, \dot{q}_0'^G$.

Now suppose $q_1 \in H$ and $q_1 \leqslant q_0$. Let $\dot{q}_1$ and p_1 be such that $\dot{q}_1^G = q_1$ and $\langle p_1, [\dot{q}_1]_{p_1} \rangle \in I$. Note that $p_1 \in G$. Let $\dot{q}_0$ be such that $\dot{q}_0^G = q_0$ and $\Vdash \dot{q}_0 \in |\dot{\mathbb{Q}}|$, and let $p \in G$ be such that $p \Vdash \dot{q}_1 \leqslant \dot{q}_0$. Let p' be a common extension of p_1 and p in G. Then $\langle p', [\dot{\mathbf{1}}]_{p'} \rangle \in I$. Let $\langle p_0, [\dot{q}']_{p_0} \rangle$ be a common extension of $\langle p_1, [\dot{q}_1]_{p_1} \rangle$ and $\langle p', [\dot{\mathbf{1}}]_{p'} \rangle$ in I. Then $p_0 \Vdash \dot{q}' \leqslant \dot{q}_1$ and $p_0 \Vdash \dot{q}_1 \leqslant \dot{q}_0$, so $\langle p_0, [\dot{q}']_{p_0} \rangle \leqslant \langle p_0, [\dot{q}_0]_{p_0} \rangle$. Since I is a filter, $\langle p_0, [\dot{q}_0]_{p_0} \rangle \in I$, so $q_0 = \dot{q}_0^G \in H$.

Finally, suppose $X \in M[G]$ is dense in $\mathbb{Q}$. Let $\dot{X} \in M^{\mathbb{P}}$ be such that $\dot{X}^G = X$, and let $p_0 \in G$ be such that $p_0 \Vdash \ulcorner (\dot{X})$ is dense in $(\dot{\mathbb{Q}})\urcorner$. By an adjustment similar to that used in the proof of (8.173), involving $\{p_0\}$ and $\{p_0\}^\perp$, we obtain a term $\dot{X}' \in M^{\mathbb{P}}$ such that for any M-generic filter G' on $\mathbb{P}$, if $p_0 \in G'$ then $\dot{X}'^{G'} = \dot{X}^{G'}$, and if $p_0 \notin G'$ then $\dot{X}'^{G'} = |\dot{\mathbb{Q}}^{G'}|$. Then $\dot{X}'^G = X$ and $\Vdash^{\mathbb{P}} \ulcorner (\dot{X}')$ is dense in $(\dot{\mathbb{Q}})\urcorner$. Thus, $Y = \{\langle p, [\dot{q}]_p \rangle \mid p \Vdash \dot{q} \in \dot{X}'\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$,[8.173] so there exists $\langle p, [\dot{q}]_p \rangle \in I \cap Y$. Since $\langle p, [\dot{q}]_p \rangle \in I$, $p \in G$ and $\dot{q}^G \in H$. Since $\langle p, [\dot{q}]_p \rangle \in Y$ and $p \in G$, $\dot{q}^G \in \dot{X}'^G = X$. Hence, H meets X.

2.3 Clearly, if $\langle p, [\dot{q}]_p \rangle \in I$ then $\langle p, [\dot{q}]_p \rangle \in G * H$. Conversely, suppose $\langle p, [\dot{q}]_p \rangle \in G * H$. Then $p \in G$, i.e., $\langle p, [\dot{\mathbf{1}}]_p \rangle \in I$, and $\dot{q}^G \in H$. Hence, $\exists \dot{q}'$ such that $\dot{q}'^G = \dot{q}^G$ and $\exists p_1\, \langle p_1, [\dot{q}']_{p_1} \rangle \in I$. Let $p_2 \in G$ be such that $p_2 \leqslant p$ and $p_2 \Vdash \dot{q}' = \dot{q}$. Then $\langle p_2, [\dot{\mathbf{1}}]_{p_2} \rangle \in I$, so there exists $\langle p_3, [\dot{q}'']_{p_3} \rangle \in I$ such that p_3 extends both p_1 and p_2, and $p_3 \Vdash \dot{q}'' \leqslant \dot{q}'$. It follows that $p_3 \Vdash \dot{q}'' \leqslant \dot{q}$, so $\langle p_3, [\dot{q}'']_{p_3} \rangle \leqslant \langle p, [\dot{q}]_p \rangle$; hence, $\langle p, [\dot{q}]_p \rangle \in I$. $\square$[8.174]

In the context of Theorem 8.174 we have for each $x \in M[G]$ the canonical $\dot{\mathbb{Q}}^G$-term $\check{x}$ such that $\check{x}^H = x$ for any $M[G]$-generic filter H on $\dot{\mathbb{Q}}^G$. We now define for each $\mathbb{P}$-term $\dot{x}$ a canonical $\mathbb{P} * \dot{\mathbb{Q}}$-term $\iota\dot{x}$ such that for any M-generic filter $G * H$ on $\mathbb{P} * \dot{\mathbb{Q}}$, $(\iota\dot{x})^{G*H} = \dot{x}^G$.

(8.175) **Definition** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and $\dot{\mathbb{Q}} \in M^{\mathbb{P}}$ is such that $\Vdash^{\mathbb{P}} \ulcorner(\dot{\mathbb{Q}})$ is a partial order$\urcorner$. We define $\iota_{\mathbb{P},\dot{\mathbb{Q}}}\dot{x}$ recursively for $\dot{x} \in M^{\mathbb{P}}$ so that*

$$\iota_{\mathbb{P},\dot{\mathbb{Q}}}\dot{x} = \left\{\left\langle \iota_{\mathbb{P},\dot{\mathbb{Q}}}\dot{y}, \langle p, [\dot{\mathbf{1}}]_p\rangle\right\rangle \,\middle|\, \langle \dot{y}, p\rangle \in \dot{x}\right\}.$$

(8.176) **Theorem** [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and $\dot{\mathbb{Q}} \in M^{\mathbb{P}}$ is such that $\Vdash^{\mathbb{P}} \ulcorner(\dot{\mathbb{Q}})$ is a partial order$\urcorner$. Then for any $\dot{x} \in M^{\mathbb{P}}$, $(\iota_{\mathbb{P},\dot{\mathbb{Q}}}\dot{x})^{G*H} = \dot{x}^G$.*

Proof By induction, letting $\iota = \iota_{\mathbb{P},\dot{\mathbb{Q}}}$:

$$\begin{aligned}(\iota\dot{x})^{G*H} &= \{z^{G*H} \mid \exists r \in G * H\ \langle z, r\rangle \in \iota\dot{x}\}\\ &= \{(\iota\dot{y})^{G*H} \mid \exists\langle p, [\dot{\mathbf{1}}]_p\rangle \in G * H\ \langle \dot{y}, p\rangle \in \dot{x}\}\\ &= \{\dot{y}^G \mid \exists p \in G\ \langle \dot{y}, p\rangle \in \dot{x}\}\\ &= \dot{x}^G.\end{aligned}$$

□8.176

The treatment of iterated forcing in terms of boolean algebras is particularly elegant and instructive. Analogously to (8.169) we will make use of canonical representatives of equivalence classes of elements of $V^{\mathfrak{A}}$ modulo the identity relation. The following lemma captures the basic idea.

(8.177) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra. Suppose $x \in M^{\mathfrak{A}}$, $X \in M$, and $\operatorname{dom} x \subseteq X \subseteq M^{\mathfrak{A}}$. Let $y \in M^{\mathfrak{A}}$ be such that $\operatorname{dom} y = X$ and $\forall z \in X\ y(z) = [\![z \in x]\!]$. Then $[\![y = x]\!] = \mathbf{1}$.*

Proof We must show[8.64.2] that

$$\bigwedge_{z \in \operatorname{dom} x} \big(x(z) \to [\![z \in y]\!]\big) \wedge \bigwedge_{z \in \operatorname{dom} y} \big(y(z) \to [\![z \in x]\!]\big) = \mathbf{1},$$

i.e., that

$$\forall z \in \operatorname{dom} x\ \big(x(z) \leqslant [\![z \in y]\!]\big)$$

and

$$\forall z \in \operatorname{dom} y\ \big(y(z) \leqslant [\![z \in x]\!]\big).$$

The first follows from the fact that for all $z \in \operatorname{dom} x$

$$x(z) \leqslant [\![z \in x]\!] = y(z) \leqslant [\![z \in y]\!],$$

and the second follows from the fact that $y(z) = [\![z \in x]\!]$. □8.177

(8.178) **Definition** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra. An element $x \in M^{\mathfrak{A}}$ is* canonical $\overset{\text{def}}{\iff}$ *there exists $\alpha \in \mathrm{Ord}^M$ such that*

1. $\operatorname{dom} x = M^{\mathfrak{A}}_{\alpha}$;
2. $\forall z \in M^{\mathfrak{A}}_{\alpha}\ x(z) = [\![z \in x]\!]$; *and*
3. $\forall y \in M^{\mathfrak{A}}_{\alpha}\ [\![x = y]\!] \neq \mathbf{1}$.

(8.179) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra. Suppose $x \in M^{\mathfrak{A}}$.*

1. *Then there exists a unique canonical x' such that $[\![x = x']\!] = \mathbf{1}$.*
2. *Let X be the class of canonical terms y such that $[\![y \in x]\!] = \mathbf{1}$. Then X is a set (in M).*

Proof 1 Let $\alpha \in \mathrm{Ord}^M$ be least such that there exists $y \in M^{\mathfrak{A}}$ such that $[\![x = y]\!] = \mathbf{1}$ and $\operatorname{dom} y \subseteq M^{\mathfrak{A}}_{\alpha}$, let $y \in M^{\mathfrak{A}}$ be such that $[\![x = y]\!] = \mathbf{1}$ and $\operatorname{dom} y \subseteq M^{\mathfrak{A}}_{\alpha}$, and let x' be the function with domain $M^{\mathfrak{A}}_{\alpha}$ such that $\forall z \in M^{\mathfrak{A}}_{\alpha}\ x'(z) = [\![z \in y]\!]$. Then[8.177] $[\![x' = y]\!] = \mathbf{1}$, so $[\![x' = x]\!] = \mathbf{1}$, and $\forall z \in M^{\mathfrak{A}}_{\alpha}\ x'(z) = [\![z \in x']\!]$. By virtue of the minimality of α, x' is canonical.

To show uniqueness, it is enough to show that if x_0 and x_1 are canonical and $[\![x_0 = x_1]\!] = \mathbf{1}$, then $x_0 = x_1$. Suppose $\operatorname{dom} x_0 = M^{\mathfrak{A}}_{\alpha_0}$ and $\operatorname{dom} x_1 = M^{\mathfrak{A}}_{\alpha_1}$. If $\alpha_0 = \alpha_1$ then $x_0 = x_1$, so we may suppose without loss of generality that $\alpha_0 < \alpha_1$. Then (8.178.3) is violated for $\alpha = \alpha_1$.

2 It is enough to show that an element y of X is uniquely determined by the function $\langle [\![y = z]\!] \mid z \in \operatorname{dom} x\rangle$, because any such function is in M, and the set of such functions is an M-definable subset of ${}^{\operatorname{dom} x}|\mathfrak{A}| \cap M = \ulcorner {}^{\operatorname{dom} x}|\mathfrak{A}| \urcorner^M$, which is a set in M. Thus, suppose $y_0, y_1 \in X$ and $\forall z \in \operatorname{dom} x\ [\![y_0 = z]\!] = [\![y_1 = z]\!]$. By virtue of the uniqueness of canonical representatives[8.179.1] it suffices to show that $[\![y_0 = y_1]\!] = \mathbf{1}$. Arguing with generic extensions, it is enough to show that for any M-generic G on $\mathfrak{A}$, $y_0^G = y_1^G$. Since $[\![y_0 \in x]\!] = \mathbf{1}$, $y_0^G \in x^G$, so $y_0^G = z^G$ for some $z \in \operatorname{dom} x$. It follows that $[\![y_0 = z]\!] \in G$, so $[\![y_1 = z]\!] \in G$, so $y_1^G = z^G = y_0^G$. $\square$[8.179]

(8.180) **Definition** [GB] *Suppose M is a transitive model of* ZFC*, $\mathfrak{A} \in M$ is an M-complete boolean algebra and $\dot{\mathfrak{B}} \in M^{\mathfrak{A}}$ is such that*

$$[\![(\dot{\mathfrak{B}}) \textit{ is a complete boolean algebra}]\!]^{\mathfrak{A}} = \mathbf{1}.$$

We define the boolean algebra $\mathfrak{A} * \dot{\mathfrak{B}}$ as follows:

1. $|\mathfrak{A} * \dot{\mathfrak{B}}| = \left\{ \dot{b} \,\middle|\, \dot{b} \in M^{\mathfrak{A}} \wedge \dot{b} \textit{ is canonical} \wedge [\![\dot{b} \in |\dot{\mathfrak{B}}|]\!]^{\mathfrak{A}} = \mathbf{1} \right\}$. *Note that $|\mathfrak{A} * \dot{\mathfrak{B}}|$ is a set.*[8.179.2]
2. *The boolean operations are defined in the natural way. For example, letting $\mathfrak{C} = \mathfrak{A} * \dot{\mathfrak{B}}$, so that $|\mathfrak{C}|$ consists of canonical $\mathfrak{A}$-terms for members of $\dot{\mathfrak{B}}$, given $c_0, c_1 \in |\mathfrak{C}|$, $c_0 \vee^{\mathfrak{C}} c_1$ is the unique canonical $\mathfrak{A}$-term c that with $\mathfrak{A}$-value $\mathbf{1}$ is the join of c_0 and c_1 in $\dot{\mathfrak{B}}$.*[49] *The same goes for $\wedge$ and $\neg$, and we also let $c_0 \leqslant^{\mathfrak{C}} c_1$ iff $[\![c_0 \leqslant^{\dot{\mathfrak{B}}} c_1]\!] = \mathbf{1}$.*

We will often work with ground models of ZFC so as to have (8.109) available, and we will often work in GBC, so V models ZFC and is a suitable ground model.

[49]In terms of generic extensions: c is such that for any M-generic G on $\mathfrak{A}$, c^G is the join of c_0^G and c_1^G in $\dot{\mathfrak{B}}^G$. With a little work we can design a term from $\dot{\mathfrak{B}}$, c_0, and c_1, and any two such terms are equal with $\mathfrak{A}$-value $\mathbf{1}$, so there is a unique canonical such term.

(8.181) **Theorem** [GBC] *In the terminology of (8.180), $\mathfrak{C} = \mathfrak{A} * \dot{\mathfrak{B}}$ is a complete boolean algebra, and $\mathfrak{A}$ is completely embeddable in $\mathfrak{C}$, i.e., is isomorphic to a complete subalgebra*[3.166] *of $\mathfrak{C}$.*

Proof Suppose $X \subseteq |\mathfrak{C}|$. Let $\dot{X} = \{(\dot{b}, \mathbf{1}) \mid \dot{b} \in X\}$. Then $[\![\dot{X} \subseteq |\dot{\mathfrak{B}}|]\!] = \mathbf{1}$. Since $\ulcorner \dot{B}$ is a complete boolean algebra$\urcorner$ is also $\mathfrak{A}$-valid, by (8.109) there exists $\dot{b} \in M^{\mathfrak{A}}$ such that $\ulcorner \dot{b} = \bigvee \dot{X}\urcorner$ is $\mathfrak{A}$-valid. Let c be the canonical term such that $[\![c = \dot{b}]\!] = \mathbf{1}$. Then it is straightforward to show that c is the least upper bound of X in $\mathfrak{C}$.

(8.182) *We define the canonical embedding j of $\mathfrak{A}$ in $\mathfrak{C}$ by letting $ja = c$, where c is the unique canonical $\mathfrak{A}$-term such that*

$$[\![c = \mathbf{1}^{\dot{\mathfrak{B}}}]\!]^{\mathfrak{A}} = a \text{ and } [\![c = \mathbf{0}^{\dot{\mathfrak{B}}}]\!]^{\mathfrak{A}} = \neg\, a.$$

It is a routine exercise to show that j is a complete embedding of $\mathfrak{A}$ in $\mathfrak{C}$. □[8.181]

To relate this to the previous definition of iteration in terms of partial orders $\mathbb{P}$ and $\dot{\mathbb{Q}}$, we note that $\mathbb{P} * \dot{\mathbb{Q}}$ embeds densely in $\mathfrak{A} * \dot{\mathfrak{B}}$, where $\mathfrak{A} = \mathfrak{R}\,\mathbb{P}$ and $\dot{\mathfrak{B}}$ is such that $\ulcorner \dot{\mathfrak{B}} = \mathfrak{R}\,\dot{\mathbb{Q}}\urcorner$ is $\mathfrak{A}$-valid. (To make this legitimate, we use the canonical correspondence of $V^{\mathbb{P}}$ and $V^{\mathfrak{R}\,\mathbb{P}}$.[8.78])

(8.181) has an important converse.[8.184] The following definition will be used in its proof.

(8.183) **Definition** [GB] *We suppose Definition 8.90 extended in the natural way to finite sequences and to structures represented in the informal way we have been using, according to which $(M; A, \dots)$, for example, is a structure with domain M and predicates/operations $A, \dots$.*[50]

(8.184) **Theorem** [GBC] *Suppose $\mathfrak{C}$ is a complete boolean algebra, and $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$. Then ($\mathfrak{A}$ is of course complete and) there exists $\dot{\mathfrak{B}} \in V^{\mathfrak{A}}$ such that $\ulcorner \dot{\mathfrak{B}}$ is a complete boolean algebra$\urcorner$ is $\mathfrak{A}$-valid, and $\mathfrak{C} \cong \mathfrak{A} * \dot{\mathfrak{B}}$.*

Proof Let $\dot{F} \in \mathsf{V}^{\mathfrak{A}}$ be such that $\operatorname{dom} \dot{F} = \{\check{c} \mid c \in |\mathfrak{C}|\}$ and for all $c \in |\mathfrak{C}|$,

$$\dot{F}(\check{c}) = \bigvee \{a \in |\mathfrak{A}| \mid a \leqslant c\}.$$

Since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$ we need not specify whether the join is to be taken in $\mathfrak{A}$ or $\mathfrak{C}$; and since $\mathfrak{A}$ is complete, $\dot{F}(\check{c})$ is the greatest element of $\mathfrak{A}$ below c.

Arguing in a generic extension it is easy to show that $\ulcorner (\dot{F})$ is the upward closure of G in $(\check{\mathfrak{C}})\urcorner$ is $\mathfrak{A}$-valid. For suppose G is V-generic on $\mathfrak{A}$. Then

$$\begin{aligned} \dot{F}^G &= \bigl\{c \in |\mathfrak{C}| \bigm| \bigvee \{a \in |\mathfrak{A}| \mid a \leqslant c\} \in G\bigr\} \\ &= \{c \in |\mathfrak{C}| \mid \exists a \in G\ a \leqslant c\}. \end{aligned}$$

(8.185) *For each $c \in |\mathfrak{C}|$ let $\tilde{c} \in V^{\mathfrak{A}}$ be such that* $\operatorname{dom} \tilde{c} = \{\check{c}' \mid c' \in |\mathfrak{C}|\}$ *and for all $c' \in |\mathfrak{C}|$*

$$\tilde{c}(\check{c}') = \bigvee \{a \in |\mathfrak{A}| \mid a \leqslant (c \leftrightarrow c')\}.$$

[50] If we are to be complete, we must have a rule for creating a signature and assigning indices to $A, \dots$ appropriately, but this may safely be left to the reader's imagination.

Then for any V-generic G on $\mathfrak{A}$,

$$\tilde{c}^G = \{c' \in |\mathfrak{C}| \mid (c \leftrightarrow c') \in \dot{F}^G\},$$

so $\ulcorner(\tilde{c})$ is the equivalence class of $(\check{c})$ mod $(\dot{F})\urcorner$ is $\mathfrak{A}$-valid. (Strictly speaking, we refer to the equivalence relation $\equiv^{\dot{F}}$ that identifies $c_0, c_1 \in |\check{\mathfrak{C}}|$ iff $(c_0 \leftrightarrow c_1) \in \dot{F}$. Note that $\dot{F}^G \cap |\mathfrak{A}|$ is an ultrafilter (viz., G), so $\mathfrak{A}$ is reduced to the 2-element algebra by this quotient operation.)

(8.186)

1. *Let $\dot{B} \in V^{\mathfrak{A}}$ be such that* $\operatorname{dom} \dot{B} = \{\tilde{c} \mid c \in |\mathfrak{C}|\}$*, and for every $c \in |\mathfrak{C}|$, $\dot{B}(\tilde{c}) = \mathbf{1}$. Then in any $\mathfrak{A}$-generic extension $V[G]$, $B^G = |\mathfrak{C}/\dot{F}^G|$.*
2. *Let $\dot{\leqslant} \in V^{\mathfrak{A}}$ be such that* $\operatorname{dom} \dot{\leqslant} = \left\{\langle \tilde{c}, \tilde{c}' \rangle^{\mathfrak{A}} \,\middle|\, c, c' \in |\mathfrak{C}|\right\}$[8.181] *and for all $c, c' \in |\mathfrak{C}|$*

$$\dot{\leqslant}\big(\langle \tilde{c}, \tilde{c}' \rangle^{\mathfrak{A}}\big) = \bigvee\{a \in |\mathfrak{A}| \mid a \leqslant (c \rightarrow c')\}.$$

Then in any $\mathfrak{A}$-generic extension $V[G]$, $\dot{\leqslant}^G$ is the binary relation on $\dot{B}^G$ such that for any $c, c' \in |\mathfrak{C}|$, $(c/\dot{F}^G)\dot{\leqslant}^G(c'/\dot{F}^G)$ iff $(c \rightarrow c') \in \dot{F}^G$, i.e., $c \leqslant c'$ mod $\dot{F}^G$.

3. *Let $\dot{\mathfrak{B}} = (\dot{B}; \dot{\leqslant})^{\mathfrak{A}}$.*[8.183] *Then $\ulcorner(\dot{\mathfrak{B}}) = (\check{\mathfrak{C}})/(\dot{F})\urcorner$ is $\mathfrak{A}$-valid.*

To define $\dot{\mathfrak{B}}$ we have for convenience treated boolean algebras as structures with an order predicate and no operations. The corresponding operations are definable from the order relation, and we have the following identities. Any references to boolean operations and relations are to be interpreted in $\mathfrak{A}$, $\mathfrak{C}$, or $\dot{\mathfrak{B}}^G$, as appropriate, where G is V-generic on $\mathfrak{A}$.

(8.187) *Suppose $c, c_0, c_1 \in |\mathfrak{C}|$.*

1. *If $c_1 = \neg\, c_0$ then $\tilde{c}_1^G = \neg\, \tilde{c}_0^G$.*
2. *If $c = c_0 \vee c_1$ then $\tilde{c}^G = \tilde{c}_0^G \vee \tilde{c}_1^G$.*
3. *If $c = c_0 \wedge c_1$ then $\tilde{c}^G = \tilde{c}_0^G \wedge \tilde{c}_1^G$.*

The following identities provide useful alternative formulations of key relations.

(8.188) *Suppose $c, c_0, c_1 \in |\mathfrak{C}|$.*

1. *$\tilde{c}_0^G = \tilde{c}_1^G$ iff $\exists a \in G\ c_0 \wedge a = c_1 \wedge a$.*
2. *$\tilde{c}_0^G \leqslant \tilde{c}_1^G$ iff $\exists a \in G\ c_0 \wedge a \leqslant c_1 \wedge a$.*
3. *In particular, $\tilde{c}^G = \mathbf{0}$ iff $\exists a \in G\ c \wedge a = \mathbf{0}$.*

Note that

$$[\![\tilde{c}_0 = \tilde{c}_1]\!] = \bigvee\{a \in |\mathfrak{A}| \mid a \leqslant (c_0 \leftrightarrow c_1)\} \leqslant (c_0 \leftrightarrow c_1),$$

since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$. Thus, $[\![\tilde{c}_0 = \tilde{c}_1]\!] = \mathbf{1}$ iff $c_0 = c_1$.

(8.189) *It follows that $\mathfrak{C}$ is isomorphic to $\mathfrak{A} * \dot{\mathfrak{B}}$, with $c \in |\mathfrak{C}|$ corresponding to the unique canonical $\mathfrak{A}$-term equivalent to $\tilde{c}$.*

Let $\mathbf{1} = \mathbf{1}^{\mathfrak{A}} = \mathbf{1}^{\mathfrak{C}}$ and $\mathbf{0} = \mathbf{0}^{\mathfrak{A}} = \mathbf{0}^{\mathfrak{C}}$. Then $\tilde{\mathbf{1}}$ and $\tilde{\mathbf{0}}$ represent $\mathbf{1}^{\dot{\mathfrak{B}}}$ and $\mathbf{0}^{\dot{\mathfrak{B}}}$, respectively. Given $a \in |\mathfrak{A}|$,

$$[\![\tilde{a} = \mathbf{1}^{\dot{\mathfrak{B}}}]\!]^{\mathfrak{A}} = \bigvee\{a' \in |\mathfrak{A}| \mid a' \leqslant (a \leftrightarrow \mathbf{1})\} = a,$$

and

$$[\![\tilde{a} = \mathbf{0}^{\dot{\mathfrak{B}}}]\!]^{\mathfrak{A}} = \neg\, a.$$

Thus[8.182] $\tilde{a}$ is the image of a under the canonical embedding of $\mathfrak{A}$ in $\mathfrak{A} * \dot{\mathfrak{B}}$.

To show that ⌜$\dot{\mathfrak{B}}$ is complete⌝ is $\mathfrak{A}$-valid, we argue in a generic extension. Suppose G is V-generic on $\mathfrak{A}$. Let $\mathfrak{B} = \dot{\mathfrak{B}}^G$. Suppose $X \subseteq |\mathfrak{B}|$. Let $\dot{X} \in V^{\mathfrak{A}}$ be such that $\dot{X}^G = X$, and let

$$c_0 = \bigvee_{c \in |\mathfrak{C}|} \big([\![\tilde{c} \in \dot{X}]\!] \wedge c\big). \tag{8.190}$$

(8.191) **Claim** $\tilde{c}_0^G = \bigvee X$.

Proof To show that $\tilde{c}_0^G$ is an upper bound of X, suppose $x \in X$ and let $c \in |\mathfrak{C}|$ be such that $\tilde{c}^G = x$. Let $a = [\![\tilde{c} \in \dot{X}]\!]$. Then $c_0 \geqslant (a \wedge c)$,[8.190] so $c_0 \wedge a \geqslant (a \wedge c) \wedge a = c \wedge a$. Since $a \in G$, $\tilde{a}^G = \mathbf{1}^{\mathfrak{B}}$, so $x = \tilde{c}^G \leqslant \tilde{c}_0^G$.[8.187]

To show that $\tilde{c}_0^G$ is the least upper bound of X we must show that every $\tilde{c}_1^G$ that meets $\tilde{c}_0^G$ (in the sense of $\mathfrak{B}$) meets a member of X (in the same sense). To this end, suppose $c_1 \in |\mathfrak{C}|$, and suppose

$$\forall a \in G\, (c_1 \wedge c_0 \wedge a) \neq \mathbf{0}.^{8.188.3} \tag{8.192}$$

For each $c \in |\mathfrak{C}|$ let $a_c = \bigwedge\{a \in |\mathfrak{A}| \mid (c_1 \wedge [\![\tilde{c} \in \dot{X}]\!] \wedge c) \leqslant a\}$. Since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$, it doesn't matter whether we compute this meet in $\mathfrak{C}$ or $\mathfrak{A}$, and $a_c \in |\mathfrak{A}|$. Let $a_0 = \bigvee\{a_c \mid c \in |\mathfrak{C}|\}$. Then $a_0 \in |\mathfrak{A}|$ and $(c_1 \wedge c_0) \leqslant a_0$. Since $\neg\, a_0 \notin G$,[8.192] $a_0 \in G$. Since G is V-generic on $\mathfrak{A}$, for some $c \in |\mathfrak{C}|$, $a_c \in G$.

Suppose $a \in |\mathfrak{A}|$ and $(c_1 \wedge [\![\tilde{c} \in \dot{X}]\!] \wedge c \wedge a) = \mathbf{0}$. Then $a_c \leqslant \neg\, a$, so $a \notin G$ since a_c is in G. Hence

$$\forall a \in G\, (c_1 \wedge [\![\tilde{c} \in \dot{X}]\!] \wedge c \wedge a) \neq \mathbf{0}.$$

Two things follow. First, $[\![\tilde{c} \in \dot{X}]\!] \in G$, so $\tilde{c}^G \in \dot{X}^G = X$. Second, $\forall a \in G\, (c_1 \wedge c \wedge a) \neq \mathbf{0}$, so[8.188.3] $\tilde{c}_1^G \wedge \tilde{c}^G \neq \mathbf{0}$, as desired. □[8.191]

Thus, we have shown that ⌜$\dot{\mathfrak{B}}$ is complete⌝ is $\mathfrak{A}$-valid and $\mathfrak{C} \cong \mathfrak{A} * \dot{\mathfrak{B}}$,[8.189] as desired. □[8.184]

(8.193) **Definition** [GB] *Suppose $\mathfrak{C}$ is a complete boolean algebra, and $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$.*

1. *We define $\mathfrak{C} : \mathfrak{A}$ to be the $\mathfrak{A}$-term $\dot{\mathfrak{B}}$ as defined*[8.186] *in the proof of Theorem 8.184.*
2. *The* canonical projection of $\mathfrak{C}$ to $\mathfrak{C} : \mathfrak{A}$ $\overset{\text{def}}{=}$ *the map $c \mapsto \tilde{c}$.*[8.185]

(8.194) **Theorem** [GB] *Suppose $\mathfrak{C}$ is a complete boolean algebra, $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$, and $C \subseteq \mathfrak{C}^+$ is dense in $\mathfrak{C}$. Let $\dot{\mathfrak{B}} = \mathfrak{C} : \mathfrak{A}$, and let $\dot{B} = \{(\tilde{c}, \mathbf{1}) \mid c \in C\}$ be the canonical projection*[8.193.2] *of C to $\dot{\mathfrak{B}}$. Then ⌜$(\dot{B}) \setminus \{\mathbf{0}\}$ is dense in $(\dot{\mathfrak{B}})$⌝ is $\mathfrak{A}$-valid.*

Proof We will argue in an $\mathfrak{A}$-generic extension $V[G]$. Let $\mathfrak{B} = \dot{\mathfrak{B}}^G$ and $B = \dot{B}^G$. Suppose $b \in |\mathfrak{B}|$ and $b \neq \mathbf{0}$. Let $c_0 \in |\mathfrak{C}|$ be such that $\tilde{c}_0^G = b$, and let $C_0 = \{c \in C \mid c \leqslant c_0\}$. Trivially, for any $c \leqslant c_0$, $\tilde{c}^G \leqslant \tilde{c}_0^G = b$. Thus it suffices to show that there exists $c \in C_0$ such that $\tilde{c}^G \neq \mathbf{0}$.

By virtue of (8.188.3), since $b \neq \mathbf{0}$,

(8.195) *every $a \in G$ meets c_0,*

and we must show that for some $c \in C_0$, every $a \in G$ meets c. For each $c \in C_0$, let $a_c = \bigvee\{a \in |\mathfrak{A}| \mid a \wedge c = \mathbf{0}\}$. Keep in mind that since $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{C}$, meets and joins of subsets of $|\mathfrak{A}|$ are the same whether computed in $\mathfrak{A}$ or $\mathfrak{C}$. Note that $a_c \wedge c = \mathbf{0}$. Suppose toward a contradiction that for every $c \in C_0$, there exists $a \in G$ such that $a \wedge c = \mathbf{0}$. Then for every $c \in C_0$, $a_c \in G$. Since G is V-generic, $\bigwedge_{c\in C_0} a_c \in G$; however, since $a_c \wedge c = \mathbf{0}$ for all $c \in C_0$, $\bigwedge_{c\in C_0} a_c \wedge \bigvee C_0 = \mathbf{0}$. Since C is dense in $\mathfrak{C}$, $c_0 = \bigvee C_0$, so this contradicts (8.195). □[10.183]

8.13.1 Generic is not so special

(8.196) **Theorem** [GB] *Suppose M is a transitive model of ZF, $\mathfrak{A} \in M$ is an M-complete boolean algebra, and G is an M-generic filter on $\mathfrak{A}$. Suppose $x \subseteq M$ and $x \in M[G]$. Then there is a complete subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (in the sense of M) such that $x \in M[G \cap |\mathfrak{B}|]$ and for every transitive model $\mathfrak{N} = (N; \in, M)$ of ZF^{V} (with $\mathsf{V}^{\mathfrak{N}} = M$), if $x \in N$ then $M[G \cap |\mathfrak{B}|] \subseteq N$.*

Remark Note that $G \cap |\mathfrak{B}|$ is M-generic on $\mathfrak{B}$, since $\mathfrak{B}$ is a complete subalgebra of $\mathfrak{A}$[3.166] (not just a subalgebra of $\mathfrak{A}$ that is on its own complete), so that every dense $D \subseteq |\mathfrak{B}|$ is predense in $\mathfrak{A}$ (since $\bigvee D = \mathbf{1}$) . By virtue of this theorem, we may define $M[x]$, for any x in $M[G]$ with $x \subseteq M$, to be the *minimum extension of M that contains x* (with the understanding that structure $\mathfrak{M}[x] = (M[x]; \in, M)$ is intended, and $\mathfrak{M}[x] \models \mathsf{ZF}^{\mathsf{V}}$). The import of (8.196) is that minimum extensions exist in this setting and that they are generic extensions. Thus, genericity is the rule, not the exception. Note also that $M[G]$ is a generic extension of $M[x]$ by the algebra $(\mathfrak{A} : \mathfrak{B})^{G\cap|\mathfrak{B}|}$, i.e., the $\mathfrak{B}$-term $\mathfrak{A} : \mathfrak{B}$ interpreted via the M-generic filter $G \cap |\mathfrak{B}|$ on $\mathfrak{B}$.[8.193]

Proof Let $\dot{x} \in M^{\mathfrak{A}}$ be such that $\dot{x}^G = x$. Working in M, let B_0 be the set of elements of $|\mathfrak{A}|$ of the form $[\![\check{y} \in \dot{x}]\!]$ or $[\![\check{y} \notin \dot{x}]\!]$. We extend $\langle B_0 \rangle$ to an increasing sequence $\langle B_\alpha \mid \alpha \leqslant \eta \rangle$ of subsets of $|\mathfrak{A}|$ by letting B_α for $\alpha > 0$ be such that

1. if $\alpha = \beta + 1$ then
$$B_\alpha = \big\{ \bigvee X \,\big|\, X \subseteq B_\beta \big\} \cup \big\{ \bigwedge X \,\big|\, X \subseteq B_\beta \big\};$$
and
2. if $\operatorname{Lim} \alpha$ then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

η is simply the first ordinal such that $B_{\eta+1}$, if it were defined, would be B_η. By induction, each B_α is closed under complementation. Let $B = B_\eta$, and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ such that $|\mathfrak{B}| = B$. Then $\mathfrak{B}$ is a complete subalgebra of $\mathfrak{A}$.

Keep in mind that the above construction was carried out in M. Thus, $\mathfrak{B} \in M$, $\mathfrak{B}$ is M-complete, and $G \cap |\mathfrak{B}|$ is an M-generic filter on $\mathfrak{B}$. $x = \{y \in M \mid [\![\check{y} \in \dot{x}]\!] \in G \cap |\mathfrak{B}|\}$, so $x \in M[G \cap |\mathfrak{B}|]$.

Now suppose $\mathfrak{N} = (N; \in, M)$ is a transitive model of ZF^{V} and $x \in N$. Then $M \subseteq N$, so $\mathfrak{B} \in N$. Define in $\mathfrak{N}$ (from the parameter x) an increasing sequence $\langle G_\alpha \mid \alpha \leqslant \eta \rangle$ of subsets of $|\mathfrak{B}|$ by letting $G_0 = \{[\![\check{y} \in \dot{x}]\!] \mid y \in x\} \cup \{[\![\check{y} \notin \dot{x}]\!] \mid y \notin x\}$, and for all $\alpha \leqslant \eta$

1. if $\alpha = \beta + 1$ then

$$G_\alpha = \big\{ \bigvee X \,\big|\, X \in M \wedge X \subseteq B_\beta \wedge X \cap G_\beta \neq 0 \big\} \\ \cup \big\{ \bigwedge X \,\big|\, X \in M \wedge X \subseteq G_\beta \big\};$$

   and

2. if $\operatorname{Lim} \alpha$ then $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$.

Since G is M-generic, for all $\alpha \leqslant \eta$, $G_\alpha = G \cap B_\alpha$; and $G_\eta = G \cap |\mathfrak{B}|$. Thus, $G \cap |\mathfrak{B}| \in N$, so $M[G \cap |\mathfrak{B}|] \subseteq N$. $\square$[8.196]

8.13.2 Transfinite iteration

The construction described by (8.170) can obviously be repeated finitely many times without any additional consideration. To define transfinite iteration we must specify how limit stages are handled.

As usual, we assume that every partial order $\mathbb{P}$ has a maximum element, $\mathbf{1}^{\mathbb{P}}$. We may use '$\mathbf{1}$' to denote $\mathbf{1}^{\mathbb{P}}$. If $\dot{Q} \in M^{\mathbb{P}}$ is a forcing term for a partial order then[8.168] $\dot{\mathbf{1}} = \dot{\mathbf{1}}^{\mathbb{Q}} \in M^{\mathbb{P}}$ is such that $\Vdash^{\mathbb{P}} \ulcorner(\dot{\mathbf{1}})$ is the maximum element of $(\dot{\mathbb{Q}})\urcorner$.

We define *iteration of length* α for ordinals α by recursion on α. It will be obvious from the definition that an iteration of length α is a partial order whose elements are α-sequences. In the interest of uniformity of definition, we define $\mathbb{P}^0$ as the partial order with the single element 0; we identify $V^{\mathbb{P}^0}$ with V and $[x]_0$ with x; and we regard '$0 \Vdash^{\mathbb{P}^0} \phi$' as synonymous with '$\phi$'.

(8.197) **Definition** [GBC] *Suppose α is an ordinal. A partial order $\mathbb{P}$ is an* iteration of length α $\overset{\text{def}}{\iff}$ *there is a sequence $\langle \mathbb{P}_\beta \mid \beta \leqslant \alpha \rangle$ such that $\mathbb{P}_0 = \mathbb{P}^0$, $\mathbb{P}_\alpha = \mathbb{P}$,*[51] *and for each $\beta \leqslant \alpha$,*

1. *if $\beta = \gamma + 1$ then for some $\dot{\mathbb{Q}} \in V^{\mathbb{P}_\gamma}$,*[52]
   1. $\Vdash^{\mathbb{P}_\gamma} \ulcorner(\dot{\mathbb{Q}})$ *is a partial order*$\urcorner$*;*
   2. $|\mathbb{P}_\beta| = \{p ^\frown \langle [\dot{q}]_p \rangle \mid p \in |\mathbb{P}_\gamma| \wedge \Vdash^{\mathbb{P}_\gamma} \dot{q} \in |\dot{\mathbb{Q}}|\}$*; and*
   3. $p_1 ^\frown \langle [\dot{q}_1]_{p_1} \rangle \leqslant^{\mathbb{P}_\beta} p_0 ^\frown \langle [\dot{q}_0]_{p_0} \rangle \leftrightarrow p_1 \leqslant^{\mathbb{P}_\gamma} p_0 \wedge p_1 \Vdash^{\mathbb{P}_\gamma} \dot{q}_1 \leqslant^{\dot{\mathbb{Q}}} \dot{q}_0$.
2. *if β is a limit ordinal then*
   1. $|\mathbb{P}_\beta|$ *is a set of β-sequences;*
   2. *the β-sequence $\langle \mathbf{1}, \mathbf{1}, \dots \rangle$ is in $|\mathbb{P}_\beta|$; and*
   3. *for each $\gamma < \beta$,*

[51] Note that $\mathbb{P}^0$ is the (unique) iteration of length 0. The sole element of $|\mathbb{P}^0|$ being the 0-sequence 0.

[52] Note that if $\beta = 1$ then $\gamma = 0$, so for some partial order $\mathbb{Q}$,

1. $|\mathbb{P}_1| = \{\langle q \rangle \mid q \in |\mathbb{Q}|\}$; and
2. $\langle q_1 \rangle \leqslant^{\mathbb{P}_1} \langle q_0 \rangle \leftrightarrow q_1 \leqslant^{\mathbb{Q}} q_0$.

1. $|\mathbb{P}_\gamma| = \{p \restriction \gamma \mid p \in |\mathbb{P}_\beta|\}$;
2. *for any* $p \in |\mathbb{P}_\beta|$ *and* $q \in |\mathbb{P}_\gamma|$, *if* $q \leqslant^{\mathbb{P}_\gamma} p \restriction \gamma$ *then* $r \in |\mathbb{P}_\beta|$, *where* r *is such that for all* $\delta < \beta$,

$$r(\delta) = \begin{cases} q(\delta) & \text{if } \delta < \gamma \\ p(\delta) & \text{if } \delta \geqslant \gamma; \end{cases}$$

and

3. *for all* $p_0, p_1 \in |\mathbb{P}_\beta|$, $p_1 \leqslant^{\mathbb{P}_\beta} p_0 \leftrightarrow \forall \gamma < \beta\ p_1 \restriction \gamma \leqslant^{\mathbb{P}_\gamma} p_0 \restriction \gamma$.

Note that the sequence $\langle \mathbb{P}_\beta \mid \beta \leqslant \alpha \rangle$ that witnesses that $\mathbb{P}_\alpha$ is an iteration of length α is uniquely determined by $\mathbb{P}_\alpha$, inasmuch as $|\mathbb{P}_\beta| = \{p \restriction \beta \mid p \in |\mathbb{P}_\alpha|\}$ and $p_1' \leqslant^{\mathbb{P}_\beta} p_0' \leftrightarrow p_1 \leqslant^{\mathbb{P}_\alpha} p_0$, where, for each $i \in 2$ and $\gamma < \alpha$,

$$p_i(\gamma) = \begin{cases} p_i'(\gamma) & \text{if } \gamma < \beta \\ \mathbf{1} & \text{if } \gamma \geqslant \beta. \end{cases}$$

Note also that for each $\beta \leqslant \alpha$, $\mathbb{P}_\beta$ is an iteration of length β. In general an expression such as '$\mathbb{P}_\alpha$' will be taken to refer to an iteration of length α.

Definition [GBC] *Suppose* $\mathbb{P}$ *is an iteration of length* α *and* $\beta \leqslant \alpha$. *Then* $\mathbb{P}_\beta \overset{\text{def}}{=}$ *the (unique) partial order occurring in position* β *in a sequence witnessing that* $\mathbb{P}$ *is an iteration of length* α. *To simplify the notation we let* $\leqslant_\beta = \leqslant^{\mathbb{P}_\beta}$, $\Vdash_\beta = \Vdash^{\mathbb{P}_\beta}$, *etc.*

Clearly, $\mathbb{P}_\alpha$ may be specified by giving a sequence $\langle \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \rangle$ of appropriate forcing terms[53] and, for each limit $\beta \leqslant \alpha$, a rule for determining $|\mathbb{P}_\beta|$ as a subset of the set of all β-sequences p such that $\forall \gamma < \beta\ p \restriction \gamma \in |\mathbb{P}_\gamma|$. Useful such rules may be defined in terms of the notion of the *support* of a condition in a forcing iteration.

Definition [GBC] *Suppose* $\mathbb{P}$ *is an iteration of length* α.

1. *Suppose* $p \in |\mathbb{P}|$. *The* support of $p \overset{\text{def}}{=} \operatorname{supp} p \overset{\text{def}}{=} \{\beta < \alpha \mid p(\beta) \neq \mathbf{1}\}$.
2. *Suppose* $I \subseteq \mathcal{P}\alpha$ *is a nonprincipal ideal.* $\mathbb{P}$ is an I-support *iteration* $\overset{\text{def}}{\Longleftrightarrow}$ *for every limit ordinal* $\beta \leqslant \alpha$, $|\mathbb{P}_\beta|$ *is the set of* β*-sequences* p *such that*
   1. $\forall \gamma < \beta\ p \restriction \gamma \in |\mathbb{P}_\gamma|$; *and*
   2. $\operatorname{supp} p \in I$.
3. *In particular,* $\mathbb{P}$ *is a* finite-support *or* countable-support *iteration* $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{P}$ *is an* I*-support iteration where* I *is respectively the ideal of finite or countable (including finite) subsets of* α.

Given an ordinal α and a nonprincipal ideal on $\mathcal{P}\alpha$, an I-support iteration of length α is uniquely determined by a sequence $\langle \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \rangle$ via the stipulation that the term $\dot{\mathbb{Q}}$ occurring in (8.197.1) is $\dot{\mathbb{Q}}_\gamma$. Of course, in order to permit such a construction, for each $\beta < \alpha$, $\ulcorner(\dot{\mathbb{Q}}_\beta)$ is a partial order$\urcorner$ must be $\mathbb{P}_\beta$-valid.

Given an iteration $\mathbb{P}$ of length α and $\beta < \alpha$, let γ be the (unique) ordinal such that $\beta + \gamma = \alpha$. Then $\mathbb{P}$ is naturally equivalent to the iteration of length $1 + \gamma$ defined by the sequence $\langle \mathbb{P}_\beta \rangle ^\frown \langle \dot{\mathbb{Q}}_{\beta+\delta} \mid \delta < \gamma \rangle$, with the same rules at limit stages as for $\mathbb{P}$.

[53] Note that $\dot{\mathbb{Q}}_0$ is a $\mathbb{P}_0$-term, i.e., a $\mathbb{P}^0$-term, where $\mathbb{P}^0$ is the 1-element partial order, for which forcing is trivial, and $\dot{\mathbb{Q}}_0$ is a partial order *per se*, as described in the remarks preceding (8.197).

We may also define in a natural way a $\mathbb{P}_\beta$-term $\dot{\mathbb{P}}_{\beta,\alpha}$ that describes a partial order (in a hypothetical $\mathbb{P}_\beta$-generic extension) related to the sequence $\langle \dot{\mathbb{Q}}_{\beta+\delta} \mid \delta < \gamma \rangle$ as $\mathbb{P}$ is related to $\langle \dot{\mathbb{Q}}_\delta \mid \delta < \alpha \rangle$. $\mathbb{P}$ is then naturally equivalent to $\mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\alpha}$.[54]

(8.198) **Definition** [GB] *The operation corresponding to* $\iota_{\mathbb{P}_\beta,\dot{\mathbb{P}}_{\beta,\alpha}}$[8.175] *is* $\iota_{\beta,\alpha}$*, which is defined recursively so that for any* $\mathbb{P}_\beta$*-term* $\dot{x}$*,*

$$\iota_{\beta,\alpha}\dot{x} = \left\{ \left\langle \iota_{\beta,\alpha}\dot{y}, p ^\frown \vec{\mathbf{1}} \right\rangle \middle| \langle \dot{y}, p \rangle \in \dot{x} \right\},$$

where $\vec{\mathbf{1}}$ *is a sequence of* $\mathbf{1}$*s of length* δ*, such that* $\beta + \delta = \alpha$*.*

8.13.3 Martin's axiom

The first example of transfinite forcing iteration employed finite support and was used by Solovay and Tennenbaum to show the consistency of $\mathsf{ZFC} + \mathsf{SH}$ (Suslin's hypothesis) relative to ZF. Martin noted that SH is a consequence of a general principle whose consistency can be proved by the same method.

Definition [ZFC]

1. *Suppose* κ *is a cardinal.* $\mathsf{MA}_\kappa \overset{\text{def}}{\iff}$ *for every ccc partial order* $\mathbb{P}$ *and set* $\mathcal{D}$ *of at most* κ *dense subsets of* $\mathbb{P}$ *there exists a* $\mathcal{D}$*-generic filter on* $\mathbb{P}$*.*
2. Martin's axiom $\overset{\text{def}}{\iff} \mathsf{MA} \overset{\text{def}}{\iff} \forall \kappa < 2^\omega\ \mathsf{MA}_\kappa$.

(8.199) **Theorem** [ZFC]

1. MA_ω.
2. *For any cardinal* κ*,* $\mathsf{MA}_\kappa \to \kappa < 2^\omega$*.*

Proof (8.199.1) is the familiar and simple observation[8.11] that a filter can be constructed to meet any countable collection of dense sets. To prove (8.199.2) let $\mathbb{P}$ be the Cohen order.[8.118] $\mathbb{P}$ is countable, hence, ccc. For any $f \in {}^\omega 2$, the set $A_f = \{p \in |\mathbb{P}| \mid p \nsubseteq f\}$ is dense. Let $\mathcal{A} = \{A_f \mid f \in {}^\omega 2\}$, and let $\mathcal{B} = \{B_n \mid n \in \omega\}$, where $B_n = \{p \in |\mathbb{P}| \mid n \in \operatorname{dom} p\}$. The B_ns are also dense, and as shown in the proof of (8.119), if G is an $\mathcal{B}$-generic filter on $\mathbb{P}$ then there exists $f \in {}^\omega 2$ such that $G = \{p \in |\mathbb{P}| \mid p \subseteq f\}$. It follows that G does not meet A_f. Let $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$. Then $|\mathcal{D}| = 2^\omega$ and there is no $\mathcal{D}$-generic filter on $\mathbb{P}$. □[8.199]

Thus, the continuum hypothesis (CH) implies MA, and for any uncountable cardinal κ, MA_κ is incompatible with CH. Many consequences of CH have been shown to follow from MA without any assumption about the size of the continuum. Many additional consequences follow from the assumption of MA and $\neg\,\mathsf{CH}$. MA_{ω_1} is of particular importance. The Solovay-Tennenbaum result is an example.

(8.200) **Theorem** [ZFC] $\mathsf{MA}_{\omega_1} \to \mathsf{SH}$.

Proof Recall that Suslin's hypothesis[5.186] (SH) may be formulated[7.38] as the statement that there does not exist a Suslin tree.[7.35] Recall also that if there exists a Suslin tree then there exists a normal Suslin tree.[7.37] Suppose $\mathbb{T}$ is a normal tree[7.36]

[54] Keeping in mind that $\mathbb{P}_0$ is the trivial partial order $\mathbb{P}^0$, we note that $\mathbb{P}$ is $\dot{\mathbb{P}}_{0,\alpha}$ and $\mathbb{P}_\beta$ is $\dot{\mathbb{P}}_{0,\beta}$, so we may employ a more uniform notation and say that $\dot{\mathbb{P}}_{0,\alpha}$ is naturally equivalent to $\dot{\mathbb{P}}_{0,\beta} * \dot{\mathbb{P}}_{\beta,\alpha}$.

of height ω_1 with no uncountable antichain. We will assume MA_{ω_1} and show that $\mathbb{T}$ has an uncountable branch. Let $\mathbb{P}$ be the partial order such that $|\mathbb{P}| = |\mathbb{T}|$ and $\leqslant^{\mathbb{P}} = \geqslant^{\mathbb{T}}$, i.e, $\mathbb{P}$ is $\mathbb{T}$ turned upside down. Recall that an antichain in a tree is a set of pairwise incomparable elements. Clearly, elements are compatible in $\mathbb{P}$ iff they are comparable. Thus, $\mathbb{P}$ is ccc. For each $\alpha < \omega_1$ let D_α be the set of elements of $|\mathbb{P}|$ ($= |\mathbb{T}|$) of order $> \alpha$. Since $\mathbb{T}$ is normal, D_α is dense in $\mathbb{P}$. Let $\mathcal{D} = \{D_\alpha \mid \alpha < \omega_1\}$. Clearly, a filter in $\mathbb{P}$ is a set of comparable elements, and a $\mathcal{D}$-generic filter is an uncountable branch of $\mathbb{T}$. Thus, $\mathbb{T}$ has an uncountable branch. $\square$[8.200]

The following theorem may be viewed as an example of the phenomenon mentioned above: it states that a certain consequence of CH follows from MA. The consequence is the Baire category theorem—that any countable intersection of open dense subsets of $\mathbb{R}$ has nonempty intersection—reformulated by substituting 'of power $< 2^\omega$' for 'countable'. Assuming CH, of course, these are equivalent.

(8.201) **Theorem** [ZFC] *Suppose* MA, $\kappa < 2^\omega$, *and for each* $\alpha < \kappa$, $X_\alpha \subseteq \mathbb{R}$ *is open dense. Then* $\bigcap_{\alpha<\kappa} X_\alpha$ *is nonempty.*

Proof Let $\mathbb{P}$ be the partial order such that $|\mathbb{P}|$ is the set of nonempty open intervals in $\mathbb{R}$, and $\leqslant^{\mathbb{P}} = \subseteq$. Note that $\mathbb{P}$ is ccc. (This is the observation that led Suslin to his eponymous hypothesis.) For each $\alpha < \kappa$ let D_α consist of the open intervals p such that the closure $\overline{p}$ of p is included in X_α. Clearly, each D_α is dense. Let G be a $\{D_\alpha \mid \alpha < \kappa\}$-generic filter on $\mathbb{P}$. Since G is a filter, $\{\overline{p} \mid p \in G\}$ is closed under finite intersections, so $C = \bigcap_{p\in G} \overline{p}$ is nonempty. Any element of C is in every $\overline{p}$ for $p \in G$, so it is in every X_α. $\square$[8.201]

The consistency of MA with $\neg\,\mathsf{CH}$ (which implies MA_{ω_1}) may be proved by a finite-support iteration of ccc forcing. The following theorems pave the way by showing that (for any regular uncountable cardinal κ, in particular for ω_1) the κ-chain condition is preserved by finite-support iteration.

(8.202) **Theorem** [ZFC] *Suppose κ is an uncountable regular cardinal. If $\mathbb{P}$ is κ-cc and* $\Vdash^{\mathbb{P}} \ulcorner(\dot{\mathbb{Q}})$ *is* $(\check{\kappa})$*-cc*$\urcorner$ *then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is* κ*-cc.*

Proof Suppose toward a contradiction that $\big\langle\langle p_\alpha, [\dot{q}_\alpha]_{p_\alpha}\rangle \,\big|\, \alpha < \kappa\big\rangle$ is an antichain in $\mathbb{P} * \dot{\mathbb{Q}}$. Suppose α, β are distinct elements of κ. Then for any $\dot{q} \in V^{\mathbb{P}}$ and any p extending both p_α and p_β, $p \nVdash (\dot{q} \leqslant^{\dot{\mathbb{Q}}} \dot{q}_\alpha, \dot{q}_\beta)$. Thus, for any $p \leqslant p_\alpha, p_\beta$, and any $\dot{q} \in V^{\mathbb{P}}$, $p \Vdash \neg(\dot{q} \leqslant^{\dot{\mathbb{Q}}} \dot{q}_\alpha, \dot{q}_\beta)$; hence, $p \Vdash \forall v\ \neg(v \leqslant^{\dot{\mathbb{Q}}} \dot{q}_\alpha, q_\beta)$, i.e., $p \Vdash (\dot{q}_\alpha \perp \dot{q}_\beta)$. In short,

$$p \leqslant p_\alpha, p_\beta \rightarrow p \Vdash (\dot{q}_\alpha \perp \dot{q}_\beta). \tag{8.203}$$

Let $\dot{X} = \{\langle\check{\alpha}, p_\alpha\rangle \mid \alpha < \kappa\}$. Thus, $\dot{X} \in V^{\mathbb{P}}$ and for all $\alpha < \kappa$, $[\![\check{\alpha} \in \dot{X}]\!]^{\mathbb{P}} = p_\alpha$.[55]

Arguing in a generic extension, suppose G is a (V-)generic filter on $\mathbb{P}$. Let $\mathbb{Q} = \dot{\mathbb{Q}}^G$ and $X = \dot{X}^G$. Then $X = \{\alpha < \kappa \mid p_\alpha \in G\}$. Suppose α and β are distinct elements of X. Then $p_\alpha, p_\beta \in G$, so there exists $p \leqslant p_\alpha, p_\beta$, such that $p \in G$. Since[8.203] $p \Vdash (\dot{q}_\alpha \perp \dot{q}_\beta)$, it follows that $\dot{q}_\alpha^G \perp \dot{q}_\beta^G$. Since $\mathbb{Q}$ is κ-cc, $|X| < \kappa$.

Thus, $\Vdash^{\mathbb{P}} |\dot{X}| < \check{\kappa}$. Since[8.127] $\Vdash^{\mathbb{P}} \ulcorner(\check{\kappa})$ is regular$\urcorner$, there exists[8.109] $\dot{\gamma} \in V^{\mathbb{P}}$ such that $\Vdash^{\mathbb{P}} \dot{\gamma} \in \check{\kappa}$ and $\Vdash^{\mathbb{P}} |\dot{X}| \subseteq \dot{\gamma}$. Any condition in $\mathbb{P}$ has an extension that forces $\dot{\gamma} = \check{\alpha}$ for some $\alpha \in \kappa$. Let Y be a maximal antichain in $\mathbb{P}$ such that for every

[55] This is of course to be understood in terms of $\mathfrak{R}\,\mathbb{P}$.

$p \in Y$, there exists α_p such that $p \Vdash \dot\gamma = \check\alpha_p$. Since $\mathbb{P}$ is κ-cc, $|Y| < \kappa$, so there exists $\alpha < \kappa$ such that $\forall p \in Y\ \alpha_p < \alpha$. Then $\Vdash \dot X \subseteq \check\alpha$. Thus, $\Vdash \check\alpha \notin \dot X$. But $p_\alpha \Vdash \check\alpha \in \dot X$; contradiction. $\square$[8.202]

(8.204) **Theorem** [GBC] *Suppose κ is an uncountable regular cardinal, $\alpha > 0$, $\mathbb{P}_\alpha$ is the finite-support iteration of $\langle \dot Q_\beta \mid \beta < \alpha\rangle$, and $\forall \beta < \alpha \Vdash_\beta \ulcorner(\dot Q_\beta)$ is $(\check\kappa)$-cc$\urcorner$. Then $\mathbb{P}_\alpha$ is κ-cc.*

Proof By induction on α. If $\alpha = \beta + 1$ then $\mathbb{P}_\alpha \cong \mathbb{P}_\beta * \dot Q_\beta$, and the result follows from (8.202). Thus, suppose $\operatorname{Lim}\alpha$, and suppose X is a subset of $|\mathbb{P}_\alpha|$ of size κ. We will show that there are compatible elements in X.

Suppose first that $\operatorname{cf}\alpha \neq \kappa$. For each $\beta < \alpha$ let $X_\beta = \{p \in X \mid \operatorname{supp} p \subseteq \beta\}$. Then there exists $\beta < \alpha$ such that $|X_\beta| = \kappa$; otherwise, if $\operatorname{cf}\alpha < \kappa$ then κ would be a union of fewer than κ sets each of which is smaller than κ, which is impossible, whereas if $\operatorname{cf}\alpha > \kappa$ then there would be a κ-sequence in $\operatorname{cf}\alpha$ cofinal in $\operatorname{cf}\alpha$, which is also impossible, since $\operatorname{cf}\alpha$ is regular. By induction hypothesis, $\mathbb{P}_\beta$ is κ-cc, so there exist $p, p' \in X_\beta$ such that $p \restriction \beta$ and $p' \restriction \beta$ are compatible (in $\mathbb{P}_\beta$), whence p and p' are compatible, since $\operatorname{supp} p$, $\operatorname{supp} p' \subseteq \beta$.

Now suppose $\operatorname{cf}\alpha = \kappa$. Let $\langle p_\beta \mid \beta < \kappa\rangle$ enumerate X. Let $\langle \alpha_\beta \mid \beta < \kappa\rangle$ be a continuous increasing sequence in α with limit α. Let $C = \{\gamma < \kappa \mid \forall \beta < \gamma\ \operatorname{supp} p_\beta \subseteq \alpha_\gamma\}$. Then C is closed unbounded in κ. For each limit element γ of C (i.e., $C \cap \gamma$ is unbounded in γ) let $\delta(\gamma) < \gamma$ be such that $\operatorname{supp} p_\gamma \cap \alpha_\gamma \subseteq \alpha_{\delta(\gamma)}$. By Fodor's lemma[3.173] there exist $\delta < \kappa$ and a stationary subset $S \subseteq C$ such that for all $\gamma \in S$, $\delta(\gamma) = \delta$, so $\operatorname{supp} p_\gamma \cap \alpha_\gamma \subseteq \alpha_\delta$.

Let $Y = \{p_\gamma \restriction \alpha_\delta \mid \gamma \in S\}$. Since $|S| = \kappa$ and $\mathbb{P}_{\alpha_\delta}$ is κ-cc, there exist $\gamma, \gamma' \in S$ such that $\delta < \gamma < \gamma'$, and $p_\gamma \restriction \alpha_\delta$ and $p_{\gamma'} \restriction \alpha_\delta$ are compatible. Let $p \in |\mathbb{P}_{\alpha_\delta}|$ be a common extension of $p_\gamma \restriction \alpha_\delta$ and $p_{\gamma'} \restriction \alpha_\delta$, and let q be the α-sequence such that for all $\beta < \alpha$

$$q(\beta) = \begin{cases} p(\beta) & \text{if } \beta < \alpha_\delta \\ p_\gamma(\beta) & \text{if } \alpha_\delta \leqslant \beta < \alpha_{\gamma'} \\ p_{\gamma'}(\beta) & \text{if } \alpha_{\gamma'} \leqslant \beta. \end{cases}$$

Then $q \in |\mathbb{P}_\alpha|$. Since $\operatorname{supp} p_\gamma \subseteq \alpha_{\gamma'}$ and $\operatorname{supp} p_{\gamma'} \cap \alpha_{\gamma'} \subseteq \alpha_\delta$, q extends both p_γ and $p_{\gamma'}$. Hence, p_γ and $p_{\gamma'}$ are compatible. $\square$[8.204]

One can easily imagine that MA may be achieved in a generic extension by repeatedly shooting generic filters through ccc partial orders, but it is necessary to place a reasonable bound on the number of partial orders that must be attended to. This is accomplished by the following theorem.

(8.205) **Theorem** [ZFC] *Suppose* MA *holds for all partial orders smaller than 2^ω. Then* MA *holds in general.*

Proof Suppose $\mathbb{P}$ is a ccc partial order and $\mathcal{D}$ is a set of dense sets in $\mathbb{P}$ of size $\kappa < 2^\omega$. For each $D \in \mathcal{D}$, let X_D be a maximal antichain in D. Since D is dense, X_D is a maximal antichain in $\mathbb{P}$ and is therefore predense. Let $Y_0 = \bigcup_{D \in \mathcal{D}} X_D$. Each X_D is countable, so $|Y| \leqslant \kappa$. We now let $Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq \cdots$ $(n \in \omega)$ be subsets of $|\mathbb{P}|$ of size $\leqslant \kappa$ such that for any $p, q \in Y_n$, if $p \parallel q$ in $\mathbb{P}$ then there exists a common extension of p and q in Y_{n+1}. Let $Y = \bigcup_{n \in \omega} Y_n$. Then $|Y| \leqslant \kappa$.

Let $\mathbb{Y}$ be the partial order with $|\mathbb{Y}| = Y$ and the order relation inherited from $\mathbb{P}$. For each $D \in \mathcal{D}$, let $E_D = \{q \in Y \mid \exists p \in X_D\ q \leqslant p\}$; and let $\mathcal{E} = \{E_D \mid D \in \mathcal{D}\}$.

Then $\mathcal{E}$ is a set of dense sets in $\mathbb{Y}$ of size $\leqslant \kappa$, so by hypothesis there exists an $\mathcal{E}$-generic filter G on $\mathbb{Y}$. $\{p \in |\mathbb{P}| \mid \exists q \in G\ q \leqslant p\}$ is a $\mathcal{D}$-generic filter on $\mathbb{P}$. $\square$[8.205]

We will make use of (8.205) by restricting our attention to partial orders $\mathbb{Q}$ such that $|\mathbb{Q}| = \lambda < 2^\omega$ (with '$|\mathbb{Q}|$' here referring to the domain of $\mathbb{Q}$, not to its cardinality).

(8.206) **Theorem** [ZFC] *Suppose* GCH, *and suppose $\kappa > \omega_1$ is a regular cardinal. Then there exists a ccc partial order $\mathbb{P}$ such that* $\ulcorner\mathsf{MA} + 2^\omega = (\check\kappa)\urcorner$ *is $\mathbb{P}$-valid.*

Proof We posit a fixed appropriate choice function. $\mathbb{P}$ will be the finite-support iteration of a sequence $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$. The construction will be such that for all $\alpha < \kappa$, $\Vdash_\alpha \ulcorner(\dot{\mathbb{Q}}_\alpha)$ is ccc$\urcorner$, so[8.204] $\mathbb{P}_\alpha$ will be ccc for every $\alpha \leqslant \kappa$. We will also arrange that for each $\alpha < \kappa$, for some cardinal $\lambda < \kappa$, $\Vdash_\alpha \ulcorner|(\dot{\mathbb{Q}}_\alpha)| = (\check\lambda)\urcorner$, i.e., $\mathbb{Q}_\alpha$ is a partial ordering of the ordinals $< \lambda$.

(8.207) **Claim** *Suppose $\alpha \leqslant \kappa$. Then $||\mathbb{P}_\alpha|| \leqslant \kappa$.*

Proof By induction. The argument at limits is easy, given GCH and the fact that the iteration has finite support. Thus, suppose $\alpha < \kappa$ and $||\mathbb{P}_\alpha|| \leqslant \kappa$. By design, for some cardinal $\lambda < \kappa$, for any $\dot q$ such that $\Vdash_\alpha \dot q \in |\dot{\mathbb{Q}}_\alpha|$, $\Vdash_\alpha \dot q \in \check\lambda$. Given such a term $\dot q$, the set of conditions forcing $\dot q = \check\gamma$ for some $\gamma < \lambda$ is dense in $\mathbb{P}_\alpha$. Let X be a maximal antichain of such conditions, and let $f_{\dot q} : X \to \lambda$ be such that for each $p \in X$, $p \Vdash_\alpha \dot q = \check\gamma$, where $\gamma = f_{\dot q} p$. If $\dot q$ and $\dot q'$ are two such terms and $f_{\dot q} = f_{\dot q'}$ then $\Vdash_\alpha \dot q = \dot q'$. Given GCH, since κ is regular and uncountable, and $\mathbb{P}_\alpha$ is ccc there are no more than $\kappa^\omega = \kappa$ such sets X and for each X no more than $\lambda^\omega \leqslant \kappa$ functions $f : X \to \lambda$. Thus, there are no more than κ relevant equivalence classes $[\dot q]_{\mathbf{1}}$, *a fortiori*, no more than κ relevant equivalence classes $[\dot q]_p$ for any $p \in |\mathbb{P}_\alpha|$ (these being larger). Hence, $||\mathbb{P}_{\alpha+1}|| \leqslant \kappa \cdot \kappa = \kappa$. $\square$[8.207]

By a similar computation as in the proof of (8.140), since $\mathbb{P} = \mathbb{P}_\kappa$ is a ccc partial order of size $\leqslant \kappa$, $\Vdash^{\mathbb{P}} \ulcorner 2^{(\check\lambda)} \leqslant (\check\kappa)\urcorner$ for every $\lambda < \kappa$, in particular,

(8.208) $$\Vdash^{\mathbb{P}} \ulcorner 2^\omega \leqslant (\check\kappa)\urcorner.$$

Now we have to deal with the bookkeeping to ensure that every ccc partial order on a cardinal $\lambda < \kappa$ in the generic extension by $\mathbb{P}$ is dealt with. Let $\pi : \kappa \overset{\text{sur}}{\to} \kappa \times \kappa$ be such that $\forall \alpha < \kappa\ (\pi\alpha)_0 \leqslant \alpha$. We will define by recursion on α, the sequence $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$ together with a system $\langle \dot{\mathbb{Q}}^\gamma_\alpha \mid \gamma, \alpha < \kappa\rangle$.

Suppose $\alpha < \kappa$, and suppose $\dot{\mathbb{Q}}^\gamma_\beta$ and $\dot{\mathbb{Q}}_\beta$ have been defined for $\beta < \alpha$. $\mathbb{P}_\alpha$ has therefore also been defined. By arguments similar to the preceding there are no more than κ equivalence classes $[\dot{\mathbb{Q}}]_{\mathbf{1}}$ of terms $\dot{\mathbb{Q}} \in V^{\mathbb{P}_\alpha}$ such that for some cardinal $\lambda < \kappa$, $\Vdash_\alpha \ulcorner(\dot{\mathbb{Q}})$ is a partial order on $(\check\lambda)\urcorner$. Let $\langle \dot{\mathbb{Q}}^\gamma_\alpha \mid \gamma < \kappa\rangle$ enumerate a set of representatives of these equivalence classes.

Now suppose $\pi\alpha = \langle \beta, \gamma\rangle$. Let $\dot{\mathbb{Q}} = \iota_{\beta,\alpha}\dot{\mathbb{Q}}^\gamma_\beta$ be the canonical $\mathbb{P}_\alpha$-term corresponding to the $\mathbb{P}_\beta$-term $\dot{\mathbb{Q}}^\gamma_\beta$,[8.198] and let $\dot{\mathbb{Q}}_\alpha$ be the canonical $\mathbb{P}_\alpha$-term such that

$$[\![\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}]\!]^{\mathbb{P}_\alpha} = [\![(\dot{\mathbb{Q}}) \text{ is ccc}]\!]^{\mathbb{P}_\alpha}$$
$$[\![\dot{\mathbb{Q}}_\alpha = \check{\mathbb{P}}^0]\!]^{\mathbb{P}_\alpha} = [\![(\dot{\mathbb{Q}}) \text{ is not ccc}]\!]^{\mathbb{P}_\alpha},$$

suitably framed in terms of partial orders. (Recall that the default value $\mathbb{P}^0$ for $\dot{\mathbb{Q}}_\alpha$ is the trivial partial order, which has essentially no forcing effect.)

This completes the description of the forcing iteration. We now argue in a generic extension to show that MA is $\mathbb{P}$-valid. Thus, we suppose M is a transitive model of ZFC + GCH, κ is a regular uncountable cardinal in M, and the above construction has been carried out in M. Suppose G is M-generic on $\mathbb{P}$. Let $G_\alpha = G \restriction \mathbb{P}_\alpha$ for each $\alpha < \kappa$.

The following claim is obviously pertinent.

(8.209) **Claim** *Suppose $\lambda < \kappa$ and $X \subseteq \lambda$ is in $M[G]$. Then $X \in M[G_\alpha]$ for some $\alpha < \kappa$.*

Proof Let $\dot{X} \in M^{\mathbb{P}}$ be such that $\dot{X}^G = X$. For each $\gamma < \lambda$ let $X_\gamma \subseteq |\mathbb{P}|$ be a maximal antichain of conditions deciding $\check{\gamma} \in \dot{X}$. Since G is M-generic it meets each X_γ. Each X_γ is countable, so there exists $\alpha < \lambda$ such that for all $\gamma < \lambda$ and $p \in X_\gamma$, $\operatorname{supp} p \subseteq \alpha$. Then for each $\gamma < \lambda$, $\gamma \in X$ iff $\exists p \in X_\gamma \, (p \restriction \alpha \in G_\alpha \wedge p \Vdash^{\mathbb{P}} \check{\gamma} \in \dot{X})$. Thus, $X \in M[G_\alpha]$. $\square$8.209

(8.210) **Claim** *Suppose $\mathbb{Q}, \mathcal{D} \in M[G]$, where, in the sense of $M[G]$, $\mathbb{Q}$ is a ccc partial order, $||\mathbb{Q}|| < \kappa$, and $|\mathcal{D}| < \kappa$. Then there exists a $\mathcal{D}$-generic filter on $\mathbb{Q}$ in $M[G]$.*

Proof By virtue of (8.209) there exists $\beta < \kappa$ such that $\mathbb{Q}, \mathcal{D} \in M[G_\beta]$. Let $\gamma < \kappa$ be such that $(\dot{\mathbb{Q}}^\gamma_\beta)^{G_\beta} = \mathbb{Q}$, where $\langle \dot{\mathbb{Q}}^\gamma_\alpha \mid \gamma, \alpha < \kappa \rangle$ is the enumeration of terms for partial orders used in the construction of $\mathbb{P}$. Let $\alpha < \kappa$ be such that $\pi\alpha = \langle \beta, \gamma \rangle$. Then by construction, $\dot{\mathbb{Q}}_\alpha{}^{G_\alpha} = \mathbb{Q}$, and $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \mathbb{Q}$. Let $H = \{\dot{q}^{G_\alpha} \mid \exists p \; p^\frown\langle [\dot{q}]_p \rangle \in G_{\alpha+1}\}$. Then $H \in M[G]$ and[8.174.2.2] H is an $M[G_\alpha]$-generic filter on $\mathbb{Q}$. Thus, H is a $\mathcal{D}$-generic filter on $\mathbb{Q}$ in $M[G]$. $\square$8.210

It follows that $M[G]$ satisfies MA_λ for each cardinal $\lambda < \kappa$ in M. Hence,[8.199.2] $2^\omega \geqslant \kappa$ in $M[G]$. By virtue of (8.208) $2^\omega = \kappa$ in $M[G]$. Thus, $M[G] \models \mathsf{MA}$.

The existence of the above argument in $M[G]$ demonstrates that $\ulcorner \mathsf{MA} + 2^\omega = (\check{\kappa})\urcorner$ is $\mathbb{P}$-valid. $\square$8.206

8.14 Some forcing constructions

The purpose of this section is to give some idea of the flexibility and power of the method of forcing.

8.14.1 A generic Suslin tree

There are a number of ways of forcing to create a Suslin tree. The one presented here is chronologically the first and is due to Tennenbaum. It is remarkable in that it was done in the summer of 1963, Cohen having presented the method of forcing only in the spring of that year. It is also surprising in that the conditions are exceptionally simple and do not appear to be customized for the purpose.[56]

(8.211) *Let P be the set of finite binary relations $p \subseteq \omega_1 \times \omega_1$ such that*

1. *p is the reflexive order relation of a tree; and*

[56]Shelah has shown, even more surprisingly, that Cohen forcing (i.e., forcing with a complete boolean algebra that has a countable dense set) also adds a Suslin tree.

2. $\forall\langle\alpha,\beta\rangle \in p\ \alpha \leqslant \beta$ *(in the usual ordering of the ordinals).*

Let $\mathbb{P} = (P; \leqslant)$*, where* $q \leqslant p$ *iff* $q \cap (\operatorname{fld} p \times \operatorname{fld} p) = p$.

Theorem [GBC] $\Vdash^{\mathbb{P}}$ ⌜$\bigcup \mathsf{G}$ *is (the reflexive order relation of) a Suslin tree*⌝.

Proof We "work in a generic extension of V". Suppose G is a V-generic filter on $\mathbb{P}$, and let $R = \bigcup G$. Clearly, R is a reflexive partial order which is "tree-like" in that the predecessors of any $\alpha \in \operatorname{fld} R$ are linearly ordered. But R also satisfies (8.211.2), so the predecessors of α are actually wellordered, and R is therefore a tree.

It is easy to see that for any $p \in P$ and $\alpha \in \omega_1$, there exists $q \leqslant p$ such that $\alpha \in \operatorname{fld} q$, so $\operatorname{fld} R = \omega_1^{\mathsf{V}}$. Another simple density argument shows that every node of R has incomparable extensions. We will show that $\omega_1^{\mathsf{V}} = \omega_1$, and that R has no uncountable antichains. It follows that every level of R is countable, so the height of R is ω_1. It also follows that R has no uncountable branch; otherwise, we could obtain an uncountable antichain by defining an increasing sequence $\langle\alpha_\gamma \mid \gamma < \omega_1\rangle$ in an uncountable branch B, together with a sequence $\langle\beta_\gamma \mid \gamma < \omega_1\rangle$, such that for each $\gamma < \omega_1$, β_γ extends α_γ and is incomparable with $\alpha_{\gamma+1}$ (since α_γ has incomparable extensions, at least one of which must not be in B). $\{\beta_\gamma \mid \gamma < \omega_1\rangle$ is an uncountable antichain. The preservation of ω_1 is a consequence of the following claim.

(8.212) **Claim** $\mathbb{P}$ *has the countable chain condition.*

Proof Suppose toward a contradiction that $X_0 \subseteq P$ is an uncountable set of pairwise incompatible conditions. By the Δ-lemma (8.128) there exist an uncountable $X_1 \subseteq X$ and a set d such that for all distinct $p, p' \in X_1$, $\operatorname{fld} p \cap \operatorname{fld} p' = d$. Let $\eta \in \omega_1$ be such that $d \subseteq \eta$. By discarding at most one member of X_1 for each $\alpha < \eta$, we obtain an uncountable $X_2 \subseteq X_1$ such that for all $p \in X_2$, $\operatorname{fld} p \cap \eta = d$. Since there are only finitely many binary relations on d, there exists an uncountable $X_3 \subseteq X_2$ such that for all $p, p' \in X_3$, $p \cap (d \times d) = p' \cap (d \times d)$.

Suppose $p, p' \in X_3$. Let $q = p \cup p'$. It is easy to check that $q \in P$ and $q \leqslant p, p'$. This contradicts the assumption that X_0 is an antichain. $\square$8.212

(8.213) **Claim** R *has no uncountable antichain.*

Proof If it did, some condition would have to force it to. Suppose $p_0 \Vdash$ ⌜$(\dot{A})$ is uncountable⌝. A simple density argument shows that there exists an uncountable set X of pairs $\langle p, \alpha_p\rangle$ such that $p \leqslant p_0$ and $p \Vdash \check{\alpha}_p \in \dot{A}$, such that if $\langle p, \alpha_p\rangle$ and $\langle p', \alpha_{p'}\rangle$ are distinct members of X then $\alpha_p \neq \alpha_{p'}$. As noted above, we may suppose that each $p \in X$ has been extended so that $\alpha_p \in \operatorname{fld} p$. Using the Δ-lemma as before, there exist an uncountable $Y \subseteq X$, $\eta \in \omega_1$, $d \subseteq \eta$, $r \subseteq (d \times d)$, and $s \subseteq d$ such that

1. $r \in P$;
2. for all distinct $\langle p, \alpha_p\rangle, \langle p', \alpha_{p'}\rangle \in Y$, $\operatorname{fld} p \cap \operatorname{fld} p' = d$; and
3. for all $\langle p, \alpha_p\rangle \in Y$,
    1. $\operatorname{fld} p \cap \eta = d$;
    2. $p \leqslant r$; and

3. $\{\alpha \in d \mid \langle \alpha, \alpha_p \rangle \in p\} = s$.

Now suppose $\langle p', \alpha_{p'} \rangle$ and $\langle p'', \alpha_{p''} \rangle$ are distinct elements of Y. Let q be the smallest partial order that includes p', p'', and the set of $\langle \alpha, \beta \rangle$ such that $\alpha < \beta$ and either $\langle \alpha, \alpha_{p'} \rangle \in p'$ and $\langle \beta, \alpha_{p''} \rangle \in p''$ or *vice versa*.[57] It is easy to see that q is a tree and $q \leqslant p', p''$. Since $\alpha_{p'}$ and $\alpha_{p''}$ are comparable in q, $q \Vdash \ulcorner (\check{\alpha}_{p'})$ and $(\check{\alpha}_{p''})$ are comparable in $R \urcorner$. Since $q \leqslant p_0$, $p_0 \nVdash \ulcorner (\dot{A})$ is an antichain$\urcorner$. Hence no condition forces the existence of an uncountable antichain, so R has no uncountable antichain. $\square^{8.213}$

As argued above, these claims suffice to establish that R is a Suslin tree. $\square^{8.211}$

8.14.2 A generic $\Diamond$-sequence

We have previously derived the existence of a Suslin tree from the $\Diamond$ principle,[7.41] which holds in L. The following theorem shows that $\Diamond$-sequences may also be obtained generically.

(8.214) **Theorem** [ZFC] *Let $\mathbb{P}$ be the partial order of countable partial functions from ω_1 to 2. Then $\Diamond$ is $\mathbb{P}$-valid.*

Proof Rather than work directly with $\mathbb{P}$ as defined above, we will work with the isomorphic partial order $\mathbb{P} = (|\mathbb{P}|; \leqslant)$ defined as follows.

1. $p \in |\mathbb{P}|$ iff
    1. p is a countable function;
    2. $\operatorname{dom} p \subseteq \{\langle \alpha, \beta \rangle \mid \beta < \alpha < \omega_1\}$; and
    3. $\operatorname{im} p \subseteq 2$.
2. $q \leqslant p \leftrightarrow q \supseteq p$.

As usual we will make free use of the method of arguing in generic extensions. Thus, suppose M is a transitive model of ZFC and G is an M-generic filter on $\mathbb{P}^M$. Note that $M \models \ulcorner [\mathbb{P}]$ is ω-closed$\urcorner$, so $\mathbb{P}$-forcing does not add any new functions from ω into M and therefore does not collapse any cardinals. In particular, $\omega_1^{M[G]} = \omega_1^M$.

A simple density argument shows that $\bigcup G$ is a function from $\{\langle \alpha, \beta \rangle \mid \beta < \alpha < \omega_1^M\}$ to 2. For any such function F and any $\alpha < \omega_1$, let us define for the nonce, F_α to be $\{\beta < \alpha \mid F\langle \alpha, \beta \rangle = 1\}$. Thus, for each $\alpha < \omega_1^M$, $\left(\bigcup G\right)_\alpha \subseteq \alpha$.

We claim that in $M[G]$, $\langle \left(\bigcup G\right)_\alpha \mid \alpha < \omega_1 \rangle$ is a $\Diamond$-sequence. Suppose toward a contradiction that this is not the case. Let $X, C \in (\mathcal{P}\omega_1)^{M[G]}$ be such that C is closed unbounded (club) in ω_1^M, and for all $\alpha \in C$, $X \cap \alpha \neq (\bigcup G)_\alpha$. Let $\dot{X}$ and $\dot{C}$ be such that $\dot{X}^G = X$ and $\dot{C}^G = C$, and let $p_0 \in G$ be such that

$$p_0 \Vdash \ulcorner (\dot{X}) \subseteq \omega_1 \wedge (\dot{C}) \text{ is club in } \omega_1 \wedge \forall \alpha \in (\dot{C})\ (\dot{X}) \cap \alpha \neq (\textstyle\bigcup \mathsf{G})_\alpha \urcorner. \tag{8.215}$$

In M we construct sequentially, for $n \in \omega$, $p_n \in |\mathbb{P}|$, $\alpha_n \in \omega_1$, and $S_n \subseteq \alpha_n$, such that for each $n \in \omega$,

[57] Let $p = p' \cup p''$, which is the smallest tree extending p' and p''. Let s' and s'' be respectively the sets of predecessors of $\alpha_{p'}$ in p' and of $\alpha_{p''}$ in p''. We may regard s' (resp., s'') as the "trunk" of p' (resp., p''), with subtrees sprouting from it (as we may do with any initial segment of any branch of any finite tree). s is the initial segment of both s' and s'' in d, and the remainders of s' and s'' are disjoint. q is formed by merging the portions of s' and s'' above s (with the ordering mandated by (8.211.2)) and then filling out the relation to satisfy the transitivity condition. The subtrees formerly sprouting from s' and s'' above s now sprout from the merged trunk.

1. $\operatorname{dom} p_n \subseteq \alpha_n \times \mathrm{Ord}$;
2. $p_{n+1} \leqslant p_n$;
3. $p_{n+1} \Vdash \check{\alpha}_n \in \dot{C}$;
4. $p_{n+1} \Vdash \dot{X} \cap \check{\alpha}_n = \check{S}_n$; and
5. $\alpha_{n+1} > \alpha_n$.

We accomplish this as follows. Suppose $n \in \omega$ and we have p_n;[58]; if $n > 0$, suppose we have α_{n^-} and S_{n^-} as well; and suppose the above conditions are satisfied up to this point. Let $\alpha \in \omega_1$ be such that $\operatorname{dom} p_n \subseteq \alpha \times \alpha$ (and $\alpha > \alpha_{n^-}$ if $n > 0$). Since $p_n \leqslant p_{n^-} \leqslant \cdots \leqslant p_0$ by design, $p_n \Vdash \ulcorner(\dot{C})$ is unbounded in $\omega_1\urcorner$, so there exist $p \leqslant p_n$ and $\alpha' > \alpha$ such that $p \Vdash \check{\alpha}' \in \dot{C}$. Since $\mathbb{P}$ is ω-closed, $\Vdash^{\mathbb{P}} \dot{X} \cap \check{\alpha}' \in \mathsf{V}$, so there exist $p' \leqslant p$ and $S \subseteq \alpha'$ such that $p' \Vdash \dot{X} \cap \check{\alpha}' = \check{S}$. Let $\alpha_n = \alpha$, $S_n = S$, and $p_{n+1} = p'$.

Let $p = \bigcup_{n\in\omega} p_n$, $\alpha = \bigcup_{n\in\omega} \alpha_n$, and $S = \bigcup_{n\in\omega} S_n$. Then $\operatorname{dom} p \subseteq \alpha \times \mathrm{Ord}$ and $S \subseteq \alpha$. Let p' be the extension of p by the addition of the characteristic function of S at position α:

$$p' = p \cup \{(\langle\alpha,\beta\rangle, 1) \mid \beta \in S\} \cup \{(\langle\alpha,\beta\rangle, 0) \mid \beta \in \alpha\backslash S\}.$$

Then $p' \Vdash \dot{X} \cap \check{\alpha} = \check{S}$. Since $p' \Vdash \ulcorner(\dot{C})$ is closed$\urcorner$, $p' \Vdash \check{\alpha} \in \dot{C}$. This contradicts (8.215). $\square$[8.214]

8.14.3 Silver's singular cardinals theorem

As discussed above[§ 3.9.2] the following theorem of Silver was a stunning breakthrough in the theory of singular cardinal arithmetic. The original proof involves a nonwell-founded ultrapower of a transitive model M constructed in a generic extension of M. A direct proof of Silver's theorem—without any model-theoretic considerations—was soon formulated by Baumgartner and Prikry, but we will present Silver's proof here as an instructive example of the incorporation of forcing within a larger context, i.e., not as the centerpiece of a straightforward relative consistency argument.

(8.216) **Theorem** [ZFC] *Suppose κ is a singular cardinal of uncountable cofinality and the set of cardinals $\lambda < \kappa$ such that $2^\lambda = \lambda^+$ is stationary in κ. Then $2^\kappa = \kappa^+$.*

Remark As noted above the forcing component of the following proof is not a *straightforward* relative consistency argument such as we have seen so far, but it may nevertheless be formulated in terms of relative consistency. We will show that if ZFC plus the negation of the theorem is consistent then ZFC + $\ulcorner 0 = 1 \urcorner$[59] is consistent. Since the latter is false, the former is false, i.e., ZFC proves the theorem. We will show that ZFC + $\ulcorner 0 = 1\urcorner$ is consistent by showing that it is $\mathbb{P}$-valid for an appropriate partial order $\mathbb{P}$, and we will do this by arguing in a $\mathbb{P}$-generic extension using the theory Θ',[8.106] which incorporates GB and can deal with satisfaction in proper class structures. We will give the proof in its conventional form. To obtain a proof that is strictly in the above format, the reader may substitute V, P, and G for M, $\mathbb{P}$, and G, respectively, and may suppose that $V = \mathsf{V}[\mathsf{G}]$, as in Θ'.

[58] Note that we have p_0 at the outset.[8.215]

[59] $\ulcorner 0 = 1\urcorner$ is just a convenient sentence in the language of set theory that is disprovable. Its use for this purpose is traditional.

Proof Suppose M is a transitive model of ZFC, and suppose toward a contradiction that $M \models \ulcorner [\kappa]$ is a singular cardinal of uncountable cofinality, the set of cardinals $\lambda < [\kappa]$ such that $2^\lambda = \lambda^+$ is stationary in $[\kappa]$, and $2^{[\kappa]} \neq [\kappa]^{+}\urcorner$.

Working in M, let $\nu = \operatorname{cf} \kappa$. Note that ν is an uncountable regular cardinal. Let $h : \nu \to \kappa$ be a strictly increasing continuous sequence of cardinals cofinal in κ. Let $X = \{\alpha < \nu \mid 2^{h\alpha} = (h\alpha)^+\}$. Then X is stationary in ν. Let $\mu = 2^\nu$, and let $\mathbb{P}$ be the partial order of finite functions from ω to μ ordered by reverse inclusion. We will argue in a generic extension by $\mathbb{P}$.

Thus, suppose G is M-generic on $\mathbb{P}$. Since M models AC, so does $M[G]$. Clearly, $\bigcup G$ is a function mapping ω onto μ, so μ is countable in $M[G]$. Since $\ulcorner 2^\nu = \mu \urcorner^M$, it follows that $({}^\nu\nu)^M$ is countable in $M[G]$. Let $\langle f_n \mid n \in \omega\rangle \in M[G]$ be an enumeration of the regressive functions in $({}^\nu\nu)^M$. Since $||\mathbb{P}||^M = \mu$, $M \models \ulcorner [\mathbb{P}]$ satisfies the $[\mu]^+$-chain condition$\urcorner$, so[8.127] cardinals in M above μ remain cardinals in $M[G]$.

We now define a decreasing sequence $\langle X_n \mid n \in \omega\rangle \in M[G]$, where each X_n is a stationary subset of ν in M. Let $X_0 = X$. For each $n \in \omega$, given X_n, let[3.173] X_{n+1} be a stationary subset of X_n such that f_n is constant on X_{n+1}. Let $U = \{Y \in \mathcal{P}\nu \cap M \mid \exists n \in \omega\ X_n \subseteq Y\}$. It is easy to see that U is an ultrafilter on $\mathcal{P}\nu \cap M$.[60]

Let $(A; E) = \big({}^\nu(M; \in)\big)^M / U$ be the ultrapower[2.168] of $(M; \in)$ mod U using only functions from ν to M that are in M. Thus, $A = \{[f] \mid f \in M \wedge f : \nu \to M\}$, and $[f]\, E\, [g] \leftrightarrow \{\alpha < \nu \mid f\alpha \in g\alpha\} \in U$.[61]

(8.217) *Łoś's theorem*[2.164] *holds for this construction because* $(M; \in) \models$ AC, *so for any formula* ϕ *with free variables* $v, v_0, \ldots, v_{n^-}$ *and any* $f_0, \ldots, f_{n^-} \in {}^\nu M \cap M$, *there exists* $f \in {}^\nu M \cap M$ *such that for all* $\alpha \in \nu$, *if* $(M; \in) \models (\exists v\, \phi)\big[\begin{smallmatrix} v_0 & \cdots & v_{n^-} \\ f_0\alpha & \cdots & f_{n^-}\alpha \end{smallmatrix}\big]$ *then* $(M; \in) \models \phi\big[\begin{smallmatrix} v & v_0 & \cdots & v_{n^-} \\ f\alpha & f_0\alpha & \cdots & f_{n^-}\alpha \end{smallmatrix}\big]$.[62]

For each $a \in A$, let $a_E = \{b \in A \mid b\, E\, a\}$, the *$E$-extension of a*. Let $j : M \to A$ be the canonical injection: $j a \overset{\text{def}}{=} [f_a]$, where $\forall \alpha < \nu\ f_a\alpha = a$. Let $e = [i]$, where i is the identity function: $\forall \alpha < \nu\ i\alpha = \alpha$.

Suppose $[f] \in A$ and $[f]\, E\, e$. Then there exists $Y \in U$ such that f is regressive on Y. Let $f' \in {}^\nu\nu \cap M$ be such that

$$f'\alpha = \begin{cases} f\alpha & \text{if } \alpha \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Then f' is regressive and $[f'] = [f]$. Let $n \in \omega$ be such that $f_n = f'$. Then there exists $\beta \in \nu$ such that $\forall \alpha \in X_{n+1}\ f_n\alpha = \beta$. Hence $[f] = [f'] = [f_\beta] = j\beta$. Thus, $\{a \in A \mid a\, E\, e\} = j^{\to}\nu$.

Let $d = (jh)e$, and let $D = d_E$ $(= \{a \in A \mid a\, E\, d\})$. Since h is strictly increasing and continuous, jh is strictly increasing and continuous in $(A; E)$, so d

[60] Suppose $Y \in \mathcal{P}\nu \cap M$. If $1 \in Y$ let $f : \nu \to 2$ be such that $f\alpha = 0 \leftrightarrow \alpha \in Y$; otherwise, let $f\alpha = 1 \leftrightarrow \alpha \in Y$. f is regressive, so $f = f_n$ for some n, and $X_{n+1} \subseteq Y$ or $X_{n+1} \subseteq \nu \backslash Y$.

[61] Note that $(A; E)$ is a proper class structure. This is legitimate because, as noted above, when "working in $M[G]$" we are using the theory Θ', which incorporates GB. As used here, '$[f]$' denotes the reduced equivalence class $[f]^*_U$ of a function $f : \nu \to M$ as defined in (2.167). Alternatively, we could use the ultrapower of M_η for some sufficiently large η, in which case $[f]$ may be taken to be the full equivalence class $[f]_U$.

[62] It is perhaps worth pointing out that Łoś's theorem in general depends on AC holding in "the real world", whereas in the present application it depends on AC holding in $(M; \in)$, while the "real world" is $(M[G]; \in)$.

is the supremum in $(A;E)$ of elements $(jh)(j\alpha) = j(h\alpha)$, $\alpha < \nu$. Thus, d is the supremum in $(A;E)$ of elements $j\eta$, where $\eta < \kappa$ is a cardinal in M. For each such η, let $D_\eta = (j\eta)_E$. The elements of D_η are represented by functions in ${}^\nu\eta \cap M$. Since κ is a strong limit cardinal in M, there are fewer than κ such functions in the sense of M, so there exist $\theta < \kappa$ and $g \in M$ such that $g : \theta \overset{\text{sur}}{\to} {}^\nu\eta \cap M$. $\gamma \mapsto [g\gamma]$ is in $M[G]$ and maps θ onto D_η. $(D; E \cap D \times D)$ is therefore (in $M[G]$) a linear order all of whose initial segments have size $< \kappa$. (Recall that cardinals in M above μ remain cardinals in $M[G]$, so κ is an uncountable cardinal in $M[G]$.) It follows easily that (in $M[G]$) $|D| \leqslant \kappa$, and clearly $|D| \geqslant \kappa$, so $|D| = \kappa$.

Recall that we have assumed (toward a contradiction) that $M \models 2^{[\kappa]} \neq [\kappa]^+$. It follows that $M \models \ulcorner$there exist $[\kappa]^{++}$ distinct subsets of $[\kappa]\urcorner$. Recall again that cardinals in M above μ remain cardinals in $M[G]$, so $\ulcorner[\kappa]^{++}\urcorner^M = \kappa^{++}$. Thus, $M[G] \models \ulcorner$there exist $[\kappa]^{++}$ distinct subsets of $[\kappa]$ in $M\urcorner$.

In $M[G]$, for each $C \in \mathcal{P}\kappa \cap M$, let a^C be the (unique) $a \in A$ such that $(A;E) \models [a] = [d] \cap [jC]$. Note that $a^C_E = (jC) \cap d_E = (jC) \cap D$. Suppose C, C' are distinct elements of $\mathcal{P}\kappa \cap M$. Without loss of generality, suppose there exists $\gamma \in C \backslash C'$. Then $(j\gamma)\, E\, (jC)$, $\neg\big((j\gamma)\, E\, (jC')\big)$, and $(j\gamma)\, E\, d$, so $(j\gamma)\, E\, a^C$ and $\neg\big((j\gamma)\, E\, a^{C'}\big)$. Hence, a^C and $a^{C'}$ are distinct elements of $(A;E)$, and their E-extensions a^C_E and $a^{C'}_E$ are distinct subsets of $d_E = D$.

Thus, $M[G] \models \ulcorner$there exist $[\kappa]^{++}$ distinct subsets of $[D]\urcorner$. Since $\{\alpha < \nu \mid 2^{h\alpha} = (h\alpha)^+\} = X \in U$, $(A;E) \models 2^{[d]} = [d]^+$.[8.217] Let $b = \ulcorner[d]^+\urcorner^{(A;E)}$, and let $B = b_E$. In $(A;E)$ there exists an injection of the powerset of d into b, which can be used in $M[G]$ to show that $M[G] \models \ulcorner$there exist $[\kappa]^{++}$ distinct elements of $[B]\urcorner$, i.e.,

$$M[G] \models |[B]| \geqslant [\kappa]^{++}. \tag{8.218}$$

However, for every $c\, E\, b$, there is an injection in $(A;E)$ of c_E into $d_E = D$, and we have seen that $M[G] \models |[D]| = [\kappa]$, so in $M[G]$ every initial segment of B has size $\leqslant \kappa$. It follows as before, that $M[G] \models |[B]| \leqslant [\kappa]^+$, contradicting (8.218).

As remarked above, this argument in the generic extension $M[G]$ shows that the supposition made of M in the first paragraph leads to the existence of a partial order $\mathbb{P}$ such that two contradictory sentences are $\mathbb{P}$-valid. This is the contradiction that refutes the supposition. Letting $M = V$, we have the theorem *per se*. $\square$[8.216]

8.14.4 A model in which every set of reals has the Baire property, is Lebesgue-measurable, and has the perfect set property

8.14.4.1 The Levy collapse

(8.219) **Definition** [ZF] *Suppose κ is a regular cardinal.*

1. *Suppose λ is an ordinal. Then $\mathbb{C}(\kappa, \lambda) \overset{\text{def}}{=}$ the partial order $\mathbb{C}$ such that $|\mathbb{C}|$ is the set of functions p such that*
   1. $\operatorname{dom} p \subseteq \kappa$*;*
   2. $\operatorname{im} p \subseteq \lambda$*; and*
   3. $|p| < \kappa$*;*

   and $\leqslant^{\mathbb{C}} = \supseteq$.

2. *Suppose S is a set of ordinals other than an ordinal. Then* $\mathbb{C}(\kappa, S) \stackrel{\text{def}}{=}$ *the partial order* $\mathbb{C}$ *such that* $|\mathbb{C}|$ *is the set of functions p such that*
   1. $\operatorname{dom} p \subseteq S \times \kappa$*;*
   2. *for each* $\langle\alpha, \beta\rangle \in \operatorname{dom} p$, $p\langle\alpha, \beta\rangle \in \alpha$*; and*
   3. $|p| < \kappa$*;*

   and $\leqslant^{\mathbb{C}} = \supseteq$.
3. *Suppose λ is an ordinal. Then* $\mathbb{C}(\kappa, {<}\lambda) \stackrel{\text{def}}{=}$ *the partial order defined as in (8.219.2) for* $S = \lambda$.
4. $\mathfrak{C}(\kappa, \lambda) \stackrel{\text{def}}{=} \mathfrak{R}(\mathbb{C}(\kappa, \lambda))$, $\mathfrak{C}(\kappa, S) \stackrel{\text{def}}{=} \mathfrak{R}(\mathbb{C}(\kappa, S))$, $\mathfrak{C}(\kappa, {<}\lambda) \stackrel{\text{def}}{=} \mathfrak{R}(\mathbb{C}(\kappa, {<}\lambda))$.

Note that $\mathbb{C}(\omega, \lambda)$ as defined in (8.219.1) is isomorphic to $\mathbb{C}(\omega, \{\lambda\})$ as defined in (8.219.2). The commonest use of (8.219.2) for a set S that is not an ordinal is when S is an interval, usually of the form $[\nu, \lambda)$. Specifically, we have the following factorization:

$$\mathbb{C}(\kappa, {<}\lambda) \cong \mathbb{C}(\kappa, {<}\nu) \times \mathbb{C}(\kappa, [\nu, \lambda)). \tag{8.220}$$

(8.221) **Theorem** [ZFC] *Suppose κ is a regular cardinal and $\lambda > \kappa$ is a cardinal.*

1. $\mathbb{C}(\kappa, \lambda)$ *is* ${<}\kappa$*-closed.*
2. *If* $\lambda^{<\kappa} = \lambda$ *then* $||\mathbb{C}(\kappa, \lambda)|| = \lambda$, *so* $\mathbb{C}(\kappa, \lambda)$ *trivially has the* λ^+*-chain condition.*

Proof Straightforward. $\square$[8.221]

$\mathbb{C}(\kappa, \lambda)$ ($\mathfrak{C}(\kappa, \lambda)$) are referred to collectively as *collapsing* partial orders (algebras), by virtue of the following theorem.

(8.222) **Theorem** [GB] *Suppose M is a transitive model of* ZFC, *κ is a regular cardinal in M, $\lambda > \kappa$ is a cardinal in M, and G is an M-generic filter on* $\mathbb{C}(\kappa, \lambda)$.

1. *Every cardinal in $M \leqslant \kappa$ is a cardinal in $M[G]$.*
2. $|\lambda|^{M[G]} = \kappa$.
3. *If* $M \models \ulcorner[\lambda]^{<[\kappa]} = [\lambda]\urcorner$ *then every cardinal in $M > \lambda$ is a cardinal in $M[G]$.*

Proof Clearly, $\bigcup G : \kappa \stackrel{\text{sur}}{\to} \lambda$. The cardinal-preservation properties follow from (8.221) with (8.127) and (8.134). $\square$[8.222]

The partial orders $\mathbb{C}(\kappa, {<}\lambda)$ clearly collapse all cardinals $< \lambda$ to κ. A case of particular interest is that in which λ is a inaccessible cardinal.

Definition [ZFC] *Suppose κ is a regular cardinal and λ is an inaccessible cardinal $> \kappa$. The* Levy collapse of λ to κ^+ $\stackrel{\text{def}}{=} \mathbb{C}(\kappa, {<}\lambda)$.

(8.223) **Theorem** [ZFC] *Suppose κ is a regular cardinal and λ is an inaccessible cardinal $> \kappa$. Then* $\mathbb{C}(\kappa, {<}\lambda)$ *satisfies the λ-chain condition.*

Proof Given $p \in \mathbb{C}(\kappa, {<}\lambda)$, let $\hat{p}$ be the function whose domain is $\{\alpha \in \lambda \mid \exists \beta \in \kappa\ \langle\alpha, \beta\rangle \in \operatorname{dom} p\}$, such that for any $\alpha \in \operatorname{dom} \hat{p}$, $\hat{p}\alpha = \{(\beta, \gamma) \mid (\langle\alpha, \beta\rangle, \gamma) \in p\}$. Note that

1. $|\operatorname{dom}\hat{p}| < \kappa$;
2. $\forall\alpha \in \operatorname{dom}\hat{p}\ \hat{p}\alpha : \kappa \rightharpoonup \alpha$;
3. $|{}^{\kappa}\alpha| < \lambda$.

Note also that if p and p' are incompatible in $\mathbb{C}(\kappa,{<}\lambda)$ then $\exists\alpha \in \operatorname{dom}\hat{p}\cap\operatorname{dom}\hat{p}'\ \hat{p}\alpha \neq \hat{p}'\alpha$.

Suppose X is an antichain in $\mathbb{C}(\kappa,{<}\lambda)$. Let $P = \{\hat{p} \mid p \in X\}$. We will carry out a construction similar to that in the proof of (8.135) to show that $|P| < \lambda$, so $|X| < \lambda$. Thus, we let $\bar{P} = \{\hat{p} \mid p \in \mathbb{C}(\kappa,<\lambda) \wedge \exists\hat{p}' \in P\ \hat{p} \subseteq \hat{p}'\}$, and we construct a κ-sequence $0 = P_0 \subseteq P_1 \subseteq \cdots$ of subsets of P such that, letting $B_\alpha = \bigcup_{\hat{p}\in P_\alpha}\operatorname{dom}\hat{p}$,

1. $|P_\alpha| < \lambda$;
2. $\forall\hat{p} \in \bar{P}\left(\operatorname{dom}\hat{p} \subseteq B_\alpha \to \left(\exists\hat{q} \in P\,(\hat{p} = \hat{q}\restriction B_\alpha) \to \exists\hat{q} \in P_{\alpha+1}\,(\hat{p} = \hat{q}\restriction B_\alpha)\right)\right)$;
3. if α is a limit ordinal then $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$;

and $P = \bigcup_{\alpha<\kappa} P_\alpha$.

Clearly Property 1 is maintained at limit stages. We will show that it can be maintained at successor stages. Suppose $|P_\alpha| < \lambda$. Then $|B_\alpha| < \lambda$. It follows that $|A| < \lambda$, where $A = \{\hat{p} \in \bar{P} \mid \operatorname{dom}\hat{p} \subseteq B_\alpha\}$, and no more than $|A|$ elements have to be added to P_α to form $P_{\alpha+1}$ so as to satisfy Property 2; hence, $P_{\alpha+1}$ may be chosen so that $|P_{\alpha+1}| < \lambda$, as desired.

It remains to show that $P = \bigcup_{\alpha<\kappa} P_\alpha$. Suppose $\hat{p} \in P$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. Since $|\operatorname{dom}\hat{p}| < \kappa$ and κ is regular, for some $\alpha < \kappa$, $\operatorname{dom}\hat{p} \cap B_\alpha = \operatorname{dom}\hat{p} \cap B$. Let $\hat{p}' = \hat{p}\restriction B_\alpha$. Then $\hat{p}' \in \bar{P}$, so there exists $\hat{q} \in P_{\alpha+1}$ such that $\hat{q}\restriction B_\alpha = \hat{p}' = \hat{p}\restriction B_\alpha = \hat{p}\restriction B$. Since $\operatorname{dom}\hat{q} \subseteq B_{\alpha+1} \subseteq B$, $\hat{p}$ and $\hat{q}$ agree on their common domain, so they are compatible. Since X is an antichain, $\hat{q} = \hat{p}$.

Hence $|P| = \sum_{\alpha<\kappa} |P_\alpha| < \lambda$. $\square$ 8.223

(8.224) **Theorem** [GB] *Suppose M is a transitive model of* ZFC*, κ is a regular cardinal in M, $\lambda > \kappa$ is an inaccessible cardinal in M, and G is an M-generic filter on $\mathbb{C}(\kappa,{<}\lambda)$. Then*

1. *every cardinal in M that is $\leqslant \kappa$ or $\geqslant \lambda$ remains a cardinal in $M[G]$; and*
2. $M[G] \models \ulcorner[\kappa]^+ = [\lambda]\urcorner$.

Proof Since $M \models \ulcorner\mathbb{C}([\kappa],{<}[\lambda])$ is ${<}[\kappa]$-closed$\urcorner$, cardinals in $M \leqslant \kappa$ remain cardinals in $M[G]$. It follows from (8.223) that cardinals in $M \geqslant \lambda$ remain cardinals in $M[G]$. Clearly all cardinals η in M such that $\kappa < \eta < \lambda$ are collapsed to κ in $M[G]$, so $M[G] \models \ulcorner[\kappa]^+ = [\lambda]\urcorner$. $\square$ 8.224

8.14.4.2 Solovay's theorem

Suppose M is a transitive model of ZFC and $M \models \ulcorner[\kappa]$ is inaccessible$\urcorner$. Let $\mathbb{C} = \mathbb{C}(\omega,{<}\kappa)$, and suppose G is an M-generic filter on $\mathbb{C}$, then $\omega_1^{M[G]} = \kappa$. Let S be the class of ω-sequences of ordinals in $M[G]$ and let N be the class $\mathrm{HOD}(S)$ of sets hereditarily ordinal-definable from S in the sense of $M[G]$.

(8.225) **Theorem** [GBC] *Under the above conditions, $N \models \ulcorner$for every $X \subseteq \mathbb{R}$*

1. *X has the Baire property;*
2. *X is Lebesgue measurable; and*
3. *X has the perfect set property.*⌝

The proof of this theorem is a big step forward in the sophistication of its use of forcing ideas, and it is worth our while to get an overview of the argument before plunging into the details. In broad strokes the proof goes as follows. We start with a transitive model M of ZFC with $\kappa \in \mathrm{Ord}^M$ such that $M \models \ulcorner[\kappa]$ is inaccessible$\urcorner$, and an M-generic filter G on $\mathbb{C}(\omega, {<}\kappa)^M$. We let N be $\mathrm{HOD}(S)$ in the sense of $M[G]$, where S is the class of ω-sequences of ordinals in $M[G]$. Then[10.186] $N \models \mathsf{ZF} + \mathsf{DC}$ and $\mathbb{R}^N = \mathbb{R}^{M[G]}$. Suppose $X \in N$ is a set of reals. Then there exists $t \in S$ such that X is definable from t in $M[G]_\alpha$ for some $\alpha \in \mathrm{Ord}^M$. t, α, and the defining formula (which as an hereditarily finite set is naturally coded as a finite ordinal) may be incorporated into a single $s \in S$, and

$$X = \{x \in \mathbb{R}^{M[G]} \mid M[G] \models \Delta[s, x]\}, \tag{8.226}$$

where Δ is a fixed s-formula.

Let $M' = M[s]$. We wish to characterize the membership of a real x in X as a property of x in $M'[x]$. For this we use the homogeneity[10.180] and the factorization property[10.184] of the Levy algebra $\mathfrak{C}(\omega, {<}\kappa)$, by virtue of which[10.200]

$$x \in X \leftrightarrow M'[x] \models \Delta^{\Vdash}[\mathbb{C}, 0, \check{s}, \check{x}],$$

where $\Delta^{\Vdash}$ is the s-formula that expresses the forcing relation for sentences derived by substitution of forcing terms for the (two) free variables of Δ, 0 is the empty condition in $\mathbb{C}$, and $\check{s}, \check{x}$ are the canonical terms for s, x in $M'[x]^{\mathbb{C}}$. Via Definition 10.201 we have

$$x \in X \leftrightarrow M'[x] \models D[\mathbb{C}, s, x].$$

We now consider the quotient algebras $\mathfrak{B} = \mathbf{Borel}/\mathfrak{m}$ and $\mathfrak{L} = \mathbf{Borel}/\mathfrak{n}$ constructed in M', where **Borel** is the boolean algebra of Borel sets[5.88] and $\mathfrak{m}$ and $\mathfrak{n}$ are the ideals of meager[5.143] and null[5.161] Borel sets, respectively. There is a natural bijection $F \mapsto x_F$ between M'-generic filters on $\mathfrak{B}$ ($\mathfrak{L}$) and reals such that for any Borel code $\epsilon \in M'$

$$\left[(B\epsilon)^{M'}\right] \in F \leftrightarrow x_F \in (B\epsilon)^N.$$

The reals of the form x_F for M'-generic F are exactly those that are not in any meager (null) Borel set with a Borel code[5.89] in M'. We call these reals *Cohen* or *random* over M', respectively. Let $\dot{x} \in M'^{\mathfrak{B}}$ $\left(M'^{\mathfrak{L}}\right)$ be the canonical term for x_F in $M'[F]$ (in effect, $\dot{x}$ is x_{G}).

We now use the fact that $\mathbb{R}^{M'}$ is countable in N[10.198] to infer that almost every real in N is Cohen (random) over M'. If $x \in N$ is Cohen (random) over M' and F is such that $x = x_F$, then

$$\begin{aligned} M'[x] \models D[\mathbb{C}, s, x] &\leftrightarrow [\![D(\check{\mathbb{C}}, \check{s}, \dot{x})]\!] \in F \\ &\leftrightarrow x \in (B\epsilon)^N, \end{aligned}$$

where $\epsilon \in M'$ is a Borel code such that $[\![D(\check{\mathbb{C}}, \check{s}, \dot{x})]\!] = \left[(B\epsilon)^{M'}\right]$. Thus, for almost every real x in N, membership in X is equivalent to membership in the Borel set $B\epsilon$. Hence, X has the Baire property (is Lebesgue measurable).

Now suppose X is uncountable in N. We must show that X has a perfect subset in N. It will be convenient to use the Baire space ${}^{\omega}\omega$ rather than the real line for this purpose. (Note that if $X \subseteq \mathbb{R}$ is uncountable then $X \backslash \mathbb{Q}$ is uncountable, and we may use the homeomorphism[5.77.2] of $\mathbb{R} \backslash \mathbb{Q}$ with ${}^{\omega}\omega$ to obtain a topologically equivalent uncountable subset of ${}^{\omega}\omega$.) Let $x \in X \backslash M'$. There exists $\nu < \kappa$ such that $x \in M'[G_\nu]$, where $G_\nu = G \cap |\mathbb{C}_\nu|$ and $\mathbb{C}_\nu = \mathbb{C}(\omega, {<}\nu)$. G_ν is an M'-generic filter on $\mathbb{C}_\nu$. By the factorization and homogeneity properties, again, since $x \in X$,

$$M'[G_\nu] \models \Delta^{\Vdash}[\mathbb{C}, 0, \check{s}, \check{x}],$$

so[10.201]

$$M'[G_\nu] \models D[\mathbb{C}, s, x].$$

Let $\dot{x} \in M'^{\mathbb{C}_\nu}$ be such that $\dot{x}^{G_\nu} = x$, and let $p \in \mathbb{C}_\nu$ be such that

$$p \Vdash^{M', \mathbb{C}_\nu} D(\check{\mathbb{C}}, \check{s}, \dot{x}) \wedge \neg \mathsf{V}(\dot{x}).^{63}$$

$(\mathcal{P}\,\mathbb{C}_\nu)^{M'}$ is countable in $M[G]$, and we let $\langle D_n \mid n \in \omega \rangle$ be an enumeration in $M[G]$ of the dense subsets of $\mathbb{C}_\nu$ in M'. By (10.186.1) $\langle D_n \mid n \in \omega \rangle \in N$.

We now define in N for each $t \in {}^{<\omega}2$ a condition $p_t \in |\mathbb{C}_\nu|$ and a sequence $\xi_t \in {}^{<\omega}\omega$ such that for all $t, t' \in {}^{<\omega}2$

1. $p_0 \leqslant p$;
2. $p_t \in D_{|t|}$;
3. $t' \supseteq t \leftrightarrow \xi_{t'} \supseteq \xi_t$;
4. $t' \supseteq t \rightarrow p_{t'} \leqslant p_t$; and
5. $p_t \Vdash \ulcorner (\check{\xi}_t) \subseteq (\dot{x}) \urcorner$.

To accomplish this, we first let p_0 be any extension of p in D_0 and let $\xi_0 = 0$. Given p_t and ξ_t we obtain $p_{t ^\frown \langle i \rangle}$ and $\xi_{t ^\frown \langle i \rangle}$ for $i \in 2$ as follows. Since $p_t \leqslant p$, p_t forces $\dot{x}$ not to be in M', so there must be extensions of p_t forcing incompatible information about $\dot{x}$. Let $n, m_0, m_1 \in \omega$ and $q_0, q_1 \leqslant p_t$ be such that $m_0 \neq m_1$, $q_0 \Vdash \dot{x}(\check{n}) = \check{m}_0$, and $q_1 \Vdash \dot{x}(\check{n}) = \check{m}_1$. Let q_0', q_1' be extensions of q_0, q_1, respectively, that are in $D_{|t|+1}$. For $i \in 2$, let $p_{t ^\frown \langle i \rangle} \leqslant q_i'$ and $\xi_{t ^\frown \langle i \rangle} \in {}^{n+1}\omega$ be such that $\xi_{t ^\frown \langle i \rangle}(n) = m_i$ and $p_{t ^\frown \langle i \rangle} \Vdash \check{\xi}_{t ^\frown \langle i \rangle} \subseteq \dot{x}$.

No Choice axiom is necessary to obtain p_t and ξ_t, as $|\mathbb{C}|$ and ${}^{<\omega}\omega$ have definable wellorderings; however, we do need a Choice axiom to obtain $\langle D_n \mid n \in \omega \rangle$. This is why we obtained this enumeration initially in $M[G]$, which satisfies AC.

Reasoning now in N, suppose $z \in {}^{\omega}2$. Let $F_z = \{r \in |\mathbb{C}_\nu| \mid \exists n \in \omega\ r \geqslant p_{z \restriction n}\}$. Then F_z is a filter on $\mathbb{C}_\nu$ and meets every D_n, so it is M'-generic. Obviously, $\dot{x}^{F_z} = \bigcup_{n \in \omega} \xi_{z \restriction n}$. Since $p \in F_z$ and $p \Vdash D(\check{\mathbb{C}}, \check{s}, \dot{x})$, $\dot{x}^{F_z} \in X$. $\{\dot{x}^{F_z} \mid z \in {}^{\omega}2\}$ is therefore a perfect subset of X.

In Note 10.31 we provide a detailed proof of Theorem 8.225.

---

[63] Recall that V is the predicate comprehending the ground model for the relevant generic extension, so in this case $\mathsf{V}^{M'[G_\nu]}$ is M'.

8.14.4.3 Relative consistency

To obtain a finitary proof of the consistency of $\mathsf{ZF} + \mathsf{DC} + \ulcorner$all sets of reals have the Baire property, etc.$\urcorner$ relative to that of $\mathsf{ZFC} + \ulcorner$there exists an inaccessible cardinal$\urcorner$, we may use any of the methods described in Section 8.6. Note that we have used Definition 6.22 of ordinal-definability, rather than (6.21) in the above argument. As the discussion in Section 6.5 makes clear, this choice is obligatory if M is allowed to be a proper class, as it must be if we are to use Method 3 of Section 8.6, i.e., "arguing in a generic extension (of V)". Note also that our use of the notion of satisfaction for $M[G]$ and its submodels is legitimate, because we have restricted it to the specific formulas Δ, $\Delta^{\Vdash}$ and D. For example, we could have written these out in full and replaced satisfaction statements by relativizations. This would not have been the case if we had used (6.21).

Suppose, however, we specified that the ground model M be countable. Then M-generic filters G would actually exist, $M[G]$ would be countable, and we could define $N \subseteq M[G]$ using Definition 6.21 of ordinal-definability. Likewise, satisfaction relations would exist for all structures under consideration. The above argument would then show that if there is a transitive model of $\mathsf{ZFC} + \ulcorner$there exists an inaccessible cardinal$\urcorner$ then there is a transitive model of $\mathsf{ZF} + \mathsf{DC} + \ulcorner$all sets of reals have the Baire property, etc.$\urcorner$. We would not, however, be able to obtain a finitary proof of relative consistency by replacing ZF by a finite fragment as required by Method 1 of Section 6.5, because we must allow arbitrary formulas ϕ in definitions of sets X of reals in N, which now take the form

$$X = \{x \in \mathbb{R}^{M[G]} \mid M[G] \models \phi[s, x]\},$$

rather than (8.226).

8.15 Summary

The problem of extending a transitive model M of ZF without the addition of new ordinals was solved by Paul Cohen with the concept of a generic extension. Supposing M is countable, $\mathbb{P} \in M$ is a partial order, ϕ is a $\mathbb{P}$-forcing sentence, and $p \in |\mathbb{P}|$, we say that $p \Vdash^{*} \phi \overset{\text{def}}{\iff} M[G] \models \phi$ for every M-generic filter G on $\mathbb{P}$ such that $p \in G$. We establish a simple set of identities for $\Vdash^{*}$, whose derivation depends on the existence of M-generic filters, by which we mean that for every $p \in |\mathbb{P}|$ there exists an M-generic filter G on $\mathbb{P}$ such that $p \in G$; this follows from the countability of M.

Once derived, these identities are seen to provide a recursive definition of a relation $\Vdash^{M,\mathbb{P}}$ over any transitive class model M of ZF containing $\mathbb{P}$, including V. If M-generic filters exist then $\Vdash^{M,\mathbb{P}} \; = \; \Vdash^{*}$. A very useful technique in forcing is to imagine that M-generic filters exist even when they don't. This technique may be justified by the consideration of countable transitive models of sufficiently large finite fragments of ZF, which exist by the reflection principle. We call this "arguing with generic extensions".

Defining the boolean value of a forcing sentence to be the set of conditions that force it leads naturally to the notion of boolean-valued structures, which can also be used as a primary framework in which to develop the theory. In practice partial orders and boolean algebras are used interchangeably as convenience dictates.

The same difficulties arise in the definition of forcing and boolean valuation for proper class models as in the definition of satisfaction for these models, and they are dealt with in the same way, and similar logical considerations apply.

The definability of forcing (or boolean valuation) leads to a straightforward proof that if M is a transitive model of ZF, $\mathbb{P} \in M$ is a partial order, and G is an M-generic filter on $\mathbb{P}$, then $\mathfrak{M}[G]$ is a transitive model of ZF with the same ordinals as M, and $\mathfrak{M}[G] \models \ulcorner [G]$ is a V-generic filter on $[\mathbb{P}]\urcorner$, where $\mathfrak{M}[G]$ is $M[G]$ with the additional predicate symbol V denoting M. The method of "arguing in a generic extension" using the theory Θ[8.101] achieves the advantages of reasoning in $\mathfrak{M}[G]$ even when M-generic filters do not exist, e.g., when $M = V$.

Θ is a pure set theory that implements the assumption that M models ZF as the set of relativizations θ^{V} of the axioms θ of ZF to V. A conservative extension result[8.108] allows us to use instead the class theory Θ', which implements the assumption that M models ZF as the single sentence $\ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$. One may also argue directly in a boolean-valued universe $V^{\mathfrak{A}}$.[§ 8.5.4]

Several ways to obtain a relative consistency result by a forcing argument are given in Section 8.6. They are all essentially equivalent.

Certain properties of partial orders and boolean algebras are particularly relevant to the properties of generic extensions. Among these are chain (or saturation), closure, and distributivity conditions. We show how these conditions may be employed to control cardinalities and powersets in extensions, and we show the relative consistency of the violation of the generalized continuum hypothesis (GCH) at any regular cardinal.

We adapt the Fraenkel-Mostowski method of symmetric models to generic extensions to show the consistency of $\mathsf{ZF} + \neg\, \mathsf{AC}$ (without the use of urelements).

We discuss the basic forcing properties of products of partial orders, and apply this theory to prove a limited version of Easton's result on adjusting the size of 2^κ for a set of regular cardinals κ.

Up to this point the discussion has been limited to "set forcing", i.e., to the case of partial orders that are sets. We now indicate the issues that must be dealt with to extend the discussion to "class forcing", and we show that the *tame* partial orders are the appropriate ones for class forcing. We use this theory to prove the full Easton result on adjusting the size of 2^κ for all regular cardinals κ.

We show how a generic extension of a generic extension may be modeled as a single forcing construction, with particular attention to the elegant analysis in terms of complete subalgebras. This theory is important in its own right, but it is also the basis for the theory of forcing iterations of arbitrary length. Here we limit our discussion to iteration with finite support, and we give the application for which it was invented, viz., the relative consistency of Suslin's hypothesis and more generally Martin's axiom.

We conclude with several arguments giving some indication of the diversity of uses of forcing. We begin with Tennenbaum's clever method of obtaining a Suslin tree by forcing with finite conditions, followed by the construction of a $\diamondsuit$-sequence using countable conditions. We then present Silver's theorem on the powersets of singular cardinals of uncountable cofinality. Finally, we present Solovay's model of $\mathsf{ZF} + \mathsf{DC}$ in which all sets of reals have the Baire, Lebesgue measurability, and perfect set properties.

Chapter 9

On Beyond ZF

Contents

9.1 Introduction **579**
9.1.1 Comparison of theories 580
9.2 Large cardinals **580**
9.2.1 Somewhat large cardinals 580
9.2.2 Measurable cardinals 582
9.2.3 Elementary embeddings of transitive classes 589
9.2.4 Ultrafilters from embeddings 594
9.2.5 On the largeness of measurable cardinals 596
9.2.6 Iterated ultrapowers 597
9.2.7 Compactness 602
9.2.8 Trees 606
9.2.9 Partitions 608
9.3 Large cardinals and constructibility **615**
9.3.1 Weak compactness 615
9.3.2 Strong compactness 618
9.3.3 Measurability: $L[U]$ 619
9.3.4 Indiscernibles 621
9.4 Ideals over cardinals **629**
9.4.1 Saturation 629
9.4.2 Precipitousness and normality 632
9.4.3 Real-valued and 2-valued measurability 640
9.5 Larger cardinals **644**
9.5.1 Supercompactness 644
9.5.2 An upper limit 647
9.5.3 Vopěnka's principle and extendibility 649
9.6 The continuum problem, continued **653**
9.7 Determinacy **654**
9.7.1 Regularity properties of pointsets 654
9.7.2 Structural properties of pointclasses 659
9.7.3 The Wadge hierarchy 663
9.7.4 AD 673
9.7.5 Large cardinals from determinacy 674

9.7.6 Determinacy from large cardinals 686

9.8 Summary 697

Said Conrad Cornelius o'Donald o'Dell,
My very young friend who is learning to spell:
"The A is for Ape. And the B is for Bear.
"The C is for Camel. The H is for Hare.
. . .
". . . Through Z is for Zebra. I know them all well."
Said Conrad Cornelius o'Donald o'Dell.
"So now I know everything *any*one knows
"From beginning to end. From the start to the close.
"Because Z is as far as the alphabet goes."
Then he almost fell flat on his face on the floor
When I picked up my chalk and drew one letter more!
A letter he never had dreamed of before!
And I said, "You can stop, if you want, with the Z
"Because most people stop with the Z
"*But not me!*
. . .
So, on beyond Zebra!
Explore!
Like Columbus!
. . .

On Beyond Zebra by Theodor Seuss Geisel (Dr. Seuss)

. . . all experience is an arch wherethro'
Gleams that untravell'd world whose margin fades
For ever and for ever. . .

Ulysses by Alfred, Lord Tennyson

'Beauty is truth, truth beauty, . . . '

Ode on a Grecian Urn by John Keats

Ad pulcritudinem tria requiruntur, integritas, consonantia, claritas.[1]

Summa Theologica by Thomas Aquinas
as paraphrased by Stephen Dedalus in *A Portrait of the Artist as a Young Man* by James Joyce

[Aki Kanamori's *The Higher Infinite*[14] is an excellent source for the topics of this chapter, particularly large cardinals. As we have previously stated, Thomas Jech's *Set Theory*[12] is an excellent source for all aspects of set theory, including this these topics. Both books provide an entrée into the world of active research in set theory. The *Handbook of Set Theory*[5], edited by Matthew Foreman and Aki Kanamori, is a collection of fairly self-contained monographs by leading mathematicians in the field; an "unofficial index" is available at http://handbook.assafrinot.com/ with links to the authors' websites, where the chapters can be downloaded.]

[1] Three things are needed for beauty: wholeness, harmony, radiance.

9.1 Introduction

The limitations of ZF have been extensively studied by the use of inner models and generic extensions. Many statements are known to be neither provable nor disprovable using the axioms of ZFC alone. If we take the position that sets exist in some ideal realm, then each of these statements is either true or false, and we are motivated to determine which is which, just as we are motivated to determine which statements are true concerning the physical world.

I am not at all convinced of the validity of this so-called *platonist* position, but then again, I am also uncertain as to the nature of physical reality. Perhaps after we have explored the foundations of physics you will join me in the latter skepticism. In any event, a good many students of the foundations of mathematics are uncomfortable with the notion of absolute truth as regards statements of set theory.

But it is a narrow conception of mathematics to regard its value as solely descriptive. Indeed, mathematics, more than any other human endeavor, reveals that 'beauty is truth, truth beauty'. Before the foundations of mathematics—and concomitantly set theory—were investigated, mathematics had produced many examples of the beauty of truth, and it continues to do so; but perhaps now it is time also to embrace the bolder half of Keats's thesis: the truth of beauty. Given the inability of the Zermelo-Fraenkel axioms to answer many mathematical questions, may we not undertake, in the words of Tennyson's Ulysses, 'to seek a newer world ... To sail beyond the sunset ... ', and find a larger truth? We of the present era have an opportunity unsuspected by mathematicians of former times, the opportunity to create mathematical truth even as we discover it. And what better guide in this endeavor—indeed, what *other* guide—than that of beauty?

The simplest option that presents itself for extending ZF is the axiom of constructibility, $\ulcorner V = L \urcorner$. This is a very powerful assumption, which settles many questions left open by ZF. The theory of L has real aesthetic merit, but I think it is fair to say that it is a rare set theorist in whose mind L is anything but a very thin sliver of the universe. The reason is this: once one has conceived of a mathematical object, then it exists. Unlike Descartes, who was not content to be someone's dream content, sets are happy to say 'I am thought of, therefore I am.'. At least, if an object can conceivably (consistently) exist, then it seems unwarranted to deny its existence without good reason. In terms of Aquinas's three conditions of beauty, we therefore conclude that while the theory of L possesses harmony, it lacks wholeness and consequently radiance.

If L is the safe and ordered world of Ithaca after Ulysses's return (and after a little messy business was attended to), the world outside of L is just as wild and wonderful as the one Ulysses longed to explore again. The challenge of defining truth out there is a daunting one, but there are few intellectual adventures to match it. In this chapter we will see how the expansive impulse has led to a truly beautiful theory of the countably infinitary.

All of the above makes sense from a certain point of view, but I would be remiss if I did not point out that instead of viewing $\boldsymbol{V = L}$ as saying that V is so thin that every set is constructible, we may view it as saying that Ord is so long that every set gets constructed. From this point of view $\boldsymbol{V = L}$ is a strong axiom of infinity. That said, $\mathsf{ZF} \cup \{\boldsymbol{V = L}\}$ can only be considered to "generate" the sets in the minimum model of it. This model certainly does not contain any transitive set models with large cardinals. We should also note that a measure of a theory's power is its ability to interpret other theories. $\mathsf{ZF} \cup \{\boldsymbol{V} = \boldsymbol{L}\}$ cannot interpret any theory with large (enough) cardinals, as it cannot interpret any theory that $\mathsf{ZF} \cup \{\mathrm{Con}\,\mathsf{ZF}\}$ cannot prove consistent.

9.1.1 Comparison of theories

The *strength* of a theory is its position in the partial ordering of theories by the relation $\vdash$. Thus, Θ' is as strong as Θ iff $\Theta' \vdash \Theta$. The *consistency strength* of a theory Θ is the position of $\mathrm{Con}\,\Theta$ in the partial ordering of sentences by $\vdash$ modulo a suitable base theory such as S. Thus, the consistency strength of Θ' is as great as that of Θ iff $\mathsf{S} \vdash \ulcorner \mathrm{Con}\,\Theta' \to \mathrm{Con}\,\Theta \urcorner$. We may also compare theories by the relation of interpretability; however, the first two modes of comparison are the most used.

9.2 Large cardinals

One of the most potent ways of extending ZF is by asserting the existence of sets with some attribute that implies that they are large in a suitable sense. In the context of AC, every set is equipollent with a cardinal, and we refer generally to assumptions of this sort as *large cardinal hypotheses.*

The axiom of $\mathsf{Infinity}$ is the paradigm of large cardinal hypotheses, and in terms of the qualitative difference between the theory of membership with and without it, it is the most powerful large cardinal axiom. One way to formulate a statement that a cardinal is "large" is to say that it stands in a similar relation to all smaller cardinals as ω does to all finite cardinals.

There is a great diversity of assertions that imply that a cardinal is large, and it is a remarkable fact that the large cardinal hypotheses that have been proposed are for the most part linearly ordered by strength. Figure 9.1 compares the strengths of some of the large cardinal hypotheses considered here.

9.2.1 Somewhat large cardinals

(9.1) **Definition** [ZFC] *Suppose κ is an uncountable cardinal.*

1. κ *is* weakly inaccessible $\overset{\text{def}}{\iff}$ *κ is a regular limit cardinal.*
2. κ *is* (strongly) inaccessible $\overset{\text{def}}{\iff}$ *κ is weakly inaccessible and is also a strong limit cardinal, i.e, for every cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.*

Note that apart from the condition of uncountability, ω satisfies the definition of inaccessibility. Thus, inaccessibility is an example of asserting that a cardinal is large by attributing to it a property that ω enjoys *vis-à-vis* smaller cardinals, viz., that its cofinality exceeds any smaller cardinal and that it exceeds the size of the powerset of any smaller cardinal.

(9.2) **Theorem** [ZFC] *Suppose κ is a strongly inaccessible cardinal. Then $V_\kappa \models$ ZFC.*

Proof For any limit ordinal $\alpha > \omega$, V_α is easily seen to be a model of ZFC without the $\mathsf{Collection}$ schema. To show that $V_\kappa \models \mathsf{Collection}$ it suffices to show that for any $x \in V_\kappa$ and $f : x \to V_\kappa$, $\mathrm{im}\, f \subseteq V_\alpha$ for some $\alpha < \kappa$. Since κ is weakly inaccessible, it is enough to show that $|x| < \kappa$, and this follows from the fact that κ is a strong limit cardinal. $\square^{9.2}$

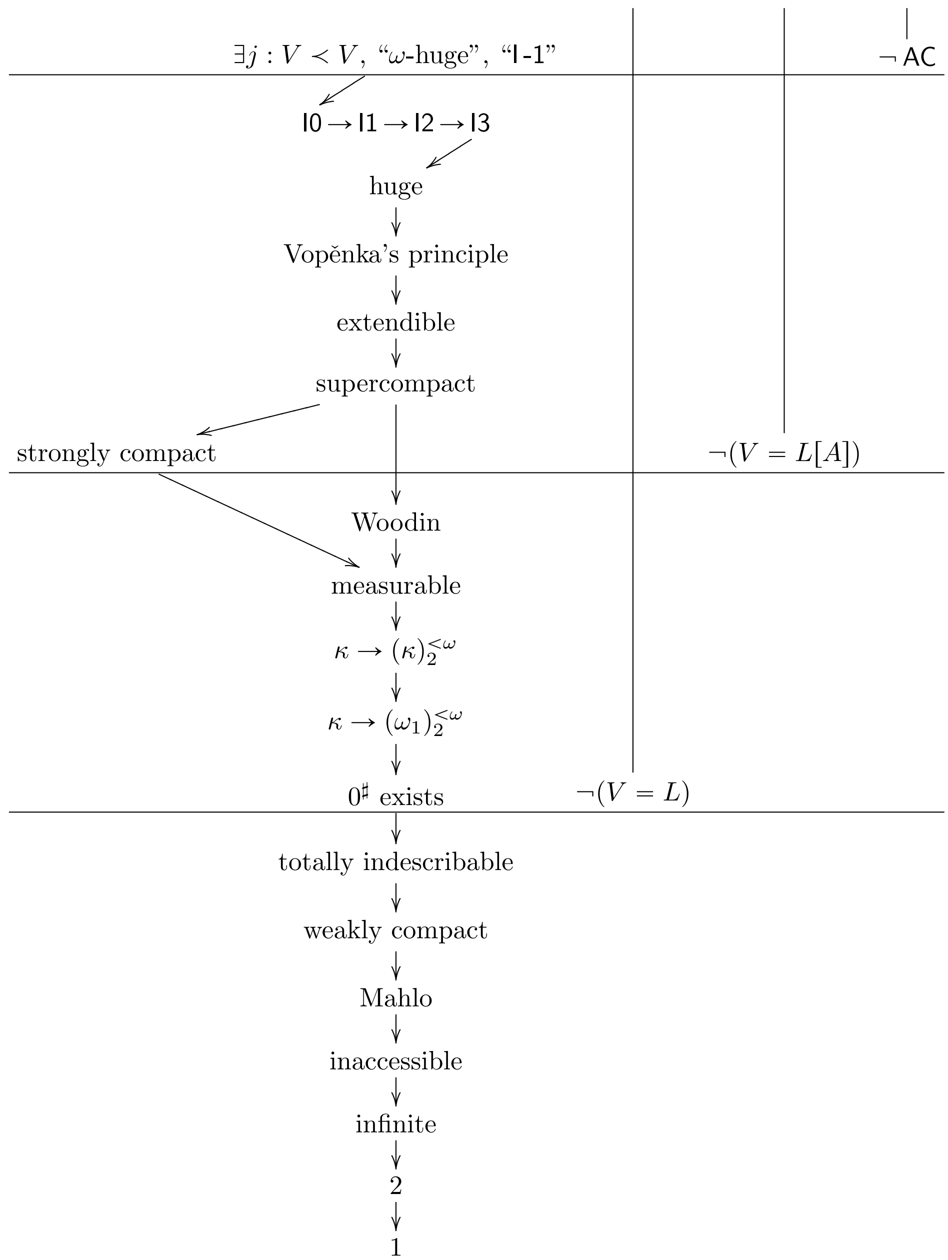

Figure 9.1: Large cardinal hypotheses. An arrow joining A to B means variously that A^* implies B^* or that $\mathrm{Con}(\mathsf{GBC} + A^*)$ implies $\mathrm{Con}(\mathsf{GBC} + B^*)$, where A^* is either the statement that a given cardinal κ has property A or that there exists a cardinal with property A or that $V \models A$, as appropriate.

(9.3) **Theorem** [S] *Suppose* ZF *is consistent. Then* $\mathsf{ZFC} + \mathsf{GCH} \nvdash$ ⌜*there exists a weakly inaccessible cardinal*⌝.

Proof Suppose $\mathsf{ZFC} + \mathsf{GCH} \vdash$ ⌜there exists a weakly inaccessible cardinal⌝. In the context of ZFC, GCH implies that every limit cardinal is a strong limit cardinal, so every weakly inaccessible cardinal is strongly inaccessible. GCH also implies that $V_\kappa \models \mathsf{GCH}$ for every limit cardinal $\kappa > \omega$. Thus, $\mathsf{ZFC} + \mathsf{GCH} \vdash$ ⌜there exists a strongly inaccessible cardinal κ, and[9.2] $V_\kappa \models \mathsf{ZFC} + \mathsf{GCH}$⌝.

It follows that ZFC + GCH proves its own consistency, so by Gödel's second incompleteness theorem, it is inconsistent. We've already seen that if ZF is consistent then ZFC + GCH is consistent, so ZF is inconsistent, contrary to hypothesis. □[9.3]

By virtue of (9.3) inaccessibility qualifies as a large cardinal property; it is in some sense the weakest such property. We can easily manufacture stronger large cardinal properties by saying that a cardinal is 0-*inaccessible* $\overset{\text{def}}{\iff}$ it is inaccessible; $(\alpha+1)$-*inaccessible* $\overset{\text{def}}{\iff}$ it is α-inaccessible and is a limit of α-inaccessibles; and α-*inaccessible* for a limit ordinal α $\overset{\text{def}}{\iff}$ it is β-inaccessible for all $\beta < \alpha$. The following definition is a more sophisticated version of this, and it hints at the sort of relationship that often holds between large cardinal properties of differing strengths.

(9.4) **Definition** [ZFC] *Suppose κ is a cardinal and α is an ordinal. The following definition is to be understood as a recursion on α.*

1. κ *is* 0-Mahlo $\overset{\text{def}}{\iff}$ κ *is (strongly) inaccessible.*
2. κ *is* $(\alpha+1)$-Mahlo $\overset{\text{def}}{\iff}$ κ *is α-Mahlo and* $\{\lambda < \kappa \mid \lambda \text{ is } \alpha\text{-Mahlo}\}$ *is stationary in* κ.
3. κ *is* α-Mahlo *for α a limit ordinal* $\overset{\text{def}}{\iff}$ κ *is β-Mahlo for all $\beta < \alpha$.*

κ *is* Mahlo $\overset{\text{def}}{\iff}$ *it is* 1*-Mahlo.* κ *is* hyper-Mahlo $\overset{\text{def}}{\iff}$ κ *is α-Mahlo for all $\alpha < \kappa$.*

Clearly we can extend this sort of thing as long as we wish, defining *1-hyper-Mahlo* to mean 'hyper-Mahlo', α-*hyper-Mahlo* by recursion as in (9.4.2, 3), *hyper-hyper-Mahlo* by diagonalization as for 'hyper-Mahlo', etc.

9.2.2 Measurable cardinals

Although each of the successive notions of largeness defined in the last section is stronger than each of the preceding ones, none of them is very large by modern standards. In particular, the existence of any of them is consistent with ⌜$V = L$⌝.[2] In other words, their existence implies the universe V of sets is tall, but not that it is fat.

Inconsistency with ⌜$V = L$⌝ is something of a threshold for large cardinal hypotheses. Remarkably, hypotheses above this threshold often have consequences concerning the existence and behavior of relatively small sets, in particular, reals, i.e., subsets of ω. The paradigm of "large" large cardinal properties is that of measurability.

[2] This is to be understood in terms of consistency strength, i.e., if it is consistent that such a cardinal exists then it is consistent that such a cardinal exists and $V = L$. In fact, in the cases mentioned, any cardinal with the given property continues to have that property in L.

Recall the notion of *measure* that we have seen is so important in analysis and descriptive set theory. A *finite*[3] *measure space* is a set X, a set $\mathfrak{M}$ of subsets of X that is a countably complete subalgebra of $\mathcal{P}X$ regarded as a boolean algebra, and a map $\mu : \mathfrak{M} \to [0, \infty)$ that is *countably additive*:

(9.5) *For any sequence $\langle A_n \mid n \in \omega \rangle$ of disjoint sets in $\mathfrak{M}$,*

$$\mu\left(\bigcup_{n\in\omega} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n).$$

This sort of measure is called *real-valued* for the obvious reason.

As we have already seen, the existence of real-valued measures is a matter of considerable importance to the foundations of mathematics. The *measure problem* as originally posed by Lebesgue was whether there exists a nontrivial translationally invariant measure on $\mathcal{P}\mathbb{R}$; 'nontrivial' in this context means not identically 0. As we have seen,[§ 5.7] Lebesgue measure is the unique solution to this problem (up to normalization) if we restrict our attention to certain countably closed subalgebras of $\mathcal{P}\mathbb{R}$, such as the algebra of Borel sets, or more generally, the closure of the class of analytic sets under the Suslin operation $\mathcal{A}$. Vitali showed[5.162] that a complete solution is incompatible with ZFC. Solovay—assuming the consistency of ZFC+ ⌜there exists an inaccessible cardinal⌝—showed[8.225] that the existence of a complete solution of Lebesgue's problem is consistent with ZF + DC.

Following Vitali's result, Banach posed the following variant of Lebesgue's problem: whether there exists any nontrivial measure on $\mathcal{P}\mathbb{R}$, not necessarily translation-invariant. For this problem, 'nontrivial' means not identically 0 but assigning measure 0 to singletons $\{x\}$. Banach and Kuratowski showed that a solution of this problem is incompatible with ZFC + CH. Note that existence of a solution to Banach's measure problem for subsets of any bounded interval is equivalent to a full solution, and it is acceptable to standardize the problem by restricting attention to *unit measures*, i.e. those whose values are bounded by 1. Also, given that translation-invariance is not required, the structure of $\mathbb{R}$ is irrelevant, and any set will serve.

(9.6) *Suppose S is a set. μ is a* unit real-valued measure over S $\overset{\text{def}}{\iff}$

1. $\mu : \mathcal{P}S \to [0, 1]$*;*
2. $\mu S = 1$*;*
3. *for each $x \in S$, $\mu\{x\} = 0$; and*
4. *μ is countably additive.*[9.5]

In this discussion, we will use measure *to mean* unit real-valued measure*.*

In general, an object defined *on* a boolean algebra $\mathfrak{A}$ of subsets of a given set S, is said to be defined *over* S when $|\mathfrak{A}| = \mathcal{P}S$. (This usage is subject to variation, and we will rely on context to resolve any potential ambiguity.)

Banach's problem is as follows:

(9.7) *Does there exist a (unit real-valued) measure over a nonempty set S?*

Given that the structure of the set S in (9.7) does not matter, in the setting of ZFC we may suppose S is a cardinal.

[3]'Finite' here refers to the fact that every set in $\mathfrak{M}$ has finite measure.

Clearly, S must be uncountable.

Banach observed that it is not necessary to restrict the additivity of measures to countable sets. At first this would not seem to be a useful generalization, since uncountable sums of real numbers are rather trivial: We define the sum of an arbitrary set of nonnegative reals as the supremum of the sums of its finite subsets, which is either a nonnegative real number or ∞; and the sum of an uncountable set of positive real numbers is always ∞, since for some $n > 0$, there are uncountably many $> 1/n$.

Nevertheless, we do generalize this way, and we find that it does not make the existence problem any harder.[9.8]

Definition [ZFC] *Suppose κ is a cardinal. A measure μ over a set S is κ-additive $\overset{\text{def}}{\iff}$ for every $\lambda < \kappa$, if A_α $(\alpha \in \lambda)$ are disjoint subsets of S then*

$$\mu \bigcup_{\alpha\in\lambda} A_\alpha = \sum_{\alpha\in\lambda} \mu A_\alpha.$$

Note that ω_1-additivity is countable additivity.

(9.8) **Theorem** [ZFC] *Suppose κ is the least cardinal such that there is a (countably additive) measure over κ. Then every (countably additive) measure over κ is κ-additive.*

Proof Suppose toward a contradiction that μ is a measure over κ that is not κ-additive. Let $\lambda < \kappa$ and A_α $(\alpha \in \lambda)$ be disjoint subsets of κ such that $\mu \bigcup_{\alpha\in\lambda} A_\alpha \neq \sum_{\alpha\in\lambda} \mu A_\alpha$. There exist only countably many $\alpha \in \lambda$ such that $\mu A_\alpha > 0$, and if we remove these, by countable additivity, we still have the foregoing inequality, so we may assume without loss of generality that for all $\alpha \in \lambda$, $\mu A_\alpha = 0$, yet $\mu \bigcup_{\alpha\in\lambda} A_\alpha = r > 0$. Let $\mu' : \mathcal{P}\,\lambda \to [0,1]$ be such that for every $X \subseteq \lambda$,

$$\mu' X = \frac{\mu \bigcup_{\alpha\in X} A_\alpha}{r}.$$

Then μ' is a measure over λ. $\square$[9.8]

This leads to the following definition.

(9.9) **Definition** [ZFC] *A cardinal κ is* real-valued measurable $\overset{\text{def}}{\iff}$ *there exists a κ-additive measure over κ.*

Banach showed under the hypothesis of GCH that real-valued measurability is a large cardinal property; specifically, if κ is real-valued measurable then κ is weakly inaccessible.

A more discriminating analysis was achieved by Ulam.

(9.10) **Theorem** [ZFC] *For any cardinal λ there exists an* Ulam matrix, *i.e., $\langle A^\eta_\alpha \mid \alpha < \lambda^+ \wedge \eta < \lambda\rangle$ such that for all $\alpha < \beta < \lambda^+$ and $\eta < \lambda$*

1. $A^\eta_\alpha \subseteq \lambda^+$;
2. $A^\eta_\alpha \cap A^\eta_\beta = 0$; *and*
3. $|\lambda^+ \setminus \bigcup_{\eta<\lambda} A^\eta_\alpha| \leqslant \lambda$.

Proof For each $\gamma < \lambda^+$ let $f_\gamma : \lambda \overset{\text{sur}}{\to} \gamma + 1$, and for $\alpha < \lambda^+$ and $\eta < \lambda$, let $A^\eta_\alpha = \{\gamma < \lambda^+ \mid f_\gamma \eta = \alpha\}$. Note that if $\gamma \geqslant \alpha$ then $\exists \eta < \lambda\ f_\gamma \eta = \alpha$, so $\gamma \in \bigcup_{\eta<\lambda} A^\eta_\alpha$. $\square$[9.11]

(9.11) **Theorem** [ZFC] *Suppose κ is real-valued measurable. Then κ is weakly inaccessible.*

Proof Let μ be a κ-additive measure over κ. We first show that κ is regular. To this end, suppose to the contrary that $\langle \gamma_\alpha \mid \alpha < \lambda \rangle$ is unbounded in κ, where $\lambda < \kappa$. Since singletons have measure 0,[9.6.3] and μ is κ-additive,

$$\mu\kappa \leqslant \sum_{\alpha<\lambda} \mu\gamma_\alpha = \sum_{\alpha<\lambda} \sum_{\beta<\gamma_\alpha} \mu\{\beta\} = 0.$$

To show that κ is a limit cardinal, suppose toward a contradiction that $\kappa = \lambda^+$. Let $\langle A^\eta_\alpha \mid \alpha < \lambda^+ \wedge \eta < \lambda \rangle$ be an Ulam matrix[9.11] for λ. Then for each $\alpha < \kappa$ there exists $\eta_\alpha < \lambda$ such that $\mu A^{\eta_\alpha}_\alpha > 0$. For some $\eta < \lambda$, $|\{\alpha < \kappa \mid \eta_\alpha = \eta\}| = \kappa$. $\{A_\alpha \mid \eta_\alpha = \eta\}$ is therefore a set of κ pairwise disjoint sets of positive measure, which is not possible, since κ is uncountable. $\square$[9.11]

Ulam pointed out the following distinction, which proves to be far deeper than it first appears.

(9.12) **Definition** [ZFC] *Suppose μ is a measure over a set S.*

1. *$A \subseteq S$ is an* atom for μ $\overset{\text{def}}{\iff}$ *$\mu A > 0$ and for any $B \subseteq A$, $\mu B = 0$ or $\mu B = \mu A$.*
2. *μ is* atomless $\overset{\text{def}}{\iff}$ *no $A \subseteq S$ is an atom for μ.*

As it turns out, it is the atomless case that is directly relevant to the original measure problem of Lebesgue and Banach. The existence of a κ-additive measure with an atom is, on the other hand, a very powerful large cardinal hypothesis, which we will use as our entrée into the world of truly large cardinals. Importantly, from the standpoint of consistency strength the two cases are equivalent, and Solovay's proof[9.113] of this is a beautiful synthesis of some of the major themes in set theory: constructibility, genericity, and large cardinals.

(9.13) **Theorem** [ZFC] *Suppose there is an atomless κ-additive measure over κ.*

1. *$\kappa \leqslant 2^\omega$.*
2. *There is a measure over $\mathbb{R}$ extending Lebesgue measure.*

Proof 1 Suppose μ is an atomless κ-additive measure over κ. We first observe that for any $X \subseteq \kappa$ such that $\mu X > 0$ and any $\varepsilon > 0$, there exists $Y \subseteq X$ such that $0 < \mu Y < \varepsilon$. To prove this, construct a sequence $X = X_0 \supseteq X_1 \supseteq \cdots$ such that for each $n \in \omega$, $0 < \mu X_{n+1} \leqslant \frac{1}{2}\mu X_n$. To accomplish this we use the fact that X_n is not an atom[9.12] to let $X' \subseteq X_n$ be such that $0 < \mu X' < \mu X_n$. Then let X_{n+1} be either X' or $X_n \backslash X'$, whichever is smaller. Clearly, for some $n \in \omega$, $\mu X_n < \varepsilon$. Let $Y = X_n$.

Next we observe that any $X \subseteq \kappa$ may be divided precisely in half. We may assume that $\mu X > 0$. We will use the preceding observation to construct a sequence of disjoint subsets X_α of X for $\alpha < \eta$, where η is a countable ordinal to be determined, such that for each $\alpha < \eta$, $\mu \bigcup_{\beta<\alpha} X_\beta < \frac{1}{2}\mu X$, and $\mu \bigcup_{\beta<\eta} X_\beta = \frac{1}{2}\mu X$. Start by letting $X_0 \subseteq X$ be such that $\mu X_0 \leqslant \frac{1}{2}\mu X$. If, having defined X_β for all $\beta < \alpha$, we find that $\mu \bigcup_{\beta<\alpha} X_\beta = \frac{1}{2}\mu X$, we let $\eta = \alpha$ and we are finished. Otherwise, $\mu \bigcup_{\beta<\alpha} X_\beta < \frac{1}{2}\mu X$, and we let $X_\alpha \subseteq X \setminus \bigcup_{\beta<\alpha} X_\beta$ be such that $\mu \bigcup_{\beta<\alpha} X_\beta < \mu \bigcup_{\beta\leqslant\alpha} X_\beta \leqslant \frac{1}{2}\mu X$. At some countable stage, equality must obtain; otherwise, we would have an uncountable set of disjoint sets of positive measure, some countable subset of which would necessarily have total measure > 1.

Now define for each $s \in {}^{<\omega}2$ a subset X_s of κ, such that

1. $X_0 = \kappa$;
2. $\forall s \in {}^{<\omega}2 \; \mu X_s = 2^{-|s|}$, where $|s|$ is the size of s, which is also the length of s;
3. for all $s \in {}^{<\omega}2$, X_s is the disjoint union of $X_{s^\frown\langle 0\rangle}$ and $X_{s^\frown\langle 1\rangle}$.

For each $f \in {}^\omega 2$, let $X_f = \bigcap_{n\in\omega} X_{f\restriction n}$. Note that each $\mu X_f = 0$, and $\kappa = \bigcup_{f\in{}^\omega 2} X_f$. It follows that μ is not $(2^\omega)^+$-additive, so $\kappa \leqslant 2^\omega$.

2 We use μ and the system of X_ss to define an extension of Lebesgue measure as follows. Let I be the set of real numbers $x \in [0,1]$ whose binary representation does not terminate. Note that I simply omits the countable set of numbers in $[0,1]$ that are integer multiples of 2^{-n} for some $n \in \omega$. Similarly, let J be the set of $f \in {}^\omega 2$ that are not eventually constant. J omits only a countable set of functions. Let $\iota : J \xrightarrow{\text{bij}} I$ be such that

$$\iota f = \sum_{n\in\omega} f(n)\, 2^{-n-1}.$$

Define $\mu' : \mathcal{P}[0,1] \to [0,1]$ so that for each $R \subseteq [0,1]$,

$$\mu' R = \mu\big(\bigcup_{f\in(\iota\leftarrow R)} X_f\big).$$

Note that for each interval of the form $[k\cdot 2^{-n}, (k+1)\cdot 2^{-n}]$, μ' assigns the usual Lebesgue measure 2^{-n}. Since these intervals form a base for the topology on $[0,1]$, μ' agrees with Lebesgue measure on all Borel sets. Thus, it also has the same null sets, and it therefore agrees with Lebesgue measure on all Lebesgue-measurable sets. It therefore extends Lebesgue measure.

μ' satisfies the theorem if we interpret $\mathbb{R}$ to be $[0,1]$, but we can also extend μ' by translation to all of $\mathbb{R}$ in the usual sense. Note that this is only translation by integers; the resulting measure cannot be translation invariant *per se*, by virtue of Vitali's theorem,[5.162] despite the fact that it extends Lebesgue measure, which is translation invariant. □[9.13]

Of course, Theorem 9.13 could be moot, as in ZFC we cannot prove that there exists a measure, since the least cardinal κ that supports a measure is weakly inaccessible,[9.8, 9.11] so $L_\kappa \models$ ZFC, and Gödel's incompleteness theorem applies. Nevertheless, it appears unlikely that in ZFC one can disprove the existence of a measure (which is not to say that people haven't tried,[4] but it would not be a good choice of dissertation topic), and (9.13) is a good early example of a plausible hypothesis implying a drastic failure of the continuum hypothesis.

[4] Indeed, it is because very smart people have tried hard and failed that it appears unlikely.

We now turn to the other branch[9.12.1] of Ulam's investigation of measure. Suppose $A \subseteq \kappa$ is an atom for a κ-additive measure μ over κ. Let μ' be defined on $\mathcal{P}\kappa$ so that for every $X \subseteq \kappa$

$$\mu' X = \frac{\mu(X \cap A)}{\mu A}.$$

Note that μ' is κ-additive. For every $X \subseteq \kappa$, $\mu' X$ is 0 or 1, and we say that μ' is 2-*valued*, where 2 is understood as the von Neumann ordinal $\{0, 1\}$. In this context, the character of 0 and 1 as real numbers is no longer relevant.

For any measure μ over a set S, the sets $\{X \subseteq S \mid \mu X = 0\}$ and $\{X \subseteq S \mid \mu X = 1\}$ are respectively an ideal and a filter on $\mathcal{P}S$.[5] If μ is 2-valued, its filter and its ideal are each maximal in the sense that neither can be enlarged without encompassing the entire algebra. A maximal filter on a boolean algebra is usually referred to as an *ultrafilter*. The topic of 2-valued measures is therefore coextensive with that of ultrafilters, and the discussion usually focuses on the latter.

Definition [ZF] *A filter F over a set X is* principal $\overset{\text{def}}{\iff} \exists x \in X \, \forall A \in F \; x \in A$.

Given $x \in X$, let $U^x \overset{\text{def}}{=} \{A \subseteq X \mid x \in A\}$. U^x is clearly a principal ultrafilter. The use of a principal ultrafilter trivializes the ultraproduct construction described in Section 2.7.5, inasmuch as $\prod \langle \mathfrak{A}_y \mid y \in X \rangle / U^x \cong \mathfrak{A}_x$. The existence of nonprincipal ultrafilters, on the other hand, is a matter of considerable interest.

Definition [ZF] *Suppose κ is a cardinal. A filter F over a set X is* κ-complete $\overset{\text{def}}{\iff}$ *for all $\lambda < \kappa$, for all $\langle A_\alpha \mid \alpha < \lambda \rangle$, if $A_\alpha \in F$ for all $\alpha < \lambda$ then $\bigcap_{\alpha < \lambda} A_\alpha \in F$.*

Note that 'countably complete' is synonymous with 'ω_1-complete'.

Nothing we have to say apropos of measures over a set X has anything to do with any attribute of X other than its size, and in the presence of AC it is convenient to restrict our attention to the canonical representatives of sizes of sets, viz., cardinals.

(9.14) **Definition** [ZF] *A cardinal κ is* measurable $\overset{\text{def}}{\iff}$ *κ is uncountable and there exists a nonprincipal κ-complete ultrafilter over κ. In the context of a measurable cardinal κ, the term 'measure', unqualified, will be taken to mean nonprincipal κ-complete measure.*

Definition [ZF] *The* completeness *of a nonprincipal ultrafilter U over some set S $\overset{\text{def}}{=}$ the greatest cardinal κ such that U is κ-complete, i.e., the least cardinal κ such that there exist $X_\alpha \in U$ ($\alpha < \kappa$) such that $\bigcap_{\alpha<\kappa} X \notin U$. Note that $\kappa \leqslant |S|$. The corresponding term for measures is* additivity.

Note that the completeness of a principal ultrafilter U is undefined because U is κ-complete for all κ.

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[5] Recall that a *filter* on a boolean algebra $\mathfrak{M}$ is a nonempty set $F \subseteq |\mathfrak{M}|$ such that

1. $0 \notin F$;
2. $\forall a \in F \, \forall b \in |\mathfrak{M}| \, (b \geqslant a \implies b \in F)$;
3. $\forall a, b \in F \, (a \wedge b \in F)$.

An *ideal* is *dual* to a filter, i.e., it is a nonempty subset I of $|\mathfrak{M}|$ such that

1. $1 \notin I$;
2. $\forall a \in I \, \forall b \in |\mathfrak{M}| \, (b \leqslant a \implies b \in I)$;
3. $\forall a, b \in I \, (a \vee b \in I)$.

(9.15) **Theorem** [ZFC]

1. *Suppose U is a countably complete nonprincipal ultrafilter over a set S. Then the completeness of U is measurable.*
2. *Suppose κ is the least cardinal such that there is a countably complete nonprincipal ultrafilter over κ. Then every countably complete nonprincipal ultrafilter over κ is κ-complete. Hence, κ is measurable.*
3. *If a cardinal κ is measurable then there exists a normal*[3.175] *ultrafilter over κ.*

Proof 1 Essentially the same as the proof of (9.8). Suppose U is a countably complete nonprincipal ultrafilter over S. Let λ be the completeness of U. Note that λ is uncountable. Let $X_\alpha \subseteq \kappa$ $(\alpha < \lambda)$ be such that $\forall\alpha < \lambda\ X_\alpha \notin U$, but $\bigcup_{\alpha<\lambda} X_\alpha \in U$. Let $U' = \{Y \subseteq \lambda \mid \bigcup_{\alpha\in Y} X_\alpha \in U\}$. Then U' is a λ-complete nonprincipal ultrafilter over λ, so λ is measurable.[9.14]

2 This is a corollary of (9.15.1) more along the lines of (9.8). Suppose κ is the least cardinal such that there is a countably complete nonprincipal ultrafilter over κ, and suppose U is a countably complete nonprincipal ultrafilter over κ. Let λ be the completeness of U. Then λ is measurable,[9.15.1] so $\lambda \geqslant \kappa$. Hence U is κ-complete.

3 Suppose κ is uncountable and U is a κ-complete nonprincipal ultrafilter over κ. Let $\prec^U$ be the relation on ${}^\kappa\kappa$ given by

$$f \prec^U g \leftrightarrow \{\alpha \in \kappa \mid f(\alpha) < g(\alpha)\} \in U.$$

(9.16) **Claim** $\prec^U$ *is wellfounded.*

Proof Suppose toward a contradiction that $f_0 \succ^U f_1 \succ^U \cdots$ is a descending ω-sequence in $\prec^U$. (We are assuming AC, which gives us a descending ω-sequence $E_0, E_1, \ldots$ of $\prec^U$-equivalence classes if $\prec^U$ is nonwellfounded, and, once we have the equivalence classes, a member f_n of each class E_n.) Let $X_n = \{\alpha \in \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\}$. Then each $X_n \in U$. Let $X = \bigcap_{n\in\omega} X_n$. Since U is countably complete, $X \in U$. Suppose $\alpha \in X$. Then $f_0(\alpha) > f_1(\alpha) > \cdots$ is a descending ω-sequence of ordinals, which is impossible. $\square$[9.16]

Let f be $\prec^U$-minimal such that for all $\alpha \in \kappa$, $f \not\equiv^U \bar\alpha$, where $f \equiv^U g \overset{\text{def}}{\Longleftrightarrow} \{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\} \in U$, and $\bar\alpha \in {}^\kappa\kappa$ is the constant function with value α.[2.166] Define an ultrafilter U' by:

$$A \in U' \overset{\text{def}}{\Longleftrightarrow} f^{\leftharpoonup}A \in U \tag{9.17}$$

for $A \subseteq \kappa$. Observe that U' is nonprincipal and κ-complete.

(9.18) **Claim** U' *is closed under diagonal intersection.*[3.168]

Proof Suppose $\forall\alpha \in \kappa\ A_\alpha \in U'$ and suppose toward a contradiction that $\{\beta \in \kappa \mid \forall\alpha < \beta\ \beta \in A_\alpha\} \notin U'$. Then

$$\{\gamma \in \kappa \mid \exists\alpha < f(\gamma)\ f(\gamma) \notin A_\alpha\} = f^{\leftharpoonup}\{\beta \in \kappa \mid \exists\alpha < \beta\ \beta \notin A_\alpha\} \in U.$$

For $\gamma \in \kappa$, let $g(\gamma)$ be the least $\alpha < f(\gamma)$ such that $f(\gamma) \notin A_\alpha$ if there is such an α; 0 otherwise (which only happens for γ in a U-small set). Then $g \prec^U f$, so by the minimality assumption on f, there exists $\alpha \in \kappa$ such that $g \equiv^U \bar{\alpha}$, whence $\{\gamma \in \kappa \mid f(\gamma) \notin A_\alpha\} \in U$, and $A_\alpha \notin U'$, contrary to hypothesis. $\square$[9.18] $\square$[9.15]

The following theorem of Tarski and Ulam complements (9.11) and (9.13.1).

(9.19) **Theorem** [ZFC] *A measurable cardinal is (strongly) inaccessible.*

Proof Suppose U is a nonprincipal κ-complete ultrafilter over an uncountable cardinal κ. If κ is singular then for some $\lambda < \kappa$ there is a λ-sequence $\langle \alpha_\beta \mid \beta \in \lambda\rangle$ of ordinals $< \kappa$ such that $\kappa = \bigcup_{\beta\in\lambda} \alpha_\beta$. Since U is nonprincipal and κ-complete, $\alpha_\beta \notin U$ for all $\beta \in \lambda$, so, since U is κ-complete, $\bigcup_{\beta\in\lambda} \alpha_\beta \notin U$, a contradiction.

Now it only remains to show that κ is a strong limit cardinal, so suppose toward a contradiction that for some $\lambda < \kappa$, $2^\lambda \geqslant \kappa$. Let $h : \kappa \xrightarrow{\text{inj}} {}^\lambda 2$, and let

$$U' = \{A \subseteq {}^\lambda 2 \mid h^{\leftarrow} A \in U\}.$$

Then U' is a nonprincipal κ-complete ultrafilter on $\mathcal{P}({}^\lambda 2)$. Define a function $f \in {}^\lambda 2$ recursively so that for all $\alpha \leqslant \lambda$,

$$\{g \in {}^\lambda 2 \mid g \restriction \alpha = f \restriction \alpha\} \in U'. \tag{9.20}$$

(9.20) is trivially true for $\alpha = 0$. Suppose we have constructed $f \restriction \alpha$ so that (9.20) is true for α, i.e., $G = \{g \in {}^\lambda 2 \mid g \restriction \alpha = f \restriction \alpha\} \in U'$. Then there exists $i \in 2$ such that $\{g \in G \mid g(\alpha) = i\} \in U'$, and we continue the construction of f by letting $f(\alpha)$ be that i. If $\eta \leqslant \lambda$ is a limit ordinal and (9.20) holds for all $\alpha < \eta$, then (9.20) holds for $\alpha = \eta$ by virtue of the κ-completeness of U'. In particular, therefore, (9.20) holds for $\alpha = \lambda$. But $\{g \in {}^\lambda 2 \mid g \restriction \lambda = f \restriction \lambda\} = \{f\}$, so U' is principal, a contradiction. $\square$[9.19]

We will soon obtain much stronger "largeness" consequences of (2-valued) measurability, which make it clear that it is a much stronger assumption than real-valued measurability. As we will show, however, from the standpoint of relative consistency it is equivalent.[9.113]

9.2.3 Elementary embeddings of transitive classes

The following theorem reveals the source of the great strength of measurable cardinals. Recall the construction of the ultrapower of a structure that may be a proper class.[2.169]

(9.21) **Theorem** [GBC] *Suppose U is a countably complete ultrafilter over a cardinal κ, and $\mathfrak{A}$ is a structure whose similarity type contains a binary relation index R such that $R^{\mathfrak{A}}$ is wellfounded. Then $R^{{}^\kappa\mathfrak{A}/U}$ is wellfounded.*

Proof Suppose toward a contradiction that $R^{{}^\kappa\mathfrak{A}/U}$ is not wellfounded. As in the proof of (9.16), using AC, suppose $\langle a_n \mid n \in \omega\rangle$ is a descending sequence in $R^{{}^\kappa\mathfrak{A}/U}$ and $\langle f_n \mid n \in \omega\rangle$ is such that $\forall n \in \omega\ f_n \in a_n$; so for all $n \in \omega$, $\{\alpha \in \kappa \mid \langle f_{n+1}\alpha, f_n\alpha\rangle \in R^{\mathfrak{A}}\} \in U$. Since U is countably complete, $\{\alpha \in \kappa \mid \forall n \in \omega\ \langle f_{n+1}\alpha, f_n\alpha\rangle \in R^{\mathfrak{A}}\} \in U$, but $R^{\mathfrak{A}}$ is well founded, so this set must be empty. $\square$[9.21]

(9.22) **Definition** [GBC] *Suppose U is a countably complete ultrafilter over a cardinal κ. Let $\pi : {}^{\kappa}(V;\in)/U \cong (M;\in)$ be the transitive collapse.*[9.21] *The* canonical injection *is the map $j : V \to M$ given by*

$$j(x) = \pi([\bar{x}]^*),$$

where $\bar{x}$ is the function on κ with constant value x and $[\bar{x}]^*$ is its reduced equivalence class[2.167] *mod U.*

Since $x \mapsto \bar{x}$ is an elementary embedding,[2.173] and π is an isomorphism, j is an elementary embedding.

The following theorem provides important information about the canonical injection for a measurable cardinal.

(9.23) **Theorem** [GBC] *Suppose U is a κ-complete nonprincipal ultrafilter over an uncountable cardinal κ. Let $\pi : {}^{\kappa}V/U \to M$ be the transitive collapse, and let $j : V \to M$ be the canonical injection. Let i be the identity function on κ, and let $\eta = \pi[i]^*$.*

1. $\forall x \in V_\kappa\ jx = x$.
2. $\kappa \leqslant \eta < j\kappa$.
3. $U = \{X \subseteq \kappa \mid \eta \in jX\}$.
4. *For any $f : \kappa \to V$, $\pi[f]^* = (jf)(\eta)$.*
5. ${}^{\kappa}M \subseteq M$. *In particular, $V_{\kappa+1} \subseteq M$.*
6. *U is normal iff $\eta = \kappa$.*

Proof 1 Since κ is inaccessible,[9.19] for all $x \in V_\kappa$, $|x| < \kappa$. By virtue of the κ-completeness of U, if $x \in V_\kappa$, $f : \kappa \to V$, and $[f]^* \in [\bar{x}]^*$, i.e., $f(\alpha) \in x$ for almost every α,[6] then for some $y \in x$, $f(\alpha) = y$ for almost every α, i.e., $[f]^* = [\bar{y}]^*$. Hence $jx = \{jy \mid y \in x\}$, and (9.23.1) follows by $\in$-induction on x.

2 Suppose $\beta < \kappa$. Since U is κ-complete and nonprincipal, $\beta \notin U$, so $\kappa \backslash \beta \in U$. Hence, for almost every $\alpha \in \kappa$, $\bar{\beta}(\alpha) = \beta < i(\alpha) < \kappa = \bar{\kappa}(\alpha)$, so $\beta = j\beta = \pi[\bar{\beta}]^* < \pi[i]^* < \pi[\bar{\kappa}]^* = j\kappa$. It follows that $\kappa \leqslant \eta < j\kappa$.

3 Suppose $X \subseteq \kappa$. Then

$$\begin{aligned} \eta \in jX &\leftrightarrow [i]^* \in [\bar{X}]^* \leftrightarrow \{\alpha \in \kappa \mid i(\alpha) \in X\} \in U \\ &\leftrightarrow X \in U. \end{aligned}$$

4 Suppose $f, g : \kappa \to V$. Let $X = \{\alpha < \kappa \mid g(\alpha) \in f(\alpha)\}$. Then $jX = \{\alpha < j\kappa \mid (jg)(\alpha) \in (jf)(\alpha)\}$, and

$$\begin{aligned} \pi[g]^* \in \pi[f]^* &\leftrightarrow X \in U \leftrightarrow \eta \in jX \\ &\leftrightarrow (jg)(\eta) \in (jf)(\eta). \end{aligned}$$

[6] We will use the conventional expressions *large* and *small* to refer to subsets of κ that are respectively in and not in U, and *almost always*, *almost every*, and the like, to mean *for a large subset of κ*.

Recall that $M = \{\pi[f]^* \mid f \in {}^\kappa V\}$. Suppose $f \in {}^\kappa V$ and for all $g \in {}^\kappa V$, if $\pi[g]^* \in \pi[f]^*$ then $\pi[g]^* = (jg)(\eta)$. Then

$$\begin{aligned}\pi[f]^* &= \{\pi[g]^* \mid g \in {}^\kappa V \wedge \pi[g]^* \in \pi[f]^*\}\\ &= \{\pi[g]^* \mid g \in {}^\kappa V \wedge (jg)(\eta) \in (jf)(\eta)\} = \{x \in M \mid x \in (jf)(\eta)\}\\ &= (jf)(\eta).\end{aligned}$$

It follows by $\in$-induction that for all $f \in {}^\kappa V$, $\pi[f]^* = (jf)(\eta)$.

5 Let $k : \kappa \to \mathrm{Ord}$ be such that $\pi[k]^* = \kappa$. Suppose $f : \kappa \to M$. For each $\alpha \in \kappa$ let $h_\alpha : \kappa \to V$ be such that $\pi([h_\alpha]^*) = f(\alpha)$. Define $F : \kappa \to V$ so that for each $\beta \in \kappa$, $F(\beta) : k(\beta) \to V$ and for each $\alpha \in k(\beta)$,

$$F(\beta)(\alpha) = h_\alpha(\beta).$$

For all $\beta \in \kappa$, $\mathrm{dom}(F(\beta)) = k(\beta)$, so $\mathrm{dom}(\pi[F]^*) = \pi[k]^* = \kappa$. Suppose $\alpha < \kappa$. Then for almost all $\beta < \kappa$, $\alpha \in \mathrm{dom}(F(\beta))$ and $F(\beta)(\alpha) = h_\alpha(\beta)$, so $[F]^*(\alpha) = [h_\alpha]^*$, and $\pi[F]^*(\alpha) = \pi[h_\alpha]^* = f(\alpha)$; hence, $\pi[F]^* = f$, so $f \in M$.

6 As above, let $k : \kappa \to \kappa$ be such that $\pi[k]^* = \kappa$. Since $\kappa \leqslant \eta$, $[k]^* \leqslant [i]^*$. Suppose U is normal, and suppose toward a contradiction that $\kappa < \eta$. Let $X = \{\alpha < \kappa \mid k(\alpha) < \alpha\}$. Then $X \in U$. Let $g = (k \restriction X) \cup \{(\alpha, 0) \mid \alpha \in \kappa \backslash X\}$. Then $[g]^* = [k]^*$ and g is regressive, so for some $\beta < \kappa$, $\{\alpha < \kappa \mid g(\alpha) = \beta\} \in U$. Hence $[k]^* = [g]^* = [\bar{\beta}]^*$, and $\kappa = \pi[k]^* = \pi[g]^* = \pi[\bar{\beta}]^* = \beta$, a contradiction. Thus $\kappa = \eta$.

Conversely, suppose $\kappa = \eta$. Then $[k]^* = [i]^*$. Suppose $h : \kappa \to \kappa$ is a regressive function. Then $[h]^* < [i]^*$, so $[h]^* < [k]^*$, and $\pi[h]^* < \pi[k]^* = \kappa$. Let $\beta = \pi[h]^*$. Then $[h]^* = [\bar{\beta}]^*$, so $\{\alpha < \kappa \mid h(\alpha) = \beta\} \in U$. Thus, U is normal. □[9.23]

Definition [GB] *Suppose $j : N \to M$ is a Δ_0-elementary embedding of transitive classes. If j is not the identity on $\mathrm{Ord} \cap N$, the* critical point of $j \overset{\text{def}}{=} \mathrm{crit}\, j \overset{\text{def}}{=}$ *the first ordinal moved by j.*

(9.23.2) states that κ is the critical point of the embedding associated with a κ-complete measure over κ.

For most purposes we may restrict our attention to normal measures,[9.15.2] and the following theorem is specific to this case. Note that since it deals with an arbitrary s-formula ϕ and proper class structures, the existence of $\{\phi\}$-satisfaction relations must be explicitly assumed.

(9.24) **Theorem** [GBC] *Suppose U is a normal ultrafilter over a cardinal κ. Let $j : V \to M$ be the canonical injection.[9.22] Suppose ϕ is an s-formula with one free variable and the $\{\phi\}^V$-satisfaction relation exists. Then the $\{\phi\}^M$-satisfaction relation exists and, letting S and S' be these respective relations,*

$$M \models^{S'} \phi[\kappa] \leftrightarrow \{\alpha < \kappa \mid V \models^S \phi[\alpha]\} \in U.$$

Remark We could omit 'S'' above, and still have a meaningful and true statement. It would not, however, be proper to write '$\{\alpha < \kappa \mid V \models \phi[\alpha]\}$' in place of '$\{\alpha < \kappa \mid V \models^S \phi[\alpha]\}$', because the definition[1.61.1] of '$\models$' involves quantification over classes (partial satisfaction relations for $(V; \in)$), and the Comprehension axiom of GB is restricted to formulas with only set quantification, so the abstraction term is not well formed. We may nevertheless lapse into the former usage, relying on the reader to make the necessary adjustment.

Proof This is Łoś's theorem for ultrapowers of proper class structures[2.170] carried through from ${}^{\kappa}V/U$ to M via the transitive collapse isomorphism. $\square$[9.24]

We now turn our attention to elementary embeddings of transitive classes—in particular of V into a class M—without presupposing that the embedding comes from a countably complete ultrafilter. Recall[6.3] that we can prove in GB that $\{\phi\}$-satisfaction relations exist for proper classes for any particular ϕ, or for all formulas with any given bound on their complexity, say, for all Σ^{s}_1 formulas, for all Σ^{s}_2 formulas, etc. This can be combined with the absoluteness of Δ_0-formulas[6.6.1] for transitive classes to obtain the following theorem.

(9.25) **Theorem** [S] *Suppose ϕ is a Δ^{ZF}_1 s-formula and $\langle u_0, \ldots, u_{n^-} \rangle$ enumerates* $\operatorname{Free}\phi$. *Let $\hat{\phi}, \hat{u}_0, \ldots, \hat{u}_{n^-}$ be the corresponding canonical terms. Suppose M, j, v_0, $\ldots$, v_{n^-}, are distinct variables not in ϕ. Then* $\mathsf{GB} \vdash$ $\ulcorner$*Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Then*

$$(9.26)\qquad (\phi\bigl(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-} \end{smallmatrix}\bigr)) \leftrightarrow (\phi\bigl(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ j\bar{v}_0 & \cdots & j\bar{v}_{n^-} \end{smallmatrix}\bigr)) \leftrightarrow (M) \models (\hat{\phi})\bigl[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-}) \end{smallmatrix}\bigr].\urcorner$$

Proof For any ϕ, with no restriction on its complexity, we can reason as follows in GB:

$\ulcorner$Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Let (v_0), $\ldots$, $(v_{n^-}) \in V$ be given. Then

$$\begin{aligned} (\phi\bigl(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-} \end{smallmatrix}\bigr)) &\leftrightarrow V \models (\hat{\phi})\bigl[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (\bar{v}_0) & \cdots & (\bar{v}_{n^-}) \end{smallmatrix}\bigr] \\ &\leftrightarrow (M) \models (\hat{\phi})\bigl[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-}) \end{smallmatrix}\bigr].\urcorner, \end{aligned}$$

where the existence of a GB-proof of the first equivalence is an instance of (3.98) with V for C,[7] and the second equivalence follows from the elementarity of j and the fact that there is a GB-proof of the existence of $\{\phi\}$-satisfaction relations.

Now suppose ϕ is Δ^{s}_0. Then we can reason as follows in GB:

$\ulcorner$Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Let (v_0), $\ldots$, $(v_{n^-}) \in V$ be given. Then

$$\begin{aligned} (\phi\bigl(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-} \end{smallmatrix}\bigr)) &\leftrightarrow (M) \models (\hat{\phi})\bigl[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-}) \end{smallmatrix}\bigr] \\ &\leftrightarrow V \models (\hat{\phi})\bigl[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-}) \end{smallmatrix}\bigr] \\ &\leftrightarrow (\phi\bigl(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ j\bar{v}_0 & \cdots & j\bar{v}_{n^-} \end{smallmatrix}\bigr)).\urcorner, \end{aligned}$$

where the second equivalence is an instance of the absoluteness of Δ^{s}_0 formulas for transitive classes,[6.6.1] and the existence of a GB-proof of the last equivalence is again an instance of (3.98).

Now suppose ϕ is Δ^{ZF}_1. Let $\phi_0 = \exists u\, \psi_0$ and $\phi_1 = \forall u\, \psi_1$ be ZF-equivalent to ϕ, where ψ_0 and ψ_1 are Δ^{s}_0, and u is distinct from $u_0, \ldots, u_{n^-}$. Let $\hat{\phi}_0$, $\hat{\phi}_1$, $\hat{\psi}_0$, $\hat{\psi}_1$, and

---

[7] Remember that the incorporation of an instance of a metatheorem such as (3.98) into a GB-proof π does not consist of quoting the metatheorem, but rather of inserting into π a GB-proof whose existence (3.98) asserts.

$\hat{u}$ be the corresponding canonical terms. Let v be a variable distinct from all the others we've mentioned. Then we can reason as follows in GB:

$\ulcorner$Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Let $(v_0), \ldots, (v_{n^-}) \in V$ be given. Then

$$\begin{aligned}
\left(\phi_0\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-}\end{smallmatrix}\right)\right) &\leftrightarrow (M) \models (\hat{\phi}_0)\left[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-})\end{smallmatrix}\right] \\
&\leftrightarrow \exists (v) \in (M)\ (M) \models (\hat{\psi}_0)\left[\begin{smallmatrix} (\hat{u}) & (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (\bar{v}) & (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-})\end{smallmatrix}\right] \\
&\leftrightarrow \exists (v) \in (M)\ V \models (\hat{\psi}_0)\left[\begin{smallmatrix} (\hat{u}) & (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (\bar{v}) & (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-})\end{smallmatrix}\right] \\
&\to \exists (v) \in V\ V \models (\hat{\psi}_0)\left[\begin{smallmatrix} (\hat{u}) & (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (\bar{v}) & (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-})\end{smallmatrix}\right] \\
&\leftrightarrow V \models (\hat{\phi}_0)\left[\begin{smallmatrix} (\hat{u}_0) & \cdots & (\hat{u}_{n^-}) \\ (j\bar{v}_0) & \cdots & (j\bar{v}_{n^-})\end{smallmatrix}\right] \\
&\leftrightarrow \left(\phi_0\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ j\bar{v}_0 & \cdots & j\bar{v}_{n^-}\end{smallmatrix}\right)\right).\urcorner,
\end{aligned}$$

by cobbling together proofs of the previous two sorts for the bi-implications, and using the fact that $M \subseteq V$ for the implication.

Similarly, in GB we can prove:

$\ulcorner$Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Let $(v_0), \ldots, (v_{n^-}) \in V$ be given. Then

$$\left(\phi_1\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ j\bar{v}_0 & \cdots & j\bar{v}_{n^-}\end{smallmatrix}\right)\right) \to \left(\phi_1\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-}\end{smallmatrix}\right)\right).\urcorner$$

Since $\mathsf{GB} \vdash \phi_0 \leftrightarrow \phi_1 \leftrightarrow \phi$, $\mathsf{GB} \vdash$

$\ulcorner$Suppose (M) is transitive and $(j) : V \to (M)$ is elementary. Let $(v_0), \ldots, (v_{n^-}) \in V$ be given. Then

$$\left(\phi\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \bar{v}_0 & \cdots & \bar{v}_{n^-}\end{smallmatrix}\right)\right) \leftrightarrow \left(\phi\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ j\bar{v}_0 & \cdots & j\bar{v}_{n^-}\end{smallmatrix}\right)\right).\urcorner$$

This is the first equivalence of (9.26), and the second equivalence follows from the preceding discussion. $\square^{9.25}$

It may be helpful to review some of our previous complexity classifications: (4.16), (4.18), and (4.27). In addition,

(9.27) **Theorem** [S] *The following* s*-formulas are* Δ_1^{ZF}*:*

1. $\ulcorner$*S is the satisfaction relation for $\mathfrak{S}$*$\urcorner$.
2. $\ulcorner \mathfrak{S} \models \phi[A] \urcorner$.

Remark Note that the theorem refers to the ZF-equivalence of s-expressions. The variables $\ulcorner S \urcorner$ and $\ulcorner \mathfrak{S} \urcorner$ are therefore understood as referring to sets, and the existence of satisfaction relations is provable in ZF.

Proof Straightforward. $\square^{9.27}$

The following theorem lists some of the features of elementary embeddings of the universe into transitive classes.

(9.28) **Theorem** [GBC] *Suppose M is a transitive class and $j : V \to M$ is elementary. Then for all $x, y, x_0, \ldots, x_{n^-}, \langle x_n \mid n \in \omega \rangle \in V$ and $\alpha \in \mathrm{Ord}$,*

1. $x \in y \leftrightarrow jx \in jy$;

2. *$j : x \to jx$;*
3. *$j\{x, y\} = \{jx, jy\}$;*
4. *$j(x, y) = (jx, jy)$;*
5. *$\forall n \in \omega\ jn = n$;*
6. *$j\omega = \omega$;*
7. *$j\langle x_0, \ldots, x_{n\text{-}}\rangle = \langle jx_0, \ldots, jx_{n\text{-}}\rangle$;*
8. *$j\langle x_n \mid n \in \omega\rangle = \langle jx_n \mid n \in \omega\rangle$;*
9. *$j(V_\alpha) = V_{j\alpha} \cap M$;*
10. *$j \restriction x$ is an elementary embedding of x into jx (i.e., of $(x; \in)$ into $(jx; \in)$).*

Proof Straightforward. $\square$[9.28]

We make frequent use of these and other similar identities for elementary embeddings of transitive classes, often without explicit recognition.

9.2.4 Ultrafilters from embeddings

> The whole world having been into its ultrapower injected—
> The latter being founded well if all goes as expected—
> The sets whose images contain the point of criticality
> Return an ultrafilter with a dividend: normality!
>
> *Plus Ultra* by Robert A. Van Wesep

Suppose U is a κ-complete nonprincipal ultrafilter over an uncountable cardinal κ, and $j : V \to M$ is the canonical injection. Let $j' = j \restriction V_{\kappa+1}$, and let $M' = V_{j\kappa+1} \cap M$. Then j' is an elementary embedding of $V_{\kappa+1}$ in M'[9.28.10] with critical point κ,[9.23.1, 2] and

$$U = \{X \subseteq \kappa \mid \eta \in j'X\},$$

where η is as in (9.23). So U is recoverable from the "initial segment" j' of j. It is also straightforward to show that U generates j' directly, in that $j'x = \pi'[\bar{x}]$,[8] where $\pi' : {}^\kappa V_{\kappa+1}/U \to M'$ is the transitive collapse. We can to some extent avoid the consideration of proper class structures by working with j' instead of j.

(9.29) **Theorem** [GB] *Suppose $j : N \to M$ is an elementary embedding, where N, M are transitive classes, $\kappa = \operatorname{crit} j$, and $V_{\kappa+1} \subseteq N$.*

1. *κ is an uncountable cardinal.*
2. *$\{A \subseteq \kappa \mid \kappa \in jA\}$ is a normal ultrafilter over κ.*

Hence κ is a measurable cardinal.

Remark Note that if $N = V$ then trivially $V_{\kappa+1} \subseteq N$, so we have the simple statement: If $j : V \to M$ is a nontrivial elementary embedding of V into a transitive class M then its critical point is a measurable cardinal.

[8] Since $V_{\kappa+1}$ is a set, the U-equivalence classes in ${}^\kappa V_{\kappa+1}$ are sets, and we do not have to resort to reduced U-equivalence classes $[f]^*$, although we could.

Proof 1 Since $\kappa = \operatorname{crit} j$, $j\kappa > \kappa$. Clearly, κ is a limit ordinal. If $\kappa = \omega$ then $M \models \ulcorner [j\omega]$ is the least infinite ordinal, and $[\omega] < [j\omega]$, so $[\omega]$ has an immediate predecessor$\urcorner$, which is impossible. So $\kappa > \omega$.

If κ is not a cardinal, let $f : \lambda \overset{\text{sur}}{\to} \kappa$ for some $\lambda < \kappa$. Since $V_{\kappa+1} \subseteq N$, $f \in N$. Since $\kappa = \operatorname{crit} j$, $j\lambda = \lambda$, so $M \models \ulcorner [jf] : [\lambda] \overset{\text{sur}}{\to} [j\kappa]\urcorner$. Since $\kappa \in j\kappa$, there exists $\alpha \in \lambda$ such that

$$M \models \ulcorner [jf]([\alpha]) = [\kappa]\urcorner. \tag{9.30}$$

Let $\beta = f(\alpha) < \kappa$. Then $j\alpha = \alpha$, and $j\beta = \beta$, so $M \models \ulcorner [jf]([\alpha]) = [\beta]\urcorner$, contradicting (9.30). Hence, κ is a cardinal, and since $\kappa > \omega$, it is an uncountable cardinal.

2 Let $U = \{A \subseteq \kappa \mid \kappa \in jA\}$. U is easily seen to be an ultrafilter. Since, by hypothesis, for any $\alpha \in \kappa$, $j\alpha = \alpha$, and j is elementary, $j\{\alpha\} = \{\alpha\}$, so $\kappa \notin j\{\alpha\}$, and U is therefore not principal.

We now show that U is κ-complete. Suppose $\lambda < \kappa$ and for each $\alpha \in \lambda$, $A_\alpha \in U$. Let $A = \bigcap_{\alpha\in\lambda} A_\alpha$. Let $B = \{(\alpha, \beta) \mid \alpha \in \lambda \wedge \beta \in A_\alpha\}$. Note that

$$A = \{\beta \mid \forall\alpha \in \lambda\ (\alpha, \beta) \in B\}. \tag{9.31}$$

V_κ is closed under the formation of pairs and ordered pairs; hence, $B \subseteq V_\kappa$, so $B \in V_{\kappa+1} \subseteq N$. Since j is the identity on κ, j is the identity on B and $j\lambda = \lambda$.

jB consists of pairs (α, β) of ordinals such that $\alpha \in j\lambda = \lambda$. For each $\alpha \in \lambda$, $\forall\beta\,((\alpha, \beta) \in B \leftrightarrow \beta \in A_\alpha)$. By elementarity, since $j\alpha = \alpha$, $\forall\beta\,((\alpha, \beta) \in jB \leftrightarrow \beta \in jA_\alpha)$, i.e.,

$$jB = \{(\alpha, \beta) \mid \alpha \in \lambda \wedge \beta \in jA_\alpha\}.$$

Thus,[9.31]

$$\begin{aligned} jA &= \{\beta \mid \forall\alpha \in \lambda\ (\alpha, \beta) \in jB\} \\ &= \{\beta \mid \forall\alpha \in \lambda\ \beta \in jA_\alpha\}. \end{aligned}$$

By the definition of U, $\kappa \in jA_\alpha$ for every $\alpha \in \lambda$, so $\kappa \in jA$, so $A \in U$.

Finally, we show that U is normal. Suppose $f : \kappa \to \kappa$ is regressive. Then $jf : j\kappa \to j\kappa$ is regressive, and since $j\kappa > \kappa$, jf is defined at κ and $(jf)(\kappa) < \kappa$. Let $\alpha = (jf)(\kappa)$. Let $A = \{\beta \in \kappa \mid f(\beta) = \alpha\}$. Then $jA = \{\beta \in j\kappa \mid (jf)(\beta) = j\alpha\}$. Since $j\alpha = \alpha$, we find that $\kappa \in jA$, so $A \in U$. In other words, f is constant (with value α) on a large set. □[9.29]

Definition *Suppose $j : N \to M$ is an elementary embedding, where N, M are transitive classes, $\kappa = \operatorname{crit} j$, and $V_{\kappa+1} \subseteq N$. Then $\{A \subseteq \kappa \mid \kappa \in jA\}$ is the* canonical ultrafilter *over κ (for j).*

(9.32) **Theorem** [GBC] *Under the conditions of Theorem 9.29, suppose additionally that ${}^\kappa N \subseteq N$. Let $M' = \{(jf)(\kappa) \mid f \in {}^\kappa N\}$. Then $M' \cong {}^\kappa N/U$, where $U = \{A \subseteq \kappa \mid \kappa \in jA\}$.*

Proof Given $f, f' \in {}^\kappa N$,

$$\begin{aligned} [f]^* = [f']^* &\leftrightarrow \{\alpha < \kappa \mid f(\alpha) = f'(\alpha)\} \in U \\ &\leftrightarrow \kappa \in j\{\alpha < \kappa \mid f(\alpha) = f'(\alpha)\} \\ &\leftrightarrow \kappa \in \{\alpha < j\kappa \mid (jf)(\alpha) = (jf')(\alpha)\} \\ &\leftrightarrow (jf)(\kappa) = (jf')(\kappa). \end{aligned}$$

Likewise,

$$[f]^* \, E \, [f']^* \leftrightarrow (jf)(\kappa) \in (jf')(\kappa),$$

where E is the membership relation of ${}^\kappa N/U$. $\square^{9.32}$

(9.33) **Theorem** [GBC] *Suppose U is a nonprincipal κ-complete ultrafilter over an uncountable cardinal κ, and $j : V \to M$ is the canonical embedding. Then $U \notin M$.*

Proof If $U \in M$, then, since ${}^\kappa \mathrm{Ord} \subseteq {}^\kappa M \subseteq M$,[9.23.5] the prewellorder of ${}^\kappa \mathrm{Ord}$ defined by $f \preceq g \iff \{\alpha \in \kappa \mid f(\alpha) \leqslant g(\alpha)\} \in U$ is definable in M, so M contains the canonical map of ${}^\kappa\kappa$ onto $j\kappa$. On the other hand, since κ is measurable (in V) and j is elementary, $j\kappa$ is measurable in M. M therefore believes that $j\kappa$ is strongly inaccessible and is therefore not the surjective image of ${}^\alpha\alpha$ for any $\alpha < j\kappa$, in particular for $\alpha = \kappa$. ($|{}^\lambda\lambda| = 2^\lambda$ for any infinite cardinal λ.) $\square^{9.33}$

(9.33) immediately yields the following theorem of Scott.

(9.34) **Theorem** [S] $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L} \vdash$ ⌜*Measurable cardinals do not exist.*⌝.

Proof Reason as follows in $\mathsf{GB} + \boldsymbol{V} = \boldsymbol{L}$ (remember that AC follows from $\boldsymbol{V} = \boldsymbol{L}$):

⌜Suppose $V = L$ and there exists a measurable cardinal. Let κ be the least such. Let $j : V \to M$ be the canonical embedding. Since j is elementary, $M \models$ ⌜$[j\kappa]$ is the smallest measurable cardinal⌝. But since $V = L$ and L is the smallest model of ZF that contains all ordinals, $M = V$. So $j\kappa$ is the smallest measurable cardinal, which is a contradiction.⌝

Since GB is a conservative extension of ZF, we can also obtain the contradiction from $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}$. $\square^{9.34}$

Contrapositively, if κ is measurable then there exists a nonconstructible set. In particular, any nonprincipal κ-complete measure over κ is nonconstructible.

9.2.5 On the largeness of measurable cardinals

The characterization of measurability in terms of elementary embeddings given by Theorem 9.29 is an indication of the deep significance of measurability and is the starting point for the study of large large cardinal properties (as opposed to small large cardinal properties like inaccessibility).

In the setting of (9.24), any formula ϕ that holds at κ and is downward absolute at κ from V to M holds at almost every $\alpha < \kappa$. In this sense, κ is large. The following theorem states this for an important class of formulas.

(9.35) **Theorem** [GBC] *Suppose U is a normal ultrafilter over a cardinal κ, $R \subseteq V_\kappa$, ϕ is a Π_1^{s}-formula with two free variables, and $V \models \phi[V_{\kappa+1}, R]$. Then $\{\alpha < \kappa \mid V \models^S \phi[V_{\alpha+1}, R \cap V_\alpha]\} \in U$, where S is the $\{\phi\}^V$-satisfaction relation.*

Remark We do not need to posit the existence of the $\{\phi\}^V$-satisfaction relation as an hypothesis in (9.35), as we have done in (9.24), because we have specified that ϕ is Π_1, and it is a theorem of GB that for every Π_1 formula ψ, the $\{\psi\}^M$-satisfaction relation exists for any class M.[9]

[9]This is a straightforward exercise. The essential thing is to show that the Δ_0-satisfaction relation for structures $(M; \in)$ is uniformly definable, despite the unlimited quantifier depth of Δ_0 formulas.

Proof Let $j : V \to M$ be the canonical injection. $V_{\kappa+1}, R \in M$,[9.23.5] and j is the identity on V_κ,[9.23.1] so $R = j(R) \cap V_\kappa$. Π_1 formulas are absolute downward for transitive classes, so $M \models \phi[V_{\kappa+1}, j(R) \cap V_\kappa]$. Hence[9.24]

$$\{\alpha < \kappa \mid V \models^S \phi[V_{\alpha+1}, R \cap V_\alpha]\} \in U,$$

as claimed. □[9.35]

(9.36) **Theorem** [GBC] *Measurable cardinals are Mahlo, hyper-Mahlo, etc.*[9.4]

Proof Suppose κ is measurable, with a normal ultrafilter U and canonical embedding $j : V \to M$. Inaccessibility of an ordinal η is a Δ_0 property of $V_{\eta+1}$, inasmuch as η is an inaccessible cardinal iff

1. η is a limit ordinal $(\forall \alpha \in \eta \, \exists \beta \in \eta \; \alpha \in \beta)$;

2. there exists $\lambda \in \eta$ such that λ is a limit ordinal $(\eta > \omega)$;

3. for all functions $f \in V_{\eta+1}$, $\forall \lambda \in \eta \, \exists \alpha \in \eta \, \forall (\beta, \gamma) \in f \, (\beta \in \lambda \to \gamma \in \alpha)$ (no function maps λ cofinally into η, so η is a weakly inaccessible cardinal); and

4. for all functions $f \in V_{\eta+1}$, $\forall \lambda \in \eta \, \exists \alpha \in \eta \, \forall (x, \alpha) \in f \, \exists y \in x \; y \notin \lambda$ (no function maps $\mathcal{P}\lambda$ onto η, so η is a strong limit cardinal).

Since κ is inaccessible,[9.19] it follows from (9.35) that almost every cardinal less than κ is inaccessible. By Theorem 3.178 κ is therefore 1-Mahlo.

Suppose γ is inaccessible. Clearly, for any $\alpha \leqslant \beta < \gamma$, $V_\gamma \models \ulcorner [\beta]$ is $[\alpha]$-Mahlo$\urcorner$ iff β is α-Mahlo. Hence, γ is α-Mahlo iff γ is inaccessible and for every closed unbounded $C \subseteq \gamma$, for every $\alpha' < \alpha$, for some $\beta \in C$, $V_\gamma \models \ulcorner [\beta]$ is $[\alpha']$-Mahlo$\urcorner$. This is a Δ_0 statement about $V_{\gamma+1}$. It follows that if $\alpha \leqslant \kappa$ and κ is α-Mahlo then almost every $\beta < \kappa$ is α-Mahlo, so[3.178] κ is $(\alpha + 1)$-Mahlo. By induction, therefore, κ is κ-Mahlo, i.e., κ is hyper-Mahlo.

Etc. □[9.36]

9.2.6 Iterated ultrapowers

The ultrapower construction applied to transitive models of sufficient fragments of set theory lends itself naturally to iteration. Suppose M is a transitive model of ZFC^-, κ is a cardinal in M, and $U \in M$ is such that $M \models \ulcorner [U]$ is a countably complete ultrafilter over $[\kappa]\urcorner$. Let $\ulcorner {}^{[\kappa]}[M]/[U] \urcorner^M$ be the ultrapower of M mod U formed within M, whose individuals are equivalence classes of elements of ${}^\kappa M \cap M$. Then $\ulcorner {}^{[\kappa]}[M]/[U] \urcorner^M$ is wellfounded in the sense of M and is therefore wellfounded, since M is wellfounded. Let $\pi : \ulcorner {}^{[\kappa]}[M]/[U] \urcorner^M \to N$ be the transitive collapse. Since M models AC, the map $x \mapsto [\bar{x}]^*$, where $\bar{x}$ is the constant map on κ with value x, is an elementary embedding to M into $\ulcorner {}^{[\kappa]}[M]/[U] \urcorner^M$, and the map $j : M \to N$ given by

$$jx = \pi[\bar{x}]^*$$

is likewise elementary. The following definition formalizes (9.22) in the present more general setting and provides a name for N.

(9.37) **Definition** [GBC] *Given M and U as above,* $\mathrm{Ult}_U\, M \overset{\text{def}}{=}$ *the transitive collapse of the ultrapower* $\ulcorner {}^{[\kappa]}[M]/[U] \urcorner^M$ *of M mod U as defined in M. The map j*[9.37] *is the* canonical embedding *or* injection.

Suppose U is a κ-complete nonprincipal ultrafilter over κ. Let $j : V \to \mathrm{Ult}_U\, V$ be the canonical embedding.[9.37] Let $\kappa_1 = j\kappa$ and $U_1 = jU$. Then by elementarity, $M \models \ulcorner [U_1]$ is a $[\kappa_1]$-complete nonprincipal ultrafilter over $[\kappa_1]\urcorner$. It follows that within the structure $(M; \in)$ we may carry out the canonical embedding construction with U_1 and κ_1, and we may continue this process *ad infinitum*.

In the interest of uniformity of notation, let $M_0 = V$, $\kappa_0 = \kappa$, and $U_0 = U$, and define indexed families $[M_\alpha \mid \alpha < \omega]$ and $[j_\alpha \mid \alpha < \omega]$, together with the sequences $\langle U_\alpha \mid \alpha < \omega \rangle$ and $\langle \kappa_\alpha \mid \alpha < \omega \rangle$, such that for each $\alpha < \omega$

1. M_α is a transitive model of ZFC;
2. $M_\alpha \models \ulcorner [U_\alpha]$ is a normal ultrafilter over $[\kappa_\alpha]\urcorner$;
3. $j_\alpha : M_\alpha \to M_{\alpha+1} = \mathrm{Ult}_{U_\alpha}\, M_\alpha$ is the canonical injection;
4. $j_\alpha \kappa_\alpha = \kappa_{\alpha+1}$; and
5. $j_\alpha U_\alpha = U_{\alpha+1}$.

For each $\alpha \leqslant \beta < \omega$, let $i_{\alpha\beta} = j_{\beta\text{-}} \circ \cdots \circ j_{\alpha+1} \circ j_\alpha$.[10] Then $[i_{\alpha\beta} \mid \alpha \leqslant \beta < \omega]$ is a directed system of elementary embeddings, i.e., $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$, with $i_{\alpha\beta} : M_\alpha \to M_\beta$.

To show the existence of these ω- and $\omega\times\omega$-indexed families of proper classes, we cannot use a straightforward recursion argument, because it would rely on quantification over classes. Note, however, that if we let $M_0 = V_\eta$ for any ordinal $\eta \geqslant \kappa + \omega$ then $U \in V_\eta$, and we can carry out the above construction using only set quantification. Let M_α^η, j_α^η, etc. be the sets obtained in this way. Note that $\kappa_\alpha^\eta = \kappa_\alpha^{\eta'}$ and $U_\alpha^\eta = U_\alpha^{\eta'}$ for all $\eta, \eta' \geqslant \kappa + \omega$. Let $M_\alpha = \bigcup_{\eta \geqslant \kappa+\omega} M_\alpha^\eta$, $j_\alpha = \bigcup_{\eta \geqslant \kappa+\omega} j_\alpha^\eta$, etc. Then $[M_\alpha \mid \alpha < \omega]$, $[j_\alpha \mid \alpha < \omega]$, $[i_{\alpha\beta} \mid \alpha \leqslant \beta < \omega]$, $\langle \kappa_\alpha \mid \alpha < \omega \rangle$, and $\langle U_\alpha \mid \alpha < \omega \rangle$ have the requisite properties.

Satisfaction in $(M_\alpha; \in)$ is to be understood in the usual way. We will be concerned only with satisfaction for specific formulas, and we may suppose that we have proved the existence of partial satisfaction relations for these. At no time will we have to consider satisfaction for arbitrary formulas. The entire argument could be formulated in ZFC by replacing references to defined classes by their definitions.

The natural extension of the construction to ω and beyond is by means of the direct limit construction.[2.156] The critical issue is the preservation of wellfoundedness at limits. To address this question without prejudice, suppose $\mathfrak{M} = (M; E, k, U)$ is such that $\mathfrak{M} \models \mathsf{ZFC}+ \ulcorner [U]$ is a $[k]$-complete nonprincipal ultrafilter over the cardinal $[k]\urcorner$. We define structures $\mathrm{Ult}_U^\alpha(\mathfrak{M}) = (M^\alpha; E^\alpha, k^\alpha, U^\alpha)$ ($\alpha \in \mathrm{Ord}$) such that $\mathrm{Ult}_U^\alpha(\mathfrak{M}) \models \mathsf{ZFC}+ \ulcorner [U_\alpha]$ is a $[k^\alpha]$-complete nonprincipal ultrafilter over $[k^\alpha]\urcorner$, and a directed system of elementary embeddings $i_{\alpha,\beta} : \mathrm{Ult}_U^\alpha(\mathfrak{M}) \to \mathrm{Ult}_U^\beta(\mathfrak{M})$ ($\alpha \leqslant \beta \in \mathrm{Ord}$), as follows.

1. $\mathrm{Ult}_U^0(\mathfrak{M}) = \mathfrak{M}$. Thus, $M^0 = M$, $E^0 = E$, $k^0 = k$, and $U^0 = U$.

[10] $i_{\alpha(\alpha+1)} = j_\alpha$ and $i_{\alpha\alpha}$ is the identity on M_α.

2. $\mathrm{Ult}_U^{\alpha+1}(\mathfrak{M}) = \mathrm{Ult}_{U^\alpha}\big(\mathrm{Ult}_U^\alpha(\mathfrak{M})\big) \overset{\text{def}}{=} \ulcorner \mathrm{Ult}_{[U^\alpha]} \urcorner^{\mathfrak{M}}$; and $i_{\alpha,\alpha+1} : \mathrm{Ult}_U^\alpha(\mathfrak{M}) \overset{\text{inj}}{\to} \mathrm{Ult}_U^{\alpha+1}(\mathfrak{M})$ is the elementary embedding $x \mapsto [\bar{x}]^*$. Note that $\mathrm{Ult}_{U^\alpha}\big(\mathrm{Ult}_U^\alpha(\mathfrak{M})\big)$ is the ultrapower of the universe by U^α as understood by $\mathrm{Ult}_U^\alpha(\mathfrak{M})$ $\big(= (M^\alpha; E^\alpha, k^\alpha, U^\alpha)\big)$. In particular, its elements are—in effect—reduced equivalence classes mod U^α of functions from k^α to M^α that are in M^α.

3. If $\mathrm{Lim}\,\eta$ then $\mathrm{Ult}_U^\eta(\mathfrak{M})$, together with $[i_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \eta]$, is the direct limit of $\big[[\mathrm{Ult}_U^\alpha(\mathfrak{M}) \mid \alpha < \eta], [i_{\alpha,\beta} \mid \alpha \leqslant \beta < \eta]\big]$.

If $\mathfrak{M} = (V; \in, \kappa, U)$, we let $\mathrm{Ult}_U^\alpha = \mathrm{Ult}_U^\alpha(\mathfrak{M})$.

(9.38) **Theorem** [GBC] *Given $\alpha \in \mathrm{Ord}$ and $b \in \mathrm{Ord}^{\mathrm{Ult}_U^\alpha}$, the above construction in the sense of Ult_U^α yields a structure $\ulcorner \mathrm{Ult}_{[U^\alpha]}^{[b]} \urcorner^{\mathrm{Ult}_U^\alpha}$. If Ult_U^α is wellfounded (i.e., E^α is wellfounded) then b corresponds to an ordinal β, and it is reasonable to expect that $\ulcorner \mathrm{Ult}_{[U^\alpha]}^{[b]} \urcorner^{\mathrm{Ult}_U^\alpha} \cong \mathrm{Ult}_{U^\alpha}^\beta(\mathrm{Ult}_U^\alpha) \cong \mathrm{Ult}_U^{\alpha+\beta}$. This is indeed the case.*

Proof Fairly straightforward by induction on β. $\square^{9.38}$

(9.39) **Theorem (Gaifman)** [GBC] *Suppose U is a κ-complete nonprincipal ultrafilter over κ. Ult_U^α is wellfounded for all $\alpha \in \mathrm{Ord}$.*

Proof Suppose not. Let α be least such that Ult_U^α is not wellfounded. α is necessarily a limit ordinal. Let β be least such that the ordinals preceding $i_{0,\alpha}\beta$ in Ult_U^α are not wellordered. Let $b = i_{0,\alpha}\beta$, and let $c \in |\mathrm{Ult}_U^\alpha|$ be such that $c <^\alpha b$ and the ordinals preceding c in $\mathrm{Ord}^{\mathrm{Ult}_U^\alpha}$ are not wellordered. Since α is limit, there exists $\alpha' < \alpha$ and $c' \in \mathrm{Ult}_U^{\alpha'}$ such that $c = i_{\alpha',\alpha}c'$. Note that $c' <^{\alpha'} i_{0,\alpha'}\beta$. Let α'' be such that $\alpha = \alpha' + \alpha''$. $\mathrm{Ult}_U^{\alpha'}$ is wellfounded by hypothesis. Let a'' be the α''th ordinal in $\mathrm{Ult}_U^{\alpha'}$. By virtue of (9.38), $\mathrm{Ult}_U^{\alpha'} \models \ulcorner$the predecessors of $i_{0,[a'']}[c']$ in $\mathrm{Ult}_{[U^{\alpha'}]}^{[a'']}$ are not wellordered$\urcorner$. Since $\alpha'' \leqslant \alpha$, $a'' \leqslant^{\alpha'} i_{0,\alpha'}\alpha$. As noted, $c' <^{\alpha'} i_{0,\alpha'}\beta$. Hence $\mathrm{Ult}_U^{\alpha'} \models \ulcorner$there exist $\eta \leqslant [i_{0,\alpha'}\alpha]$ and $\zeta < [i_{0,\alpha'}\beta]$ such that the predecessors of $i_{0,\eta}\zeta$ in $\mathrm{Ult}_{[U^{\alpha'}]}^\eta$ are not wellordered$\urcorner$. Since $i_{0,\alpha'}$ is elementary, $(V; \in) \models \ulcorner$there exist $\eta \leqslant [\alpha]$ and $\zeta < [\beta]$ such that the predecessors of $i_{0,\eta}\zeta$ in $\mathrm{Ult}_{[U]}^\eta$ are not wellordered$\urcorner$, contrary to the minimality of α and β. $\square^{9.39}$

Suppose U is a κ-complete nonprincipal ultrafilter over an uncountable cardinal κ. In light of (9.39) we now replace Ult_U^α by its transitive collapse, to which we give the same name. $i_{0,\alpha} : V \to \mathrm{Ult}_U^\alpha$ is therefore an elementary embedding of V into a transitive class. It is possible to define each Ult_U^α and $i_{0,\alpha}$ directly in terms of an ultrafilter U_α over a subalgebra of $\mathcal{P}^\alpha\kappa$. The ultrafilters U_α form a coherent system that reflects the structure of $[i_{\alpha,\beta} \mid \alpha \leqslant \beta \in \mathrm{Ord}]$. Generalizations of this line of development are important in the definition of inner models of large cardinals, but are largely beyond the scope of this book. We will limit our attention at present to the important case of finite iterations.

The following notation will be convenient.

Definition *Suppose U is an ultrafilter over a set S and ϕ is a formula. Then '$\mathsf{Q}^U x\ \phi$' abbreviates '$\{x \in S \mid \phi\} \in U$'.*

'Q^U' is a quantifier in the general sense. Like all non-null quantifiers, it is somewhere between '$\forall$' and '$\exists$'; in a sense it is halfway between, inasmuch as $\neg\,\mathsf{Q}^U x\ \phi \leftrightarrow \mathsf{Q}^U x\ \neg\phi$.

(9.40) **Definition** [ZFC] *Suppose U is a κ-complete nonprincipal ultrafilter over κ and $n \in \omega$. Let $U_n = \{X \subseteq {}^n\kappa \mid \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X\}$.*

(9.41) **Theorem** [ZFC] *Suppose U is a κ-complete nonprincipal ultrafilter over κ and $n \in \omega$. Then U_n[9.40] is a κ-complete nonprincipal ultrafilter over ${}^n\kappa$.*

Proof To show that U_n is an ultrafilter we note that for any $X \subseteq {}^n\kappa$

$$\begin{aligned} X \notin U_n &\leftrightarrow \neg\, \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X \\ &\leftrightarrow \mathsf{Q}^U \xi_0\ \neg\, \mathsf{Q}^U \xi_1 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X \\ &\ \ \vdots \\ &\leftrightarrow \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \notin X \\ &\leftrightarrow \kappa \backslash X \in U_n. \end{aligned}$$

It is trivial that U_n is nonprincipal. To show that U_n is κ-complete, suppose $\lambda < \kappa$ and $\forall \alpha < \lambda\ X_\alpha \in U_n$. Since U is κ-complete,

$$\begin{aligned} &\forall \alpha < \lambda\, \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X_\alpha \\ &\quad \to \mathsf{Q}^U \xi_0\, \forall \alpha < \lambda\, \mathsf{Q}^U \xi_1 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X_\alpha \\ &\quad\ \ \vdots \\ &\quad \to \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\, \forall \alpha < \lambda\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X_\alpha. \end{aligned}$$

Hence, $\bigcap_{\alpha<\lambda} X_\alpha \in U_n$. $\square$[9.41]

(9.42) **Definition** [ZFC] *Let ${}^{n\uparrow}\kappa$ be the set of strictly increasing elements of ${}^n\kappa$. Note that ${}^{n\uparrow}\kappa \in U_n$, so $U_n \cap \mathcal{P}({}^{n\uparrow}\kappa)$ is an ultrafilter over ${}^{n\uparrow}\kappa$ which is essentially equivalent to U_n. Let $[\kappa]^n$ be the set of n-element subsets of κ. The function* im, *i.e., $f \mapsto \operatorname{im} f$, is a natural bijection between ${}^{n\uparrow}\kappa$ and $[\kappa]^n$, and $\{\operatorname{im}^{\to} X \mid X \in U_n \cap \mathcal{P}({}^{n\uparrow}\kappa)\}$ is the corresponding ultrafilter over $[\kappa]^n$. We will refer to all of these ultrafilters as U_n.*

(9.43) **Theorem** [GBC] *Suppose U is normal. Let Ult_U^α, $i_{\alpha\beta} : \mathrm{Ult}_U^\alpha \to \mathrm{Ult}_U^\beta$, κ^α, and U^α be as above. Suppose $n \in \omega$.*

1. *Let*[9.40] *$U_n = \{X \subseteq {}^n\kappa \mid \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \langle \xi_0, \ldots, \xi_{n^-} \rangle \in X\}$. Then*

$$U_n = \{X \subseteq {}^n\kappa \mid \langle \kappa^0, \ldots, \kappa^{n^-} \rangle \in i_{0n} X\}.$$

2. *Equivalently (*per *the preceding discussion), letting $U_n = \{X \subseteq [\kappa]^n \mid \mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\ \{\xi_0, \ldots, \xi_{n^-}\} \in X\}$,*

$$U_n = \{X \subseteq [\kappa]^n \mid \{\kappa^0, \ldots, \kappa^{n^-}\} \in i_{0n} X\}.$$

Proof Since $U = U^0$ is a normal ultrafilter over $\kappa = \kappa^0$ in $(V; \in) = \mathrm{Ult}_U^0$, for each $m \in \omega$, U^m is a normal ultrafilter over κ^m in Ult_U^m. Suppose $m < n$ and let Y be the set of $\xi_m \in \kappa^m$ such that

$$\mathsf{Q}^{U^m} \xi_{m+1} \cdots \mathsf{Q}^{U^m} \xi_{n^-}\ \langle \kappa^0, \ldots, \kappa^{m^-}, \xi_m, \xi_{m+1}, \ldots, \xi_{n^-} \rangle \in i_{0m} X.$$

Then

$$\begin{aligned}
&\mathsf{Q}^{U^m}\xi_m\,\mathsf{Q}^{U^m}\xi_{m+1}\cdots\mathsf{Q}^{U^m}\xi_{n-}\ \langle\kappa^0,\ldots,\kappa^{m-},\xi_m,\xi_{m+1},\ldots,\xi_{n-}\rangle\in i_{0m}X\\
&\qquad\leftrightarrow\mathsf{Q}^{U^m}\xi_m\,(\xi_m\in Y)\leftrightarrow Y\in U^m\leftrightarrow\kappa^m\in i_{m,m+1}Y\\
&\quad\leftrightarrow\mathsf{Q}^{U^{m+1}}\xi_{m+1}\cdots\mathsf{Q}^{U^{m+1}}\xi_{n-}\ \langle\kappa^0,\ldots,\kappa^{m-},\kappa^m,\xi_{m+1},\ldots,\xi_{n-}\rangle\in i_{0(m+1)}X.
\end{aligned}$$

Applying this repeatedly, for any $X\subseteq{}^n\kappa$

$$\begin{aligned}
X\in U_n&\leftrightarrow\mathsf{Q}^{U^0}\xi_0\cdots\mathsf{Q}^{U^0}\xi_{n-}\ \langle\xi_0,\ldots,\xi_{n-}\rangle\in X\\
&\leftrightarrow\mathsf{Q}^{U^1}\xi_1\cdots\mathsf{Q}^{U^1}\xi_{n-}\ \langle\kappa^0,\xi_1,\ldots,\xi_{n-}\rangle\in i_{01}X\\
&\quad\vdots\\
&\leftrightarrow\mathsf{Q}^{U^{n-}}\xi_{n-}\ \langle\kappa^0,\ldots,\kappa^{n-2},\xi_{n-}\rangle\in i_{0n-}X\\
&\leftrightarrow\langle\kappa^0,\ldots,\kappa^{n-}\rangle\in i_{0n}X.
\end{aligned}$$

□[9.43]

The following theorem gives another characterization of U_n.

(9.44) **Theorem** [ZFC] *Suppose U is a normal ultrafilter over κ, $n\in\omega$, and $X\subseteq[\kappa]^n$. Then $X\in U_n\leftrightarrow\exists Y\in U\ [Y]^n\subseteq X$.*

Proof As above, let $i_{0m}:V\to\mathrm{Ult}_U^m(V)$ be the canonical elementary embedding of V into its mth U-ultrapower. Let $\kappa_m=i_{0m}\kappa$. (Thus, $\kappa_0=\kappa$.)

$\leftarrow$ Suppose $Y\in U$. Then for each $m<m'<\omega$, $\kappa_m\in i_{0m'}Y$; in particular, $\kappa_0,\ldots,\kappa_{n-}\in i_{0n}Y$. Suppose $[Y]^n\subseteq X$. By elementarity, $[i_{0n}Y]^n\subseteq i_{0n}X$; in particular, $\{\kappa_0,\ldots,\kappa_{n-}\}\subseteq i_{0n}X$. Hence,[9.43.2] $X\in U_n$.

$\rightarrow$ Suppose $X\in U_n$. By definition,

$$\mathsf{Q}^U\xi_0\ \cdots\ \mathsf{Q}^U\xi_{n-}\ \{\xi_0,\ldots,\xi_{n-}\}\in X.$$

Let

$$A_0=\{\xi_0\in\kappa\mid\mathsf{Q}^U\xi_1\ \cdots\ \mathsf{Q}^U\xi_{n-}\ \{\xi_0,\ldots,\xi_{n-}\}\in X\}.$$

By the definition of 'Q^U', $A_0\in U$. For each $\xi_0\in\kappa$, let

$$B_{\xi_0}=\begin{cases}\{\xi_1\in A_0\mid\mathsf{Q}^U\xi_2\ \cdots\ \mathsf{Q}^U\xi_{n-}\ \{\xi_0,\xi_1,\ldots,\xi_{n-}\}\in X\} & \text{if } \xi_0\in A_0\\ A_0 & \text{otherwise.}\end{cases}$$

Then $B_{\xi_0}\in U$ and $B_{\xi_0}\subseteq A_0$. Let

$$A_1=\Delta_{\xi_0<\kappa}B_{\xi_0}.$$

Since U is normal, $A_1\in U$. For any $\xi_0,\xi_1\in A_1$, if $\xi_0<\xi_1$ then

$$\mathsf{Q}^U\xi_2\ \cdots\ \mathsf{Q}^U\xi_{n-}\ \{\xi_0,\ldots,\xi_{n-}\}\in X.$$

Continuing, we obtain $A_0\supseteq A_1\supseteq\cdots\supseteq A_{n-}$ such that for each $m<n$, $A_m\in U$ and for all $\xi_0,\ldots,\xi_m\in A_m$, if $\xi_0<\cdots<\xi_m$ then

$$\mathsf{Q}^U\xi_{m+1}\ \cdots\ \mathsf{Q}^U\xi_{n-}\ \{\xi_0,\ldots,\xi_{n-}\}\in X.$$

In particular, if $m = n-1$ then $\{\xi_0, \ldots, \xi_{n^-}\} \in X$. In other words,

$$[A_{n^-}]^n \subseteq X.$$

$\square^{9.44}$

9.2.7 Compactness

We have seen how the examination of the existence of measures leads to a large cardinal property, viz., measurability; and we have seen how measurable cardinals may be characterized metamathematically, in terms of elementary embeddings of transitive models. Large cardinals also arise naturally in another metamathematical context.

Definition [ZFC] *Suppose κ, λ are infinite cardinals and ρ is a signature. $\mathcal{L}^{\rho}_{\kappa\lambda}$ $\overset{\text{def}}{=}$ the language in the signature ρ whose expressions are generated in the usual way, with the following modifications.*

1. *The size of the set $\mathcal{V}$ of variables is* $\max\{\kappa, \lambda\}$.
2. *The terms are formed in the usual way, as are atomic formulas, but it is permitted to form disjunctions and conjunctions of potentially infinite sets of formulas, as long as the size of such a set is $< \kappa$ and the size of the set of free variables in the resulting expression is $< \lambda$. Thus, we permit operations that may be represented as $\bigvee_{\alpha<\mu}$ and $\bigwedge_{\alpha<\mu}$, where $\mu < \kappa$.*
3. *It is permitted to quantify existentially or universally over infinite sets of variables of size $< \lambda$. These operations may be represented as $\exists\langle u_\alpha \mid \alpha < \mu\rangle$ and $\forall\langle u_\alpha \mid \alpha < \mu\rangle$, where $\mu < \lambda$ and u_α $(\alpha < \mu)$ are variables. These may be regarded as potentially infinitary quantifier prefixes $\exists u_0\, \exists u_1 \cdots \exists u_\alpha \cdots$ and $\forall u_0\, \forall u_1 \cdots \forall u_\alpha \cdots$.*

Ordinary languages are of the form $\mathcal{L}^{\rho}_{\omega\omega}$. The interpretation of $\mathcal{L}^{\rho}_{\kappa\lambda}$-expressions is done in the natural way (in ρ-structures of the usual type).

The study of these languages was initiated by Tarski and pursued by Tarski and his students William Hanf and Jerome Keisler, among others. The issue of compactness of these languages was found to be particularly interesting. We say that a set Σ of $\mathcal{L}^{\rho}_{\kappa\lambda}$-sentences is ν-satisfiable $\overset{\text{def}}{\iff}$ every subset of Σ of size $< \nu$ is satisfiable. The compactness question is whether ν-satisfiability implies satisfiability.

Definition [ZFC] *Suppose κ is an uncountable cardinal.*

1. *κ is* weakly compact $\overset{\text{def}}{\iff}$ *for any signature ρ of size $\leqslant \kappa$ and any set Σ of $\mathcal{L}^{\rho}_{\kappa\kappa}$ sentences, if Σ is κ-satisfiable then Σ is satisfiable.*
2. *κ is* strongly compact $\overset{\text{def}}{\iff}$ *for any signature ρ (with no restriction on its size) and any set Σ of $\mathcal{L}^{\rho}_{\kappa\kappa}$ sentences, if Σ is κ-satisfiable then Σ is satisfiable.*

Note that if we eliminate the requirement that κ be uncountable then ω is strongly (and therefore also weakly) compact; so weak and strong compactness assert for an uncountable cardinal a property enjoyed by ω. Just as the compactness property of ω may be viewed as relating to its great size compared to smaller (i.e., finite) cardinals, compactness of an uncountable cardinal also implies it is quite large compared to smaller cardinals.

(9.45) **Theorem** [ZFC] *Suppose κ is a weakly compact cardinal. Then κ is inaccessible.*

Proof To show that κ is regular, suppose toward a contradiction that $X \subseteq \kappa$, $|X| < \kappa$, and X is unbounded in κ. Let the signature ρ have κ distinct constants c_α $(\alpha < \kappa)$ and an additional constant c. Let

$$\Sigma = \{c \neq c_\alpha \mid \alpha < \kappa\} \cup \Big\{ \bigvee_{\beta \in X} \bigvee_{\alpha < \beta} c = c_\alpha \Big\}.$$

Σ is clearly κ-satisfiable but not satisfiable.

To show that κ is a strong limit cardinal, suppose toward a contradiction that $\lambda < \kappa$ and $2^\lambda \geqslant \kappa$. Let ρ have distinct constants c_α $(\alpha < \lambda)$ and d_i $(i \in 2)$. Let

$$\Sigma = \{d_0 \neq d_1\} \cup \{c_\alpha = d_0 \vee c_\alpha = d_1 \mid \alpha < \lambda\} \cup \Big\{ \bigwedge_{\alpha < \lambda} c_\alpha \neq d_{f(\alpha)} \,\Big|\, f \in {}^\lambda 2 \Big\}.$$

If $T \subseteq \Sigma$ and $|T| < \kappa$, let $f \in {}^\lambda 2$ be such that $\bigwedge_{\alpha<\lambda} c_\alpha \neq d_{f(\alpha)} \notin T$. Then we can satisfy T in a structure $\mathfrak{A}$ by letting $c_\alpha^{\mathfrak{A}} = \big(d_{f(\alpha)}\big)^{\mathfrak{A}}$. On the other hand, Σ is clearly not satisfiable. $\square^{9.45}$

(9.46) *Note that the expressions used in the preceding proof have no quantifiers, so they are actually in $\mathcal{L}_{\kappa\omega}$, even in the simpler language $\mathcal{L}_\kappa$, which has disjunctions and conjunctions of length $< \kappa$, but no quantification.*

It can be shown that there are many inaccessible cardinals below any weakly compact cardinal, so weak compactness is strictly stronger than inaccessibility. On the other hand, we have the following theorem of Erdös and Tarski comparing weak compactness with measurability.

(9.47) **Theorem** [ZFC] *If κ is measurable then κ is weakly compact.*

Proof Suppose $|\rho| \leqslant \kappa$ and Σ is a κ-satisfiable set of $\mathcal{L}^\rho_{\kappa\kappa}$-sentences. Clearly, $|\Sigma| \leqslant \kappa^{<\kappa}$, so $|\Sigma| \leqslant \kappa$, since κ is inaccessible. Let $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ enumerate Σ. For each $\beta < \kappa$ let $\mathfrak{A}$ be a ρ-structure such that $\mathfrak{A} \models \bigwedge_{\alpha<\beta} \sigma_\alpha$. Let U be a nonprincipal κ-complete ultrafilter over κ, and let $\mathfrak{A} = \prod_{\beta<\kappa} \mathfrak{A}_\beta / U$. Then for any $\alpha < \kappa$

$$\{\beta < \kappa \mid \mathfrak{A}_\beta \models \sigma_\alpha\} \supseteq (\alpha, \kappa) \in U,$$

so $\mathfrak{A} \models \sigma_\alpha$. Hence $\mathfrak{A}$ satisfies Σ. $\square^{9.47}$

Thus, weak compactness is weaker than or equivalent to measurability. In fact, it is strictly weaker.

(9.48) **Theorem** [ZFC] *Suppose U is a normal ultrafilter on κ. Then the set of weakly compact cardinals $< \kappa$ is in U.*

Proof This follows from (9.35) and the fact that weak compactness of κ is a Δ_0 property of $V_{\kappa+1}$. $\square^{9.48}$

Turning now to strong compactness, we have the following characterization in terms of ultrafilters, due to Keisler and Tarski. We begin with a definition.

(9.49) **Definition** [ZFC] *Suppose κ is a cardinal. $\mathcal{P}_\kappa X \overset{\text{def}}{=} \{Y \subseteq X \mid |Y| < \kappa\}$.*

(9.50) **Theorem** [ZFC] *An uncountable cardinal κ is strongly compact iff for any set Z, every κ-complete filter over Z can be extended to a κ-complete ultrafilter over Z.*

Proof $\rightarrow$ Suppose κ is strongly compact and F is a κ-complete filter over a set Z. Since the theorem depends only on the size of Z we may assume without loss of generality that $Z \cap \mathcal{P} Z = 0$. Let $P = \mathcal{P} Z$. Let ρ be a signature consisting of two unary relation symbols I and S, a binary relation symbol E and a distinct constant symbol $\dot{X}$ for each $X \subseteq Z$. Let $\mathfrak{A}$ be the ρ-structure with

1. $|\mathfrak{A}| = Z \cup P$;
2. $I^{\mathfrak{A}} = Z$;
3. $S^{\mathfrak{A}} = P$;
4. $E^{\mathfrak{A}} = \{\langle z, X\rangle \mid X \in P \wedge z \in X\}$; and
5. $\dot{X}^{\mathfrak{A}} = X$ for each $X \in P$.

Let ρ' be the extension of ρ by the addition of a new constant symbol c. Let T be the $\mathcal{L}_{\kappa\kappa}$-theory of $\mathfrak{A}$, and let

$$\Sigma = T \cup \{\ulcorner (c)\, E\, (\dot{X})\urcorner \mid X \in F\}.$$

Suppose $\Sigma' \subseteq \Sigma$ and $|\Sigma'| < \kappa$. Since F is κ-complete, Σ' is satisfied in a ρ'-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ obtained by letting $c^{\mathfrak{A}'} = z$, where $z \in \bigcap\{X \in F \mid \ulcorner (c)\, E\, (\dot{X})\urcorner \in \Sigma'\}$. Thus, Σ is κ-satisfiable.

In general, if $\mathfrak{A}' \models T$ then $|\mathfrak{A}'|$ is the union of a set $Z' = I^{\mathfrak{A}'}$ of "individuals" and a disjoint set $P' = S^{\mathfrak{A}'}$ of "sets", with the "membership relation" $E' = E^{\mathfrak{A}'}$. Replacing $\mathfrak{A}'$ by a suitable isomorph we may arrange that $P' \subseteq \mathcal{P} Z'$ and E' is the membership relation *per se*. If $\dot{X}^{\mathfrak{A}} = X \in P$ is a singleton $\{z\}$, then $\dot{X}^{\mathfrak{A}'}$ is necessarily a singleton $\{z'\}$ for some $z' \in Z'$. By replacing $\mathfrak{A}'$ again by a suitable isomorph, we may arrange that $z' = z$ in each such case, so that $Z' \supseteq Z$ and $P' \subseteq \mathcal{P} Z'$. Note that $\dot{X}^{\mathfrak{A}'} \cap Z = X$ for each $X \in P$.

(9.51) *Additional structural properties of $\mathfrak{A}$ also carry over to $\mathfrak{A}'$:*

1. *Suppose $\{X, Y\}$ is a partition of Z, i.e., $X \cup Y = Z$ and $X \cap Y = 0$. Then $\{\dot{X}^{\mathfrak{A}'}, \dot{Y}^{\mathfrak{A}'}\}$ is likewise a partition of Z'.*
2. *Suppose $\beta < \kappa$ and $\langle X_\alpha \mid \alpha < \beta\rangle \subseteq P$. Let $X = \bigcap_{\alpha<\beta} X_\alpha$.*
   1. *Then*
   $$\forall z' \in Z' \left(z' \in \dot{X}^{\mathfrak{A}'} \leftrightarrow \bigwedge\nolimits_{\alpha<\beta} z' \in \dot{X_\alpha}^{\mathfrak{A}'}\right).$$
   2. *Moreover, if $X = 0$ then $\dot{X}^{\mathfrak{A}} = 0$, so $\dot{X}^{\mathfrak{A}'} = 0$.*

Since κ is strongly compact, there exists a model $\mathfrak{A}'$ of Σ. Let $Z' = I^{\mathfrak{A}'}$ and $P' = S^{\mathfrak{A}'}$. By a suitable choice of isomorph as above, we may suppose that $Z' \supseteq Z$ and $P' \subseteq \mathcal{P} Z'$. Let U be the set of $X \in P$ such that $\mathfrak{A}' \models c\, E\, \dot{X}$. Then[9.51.1] U is an ultrafilter over Z extending F.

We will show that U is κ-complete. To this end, suppose $\beta < \kappa$ and for each $\alpha < \beta$, $X_\alpha \in U$. Let $X = \bigcap_{\alpha<\beta} X_\alpha$, and let $X' = \dot{X}^{\mathfrak{A}'}$. Then $X' \in P'$, and[9.51.2.1] $\forall z' \in Z' \left(z' \in X' \leftrightarrow \bigwedge_{\alpha<\beta} z' \in \dot{X_\alpha}^{\mathfrak{A}'}\right)$. Since $c^{\mathfrak{A}'} \in \dot{X_\alpha}^{\mathfrak{A}'}$ for each $\alpha < \beta$, $c^{\mathfrak{A}'} \in X'$, so $X' \neq 0$. Hence[9.51.2.2] $X \neq 0$.

$\leftarrow$ Suppose κ is uncountable and for any set Z, every κ-complete filter over Z can be extended to a κ-complete ultrafilter over Z.

(9.52) **Claim** *κ is regular.*

Proof Let $F = \{X \subseteq \kappa^+ \mid |\kappa^+ \backslash X| < \kappa^+\}$. Then F is a κ-complete filter over κ^+. Let U be a κ-complete ultrafilter over κ^+ extending F. Now observe that if κ is singular then (without any additional assumptions) any κ-complete filter is κ^+-complete; hence, U is nonprincipal κ^+-complete ultrafilter over κ^+, so κ^+ is measurable, contradicting (9.19). $\square$[9.52]

Suppose Σ is a κ-satisfiable set of $\mathcal{L}^{\rho}_{\kappa\kappa}$-sentences. For each $s \in \mathcal{P}_\kappa \Sigma$ let $\mathfrak{A}_s$ be a ρ-structure such that $\mathfrak{A}_s \models s$, i.e., $\mathfrak{A}_x \models \sigma$ for all $\sigma \in s$. Let $F = \{X \subseteq \mathcal{P}_\kappa \Sigma \mid \exists x \in \mathcal{P}_\kappa \Sigma \; \forall y \in \mathcal{P}_\kappa \Sigma \, (y \supseteq x \rightarrow y \in X\}$. Clearly F is a κ-complete (since κ is regular) filter over $\mathcal{P}_\kappa \Sigma$.

Let U be a κ-complete ultrafilter over Σ extending F, and let

$$\mathfrak{A} = \prod_{s \in \mathcal{P}_\kappa \Sigma} \mathfrak{A}_s / U.$$

By virtue of the κ-completeness of U, Łoś's theorem[2.164] applies to $\mathcal{L}_{\kappa\kappa}$. For each $\sigma \in \Sigma$

$$\{s \in \mathcal{P}_\kappa \Sigma \mid \mathfrak{A}_s \models \sigma\} \supseteq \{s \in \mathcal{P}_\kappa \Sigma \mid \sigma \in s\} \in F,$$

so $\{s \in \mathcal{P}_\kappa \Sigma \mid \mathfrak{A}_s \models \sigma\} \in U$, whence $\mathfrak{A} \models \sigma$. Thus $\mathfrak{A}$ satisfies Σ. $\square$[9.50]

Note that for any infinite cardinal κ and $\gamma \geqslant \kappa$, the filter F generated by the sets $\{x \in \mathcal{P}_\kappa \gamma \mid \alpha \in x\}$ for $\alpha \in \gamma$ is κ-complete. If κ is strongly compact then F is extendible to a κ-complete ultrafilter. Conversely, if we examine the proof of (9.50) we see that the existence of such an ultrafilter for each $\gamma \geqslant \kappa$ implies that κ is strongly compact. This motivates the following definition.

(9.53) **Definition** [ZFC] *Suppose κ is an uncountable cardinal and $\gamma \geqslant \kappa$.*

1. *A filter F over $\mathcal{P}_\kappa \gamma$ is* fine $\overset{\text{def}}{\iff}$ *for every $\alpha \in \gamma$, $\{x \in \mathcal{P}_\kappa \gamma \mid \alpha \in x\} \in F$.*
2. *κ is* γ-compact $\overset{\text{def}}{\iff}$ *there exists a fine κ-complete ultrafilter over $\mathcal{P}_\kappa \gamma$.*

By the preceding remarks:

(9.54) **Theorem** [ZFC] *An uncountable cardinal κ is strongly compact iff for every $\gamma \geqslant \kappa$, κ is γ-compact.*

The following theorem is an immediate corollary of (9.50).

(9.55) **Theorem** [ZFC] *If κ is strongly compact then κ is measurable.*

Proof Let $F = \{X \subseteq \kappa \mid |\kappa \backslash X| < \kappa\}$. Then F is clearly a filter over κ, and since κ is (at least) weakly compact, κ is regular,[9.45] so F is κ-complete. Thus[9.50] there is a κ-complete ultrafilter U over κ extending F. Since F contains $\kappa \backslash \{\alpha\}$ for every $\alpha \in \kappa$, U is nonprincipal. $\square$[9.55]

Thus, strong compactness is at least as strong as measurability.

9.2.8 Trees

Several sorts of "combinatorial" objects figure importantly in the theory of large cardinals. Principal among these are trees and partitions. We have encountered trees already in several contexts. To introduce trees in the context of large cardinals, we begin with the following theorem of König.

(9.56) **Theorem: König's lemma** [ZFC] *Suppose* $(T;<)$ *is a tree*[3.179] *of height* ω *all of whose levels are finite. Then* T *has an infinite branch.*

Proof We proceed recursively. Let a_0 be an element of T in level 0 such that there are infinitely many elements of T above a_0. Such an element exists because level 0 is finite and T is infinite. Let a_1 be an element of level 1 above a_0 such that there are infinitely many elements of T above a_1, and continue in this fashion to construct a sequence $a_0 < a_1 < \cdots$ of length ω. $\{a_n \mid n \in \omega\}$ is clearly an infinite branch in T. □[9.56]

This motivates the following definition.

(9.57) **Definition** [ZFC] *A cardinal* κ *has the* tree property $\overset{\text{def}}{\iff}$ *every tree of height* κ*, all of whose levels have size* $< \kappa$*, has a branch of length* κ*.*

König's lemma[9.56] states that ω has the tree property. Our present interest is in the conjunction of the tree property with inaccessibility, which is equivalent to weak compactness.

(9.58) **Theorem** [ZFC] *Suppose* κ *is an uncountable cardinal. Then* κ *is weakly compact iff* κ *is inaccessible and has the tree property.*

Proof $\rightarrow$ By (9.45) κ is inaccessible. Suppose $(T;<_T)$ is a tree of height κ each of whose levels is of size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$. For each $t \in T$, let P_t be a proposition symbol and let Σ consist of the following sentences of $\mathcal{L}_{\kappa\kappa}$ (which are actually in $\mathcal{L}_\kappa$[9.46]):

1. $\bigvee\{P_t \mid \mathrm{o}\, t = \alpha\}$ for each $\alpha < \kappa$;
2. $\neg(P_t \wedge P_{t'})$ for each $t, t' \in T$ such that t and t' are $<_T$-incomparable.

Since T has height κ, Σ is κ-satisfiable. By weak compactness there is an interpretation $\mathfrak{A}$ of Σ. $\{t \in T \mid \mathfrak{A} \models P_t\}$ is a branch of T of length κ.

$\leftarrow$ For this we adapt the proof of the completeness theorem. Note that we do not seek to establish a system of deduction for $\mathcal{L}_{\kappa\kappa}$, and we use satisfiability rather than consistency throughout. Keep in mind that we assume that κ is inaccessible. Suppose ρ is a signature of size κ, and Σ is a set of $\mathcal{L}^\rho_{\kappa\kappa}$-sentences that is κ-satisfiable. We will construct a κ-satisfiable complete extension Σ' of Σ with witnesses in an expanded signature ρ', and then use the tree property to obtain a ρ'-model of Σ', whose reduct to ρ is a model of Σ.

Let ρ' be an expansion of ρ by the addition of κ new constant symbols. Let $\langle \sigma_\alpha \mid \alpha < \kappa \rangle$ enumerate the $\mathcal{L}^{\rho'}_{\kappa\kappa}$-sentences. Let $\langle e_\alpha \mid \alpha < \kappa \rangle$ enumerate the existential sentences in the same order. (A formula is existential iff it begins with an existential quantifier.) For each $\alpha < \kappa$, let ϕ_α be the (unique) non-existential formula ϕ such that $e_\alpha = \exists \langle u_\xi \mid \xi < \mu \rangle\, \phi$ (for some ordinal μ). By recursion on $\alpha < \kappa$, let w_α be the Skolem sentence $e_\alpha \rightarrow \phi_\alpha(c^\alpha_\xi \mid \xi < \mu)$, where '$(c^\alpha_\xi \mid \xi < \mu)$'

indicates the substitution $\left\langle \binom{u_\xi}{c^\alpha_\xi} \mid \xi < \mu \right\rangle$, and c^α_ξ $(\xi < \mu)$ are the first μ new constants that do not occur in any w_β $(\beta < \alpha)$ or in ϕ_α. Let $W = \{w_\alpha \mid \alpha < \kappa\}$.

For $\alpha < \kappa$ let ρ_α be the expansion of ρ by the new constants occurring in $\{w_\beta \mid \beta < \alpha\}$. Given a ρ-structure $\mathfrak{A}$, there is a sequence $\langle \mathfrak{A}_\alpha \mid \alpha \leqslant \kappa \rangle$ of successive expansions, where $\mathfrak{A}_\alpha$ is a ρ_α-structure, such that $\mathfrak{A}_0 = \mathfrak{A}$ and for each for each $\alpha \leqslant \kappa$, $\mathfrak{A}_\alpha \models \{w_\beta \mid \beta < \alpha\}$. Given $\mathfrak{A}_\alpha$, $\alpha < \kappa$, to obtain $\mathfrak{A}_{\alpha+1}$ we first assign arbitrary denotations to any new constants in ϕ_α to obtain an expansion $\mathfrak{A}'$; then

1. if $\mathfrak{A}' \models e_\alpha$ we obtain $\mathfrak{A}_{\alpha+1}$ by assigning denotations to the witnesses c^α_ξ $(\xi < \mu)$ so as to satisfy $\phi(c^\alpha_\xi \mid \xi < \mu)$; and

2. if $\mathfrak{A}' \not\models e_\alpha$ we obtain $\mathfrak{A}_{\alpha+1}$ by assigning arbitrary denotations to the witnesses.

We make the obvious definition at limit ordinals, including κ. Then any ρ-structure $\mathfrak{A}$ has a ρ'-expansion $\mathfrak{A}'$ that satisfies W.

Let T be the set of functions $f \in {}^{<\kappa}2$ such that

1. for all $\alpha \in \operatorname{dom} f$, if $\sigma_\alpha \in \Sigma \cup W$ then $f\alpha = 1$; and

2. $\{\sigma_\alpha \mid f\alpha = 1\} \cup \{\neg\,\sigma_\alpha \mid f\alpha = 0\}$ is satisfiable.

Let $<_T = \subseteq$. Since κ is inaccessible, each level of T is smaller than κ. It is easy to see that T has height κ. For suppose $\alpha < \kappa$. Let $\Sigma_\alpha = \{\sigma_\beta \mid \beta < \alpha \wedge \sigma_\beta \in \Sigma\}$. Since Σ is κ-satisfiable, Σ_α is satisfiable. Let $\mathfrak{A}$ be a ρ-structure satisfying Σ_α. Let $\mathfrak{A}'$ be an expansion of $\mathfrak{A}$ satisfying W. Let $f : \alpha \to 2$ be such that for each $\beta < \alpha$, $f\beta = 1$ iff $\mathfrak{A}' \models \sigma_\beta$. Then $f \in T$ (at level α).

By the tree property, there exists $f : \kappa \to 2$ such that $\forall \alpha < \kappa\ f \restriction \alpha \in T$. We use f to define a structure $\mathfrak{A}$ as in the proof of the completeness theorem. Let $\langle c_\alpha \mid \alpha < \kappa \rangle$ enumerate the set C of "new" constant symbols. (There is no need to use "old" constant symbols, because each of these is stated to be equal to a new one by a sentence in W.) Define an equivalence relation $\equiv$ on C by setting $c_\alpha \equiv c_\beta$ iff $f(c_\alpha = c_\beta) = 1$. The satisfiability requirement on $f \restriction \gamma$ for every $\gamma < \kappa$ ensures that $\equiv$ is in fact an equivalence relation. Let A be the set of $\equiv$-equivalence classes, and let $|\mathfrak{A}'| = A$. For each n-ary predicate symbol R of ρ, let $R^{\mathfrak{A}'}$ be the set of n-tuples $\langle a_0, \ldots, a_{n\text{-}} \rangle \in {}^nA$ such that $f(\tilde{R}\langle b_0, \ldots, b_{n\text{-}} \rangle) = 1$, where for each $k < n$, $[b_k] = a_k$. Again, the satisfiability requirement on $f \restriction \gamma$ for every $\gamma < \kappa$ ensures that it does not matter which representatives b_k we use for this determination. Similarly, for each n-ary operation symbol F of ρ, let $F^{\mathfrak{A}'}$ be the function that maps each n-tuple $\langle a_0, \ldots, a_{n\text{-}} \rangle \in {}^nA$ to that unique $a \in A$ such that $f(\tilde{F}\langle b_0, \ldots, b_{n\text{-}} \rangle = b) = 1$, where $[b] = a$ and for each $k < n$, $[b_k] = a_k$. Again we use the satisfiability requirement to justify this definition.

It is now straightforward to show by induction on logical complexity that for every $\alpha < \kappa$, $\mathfrak{A}' \models \sigma_\alpha$ iff $f\alpha = 1$. The existence of witnesses is of course used to justify the quantification steps in this induction. Since $f\alpha = 1$ whenever $\sigma_\alpha \in \Sigma$, $\mathfrak{A}' \models \Sigma$. Let $\mathfrak{A}$ be the reduct of $\mathfrak{A}'$ to the signature ρ. Then $\mathfrak{A} \models \Sigma$. $\square$[9.58]

It is of interest that by virtue of (9.46) and the parenthetical remark made early in the above proof, the weak compactness of the quantifier-free language $\mathcal{L}_\kappa$ is equivalent to the weak compactness of $\mathcal{L}_{\kappa\kappa}$. For this reason, the original characterization of weak compactness has largely been supplanted in modern treatments by more overtly combinatorial definitions: either the tree property with inaccessibility, or—even more directly—an equivalent partition property to be defined presently. We have followed the above historically representative approach in part as a way of recalling to mind the fundamental logical principles that got us going on this journey way back in Chapter 2.

The tree property for a cardinal κ without the assumption of inaccessibility has been extensively investigated. The original result is the following theorem of Nathan Aronszajn.

Theorem ω_1 *does not have the tree property.*

In general, a counterexample to the tree property for κ is called a *κ-Aronszajn tree*, and an ω_1-Aronszajn tree is called simply an *Aronszajn tree*. Thus, Aronszajn's theorem is that there exists an Aronszajn tree. It is easy to show that for any singular cardinal κ there exists a κ-Aronszajn tree. Thus, the issue centers on the existence of κ-Aronszajn trees for regular cardinals $> \omega_1$. Mitchell and Silver showed that for any weakly compact cardinal κ above a regular uncountable cardinal λ there is a generic extension in which all cardinals $\leqslant \lambda$ are preserved and κ becomes λ^+ and retains the tree property. Thus, if it is consistent (with ZFC) that there exists a weakly compact cardinal, then it is consistent that ω_2 has the tree property (no ω_2-Aronszajn tree). On the other hand, Silver showed that if $\kappa > \omega_1$ and κ has the tree property then κ is weakly compact in L. Hence the tree property for ω_2 is equiconsistent with the existence of a weakly compact cardinal.

9.2.9 Partitions

The theory of partitions originated in a theorem of Ramsey, which we introduce by giving some definitions of general utility.

Definition [ZF] *A* partition *P of a set S is a set P of disjoint nonempty subsets of S such that $\bigcup P = S$.*

Note that any function with domain S defines a partition of S, viz., $\{f^{\leftarrow}\{y\} \mid y \in \operatorname{im} f\}$, and any partition P of S arises from a function in this way, e.g. the function $f : S \to P$ defined by the condition that for all $s \in S$, $s \in f(s)$. We frequently want to limit the cardinality of partitions, which we can conveniently do in the context of AC by simply specifying, for some cardinal κ, that $f : S \to \kappa$.

(9.59) **Definition** [ZFC]

1. *Suppose X is a set and $n \in \omega$.*
   1. $[X]^n \overset{\text{def}}{\iff}$ *the set of subsets of X of size n.*
   2. $[X]^{<\omega} \overset{\text{def}}{\iff}$ *the set of finite subsets of X.*[11]
2. *Suppose X is a set of ordinals and γ is an ordinal.*
   1. $[X]^\gamma \overset{\text{def}}{\iff}$ *the set of subsets of X of order type γ.*
   2. $[X]^{<\gamma} \overset{\text{def}}{\iff}$ *the set of subsets of X of order type $< \gamma$.*

   Note that (9.59.2) agrees with (9.59.1) when $\gamma \in \omega$, and when $\gamma = \omega$ in (9.59.2.2).
3. *Given a function f with domain $[X]^\kappa$ in either of the above senses, a subset A of X is* homogeneous for f $\overset{\text{def}}{\iff}$ *f is constant on $[A]^\kappa$.*

[11] Thus,[9.49] '$[X]^{<\omega}$' is synonymous with '$\mathcal{P}_\omega X$'.

The matter of the existence of homogeneous sets as above, with various constraints on the sizes of X, κ, A and $\operatorname{im} f$, is a very fertile source of ideas in set theory. The paradigm and progenitor of the theory of partition relations is the following theorem of Ramsey mentioned above.

(9.60) **Theorem** [ZFC] *If X is infinite, and m and n are finite, then for any $f : [X]^m \to n$ then there exists an infinite homogeneous set for f.*

Proof The proof is by induction on m. The case $m = 1$ is trivial. We now assume the result for $m = M$ and prove it for $m = M + 1$. We need the axiom of choice to carry out the following construction, and we posit at the outset a choice function for $\mathcal{P}\mathcal{P}X$. We also need a choice function for $\mathcal{P}X$, although we can obviate this by observing that it is enough to deal with the case $X = \omega$, using the fact that any infinite set has a subset that is equipollent with ω (a consequence of AC). We then have the definable choice function $S \mapsto \inf S$ for $\mathcal{P}\omega\backslash\{0\}$. We will take this route, and we now suppose that $X = \omega$. We will define sequences $\langle X_k \mid k \in \omega\rangle$, $\langle x_k \mid k \in \omega\rangle$, and $\langle i_k \mid k \in \omega\rangle$ recursively to have the following properties.

1. $\omega = X_0 \supseteq X_1 \supseteq \cdots$, and X_k is infinite for all $k \in \omega$.
2. x_k is the least element of X_k, and $x_k \notin X_{k+1}$.
3. $\forall k \in \omega\, \forall s \in [X_{k+1}]^M\ f(\{x_k\} \cup s) = i_k$.

Given X_k, we let x_k be its least element, and we obtain i_k and X_{k+1} as follows. Let $X' = X_k\backslash\{x_k\}$ and define $g : [X']^M \to n$ by

$$g(s) = f(\{x_k\} \cup s).$$

Use the induction hypothesis to conclude that there is an infinite subset of X' homogeneous for g, and let i_k be the least $i \in n$ for which an infinite homogeneous set exists,[12] and use our initial choice function to pick one such set to be X_{k+1}. This construction clearly satisfies the three conditions stated above.

Now let $i \in n$ and $K \subseteq \omega$ be such that K is infinite and $(\forall k \in K)\, i_k = i$. Let $A = \{x_k \mid k \in K\}$. Note that for any $k, k' \in K$, if $k < k'$ then $x_{k'} \in X_{k+1}$. Let s be an arbitrary element of $[A]^{M+1}$. Let x be the least element of s. Since $x \in A$, $x = x_k$ for some $k \in K$. Let $s' = s\backslash\{x\}$. Each element of s' is $x_{k'}$ for some $k' > k$, so $s' \subseteq X_{k+1}$, and $s' \in [X_{k+1}]^M$. By construction, $f(s) = f\big(\{x_k\} \cup s'\big) = i_k = i$. In other words, A is homogeneous for f. $\square$[9.60]

Although we have proved this theorem in ZFC, it is not hard to prove it in ZF if we assume that X has a subset equipollent with ω, or, without significant loss of generality, if $X = \omega$. It is amusing to note, however, that a metatheoretical argument may be used to achieve the same end:

> Recall that for any set A, $L[A] \models$ ZFC.[7.28] Thus $L[A] \models$ ⌜Ramsey's theorem⌝ by the above argument. We may therefore reason in ZF as follows. Suppose $m, n \in \omega$ and $f : [\omega]^m \to n$. It is easy to see that $f \in L[f]$, so in $L[f]$ there exists an infinite $X \subseteq \omega$ that is homogeneous for f in the sense of $L[f]$. It is easy to see that X is homogeneous for f *per se*.

[12] We do not have to choose i_k to be the smallest acceptable value of i; we just want to have a definite choice. Since there are definable choice functions for $\mathcal{P}n$, we might as well use one—the least-element function in this case—rather than invoke the axiom of choice yet again.

(9.61) **Definition** [ZFC] *Suppose α and β are ordinals and ν and λ are cardinals. Then $\alpha \to (\beta)^{\nu}_{\lambda} \overset{\text{def}}{\Longleftrightarrow}$ for any $f : [\alpha]^{\nu} \to \lambda$ there exists $A \in [\alpha]^{\beta}$ such that $|f^{\to}[A]^{\nu}| = 1$.*[13]

In other words, for any partition of $[\alpha]^{\nu}$ into λ parts there is a homogeneous set of order type β.

In the terminology of (9.61) Ramsey's theorem states that for all $m, n \in \omega$, $\omega \to (\omega)^m_n$. In Ramsey's original article[20] this was only a prelude to the following finitary version, which he used to prove the decidability of a class of first-order predicate theories. Although it is not germane to the present discussion, it is worth noting that there is an extensive literature focused on finitary partition theory. For our purposes, the following derivation of the finitary version from the infinitary version of Ramsey's theorem is instructive, as it brings into play the compactness theorem (of ordinary logic, i.e., $\mathcal{L}_{\omega\omega}$) and thus illustrates in a relatively tame environment the role of metatheoretical considerations in the theory of membership. In this connection, we note that König's lemma is itself more or less equivalent to the completeness theorem for $\mathcal{L}_{\omega\omega}$.[14]

(9.62) **Theorem** [ZF] *Suppose $m, n \in \omega$. For each $M \in \omega$ there exists $N \in \omega$ such that $N \to (M)^m_n$.*

Proof Suppose $m, n \in \omega$. We consider a signature ρ for logic with equality that has a unary predicate symbol P and an m-ary operation symbol F. For each $k \in \omega$ let δ_k be the formula $\bigwedge_{l<l'<k} \mathrm{v}_l \neq \mathrm{v}_{l'}$. Let $\mathcal{S}_m$ be the set of permutations of m, i.e., functions $\pi : m \overset{\text{bij}}{\to} m$. Let T be the ρ-theory consisting of the sentences

1. $\exists \mathrm{v}_0, \dots, \mathrm{v}_{n^-} \left(\delta_n \wedge \bigwedge_{k<n} \tilde{P}\langle \mathrm{v}_k \rangle \wedge \forall \mathrm{v}_n \left(\tilde{P}\langle \mathrm{v}_n \rangle \to \bigvee_{k<n} \mathrm{v}_n = \mathrm{v}_k\right)\right)$;
2. $\forall \mathrm{v}_0, \dots, \mathrm{v}_{m^-}\, \tilde{P}\langle \tilde{F}\langle \mathrm{v}_0, \dots, \mathrm{v}_{m^-} \rangle\rangle$;
3. $\forall \mathrm{v}_0, \dots, \mathrm{v}_{m^-} \bigwedge_{\pi \in \mathcal{S}_m} \tilde{F}\langle \mathrm{v}_0, \dots, \mathrm{v}_{m^-} \rangle = \tilde{F}\langle \mathrm{v}_{\pi 0}, \dots, \mathrm{v}_{\pi m^-} \rangle$; and
4. $\neg \exists \mathrm{v}_0, \dots, \mathrm{v}_{M^-} \left(\delta_M \wedge \exists \mathrm{v}_M \bigwedge_{k_0 < \cdots k_{m^-} < M} \tilde{F}\langle \mathrm{v}_{k_0}, \dots, \mathrm{v}_{k_{m^-}} \rangle = \mathrm{v}_M\right)$.

Succinctly, if $\mathfrak{A} \models T$ then, letting $A = |\mathfrak{A}|$, $p = P^{\mathfrak{A}}$ and $f = F^{\mathfrak{A}}$,

1. p is a set of exactly n elements of A;
2. $f : {}^m A \to p$;
3. the value of f is independent of the order of its arguments, so that—restricted to distinct arguments—f is in effect a function from $[A]^m$ to p; and
4. there is no homogeneous subset of A for f of size M.

For each $N \in \omega$, let $\alpha_N = \exists \mathrm{v}_0, \dots, \mathrm{v}_{N^-}\ \delta_N$. Thus, $\mathfrak{A} \models \alpha_N$ iff there are at least N distinct elements in A. Let

$$S = T \cup \{\alpha_N \mid N \in \omega\}.$$

[13] Recall[9.59.2] that if X is a set of ordinals and γ is an ordinal then $[X]^{\gamma}$ is the set of subsets of X of order type γ, and $[X]^{<\gamma}$ is the set of subsets of X of order type $< \gamma$.

[14] This may be understood informally as the impression that they say roughly the same thing, or that each is a "corollary" the other. More precisely, a form of König's lemma known as *weak König's lemma* is equivalent to the completeness theorem over a certain well developed theory of second-order arithmetic, RCA_0, which is itself too weak to prove the completeness theorem.

Suppose toward a contradiction that there does not exist $N \in \omega$ such that $N \to (M)^m_n$. Then any finite subset of S is satisfiable. By the compactness theorem, S is satisfiable. Let $\mathfrak{A}$ be a model of S, and let $A = |\mathfrak{A}|$. Then A is infinite, $F^{\mathfrak{A}}$ is (in effect) a partition of $[A]^m$ into n sets, and there is no M-element subset of A homogeneous for f, contradicting (the infinitary version of) Ramsey's theorem. $\square^{9.62}$

We take this opportunity to introduce the notion of indiscernibility in model theory, which will be very important later on.

Definition [GB] *Suppose $\mathfrak{S}$ is a ρ-structure and $(X;<)$ is a linear order. $(X;<)$ is a* class of indiscernibles for $\mathfrak{S}$[15] $\overset{\text{def}}{\iff}$ $X \subseteq |\mathfrak{S}|$,[16] *and for every $n \in \omega$, every ρ-formula ϕ with n free variables, and every pair $\langle x_m \mid m \in n\rangle$ and $\langle x'_m \mid m \in n\rangle$ of increasing n-sequences from $(X;<)$,*

$$\mathfrak{S} \models \phi[x_0, \ldots, x_{n^-}] \iff \mathfrak{S} \models \phi[x'_0, \ldots, x'_{n^-}].$$

The concept of indiscernibility was introduced by Ehrenfeucht and Mostowski in the following seminal theorem, with the original purpose of obtaining models with many automorphisms. Its proof exploits the evident analogy of indiscernibility for models and homogeneity for partitions.

(9.63) **Theorem** [ZF] *Suppose Θ is a theory with an infinite model and $(X;<)$ is a linear order. Then there exists a model $\mathfrak{S}$ of Θ such that $(X;<)$ is a set of indiscernibles for $\mathfrak{S}$.*

Proof Let ρ be the signature of Θ, and let ρ' expand ρ by the addition of a distinct constant symbol $\dot{x}$ for each $x \in X$. Let Θ' be the extension of Θ by the addition of the sentences

1. $\dot{x} \neq \dot{y}$ for every $x, y \in X$ such that $x \neq y$; and
2. $\phi(\dot{x}_0, \ldots, \dot{x}_{n^-}) \leftrightarrow \phi(\dot{x}'_0, \ldots, \dot{x}'_{n^-})$ for every ρ-formula ϕ with n free variables and every pair of increasing sequences $\langle x_0, \ldots, x_{n^-}\rangle$ and $\langle x'_0, \ldots, x'_{n^-}\rangle$ in $(X;<)$.

Clearly, any model $\mathfrak{S}$ of Θ' has an isomorph $\mathfrak{S}'$ such that $\dot{x}^{\mathfrak{S}'} = x$ for all $x \in X$, and $(X;<)$ is a set of indiscernibles for $\mathfrak{S}'$.

Thus it suffices to show that Θ' is satisfiable. By the compactness theorem, it is enough to show that every finite subset of Θ' is satisfiable. To this end, suppose Σ is a finite subset of Θ'. Let $\mathfrak{S}$ be an infinite model of Θ, and let $\langle a_n \mid n \in \omega\rangle$ enumerate a subset of $|\mathfrak{S}|$.[17] Let C be the (finite) set of new constants occurring in Σ, and let Φ be the (finite) set of formulas ϕ occurring in Σ in sentences of the second type above. Let $M = |C|$ and let $n = 2^{|\Phi|}$, the number of subsets of Φ. For each nonzero $m \leqslant M$ let f_m be the function with domain $[\omega]^m$ such that for each $s \in [\omega]^m$, letting $\langle i_l \mid l < m\rangle$ be the increasing enumeration of s,

$$f_m s = \{\phi \in \Phi \mid |\operatorname{Free}\phi| = m \wedge \mathfrak{S} \models \phi[a_{i_0}, \ldots, a_{i_{m^-}}]\}.$$

[15] Referring to $(X;<)$ as a *class of indiscernibles* appears to ignore $<$ in favor of X and is not altogether appropriate in general; but in most of our applications X is a class of ordinals and $<$ is the usual ordering, so the emphasis on X is proper.

[16] X and $<$ are not necessarily definable over $\mathfrak{S}$.

[17] No choice is required for this, as we may invoke the compactness theorem to show that there is a model of $\Theta \cup \{\dot{a}_m \neq \dot{a}_n \mid m < n < \omega\}$, where $\langle \dot{a}_n \mid n \in \omega\rangle$ is an ω-sequence of distinct new constant symbols.

Since $\operatorname{im} f_m \subseteq \mathcal{P}\Phi$, f_m partitions $[\omega]^m$ into n components (some empty, but no matter). We will show that there exists $H \subseteq \omega$ of size M such that H is homogeneous for f_m for all nonzero $m \leqslant M$. The infinitary version of Ramsey's theorem is available to us, and with it we can actually obtain an infinite homogeneous H, but the finitary version is also adequate. Let $M = N_0 < N_1 < \cdots < N_M$ be such that for each $m < M$,[9.62] $N_{m+1} \to (N_m)_n^{m+1}$. Now working backward, let $N_M = H_M \supseteq H_{M^-} \supseteq \cdots \supseteq H_0$ be such that $\forall m \leqslant M\ |H_m| = N_m$ and for each nonzero $m \leqslant M$, H_{m^-} is homogeneous for f_m. Let $H = H_0$.

Let $\langle i_0, \ldots, i_{M^-} \rangle$ be the increasing enumeration of H, and let $\langle c_0, \ldots, c_{M^-} \rangle$ be the increasing enumeration of C $\big($in the order inherited from $(X; <)\big)$. Expand $\mathfrak{S}$ to a ρ'-structure $\mathfrak{S}'$ by letting $c_l^{\mathfrak{S}'} = a_{i_l}$ for each $l < M$, and assigning arbitrary denotations to new constant symbols that do not occur in Σ. Clearly, $\mathfrak{S}' \models \Sigma$.□[9.63]

The grouping of arguments of the arrow symbol reflects that fact that a given partition relation $\alpha \to (\beta)_\lambda^\nu$ implies the corresponding relation with α increased, or with β, ν, or λ decreased. Although we have defined the relation for an arbitrary cardinal ν, we are only interested in the case of finite ν—indeed, ZFC proves that $\alpha \to (\omega)_2^\omega$ does not hold for any α.[18] The following definition states a property that is as close as we can get to an infinite exponent without knowingly contradicting ZFC.

Definition [ZFC] *Suppose α and β are ordinals and λ is a cardinal. Then $\alpha \to (\beta)_\lambda^{<\omega} \overset{\text{def}}{\iff}$ for any $\langle f_n \mid n \in \omega \rangle$, where $f_n : [\alpha]^n \to \lambda$, there exists $A \subseteq \alpha$ of order type β such that for all $n \in \omega$, $|f_n{}^{\to}[A]^n| = 1$.*

In other words A is homogeneous for all the partitions f_n $(n \in \omega)$ simultaneously.

(9.64) **Definition** [ZFC] *A cardinal κ is* Ramsey $\overset{\text{def}}{\iff} \kappa \to (\kappa)_2^{<\omega}$.

Although this property is named in recognition of Ramsey's work on partitions of ω, ω is not Ramsey. For example, define $f_n : [\omega]^n \to 2$ so that

$$f_n s = \begin{cases} 0 & \text{if } \min s \leqslant n \\ 1 & \text{otherwise.} \end{cases}$$

There is no infinite homogeneous set for $\langle f_n \mid n \in \omega \rangle$.

The following simple combinatorial result is worth noting.

(9.65) **Theorem** [ZFC] *If κ is Ramsey then $\kappa \to (\kappa)_\lambda^{<\omega}$ for all $\lambda < \kappa$.*

Proof Suppose for each $n \in \omega$, $f_n : [\kappa]^n \to \lambda$. For $m \in \omega$ and $s \in [\kappa]^{2m}$, let s^0 and s^1 be respectively the lower and upper halves of s. For each $n \in \omega$ define $g : [\kappa]^n \to 2$ so that

[18] Given $s \in [\omega]^\omega$ let e_s be its enumeration in increasing order. Define an equivalence relation on $[\omega]^\omega$ by letting $s \equiv t$ iff $\{n \in \omega \mid e_s n \neq e_t n\}$ is finite. Pick a representative from each $\equiv$-equivalence class. Define $f : [\omega]^\omega$ by letting $fs = 0$ if e_s differs from the enumeration of the representative of its class at an even number of places; otherwise, $fs = 1$. Suppose $X \in [\omega]^\omega$. We claim that X is not homogeneous for f. To this end, let $s = \{e_X(2n) \mid n \in \omega\}$, i.e., s consists of every other element of X. Let u be the representative of the equivalence class of s. Let $N \in \omega$ be such that $e_s n = e_u n$ for all $n \geqslant N$. Let t be obtained from s by removing $e_s N = e_X(2N)$ and adding $e_X(2N+1)$. Then $e_s n = e_t n$ for all n other than N. e_t differs from e_u at exactly the same places as e_s does, except the Nth place, where e_s agrees with e_u and e_t does not. Hence, the number of places of disagreement with e_u is even for one of e_s and e_t and odd for the other. Thus, X is not homogeneous for f.

1. if $n = 2m$ then
$$g_n s = \begin{cases} 0 & \text{if } f s^0 = f s^1 \\ 1 & \text{otherwise;} \end{cases}$$
2. if n is odd, $g_n s = 1$.

Let $X \in [\kappa]^\kappa$ be homogeneous for every g_n. Suppose $m \in \omega$. Since $\lambda < \kappa$, there exist $s, t \in [X]^m$ such that $\max s < \min t$ and $f s = f t$. Then $s \cup t \in [X]^{2m}$ and $g_{2m}(s \cup t) = 0$. Now, given any $s, s' \in [X]^m$, let $t \in [X]^m$ be such that $\max(s \cup s') < \min t$. Since X is homogeneous for g_{2m},
$$f s = f t = f s'.$$
Thus, X is homogeneous for f_m. □[9.65]

(9.66) **Theorem** [ZFC] *Suppose U is a normal ultrafilter over a cardinal κ, $\lambda < \kappa$, and $f : [\kappa]^{<\omega} \to \lambda$. Then there exists $X \in U$ homogeneous for f in the sense that for each $n \in \omega$, $|f^{\rightarrow}[X]^n| = 1$. Thus, every measurable cardinal is Ramsey.*

Remark Note that if X is homogeneous for f then $|f^{\rightarrow}[X]^{<\omega}|$ is countable, which is all that is required for some applications.

Proof Suppose $n \in \omega$. Let U_n be the ultrafilter over $[\kappa]^n$ corresponding to U.[9.40, 9.42] Since U_n is κ-complete[9.41] there exists a unique $\gamma < \lambda$ such that $f^{\leftarrow}\{\gamma\} \in U_n$.[19] Thus
$$\mathsf{Q}^U \xi_0 \cdots \mathsf{Q}^U \xi_{n^-}\; f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma.$$
Let
$$A_0 = \bigl\{\xi_0 \in \kappa \bigm| \mathsf{Q}^U \xi_1 \cdots \mathsf{Q}^U \xi_{n^-}\; f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma\bigr\}.$$
By the definition of 'Q^U', $A_0 \in U$. For each $\xi_0 \in \kappa$, let
$$B_{\xi_0} = \begin{cases} \bigl\{\xi_1 \in A_0 \bigm| \mathsf{Q}^U \xi_2 \cdots \mathsf{Q}^U \xi_{n^-}\; f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma\bigr\} & \text{if } \xi_0 \in A_0 \\ A_0 & \text{otherwise.} \end{cases}$$
Then $B_{\xi_0} \in U$ and $B_{\xi_0} \subseteq A_0$. Let
$$A_1 = \Delta_{\xi_0 < \kappa} B_{\xi_0}.$$
Since U is normal, $A_1 \in U$. For any $\xi_0, \xi_1 \in A_1$, if $\xi_0 < \xi_1$ then
$$\mathsf{Q}^U \xi_2 \cdots \mathsf{Q}^U \xi_{n^-}\; f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma.$$
Continuing, we obtain $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{n^-}$ such that for each $m < n$, $A_m \in U$ and for all $\xi_0, \ldots, \xi_m \in A_m$, if $\xi_0 < \cdots < \xi_m$ then
$$\mathsf{Q}^U \xi_{m+1} \cdots \mathsf{Q}^U \xi_{n^-}\; f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma.$$
In particular, if $m = n - 1$ then $f\{\xi_0, \ldots, \xi_{n^-}\} = \gamma$. In other words,
$$f^{\rightarrow}[A_{n^-}]^n = \{\gamma\}.$$
Thus, for each $n \in \omega$ there exists $X_n \in U$ homogeneous for $f \restriction [\kappa]^n$. Let $X = \bigcap_{n \in \omega} X_n$. Then X is homogeneous for f. □[9.66]

The following theorem in conjunction with (9.58) provides another characterization of weak compactness.

[19] It is worth noting that by virtue of (9.43) $\gamma = (i_{0n} f)\{\kappa_0, \ldots, \kappa_{n^-}\}$, where i_{0n} is the canonical injection of V into its n-fold U-ultrapower, which is also its U_n-ultrapower. (Keep in mind that $\kappa_0 = \kappa$ and i_{0n} is the identity on κ, so $i_{0n} f : [\kappa_n]^n \to \lambda$.)

(9.67) **Theorem** [ZFC] *Suppose κ is an uncountable regular cardinal. Then κ is inaccessible and has the tree property iff $\kappa \to (\kappa)^2_2$.*

Proof $\to$ It's no harder to show that $\kappa \to (\kappa)^2_\nu$ for any $\nu < \kappa$, so we'll do that. Suppose $f : [\kappa]^2 \to \nu$. We will construct a set T of functions $s \in {}^{<\kappa}\nu$ so that $(T; \subseteq)$ is a tree of height κ with levels smaller than κ to which we can apply the tree property to obtain a homogeneous set for f. We will define T recursively as $\{s_\alpha \mid \alpha < \kappa\}$, where the map $\alpha \mapsto s_\alpha$ is injective.

Let $s_0 = 0$. Suppose we have $\langle s_\beta \mid \beta < \alpha\rangle$. We will define $t = \langle t_\xi \mid \xi < \eta\rangle$ by recursion on ξ, where the length η of the construction is to be determined. Let $t_0 = 0$. At stage ξ

1. if $t \restriction \xi$, i.e., $\langle t_{\xi'} \mid \xi' < \xi\rangle$, is not s_β for any $\beta < \alpha$, let $\eta = \xi$, and the construction of t is complete; but
2. if $t \restriction \xi = s_\beta$ for some $\beta < \alpha$ then let $t_\xi = f\{\beta, \alpha\}$.

At some η the construction of t must terminate, since we only have $|\alpha|$ s_β's to which to compare $t \restriction \xi$. Let $s_\alpha = t$ $(= t \restriction \eta)$. The construction guarantees that s_α is not s_β for any $\beta < \alpha$, but that every proper initial segment of s_α is s_β for some $\beta < \alpha$. Thus, $\alpha \mapsto s_\alpha$ is injective, and the elements of T at level α are elements of ${}^\alpha\nu$. Since κ is inaccessible, the levels are smaller than κ, and since $\alpha \mapsto s_\alpha$ is injective, $|T| = \kappa$, so the height of T is κ.

Invoking the tree property, let $F : \kappa \to \nu$ be a branch of T of length κ. For each $\mu \in \nu$, let $X_\mu = \{\beta < \kappa \mid s_\beta \subseteq F \wedge F(\operatorname{dom} s_\beta) = \mu\}$. By construction, for any $\alpha, \beta \in \kappa$, if $s_\beta \subsetneq s_\alpha$ then $\beta < \alpha$ and $s_\alpha(\operatorname{dom} s_\beta) = f\{\beta, \alpha\}$. For each $\mu < \nu$, for any distinct $\beta, \alpha \in X_\mu$, s_β and s_α are initial segments of F, hence comparable, so supposing without loss of generality that $\beta < \alpha$, $s_\beta \subsetneq s_\alpha$, so $f\{\beta, \alpha\} = s_\alpha(\operatorname{dom} s_\beta) = F(\operatorname{dom} s_\beta) = \mu$, so X_μ is homogeneous for f. Since κ is inaccessible, there exists μ such that $|X_\mu| = \kappa$.

$\leftarrow$ Suppose κ is inaccessible and $\mathbb{T} = (T; <_T)$ is a tree of height κ all of whose levels have size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$, and we will assume that $T = \kappa$. Let $\prec$ be the lexicographic extension of $<_T$ to a total order, i.e., for any $\alpha, \beta \in \kappa$, $\alpha \prec \beta$ iff

1. $\alpha <_T \beta$; or
2. α and β are $\leqslant_T$-incomparable, and letting ξ be the first level of T at which the respective predecessors α_ξ and β_ξ of α and β are different, $\alpha_\xi < \beta_\xi$ (in the usual ordering of ordinals).

Define $f : [\kappa]^2 \to 2$ so that $f\{\alpha, \beta\} = 1$ iff $(\alpha \prec \beta \leftrightarrow \alpha < \beta)$. Let X of size κ be homogeneous for f. Let B be the set of γ such that $|\{\alpha \in X \mid \gamma <_T \alpha\}| = \kappa$. Since the levels of $\mathbb{T}$ are smaller than κ there exists a member of B at every level. We will show that there is only one member of B at each level, i.e., that any two members of B are $<_T$-comparable, so B is a branch of $\mathbb{T}$ of length κ.

Suppose toward a contradiction that $\gamma_0, \gamma_1 \in B$ are $<_T$-incomparable, and suppose $\gamma_0 \prec \gamma_1$. Pick $\alpha_0 \in X$ such that $\gamma_0 <_T \alpha_0$, then pick $\alpha_1 > \alpha_0$ (in the usual ordering of ordinals) such that $\alpha_1 \in X$ and $\gamma_1 <_T \alpha_1$, and then pick $\beta_0 > \alpha_1$ such that $\beta_0 \in X$ and $\gamma_0 <_T \beta_0$. Then α_1 is $<_T$-incomparable to α_0 and β_0, so $\alpha_0, \beta_0 \prec \alpha_1$, since $\gamma_0 \prec \gamma_1$. Hence, $f\{\alpha_0, \alpha_1\} = 1$ and $f\{\alpha_1, \beta_0\} = 0$; a contradiction, since X is homogeneous for f. $\square$ 9.67

As mentioned above (following the proof of (9.58)), the partition property $\kappa \to (\kappa)^2_2$ is often taken as the definition of weak compactness. As the preceding proof makes clear, $\kappa \to (\kappa)^2_2$ implies $\kappa \to (\kappa)^2_\lambda$ for any $\lambda < \kappa$. It also implies $\kappa \to (\kappa)^m_\lambda$ for any $m \in \omega$ (proof omitted). It does not, however, imply $\kappa \to (\kappa)^{<\omega}_2$, which is much stronger.

9.3 Large cardinals and constructibility

We have already derived the seminal result[9.34] of Scott that measurable cardinals do not exist in L, which is to say, there does not exist an ordinal that L thinks is a measurable cardinal. In general, the interpretation of large cardinal properties in L and in other inner models with a similar mode of construction is an important theme in the subject. At the simplest level, consistency with $\boldsymbol{V=L}$ is a natural threshold in the assessment of the strength of large cardinal properties.

9.3.1 Weak compactness

As we have noted, measurability has crossed the constructibility threshold. It is quite easy to see that inaccessibility has not, as a cardinal that is inaccessible (in the "real world" of V) is clearly inaccessible in L. The following series of theorems shows that weak compactness is also consistent with $\boldsymbol{V=L}$.

We begin with an extendibility property, followed by a reflection property.

(9.68) **Theorem** [ZFC] *Suppose κ is weakly compact, and $R \subseteq V_\kappa$. Then there exist a transitive set $M \supsetneq V_\kappa$ and $R' \subseteq M$ such that $(V_\kappa; \in, R) \prec (M; \in, R')$.*

Proof Let ρ_0 be an expansion of s by the addition of a unary predicate symbol $\dot{R}$ and a constant symbol $\dot{x}$ for each $x \in V_\kappa$. Let $\mathfrak{A} = (V_\kappa; \in, R, x)_{x\in V_\kappa}$ be the ρ_0-structure with $\dot{R}^{\mathfrak{A}} = R$ and $\dot{x}^{\mathfrak{A}} = x$ for all $x \in V_\kappa$. For convenience, let $\dot{\in}$ be the predicate symbol in s denoting membership. Thus $\mathfrak{A} = (V_\kappa; \dot{\in}^{\mathfrak{A}}, \dot{R}^{\mathfrak{A}}, \dot{x}^{\mathfrak{A}})_{x\in V_\kappa}$.

Let Σ_0 be the $\mathcal{L}^{\rho_0}_{\kappa\kappa}$-theory of $\mathfrak{A}$ $\big($the *complete $\mathcal{L}_{\kappa\kappa}$-theory* of $(V; \in, R)\big)$. Let ρ be ρ_0 with one more constant symbol c. Let

$$\Sigma = \Sigma_0 \cup \{c \neq \dot{x} \mid x \in V_\alpha\}.$$

$|\rho_0| = \kappa$ and Σ is κ-satisfiable, since for any subset S of Σ smaller than κ, we may expand $\mathfrak{A}$ to a ρ-structure by letting c denote any $x \in V_\kappa$ such that $\dot{x}$ does not occur in S.

Let $\mathfrak{B} = (B; \dot{\in}^{\mathfrak{B}}, \dot{R}^{\mathfrak{B}}, \dot{x}^{\mathfrak{B}})_{x\in V_\kappa}$ be a model of Σ. Clearly we may assume that $V_\kappa \subseteq B$, that $\dot{\in}^{\mathfrak{B}}$ agrees with $\dot{\in}^{\mathfrak{A}}$ $(= \in)$ on V_κ, and that $\dot{x}^{\mathfrak{B}} = x$ for all $x \in V_\kappa$; and then clearly $\mathfrak{A} \prec \mathfrak{B}$. Suppose $\dot{\in}^{\mathfrak{B}}$ is wellfounded. Then there is a (unique) transitive set M and (unique) isomorphism $\pi : (B; \dot{\in}^{\mathfrak{B}}) \to (M; \dot{\in}^{\mathfrak{M}})$, where $\dot{\in}^{\mathfrak{M}}$ is necessarily $\in$. We let $R' = \pi^{\to}\dot{R}^{\mathfrak{B}}$. Then π is also an isomorphism of $(B; \dot{\in}^{\mathfrak{B}}, \dot{R}^{\mathfrak{B}})$ with $(M; \in, R')$. Since $V_\kappa \subseteq B$, π is the identity on V_κ, so $V_\kappa \subseteq M$ and $(V_\kappa; \in, R) \prec (M; \in, R')$. Since $c^{\mathfrak{B}} \notin V_\kappa$, $B \neq V_\kappa$, so $M \neq V_\kappa$.

Thus it suffices to show that $\dot{\in}^{\mathfrak{B}}$ is wellfounded. The sentence

$$\neg\, \exists \langle \mathrm{v}_n \mid n \in \omega \rangle \ \bigwedge\nolimits_{n\in\omega} \mathrm{v}_{n+1} \,\dot{\in}\, \mathrm{v}_n$$

is in Σ since $\in$ is wellfounded, so it holds in $(B; \dot{\in}^{\mathfrak{B}})$, so $\dot{\in}^{\mathfrak{B}}$ is wellfounded. $\square$[9.68]

Note that this is the first time we have used quantification in $\mathcal{L}_{\kappa\kappa}$, but even here $\mathcal{L}_{\kappa\omega_1}$ suffices.

(9.69) **Theorem** [ZFC] *Suppose κ is weakly compact and $U \subseteq V_\kappa$. Suppose ρ is the expansion of* s *by the addition of two unary predicate symbols $\dot{R}$ and $\dot{X}$, and suppose σ is a ρ-sentence such that*

$$\forall X \subseteq V_\kappa\ (V_\kappa; \in, R, X) \models \sigma,$$

where, here and elsewhere, it is to be understood that $\dot{R}$ and $\dot{X}$ respectively denote R and X. Then there exists $\alpha < \kappa$ such that

$$\forall X \subseteq V_\alpha\ (V_\alpha; \in, R \cap V_\alpha, X) \models \sigma.$$

Proof Let[9.68] $(M; \in, R')$ be an elementary extension of $(V; \in, R)$, where M is transitive and $M \neq V_\kappa$. Note that $\kappa \in M$, and $\ulcorner V_{[\kappa]} \urcorner^M = V_\kappa$. Note also that for any structure $\mathfrak{A}$ in V_κ, the satisfaction relation for $\mathfrak{A}$ is in V_κ and is recognized as such in $(V_\kappa; \in)$, along with all the properties appertaining thereto. With a mild abuse of notation that is easily made right, $(M; \in, R')$ may refer to R as $\ulcorner \dot{R} \cap V_{[\kappa]} \urcorner$. Since

$$\forall X \subseteq V_\kappa\ (V_\kappa; \in, R, X) \models \sigma,$$

$$(M; \in, R') \models \ulcorner \forall X \subseteq V_{[\kappa]}\ (V_{[\kappa]}; \in, \dot{R} \cap V_{[\kappa]}, X) \models \sigma \urcorner.$$

Hence

$$(M; \in, R') \models \ulcorner \exists_{\mathrm{Ord}} \alpha\, \forall X \subseteq V_\alpha\ (V_\alpha; \in, \dot{R} \cap V_\alpha, X) \models \sigma \urcorner.$$

By elementarity,

$$(V_\kappa; \in, R) \models \ulcorner \exists_{\mathrm{Ord}} \alpha\, \forall X \subseteq V_\alpha\ (V_\alpha; \in, \dot{R} \cap V_\alpha, X) \models \sigma \urcorner.$$

Hence, there exists $\alpha < \kappa$ such that

$$\forall X \subseteq V_\alpha\ (V_\alpha; \in, R \cap V_\alpha, X) \models \sigma.$$

□[9.69]

Indescribability The property of weakly compact cardinals just demonstrated[9.69] is referred to as Π^1_1-*indescribability*. The notion of indescribability of cardinals was introduced by Hanf and Scott. It is most naturally defined in terms of second- and higher-order predicate languages. As indicated in the discussion following (1.19), first-order predicate languages are the usual languages we have been discussing all along. For the purpose of generalization, the names of the complexity classes Σ_n and Π_n are given the superscript '0'. Thus, first-order expressions are classified as Σ^0_n or Π^0_n. To interpret a first-order quantifier phrase in a structure $\mathfrak{S}$, we allow the quantified variable to range over the members of $|\mathfrak{S}|$. A second-order variable ranges over $\mathcal{P}|\mathfrak{S}|$, a third-order variable over $\mathcal{P}\mathcal{P}|\mathfrak{S}|$, etc. Σ^m_n and Π^m_n refer to syntactical and semantical classes corresponding to quantification over variables of order $m + 1$ or less.[20]

In this system we would render

$$\ulcorner \forall X \subseteq |\mathfrak{S}|\ \mathfrak{S} \models \sigma \urcorner,$$

[20] The terminological adjustment from m to $m + 1$ is unfortunate but entrenched. Recall that we have used the related concept of *type* in our discussion[5.1] of the theory of V_ω and $V_{\omega+1}$. Type n corresponds to order $n + 1$, and we do not have the above discrepancy. There are no objects of order 0.

as

$$\ulcorner \mathfrak{S} \models \forall^1 X\ \sigma \urcorner,$$

with the superscript '1' indicating second-order quantification. $\ulcorner \forall^1 X\ \sigma \urcorner$ is a Π^1_1 sentence. In general, κ is respectively Π^m_n- or Σ^m_n-indescribable $\overset{\text{def}}{\iff}$ the analog of (9.69) holds for an arbitrary Π^m_n or Σ^m_n sentence in place of $\forall^1 X\ \sigma$. A cardinal is *totally indescribable* $\overset{\text{def}}{\iff}$ it is Σ^m_n-indescribable for all $m, n \in \omega$.

The application of 'indescribability' to the property stated in Theorem 9.69 derives from its implication that weak compactness of a cardinal κ is not a Π^1_1 property of the structure $(V_\kappa; \in)$.[21] In their seminal work on indescribability, Hanf and Scott proved, *inter alia*, the above theorem and its converse, giving yet another characterization of weak compactness and further evidence of the robustness of this concept. In this paper they also showed that measurable cardinals are Π^2_1-indescribable. This is, of course, a specialization of (9.35).

Since the existence of a nonprincipal κ-complete ultrafilter over κ is a Σ^2_1 property of V_κ, the least measurable cardinal is not Σ^2_1-indescribable. Indescribability is not, however, stronger than measurability, as shown by the following theorem of Vaught.

(9.70) **Theorem** [ZFC] *Suppose U is a normal ultrafilter over κ. Then the set of totally indescribable cardinals below κ is in U.*

Proof We will carry out the proof in GBC so that we can treat proper classes properly. Let $j : V \to M$ be the canonical elementary embedding. Note that since $U \notin M$,[9.33] M cannot reason as above that κ is not Σ^2_1-indescribable. The key observation is that in fact $M \models \ulcorner [\kappa]$ is totally indescribable$\urcorner$. To show this, suppose ϕ is a Σ^m_n-sentence for some $m, n \in \omega$, $R \subseteq V_\kappa$, and

$$M \models \ulcorner (V_{[\kappa]}; \in, [R]) \text{ satisfies } \phi \urcorner.$$

Since $\kappa < j\kappa$ and j is the identity on V_κ so that $R = jR \cap V_\kappa$,

$$M \models \ulcorner \text{there exists } \alpha < [j\kappa] \text{ such that } (V_\alpha; \in, [jR] \cap V_\alpha) \text{ satisfies } \phi \urcorner.$$

By the elementarity of j,

$$V \models \ulcorner \text{there exists } \alpha < [\kappa] \text{ such that } (V_\alpha; \in, [R] \cap V_\alpha) \text{ satisfies } \phi \urcorner.$$

Since $\ulcorner (V_\alpha; \in, R \cap V_\alpha)$ satisfies $\phi \urcorner$ is a Δ_0 property of $\langle V_{\alpha+\omega}, R\rangle$ and $V_\kappa \subseteq M$,

$$M \models \ulcorner \text{there exists } \alpha < [\kappa] \text{ such that } (V_\alpha; \in, [R] \cap V_\alpha) \text{ satisfies } \phi \urcorner.$$

Thus, $M \models \ulcorner [\kappa]$ is totally indescribable$\urcorner$.

As usual, since κ is represented in ${}^\kappa V/U$ by the identity function on κ, $\{\alpha < \kappa \mid V \models \ulcorner [\alpha]$ is totally indescribable$\urcorner\}$ is in U.[22] Again using the equivalence of satisfaction in $(V; \in)$ and reality, we conclude that $\{\alpha < \kappa \mid [\alpha]$ is totally indescribable$\} \in U$. □[9.70]

As an easy corollary of (9.69) we have the following theorem.

[21]Note that it is necessary to refer to $(V_\kappa; \in)$ rather than $(\kappa; \in)$, as the structure of $(\kappa; \in)$ does not in general embody the structure of $(V_\kappa; \in)$ in the way that $(\omega; \in)$ embodies that of $(V_\omega; \in)$—as has been amply demonstrated in the preceding chapter.

[22]Keeping in mind that total indescribability of an ordinal α is a Δ_0 property of $V_{\alpha+\omega}$, we only need the Δ_0-satisfaction relation for $(V; \in)$, whose existence is GB-demonstrable, to explicate and justify this statement and the next.

(9.71) **Theorem** [ZFC] *Suppose κ is weakly compact. Then κ is Mahlo.*[9.4]

Proof We have already shown[9.45] that κ is inaccessible. Suppose R is closed unbounded in κ. Then $(V_\kappa; \in, R)$ satisfies the Π^1_1 sentence $\ulcorner \forall^1 F\ \big(F$ is not a function from some ordinal cofinal in Ord, the Power axiom holds, and $\dot{R}$ is unbounded (in Ord)$\big)\urcorner$, where $\dot{R}$ is the constant symbol denoting R in $(V_\kappa; \in, R)$. By Π^1_1-indescribability, there exists $\alpha < \kappa$ such that α is a regular strong limit cardinal, hence inaccessible, and $R \cap \alpha$ is unbounded in α, so $\alpha \in R$. $\square$[9.71]

This argument is easily adapted to show that a weakly compact cardinal κ is hyper-Mahlo, etc. The proof is left for the energetic reader.[23]

(9.72) **Theorem** [ZFC] *Suppose κ is weakly compact, $R \subseteq \kappa$, and $X \cap \alpha$ is constructible for every $\alpha < \kappa$. Then R is constructible.*

Proof Let[9.68] $(M; \in, R')$ be a transitive proper elementary extension of $(V_\kappa; \in, R)$. The constructible hierarchy is given by the standard definition interpreted over any transitive model of ZF, so the sentence $\ulcorner$for all ordinals α, $(\dot{R}) \cap \alpha$ is constructible$\urcorner$ is true in $(V_\kappa; \in, R)$ (where $\dot{R}$ is the predicate symbol for R). It is therefore true in $(M; \in, R')$. Hence $\ulcorner(\dot{R}) \cap [\kappa]$ is constructible$\urcorner$ is true in $(M; \in, R')$. $\dot{R}^{(M; \in, R')} \cap \kappa = R' \cap \kappa = R$, so we may invoke the absoluteness of constructibility again to conclude that R is constructible. $\square$[9.72]

Finally we have the promised result about constructibility.

(9.73) **Theorem** [ZFC] *Suppose κ is weakly compact. Then κ is weakly compact in the sense of L.*

Proof Obviously κ is inaccessible in L. Suppose that, in L, $\mathbb{T}$ is a tree of height κ with all levels smaller than κ. In V, $\mathbb{T}$ has a branch B of length κ. By (9.72) $B \in L$. $\square$[9.73]

9.3.2 Strong compactness

Just as weak compactness is in the sense of (9.73) weak, strong compactness is in a similar sense strong, as shown by Vopěnka and Hrbáček.

(9.74) **Theorem** [GB] *Suppose there is a set A such that $V = L[A]$. Then no cardinal is strongly compact.*

Proof Since AC holds in $L[A]$, AC holds. We can use a bijection of $\operatorname{tc} A$ with a set of ordinals to obtain a relation on ordinals isomorphic to $(\operatorname{tc} A; \in)$, and then use a definable pairing function on ordinals to obtain a set A' of ordinals such that $V = L[A']$. Thus, without loss of generality, we assume that $A \subseteq \mathrm{Ord}$.

(9.75) *Suppose toward a contradiction that κ is a strongly compact cardinal.*

Let $\lambda \geqslant \kappa$ be a cardinal such that $A \subseteq \lambda$. Let $F = \{X \subseteq \lambda^+ \mid |\lambda^+ \backslash X| \leqslant \lambda\}$. Then F is a κ-complete filter over λ^+. Let $U \supseteq F$ be a κ-complete ultrafilter over λ^+. Note that for any $X \subseteq \lambda^+$ smaller than λ^+, $\lambda^+ \backslash X \in F$, so $X \notin U$. Briefly, $U \cap \mathcal{P}_{\lambda^+} \lambda^+ = 0$.

[23] Given that the reader is still reading, 'energetic' may be taken to be descriptive rather than restrictive.

For $f : \lambda^+ \to V$, let $[f]^*$ be the reduced equivalence class[2.167] of f mod U. Let $\mathrm{Ult}_U V = {}^{\lambda^+}V/U$ be the ultrapower. Since U is κ-complete, it is ω_1-complete, so $\mathrm{Ult}_U V$ is wellfounded. Let $\pi : \mathrm{Ult}_U V \overset{\text{sur}}{\to} M$ be its transitive collapse, and let $j : V \to M$ be the canonical embedding. $jx = \pi[\bar{x}]^*$, where $\bar{x} : \lambda^+ \to \{x\}$.

Now construct another ultrapower of V mod U, with the following variation: use only functions $f : \lambda^+ \to V$ such that $|\operatorname{im} f| \leqslant \lambda$. Let $\mathcal{F}$ be the class of all such functions. Let $[f]^{*'}$ be the reduced equivalence class of f in $\mathcal{F}$ mod U, and let $\mathrm{Ult}'_U V$ be the ultrapower formed in this way. We will use $[f]$ generically for $[f]^*$ or $[f]^{*'}$. Clearly, $\mathrm{Ult}'_U V \subseteq \mathrm{Ult}_U V$, and $\mathrm{Ult}'_U V$ is wellfounded. Let $\pi' : \mathrm{Ult}'_U V \overset{\text{sur}}{\to} M'$ be its transitive collapse, and let $j' : V \to M'$ be the canonical injection. $j'x = \pi'[\bar{x}]^{*'}$. Łoś's theorem[2.164] holds for $\mathrm{Ult}'_U V$[24] as for $\mathrm{Ult}_U V$.

For any ordinal γ, $j\gamma$ and $j'\gamma$ are respectively the order types of the predecessors of $[\bar{\gamma}]$ in $\mathrm{Ult}_U V$ and $\mathrm{Ult}'_U V$. In either ultrapower, any predecessor of $[\bar{\gamma}]$ is $[f]$ for some $f : \lambda^+ \to \gamma$, where in the latter case f must be in $\mathcal{F}$. Suppose $\gamma < \lambda^+$, and $f : \lambda^+ \to \gamma$. Then $f \in \mathcal{F}$. Hence, $[\bar{\gamma}]$ has exactly the same predecessors in $\mathrm{Ult}_U V$ as in $\mathrm{Ult}'_U V$, so $j\gamma = j'\gamma$, i.e., $j \upharpoonright \lambda^+ = j' \upharpoonright \lambda^+$. In particular, $j'\lambda = j\lambda$.

Now consider $j'\lambda^+$ vs. $j\lambda^+$. If $f : \lambda^+ \to \lambda^+$ and $f \in \mathcal{F}$, then $f : \lambda^+ \to \gamma$ for some $\gamma < \lambda^+$, so $j'(\lambda^+) = \sup_{\gamma<\lambda^+} j'\gamma = \sup_{\gamma<\lambda^+} j\gamma$.

On the other hand, suppose f is the identity function on λ^+. Then for any $\gamma < \lambda^+$, $\{\alpha < \lambda^+ \mid f\alpha > \gamma\} \in U$, since U does not have any members of size $< \lambda^+$. Hence, $[\overline{\lambda^+}]^* > [f]^* > [\bar{\gamma}]^*$ in $\mathrm{Ult}_U V$, and therefore $j(\lambda^+) > \pi[f]^* \geqslant \sup_{\gamma<\lambda^+} j\gamma = j'(\lambda^+)$.

Now observe that if $f : \lambda^+ \to A$ then $f \in \mathcal{F}$, so $jA = j'A$. Since $V = L[A]$, $M = L[jA] = L[j'A] = M'$.

Since $V \models \ulcorner[\lambda^+]$ is the next cardinal after $[\lambda]\urcorner$,

$$M \models \ulcorner[j(\lambda^+)] \text{ is the next cardinal after } [j\lambda]\urcorner,$$

whereas

$$M' \models \ulcorner[j'(\lambda^+)] \text{ is the next cardinal after } [j'\lambda]\urcorner,$$

which is contradictory, since $M = M'$, $j\lambda = j'\lambda$, and $j(\lambda^+) > j'(\lambda^+)$; hence, (9.75) is untenable. □[9.74]

9.3.3 Measurability: $L[U]$

The following theorem, together with (9.74), shows immediately that measurability does not imply strong compactness.

(9.76) **Theorem** [GB] *Suppose U is a nonprincipal κ-complete ultrafilter over an uncountable cardinal κ. Let $U' = U \cap L[U]$. Note that $L[U] \models$ ZFC, $U' \in L[U]$, $L[U] = L[U']$, and $L[U] \models \ulcorner V = L[[U']]\urcorner$ (appropriately formulated in ZF).*

1. *$L[U] \models \ulcorner[U']$ is a nonprincipal $[\kappa]$-complete ultrafilter over $[\kappa]\urcorner$; hence, $L[U] \models \ulcorner[\kappa]$ is measurable$\urcorner$.*
2. *If U is normal then $L[U] \models \ulcorner[U']$ is normal$\urcorner$.*

Proof Straightforward. □[9.76]

The following theorem is due to Solovay.

[24] A witness $[f]^{*'}$ for an existential formula at a given assignment $[[f_0]^{*'}, \ldots, [f_{n^-}]^{*'}]$, with $f_0, \ldots, f_{n^-} \in \mathcal{F}$, can be obtained by choosing a fixed witness for each assignment $[f_0\alpha, \ldots, f_{n^-}\alpha]$, $\alpha < \lambda^+$, and these are all in $(\operatorname{im} f_0) \times \cdots \times (\operatorname{im} f_{n^-})$, so there are $\leqslant \lambda$ of them.

(9.77) **Theorem** [GB] *Suppose U is a normal ultrafilter over κ.*

1. $L[U] \models \ulcorner \forall\lambda \geqslant [\kappa]\ 2^\lambda = \lambda^+ \urcorner$.
2. $L[U] \models \ulcorner [\kappa]$ *is the only measurable cardinal*$\urcorner$.

Proof 1 This is an easy modification of the proof[7.27] of GCH in L.

2 We will "work in $L[U]$" in GBC. Thus, we assume GBC+ $\ulcorner U$ is a normal ultrafilter over a cardinal κ and $V = L[U]\urcorner$.

Suppose toward a contradiction that W is a normal ultrafilter over a cardinal $\lambda \neq \kappa$. Let $\pi : {}^\lambda V/W \to M$ be the transitive collapse, and let $j : V \to M$ be the canonical injection. We will show that $M = V$.

We will use $\ulcorner$is constructible from $U\urcorner$ as an s-equivalent of $\ulcorner$is in $L[U]\urcorner$. Since $V \models \ulcorner$every set is constructible from $[U]\urcorner$, by elementarity, $M \models \ulcorner$every set is constructible from $[jU]\urcorner$. Thus

$$M = L[jU]. \tag{9.78}$$

We will show that $M = L[U] = V$. Hence, $W \in M$, contrary to (9.33).

If $\lambda > \kappa$ then $jU = U$, so $M = L[jU] = L[U] = V$. Suppose therefore that $\lambda < \kappa$. Let I be the set of inaccessible cardinals μ such that $\lambda < \mu < \kappa$. Then $I \in U$,[9.36] and it is easy to show that for each $\mu \in I$, $j\mu = \mu$.

By the same token, $j\kappa = \kappa$, so jU is an ultrafilter on κ in M. We will show that $jU = U \cap M$. Since jU is in M and is an ultrafilter in the sense of M, it suffices to show that $jU \subseteq U$. To this end, suppose $X \in jU$. Let $f : \lambda \to U$ be such that $\pi[f]^* = X$. Let $Y = \bigcap_{\alpha<\lambda} f\alpha$. Then $Y \in U$ and $jY \subseteq X$. For all $\mu \in Y \cap I$, $\mu = j\mu \in j(Y \cap I)$. Hence, $X \supseteq jY \supseteq j(Y \cap I) \supseteq Y \cap I \in U$, so $X \in U$. Hence,[9.78]

$$M = L[jU] = L[U \cap M] = L[U],^{7.28.2}$$

as claimed. $\square$[9.77]

The following theorem of Silver extends the previous theorem to show that GCH holds in $L[U]$ for a measure ultrafilter U. As a relative consistency result it yields priority to a theorem of Jensen, who showed how to carry out an Easton-style product forcing to collapse a proper class of cardinals so as to make GCH true while retaining the measurability of a given cardinal. Like Gödel's original proof of GCH^L, however, Theorem 9.79 has the additional significance of the insight it provides into the structure of an inner model of a measurable cardinal.

(9.79) **Theorem** [GB] *Suppose U is a normal ultrafilter over κ. Then* $L[U] \models$ GCH.

Proof Again we will "work in $L[U]$" in GBC. Given (9.77), we have only to show that for any infinite cardinal $\lambda < \kappa$, $2^\lambda = \lambda^+$. Let $<^*$ be the canonical wellordering of $L[U]$, which is uniformly definable in models of the form $(L_\eta[U]; \in, U \cap L_\eta[U])$ for limit ordinals η. To prove that $2^\lambda = \lambda^+$ it suffices to show that the $<^*$-order type of $\mathcal{P}\lambda$ is λ^+. (Indeed, it would suffice merely to show that it is $< \lambda^{++}$, but in fact it is λ^+.) For this it suffices to show that for any $X \in \mathcal{P}\lambda$, the order type of the $<^*$-predecessors of X in $\mathcal{P}\lambda$ is $< \lambda^+$.

(9.80) *Suppose to the contrary that $X \in \mathcal{P}\lambda$ and the order type of the set $R = \{Y \in \mathcal{P}\lambda \mid Y <^* X\}$ of $<^*$-predecessors of X in $\mathcal{P}\lambda$ is λ^+.*

Let η be a limit ordinal $\geqslant \kappa$ such that $U \subseteq L_\eta[U]$ and $X \in L_\eta[U]$. Let $A = L_\eta[U]$. Note that $R \subseteq A$. Let ρ be a signature expanding s by the addition of unary predicates $\dot{U}$ and $\dot{R}$, a constant symbol $\dot{X}$, and for each $\gamma \in \lambda$, a constant symbol $\dot{\gamma}$. Note that $|\rho| = \lambda$. Let $\mathfrak{A}$ be the ρ-structure $(A; \in, U, R, X, \gamma)_{\gamma\in\lambda}$, with the obvious correspondences. Let F be a complete set of Skolem functions for $\mathfrak{A}$.[2.160] Note that $|F|$ may be taken to be λ. For each $f \in F$, let k_f be the arity of f.

For each $f \in F$ let $h_f : [\kappa]^{k_f} \to R$ be such that for each $s \in [\kappa]^{k_f}$, letting $\langle \alpha_0, \ldots, \alpha_{k_f{}^-}\rangle$ be the increasing enumeration of s,

$$h_f s = \begin{cases} f\langle \alpha_0, \ldots, \alpha_{k_f{}^-}\rangle & \text{if this is in } R \\ 0 & \text{otherwise.}^{25} \end{cases}$$

Since $|R| = \lambda^+ < \kappa$, (9.66) applies. For each $f \in F$, let $r_f \in R$ and $Z_f \in U$ be such that $h_f{}^{\to}[Z_f]^{k_f} = \{r_f\}$. Let $Z = \bigcap_{f\in F} Z_f$. Since $|F| = \lambda < \kappa$, $Z \in U$.

Let

$$B = \{f\langle \alpha_0, \ldots, \alpha_{k_f{}^-}\rangle \mid f \in F \wedge \alpha_0, \ldots, \alpha_{k_f{}^-} \in Z \wedge \alpha_0 < \cdots < \alpha_{k_f{}^-}\},$$

and let $\mathfrak{B}$ be the corresponding substructure of $\mathfrak{A}$. Then[2.162] $\mathfrak{B} \prec \mathfrak{A}$.[26] Note that $\mathfrak{B} = (B; \in, U \cap B, R \cap B, X, \gamma)_{\gamma\in\lambda}$. Clearly $R \cap B = R \cap \{r_f \mid f \in F\}$, so $|B \cap R| \leqslant \lambda$. Also, $\lambda \subseteq B$, and $Z \subseteq B$.

Let $\pi : (B; \in, U \cap B) \to (M; \in, W)$ be the transitive collapse. Then π is the identity on λ, and for every $Y \in B \cap \mathcal{P}\lambda$, $\pi Y = Y$. Hence $X \in M$ and $M \cap R = B \cap R$. Also $\pi\kappa = \kappa$.

(9.81) **Claim** *$W = U \cap M$.*

Proof Let $Z' = \{\alpha \in Z \mid \pi\alpha = \alpha\}$. Since π is a collapsing map, $\forall \alpha \in Z\ \pi\alpha \leqslant \alpha$, so π is regressive on $Z \backslash Z'$. Thus, since U is normal, if $Z \backslash Z' \in U$ then for some $\beta \in \kappa$, $\pi^{\leftarrow}\{\beta\} \cap Z \in U$, which is not the case, since π is injective. Thus, $Z' \in U$. Given $S \in M \cap \mathcal{P}\kappa$ we let $S' = \pi^{-1}S$. Then $S \in W \leftrightarrow S' \in U$, and $S \cap Z' = S' \cap Z'$. Thus, $S \in W \leftrightarrow S' \in U \leftrightarrow S' \cap Z' \in U \leftrightarrow S \cap Z' \in U \leftrightarrow S \in U$. $\square$[9.81]

By elementarity, $M = L_\zeta[W] = L_\zeta[U]$[27] for some limit ordinal ζ. Since $X \in M$, every $<^*$-predecessor of X is in M, i.e., $R \subseteq M$. Hence $R \subseteq B$, so $\lambda^+ = |R| = |B \cap R| \leqslant \lambda$; which contradiction invalidates (9.80). $\square$[9.79]

9.3.4 Indiscernibles

We have previously introduced the concept of indiscernibility in the setting of the Ehrenfeucht-Mostowski theorem,[9.63] deriving the existence of models with indiscernibles from Ramsey's theorem: $\forall m, n \in \omega\ \omega \to (\omega)^m_n$. Haim Gaifman observed that if κ is a measurable cardinal then by iterating the ultrapower construction one can obtain a proper class C of indiscernibles for L.

Using the terminology of Section 9.2.6, we begin with the observation that, for any $\alpha \leqslant \beta$, $i_{\alpha\beta} \upharpoonright L$ is an elementary embedding of L into L. Note that if $\gamma < \alpha$ then

[25] The choice of 0 for this case is arbitrary. It conveniently happens that $0 \in R$, but this is immaterial.

[26] Note that this is not a direct quotation of (2.162), since we apply the Skolem functions only to increasing sequences of ordinals in Z, not to all sequences; however, it follows from the closure of F under substitution[2.161.2] that this does not actually reduce the set of values obtained.

[27] By the level-specific version of (7.28.2).

$i_{\alpha\beta}\kappa_\gamma = \kappa_\gamma$, and $i_{\alpha\beta}\kappa_\alpha = \kappa_\beta$. Also, for any $n \in \omega$, $i_{\alpha\beta}\kappa_{\alpha+n} = \kappa_{\beta+n}$. With this information it is evident that for any increasing sequence $\langle \alpha_m \mid m < n \rangle$ of ordinals, the composition $i = i_{(\alpha_{n-2}+1)\alpha_{n-1}} \circ \cdots \circ i_{(\alpha_0+1)\alpha_1} \circ i_{0\alpha_0}$ has the property that for each $m < n$, $i\kappa_m = \kappa_{\alpha_m}$. Since i is elementary, for any s-formula ϕ with n free variables, $L \models \phi[\kappa_0, \ldots, \kappa_{n\text{-}}]$ iff $L \models \phi[\kappa_{\alpha_0}, \ldots, \kappa_{\alpha_{n\text{-}}}]$. It follows that $\{\kappa_\alpha \mid \alpha \in \mathrm{Ord}\}$ is a class of indiscernibles for L.

Letting x be the theory of $(L; \in, \kappa_\alpha)_{\alpha \in \mathrm{Ord}}$, we have the following theorem of Gaifman. We present this as a prelude to Silver's theory of indiscernibles and refrain from any further justification by Gaifman's methods.

(9.82) **Theorem** *Suppose there exists a measurable cardinal. Then there exists a closed unbounded class C of ordinals such that*

1. $\forall \alpha \in C\ L_\alpha \prec L$;
2. *every uncountable cardinal is a limit point of C;*
3. *there exists $x \subseteq \omega$ such that C is definable over $L[x]$.*

Note that (assuming the existence of a measurable cardinal) every uncountable cardinal λ is inaccessible in L (since $L_\lambda \prec L$). Note also that arbitrarily large cardinals in L are "collapsed" in $L[x]$, where x is as in (9.82.3).

Silver realized that the existence of indiscernibles for L is the essential implication of large cardinal hypotheses regarding L, with the same holding *mutatis mutandis* for other similarly structured inner models. Silver began with a variation on the Ehrenfeucht-Mostowski construction.[9.63] Recall that the purpose of that construction was to obtain models of finite subtheories of a theory T of indiscernibles in order to infer the existence of a model of T. The purpose of Silver's construction is to obtain indiscernibles for a pre-existing structure, viz., L or an initial segment of it. For the remainder of this discussion we will be concerned exclusively with ordinal indiscernibles. For these the order is always the natural order, so it is not necessary to specify it.

The following theorem is the starting point.

(9.83) **Theorem** [ZF] *Suppose α is a limit ordinal $\leqslant \kappa$. Then $\kappa \to (\alpha)_2^{<\omega}$ iff for every structure $\mathfrak{S}$ with a countable signature such that $\kappa \subseteq |\mathfrak{S}|$ there exists $X \subseteq \kappa$ of order type α such that $(X; <)$ is a set of indiscernibles for $|\mathfrak{S}|$.*

Proof Straightforward. $\square$[9.83]

A key idea complementing the indiscernibility in L of the Silver indiscernibles is the *definability* in L of every element of L from them. We formulate this in terms of *canonical Skolem terms*. This is a bit of a misnomer, as these terms are merely operation indices. What makes them canonical Skolem are the definitions by which they are introduced. Recall[7.29] the formula ϕ_0 that defines the canonical wellordering of L in any limit L_α.

(9.84) **Definition** [S] *Suppose ϕ is an* s*-formula with $n+1$ free variables. Let $\langle v_0, v_1, \ldots, v_n \rangle$ enumerate* Free ϕ *in the standard ordering of $\mathcal{V}$, and let u be the first variable not occurring in ϕ. The* canonical Skolem term τ_ϕ *is introduced with the definition*

$$\tau_\phi(v_1, \ldots, v_n) = v_0 \leftrightarrow \Big((\forall v_0\ \neg\, \phi(v_0, v_1, \ldots, v_n) \wedge \forall u\ u \notin v_0) \\ \vee \big(\phi(v_0, v_1, \ldots, v_n) \wedge \forall u\, (\phi_0(u, v_0) \to \neg\, \phi(u, v_1, \ldots, v_n))\big)\Big).$$

For completeness, if ϕ is an s*-sentence, let τ_ϕ be introduced with the definition*

$$\tau_\phi = \mathrm{v}_0 \leftrightarrow \forall \mathrm{v}_1\ \mathrm{v}_1 \notin \mathrm{v}_0.$$

Let sk *be the expansion of* s *by the addition of the canonical Skolem terms.*

Suppose $\mathfrak{M} = (M; E)$ is an s-structure such that ϕ_0 defines a linear ordering of M and for every s-formula ϕ,

$$\mathfrak{M} \models \forall v_1, \ldots, v_n \Big(\exists v_0\ \phi(v_0, v_1, \ldots, v_n) \\ \to \exists v_0 \left(\phi(v_0, v_1, \ldots, v_n) \wedge \forall u \left(\phi_0(u, v_0) \to \neg\, \phi(u, v_1, \ldots, v_n)\right)\right)\Big).$$

Then $\mathfrak{M}^* \overset{\text{def}}{=} (M; E, \tau_\phi^{\mathfrak{M}^*})_{\phi \in \mathcal{F}^{\mathsf{s}}}$[28] is the expansion of $(M; E)$ to an sk-structure with the obvious assignment of values to $\tau_\phi^{\mathfrak{M}^*}$. If $\mathfrak{M}$ happens to be $(L_\alpha; \in)$ for a limit ordinal α, then everything has the correct set-theoretical meaning, but to define $\mathfrak{M}^*$ it is sufficient that $\mathfrak{M}$ be elementarily equivalent to (i.e., have the same elementary, i.e., first-order, theory as) such a model (and this is not necessary).

Clearly $\{\tau_\phi^{\mathfrak{M}^*} \mid \phi \in \mathcal{F}^{\mathsf{s}}\}$ is complete set of Skolem functions for $\mathfrak{M}$.

Definition [ZF] *For $\mathfrak{M} = (M; E)$ and $\mathfrak{M}^*$ as above, and for $X \subseteq M$, the* Skolem hull of X in $\mathfrak{M}$ $\overset{\text{def}}{=} \mathrm{H}^{\mathfrak{M}} X \overset{\text{def}}{=}$ *the set of values of the canonical Skolem terms interpreted in $\mathfrak{M}$ at arguments in X.*

By the Tarski-Vaught criterion, the Skolem hull of any $X \subseteq M$ is the smallest elementary substructure of $\mathfrak{M}$ including X. Obviously, for every x in the Skolem hull of X, $\{x\}$ is definable over $\mathfrak{M}$ from parameters in X; and conversely, every such x is in the Skolem hull of X. Note that this provides an absolute notion of Skolem hull, applicable in any structure $\mathfrak{S}$ with an $\mathfrak{S}$-definable linear ordering that is a wellordering for $\mathfrak{S}$-definable classes. The specific choice of Skolem terms is not relevant to this consideration.

Let em be the signature s expanded by the addition of constant symbols $\{c_n \mid n \in \omega\}$.

(9.85) **Definition** [ZF] *An* EM-set $\overset{\text{def}}{=}$ *the theory of an* em*-structure $(L_\alpha; \in, x_n)_{n \in \omega}$ for some limit ordinal α and indiscernibles $x_0 < x_1 < \cdots$ for $(L_\alpha; \in)$.*

(9.86) **Theorem** [ZF] *Suppose Σ is an* EM*-set and α is an infinite ordinal. Let Σ^- be the* s*-reduction of Σ (i.e., the set of* s*-sentences in Σ).*

1. *There exists an* s*-model $\mathfrak{M}$ of Σ^- and a set X of ordinal indiscernibles for $\mathfrak{M}$ of order type α, such that*
   1. *for any* s*-formula ϕ with n free variables and any $x_0 <^{\mathfrak{M}} \cdots <^{\mathfrak{M}} x_{n^-}$ in X,*
   $$\mathfrak{M} \models \phi[x_0, \ldots, x_{n^-}] \leftrightarrow \phi(c_0, \ldots, c_{n^-}) \in \Sigma.$$
   2. *$M = \mathrm{H}^{\mathfrak{M}} X$.*
2. *If $(M; E)$ and $(M'; E')$ are two such structures with respective sets X and X' of ordinal indiscernibles (both of order type α) then $(M; E, X) \cong (M'; E', X')$.*

[28] Recall that $\mathcal{F}^{\mathsf{s}}$ is the set of s-formulas.

Proof The existence of $(M;E)$ and $X \subseteq M$ satisfying (9.86.1.1) is proved by a simple compactness argument. Let $(L_\gamma;\in,x_n)_{n\in\omega}$ be an em-structure with theory Σ. Let em_α be the expansion of em by the addition of constants c_β $(\omega \leqslant \beta < \alpha)$. $(\mathsf{em}_\omega = \mathsf{em}.)$ Let Σ_α be the em_α-theory obtained by adding to Σ all sentences $\phi(c_{\beta_0},\ldots,c_{\beta_{n^-}})$, where ϕ is an s-formula with n free variables, $\beta_0 < \cdots < \beta_{n^-} < \alpha$, and $\phi(c_0,\ldots,c_{n^-}) \in \Sigma$. Any finite $S \subseteq \Sigma_\alpha$ involves only a finite set of constants, say $C = \{c_{\beta_n} \mid n < N\}$, where $\beta_0 < \cdots < \beta_{N^-} < \alpha$. Let em_C be the reduction of em_α by the removal of all constants not in C. Let $\mathfrak{M}$ be the expansion of $(L_\gamma;\in)$ to an em_C-structure by letting $c^{\mathfrak{M}}_{\beta_n} = x_n$ for each $n \in N$. Then $\mathfrak{M} \models S$. By the compactness theorem, Σ_α has a model $\mathfrak{M}$, which, letting $X = \{c^{\mathfrak{M}}_\beta \mid \beta < \alpha\}$, is easily seen to satisfy the requirements of (9.86.1.1).

To obtain (9.86.1.2) we let $M' = \mathrm{H}^{(M;E)}\,X$, and let $(M';E')$ be the corresponding (elementary) substructure. Then $\mathrm{H}^{(M';E')}\,X = M'$.

Given $(M;E), X$ and $(M';E'), X'$ satisfying (9.86.1.1, 2) we extend the (unique) order-isomorphism of X and X' to an isomorphism of $(M;E)$ and $(M';E')$ using the fact that in both models every element is the value of a Skolem term applied to indiscernibles. $\square^{9.86}$

If the structures corresponding to an EM-set Σ and an ordinal α are wellfounded then there is a unique structure $(M;\in)$ of this isomorphism type, where M is a transitive set, which must by definition[9.85] be of the form $(L_\delta;\in)$ for a limit ordinal δ.

(9.87) *We refer to a model constructed as above from an* EM-*set* Σ *and ordinal* α *as* $\mathfrak{M}(\Sigma,\alpha)$ *with the understanding that it is in general only the isomorphism type of* $\mathfrak{M}(\Sigma,\alpha)$ *that is defined. If it is wellfounded then we define* $\mathfrak{M}(\Sigma,\alpha)$ *to be specifically the transitive set representative of its type, i.e., the appropriate* L_δ*, and we let* $I(\Sigma,\alpha)$ *be the set of ordinal indiscernibles in* L_δ*.*

(9.88) **Theorem** [ZF] *Suppose* Σ *is an* EM-*set. If* $\mathfrak{M}(\Sigma,\alpha)$ *is wellfounded for every* $\alpha < \omega_1$*, then* $\mathfrak{M}(\Sigma,\alpha)$ *is wellfounded for every* α*.*

Proof A straightforward descending chain argument. Choice is not needed, since $(M;E) = \mathfrak{M}(\Sigma,\alpha)$ may be wellordered by consideration of the representation of its elements as Skolem terms applied to n-sequences from α, regardless of whether E is wellfounded. Let $X \subseteq \mathrm{Ord}^{(M;E)}$ be the set of indiscernibles (of order type α). If $a_0\,E^{-1}\,a_1\,E^{-1}\,\cdots$ is an descending ω-sequence in $(M;E)$ then there is a countable $X' \subseteq X$ such that, letting $M' = \mathrm{H}^{(M;E)}\,X'$, $\forall n \in \omega\ a_n \in M'$. Letting $E' = E \cap M' \times M'$, $(M';E') \cong \mathfrak{M}(\Sigma,\alpha')$, where α' is the order type of X'. Since X' is countable, $\alpha' < \omega_1$. $\square^{9.88}$

(9.89) **Theorem** [ZF] *Suppose* $\kappa \to (\omega_1)^{<\omega}_2$*. Then there exists an* EM-*set* Σ *with the following properties.*

1. *For all ordinals* α*,* $\mathfrak{M}(\Sigma,\alpha)$ *is wellfounded.*
2. *For every* m*-ary Skolem term* τ*,* Σ *contains*
$$\mathrm{Ord}\,\tau(c_0,\ldots,c_{m^-}) \rightarrow \tau(c_0,\ldots,c_{m^-}) < c_m.$$
3. *For every* $(m+n)$*-ary Skolem term* τ*,* Σ *contains*
$$\tau(c_0,\ldots,c_{m+n^-}) < c_m \\ \rightarrow \tau(c_0,\ldots,c_{m+n^-}) = \tau(c_0,\ldots,c_{m^-},c_{m+n},c_{m+n+1},\ldots,c_{m+2n^-}).$$

Note that by indiscernibility the upper constants may be arbitrary, i.e., for any $m \leqslant k_0 < \cdots < k_{n^-}$,

$$\tau(c_0, \dots, c_{m+n^-}) < c_m \\ \rightarrow \tau(c_0, \dots, c_{m+n^-}) = \tau(c_0, \dots, c_{m^-}, c_{k_0}, \dots, c_{k_{n^-}}).$$

Proof By (9.83) there exists a set of indiscernibles in L_κ of order type ω_1. Let δ be least such that L_δ has a set of indiscernibles I of length ω_1, and let I be a set of indiscernibles in L_δ with the least possible ωth element. Let Σ be the corresponding EM-set.

By virtue of (9.88), Σ satisfies (9.89.1). To show that Σ satisfies (9.89.2), suppose to the contrary that τ is a Skolem term and

$$c_m \leqslant \tau(c_0, \dots, c_{m^-})$$

is in Σ. Note that by indiscernibility $c_{m'} \leqslant \tau(c_0, \dots, c_{m^-})$ is also in Σ for any $m' > m$. It follows that if Σ says that $\tau(c_0, \dots, c_{m^-})$ is a successor ordinal then we may let τ' be the Skolem term for the greatest limit ordinal below τ, and $c_m \leqslant \tau'(c_0, \dots, c_{m^-})$ will also be in Σ. Thus, we may suppose without loss of generality that $\mathbf{Lim}\, \tau(c_0, \dots, c_{m^-})$ is in Σ.

Let $\alpha_0 < \cdots < \alpha_{m^-}$ be the first m elements of I, and let $\nu = \tau^{L_\delta}(\alpha_0, \dots, \alpha_{m^-})$. Then ν is a limit ordinal and $I \subseteq \nu$. Let $I' = I \backslash \{\alpha_0, \dots, \alpha_{m^-}\}$. Now suppose $\beta_0 < \cdots < \beta_{n^-}$ and $\gamma_0 < \cdots < \gamma_{n^-}$ in I', and suppose ϕ is an s-formula ϕ with n free variables. Let $\phi'(u_0, \dots, u_{m^-}, v_0, \dots, v_{n^-})$ be

$$\ulcorner L_{\tau(u_0, \dots, u_{m^-})} \models \phi[v_0, \dots, v_{n^-}] \urcorner.$$

Then

$$\begin{aligned} L_\nu \models \phi[\beta_0, \dots, \beta_{n^-}] &\leftrightarrow L_\delta \models \phi'[\alpha_0, \dots, \alpha_{m^-}, \beta_0, \dots, \beta_{n^-}] \\ &\leftrightarrow L_\delta \models \phi'[\alpha_0, \dots, \alpha_{m^-}, \gamma_0, \dots, \gamma_{n^-}] \\ &\leftrightarrow L_\nu \models \phi[\gamma_0, \dots, \gamma_{n^-}]. \end{aligned}$$

Thus I' is a set of indiscernibles for L_ν. But the order type of I' is ω_1 and $\nu < \delta$, contradicting the minimality of δ.

To verify (9.89.3), we may suppose

$$\tau(c_0, \dots, c_{m+n^-}) < c_m \in \Sigma; \tag{9.90}$$

otherwise, it is satisfied trivially, since Σ is a complete theory. We may also suppose $n > 0$. As before, let $\alpha_0 < \cdots < \alpha_{m^-}$ be the first m elements of I, and let $I' = I \backslash \{\alpha_0, \dots, \alpha_{m^-}\}$. Let $\langle \beta_\gamma \mid \gamma < \omega_1 \rangle$ be the increasing enumeration of I'. Keeping in mind that any ordinal γ is uniquely of the form $\eta + k$, where η is not a successor ordinal and $k \in \omega$, for each ordinal $\gamma = \eta + k$ in this form with $\gamma < \omega_1$, let

$$\nu_\gamma = \ulcorner \tau(\alpha_0, \dots, \alpha_{m^-}, \beta_{\eta+kn}, \beta_{\eta+kn+1}, \dots, \beta_{\eta+kn+n^-}) \urcorner^{L_\delta}.$$

If

$$\tau(c_0, \dots, c_{m+n^-}) = \tau(c_0, \dots, c_{m^-}, c_{m+n}, c_{m+n+1}, \dots, c_{m+2n^-})$$

is in Σ, of course, $\langle \nu_\gamma \mid \gamma < \omega_1 \rangle$ is constant. If not then either

$$\begin{aligned} &\tau(c_0, \dots, c_{m+n^-}) > \tau(c_0, \dots, c_{m^-}, c_{m+n}, c_{m+n+1}, \dots, c_{m+2n^-}) \\ \text{or } &\tau(c_0, \dots, c_{m+n^-}) < \tau(c_0, \dots, c_{m^-}, c_{m+n}, c_{m+n+1}, \dots, c_{m+2n^-}) \end{aligned}$$

is in Σ. In the former case, $\langle \nu_\gamma \mid \gamma < \omega_1 \rangle$ is strictly decreasing, which is impossible. In the latter case, $\langle \nu_\gamma \mid \gamma < \omega_1 \rangle$ is strictly increasing, and it is easy to see that $\{\nu_\gamma \mid \gamma < \omega_1\}$ is a set of indiscernibles for L_δ. But[9.90]

$$\nu_\omega = \ulcorner \tau(\alpha_0, \ldots, \alpha_{m^\cdot}, \beta_\omega, \ldots, \beta_{\omega+n^\cdot}) \urcorner^{L_\delta} < \beta_\omega,$$

which is the ωth element of I. This contradicts the choice of I as a set of indiscernibles in L_δ with the least ωth element. $\square$[9.89]

In the original publication an EM-set was defined as *remarkable* $\overset{\text{def}}{\iff}$ it satisfies (9.89.3).

Definition [ZF] *For terminological convenience we will say that Σ is* remarkable $\overset{\text{def}}{\iff}$ *Σ is an* EM-*set satisfying (9.89.1–3). For the nonce, if Σ is remarkable and α is an ordinal then $\iota_\gamma^{\Sigma,\alpha} \overset{\text{def}}{=}$ the γth element of the set I of indiscernibles in $\mathfrak{M}(\Sigma, \alpha)$, which is well defined by virtue of (9.86.2).*

The aptness of the appellation ‘remarkable’ is illustrated by the following series of theorems and definitions.

(9.91) **Theorem** [ZF] *Suppose Σ is remarkable, α is a limit ordinal, and $\alpha < \beta$.*

1. $\iota_\alpha^{\Sigma,\beta} = \sup_{\gamma<\alpha} \iota_\gamma^{\Sigma,\beta}$.
2. $\mathfrak{M}(\Sigma, \alpha) = \mathrm{H}^{\mathfrak{M}(\Sigma,\beta)}(I(\Sigma, \beta) \cap \iota_\alpha^{\Sigma,\beta}) = L_{\iota_\alpha^{\Sigma,\beta}}$.[29]
3. $I(\Sigma, \alpha) = I(\Sigma, \beta) \cap \iota_\alpha^{\Sigma,\beta}$.

Proof Let $L_\delta = \mathfrak{M}(\Sigma, \beta)$.[9.87] Then[9.86.1.2] $L_\delta = H^{L_\delta}(I(\Sigma, \beta))$. By remarkability[9.89.3] and the fact that α is limit, $L_{\iota_\alpha^{\Sigma,\beta}} \subseteq H^{L_\delta}(I(\Sigma, \beta) \cap \iota_\alpha^{\Sigma,\beta})$, and from (9.89.2) it follows that $H^{L_\delta}(I(\Sigma, \beta) \cap \iota_\alpha^{\Sigma,\beta}) \subseteq L_\nu$, where $\nu = \sup_{\gamma<\alpha} \iota_\gamma^{\Sigma,\beta} \leqslant \iota_\alpha^{\Sigma,\beta}$.

Thus, $\iota_\alpha^{\Sigma,\beta} = \nu = \sup_{\gamma<\alpha} \iota_\gamma^{\Sigma,\beta}$, and $L_\nu = \mathrm{H}^{\mathfrak{M}(\Sigma,\beta)}(I(\Sigma, \beta) \cap \nu)$. As a Skolem hull in L_δ, L_ν is automatically an elementary substructure of L_δ, and since $\iota_\gamma^{\Sigma,\beta}$ $(\gamma \leqslant \alpha)$ are indiscernible in L_δ, they are indiscernible in L_ν. Hence, $\mathfrak{M}(\Sigma, \alpha) = L_\nu$ and $I(\Sigma, \alpha) = I(\Sigma, \beta) \cap \nu$. $\square$[9.91]

Definition [GB] *For the nonce, with (9.91) as justification, if Σ is remarkable then*

1. $\iota_\gamma^\Sigma \overset{\text{def}}{=} \iota_\gamma^{\Sigma,\alpha}$ *for any (equivalently, for all) limit $\alpha > \gamma$; and*
2. $I(\Sigma) \overset{\text{def}}{=} \{\iota_\gamma^\Sigma \mid \gamma \in \mathrm{Ord}\}$.

(9.92) **Theorem** [GB]

1. *Suppose Σ is remarkable.*
   1. *I^Σ is closed and unbounded in* Ord.
   2. *For all $\alpha < \beta$, $L_{\iota_\alpha^\Sigma} \prec L_{\iota_\beta^\Sigma}$.*
   3. *For any uncountable cardinal κ, $I^\Sigma \cap \kappa$ has order type κ. Hence, $\iota_\kappa^\Sigma = \kappa$.*
2. *There exists at most one remarkable* EM-*set.*

[29] Remember[9.87] that $\mathfrak{M}(\Sigma, \nu)$ is an initial segment of L for any ordinal ν.

Proof (9.92.1.1) follows from (9.91.1).

Given $\alpha < \beta$, let $\gamma > \beta$ be a limit ordinal.[30] Given an s-formula ϕ with n free variables, and $x_0, \ldots, x_{n^-} \in L_{\iota^\Sigma_\alpha}$, let S be a finite set of indiscernibles in I^Σ_γ above ι^Σ_β. Let $\tau_0, \ldots, \tau_{n^-}$ be Skolem terms which, evaluated in $L_{\iota^\Sigma_\gamma}$ at indiscernibles in $I^\Sigma_\alpha \cup S$, give $x_0, \ldots, x_{n^-}$. Then by the indiscernibility of $I^\Sigma_\alpha \cup \{\iota^\Sigma_\alpha, \iota^\Sigma_\beta\} \cup S$ in $L_{\iota^\Sigma_\gamma}$, $L_{\iota^\Sigma_\gamma}$ believes that

$$L_{\iota^\Sigma_\alpha} \models \phi[x_0, \ldots, x_{n^-}] \leftrightarrow L_{\iota^\Sigma_\beta} \models \phi[x_0, \ldots, x_{n^-}],$$

so this is true by the absoluteness of satisfaction.

(9.92.1.3) follows easily from the fact that $|\mathfrak{M}(\Sigma, \alpha)| = |\alpha|$ for any infinite α.

By virtue of (9.92.1), all uncountable cardinals are indiscernibles, so any remarkable EM-set is the theory of the structure $(L_{\omega_\omega}; \in, \omega_{n+1})_{n\in\omega}$. $\square^{9.92}$

(9.92.2) is perhaps the most remarkable thing of all. It certainly concentrates the mind wonderfully. Note that, as a theory in a countable signature, the unique remarkable EM-set is a countably infinitary object, i.e., a *real* in the usual sense of that term in set theory. Second, given any real x—regarded, say, as a subset of ω—the entire preceding analysis may be undertaken with $L[x]$ in place of L, with the conclusion that there exists a unique remarkable EM-set relative to x, with a canonical class of indiscernibles in $L[x]$. Letting $x^\sharp$ be the unique EM-set relative to x, $x^\sharp$ is transcendent over x from the standpoint of constructibility in much the same way that the jump[4.103] x' of x is transcendent over x from the standpoint of computability. Let $x \leqslant_c y \overset{\text{def}}{\iff} x \in L[y]$. The relation $\leqslant_c$ of *relative constructibility* is analogous to the relation $\leqslant_T$ of relative computability (Turing reducibility). The equivalence classes of $\leqslant_c$ are the *constructibility degrees*, analogous to computability degrees, i.e., Turing degrees. $x \mapsto x^\sharp$ is clearly well defined on constructibility degrees, just as the jump operation is well defined on Turing degrees.

Definition [GB] *Suppose $x \subseteq \omega$.*

1. *If there exists an* EM-*set Σ satisfying (9.89) relative to x then x-*sharp $\overset{\text{def}}{=} x^\sharp \overset{\text{def}}{=} \Sigma$, *coded (in some uniform way) as a subset of ω.*
2. *If $x^\sharp$ exists, I^x is the class of x-indiscernibles, and ι^x_α is the αth element of I^x.*
3. *If $x^\sharp$ exists for all $x \subseteq \omega$ then $I^* \overset{\text{def}}{=} \bigcap_{x\subseteq\omega} I^x$, and for all $\alpha \in \mathrm{Ord}$, u_α is the αth element of I^*.*
4. *An ordinal ν is a* uniform indiscernible $\overset{\text{def}}{\iff}$ *$\nu \in I^*$ iff $\nu = u_\alpha$ for some α.*

We may summarize Silver's analysis of indiscernibles as follows.

Theorem [GB] *Suppose $x \subseteq \omega$.*

1. *$x^\sharp$ exists iff for some ordinal δ, $L_\delta[x]$ has an uncountable set of indiscernibles.*
2. *$x^\sharp$ exists if there exists κ such that $\kappa \to (\omega_1)^{<\omega}_2$.*
3. *If $x^\sharp$ exists then*
   1. *there exists a unique class I^x of indiscernibles for $L[x]$ such that I^x is closed and unbounded in* Ord *and $H^{L[x]}(I^x) = L[x]$;*

[30] For limit α, (9.92.1.2) follows from the fact that $L_{\iota^\Sigma_\alpha} = \mathfrak{M}(\Sigma, \alpha) = H^{L_{\iota^\Sigma_\beta}}(I^\Sigma_\alpha) \prec L_{\iota^\Sigma_\beta}$, but this argument doesn't generalize.

2. *if* $\alpha \leqslant \beta$ *then* $L_{\iota^x_\alpha}[x] \prec L_{\iota^x_\beta}[x]$;
3. *for every uncountable cardinal* κ, $I^x \cap \kappa$ *has order type* κ *and* $\iota^x_\kappa = \kappa$; *and*
4. *if* $\operatorname{Lim}\alpha$ *then* $H^{L_{\iota^x_\alpha}[x]}\{\iota^x_\gamma \mid \gamma < \alpha\} = L_{\iota^x_\alpha}[x]$.

Note that if $x^\sharp$ exists for all $x \subseteq \omega$ then every uncountable cardinal is a uniform indiscernible, and $u_0 = \omega_1$. The identity of the other uniform indiscernibles is obviously a very interesting question.

Just as measurability of a cardinal κ may be characterized in terms of elementary embeddings of V into a transitive class M, the existence of $0^\sharp$ may be characterized in terms of elementary embeddings of L into L (the only transitive model of ZF included in L). In one direction this is immediate. For suppose $I = I^0$ is the canonical class of L-indiscernibles and $j : I \to I$ is an order-preserving injection. Extend j to an embedding of L into L by letting

$$j(\tau(i_0, \ldots, i_{n^-})) = \tau(ji_0, \ldots, ji_{n^-})$$

for any Skolem term τ with n arguments and any increasing sequence $\langle i_m \mid m < n\rangle$ in I. It is easy to show that j is an elementary embedding of L into L.

This construction yields an easy proof of the following theorem, which states—perhaps unsurprisingly—that indiscernibles are indescribable.

(9.93) **Theorem** [GB] *Assume* $0^\sharp$ *exists. Then every indiscernible is totally indescribable in the sense of* L.

Proof By indiscernibility, it suffices to show that $\ulcorner\iota_0$ is totally indescribable$\urcorner^L$. Let $j : L \to L$ be the elementary embedding derived as above from a map of I into I that moves ι_0. It is easy to see using (9.89.3) (with $m = 0$) that ι_0 is the critical point of j. The argument used to prove (9.70) shows that ι_0 is totally indescribable in L. $\square$ 9.93

Note that this theorem places the consistency strength of $\ulcorner 0^\sharp$ exists$\urcorner$ between that of $\ulcorner$there exists a totally indescribable cardinal$\urcorner$ and $\ulcorner$there exists a cardinal κ such that $\kappa \to (\omega_1)_2^{<\omega}$.

The following theorem of Kunen, provides a converse to the above inference of the existence of a nontrivial elementary embedding of L into L from the existence of indiscernibles, along with an equivalent statement in terms of ultrafilters. By way of orientation, if M is a transitive model of ZF (or some adequate fragment of ZF, such as ZF^-), an M-ultrafilter over an ordinal $\kappa \stackrel{\text{def}}{=}$ an ultrafilter U on the boolean algebra $\ulcorner\mathcal{P}[\kappa]\urcorner^M$. Note that U need not be in M.

(9.94) **Theorem** [GB] *The following are equivalent.*

1. *There exists a nontrivial (not the identity) elementary embedding of* L *into* L.
2. *There exists a nontrivial elementary embedding of* L_α *into* L_β *with critical point* $< |\alpha|$.
3. *There exists an* L*-ultrafilter* U *over an ordinal* κ *such that* ${}^\kappa L/U$ *is wellfounded.*
4. $0^\sharp$ *exists.*

(9.94.2) is basically a way of stating (9.94.1) without quantification over proper classes, so that it makes sense in the context of ZF. We omit the proof of this theorem.

The perceived minimality of $0^\sharp$ as an object that is intrinsically outside the realm of constructibility led Solovay to formulate the *genericity conjecture*:

If $x \subseteq \mathrm{Ord}$ *and* $0^\sharp \notin L[x]$, *then* x *is generic over* L.

Here 'generic over L' is intended to mean that there exists a partial order $\mathbb{P} \in L$ and an L-generic filter G on $\mathbb{P}$ such that $x \in L[G]$. Since forcing with a partial order $\mathbb{P}$ preserves cardinals $\geqslant |\mathbb{P}|$, $0^\sharp$ is not generic over L.[31]

This conjecture was refuted by Jensen, who showed[8.167] how to encode the universe by a real using a proper class forcing. If $0^\sharp$ does not exist initially then $0^\sharp$ does not exist in the extension. Applying Jensen's method to a model obtained by Easton forcing in which $2^\kappa > \kappa^+$ for every regular cardinal κ, we obtain $x \subseteq \omega$ such that $x \notin L[y]$ for any set-generic y,[32] and $0^\sharp \notin L[x]$. Moreover, if $0^\sharp$ exists then there exists $x <_c 0^\sharp$ such that x is not set-generic over L.

We conclude this section with Jensen's famous *covering theorem*, which gives what is perhaps the subtlest known criterion for the existence of $0^\sharp$. The proof is beyond the scope of this book.

(9.95) **Theorem** [ZF] $0^\sharp$ *exists iff there exists an uncountable* $X \subseteq \mathrm{Ord}$ *such that there is no* $Y \in L$ *such that* $X \subseteq Y$ *and* $|Y| = |X|$.

9.4 Ideals over cardinals

9.4.1 Saturation

We have previously discussed ideals on boolean algebras.[3.164] When the algebra is $\mathcal{P}S$ for some set S, we say that the ideal is *over* S.[3.171] See Sections 3.10.3 and 3.10.4 for relevant definitions. We have also discussed the saturation of boolean algebras in general terms. We now focus on boolean algebras of the form $\mathcal{P}\kappa/I$, where I is an ideal over a cardinal κ. In this context we refer to saturation as a property of I.

Definition [ZFC] *Suppose* I *is an ideal over a set* S, *and* κ *is a cardinal.*

1. $A \subseteq \mathcal{P}S$ *is an* I-antichain $\overset{\text{def}}{\iff} A \subseteq I^{+}$[3.174.2] *and for every* $X, Y \in A$, *if* $X \neq Y$ *then* $X \cap Y \in I$ *(the members of* A *are nonsmall and pairwise almost disjoint).*
2. I *is* κ-saturated $\overset{\text{def}}{\iff} \mathcal{P}S/I$ *is* κ*-saturated,*[8.121] *i.e., every* I*-antichain is smaller than* κ.
3. *The* saturation of I $\overset{\text{def}}{=} \operatorname{sat} I \overset{\text{def}}{=}$ *the least* κ *such that* I *is* κ*-saturated.*

Note that an ideal is maximal iff it is 2-saturated.[33] It is easy to show that if I is ω-saturated then it is n-saturated for some finite n.[34] Thus, $\operatorname{sat} I$ cannot be ω.

[31] Every L-indiscernible is a cardinal in L (by indiscernibility, since every uncountable cardinal is an L-indiscernible). Thus, there are arbitrarily large cardinals in L that are not cardinals in $L[0^\sharp]$.

[32] $\mathbb{P}$-forcing does not increase 2^κ for any regular cardinal $\kappa \geqslant |\mathbb{P}|$.

[33] Any discussion of ideals or filters may be framed in terms of either ideals or filters or both. For maximal such objects the preferred terminology is that of filters, and a maximal filter is called an *ultrafilter*. This preference is probably due to the fact that the definition of ultraproducts in terms of ultrafilters rather than maximal ideals avoids a negation (or complementation) operation. On the other hand, the quotient operation on algebras is more economically described in terms of ideals than filters. The preference for ideals over filters in this and other contexts may also derive from their use in ring theory and the conventional correspondence of join with addition and meet with multiplication.

[34] Suppose to the contrary that I is ω-saturated but not n-saturated for any finite n. Let A_0 be a maximal antichain. Then $|A_0| < \omega$. Let $n = |A_0|$. Let A' be any antichain of size $n' > n$,

It is also easy to show that if $\operatorname{sat} I$ is infinite it is regular. Thus, $\operatorname{sat} I$ is finite or uncountable and regular.

Ideals over countable cardinals are for the present purpose quite simple. Any ideal over a finite set is principal. Assuming AC, every ideal over ω can be extended to a 2-saturated (i.e., maximal) ideal.[35] Hence, there exists a maximal nonprincipal ideal over ω. Since every ideal is ω-complete, we have a nonprincipal ω-complete 2-saturated ideal over ω.

For the purpose of large-cardinal studies, we are concerned specifically with ideals (over cardinals κ) that are nonprincipal and κ-complete. As just discussed, these are only of interest if κ is uncountable. In the interest of terminological brevity we make the following definition.

(9.96) **Definition** [ZFC] *Suppose I is an ideal over a cardinal κ. I is* nontrivial $\overset{\text{def}}{\Longleftrightarrow}$ *κ is uncountable and I is nonprincipal and κ-complete.*[36]

It is easy to see that if I is a nontrivial ideal over κ then κ is regular.

(9.97) **Definition** [ZFC] *Suppose I is an ideal over a set S. $A \subseteq S$ is an* atom $\overset{\text{def}}{\Longleftrightarrow}$ *$A \in I^+$ and for every $X \subseteq A$, either $X \in I$ or $A \backslash X \in I$.*[9.12.1]

If I is a nontrivial ideal and $\operatorname{sat} I \leqslant \omega$, then every antichain in I^+ is finite, so every element of I^+ includes an atom; otherwise, by successive fission we could produce an infinite antichain. If A is an atom for I then $\{X \subseteq \kappa \mid X \cap A \in I^+\}$ is a κ-complete nonprincipal ultrafilter over κ, so κ is measurable. As noted above, if $\operatorname{sat} I$ is infinite then $\operatorname{sat} I$ is regular. Obviously, $\operatorname{sat} I \leqslant (2^\kappa)^+$. To summarize:

(9.98) **Theorem** [ZFC] *Suppose I is a nontrivial ideal over a cardinal κ.*

1. $\operatorname{sat} I \leqslant (2^\kappa)^+$.
2. *If* $\operatorname{sat} I \leqslant \omega$ *then κ is measurable.*
3. *If* $\operatorname{sat} I \geqslant \omega$ *then* $\operatorname{sat} I$ *is a regular cardinal.*

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let $B = \{a \cap a' \mid a \in A_0 \wedge a' \in A'\}$, and let $A_1 = B \cap I^+$. Then A_1 is an antichain. Since A_0 is maximal, for every $a' \in A'$ there exists $a \in A_0$ such that $a \cap a' \in I^+$, so there exists $b \in A_1$ (e.g., $a \cap a'$) such that $b \subseteq a'$. Hence, $|A_1| \geqslant n' > n$, and A_1 is an antichain such that every element of A_1 is included in an element of A_0. Continue in this fashion to construct antichains $A_0, A_1, \ldots$ of progressively larger cardinality such that each element of each antichain is included in some element of each earlier antichain. By a process similar to that used in the proof of König's lemma, we can now construct sequences $\langle n_i \mid i \in \omega\rangle$, $\langle a_i \mid i \in \omega\rangle$, and $\langle b_i \mid i \in \omega\rangle$ such that

1. $n_0 < n_1 < \ldots$;
2. $a_i, b_i \in A_{n_i}$ and $a_i \neq b_i$ (so $a_i \cap b_i \in I$); and
3. $i < j \rightarrow a_i \supseteq a_j, b_j$.

Then $\{b_i \mid i \in \omega\}$ is an infinite antichain.

[35] Suppose I is an ideal over ω. Let $\langle X_\alpha \mid \alpha < \kappa\rangle$ enumerate $\mathcal{P}\omega$. Define ideals I_α, $\alpha < \kappa$, such that

1. $I_0 = I$;
2. if $\operatorname{Lim}\alpha$ then $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$; and
3. if $\alpha = \beta + 1$ then
   1. if $(\omega \backslash X_\beta) \in I_\beta$ then $I_\alpha = I_\beta$; otherwise,
   2. I_α is the smallest ideal including $I \cup \{X_\beta\}$, viz., $\{X \cup Y \mid X \subseteq X_\beta \wedge Y \in I_\beta\}$.

$\bigcup_{\alpha<\kappa} I_\alpha$ is a maximal ideal extending I.

[36] The convention is often adopted that *ideal over κ* means *nontrivial ideal over κ* in this context.

The following theorem of Ulam is an important limitation on saturation for successor cardinals.

(9.99) **Theorem** [ZFC] *Suppose κ is an infinite successor cardinal and I is a nontrivial ideal over κ. Then* $\operatorname{sat} I > \kappa$.

Proof This is essentially the proof that κ is not real-valued measurable. Thus, let λ be such that $\kappa = \lambda^+$, and let $\langle A^\eta_\alpha \mid \alpha < \lambda^+ \wedge \eta < \lambda \rangle$ be an Ulam matrix,[9.11] i.e., for all $\alpha < \beta < \lambda^+$ and $\eta < \lambda$

1. $A^\eta_\alpha \subseteq \lambda^+$;
2. $A^\eta_\alpha \cap A^\eta_\beta = 0$; and
3. $|\lambda^+ \setminus \bigcup_{\eta<\lambda} A^\eta_\alpha| \leqslant \lambda$.

Since I is nonprincipal and λ^+-complete, every subset of λ^+ of size $\leqslant \lambda$ is in I. Hence (again using λ^+-completeness) for each $\alpha < \lambda^+$ there exists $\eta_\alpha < \lambda$ such that $A^{\eta_\alpha}_\alpha \in I^+$. For some $\eta < \lambda$, $|\{\alpha < \lambda^+ \mid \eta_\alpha = \eta\}| = \lambda^+$. $\{A_\alpha \mid \eta_\alpha = \eta\}$ is a counterexample to λ^+-saturation.[37] $\square$[9.99]

Thus, for successor cardinals κ, the existence of a κ^+-saturated nontrivial ideal over κ is the primary question. The most fundamental issue is whether there exists an ω_2-saturated nontrivial ideal over ω_1, in particular, whether the nonstationary ideal over ω_1 is ω_2-saturated. We will return to these questions.

For limit cardinals κ the following theorem due to Tarski (Part 1) and Levy and Silver (Part 2) establishes the significance of relatively low values of $\operatorname{sat} \kappa$.

(9.100) **Theorem** [GBC] *Suppose I is a nontrivial ideal over a cardinal κ.*

1. *If $2^{<\lambda} < \kappa$ and I is λ-saturated then κ is measurable.*
2. *If κ is weakly compact and I is κ-saturated then κ is measurable.*

Proof It is enough to show that I has an atom[9.97] A, in which case, $\{X \subseteq \kappa \mid X \cap A \in I^+\}$ is a κ-complete nonprincipal ultrafilter over κ, so κ is measurable.

Therefore, suppose toward a contradiction that I has no atom. We will construct a tree $S \subseteq {}^{<\kappa}2$ (i.e., for any $s \in S$ and $\alpha \in \operatorname{dom} s$, $s \restriction \alpha \in S$) and a function $F : S \to \mathcal{P}\kappa$ such that F is decreasing, i.e., for $s, t \in S$, if $s \subseteq t$ then $Fs \supseteq Ft$. We define F recursively. $F0 = \kappa$. Suppose Fs is given. If $Fs \in I^+$ let $X_0, X_1 \in I^+$ be disjoint such that $X_0 \cup X_1 = Fs$, and let $F(s^\frown\langle i \rangle) = X_i$ for $i \in 2$; if $Fs \notin I^+$, i.e., $Fs \in I$, we do not extend s in S.

At limit stages we put $s \in S$ iff every initial segment of s is in S, and we let $Fs = \bigcap_{\alpha \in \operatorname{dom} s} F(s \restriction \alpha)$.

Given $\nu \leqslant \kappa$ and $s : \nu \to 2$,

$$\{F(s \restriction \alpha^\frown\langle 1 - s_\alpha \rangle) \mid \alpha < \nu \wedge s_{\alpha+1} \in S\}$$

has size $|\nu|$ and consists of pairwise disjoint elements of I^+. Note also that $\kappa = \bigcup\{Fs \mid s \text{ is maximal in } S\}$, and if s is maximal in S and $\operatorname{dom} s < \kappa$ then $Fs \in I$.

1 If I is λ-saturated the height of S is at most λ, and if $2^{<\lambda} < \kappa$ then κ is the union of fewer than κ elements of I, violating the hypothesis of κ-completeness.

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[37] The sets A_α such that $\eta_\alpha = \eta$ are actually disjoint, not merely almost disjoint, but this is not a stronger result: Given that I is κ complete, any sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of almost disjoint sets in I^+ gives rise to a κ sequence of disjoint sets in I^+—replace X_α by $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta$.

2 If κ is weakly compact then κ is inaccessible, so if the height of S is $< \kappa$ then κ is again the union of fewer than κ sets in I. Hence, $\mathrm{ht}\, S = \kappa$. The levels of S are smaller than κ, so by weak compactness, S has a branch of length κ, but this violates the κ-saturation of I. $\square$[9.100]

The following is a brief summary of the preceding discussion.

(9.101) *Suppose κ is a cardinal and I is an ideal over κ.*

1. *If κ is finite then I is principal.*
2. *If $\kappa = \omega$ then I can be extended to a 2-saturated ideal. Since every ideal is ω-complete, there is a nontrivial 2-saturated ideal over ω.*
3. *If I is nontrivial then κ is regular (and uncountable by definition[9.96]) .*
4. $\mathrm{sat}\, I$ *is either finite or uncountable and regular.*
5. *Suppose I is nontrivial and $\lambda = \mathrm{sat}\, I$.*
    1. $\lambda \leqslant (2^\kappa)^+$.
    2. *If κ is a successor cardinal then $\lambda \geqslant \kappa^+$.*
    3. *If $2^{<\lambda} < \kappa$ then κ is measurable.*

Thus, the region of interest is defined by the inequalities:

1. $\lambda \leqslant 2^\kappa$, *and*
2. $2^{<\lambda} \geqslant \kappa$.

In practice, the important cases for limit cardinals κ are $\lambda = \kappa^+$, $\lambda = \kappa$, and $\lambda < \kappa$; the important case for successor cardinals κ is $\lambda = \kappa^+$.

The remainder of this section is devoted to implications of the existence of variously saturated nontrivial ideals. Ultimately we will show that if $\lambda < \kappa$ and I is a normal λ-saturated nontrivial ideal over κ then $I^* \cap L[I]$ is a normal ultrafilter over κ in $L[I]$,[9.112] so κ is measurable in $L[I]$. Indeed, it is sufficient that I be κ^+-saturated in order that κ be measurable in some inner model, but this is somewhat harder to prove and not on the main route of our presentation. The weaker result is sufficient for Solovay's proof that a real-valued measurable cardinal is measurable in an inner model.[9.113.2]

9.4.2 Precipitousness and normality

A very useful tool in the theory of nontrivial ideals over cardinals is the observation of Solovay that some arguments predicated on the existence of a maximal (2-saturated) nontrivial ideal over a cardinal κ (i.e., a measurable cardinal) could be adapted to a broader class of ideals by making use of the fact that any ideal I over κ in a transitive model M of ZF can be extended to a maximal ideal on $\mathcal{P}\kappa \cap M$ in a generic extension $M[G]$, using the partial order $\mathbb{P}^I$ defined as follows.

(9.102) **Definition** [ZFC] *Suppose I is an ideal over a set S. $\mathbb{P}^I = (I^+; \leqslant^I)$, where $Y \leqslant^I X \leftrightarrow Y \setminus X \in I$.*

The essential observation is that if G is an M-generic filter on $\mathbb{P}^I$, then G is an ultrafilter on $\mathcal{P}\kappa \cap M$, and the dual ideal G^* extends I. Note that G is not an ultrafilter *over* κ in the sense of $M[G]$, because $G \cup G^*$ is not $\mathcal{P}\kappa$ in the sense of $M[G]$ but rather in the sense of M. Nevertheless, in $M[G]$ we can form the ultrapower $\left(M \cap {}^\kappa M\right)/G^*$, whose individuals are equivalence classes modulo G^* of functions from κ into M *that are in* M. The following definition describes this construction. It is most useful when $(M; \in)$ is a model of a sufficient fragment of ZFC, but we state it in greater generality for later convenience.

Definition [GBC] *Suppose M is a transitive class and $\kappa \in \mathrm{Ord} \cap M$.*

1. *An* M-ultrafilter over κ $\stackrel{\text{def}}{=}$ *an ultrafilter on* $\mathcal{P}\kappa \cap M$.
2. *An M-ultrafilter U is* λ-complete $\stackrel{\text{def}}{\iff}$ *for every $\alpha < \lambda$ and $f \in {}^\alpha U \cap M$, $\bigcap_{\alpha<\lambda} f\alpha \in U$.*
3. *An M-ultrafilter U over κ is* normal $\stackrel{\text{def}}{\iff}$ *for every $X \in U$ and regressive $f \in {}^X\kappa \cap M$, f is constant on some $Y \in U$.*
4. *Suppose U is an M-ultrafilter over κ.* $\mathrm{Ult}_U\, M \stackrel{\text{def}}{=} \left(M \cap {}^\kappa M\right)/U$.
5. *A* generic ultrapower $\stackrel{\text{def}}{=}$ *a structure* $\mathrm{Ult}_G\, M$*, where G is an M-generic filter on $\mathbb{P}^I$ for some ideal I over κ (and hence an M-ultrafilter, in $M[G]$).*

Unsurprisingly, a key question regarding a generic ultrapower is whether it is wellfounded.

Definition [GBC] *Suppose κ is a cardinal and I is an ideal over κ. I is* precipitous $\stackrel{\text{def}}{\iff} \Vdash^{\mathbb{P}^I}$ $\ulcorner$*the generic ultrapower* $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ *is wellfounded*$\urcorner$.

Of course, we presume that the corner-quoted text is appropriately interpreted in ZFC so that it can be an argument of the forcing predicate. Note that if we undertake to prove the forcing relation by arguing in a generic extension using the class theory Θ',[8.106] then we may use the direct interpretation of $\ulcorner\mathrm{Ult}_{\mathsf{G}}(\mathsf{V}_\alpha)$ is wellfounded$\urcorner$ in GBC.

Before moving on, we observe that $\mathbb{P}^I$ is not separative.[8.57] The corresponding separative partial order is the set $\mathfrak{B}^+$ of nonzero elements of the quotient algebra $\mathfrak{B} = \mathfrak{B}^I = \mathcal{P}\kappa/I$. The regular algebra of $\mathbb{P}^I$ is therefore the completion $\mathfrak{R}\,\mathfrak{B}^I$ of $\mathfrak{B}^I$. The following theorem of Smith and Tarski is relevant.

(9.103) **Theorem** [ZFC] *Suppose I is a κ^+-saturated nontrivial ideal over κ. Then $\mathcal{P}\kappa/I$ is a complete boolean algebra.*

The proof is not difficult, but we do not need this result, so we omit it. Conversely to (9.103) the completeness of $\mathcal{P}\kappa/I$ has implications for the saturation of I. Specifically, suppose $\mathcal{P}\kappa/I$ is complete and I is not λ-saturated. Let $\langle X_\alpha \mid \alpha < \lambda\rangle$ be a counterexample to λ-saturation. For $A \subseteq \lambda$, let $f A$ be a representative of the join in $\mathcal{P}\kappa/I$ of $\{[X_\alpha] \mid \alpha \in A\}$, where $[X] \in \mathcal{P}\kappa/I$ is the equivalence class of $X \subseteq \kappa$. Then $f : \mathcal{P}\lambda \to \mathcal{P}\kappa$ is injective, so $2^\lambda \leqslant 2^\kappa$. Hence:

If I is a nontrivial ideal over κ and $\mathcal{P}\kappa/I$ is a complete boolean algebra, then

1. *if $2^\lambda > 2^\kappa$ then I is λ-saturated;*
2. *in particular*
    1. *I is 2^κ-saturated; and*
    2. *if $2^{\kappa^+} > 2^\kappa$ then I is κ^+-saturated.*

The following theorem of Solovay establishes the precipitousness of κ^+-saturated nontrivial ideals over κ.

(9.104) **Theorem** [GBC] *Suppose I is a κ^+-saturated nontrivial ideal over a cardinal κ. Then I is precipitous.*

Proof Let $\mathbb{P} = (I^+; \leqslant^I)$. It is enough to show that it is a $\mathbb{P}$-validity that the ordinals of $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ are wellordered. In the interest of notational simplicity, we will write '$[\cdot]$' for the reduced equivalence class operation modulo an ultrafilter, rather '$[\cdot]^*$'. An ordinal of $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ is the equivalence class $[\check{f}]$ mod G of the canonical representative $\check{f}$ of an ordinal-valued function on κ in the ground model (represented by 'V'). Note that for any $X \in I^+$ and $f, g \in {}^{\kappa}\mathrm{Ord}$, $X \Vdash [\check{g}] < [\check{f}]$ iff g is almost everywhere less than f on X, i.e., $X \leqslant^I \{\alpha \mid g\alpha < f\alpha\}$; similarly, $X \Vdash [\check{g}] = [\check{f}]$ iff g is almost everywhere equal to f on X.

(9.105) **Claim** *Suppose $W \in I^+$, $\dot{a} \in V^{\mathbb{P}}$ and $W \Vdash \ulcorner(\dot{a})$ is a nonempty set of ordinals in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})\urcorner$. Then $W \Vdash \ulcorner$ there is a least ordinal in $(\dot{a})\urcorner$.*

Proof

(9.106) **Claim** *Suppose $X \leqslant^I W$. Then there exist $Y \leqslant^I X$ and $f \in {}^{\kappa}\mathrm{Ord}$ such that $Y \Vdash \ulcorner[(\check{f})]$ is the least ordinal in $(\dot{a})\urcorner$.*

Proof Suppose the contrary.

(9.107) **Claim** *Suppose $X' \leqslant^I X$, $f \in {}^{\kappa}\mathrm{Ord}$, and $X' \Vdash [\check{f}] \in \dot{a}$. Then there exists $g \in {}^{\kappa}\mathrm{Ord}$ such that $X' \Vdash [\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$.*

Proof Let $A \subseteq I^+$ be maximal subject to the conditions:

1. $\forall Y \in A\ Y \leqslant^I X'$;
2. $\forall Y, Y' \in A\ Y \cap Y' \in I$; and
3. $\forall Y \in A\, \exists g \in {}^{\kappa}\mathrm{Ord}\ \ Y \Vdash [\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$.

Then A is a maximal antichain below X' in I^+ because, by hypothesis, for every $Y \leqslant^I X'$, $Y \nVdash \ulcorner[(\check{f})]$ is the least ordinal in $(\dot{a})\urcorner$, so there exist $Z \leqslant^I Y$ and $g \in {}^{\kappa}\mathrm{Ord}$ such that $Z \Vdash [\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$.

Since I is κ^+-saturated, $|A| \leqslant \kappa$. Let $\langle Y_\xi \mid \xi < \eta \rangle$ enumerate A, where $\eta \leqslant \kappa$. For each $\xi < \eta$ let $Y'_\xi = Y_\xi \setminus \bigcup_{\xi' < \xi} Y_{\xi'}$. By virtue of the κ-completeness of I, $Y'_\xi \equiv^I Y_\xi$. Let $A' = \{Y'_\xi \mid \xi < \eta\}$. Then A' has the defining characteristics of A, but it consists of disjoint sets. For each $\xi < \eta$, let $g_\xi \in {}^{\kappa}\mathrm{Ord}$ be such that $Y_\xi \Vdash [\check{g}_\xi] \in \dot{a} \wedge [\check{g}] < [\check{f}]$. Let $g' = \bigcup_{\xi < \eta} g_\xi \restriction Y'_\xi$, and let $g \in {}^{\kappa}\mathrm{Ord}$ be any extension of g'.

For every $\xi < \eta$, $Y_\xi \Vdash [\check{g}] = [\check{g}_\xi]$, so $Y_\xi \Vdash [\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$. Thus the set of conditions forcing $[\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$ is dense below X', so $X' \Vdash [\check{g}] \in \dot{a} \wedge [\check{g}] < [\check{f}]$. $\square$[9.107]

By hypothesis[9.105] there exist $X' \leqslant^I X$ and $g_0 \in {}^{\kappa}\mathrm{Ord}$ such that $X' \Vdash [\check{g}_0] \in \dot{a}$. Using Claim 9.107 construct a sequence $\langle g_n \mid n \in \omega \rangle$ such that for each $n \in \omega$, $X' \Vdash [\check{g}_n] \in \dot{a} \wedge [\check{g}_{n+1}] < [\check{g}_n]$. For each $n \in \omega$, let $Y_n = \{\alpha < \kappa \mid g_{n+1}\alpha < g_n\alpha\}$. As noted above, for each $n \in \omega$, $X' \leqslant^I Y_n$. It follows from the ω_1-completeness of I that $X' \leqslant^I \bigcap_{n \in \omega} Y_n$, so $\bigcap_{n \in \omega} Y_n \neq 0$. Let α be any member of $\bigcap_{n \in \omega} Y_n$. Then $g_0\alpha > g_1\alpha > \cdots$, which is impossible, as these are ordinals. $\square$[9.106]

Now let $A \subseteq I^+$ be maximal subject to the conditions:

1. $\forall Y \in A\ Y \leqslant^I W$;

2. $\forall Y, Y' \in A\ Y \cap Y' \in I$; and

3. $\forall Y \in A\, \exists g \in {}^\kappa\mathrm{Ord}\ \ Y \Vdash \ulcorner[(\check{g})]$ is the least ordinal in $(\dot{a})\urcorner$.

Then A is a maximal antichain below W in I^+ by virtue of Claim 9.106. As before, since I is κ^+-saturated, $|A| \leqslant \kappa$, and we let $\langle Y_\xi \mid \xi < \eta \rangle$ enumerate A, where $\eta \leqslant \kappa$. For each $\xi < \eta$ let $Y'_\xi = Y_\xi \setminus \bigcup_{\xi' < \xi} Y_{\xi'}$. For each $\xi < \eta$, let $g_\xi \in {}^\kappa\mathrm{Ord}$ be such that $Y_\xi \Vdash \ulcorner[(\check{g})]$ is the least ordinal in $(\dot{a})\urcorner$. Let $g' = \bigcup_{\xi<\eta} g_\xi \restriction Y'_\xi$, and let $g \in {}^\kappa\mathrm{Ord}$ be any extension of g'.

For every $\xi < \eta$, $Y_\xi \Vdash [\check{g}] = [\check{g}_\xi]$, so $Y_\xi \Vdash \ulcorner[(\check{g})]$ is the least ordinal in $(\dot{a})\urcorner$. Thus the set of conditions forcing $\ulcorner[(\check{g})]$ is the least ordinal in $(\dot{a})\urcorner$ is dense below W, so $W \Vdash \ulcorner[(\check{g})]$ is the least ordinal in $(\dot{a})\urcorner$. $\square$[9.105]

Now suppose toward a contradiction that I is not precipitous. Then there exist $W \in I^+$ and $\dot{a} \in V^{\mathbb{P}}$ such that $W \Vdash \ulcorner(\dot{a})$ is a nonempty set of ordinals in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ with no least member$\urcorner$, contradicting Claim 9.105. $\square$[9.104]

The preceding theorem may be viewed as extending to κ^+-saturated nontrivial ideals over κ a property of 2-saturated nontrivial ideals over κ (i.e., a property of measurability). The next theorem, also due to Solovay, similarly generalizes (9.15.2). We preface it with a definition.

(9.108) **Definition** [ZFC] *Suppose I is a nontrivial ideal over κ, $X \subseteq \kappa$, and $f : X \to \kappa$.*

1. *f is* I-unbounded *(or I-small) $\overset{\text{def}}{\iff}$ for every $\alpha \in \kappa$, $f^{\leftarrow}\{\alpha\} \in I$. Note that by κ-completeness, this is equivalent to $f^{\leftarrow}\alpha \in I$.*
2. *f is* I-incompressible $\overset{\text{def}}{\iff}$
   1. *f is* I-unbounded*; and*
   2. *for every $Y \subseteq X$ such that $Y \in I^+$, and $g : Y \to \kappa$, if $\forall \alpha \in Y\ g\alpha < f\alpha$ then g is not I-unbounded.*

(9.109) **Theorem** [ZFC] *Suppose $\lambda \leqslant \kappa^+$, and there exists a λ-saturated nontrivial ideal over a cardinal κ. Then there exists a normal λ-saturated ideal over κ.*

Proof Let I be a κ^+-saturated nontrivial ideal over a cardinal κ. Let $\mathbb{P} = \mathbb{P}^I$ and let $\Vdash\, = \,\Vdash^{\mathbb{P}}$. For any x, let $\bar{x}$ be the constant function on κ with value x. It is easy to show that $\Vdash \ulcorner$for every $\alpha \in (\check{\kappa})$, $[\bar{\alpha}]$ is the αth ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})\urcorner$. Since I is precipitous,[9.104] $\Vdash \ulcorner$there is a least ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$. It follows that for every $X \in I^+$ there exist $Y \leqslant^I X$ and $f \in {}^\kappa\kappa$[38] such that $Y \Vdash \ulcorner[(\check{f})]$ is the least ordinal exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$.

Let $A \subseteq I^+$ be maximal subject to the conditions:

1. $\forall Y, Y' \in A\, (Y \cap Y' \in I)$; and

2. $\forall Y \in A\, \exists f \in {}^\kappa\kappa\ Y \Vdash \ulcorner[(\check{f})]$ is the least ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$.

[38]Since the identity function on κ exceeds every $\bar{\alpha}$ almost everywhere, the least ordinal exceeding the $[\bar{\alpha}]$s is represented by some $f : \kappa \to \kappa$.

Then A is a maximal antichain in I^+. As in the proof of (9.104) (twice), using the κ^+-saturation and κ-completeness of I, we may assume that the elements of A are disjoint. For each $Y \in A$, let $f_Y \in {}^\kappa\kappa$ be such that $Y \Vdash \ulcorner[(\check{f}_Y)]$ is the least ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$; let $f' = \bigcup_{Y \in A}(f_Y \restriction Y)$, and let $f \in {}^\kappa\kappa$ be an extension of f'. Then for every $Y \in A$, $Y \Vdash [(\check{f})] = [(\check{f}_Y)]$, so $Y \Vdash \ulcorner[(\check{f})]$ is the least ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$; from which it follows that $\Vdash \ulcorner[(\check{f})]$ is the least ordinal in $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ exceeding $[\bar{\alpha}]$ for every $\alpha < (\check{\kappa})\urcorner$.

It is straightforward to show that f is I-incompressible.[9.108] Let

$$I' = \{A \subseteq \kappa \mid f^{\leftarrow}A \in I\}.$$

Clearly I' is a λ-saturated ideal over κ. We will show that I' is normal. I' is nonprincipal because f is I-unbounded,[9.108.1] and κ-complete because I is κ-complete. To verify the normality condition, suppose $X' \in I'^+$ and $h : X' \to \kappa$ is regressive. Let $X = f^{\leftarrow}X'$ and $g = h \circ f$. Then $X \in I^+$, $g : X \to \kappa$, and $\forall \alpha \in X\ g\alpha = h(f\alpha) < f\alpha$, since h is regressive. Thus[9.108.1.2] for some $\alpha \in \kappa$, $g^{\leftarrow}\{\alpha\} \in I^+$. But $g^{\leftarrow}\{\alpha\} = f^{\leftarrow}(h^{\leftarrow}\{\alpha\})$, so $h^{\leftarrow}\{\alpha\} \in I'^+$. □[9.109]

It is instructive to view the above construction in terms of the generic ultrapower. Thus, in $\mathsf{V}[\mathsf{G}]$ we may use f to define a normal V-ultrafilter G' by putting[9.17]

$$A \in G' \leftrightarrow f^{\leftarrow}A \in \mathsf{G},$$

for every $A \subseteq \kappa$. This is equivalent to putting $A \in G'$ iff $\kappa \in jA$, where j is the canonical injection of V into the transitive collapse of $\mathrm{Ult}_{\mathsf{G}}(\mathsf{V})$ (since $[f]$ maps to κ in the collapse). Note that G' is V-generic over $\mathcal{P}\kappa/I'$.

The κ^+-saturation of I is used twice in the proof of (9.109): first to show that I is precipitous, and again to prove the existence of an I-incompressible function, by means of which I is converted to a normal ideal. It is an open question whether the existence of a precipitous ideal over κ implies the existence of a normal precipitous ideal over κ.

The following theorem[9.112] of Solovay is in effect a generalization of (9.76) but the proof incorporates that of (9.79). Our proof of the latter used the fact that a measurable cardinal is Ramsey.[9.64] The hypothesis of (9.112) is that there is a normal λ-saturated ideal over κ with $\lambda < \kappa$, which does not imply that κ is Ramsey; however, examination of the proof of (9.79) shows that the weaker *Rowbottom property* suffices, and this does follow from the existence of such an ideal by another theorem[9.110] of Solovay. The definition of *Rowbottom* is rather technical and is tailor-made for the model-theoretic uses to which it is put.

Definition [ZFC]

1. *Suppose $\alpha, \beta, \gamma, \delta$ are ordinals.*
   1. $\alpha \to [\beta]^\gamma_\delta \overset{\text{def}}{\iff}$ *for every $f : [\alpha]^\gamma \to \delta$ there exists $X \in [\alpha]^\beta$ such that $f^{\to}[X]^\gamma \neq \delta$. This differs from the round-bracket partition relation*[9.60] *in that it is only required that $f^{\to}[X]^\gamma$ omit a member of δ, not that it consist of just one member of δ. Note that δ may be taken to be a cardinal.*
   2. *Suppose ν is a cardinal. Then $\alpha \to [\beta]^\gamma_{\delta;<\nu} \overset{\text{def}}{\iff}$ for every $f : [\alpha]^\gamma \to \delta$ there exists $X \in [\alpha]^\beta$ such that $|f^{\to}[X]^\gamma| < \nu$.*
   3. *We also have '$< \omega$'-exponent variants:*

1. $\alpha \to [\beta]^{<\omega}_{\delta} \overset{\text{def}}{\iff}$ *for every* $f : [\alpha]^{<\omega} \to \delta$ *there exists* $X \in [\alpha]^{\beta}$ *such that* $f^{\to}[X]^{<\omega} \neq \delta$.
2. $\alpha \to [\beta]^{<\omega}_{\delta;<\nu} \overset{\text{def}}{\iff}$ *for every* $f : [\alpha]^{<\omega} \to \delta$ *there exists* $X \in [\alpha]^{\beta}$ *such that* $|f^{\to}[X]^{<\omega}| < \nu$.

2. *Suppose* κ *and* ν *are cardinals.*
   1. κ *is* ν-Rowbottom $\overset{\text{def}}{\iff} \kappa \to [\kappa]^{<\omega}_{\lambda;<\nu}$ *for every* $\lambda < \kappa$. κ *is* Rowbottom $\overset{\text{def}}{\iff} \kappa$ *is* ω_1*-Rowbottom.*
   2. *Suppose* F *is a filter over* κ. F *is* ν*-Rowbottom* $\overset{\text{def}}{\iff}$ *for any* $\lambda < \kappa$ *and* $f : [\kappa]^{<\omega} \to \lambda$ *there exists* $X \in F$ *such that* $|f^{\to}[X]^{<\omega}| < \nu$. F *is* Rowbottom $\overset{\text{def}}{\iff} F$ *is* ω_1*-Rowbottom.*

Note that a Ramsey[9.64] cardinal is Rowbottom, i.e., ω_1-Rowbottom,[9.65] and as noted in the remark following (9.66) any normal ultrafilter is Rowbottom. The following theorem generalizes this to normal filters with suitable saturation properties.

(9.110) **Theorem** [GBC] *Suppose* κ *and* ν *are cardinals,* $\omega < \nu < \kappa$, ν *is regular, and* I *is a normal* ν*-saturated ideal* I *over* κ. *Then* I^* *is* ν*-Rowbottom.*

Proof Suppose $\lambda < \kappa$. It is sufficient to show that for each $n \in \omega$ and $f : [\kappa]^n \to \lambda$ there exists $C \subseteq \lambda$ and $B \in I^*$ such that $|C| < \nu$ and $f^{\to}[B]^n \subseteq C$.[39] We imitate the proof of (9.66), with the necessary modification that instead of ultrapowers *per se*, we use generic ultrapowers.[40]

Strict adherence to the structure of the previous proof would entail the use of an n-fold iteration of the generic ultrapower construction to obtain forcing terms $\dot\kappa_\alpha$ and $\dot f_\alpha$, $\alpha \leqslant n$, and a term $\dot\gamma$ with $\Vdash \dot\gamma = \dot f_n\{\dot\kappa_0, \ldots, \dot\kappa_{n^-}\}$. We would then work our way back using ν-saturation to obtain terms $\{\dot\gamma\} = \dot C_n, \dot C_{n^-}, \ldots, \dot C_0$, such that for each $\alpha \leqslant n$, $\dot C_\alpha$ denotes a set of ordinals of size $< \nu$ in the ground model. In this construction, $\dot\kappa_0 = \check\kappa$, $\dot f_0 = \check f$, and $\dot C_0 = \check C$ for some C in the ground model; and

$$\forall^I \xi_0 \cdots \forall^I \xi_{n^-} f\{\xi_0, \ldots, \xi_{n^-}\} \in C.$$

We would now use the normality of I to obtain a homogeneous set as before.

Effecting this construction is a bit complicated on account of the forcing iteration, and in the interest of clarity we formulate the argument instead as an induction over n. To begin we let $n = 1$. Suppose $f : \kappa \to \lambda$. Let $C = \{\gamma < \lambda \mid f^{\leftarrow}\{\gamma\} \in I^+\}$. Since I is ν-saturated, $|C| < \nu$. Let $X = f^{\leftarrow}C$. Since I is κ-compete, $X \in I^*$.

(9.111) *Now suppose* $n > 1$, *and suppose for all* $m < n$ *and* $g : [\kappa]^m \to \lambda$ *there exist* $C \in \mathcal{P}_\nu \lambda$ *and* $X \in I^*$ *such that* $g^{\to}[X]^m \subseteq C$.

Suppose $f : [\kappa]^n \to \lambda$. We will work in a $\mathbb{P}^I$-generic extension.[9.102] Thus, we suppose GBC, V is an inner model of ZFC plus (9.111), and U is a V-generic filter on $\mathbb{P}^I$. U is a V-ultrafilter over κ. Since I is precipitous,[9.104] $\mathrm{Ult}_U\,\mathsf{V}$ is wellfounded. Let $\pi : \mathrm{Ult}_U\,\mathsf{V} \to M$ be its transitive collapse and let $j : \mathsf{V} \to M$ be the canonical embedding. By routine arguments, since I is κ-complete, κ is the first ordinal

[39] We make use of the fact that $\nu > \omega$ and ν is regular, so $\mathrm{cf}\,\nu > \omega$.

[40] Solovay's published proof of this theorem is entirely combinatorial, and is admittedly somewhat simpler than the proof given here. Solovay indicates in that publication, however, that he first developed the relevant theory by consideration of generic ultrapowers. In any event, we prefer the generic ultrapower approach for the insight it provides.

moved by j, and since I is normal, $\kappa = \pi[k]$, where k is the identity function on κ, so $U = \{A \in \mathcal{P}\kappa \cap \mathsf{V} \mid \kappa \in jA\}$. Also, V and M assign the same cardinalities and cofinalities to ordinals $< \kappa$.

Let $\kappa_1 = j\kappa$, $I_1 = jI$, and $f_1 = jf$. For each $\xi < \kappa$, let $f^\xi : [\kappa]^{n^-} \to \lambda$ be such that

$$f^\xi s = \begin{cases} f(\{\xi\} \cup s) & \text{if } \xi < \min s \\ 0 & \text{otherwise.} \end{cases}$$

Then

$\mathsf{V} \models \ulcorner$for every $\xi < [\kappa]$ there exist $C \in \mathcal{P}_{[\nu]}\,[\lambda]$ and $X \in [I]^*$ such that $[f]^{\xi \to}[X]^{n^-} \subseteq C\urcorner$.

Since $\kappa < \kappa_1$, there exists $\delta < \nu$ such that δ is a cardinal in V (and therefore in M), and

$M \models \ulcorner$there exist $C \subseteq [\lambda]$ and $X \in [I_1]^*$ such that $|C| = [\delta]$ and $[f_1]^{[\kappa]\to}[X]^{n^-} \subseteq C\urcorner$.

To formulate these statements in the $\mathbb{P}^I$-forcing language we must formulate them in ZF, which we can do by consideration of initial segments of V and M. Let $M_0 = \mathsf{V}_{\kappa+2}$. Then $I \in M_0$, U is an ultrafilter on $\mathcal{P}\kappa \cap M_0$ extending I^*, and $\mathrm{Ult}_U\, M_0$ is wellfounded. Let M_1 be its transitive collapse. Then $M_1 = M_{\kappa_1+2}$. The canonical embedding is $j \restriction M_0$.

For the nonce, we make the following definition:

δ *is* big enough $\overset{\text{def}}{\iff} M_1 \models \ulcorner$there exist $C \subseteq [\lambda]$ and $X \in [I_1]^*$ such that $|C| = [\delta]$ and $[f_1]^{[\kappa]\to}[X]^{n^-} \subseteq C\urcorner$.

The preceding argument pertains to any V-generic filter U on $\mathbb{P}^I$, and $\mathsf{V} \models \mathsf{AC}$, so,[8.109] returning to the ground model V, there exists $\dot\delta \in V^{\mathbb{P}^I}$ such that $\Vdash \ulcorner(\dot\delta)$ is a cardinal $< (\check\nu)$ that is big enough$\urcorner$, where $\Vdash$ is $\Vdash^{\mathbb{P}^I}$.

Let $S \subseteq I^+$ be a maximal antichain subject to the condition that for each $Y \in S$ there exists a cardinal $\delta < \nu$ such that $Y \Vdash \dot\delta = \check\delta$, and let δ_Y be the unique $\delta < \nu$ with this property. Since $\Vdash \dot\delta < \check\nu$, S is a maximal antichain in $\mathbb{P}^I$. By ν-saturation, $|S| < \nu$. Let $\delta_0 = \sup_{Y \in S} \delta_Y$. Since ν is regular by assumption, $\delta_0 < \nu$. Every condition in $\mathbb{P}^I$ is compatible with a member of S, so $\Vdash \ulcorner(\check\delta_0)$ is a cardinal $< (\check\nu)$ that is big enough$\urcorner$.

Let $\dot I_1$, j, and $\dot f_1$ be the canonical terms for I_1, j, and f_1 (derived from $\dot U$), and let $\dot h$ be such that $\Vdash \ulcorner(\dot h) : (\check\delta_0) \to (\check\lambda)$ and there exists $X \in (\dot I_1)^*$ such that $(\dot f_1)^{(\check\kappa)\to}[X]^{n^-} \subseteq \operatorname{im}(\dot h)\urcorner$. By a ν-saturation argument similar to the preceding, we show that for each $\alpha < \delta_0$ there is a set C_α of possible values for $\dot h(\alpha)$ with $|C_\alpha| < \nu$, and we let $C = \bigcup_{\alpha < \delta_0} C_\alpha$. Then $|C| < \nu$ and $\Vdash \operatorname{im} \dot h \subseteq \check C$. Thus, $\Vdash \exists X \in \dot I_1^* \, (\dot f_1^{\check\kappa \to}[X]^{n^-} \subseteq \check C)$.

Let

$$A = \{\xi < \kappa \mid \exists X \in I^* \; f^{\xi\to}[X]^{n^-} \subseteq C\}.$$

Then $\Vdash \check\kappa \in j\check A$, so $\Vdash \check A \in \dot U$. It follows that $A \in I^*$ (otherwise, letting $B = \kappa \setminus A$, $B \in I^+$ and $B \Vdash \check A \notin \dot U$).

For each $\xi < \kappa$ let $X_\xi \in I^*$ be such that if $\xi \in A$ then

$$f^{\xi\to}[X_\xi]^{n^-} \subseteq C,$$

otherwise, $X_\xi = \kappa$. Let

$$B = A \cap \Delta_{\xi<\kappa} X_\xi.$$

By normality, $B \in I^*$. Suppose $s \in [B]^n$, and let $\xi = \min s$ and $t = s \backslash \{\xi\}$. Then $\xi \in A$ and $t \subseteq X_\xi$, so $fs = f^\xi t \in C$. $\square$[9.110]

(9.112) **Theorem** [GBC] *Suppose $\lambda < \kappa$ and I is a normal λ-saturated ideal over κ. Then $L[I] \models \ulcorner [\kappa]$ is measurable$\urcorner$.*

Proof Clearly $L[I] \models \ulcorner [I \cap L[I]]$ is a normal $[\lambda]$-saturated ideal over $[\kappa]\urcorner$. We now "work in $L[I]$", which is to say that everything we say is supposed to be relativized to $L[I]$: every use of an axiom θ of ZFC—whether explicit, or implicit as a premise of a quoted theorem of ZFC—is presumed to be accompanied by an interpolated proof of $\theta^{L[I]}$. For convenience, working in $L[I]$ we will use 'I' to refer to $I \cap L[I]$. By virtue of (9.99), κ is not a successor cardinal. Hence, in particular, $\lambda^+ < \kappa$.

We now adapt Silver's proof of (9.79) to show that $2^\lambda = \lambda^+$. For the most part we simply replace U by I. Instead of invoking (9.66) to obtain for each Skolem function f with arity n a set $Z_f \in U$ such that $|f^{\to}[Z_f]^n| = 1$, we instead invoke (9.110) to obtain a set $Z_f \in I^*$ such that $|f^{\to}[Z_f]^n| < \lambda$. We let $Z = \bigcap_{f\in F} Z_f$, which is in I^*, and we generate

$$\mathfrak{B} = (B; \in, I \cap B, R \cap B, X, \gamma)_{\gamma\in\lambda} \prec \mathfrak{A}$$

from Z using the Skolem functions, so that $|R \cap B| \leqslant \lambda$, $\lambda \subseteq B$, and $Z \subseteq B$. Let $\pi : (B; \in, I \cap B) \to (M; \in, J)$ be the transitive collapse. To show that $J = I$, we define, as before, $Z' = \{\alpha \in Z \mid \pi\alpha = \alpha\}$, and use the normality of I to show that $Z' \in I^*$. Given $S \in M \cap \mathcal{P}\kappa$ we let $S' = \pi^{-1}S$. Then $S \in J \leftrightarrow S' \in I$, and $S \cap Z' = S' \cap Z'$. Thus

$$S \in J \leftrightarrow S' \in I \leftrightarrow S' \cap Z' \in I \leftrightarrow S \cap Z' \in I \leftrightarrow S \in I.$$

We complete the proof as before.

Still working in $L[I]$, we now know that $2^{<\lambda} \leqslant 2^\lambda = \lambda^+ < \kappa$, and we apply (9.100.1) to conclude that κ is measurable; hence, $L[I] \models \ulcorner [\kappa]$ is measurable$\urcorner$. $\square$[9.112]

9.4.3 Real-valued and 2-valued measurability

Any set of reals that may be covered by a set
Of intervals whose total length's as short as it can get
Shy of zero is so small that we declare that it is *null*
And is to be ignored for ev'ry purpose practical:
Thus spake Lebesgue. To differ nary one of us would dare,
Nor cavil at the comparable claim advanced by Baire
That denumerable unions of sets closed and nowhere dense
Are *of first category*, therefore of no consequence.
To be precise, we say such sets as these are *almost* naught,
Which seems at first peculiar, given what we've all been
taught—
To wit, that mathematics is the science of precision,
Which contemplates the concept of 'almost' with frank derision.
From 'this is so' to 'this is almost always so' would seem
To be a backward step, yet it's become a basic theme
Of modern math, and nowhere more so than at its foundation:
In logic and the theory of the membership relation.
It's there in Cohen's concept of genericity,
Which animates the subject with such felicity;
And—allowing that the axiom of choice receive its due—
For ultraproducts that that's *almost always* true *is* true.
Leave it to Solovay to tie this all up with a bow
In his celebrated article[23] where he proceeds to show
With generic ultrapowers and sundry arguments—
Clever, to be sure, but not prohibitively dense—
That it's equally consistent with the theory ZFC
That the answer to the measure question of Banach should be
'Yes' as that there be a cardinal that's measurable.
Please take the time to read it if it isn't too much trouble.

'Almost' Here, 'Almost' There, 'Almost' Almost Everywhere by
Robert A. Van Wesep

We conclude this brief survey of saturated ideals with Solovay's celebrated theorem[9.113] stating the equiconsistency of

1. There exists a countably additive real-valued measure over some nonempty set.[9.6]
2. 2^ω is real-valued measurable.[9.9]
3. There exists a measurable cardinal.

Note that by an earlier result, to these we may add:

4. There exists a countably additive measure over $\mathbb{R}$ extending Lebesgue measure.[9.13]

(9.113) **Theorem** [GBC]

1. *Suppose there is a countably additive real-valued measure over a cardinal κ. Then κ is measurable in an inner model.*
2. *Suppose κ is a measurable cardinal. Then there is a partial order $\mathbb{P}$ such that $\Vdash^{\mathbb{P}} \ulcorner(\check{\kappa}) = 2^\omega$ and $(\check{\kappa})$ is real-valued measurable$\urcorner$.*

Proof 1 Let I be the null ideal of a countably additive real-valued measure[9.6] over κ. Then I is a nontrivial ω_1-saturated ideal over κ, so[9.109] there is a normal ω_1-saturated nontrivial ideal I' over κ. Since $\omega_1 < \kappa$,[9.8, 9.9, 9.11] $L[I'] \models \ulcorner[\kappa]$ is a measurable cardinal$\urcorner$.[9.112]

2 Recall the use of the algebra **Borel**/$\mathfrak{n}$ of Borel sets of reals modulo the ideal of null sets (in the sense of Lebesgue measure) as a forcing algebra in the analysis of Solovay's model in which all sets of reals are Lebesgue measurable.[8.225] We now use a similar algebra, albeit in a quite different way. Let $\mathcal{B}$ be the Borel algebra of ${}^{\kappa}2$ regarded as the product of κ copies of the discrete 2-element topological space. Thus, for each finite $s : \kappa \rightharpoonup 2$, let $I_s = \{f \in {}^{\kappa}2 \mid s \subseteq f\}$. The I_ss constitute a base for the product topology on ${}^{\kappa}2$, i.e., $X \subseteq {}^{\kappa}2$ is open iff $X = \bigcup\{I_s \mid I_s \subseteq X\}$. $\mathcal{B}$ is obtained by closing under complementation and countable union (and intersection).

We let u be the uniform measure on $\mathcal{B}$, which is the product of the uniform measures on each of the factor 2-element spaces. Let $\mathfrak{n} = \{X \in \mathcal{B} \mid uX = 0\}$. Let $\mathfrak{B} = \mathcal{B}/\mathfrak{n}$, the boolean algebra of Borel subsets of ${}^{\kappa}2$ modulo the u-null sets. Let u be also the measure induced on $\mathfrak{B}$ by u, i.e., $u[X] \stackrel{\text{def}}{=} uX$, for any $X \in \mathcal{B}$, where $[X]$ is the $\mathfrak{n}$-equivalence class of X in $\mathcal{B}$. Since u is countably additive, $\mathfrak{B}$ is countably complete and countably (i.e., ω_1-)saturated. It is therefore a complete boolean algebra.

Much of the analysis of the Baire and Lebesgue algebras, $\mathfrak{B}$ (no relation) and $\mathfrak{L}$, undertaken in the course of proving (8.225), is applicable here. For convenience, we will "work in a generic extension". Thus, we suppose the above definitions are made in a transitive model V of ZFC and G is a V-generic filter on $\mathfrak{B}$. There is a unique $f \in {}^{\kappa}2$ such that for all finite $s \subseteq f$, $[I_s] \in G$. We define f_G to be this f. Conversely, G is recoverable from f_G.

Given $f \in {}^{\kappa}2$, let $f^{\alpha} = \{(n, i) \mid n \in \omega \wedge i \in 2 \wedge f(\omega \cdot \alpha + n) = i\}$. In this way any $f \in {}^{\kappa}2$ encodes a κ-sequence of reals (elements of ${}^{\omega}2$ in the current context). It follows from the genericity of G that $f_G^{\alpha} \neq f_G^{\beta}$ if $\alpha \neq \beta$. Hence $2^{\omega} \geqslant \kappa$.

To show that $2^{\omega} \leqslant \kappa$, note that any $x \in {}^{\omega}2$ is $\dot{x}^G$ for some $\dot{x} \in \mathsf{V}^{\mathfrak{B}}$, and $[\![\dot{x} \in {}^{\omega}2]\!] \in G$. x is uniquely determined via G by the function (in V)

$$n \mapsto [\![\dot{x} \in {}^{\omega}2 \wedge \dot{x}(\check{n}) = \mathbf{0}\,]\!].$$

Reasoning now in V, the number of such functions is no more than $||\mathfrak{B}||^{\omega}$, so it is enough to show that $||\mathfrak{B}|| \leqslant \kappa$, since $\kappa^{\omega} = \kappa$. For this it suffices to show that $|\mathcal{B}| \leqslant \kappa$. This is done by looking at the way $\mathcal{B}$ is generated by countable unions and complements, starting from the set of basic open sets I_s, of which there are κ. This occurs in ω_1 stages, so $|\mathcal{B}| \leqslant \omega_1 \cdot \kappa^{\omega} = \kappa$.

It follows from the above remarks that $[\![\check{\kappa} = 2^{\omega}]\!] = \mathbf{1}$. Now we show that $[\![\check{\kappa} \text{ is real-valued measurable}]\!] = \mathbf{1}$. Let U be a κ-complete nonprincipal ultrafilter over κ. Let $\mathfrak{B}^{+}$ be the partial order of nonzero elements of $\mathfrak{B}$. Given a $\mathfrak{B}$-term $\dot{X}$, let $\mu_{\dot{X}} : \mathfrak{B}^{+} \to [0, 1]$ be such that if $p \Vdash \dot{X} \subseteq \check{\kappa}$ then for any $x \in [0, 1]$,

$$\mu_{\dot{X}} p = x \leftrightarrow \left\{\alpha < \kappa \,\middle|\, \tfrac{u([\![\check{\alpha} \in \dot{X}]\!] \wedge p)}{up} = x\right\} \in U.$$

($\mu_{\dot{X}} p$ may be defined arbitrarily for p such that $p \nVdash \dot{X} \subseteq \check{\kappa}$.) Since U is κ-complete and $2^{\omega} < \kappa$ (we're still back in V!), $\mu_{\dot{X}} p$ is well defined.

(9.114) *For fixed p, $\dot{X} \mapsto \mu_{\dot{X}} p$ has the following measure-like properties:*

1. *For any $X \subseteq \kappa$, $\mu_{\check{X}} p = 1$ if $X \in U$; otherwise, $\mu_{\check{X}} p = 0$.*
2. *If $p \Vdash \dot{X} \subseteq \dot{Y} \subseteq \check{\kappa}$ then $\mu_{\dot{X}} p \leqslant \mu_{\dot{Y}} p$.*
3. *For any $\nu < \kappa$, $\dot{F}$, and $\dot{X}$, if $p \Vdash \ulcorner(\dot{F})$ is a $(\check{\nu})$-sequence of pairwise disjoint subsets of $(\check{\kappa})$ and $(\dot{X}) = \bigcup \operatorname{im}(\dot{F})\urcorner$, then $\mu_{\dot{X}} p = \sum_{\eta < \nu} \mu_{\dot{X}_{\eta}} p$, where, for each $\eta < \nu$, $p \Vdash \dot{X}_{\eta} = \dot{F}(\check{\eta})$.*

The first two of these are entirely straightforward. For the last, note that for any $\alpha < \kappa$

$$\begin{aligned} u\big([\![\check{\alpha} \in \dot{X}]\!] \wedge p\big) &= u\Big(\bigvee_{\eta<\nu} [\![\check{\alpha} \in \dot{F}(\check{\eta})]\!] \wedge p\Big) = u\Big(\bigvee_{\eta<\nu} [\![\check{\alpha} \in \dot{X}_\eta]\!] \wedge p\Big) \\ &= \sum_{\eta<\nu} u\big([\![\check{\alpha} \in \dot{X}_\eta]\!] \wedge p\big), \end{aligned}$$

by the countable additivity of u, since only countably many terms on the right are nonzero.

(9.115) **Claim** *Suppose $p \in \mathfrak{B}^+$, $p \Vdash \dot{X} \subseteq \check{\kappa}$, and $x \in [0,1]$. Then*

$$\forall q \leqslant p\, \exists r \leqslant q\, (\mu_{\dot{X}} r \leqslant x) \rightarrow \mu_{\dot{X}} p \leqslant x,$$

and the same with '$\geqslant$' for '$\leqslant$' in the last two occurrences.

Proof Let A be a maximal antichain in $\mathfrak{B}^+$ below p such that $\forall r \in A\ \mu_{\dot{X}} r \leqslant x$. Assuming the antecedent of the claim, A is predense below p by virtue of its maximality, so $p = \bigvee A$. As an antichain in $\mathfrak{B}$, A is countable, so since U is countably complete there exists $Z \in U$ such that $\forall \alpha \in Z\, \forall r \in A\ u([\![\check{\alpha} \in \dot{X}]\!] \wedge r) \leqslant x \cdot u(r)$. Since u is countably additive and $p = \bigvee A$, for any $\alpha \in Z$

$$u([\![\check{\alpha} \in \dot{X}]\!] \wedge p) = \sum_{r \in A} u([\![\check{\alpha} \in \dot{X}]\!] \wedge r) \leqslant x \cdot \sum_{r \in A} u(r) = x \cdot u(p).$$

Hence, $\mu_{\dot{X}} p \leqslant x$. $\square$[9.115]

Given a $\mathfrak{B}$-term $\dot{X}$, let $\mu^*_{\dot{X}} : \mathfrak{B}^+ \rightarrow [0,1]$ be such that if $p \Vdash \dot{X} \subseteq \check{\kappa}$ then

$$\mu^*_{\dot{X}} p = \inf_{q \leqslant p} \mu_{\dot{X}} q. \tag{9.116}$$

Using (8.109), for each $\mathfrak{B}$-term $\dot{X}$ let $m_{\dot{X}}$ be a $\mathfrak{B}$-term such that

$$\Vdash \ulcorner (\dot{X}) \subseteq (\check{\kappa}) \rightarrow (m_{\dot{X}}) = \sup_{p \in \mathsf{G}} (\check{\mu^*_{\dot{X}}}) p \urcorner. \tag{9.117}$$

(9.118) **Claim** *For any $p \in \mathfrak{B}^+$, $\mathfrak{B}$-term $\dot{X}$, and $x \in [0,1]$, if $p \Vdash \dot{X} \subseteq \check{\kappa}$ then*

$$\mu^*_{\dot{X}} p \geqslant x \leftrightarrow p \Vdash m_{\dot{X}} \geqslant \check{x}.$$

Proof $\rightarrow$ is obvious. For the converse, suppose $p \Vdash m_{\dot{X}} \geqslant \check{x}$, and suppose $y \in [0,x)$. It suffices to show that $\mu^*_{\dot{X}} p \geqslant y$. By virtue of (9.117), $p \Vdash \exists r \in \mathsf{G}\, \check{\mu^*_{\dot{X}}} r \geqslant \check{y}$. Hence, $\{r \in \mathfrak{B}^+ \mid \mu^*_{\dot{X}} r \geqslant y\}$ is predense below p. It follows that $\{r \in \mathfrak{B}^+ \mid \mu_{\dot{X}} r \geqslant y\}$ is dense below p, and therefore dense below every extension of p, so[9.115] $\mu_{\dot{X}} q \geqslant y$ for every $q \leqslant p$, from which it follows that $\mu^*_{\dot{X}} p \geqslant y$. $\square$[9.118]

(9.119) **Claim**

1. *For any $X \subseteq \kappa$, $\Vdash m_{\check{X}} = 1$ if $X \in U$; otherwise, $\Vdash m_{\check{X}} = 0$.*
2. *Suppose $\dot{X}, \dot{Y}$ are $\mathfrak{B}$-terms.*
    1. *If $p \Vdash \dot{X} \subseteq \dot{Y} \subseteq \check{\kappa}$ then $p \Vdash m_{\dot{X}} \leqslant m_{\dot{Y}}$.*

2. *Hence,* $\Vdash (\dot X = \dot Y \subseteq \check\kappa \to m_{\dot X} = m_{\dot Y})$.

Proof The first claim is immediate from (9.114.1). For the second, observe that otherwise there exists $q \leqslant p$ and $x \in [0,1]$ such that $q \Vdash m_{\dot X} \geqslant \check x > m_{\dot Y}$, whence[9.118] $\mu^*_{\dot X} q \geqslant x > \mu^*_{\dot Y} q$. From the definition of μ^* it follows that for some $r \leqslant q$, $\mu_{\dot Y} r < \mu_{\dot X} r$, contrary to (9.114.2). □[9.119]

By virtue of (9.119.2.2) there is a $\mathfrak{B}$-term $\dot\mu$ such that $\Vdash \dot\mu : \mathcal{P}\check\kappa \to [0,1]$ and for any $\mathfrak{B}$-term $\dot X$,

$$\Vdash \dot\mu \dot X = m_{\dot X}. \tag{9.120}$$

Now we'll work again in a generic extension. Let G be a V-generic filter on $\mathfrak{B}$, and let

$$\mu = \dot\mu^G.$$

(9.121) **Claim** *μ is finitely additive.*

Proof Suppose $X, Y \subseteq \kappa$ and $X \cap Y = 0$. Let $Z = X \cup Y$. We show first that $\mu Z \geqslant \mu X + \mu Y$. It is enough to show that for any $x, y \in \mathsf{V}$, if $\mu X \geqslant x$ and $\mu Y \geqslant y$ then $\mu Z \geqslant x + y$, because the reals in V are dense in $\mathbb{R}$ (i.e., $\mathbb{R}$ in the sense of the "real world", which we are taking to be $\mathsf{V}[G]$). (Every rational is in V, for example.) To this end, let $\dot X$, $\dot Y$, $\dot Z$, and $p \in G$ be such that $\dot X^G = X$, $\dot Y^G = Y$, $\dot Z^G = Z$, and $p \Vdash \dot X, \dot Y \subseteq \check\kappa$, $p \Vdash \dot Z = \dot X \cup \dot Y$, $p \Vdash m_{\dot X} \geqslant \check x$, and $p \Vdash m_{\dot Y} \geqslant \check y$. Then[9.114.3]

$$\mu^*_{\dot Z} p \geqslant \mu^*_{\dot X} p + \mu^*_{\dot Y} p,$$

and[9.118]

$$\mu^*_{\dot X} p + \mu^*_{\dot Y} p \geqslant x + y.$$

Hence[9.118] $p \Vdash m_{\dot Z} \geqslant \check x + \check y$, so $\mu Z \geqslant x + y$.

To show that $\mu Z \leqslant \mu X + \mu Y$, suppose the contrary. Then there exist $x, x', y, y' \in \mathsf{V}$ and $p \in G$ such that $p \Vdash \dot\mu \dot X < \check x < \check x'$, $p \Vdash \dot\mu \dot Y < \check y < \check y'$, and $p \Vdash \dot\mu \dot Z \geqslant \check x' + \check y'$. Hence[9.120] $p \Vdash m_{\dot X} < \check x$, so[9.118] for every $q \leqslant p$, $\mu^*_{\dot X} q < x$, so[9.116] there exists $r \leqslant q$ such that $\mu_{\dot X} r < x$. Thus[9.115] $\mu_{\dot X} p \leqslant x$. Likewise, $\mu_{\dot Y} p \leqslant y$, so

$$\mu^*_{\dot Z} p \leqslant \mu_{\dot Z} p = \mu_{\dot X} p + \mu_{\dot Y} p \leqslant x + y < x' + y',$$

whence[9.118] $p \nVdash m_{\dot Z} \geqslant \check x' + \check y'$, contrary to our assumption that $p \Vdash \dot\mu \dot Z \geqslant \check x' + \check y'$. □[9.121]

Finally we have to show κ-additivity. Thus, suppose $\nu < \kappa$ and F is a ν-sequence of disjoint subsets of κ. Let $X = \bigcup_{\eta \in \nu} F(\eta)$. Let $\dot F$ be such that $\dot F^G = F$, and let $p \in G$ be such that $p \Vdash \ulcorner (\dot F)$ is a $(\check\nu)$-sequence of pairwise disjoint subsets of $(\check\kappa)\urcorner$. There are terms $\dot X$ and $\dot X_\eta$, $\eta < \nu$, such that $p \Vdash \ulcorner (\dot X) = \bigcup \operatorname{im} (\dot F)\urcorner$, and for each $\eta < \nu$, $p \Vdash \ulcorner (\dot X_\eta) = (\dot F)(\check\eta)\urcorner$.

Recall that the sum of any set of non-negative real numbers is by definition the supremum of the sums of its finite subsets. It follows from the finite additivity of μ[9.121] that $\mu X \geqslant \sum_{\eta<\nu} \mu X_\eta$. To show that $\mu X \leqslant \sum_{\eta<\nu} \mu X_\eta$, suppose the contrary, and let $x, x' \in \mathsf{V}$ be such that

$$\sum_{\eta<\nu} \mu X_\eta < x < x' \leqslant \mu X.$$

Let $p' \in G$ be such that $p' \leqslant p$ and $p \Vdash \check{x}' \leqslant \dot{\mu}\dot{X}$. Then[9.120] $p' \Vdash m_{\dot{X}} \geqslant \check{x}'$, so[9.118]

$$\mu^*_{\dot{X}} p' \geqslant x'. \tag{9.122}$$

For any finite $s \subseteq \nu$, let $X_s = \bigcup_{\eta \in s} X_\eta$. By finite additivity, $\mu X_s < x$. Let $\dot{X}_s$ be such that $p \Vdash \ulcorner(\dot{X}_s) = (\dot{X}_{\eta_0}) \cup \cdots \cup (\dot{X}_{\eta_{n^-}})\urcorner$, where $\langle \eta_m \mid m < n\rangle$ enumerates s. Let $p'' \in G$ be such that $p'' \leqslant p'$ and $p'' \Vdash \dot{\mu}\dot{X}_s < \check{x}$. As in the proof of (9.121), it follows that $\{r \mid \mu_{\dot{X}_s} r < x\}$ is dense below p'', so[9.115] $\mu_{\dot{X}_s} p'' \leqslant x$; and[9.114.3] $\mu_{\dot{X}_s} p'' = \sum_{\eta \in s} \mu_{\dot{X}_\eta} p''$, so $\sum_{\eta \in s} \mu_{\dot{X}_\eta} p'' \leqslant x$. Since this is true for any finite $s \subseteq \nu$, by virtue of (9.114.3) $\mu_{\dot{X}} p'' = \sum_{\eta < \nu} \mu_{\dot{X}_\eta} p'' \leqslant x$.

Hence[9.116] $\mu^*_{\dot{X}} p' \leqslant x < x'$, contrary to (9.122). □[9.113.2] □[9.113]

9.5 Larger cardinals

We have now given several properties of a cardinal that imply that it is large, beginning with inaccessibility and the Mahlo hierarchy, and proceeding through weak compactness, measurability, and strong compactness. As we have seen,[9.73] weak compactness does not exceed the threshold of inconsistency with $\boldsymbol{V} = \boldsymbol{L}$; it is in that sense a "small" large cardinal property. We have presented measurability as the paradigm of a truly large cardinal property. Recall the characterization[9.23, 9.29] of measurability in terms of elementary embeddings:

κ is measurable iff there exists an elementary embedding of V into a transitive class M with critical point κ.

Recall also that ${}^{\kappa}M \subseteq M$ in this event.

9.5.1 Supercompactness

A natural way to strengthen the hypothesis of measurability is to require stronger closure properties of M. An early such enhancement, which has proven quite useful, is the notion of *supercompactness*.

(9.123) **Definition** [GBC] *Suppose κ is a cardinal.*

1. *Suppose $\gamma \geqslant \kappa$. κ is* γ-supercompact $\overset{\text{def}}{\iff}$ *there exists an elementary embedding j of V into a transitive class M with critical point κ such that*
   1. *$j\kappa > \gamma$; and*
   2. *${}^{\gamma}M \subseteq M$.*
2. *κ is* supercompact $\overset{\text{def}}{\iff}$ *κ is γ-supercompact for all $\gamma \in$* Ord.

Supercompactness is stronger than measurability:

(9.124) **Theorem** [GBC] *Suppose κ is 2^κ-supercompact. Then there is a normal ultrafilter U over κ such that the set of measurable cardinals below κ is in U.*

Proof Let $A = \{\alpha < \kappa \mid \alpha \text{ is measurable}\}$. Let $j : V \to M$ witness the 2^κ-supercompactness of κ. Let $U = \{X \subseteq \kappa \mid \kappa \in jX\}$. Then, as usual, U is a normal ultrafilter over κ. Since $\mathcal{P}\kappa \subseteq M$ and ${}^{(2^\kappa)}M \subseteq M$, $U \in M$, so $M \models \ulcorner[\kappa]$ is measurable$\urcorner$. Hence $\kappa \in jA$, so $A \in U$. □[9.124]

The origin of the term 'supercompact' is the comparison with 'strongly compact' as formulated in terms of fine κ-complete ultrafilters.[9.53, 9.54] Supercompactness adds the condition of *normality*.

Definition [ZFC] *Suppose κ is an uncountable cardinal, $\gamma \geqslant \kappa$, and F is a filter over $\mathcal{P}_\kappa \gamma$. F is* normal $\overset{\text{def}}{\iff}$ *F is κ-complete and fine and for every family $\langle X_\alpha \mid \alpha \in \gamma \rangle$ of members of F, the* diagonal intersection $\Delta_{\alpha\in\gamma} X_\alpha \overset{\text{def}}{=} \{x \in \mathcal{P}_\kappa \gamma \mid \forall \alpha \in x \; x \in X_\alpha\}$ *is in F.*

The normality condition may also be stated in terms of regressive functions on F-stationary sets. $X \subseteq \mathcal{P}_\kappa \lambda$ is *F-stationary* iff X is not in the ideal dual to F, i.e., X meets every member of F. $f : \mathcal{P}_\kappa \gamma \rightharpoonup \gamma$ is *regressive* $\overset{\text{def}}{\iff} \forall x \in \operatorname{dom} f \, (x \neq 0 \rightarrow f(x) \in x)$ (f is a choice function).

A fine κ-complete filter F over $\mathcal{P}_\kappa \gamma$ is normal iff for every F-stationary X and regressive $f : X \to \gamma$, there exists $\alpha \in \gamma$ such that $f^{\leftarrow}\{x\}$ is F-stationary.

The following theorem provides a characterization of supercompactness in terms of normal ultrafilters, along with a characterization in terms of elementary embeddings of set structures. (Cf. (9.29) and the remarks preceding it.)

(9.125) **Theorem** [GBC] *Suppose κ is an uncountable cardinal and $\gamma \geqslant \kappa$. The following are equivalent.*

1. *κ is γ-supercompact.*
2. *There exist transitive classes N and M and an elementary embedding $j : N \to M$ with critical point κ such that*
   1. *$V_{\gamma+\omega} \subseteq N$;*
   2. *$j\kappa > \gamma$; and*
   3. *$j^{\rightarrow}\gamma \in M$.*
3. *There is a normal ultrafilter over $\mathcal{P}_\kappa \gamma$.*

Remark (9.125.2.1) could be weakened, but this is sufficient for our purposes.

Proof 1 → 2 Immediate.

2 → 3 Let $\Gamma = j^{\rightarrow}\gamma$. Then $\Gamma \in M$[9.125.2.3] and

(9.126) $$M \models \ulcorner [\Gamma] \in \mathcal{P}_{[j\kappa]} [j\gamma] \urcorner.$$ [9.125.2.2]

Note that $\mathcal{P}(\mathcal{P}_\kappa \gamma) \subseteq N$.[9.125.2.1] Let

$$U = \{X \subseteq \mathcal{P}_\kappa \gamma \mid \Gamma \in jX\}.$$

Suppose $X, Y \subseteq \mathcal{P}_\kappa \gamma$, and $X \cup Y = \mathcal{P}_\kappa \gamma$. Then $M \models \ulcorner [jX] \cup [jY] = \mathcal{P}_{[j\kappa]} [j\gamma] \urcorner$, so $M \models \ulcorner [\Gamma] \in [jX] \urcorner$ or $M \models \ulcorner [\Gamma] \in [jY] \urcorner$, so $\Gamma \in jX$ or $\Gamma \in jY$. Thus, U is an ultrafilter.

Suppose $\beta < \kappa$ and $f : \beta \to U$. Then $f \in N$,[9.125.2.1] and $\operatorname{dom}(jf) = j\beta = \beta$, since j is the identity on κ; for the same reason, for each $\alpha < \beta$, $(jf)\alpha = (jf)(j\alpha) = j(f\alpha)$. Let $X = \bigcap_{\alpha<\beta} f\alpha$. Then $M \models \ulcorner [jX] = \bigcap_{\alpha<[\beta]} [jf]\alpha \urcorner$, so

$jX = \bigcap_{\alpha<\beta}(jf)\alpha$. For each $\alpha < \beta$, $f\alpha \in U$, so $\Gamma \in j(f\alpha) = (jf)\alpha)$. Hence, $\Gamma \in \bigcap_{\alpha<\beta}(jf)\alpha = jX$, so $X \in U$. U is therefore κ-complete.

Suppose $\alpha \in \gamma$. Let $X = \{x \in \mathcal{P}_\kappa\gamma \mid \alpha \in x\}$. Then $M \models \ulcorner [jX] = \{x \in \mathcal{P}_{[j\kappa]}\,[j\gamma] \mid [j\alpha] \in x\}\urcorner$. Since $j\alpha \in \Gamma$, $M \models \ulcorner [\Gamma] \in [jX]\urcorner$.[9.126] Hence, $\Gamma \in jX$, so $X \in U$. Thus, U is fine.

Finally, suppose $X \in U$ and f is regressive on X. Then jf is regressive on jX. Since $X \in U$, $\Gamma \in jX$, and $(jf)\Gamma \in \Gamma$. Let $\alpha \in \gamma$ be such that $j\alpha = (jf)\Gamma$, and let $Y = \{x \in X \mid fx = \alpha\}$. Then $jY = \{x \in jX \mid (jf)x = j\alpha\}$. By construction, $\Gamma \in jX$ and $(jf)\Gamma = j\alpha$, so $\Gamma \in jY$, and Y is therefore in U. Thus, U is normal.

3 → 1 Since U is countably complete, the ultrapower ${}^{(\mathcal{P}_\kappa\gamma)}V/U$ is wellfounded. Let $\pi : {}^{(\mathcal{P}_\kappa\gamma)}V/U \to M$ be the transitive collapse, and let $j : V \to M$ be the canonical elementary embedding. We will show that j satisfies (9.123.1).

(9.127) *The following identities are easily verified, where x ranges over $\mathcal{P}_\kappa\gamma$, and f, g range over ${}^{\mathcal{P}_\kappa\gamma}V$.*

1. $\pi[x \mapsto x] = j^{\to}\gamma$.
2. *Let* $\Gamma = j^{\to}\gamma$.
    1. $U = \{X \subseteq \mathcal{P}_\kappa\gamma \mid \Gamma \in jX\}$.
    2. $\pi[f] = \pi[g]$ *iff* $(jf)\Gamma = (jg)\Gamma$.
    3. $\pi[f] \in \pi[g]$ *iff* $(jf)\Gamma \in (jg)\Gamma$.
3. $\pi[x \mapsto \operatorname{ot}(x \cap \alpha)] = \alpha$ *for all* $\alpha \leqslant \gamma$. *In particular:*
    1. $\pi[x \mapsto \alpha] = \pi[x \mapsto x \cap \alpha] = \pi[x \mapsto \operatorname{ot}(x \cap \alpha)] = \alpha$ *for all* $\alpha < \kappa$.
    2. $\pi[x \mapsto x \cap \kappa] = \pi[x \mapsto \operatorname{ot}(x \cap \kappa)] = \kappa$.
    3. $\pi[x \mapsto \operatorname{ot} x] = \gamma$.

(9.127.3.1, 2) imply that $\operatorname{crit} j = \kappa$, and (9.127.3.3) shows that $j\kappa > \gamma$. To show that ${}^\gamma M \subseteq M$, it is enough to show that for any $f : \gamma \to M$, $\operatorname{im} f \in M$ (because we can replace f by $f' = \{(\alpha, (\alpha, f\alpha)) \mid \alpha \in \gamma\}$, and $\operatorname{im} f' = f$). For each $\alpha < \gamma$, let g_α be such that $\pi[g_\alpha] = f\alpha$. Let g be the function on $\mathcal{P}_\kappa\gamma$ such that $gx = \{g_\alpha x \mid \alpha \in x\}$. Then $\pi[g] = \operatorname{im} f$. □[9.125]

The following corollary provides another useful characterization of supercompactness.

(9.128) **Theorem** [ZFC] *Suppose κ is an uncountable cardinal and $\gamma \geqslant \kappa$. Then κ is γ-supercompact iff there exist transitive sets N and M and an elementary embedding $j : N \to M$ with critical point κ such that*

1. $V_{\gamma+\omega} \subseteq N$;
2. $j\kappa > \gamma$; *and*
3. $j^{\to}\gamma \in M$.

Proof We have stated this as a theorem of ZFC, so it may be presumed that our definition of supercompactness is (9.125.3). Nevertheless, we may use GBC to prove it, since GBC is a conservative extension of ZFC. Thus, in the forward direction, suppose j is an elementary embedding of V into a transitive class M satisfying our original definition of γ-supercompactness (9.123.1). Let $j' = j \restriction V_{\gamma+\omega}$, and let $M' = M \cap V_{j\gamma+\omega}$. Then j' is an elementary embedding of $V_{\gamma+\omega}$ into

$jV_{\gamma+\omega} = \ulcorner V_{[j\gamma]+\omega} \urcorner^M = M'$. Clearly $\operatorname{crit} j' = \operatorname{crit} j = \kappa$, and $j'\kappa = j\kappa > \gamma$. $j'^{\rightarrow}\gamma = j^{\rightarrow}\gamma$, and $j^{\rightarrow}\gamma$ is in both M and $V_{j\gamma+\omega}$, so it is in M'.

We have already proved the converse, since the hypothesis here implies (9.125.2). $\square$[9.128]

An uncountable cardinal κ is therefore supercompact iff there exists a normal fine κ-complete ultrafilter on $\mathcal{P}_\kappa \gamma$ for every $\gamma \geqslant \kappa$. Thus, supercompactness adds the condition of normality to definition of strong compactness in terms of the existence of ultrafilters. Recall that in the case of ultrafilters over κ (which is, for κ-complete filters, $\mathcal{P}_\kappa \kappa$, in effect) normality comes free (see headnote to Section 9.2.4). This led Solovay to conjecture that strong compactness is equivalent to supercompactness, which has been refuted. In fact, relative to the consistency of a strongly compact cardinal, it is consistent that the least measurable cardinal is strongly compact; and relative to the consistency of a supercompact cardinal, it is consistent that the least strongly compact cardinal is supercompact; whereas, we know already that any supercompact cardinal κ is the κth measurable.[9.124]

9.5.2 An upper limit

The characterization of large cardinal properties in terms of elementary embeddings of V into transitive classes M with various closure properties leads naturally to what is evidently the strongest possible such hypothesis: the existence of a nontrivial elementary embedding of V into V. This was briefly considered by William Reinhardt in his doctoral dissertation, and the critical point of such an embedding has been referred to as a *Reinhardt cardinal*. Kunen showed that the existence of such an embedding is inconsistent with GBC, but it has not been shown to be inconsistent with GB.

To obtain Kunen's result we first establish a combinatorial theorem of Erdös and Hajnal

(9.129) **Theorem** [ZFC] *For any infinite cardinal λ there is a function f that is* ω-Jónsson *for λ, i.e.,*

1. $f : [\lambda]^\omega \to \lambda$*; and*
2. *for any $X \in [\lambda]^\lambda$, $f^{\rightarrow}[X]^\omega = \lambda$.*[41]

Proof Define an equivalence relation on $[\lambda]^\omega$ by letting $x \equiv y$ iff for some ordinal α, $0 \neq x\backslash\alpha = y\backslash\alpha$ (x and y are *eventually equal*). Let E be the set of equivalence classes, and let g be a choice function for E. Define $f : [\lambda]^\omega \to \lambda$ so that fx is the least $\alpha \in g[x]$ such that $x\backslash(\alpha + 1) = g[x]\backslash(\alpha + 1)$.

We will show that for some $X \in [\lambda]^\lambda$, for every $Y \in [X]^\lambda$, $f^{\rightarrow}[Y]^\omega \supseteq X$. To this end, suppose to the contrary that for every $X \in [\lambda]^\lambda$ there exists $Y \in [X]^\lambda$ and $\alpha \in X$ such that $\alpha \notin f^{\rightarrow}[Y]^\lambda$. Let $\lambda = X_0 \supseteq X_1 \supseteq \cdots$ and $\alpha_0 < \alpha_1 < \cdots$ be such that for each $n \in \omega$, $X_n \in [\lambda]^\lambda$, $\alpha_n \in X_n$, and

$$\alpha_n \notin f^{\rightarrow}[X_{n+1}]^\omega. \tag{9.130}$$

(Given X_n and α_m ($m < n$), let α be the least ordinal that exceeds every α_m, $m < n$. Hence, $\alpha = 0$ if $n = 0$ and $\alpha = \alpha_{n^-} + 1$ if $n > 0$. Let $X = X_{n^-}\backslash\alpha$. Let $\alpha_n \in X$ and $X_{n+1} \in [X]^\lambda$ be such that $\alpha_n \notin f^{\rightarrow}[X_{n+1}]^\omega$.) Let $x = \{\alpha_n \mid n \in \omega\}$,

[41] Recall[9.59.2] that $[\lambda]^\alpha$ is the set of subsets of λ of order type α. Since λ is a cardinal, $X \in [\lambda]^\lambda$ iff $X \subseteq \lambda$ and $|X| = \lambda$.

and let $m \in \omega$ be such that $\{\alpha_n \mid n \geqslant m\} = g[x] \backslash \alpha_m$. Let $y = \{\alpha_n \mid n > m\}$. Since $y \equiv x$, $g[y] = g[x]$, so $fy = \alpha_m$. But $y \in [X_{m+1}]^\omega$, so $\alpha_m = fy \in f^{\to}[X_{m+1}]^\omega$, contrary to (9.130).

Thus, there exists $X \in [\lambda]^\lambda$ such that for every $Y \in [X]^\lambda$, $f^{\to}[Y]^\omega \supseteq X$. Let $f' : [X]^\omega \to X$ be such that $f'x = fx$ if $fx \in X$; otherwise $f'x =$ the least element of X. Then use the isomorphism of $(X; <)$ with $(\lambda; <)$ to define a function that is ω-Jónsson for λ. $\square^{9.129}$

(9.131) **Theorem (Kunen)** [GBC] *There does not exist a nontrivial elementary embedding $j : V \to V$.*

Proof Let $\kappa = \operatorname{crit} j$. For each $n \in \omega$, let j^n be the nth iterate of j, i.e., $j^0 x = x$, and $j^{n+1}x = j(j^n x)$. Let $\kappa_n = j^n \kappa$ for each $n \in \omega$. Let $\lambda = \sup_{n\in\omega} \kappa_n$. By elementarity, each κ_n is a cardinal, so λ is a cardinal. Also by elementarity, $j\lambda = \sup j\{\kappa_n \mid n \in \omega\} = \sup\{\kappa_{n+1} \mid n \in \omega\} = \lambda$.

Let f be ω-Jónsson for λ. By elementarity, jf is ω-Jónsson for $j\lambda = \lambda$. Since $j^{\to}\lambda \in [\lambda]^\lambda$, $(jf)^{\to}[j^{\to}\lambda]^\omega = \lambda$. Let $x \in [j^{\to}\lambda]^\omega$ be such that $(jf)x = \kappa$, and let $y \in [\lambda]^\omega$ be such that $jy = x$. Then $\kappa = (jf)x = (jf)(jy) = j(fy) \in j^{\to}\lambda$; but $\kappa \notin j^{\to}\lambda$. $\square^{9.131}$

As of this writing, it has not been shown that the existence of a nontrivial elementary embedding of V into V is inconsistent with GB (without Choice); nevertheless, Kunen's theorem places a definite ceiling on the method of defining large cardinals by elementary embeddings. This ceiling is actually a bit lower than the statement of the theorem suggests: The proof of (9.131) is easily adapted to prove the following theorem.

(9.132) **Theorem** [GBC]

1. *There does not exist an elementary embedding j of V into a transitive class M with critical point κ such that ${}^\lambda M \subseteq M$, where $\lambda = \sup_{n\in\omega} j^n\kappa$.*
2. *There does not exist a nontrivial elementary embedding of $V_{\delta+2}$ into $V_{\delta+2}$ for any δ.*

There remains the possibility of embeddings with less generous closure properties along the same lines. The following is in the vein of (9.132.1).[42]

(9.133) **Definition** [GBC] *Suppose κ is a cardinal.*

1. *For $n \in \omega$, κ is* n-huge $\overset{\text{def}}{\iff}$ *there exists an elementary embedding j of V into a transitive class M with critical point κ such that ${}^{(j^n\kappa)}M \subseteq M$.*
2. *κ is* huge $\overset{\text{def}}{\iff}$ *κ is 1-huge.*
3. *κ is* almost huge $\overset{\text{def}}{\iff}$ *there exists an elementary embedding j of V into a transitive class M with critical point κ such that ${}^{<(j\kappa)}M \subseteq M$.*

The following hypotheses are in the vein of (9.132.2), curiously named in decreasing order by strength.[43]

[42] Note that by natural extension of this definition, Kunen's theorem could be said to exclude the possibility of an ω-huge cardinal.

[43] Note that I3 and I1 begin a sequence that arithmetically continued generates the name 'I-1' for Kunen's inconsistency result in the form (9.132.2), as indicated in Figure 9.1.

Definition [GBC]

I0. *For some δ there exists an elementary embedding $j : L(V_{\delta+1}) \to L(V_{\delta+1})$ with* $\operatorname{crit} j < \delta$.

I1. *For some δ there exists a nontrivial elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$.*

I2. *For some δ there exists an elementary embedding j of V into a transitive class M such that $V_\delta \subseteq M$, $\operatorname{crit} j < \delta$, and $j\delta = \delta$.*

I3. *For some δ there exists a nontrivial elementary embedding $j : V_\delta \to V_\delta$.*

I1–I3 were defined first. I0 was inserted later by Woodin for a specific purpose.

One final definition along these lines is rather technical, but quite useful, and intimately related to determinacy.

Definition [GBC] *A cardinal κ is* Woodin $\overset{\text{def}}{\iff}$ *for every $f : \kappa \to \kappa$ there exists an elementary embedding $j : V \to M$ with critical point $\alpha < \kappa$ such that*

1. $f^{\to}\alpha \subseteq \alpha$ *and*
2. $V_{(jf)\alpha} \subseteq M$.

9.5.3 Vopěnka's principle and extendibility

We have seen that the "largeness" of a large cardinal κ is often understandable as a statement that κ is much larger than any smaller cardinal. We have noted that ω is the epitome of large cardinals in this respect. For uncountable "large" κ, this often means that—in an appropriate sense—whatever happens below κ, happens repeatedly below κ. Vopěnka proposed the following principle as a statement that Ord itself is large in this sense. It is perhaps the purest assertion that Ord goes on forever and ever.[44]

Vopěnka's principle [GB] *Suppose ρ is a signature and C is a proper class of ρ-structures. Then there exist distinct $\mathfrak{S}, \mathfrak{S}' \in C$ such that $\mathfrak{S}$ is elementarily embeddable in $\mathfrak{S}'$.*

As usual, a version of Vopěnka's principle may be stated in the context of ZF as a schema using s-formulas that define proper classes (from parameters). It is clear that the mandate of elementary embeddability is unnecessary, as we may expand each structure by the addition of its satisfaction relation. Any embedding of the expanded structures is an elementary embedding of the original structures. Indeed, we get the same strength if we limit the structures to be binary relations and require only embeddability.

We will assess the strength of Vopěnka's principle in tandem with the notion of *extendibility.*

Definition [ZF] *A cardinal κ is* extendible $\overset{\text{def}}{\iff}$ *for every $\alpha > \kappa$ there exists an ordinal β and an elementary embedding of V_α into V_β with critical point κ.*

(9.134) **Theorem** [GBC] *Assume Vopěnka's principle. Then there is an extendible cardinal.*

[44] See *Ulysses* headnote.p. 578

Proof Let ϕ be the s-formula $\ulcorner \lceil v \rceil$ is an extendible cardinal$\urcorner$. By this, of course, we mean that ϕ is some such s-formula (with some variable v). We presume that ϕ has a subformula $\phi' = \ulcorner \lceil v \rceil < \lceil v' \rceil$ and there is an elementary embedding of $V_{\lceil v' \rceil}$ into some V_δ with critical point $\lceil v \rceil\urcorner$. The reflection principle[6.12] states that $A = \{\alpha \in \mathrm{Ord} \mid V_\alpha \prec^{\{\phi\}} V\}$ is closed unbounded in Ord. Keep in mind that if $V_\alpha \prec^{\{\phi\}} V$ then $V_\alpha \prec^{\{\phi'\}} V$ also. Thus for any $\alpha \in A$

(9.135)

1. for every $\kappa < \alpha$, if $V_\alpha \models \ulcorner [\kappa]$ is extendible$\urcorner$ then κ is extendible; and
2. for every $\kappa < \gamma < \alpha$, if there exists an elementary embedding of V_γ into some V_δ with critical point κ then $V_\alpha \models \ulcorner$there exists an elementary embedding of $V_{[\gamma]}$ into some V_δ with critical point $[\kappa]\urcorner$.

Let B be the class of limit points of A of cofinality ω. Let

$$C = \{(V_{\alpha+1}; \in) \mid \alpha \in B\}.$$

By Vopěnka's principle there exists an elementary embedding $j : V_{\alpha+1} \to V_{\beta+1}$ for some $\alpha, \beta \in B$ with $\alpha < \beta$. Since $j\alpha = \beta$, j moves an ordinal. Let $\kappa = \mathrm{crit}\, j$. Then κ is measurable,[9.29] so κ is not α, which has cofinality ω.

Now suppose $\kappa < \gamma < \alpha$. Then[9.28.10] $j \restriction V_\gamma$ is an elementary embedding of V_γ into $V_{j\gamma}$ with critical point κ, so[9.135.2] $V_\alpha \models \ulcorner$there exists an elementary embedding of $V_{[\gamma]}$ into some V_δ with critical point $[\kappa]\urcorner$. Hence, $V_\alpha \models \ulcorner [\kappa]$ is extendible$\urcorner$, so[9.135.1] κ is extendible. □[9.134]

So, how strong are extendible cardinals? We will limit our remarks to the establishment of two supercompactness consequences. We begin with a lemma.

(9.136) **Theorem** [ZFC] *Suppose κ is λ-supercompact, where $\lambda \geqslant \kappa$ is regular, and suppose $\alpha < \kappa$, and α is γ-supercompact for all $\gamma < \kappa$. Then α is λ-supercompact.*

Remark In stating this as a theorem of ZFC we are obviously presuming a definition of supercompactness that does not refer to proper classes, such as (9.125.3) or (9.128). The proof is given in GBC, and we invoke the fact that GBC is a conservative extension of ZFC to infer the existence of a proof in ZFC.

Proof Let U be a normal ultrafilter over $\mathcal{P}_\kappa \lambda$. Let $j_U : V \to M_U$ be the canonical embedding of V into a transitive class M_U. Then

$$M_U \models \ulcorner [j_U \alpha] \text{ is } \gamma\text{-supercompact for all } \gamma < [j_U \kappa]\urcorner.$$

By construction, $\mathrm{crit}\, j_U = \kappa > \alpha$, so $j_U \alpha = \alpha$; and $j_U \kappa > \lambda$. Hence,

$$M_U \models \ulcorner [\alpha] \text{ is } [\lambda]\text{-supercompact}\urcorner.$$

Therefore, let $U' \in M_U$ be such that $M_U \models \ulcorner [U']$ is a normal ultrafilter over $\mathcal{P}_{[\alpha]} [\lambda]\urcorner$.

(9.137) **Claim** $\mathcal{P}\mathcal{P}_\alpha \lambda \subseteq M_U$.

Proof Suppose $x \in \mathcal{P}_\alpha \lambda$. Let $f_x : \mathcal{P}_\kappa \lambda \to V$ be such that $\forall y \in \mathcal{P}_\kappa \lambda\ f_x y = \pi_y{}^{\to} x$, where $\pi_y : y \to \mathrm{ot}\, y$ is the collapsing map (order-preserving bijection).

(9.138) **Claim** $\pi[f_x] = x$, *where* $\pi : {}^{\mathcal{P}_\kappa \lambda} V/U \to M_U$ *is the collapsing map.*

Proof Since $\pi[y \mapsto \operatorname{ot} y] = \lambda$,[9.127.3.3] $\pi[f_x] \subseteq \lambda$. Suppose $\beta \in \lambda$. For almost all $y \in \mathcal{P}_\kappa \lambda$, $x \cup \{\beta\} \subseteq y$. For any such y, $\pi_y \beta = \operatorname{ot}(y \cap \beta)$, and $\pi_y \beta \in f_x y$ $(= \pi_y{}^{\to} x)$ iff $\beta \in x$. Since[9.127.3] $\pi[y \mapsto \operatorname{ot}(y \cap \beta)] = \beta$, $\beta \in \pi[f_x]$ iff $\beta \in x$. $\square$[9.138]

Suppose $X \subseteq \mathcal{P}_\alpha \lambda$. Let $f : \mathcal{P}_\kappa \lambda \to V$ be such that for each $y \in \mathcal{P}_\kappa \lambda$, $fy = \{\pi_y{}^{\to} x \mid x \in X \wedge x \subseteq y\}$.

(9.139) **Claim** $\pi[f] = X$.

Proof For any $y \in \mathcal{P}_\kappa \lambda$, $fy \subseteq \mathcal{P}_\alpha(\operatorname{ot} y)$, so $\pi[f] \subseteq \mathcal{P}_\alpha \lambda$. Suppose $x \in \mathcal{P}_\alpha \lambda$. Then $\pi[f_x] = x$.[9.138] For almost all $y \in \mathcal{P}_\kappa \lambda$, $x \subseteq y$, and for any such y and any $z \subseteq y$, $\pi_y{}^{\to} x = \pi_y{}^{\to} z$ iff $x = z$, so $f_x y \in fy$ iff $x \in X$. It follows that $x \in \pi_f[f]$ iff $x \in X$.
$\square$[9.139] $\square$[9.137]

U' is therefore a normal ultrafilter over $\mathcal{P}_\alpha \lambda$. α is therefore λ-supercompact.
$\square$[9.136]

(9.140) **Theorem** [ZFC] *Suppose κ is extendible. Then*

1. *κ is supercompact; and*
2. *there exists a normal ultrafilter U over κ such that*

$$\{\alpha < \kappa \mid \alpha \textit{ is supercompact}\} \in U.$$

Proof 1 Using the reflection principle as in the proof of (9.134), let $\alpha > \kappa$ be a limit cardinal such that[45]

1. for any $\kappa \leqslant \lambda < \alpha$ if κ is λ-supercompact then $V_\alpha \models \ulcorner [\kappa]$ is $[\lambda]$-supercompact$\urcorner$; and
2. if $V_\alpha \models \ulcorner [\kappa]$ is supercompact$\urcorner$ then κ is supercompact.

Then it suffices to show that if $\kappa \leqslant \lambda < \alpha$ then κ is λ-supercompact, as α may be taken to be arbitrarily large.

Since κ is assumed extendible, there exists an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ. Let $\langle \kappa_n \mid n < \eta \rangle$ be such that $\eta \leqslant \omega$; $\kappa_0 = \kappa$; and for each $n < \eta$, if $j\kappa_n < \alpha$ then $n + 1 < \eta$ and $\kappa_{n+1} = j\kappa_n$. Clearly, $\eta > 0$. If $\eta < \omega$ then $j\kappa_{\eta^-} \geqslant \alpha$. In this case, let $\kappa_\eta = \alpha$. If $\eta = \omega$ then $\sup_{n \in \omega} \kappa_n = \alpha$; otherwise, the proof of (9.132.1) may be adapted to derive a contradiction.

It suffices therefore to show that κ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$ for all $n < \eta$, which we do by induction on $n < \eta$. Suppose $\lambda < \kappa_1 = \min\{j\kappa, \alpha\}$. Then $\lambda < \alpha$, and j witnesses the λ-supercompactness of κ by (9.128).

If $\eta = 1$ we are finished. Otherwise, suppose $0 < n < \eta$ and κ is λ-supercompact for all regular $\lambda < \kappa_n$. Then $V_\alpha \models \ulcorner [\kappa]$ is λ-supercompact for all regular $\lambda < [\kappa_n]\urcorner$, so

(9.141) $$V_\beta \models \ulcorner [j\kappa] \text{ is } \lambda\text{-supercompact for all regular } \lambda < [j\kappa_n]\urcorner.$$

It is also true that

(9.142) $$V_\beta \models \ulcorner [\kappa] \text{ is } \gamma\text{-supercompact for all } \gamma < [j\kappa]\urcorner,$$

[45]Note that we have necessarily formulated 'supercompact' in a way that does not refer to proper classes, e.g., as in (9.125.3) or (9.128), so these statements make sense.

since $\kappa_1 = j\kappa$, so for any $\gamma < j\kappa$ we have just established this in the real world, and any normal ultrafilter over $\mathcal{P}_\kappa\,\gamma$ is in V_β and is recognized by V_β as such. Note that here and elsewhere we make use of absoluteness of various formulas between models of the form V_μ, where μ has various closure properties. For this argument the properties in question have to do with normal ultrafilters and elementary embeddings that are elements of V_μ, and for absoluteness it suffices that μ be a limit ordinal.

Now suppose $\lambda < \kappa_{n+1} \leqslant j\kappa_n$ and λ is regular. By virtue of (9.141) and (9.142) the conditions of (9.136) are satisfied with V_β for V, $j\kappa$ for κ, and κ for α. Examination of the proof of (9.136) shows that the argument is valid with the replacement of V by V_β, and we conclude that $V_\beta \models \ulcorner[\kappa]$ is $[\lambda]$-supercompact$\urcorner$, and this clearly implies that κ is in fact λ-supercompact.

2 We know that κ is supercompact.[9.140.1] Let A be the set of $\delta < \kappa$ such that δ is γ-supercompact for all $\gamma < \kappa$. By (9.136) every $\delta \in A$ is supercompact.

Now let $j : V_\alpha \to V_\beta$ be an elementary embedding with critical point κ, where $\alpha > \kappa$ is a limit ordinal. Then $A \in V_\alpha$. Let $U = \{X \subseteq \kappa \mid \kappa \in jX\}$ be the canonical normal ultrafilter over κ. We wish to show that $A \in U$, i.e., that $\kappa \in jA$. Note that $V_\alpha \models \ulcorner$for all $\delta < [\kappa]$, if δ is γ-supercompact for all $\gamma < [\kappa]$ then $\delta \in [A]\urcorner$. Thus $V_\beta \models \ulcorner$for all $\delta < [j\kappa]$, if δ is γ-supercompact for all $\gamma < [j\kappa]$ then $\delta \in [jA]\urcorner$.

Since κ is supercompact, $V_\beta \models \ulcorner[\kappa]$ is supercompact$\urcorner$. Therefore, since $j\kappa < \beta$, $V_\beta \models \ulcorner[\kappa]$ is γ-supercompact for all $\gamma < [j\kappa]\urcorner$. Thus, $V_\beta \models \ulcorner[\kappa] \in [jA]$, so $\kappa \in jA$, as desired. □[9.140]

The following theorem places an upper bound on the strength of Vopěnka's principle.

(9.143) **Theorem** [ZFC] *Suppose κ is almost huge.*[9.133.3] *Then* $\mathfrak{V}^+_\kappa \models \ulcorner$Vopěnka's principle$\urcorner$, *where* $\mathfrak{V}^+_\kappa = (V_{\kappa+1}; \in, V_\kappa)$ *construed as a* c*-structure with* $|\mathfrak{V}^+_\kappa| = V_{\kappa+1}$ *and* $V^{\mathfrak{V}^+_\kappa} = V_\kappa$.

Remark Vopěnka's principle can only be fully expressed in a class theory. Thus, a statement that Vopěnka's principle is satisfied *vis-à-vis* V_κ must be interpreted to mean that some c-structure $\mathfrak{S} \models \ulcorner$Vopěnka's principle$\urcorner$, where $V^{\mathfrak{S}} = V_\kappa$ and $|\mathfrak{S}|$ consists of V_κ together with a set of subsets of V_κ that are the proper classes in the sense of $\mathfrak{S}$. At a minimum these must include all subsets of V_κ definable over V_κ from parameters in V_κ. We have chosen the maximum, viz., $\mathcal{P}\,V_\kappa = V_{\kappa+1}$, thereby obtaining the strongest interpretation of the statement that Vopěnka's principle is satisfied *vis-à-vis* V_κ.

Proof Suppose $\langle \mathcal{S}_\alpha \mid \alpha < \kappa\rangle \in V_{\kappa+1}$ is a sequence of ρ-structures, where $\rho \in V_\kappa$. Since κ is (strongly) inaccessible, by choosing isomorphs if necessary, we may assume that $|\mathcal{S}_\alpha|$ is an ordinal ($< \kappa$) for each $\alpha < \kappa$.

Suppose $j : V \to M$ witnesses the almost hugeness of κ, and let $U = \{X \subseteq \kappa \mid \kappa \in jX\}$ be the canonical ultrafilter for j. For each $\alpha < \kappa$ let $\mathcal{X}_\alpha$ be the set of $\beta < \kappa$ such that there is an elementary embedding of $\mathcal{S}_\alpha$ into $\mathcal{S}_\beta$. Let $\mathcal{X} = \langle \mathcal{X}_\alpha \mid \alpha < \kappa\rangle$, and let $A = \{\alpha < \kappa \mid \mathcal{X}_\alpha \in U\}$.

(9.144) **Claim** *$A \in U$.*

Proof Suppose $\alpha < \kappa$. Then $\mathcal{X}_\alpha \in U$ iff $\kappa \in j(\mathcal{X}_\alpha)$ iff there is an elementary embedding in M of $(j\mathcal{S})_\alpha$ into $(j\mathcal{S})_\kappa$. Note that any $k : |(j\mathcal{S})_\alpha| \to |(j\mathcal{S})_\kappa|$ is a function from an ordinal into an ordinal, so it is a subset of M. Since $||(j\mathcal{S})_\alpha|| < \kappa < j\kappa$ and j witnesses the almost hugeness of κ, any such function is in M. Note also that $(j\mathcal{S})_\alpha = \mathcal{S}_\alpha$. Hence, for all $\alpha < \kappa$, $\mathcal{X}_\alpha \in U$ iff there is an elementary embedding (in V) of $\mathcal{S}_\alpha$ into $(j\mathcal{S})_\kappa$.

Since j is elementary, for any $\alpha < j\kappa$, $(j\mathcal{X})_\alpha \in jU$ iff there is an elementary embedding in M of $(j\mathcal{S})_\alpha$ into $(j^2\mathcal{S})_{j\kappa}$. Therefore, $A \in U$ iff $\kappa \in jA$ iff $(j\mathcal{X})_\kappa \in jU$ iff there is an elementary embedding in M of $(j\mathcal{S})_\kappa$ into $(j^2\mathcal{S})_{j\kappa}$. Let $j' = j \restriction |(j\mathcal{S})_\kappa|$. $j\rho' = \rho'$, so j' is an elementary embedding of $(j\mathcal{S})_\kappa$ into $(j^2\mathcal{S})_{j\kappa}$. Reasoning as before, since $||(j\mathcal{S})_\kappa|| < j\kappa$, $j' \in M$. Hence, $A \in U$. $\square$[9.144]

Let $B = \{\beta \in A \mid \beta \in \bigcap_{\alpha\in\beta\cap A} \mathcal{X}_\alpha\}$. Since U is normal, $B \in U$. Let $\alpha, \beta \in B$ be such that $\alpha < \beta$. Then $\beta \in \mathcal{X}_\alpha$, so there is an elementary embedding of $\mathcal{T}_\alpha$ into $\mathcal{T}_\beta$. As noted above, any such embedding restricts to an elementary embedding of $\mathcal{S}_\alpha$ into $\mathcal{S}_\beta$. $\square$[9.143]

9.6 The continuum problem, continued

In pursuing the theory of ever larger cardinals one should not lose sight of the fact that the point of large cardinals is not simply that they are large, but rather that their existence has implications for the structure of $(V;\in)$, either actual or potential, beyond the simple fact that large cardinals exist. One of the goals of their study is to shed light on the on the behavior of the continuum function $\kappa \mapsto 2^\kappa$—which, by virtue of Easton's theorem,[8.165] boils down to the singular cardinals problem,[§ 3.9.2] which is to say, how can the singular cardinals hypothesis SCH fail? The following theorem shows that large cardinal-type hypotheses are necessary if SCH is to be violated.

(9.145) **Theorem** [ZFC] *Suppose $0^\sharp$ does not exist. Then* SCH.

Proof Suppose κ is a singular cardinal such that $2^{\operatorname{cf}\kappa} < \kappa$. It suffices to show that $|A| \leqslant \kappa^+$, where $A = \{X \subseteq \kappa \mid |X| = \operatorname{cf}\kappa\}$. Let $\lambda = \max\{\omega_1, \operatorname{cf}\kappa\}$. Since $\operatorname{cf}\kappa$ is regular, $(\operatorname{cf}\kappa)^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. Also, $\omega_1^{\operatorname{cf}\kappa} \leqslant (2^\omega)^{\operatorname{cf}\kappa} = 2^{\omega\cdot\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. Since $2^{\operatorname{cf}\kappa} < \kappa$ by hypothesis, $\lambda^{\operatorname{cf}\kappa} < \kappa$.

For any $Y \subseteq \kappa$ let $B_Y = \{X \subseteq Y \mid |X| = \lambda\}$. By the covering theorem,[9.95] for every $X \in A$ there exists a constructible set $Y \subseteq \kappa$ of size λ such that $X \in B_Y$. Since $L \models \mathsf{GCH}$, there are at most κ^+ constructible sets $Y \subseteq \kappa$, and by the previous computation, for each Y of size λ there are fewer than κ subsets of Y of size $\operatorname{cf}\kappa$. It follows that $|A| \leqslant \kappa^+$. $\square$[9.145]

By Silver's theorem,[8.216] if κ is a singular cardinal of uncountable cofinality and $2^\lambda = \lambda^+$ for a stationary set of $\lambda < \kappa$ then $2^\kappa = \kappa^+$, but this does not constrain 2^κ when $\operatorname{cf}\kappa = \omega$, nor does it have any implication for 2^κ if we only assume that κ is a singular strong limit cardinal, i.e., $2^\lambda < \kappa$ for all $\lambda < \kappa$. The following theorem of Moti Gitik (9.146.1–3) and Gitik and Woodin (9.146.4) deals with both of these issues at the level of relative consistency.

(9.146) **Theorem** [S] *The following are equiconsistent over* ZFC*:*

1. *There exists a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$.*
2. *There exists a measurable cardinal κ such that $2^\kappa > \kappa^+$.*

3. *There exists a measurable cardinal κ of Mitchell order κ^{++}.*[46]
4. *For all $n \in \omega$ $2^{\omega_n} = \omega_{n+1}$ and $2^{\omega_\omega} = \omega_{\omega+2}$.*

Much more is known—for example, the result of Woodin and James Cummings that if there is a supercompact cardinal then there is a generic extension in which $2^\kappa = \kappa^{++}$ for every cardinal κ—but this will have to suffice for us.

9.7 Determinacy

In Section 5.8 we introduced the notion of *determinacy* of infinite games of perfect information, specifically games of length ω, most commonly on a countable set, such as ω.[5.165] Thus, a subset A of ${}^\omega\omega$ is *determinate* iff there is a winning I- or II-strategy in the corresponding game. We saw there that the determinacy of Borel sets is provable in ZFC.[5.177] (Interestingly, the Power axiom is necessary for this.[§ 7.6.2]) We also saw that the existence of an indeterminate set is provable in ZFC.[5.179]

A seminal event in the history of determinacy was the proposal by Jan Mycielski and Hugo Steinhaus in 1962 that the hypothesis that all sets of reals are determinate be considered as an axiom[18].

(9.147) *The* axiom of determinacy $\stackrel{\text{def}}{=}$ AD $\stackrel{\text{def}}{=}$ *the assertion that all subsets of ${}^\omega\omega$ are determinate.*

As we will see, it is not necessary to take sides, i.e., to profess either AD or AC (or to deny both). No, we are going to have our cake and eat it, too; and it is delicious.

9.7.1 Regularity properties of pointsets

A major theme in the investigation of countable infinitarity beyond ZF is the extension of the regularity properties of definable sets of reals proved in Chapter 5. Suppose X is a Polish space and $A \subseteq X$. Recall that the following are theorems of ZFC:

1. If A is analytic then A has the perfect set property.[5.183]
2. If A is obtainable from a family of open subsets of X by the operations of complementation, countable union, and the Suslin operation $\mathcal{S}$, then A has the Baire property and is Lebesgue measurable.[5.181]
3. If A is Borel then A is determined.[5.177]

Determinacy has a central position in this investigation inasmuch as each of the other listed properties is implied by it.

(9.148) **Theorem** [ZF + $\mathsf{AC}_\omega(\mathbb{R})$] *Suppose $\Gamma \subseteq \mathcal{P}({}^\omega\omega)$, Γ contains all open sets, and Γ is closed under continuous preimage and Borel operations, i.e., complementation and countable union (and hence also countable intersection). Suppose Γ-determinacy and $A \in \Gamma$.*

[46]Suppose κ is measurable. The Mitchell ordering of normal ultrafilters over κ is given by: $U < U'$ iff $U \in \mathrm{Ult}_{U'} V$. We have shown that $U \notin \mathrm{Ult}_U V$,[9.33] so $<$ is irreflexive. It is clearly transitive. It is not hard to show that it is wellfounded. The *Mitchell order* of a measurable cardinal κ is the height of $<$.

1. *A has the perfect set property.*
2. *A has the Baire property.*
3. *A is Lebesgue measurable.*

Proof 1 For $n \in \omega$ let $Z_n = \{(m, 0) \mid m \in n\}$, i.e., the sequence consisting of n 0s. Thus, $Z_0 = 0$, $Z_1 = \langle 0 \rangle$, $Z_2 = \langle 0, 0 \rangle$, etc. Let $h : {}^{\omega}\omega \to {}^{\omega}2$ be such that

$$h \langle x_0, x_1, x_2, \dots \rangle = Z_{x_0} {}^\frown \langle 1 \rangle {}^\frown Z_{x_1} {}^\frown \langle 1 \rangle {}^\frown Z_{x_2} {}^\frown \langle 1 \rangle {}^\frown \cdots .$$

For example, $h\langle 2, 0, 3, \dots \rangle = \langle 0, 0, 1, 1, 0, 0, 0, 1, \dots \rangle$. Note that h is a bicontinuous injection. Let $C_0 = \operatorname{im} h$.[47] Then h is a homeomorphism of ${}^{\omega}\omega$ to C_0.

Let $A' = h^{\to} A$.

(9.149) **Claim** *A' has the perfect set property.*

Proof Consider the following game $\mathcal{G}_1$:

> At stage n, I plays $a_n \in {}^{<\omega}2$ and II plays $b_n \in 2$, i.e., $b_n \in \{0, 1\}$. Let $c = a_0 {}^\frown \langle b_0 \rangle {}^\frown a_1 {}^\frown \langle b_1 \rangle {}^\frown \cdots$. I wins iff $c \in A'$.

(9.150) **Claim** *$\mathcal{G}_1$ is determined.*

Remark This is a straightforward coding exercise. We provide the details in this case by way of example.

Proof Let $\mathcal{G}_2$ be the following game on ω:

> Let x_n and y_n be respectively I's and II's move at stage n. Let $a_n = \vec{B}\, x_n$, where $\vec{B} : \omega \to V_\omega$ is the canonical enumeration of V_ω,[3.213.1] and let $b_n = y_n$. The following rules apply:
>
> 1. $a_n \in {}^{<\omega}2$.
> 2. $b_n \in 2$.
>
> If either player violates one of these rules, the first player to do so loses. If both players follow these rules for all $n \in \omega$, let $c = a_0 {}^\frown \langle b_0 \rangle {}^\frown a_1 {}^\frown \langle b_1 \rangle {}^\frown \cdots$. I wins iff $c \in A'$.

Given $z \in {}^{\omega}\omega$, let $x = z^{\mathrm{I}}$ and $y = z^{\mathrm{II}}$, i.e., $x_n = z_{2n}$ and $y_n = z_{2n+1}$, so $z = x * y$. Let R_1 be the set of $z \in {}^{\omega}\omega$ such that I loses $\mathcal{G}_2$ by virtue of being the first to violate one of its rules, and let R_2 be the corresponding set for II. Note that R_1 and R_2 are open sets and are therefore in Γ. Let $R_0 = {}^{\omega}\omega \backslash (R_1 \cup R_2)$. Note that $R_0 \in \Gamma$. For $z \in R_0$, let a^z, b^z, and c^z be the sequences defined in the description of $\mathcal{G}_2$, i.e., $a^z_n = \vec{B}\, x_n$, $b^z_n = y_n$, and $c^z = a^z_0 {}^\frown \langle b^z_0 \rangle {}^\frown a^z_1 {}^\frown \langle b^z_1 \rangle {}^\frown \cdots$. Let $A'' = R_2 \cup \{z \in R_0 \mid c^z \in A'\} = R_2 \cup \{z \in R_0 \mid h^{-1} c^z \in A\}$. Note that $\mathcal{G}_2 = \langle {}^{<\omega}\omega, A'' \rangle$.

Note that $z \mapsto c^z$ is continuous on R_0, and h^{-1} is continuous on C_0, so $\{z \in R_0 \mid h^{-1} c^z \in A\} \in \Gamma$. Hence, $A'' \in \Gamma$, so A'', i.e., $\mathcal{G}_2$, is determined. It is straightforward, given a winning strategy for either I or II in $\mathcal{G}_2$, to define a winning strategy for the same player in $\mathcal{G}_1$. Hence $\mathcal{G}_1$ is determined. $\square$[9.150]

Suppose σ is a winning strategy for I in $\mathcal{G}_1$. Let $S = \{\sigma * b \mid b \in {}^{\omega}2\}$ be the set of plays according to σ. Clearly, S is a nonempty perfect closed set included in A'.

[47] C_0 is the set of sequences in ${}^{\omega}2$ that are not eventually 0.

Alternatively, suppose τ is a winning strategy for II in $\mathcal{G}_1$. Suppose $t = \langle a_0, b_0, a_1, b_1, \ldots, a_{n^-}, b_{n^-}\rangle$ is partial play according to τ in which each player has moved n times. Let $c^t = a_0 {}^\frown\langle b_0\rangle {}^\frown \cdots {}^\frown a_{n^-} {}^\frown\langle b_{n^-}\rangle$, and let $d^t \in {}^\omega 2$ be such that for each $m \in \omega$

$$d^t_m = 1 - \tau(t {}^\frown\langle d^t \restriction m\rangle).$$

Since $\tau(t {}^\frown\langle d^t \restriction m\rangle)$ is 0 or 1, d^t_m is correspondingly either 1 or 0. $c^t {}^\frown d^t$ may be thought of as the outcome that τ avoids at this stage of the game, inasmuch as for any move a_n by I,

$$a_n {}^\frown\langle \tau(t {}^\frown\langle a_n\rangle)\rangle \nsubseteq d^t.^{48}$$

(9.151) *Note, however, that this is—so to speak—the only outcome that τ avoids at this stage.*

For let C be the set of sequences $c^t {}^\frown d^t$ formed in this way, and suppose $c \notin C$. Then $c \neq d^0$, so there exists $m \in \omega$ such that $c \restriction m \subseteq d^0$ but $c_m \neq d^0_m$, so $c_m = 1 - d^0_m$. Let $a_0 = c \restriction m$ and $b_0 = c_m$. Then $b_0 = \tau\langle a_0\rangle$, so $t_0 = \langle a_0, b_0\rangle$ is a partial play according to τ. Let $c^{t_0} = a_0 {}^\frown\langle b_0\rangle \subseteq c$. Then $c^{t_0} {}^\frown d^{t_0} \in C$, and accordingly $c \neq c^{t_0} {}^\frown d^{t_0}$. Arguing as before, there exist a_1 and b_1 such that $b_1 = \tau(t_0 {}^\frown\langle a_1\rangle)$ and $c^{t_0} {}^\frown a_1 {}^\frown\langle b_1\rangle \subseteq c$. Proceeding in this fashion, we obtain a play $a = \langle a_0, a_1, \ldots\rangle$ for I such that $a * \tau = c$. In other words, if $c \notin C$ then τ cannot avoid c.[9.151]

Since τ is a winning strategy for II, $c \notin C \to c \notin A'$, i.e., $A' \subseteq C$. The set of partial plays t is countable, so C is countable. Hence, A' is countable. □[9.149]

Thus,[9.149] A' either is countable or has a nonempty perfect closed subset. Since h is a homeomorphism, the same is true for A. □[9.148.1]

2 Let $M = {}^{<\omega}\omega \backslash \{0\}$. Given $B \subseteq {}^\omega\omega$, let $\mathcal{G}_B$ be the following game:

> I and II alternate choosing elements s_0, t_0, s_1, t_1, … of M such that $s_0 \subsetneq t_0 \subsetneq s_1 \subsetneq t_1 \subsetneq \cdots$. I wins iff $\bigcup_{n\in\omega} s_n \in B$.

Via the bijection $\vec{B} : \omega \to V_\omega$, it is easy to code $\mathcal{G}_B$ as a game on ω, which is in Γ if $B \in \Gamma$, hence determined by assumption, and the determinacy of which implies that of $\mathcal{G}_B$. So $\mathcal{G}_B$ is determined for all $B \in \Gamma$.

(9.152) **Claim** *If there is a winning* II*-strategy in $\mathcal{G}_B$ then B is meager.*

Proof Let τ be a winning II-strategy in $\mathcal{G}_B$. Let E be the set of positions of $\mathcal{G}_B$ of even length. For each $p \in E$, let $\hat{p}$ be the last item in p (the most recent play by II) if $p \neq 0$; otherwise, let $\hat{p} = 0$ (nobody has yet played). Let $F_p = \{x \in {}^\omega\omega \mid \hat{p} \subseteq x \wedge \forall s \in M\, (s \supsetneq \hat{p} \to \tau(p {}^\frown\langle s\rangle) \nsubseteq x\}$.

(9.153) **Claim** $B \subseteq \bigcup_{p\in E} F_p$.

Proof Suppose toward a contradiction that $x \in B$ and $\forall p \in E\ x \notin F_p$. $x \notin F_0$, so we may let $s_0 \in M$ be such that $t_0 \subseteq x$, where $t_0 = \tau\langle s_0\rangle$. $x \notin F_{\langle s_0, t_0\rangle}$, so we may let $s_1 \supsetneq t_0$ be such that $t_1 = \tau\langle s_0, t_0, s_1\rangle \subseteq x$. Proceeding in this way[49] we obtain a play $\langle s_0, t_0, \ldots\rangle$ according to τ such that $\bigcup_{n\in\omega} s_n = x \in B$, contradicting the assumption that τ is a winning II-strategy in $\mathcal{G}_B$. □[9.153]

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[48] Either $a_n \nsubseteq d^t$ or $a_n = d^t \restriction |a_n|$, in which case $\tau(t {}^\frown\langle a_n\rangle) \neq d^t_{|a_n|}$.

[49] An axiom of choice is not required, as ${}^{<\omega}\omega$ has a definable wellordering.

Note that each F_p is closed. Since M is countable, given (9.153), it suffices to show that each F_p is nowhere dense, i.e., any open interval has an open subinterval disjoint from F_p. To this end, suppose $p \in E$ and $s \in {}^{<\omega}\omega$. If $\hat{p} \nsubseteq s$ then $I_s \cap F_p = 0$. If $\hat{p} \subseteq s$ then let $s' \supseteq s$ be such that $\hat{p} \subsetneq s'$, and let $t = \tau(p^\frown\langle s'\rangle)$. Then $I_t \subseteq I_s$, and $I_t \cap F_p = 0$. $\square^{9.152}$

(9.154) **Claim** *If there is a winning* I*-strategy in $\mathcal{G}_B$ then there exists $s \in {}^{<\omega}\omega$ such that $I_s \backslash B$ is meager.*

Proof Let σ be a winning I-strategy in $\mathcal{G}_B$, and let $s = \sigma 0$. To show that $I_s\backslash B$ is meager, we will describe a winning II-strategy τ in $\mathcal{G}_{I_s\backslash B}$:

> Let $s_0, t_0, s_1, t_1, \ldots$ be the successive moves according to the II-strategy τ we are describing. Given an initial move s_0 by I, if $s_0 \nsupseteq s$, let t_0 be any proper extension of s_0 such that t_0 is incomparable with s. At this point I has already lost $\mathcal{G}_{I_s\backslash B}$, since for any $x \in {}^{\omega}\omega$, $t_0 \subseteq x \rightarrow x \notin I_s$, so II's remaining responses $t_1, t_2, \ldots$ may be chosen arbitrarily.
>
> Suppose, on the other hand, $s_0 \supseteq s$. In this case, we will define II's strategy τ in $\mathcal{G}_{I_s\backslash B}$ in terms of I's strategy σ in $\mathcal{G}_B$. Let $s'_0, t'_0, s'_1, t'_1, \ldots$ be the successive moves in the latter so-called *auxiliary game*. Thus, $s'_0 = \sigma 0$ $(= s)$. By hypothesis, s_0 extends s'_0. Let t'_0 be an extension of s_0 that is a proper extension of s'_0, let $s'_1 = \sigma\langle s'_0, t'_0\rangle$, and let $t_0 = s'_1$. Given a proper extension s_1 of t_0, let $t'_1 = s_1$ and let $t_1 = s'_2 = \sigma\langle s'_0, t'_0, s'_1, t'_1\rangle$. Continue *ad infinitum*, so that $\langle s'_0, t'_0, s'_1, t'_1, \ldots\rangle$ is a play according to σ, and for every $n \in \omega$, $t_n = s'_{n+1}$ and $t'_{n+1} = s_{n+1}$.

Let $x = \bigcup_{n\in\omega} s_n$ $\left(= \bigcup_{n\in\omega} t_n\right)$. If $s_0 \nsupseteq s$, we have chosen t_0 so that $I_{t_0} \cap I_s = 0$, so $x \notin I_s$. If $s_0 \supseteq s$, we have created an auxiliary sequence $\langle s'_0, t'_0, \ldots\rangle$ according to σ in such a way that $x = \bigcup_{n\in\omega} s'_n$. Since σ is a winning I-strategy in $\mathcal{G}_B$, $x \in B$. In either case, $x \notin I_s\backslash B$. Thus, τ is a winning II-strategy in $\mathcal{G}_{I_s\backslash B}$. It follows[9.152] that $I_s\backslash B$ is meager. $\square^{9.154}$

Let S be the set of $s \in {}^{<\omega}\omega$ such that $I_s\backslash A$ is meager. Let $G = \bigcup_{s\in S} I_s$. Then $G\backslash A$ is meager.[50] We claim that $A\backslash G$ is meager. Suppose not. Then[9.152] there is no winning II-strategy in $\mathcal{G}_{A\backslash G}$. Since $A\backslash G \in \Gamma$, there is a winning I-strategy σ in $\mathcal{G}_{A\backslash G}$. Let[9.154] s be such that $I_s\backslash(A\backslash G)$ is meager. Then $I_s\backslash A$ is meager, so $s \in S$, so $I_s \subseteq G$, so $I_s\backslash(A\backslash G) = I_s$, which is not meager (by Baire's theorem).

Thus, $A \triangle G$ is meager. Since G is open, A has the Baire property. $\square^{9.148.2}$

3 See Section 5.7, in particular the subsection 5.7.3, for a discussion of measure and measurability. We will prove the theorem for the uniform measure on ${}^{\omega}2$. The case of Lebesgue measure on $\mathbb{R}^n$ may be handled similarly, with intervals in $\mathbb{R}^n$ in place of intervals in ${}^{\omega}2$.

Recall:

1. The uniform measure on ${}^{\omega}2$ is defined on open intervals I_s, $s \in {}^{<\omega}2$, by $\mu I_s = 2^{-|s|}$.

2. The measure of an open set is unambiguously defined as $\sum_{s\in S} \mu I_s$, where $S \subseteq {}^{<\omega}2$ is any set such that $s, t \in S \rightarrow I_s \cap I_t = 0$, and $B = \bigcup_{s\in S} I_s$.

3. The outer measure of a set $B \subseteq {}^{\omega}2$ is the infimum of the measures of the open sets that include B; and the inner measure of B is the supremum of the measures of the closed sets included in it.

[50] This is where we use $\mathsf{AC}_\omega(\mathbb{R})$, invoking (5.146).

4. $B \subseteq {}^{\omega}2$ is *measurable* iff its outer and inner measures are equal, and $\mu\, B$ is (unambiguously) defined to be this common value.

5. The outer (inner) measure of B is also the infimum (supremum) of the measures of measurable sets including (included in) B; and there is a G_δ (F_σ) set including (included in) B with this measure.

6. $B \subseteq {}^{\omega}2$ is *null* iff its outer measure is 0. Note that every null set is measurable, with measure 0.

7. The class of measurable sets contains all analytic and coanalytic sets.[5.181.2]

(9.155) **Claim** *Suppose $B \in \Gamma$, and for every measurable $X \subseteq B$, $\mu\, X = 0$. Then B is null.*

Proof Suppose $\epsilon > 0$. For each $n \in \omega$ let K_n be the set of finite unions G of open intervals such that $\mu\, G \leqslant \epsilon/4^{n+1}$. Note that K_n is countable. Let $\langle G^n_k \mid k \in \omega\rangle$ enumerate K_n. (Everything is definable, so no choice is required, although countable choice would suffice.) Consider the following game $\mathcal{G}_\epsilon$:

> I and II alternate moving, with I producing a sequence $a = \langle a_0, \dots\rangle \in {}^{\omega}2$, and II producing a sequence $b = \langle b_0, \dots\rangle \in {}^{\omega}\omega$. I wins iff $a \in B$ and $a \notin \bigcup_{n\in\omega} G^n_{b_n}$.

Given Γ-determinacy and the closure properties of Γ, since $B \in \Gamma$, $\mathcal{G}_\epsilon$ is determined.

(9.156) **Claim** *There is no winning* I*-strategy in $\mathcal{G}_\epsilon$.*

Proof Suppose toward a contradiction that σ is a winning I-strategy in $\mathcal{G}_\epsilon$. The map $b \mapsto \sigma * b$ is a continuous function from ${}^{\omega}\omega$ to ${}^{\omega}2$, so the set X of all plays $\sigma * b$ according to σ is analytic and hence measurable. Since $X \subseteq B$, by hypothesis X is null. Let $G \supseteq X$ be open with $\mu\, G \leqslant \epsilon/4$. Let $\langle J_0, J_1, \dots\rangle$ be a sequence of disjoint open intervals such that $G = \bigcup_{m\in\omega} J_m$. For $n \in \omega$, define $0 = m_0 < m_1 < \cdots$ recursively so as to maintain the following conditions, letting

$$H_n = \bigcup_{m<m_n} J_m. \tag{9.157}$$

1. $\mu(G\backslash H_n) \leqslant \mu\, G/4^n$.
2. $\mu(H_{n+1}\backslash H_n) \leqslant \epsilon/4^{n+1}$.

Suppose $m_{n'}$ has been defined for all $n' \leqslant n$ and $\mu(G\backslash H_n) \leqslant \mu\, G/4^n$. Let m_{n+1} be the least $m > m_n$ such that $\mu \bigcup_{k=m_n}^{m^-} J_k \geqslant (3/4)\mu \bigcup_{k=m_n}^{\infty} J_k$. Note that[9.157] for any n', $\bigcup_{k=m_{n'}}^{\infty} J_k = G\backslash H_{n'}$, so

$$\mu(G\backslash H_{n+1}) \leqslant (1/4)\mu(G\backslash H_n) \leqslant \mu\, G/4^{n+1}.$$

Since $(H_{n+1}\backslash H_n) \subseteq (G\backslash H_n)$ and by assumption $\mu\, G < \epsilon/4$,

$$\mu(H_{n+1}\backslash H_n) \leqslant \mu(G\backslash H_n) \leqslant \mu\, G/4^n \leqslant \epsilon/4^{n+1}.$$

For each $n \in \omega$, $H_{n+1}\backslash H_n \in K_n$; and $X \subseteq G = \bigcup_{n\in\omega}(H_{n+1}\backslash H_n)$, so if II plays $\langle b_0, \dots\rangle$, where b_n is such that $G^n_{b_n} = H_{n+1}\backslash H_n$, then II wins against σ. $\square$[9.156]

It follows from the determinacy of $\mathcal{G}_\epsilon$ that II has a winning strategy τ. For each $n \in \omega$ and each $s = \langle s_0, \ldots, s_n \rangle \in {}^{n+1}2$ considered as a partial play by I, let b^n_s be the next move of II according to τ. Let $C^n = \bigcup_{s \in {}^{n+1}2} G^n_{b^n_s}$, and let $C = \bigcup_{n\in\omega} C^n$.

For any $a \in {}^{\omega}2$, letting $\langle b_0, \ldots \rangle = a * \tau$, $\bigcup_{n\in\omega} G^n_{b_n} \subseteq C$. Since τ is a winning II-strategy, for any $a \in B$, $a \in C$, i.e., $B \subseteq C$. Since $|{}^{n+1}2| = 2^{n+1}$, and $\mu\, G^n_k \leqslant \epsilon/4^{n+1}$ for all k,

$$\mu\, C^n \leqslant \epsilon/2^{n+1},$$

so $\mu\, C \leqslant \epsilon$. Since C is open, and ϵ may be any positive real number, B is null.$\square$[9.155]

We now show that A is measurable. Let u be the outer measure of A. Let Y be a Borel set including A such that $\mu\, Y = u$. Note that $Y \backslash A \in \Gamma$, so (9.155) applies. Any measurable $X \subseteq (Y\backslash A)$ is null, because otherwise there is a non-null closed set $F \subseteq X$, so we have a Borel set $Y\backslash F$ such that $A \subseteq (Y\backslash F)$ and $\mu(Y\backslash F) < u$. Hence[9.155] $Y\backslash A$ is null. Thus, $A \triangle Y$ is null, so A is measurable. $\square$[9.148.3] $\square$[9.148]

9.7.2 Structural properties of pointclasses

9.7.2.1 The reduction property

The first use of a determinacy hypothesis to prove a structural property of pointclasses was David Blackwell's derivation of the reduction property for $\mathbf{\Pi}^1_1$ from $\mathbf{\Delta}^0_1$ determinacy.[3] Blackwell's proof goes as follows. Suppose $A, B \subseteq {}^{\omega}\omega$ are $\mathbf{\Pi}^1_1$. Using the representation (5.60), let S, T be $\mathbf{\Delta}^0_1$ trees on $\omega \times \omega$ such that for all $z \in {}^{\omega}\omega$, $z \in A$ (resp., B) iff $S_{[z]}$ (resp., $T_{[z]}$) is wellfounded. For each $z \in {}^{\omega}\omega$, let G_z be the following game:

$$\begin{array}{ccccccc} \text{I} & x_0 & & x_1 & & \cdots & x \\ \text{II} & & y_0 & & y_1 & \cdots & y \end{array}$$

I wins iff for some $n \in \omega$, $x \restriction n \in T_{[z]}$ and $y \restriction n \notin S_{[z]}$.

In effect, I is trying to show that $z \notin B$, while II is trying to show that $z \notin A$, each by playing a branch of the appropriate tree. I wins iff II fails before I fails. Equivalently, II wins iff I fails either before or simultaneously with II, or if neither fails. Note that if $z \in A \cup B$ then G_z is $\mathbf{\Delta}^0_1$, i.e., both open and closed, because in this case either I or II fails, and this is known at some finite stage of the game.

Let

$$A' = \{z \in A \mid \text{there is no winning II-strategy in } G_z\}$$
$$B' = \{z \in B \mid \text{there is no winning I-strategy in } G_z\}.$$

Clearly, $A\backslash B \subseteq A'$, because if $z \in A\backslash B$ then there is a winning I-strategy, viz., to play a fixed branch of $T_{[z]}$ regardless of what II does. Likewise, $B\backslash A \subseteq B'$.

If $z \in A \cap B$ then $z \in A' \cup B'$, because I and II cannot both have winning strategies in G_z. Since $A' \subseteq A$ and $B' \subseteq B$ by definition, $A' \cup B' = A \cup B$.

To show that $\langle A', B' \rangle$ reduces $\langle A, B \rangle$ it suffices to show that $A' \cap B' = 0$. Suppose to the contrary that $z \in A' \cap B'$. Then $z \in A \cap B$, and neither I nor II has a winning strategy in G_z, contradicting the fact that G_z is determined.

To complete the proof we must show that A' and B' are $\mathbf{\Pi}^1_1$. Recall that for $z \in A \cup B$, the winner of G_z is always known at some finite stage of the game. Thus, $z \in A'$ iff $z \in A$ and for every II-strategy τ, there exist $n \in \omega$ and $t, s \in {}^{n}\omega$

such that $t * s$ is a partial play according to τ (i.e., $t * s = t * \tau$) which is a win for I, i.e., $t \in T_{[z]}$ and $s \notin S_{[z]}$. This is a $\mathbf{\Pi}^1_1$ description of A'. Similarly, B' is Π^1_1.

Moschovakis and Addison recognized the significance of Blackwell's proof and independently generalized it to prove $\mathbf{\Pi}^1_3$-reduction from $\mathbf{\Delta}^1_2$-determinacy. Moschovakis and Martin then independently showed how to prove $\mathbf{\Pi}^1_{2n+1}$- and $\mathbf{\Sigma}^1_{2n+2}$-reduction from $\mathbf{\Delta}^1_{2n}$-determinacy. We will not present the proofs of these results *per se*, as they are superseded by the corresponding results for the prewellordering property treated in the next section.

9.7.2.2 The prewellordering property

(9.158) **Theorem: First periodicity (Martin, Moschovakis)** [$\mathsf{ZF} + \mathsf{DC}$] *Suppose Γ is an adequate*[5.109] *pointclass. Let $\Delta = \Gamma \cap \breve{\Gamma}$ and suppose $\underset{\sim}{\Delta}$-determinacy. Suppose $A \subseteq {}^\omega\omega \times {}^\omega\omega$ is in Γ and has a Γ-norm.*[5.108] *Let $B = \{a \in {}^\omega\omega \mid \forall x \in {}^\omega\omega\ \langle a, x\rangle \in A\}$. Then B has a $\forall^1\exists^1\Gamma$-norm.*

Proof Let φ be a Γ-norm on A. Given $a, b \in {}^\omega\omega$, let $G(a,b)$ be the following game on ω:

Letting x and y be I's and II's respective plays.

$$\begin{aligned} \text{I wins} &\leftrightarrow \langle b, y\rangle <^*_\varphi \langle a, x\rangle; \\ \text{equivalently, II wins} &\leftrightarrow \langle b, y\rangle \not<^*_\varphi \langle a, x\rangle \\ &\leftrightarrow \langle b, y\rangle \notin A \vee \langle a, x\rangle \leqslant^*_\varphi \langle b, y\rangle. \end{aligned}$$

Let $\leqslant$ be the binary relation on $B \times B$ such that

$$a \leqslant b \leftrightarrow \text{II has a winning strategy in } G(a,b).$$

(9.159) **Claim** $\leqslant$ *is a (weak) preorder, i.e., for all $a, b, c \in B$,*

1. *$a \leqslant a$; and*
2. *if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.*

Proof 1 II wins $G(a,a)$ by copying I's moves, i.e., playing so that $y = x$.

2 Let $\tau_{a,b}$ and $\tau_{b,c}$ be winning II-strategies for $G(a,b)$ and $G(b,c)$, respectively. Let τ be the II-strategy indicated in the following diagram by the dashed arrows in the game $G(a,c)$, where $\tau_{a,b}$ and $\tau_{b,c}$ are indicated by solid arrows in the respective games $G(a,b)$ and $G(b,c)$.

$$\begin{array}{ll|ccccc}
 & \text{I} & x_0 & & x_1 & & x \\
G(a,b) & \tau_{a,b} & & & & \cdots & \\
 & \text{II} & & y_0 & & y_1 & y \\
\hline
 & \text{I} & & y_0 & & y_1 & y \\
G(b,c) & \tau_{b,c} & & & & \cdots & \\
 & \text{II} & & z_0 & & z_1 & z \\
\hline
 & \text{I} & x_0 & & x_1 & & x \\
G(a,c) & \tau & & & & \cdots & \\
 & \text{II} & & z_0 & & z_1 & z
\end{array}$$

In other words—or, rather, in words—given a first move x_0 by I, II imagines that I has played x_0 in $G(a,b)$ and applies $\tau_{a,b}$ to obtain y_0, then imagines that I has played y_0 in $G(b,c)$ and applies $\tau_{b,c}$ to obtain z_0, which II plays as $\tau\langle x_0\rangle$. Given a response x_1 by I, II repeats this procedure to obtain z_1, *ad infinitum*.

To show that τ is a winning II-strategy in $G(a,c)$ we must show that for any x, letting y and z be constructed as above, either $\langle c,z\rangle \notin A$ or $\langle a,x\rangle \leqslant^*_\varphi \langle c,z\rangle$. Thus, suppose $\langle c,z\rangle \in A$. Since $\tau_{b,c}$ is a winning II-strategy in $G(b,c)$, $\langle b,y\rangle \leqslant^*_\varphi \langle c,z\rangle$. Hence, $\langle b,y\rangle \in A$, and since $\tau_{a,b}$ is a winning II-strategy in $G(a,b)$, $\langle a,x\rangle \leqslant^*_\varphi \langle b,y\rangle$. Thus, $\langle a,x\rangle \leqslant^*_\varphi \langle c,z\rangle$. $\square^{9.159}$

(9.160) **Claim**

1. *For all $a,b \in {}^\omega\omega$, $G(a,b)$ is determined.*
2. *Suppose $a,b \in B$. Let $a < b \overset{\text{def}}{\iff} a \leqslant b \wedge b \nleqslant a$.*
   1. *$a < b$ iff* I *wins $G(b,a)$.*
   2. *$a \leqslant b \vee b \leqslant a$, so $\leqslant$ is a (total) preorder on B.*

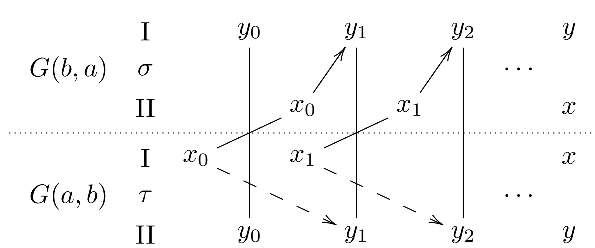
Proof 1 If $b \notin B$ then for some $y \in {}^\omega\omega$, $\langle b,y\rangle \notin A$, and II can win by playing y regardless of what I plays. If $b \in B$ then for every $y \in {}^\omega\omega$, $\langle b,y\rangle \in A$. Since φ is a Γ-norm, there exist binary relations $\leqslant_\Gamma \in \Gamma$ and $\leqslant_{\breve{\Gamma}} \in \breve{\Gamma}$ such that for all $x,y \in {}^\omega\omega$

$$\langle a,x\rangle \leqslant^*_\varphi \langle b,y\rangle \leftrightarrow \langle a,x\rangle \leqslant_\Gamma \langle b,y\rangle \leftrightarrow \langle a,x\rangle \leqslant_{\breve{\Gamma}} \langle b,y\rangle.$$

Hence, the win set for II is in $\underset{\sim}{\Delta}$ and is therefore determinate by hypothesis.

2.1 Suppose $a < b$. Then $b \nleqslant a$, so II does not win $G(b,a)$; hence, I wins $G(b,a)$.

Conversely, suppose I wins $G(b,a)$. Then II does not win $G(b,a)$, so $b \nleqslant a$. To show that $a < b$ it suffices to show that $a \leqslant b$, i.e., that II has a winning strategy in $G(a,b)$. The following diagram illustrates such a strategy τ as dashed arrows, where solid arrows indicate a winning strategy σ for I in $G(b,a)$.



Thus, in $G(a,b)$, τ initially ignores I's first move x_0 and simply plays I's first move y_0 according to σ. τ then initially ignores I's next move x_1, but imagines that II has played x_0 in response to y_0 in $G(b,a)$ and plays I's response $y_1 = \sigma\langle y_0, x_0\rangle$, etc. Since σ is a winning I-strategy in $G(b,a)$, $\langle a,x\rangle <^*_\varphi \langle b,y\rangle$, *a fortiori* $\langle a,x\rangle \leqslant^*_\varphi \langle b,y\rangle$, so τ is a winning II-strategy in $G(a,b)$.

2.2 Either II wins $G(a,b)$, in which case $a \leqslant b$, or I wins $G(a,b)$, in which case $b < a$, so $b \leqslant a$. $\square^{9.160}$

(9.161) **Claim** $\leqslant$ *is wellfounded.*

Proof If $\leqslant$ is not wellfounded then by DC there exists an ω-sequence $\langle\langle a_n, \sigma_n\rangle \mid n \in \omega\rangle$ such that for every $n \in \omega$, $a_{n+1} < a_n$ and σ_n is a I-strategy in $G(a_n, a_{n+1})$ that witnesses this.[9.160.2.1] Consider the diagram

$$\begin{tikzcd}[row sep=small]
 & \mathrm{I} & x_0^0 & & x_1^0 & & x_2^0 & & x^0 \\
G(a_0,a_1) & \sigma_0 & & & & & & \cdots & \\
 & \mathrm{II} & & x_0^1 \arrow[uur] & & x_1^1 \arrow[uur] & & & x^1 \\
 & \mathrm{I} & x_0^1 \arrow[ur] & & x_1^1 \arrow[ur] & & x_2^1 & & x^1 \\
G(a_1,a_2) & \sigma_1 & & & & & & \cdots & \\
 & \mathrm{II} & & x_0^2 \arrow[uur] & & x_1^2 \arrow[uur] & & & x^2 \\
 & \mathrm{I} & x_0^2 \arrow[ur] & & x_1^2 \arrow[ur] & & x_2^2 & & x^2 \\
G(a_2,a_3) & \sigma_2 & & & & & & \cdots & \\
 & \mathrm{II} & & x_0^3 \arrow[uur] & & x_1^3 \arrow[uur] & & & x^3 \\
 & & \arrow[ur] & & \arrow[ur] \vdots & & & &
\end{tikzcd}$$

In other words, we let $x_0^n = \sigma_n\langle\rangle$ for every $n \in \omega$; and then we use recursion on $m \in \omega$ to define $x_{m+1}^n = \sigma_n\langle x_0^n, x_0^{n+1}, x_1^n, x_1^{n+1}, \ldots, x_m^{n+1}\rangle$ for every $n \in \omega$.

As a result, for all $n \in \omega$, $\langle a_{n+1}, x_{n+1}\rangle <_\varphi^* \langle a_n, x_n\rangle$, which is impossible since $<_\varphi^*$ is wellfounded. □[9.161]

All that is left is to show that $\leqslant$ is a $\forall^1\exists^1\Gamma$ prewellordering. To this end, suppose $b \in B$ and $a \in {}^\omega\omega$. Then

$$\begin{aligned}
a \leqslant b &\leftrightarrow \mathrm{II} \text{ wins } G(a,b) \\
&\leftrightarrow \exists_{\text{II-strategy}}\tau\ \forall x \in {}^\omega\omega\ \langle a, x\rangle \leqslant_\varphi^* \langle b, (\tau * x)^{\mathrm{II}}\rangle \\
&\leftrightarrow \exists_{\text{II-strategy}}\tau\ \forall x \in {}^\omega\omega\ \langle a, x\rangle \leqslant_{\breve{\Gamma}} \langle b, (\tau * x)^{\mathrm{II}}\rangle,
\end{aligned}$$

since $\forall y \in {}^\omega\omega\ \langle b, y\rangle \in A$.[5.108] On the other hand,

$$\begin{aligned}
a \leqslant b &\leftrightarrow a \in B \text{ and I does not win } G(a,b) \\
&\leftrightarrow a \in B \text{ and } \forall_{\text{I-strategy}}\sigma\ \exists y \in {}^\omega\omega\ \langle a, (\sigma * y)^{\mathrm{I}}\rangle \leqslant_\varphi^* \langle b, y\rangle \\
&\leftrightarrow a \in B \text{ and } \forall_{\text{I-strategy}}\sigma\ \exists y \in {}^\omega\omega\ \langle a, (\sigma * y)^{\mathrm{I}}\rangle \leqslant_\Gamma \langle b, y\rangle,
\end{aligned}$$

again using the fact that $\forall y \in {}^\omega\omega\ \langle b, y\rangle \in A$. Since $\exists^1\forall^1\breve{\Gamma}$ is dual to $\forall^1\exists^1\Gamma$, $\leqslant$ is a $\forall^1\exists^1\Gamma$ prewellordering of B. □[9.158]

Combining (9.158) with (5.113) and (5.114) we have the following theorem.

(9.162) **Theorem: Prewellordering under projective determinacy** [ZF+DC] *Suppose all projective sets are determined. Then for every $n \in \omega$ and $z \in {}^\omega\omega$, $\Pi^1_{2n+1}(z)$ and $\Sigma^1_{2n+2}(z)$ have the prewellordering property.*

This alternating behavior is the origin of the term ‘periodicity’ in the name of Theorem 9.158.

9.7.2.3 The scale property and uniformization

(9.163) **Theorem** [$\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$] *Suppose* Γ *is an adequate pointclass. Let* $\Delta = \Gamma \cap \breve{\Gamma}$ *and suppose* $\underset{\sim}{\Delta}$*-determinacy. Suppose* $A \subseteq {}^\omega\omega \times {}^\omega\omega$ *is in* Γ *and has a* Γ*-scale. Let* $B = \{x \in {}^\omega\omega \mid \forall y \in {}^\omega\omega \; \langle x, y\rangle \in A\}$*. Then* B *has a* $\forall^1\exists^1\Gamma$*-scale.*

This is proved by a similar argument to that of (9.158), but of course somewhat more elaborate, and we omit it in the interest of brevity.

As in the case of the prewellordering property,[9.162] it follows from (9.163) that projective determinacy implies that the scale property "alternates" in the projective hierarchy, i.e., for every $n \in \omega$ and $z \in {}^\omega\omega$, $\Pi^1_{2n+1}(z)$ and $\Sigma^1_{2n+2}(z)$ have the scale property, and therefore also the uniformization property.

9.7.3 The Wadge hierarchy

As the reader will have noticed, complexity hierarchies of pointclasses figure prominently in descriptive set theory. We have focused on the Borel and projective hierarchies, but other hierarchies both beyond and within these have also been productively studied. For example, it is natural to define pointclasses $\mathbf{\Pi}^m_n$, $\mathbf{\Sigma}^m_n$, and $\mathbf{\Delta}^m_n$ based on quantification over type-m objects. For $m > 1$ these properly extend the projective hierarchy.

As an example of the other direction, we mention the *Hausdorff difference hierarchies* $\alpha\text{-}\mathbf{\Pi}^0_\xi$ $(0 < \xi, \alpha < \omega_1)$ within the class of Borel sets. These are based on a generalization of the usual difference operation on sets: $D_1\langle A\rangle = A$, $D_2\langle A_0, A_1\rangle = A_0 \backslash A_1$, $D_3\langle A_0, A_1, A_2\rangle = A_0\backslash(A_1\backslash A_2)$, etc. Given a class Γ and an ordinal $\alpha > 0$, $\alpha\text{-}\Gamma = \{D_\alpha A \mid A \in {}^\alpha\Gamma\}$. Hausdorff showed that $\mathbf{\Delta}^0_2 = \bigcup_{0<\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^0_1$; and Kuratowski proved the generalization: $\mathbf{\Delta}^0_{\xi+1} = \bigcup_{0<\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^0_\xi$, for all $0 < \xi < \omega_1$. The details are discussed in Note 10.32.

(9.164) *All these hierarchies* $\mathcal{H}$ *of pointclasses have several features in common:*

1. *Every* $\Gamma \in \mathcal{H}$ *is a continuously closed pointclass.*
2. *For each* $\Gamma \in \mathcal{H}$*, its dual class* $\breve{\Gamma} = \neg\,\Gamma = \{X \mid \neg\, X \in \Gamma\}$ *is also in* $\mathcal{H}$*. Note that a pointclass may or may not be selfdual.*[51]
3. *They are* semilinearly ordered *in the sense that for pointclasses* $\Gamma, \Gamma' \in \mathcal{H}$*, either* $\Gamma \subseteq \Gamma'$ *or* $\Gamma' \subseteq \Gamma$ *or* $\Gamma' = \breve{\Gamma}$*.*
4. *Each nonselfdual* $\Gamma \in \mathcal{H}$ *has a complete member, i.e.,* $A \in \Gamma$ *such that for all* $B \in \Gamma$*,* B *is a continuous preimage of* A*.*
5. *The inclusion relation on* $\mathcal{H}$ *is wellfounded.*

It is natural to wonder how pervasive this sort of organization is in the universe of continuously closed pointclasses. William Wadge made the key observation that the semilinearity of the inclusion relation on continuously closed pointclasses is an easy consequence of determinacy. Soon thereafter, Monk and Martin demonstrated the wellfoundedness of this relation.

We will only briefly sketch this theory.

Definition [ZF] *Suppose* $A, B \subseteq {}^\omega\omega$*.*

[51] In the Borel and projective hierarchies the $\mathbf{\Delta}$ classes are selfdual, while the $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ classes are nonselfdual.

1. *The* Lipschitz game $G_l(A, B)$ *is played in the conventional way with the payoff set* $(A \times \neg B) \cup (\neg A \times B)$. *Schematically,*

$$\begin{array}{lccccc} \text{I} & x_0 & & x_1 & & \cdots & x \\ \text{II} & & y_0 & & y_1 & \cdots & y \end{array}$$

   II *wins iff* $x \in A \leftrightarrow y \in B$.

2. *The* Wadge game $G_w(A, B)$ *is played as follows:*

$$\begin{array}{lcccccccccc} \text{I} & x_0 & x_1 & \cdots & x_{n_0} & & x_{n_0+1} & \cdots & x_{n_1} & & \cdots & x \\ \text{II} & & & & & y_0 & & & & y_1 & \cdots & y \end{array}$$

   I *plays* $x_0, x_1, \ldots$. *After each move by* I, II *has the option of passing or playing.* II *wins iff it plays infinitely often and* $x \in A \leftrightarrow y \in B$.

Clearly a Wadge game may be reformulated as a conventional game by requiring II to play on each turn in the usual way, but interpreting 0 as a pass and interpreting $k + 1$ as k. Given a Wadge II-strategy τ, and $x \in {}^\omega\omega$, by a minor modification of (5.164.3.2.2) we let $\vec{\tau}x$ be the sequence produced by τ in response to I playing x if this is an infinite sequence. Thus, $\vec{\tau}$ is a partial continuous function from ${}^\omega\omega$ to ${}^\omega\omega$. Conversely, if $f : {}^\omega\omega \to {}^\omega\omega$ is continuous then $f = \vec{\tau}$ for some Wadge II-strategy τ.

If τ is a Lipschitz II-strategy (i.e., a II-strategy in the usual sense) then $\vec{\tau}$ is continuous and additionally satisfies the *Lipschitz condition* with factor 1 defined in terms of a metric ρ on ${}^\omega\omega$ with the property that $\rho(x, x')$ is a decreasing function of $n(x, x')$, where $n(x, x')$ is the greatest $n \in \omega$ such that $x \restriction n = x' \restriction n$. This follows immediately from the fact that if $y = \vec{\tau}x$, $y' = \vec{\tau}x'$, and $x \restriction n = x' \restriction n$, then $y \restriction n = y' \restriction n$.

Definition [ZF] *Suppose* $A, B \subseteq {}^\omega\omega$.

1. A *is* Wadge-reducible to B $\overset{\text{def}}{\iff} A \leqslant_w B \overset{\text{def}}{\iff} A$ *is a continuous preimage of* B *iff there is a winning* II*-strategy in* $G_w(A, B)$.

2. A *is* Lipschitz-reducible to B $\overset{\text{def}}{\iff} A \leqslant_l B \overset{\text{def}}{\iff} A$ *is the preimage of* B *by a function with Lipschitz factor* 1 *iff there is a winning* II*-strategy in* $G_l(A, B)$.

Note that $A \leqslant_l B \to A \leqslant_w B$, so the Lipschitz ordering is potentially a refinement of the Wadge ordering.

We define the Lipschitz and Wadge equivalence relations in the usual way:

1. $A \equiv_l B \overset{\text{def}}{\iff} A \leqslant_l B \wedge B \leqslant_l A$.

2. $A \equiv_w B \overset{\text{def}}{\iff} A \leqslant_w B \wedge B \leqslant_w A$.

The *Lipschitz* and *Wadge degrees* are the equivalence classes of $\equiv_l$ and $\equiv_w$, respectively. The relations $\leqslant_l$ and $\leqslant_w$ are applied to degrees in the obvious way.

Wadge's observation was simply this

(9.165) **Theorem: Wadge's lemma** [ZF] *Suppose* $A, B \subseteq {}^\omega\omega$.

1. *If* $G_l(A, B)$ *is determined then* $A \leqslant_l B$ *or* $B \leqslant_l \neg A$.
2. *If* $G_w(A, B)$ *is determined then* $A \leqslant_w B$ *or* $B \leqslant_w \neg A$.

Proof Suppose II does not have a winning strategy in $G_l(A, B)$. Then I does have a winning strategy. This is easily converted to a winning II-strategy in $G_l(B, \neg A)$. The same is true for the Wadge games. $\square^{9.165}$

For the remainder of this discussion we will generally assume the *axiom of determinacy* AD.[9.147] This is largely a convenience, as many of the results are such that a limitation on the complexity of the pointsets appearing in the conclusion permits a corresponding limitation on the class of sets assumed to be determinate. These correspondences will generally be obvious.

We will also generally assume the axiom of dependent choices DC.[3.140.3]

Wadge observed that (9.165) implies that, under the hypothesis of AD, the set of all continuously closed pointclasses is a hierarchy in the sense of (9.164), with the possible exception of (9.164.5). For suppose $\Gamma, \Gamma' \subseteq \mathcal{P}\,{}^{\omega}\omega$ are continuously closed, and suppose that neither Γ nor Γ' is included in the other. Let $A \in \Gamma \backslash \Gamma'$ and suppose $A' \in \Gamma'$. Then $A \not\leqslant_w A'$, so[9.165.2] $A' \leqslant_w \neg A$. Thus, $A' \in \breve{\Gamma}$. Since A' was arbitrary in Γ', $\Gamma' \subseteq \breve{\Gamma}$. Taking duals of both sides, $\breve{\Gamma}' \subseteq \Gamma$. Similarly, $\Gamma \subseteq \breve{\Gamma}'$ and $\breve{\Gamma} \subseteq \Gamma'$; hence, $\Gamma' = \breve{\Gamma}$.

Thus, (9.164.1–3) are satisfied. Note that in the course of the proof we have also shown that if Γ is nonselfdual then any $A \in \Gamma \backslash \breve{\Gamma}$ is complete in Γ, so (9.164.4) is satisfied.

The following theorem of Martin and Leonard Monk shows that (9.164.5) is also satisfied. We make use of (the first part of) the following easy lemma.

(9.166) **Theorem** [ZF] *Suppose $X \subseteq {}^{\omega}2$.*

1. *If X has the Baire property and is not meager, then X is comeager on a basic interval, i.e., for some $s \in {}^{<\omega}2$, $I_s \backslash X$ is meager.*
2. *If X is measurable (with respect to the standard measure μ) and is not null, then for any real $p < 1$, there exists a basic interval I such that $\mu(X \cap I)/\mu(I) > p$.*

Proof 1 Let G be open such that $x \triangle G$ is meager. Since X is not meager, $G \neq 0$. Let $I \subseteq G$ be a basic interval. Then $I \backslash X$ is meager. $\square^{9.166.1}$

2 Without loss of generality, suppose $p > 0$. Let $m = \mu X > 0$. Let G be open such that $X \subseteq G$ and $\mu G < \mu X/p$. Let $\mathcal{I}$ be a set of pairwise disjoint basic intervals such that $G = \bigcup \mathcal{I}$, so $\mu G = \sum_{I \in \mathcal{I}} \mu I$ and $\mu X = \sum_{I \in \mathcal{I}} \mu(X \cap I)$. Then for some $I \in \mathcal{I}$, $\mu(X \cap I)/\mu(I) > p$. $\square^{9.166.2}$ $\square^{9.166}$

(9.167) **Theorem** [ZF + AD + DC] $\leqslant_l$ *is wellfounded.*

Proof Suppose not. By DC there exist $A_n \subseteq {}^{\omega}\omega$, $n \in \omega$, such that for all $n \in \omega$, $A_{n+1} <_l A_n$. It follows from semilinearity that I wins $G_l(A_n, A_{n+1})$ and $G_l(A_n, \neg A_{n+1})$, say by strategies σ_1^n and σ_0^n, respectively.[52]

Given $\alpha \in {}^{\omega}2$, let $x_\alpha^n \in {}^{\omega}\omega$, $n \in \omega$, be defined as indicated in Figure 9.2, which is to say, for each $n, m \in \omega$

$$x_\alpha^n(m) = \sigma_{\alpha_n}^n \langle x_\alpha^n(0), x_\alpha^{n+1}(0), \ldots, x_\alpha^n(m-1), x_\alpha^{n+1}(m-1) \rangle.$$

$$
\begin{tikzcd}[row sep=small]
& x^0_\alpha(0) & x^0_\alpha(1) & x^0_\alpha(2) & \cdots & x^0_\alpha & A_0 \\
\sigma^0_{\alpha_0} & & & & & & \\
& x^1_\alpha(0) \arrow[uur] & x^1_\alpha(1) \arrow[uur] & x^1_\alpha(2) & \cdots & x^1_\alpha & A_1 \\
\sigma^1_{\alpha_1} & & & & & & \\
& x^2_\alpha(0) \arrow[uur] & x^2_\alpha(1) \arrow[uur] & x^2_\alpha(2) & \cdots & x^2_\alpha & A_2 \\
& {} \arrow[ur] & {} \arrow[ur] & & & & \\
& & & \vdots & & & \\
& & {} & {} & {} & & \\
& x^n_\alpha(0) \arrow[ur] & x^n_\alpha(1) \arrow[ur] & x^n_\alpha(2) \arrow[ur] & \cdots & x^n_\alpha & A_n \\
\sigma^n_{\alpha_n} & & & & & & \\
& x^{n+1}_\alpha(0) \arrow[uur] & x^{n+1}_\alpha(1) \arrow[uur] & x^{n+1}_\alpha(2) & \cdots & x^{n+1}_\alpha & A_{n+1} \\
& {} \arrow[ur] & {} \arrow[ur] & & & & \\
& & & \vdots & & &
\end{tikzcd}
$$

Figure 9.2

In particular, the whole thing gets started by letting $x^n_\alpha(0) = \sigma^n_{\alpha_n}\langle\,\rangle$ (I's first move according to $\sigma^n_{\alpha_n}$, where $\langle\,\rangle = 0$ is the empty sequence) for all n. Moving to the right, each succeeding column is filled in according to the contents of the columns to its left. The result is that $x^n_\alpha = \vec{\sigma}^n_{\alpha_n} x^{n+1}_\alpha$, so

$$(x^n_\alpha \in A_n \leftrightarrow x^{n+1}_\alpha \in A_{n+1}) \leftrightarrow \alpha_n = 0. \tag{9.168}$$

Let $T = \{\alpha \in {}^\omega 2 \mid x^0_\alpha \in A_0\}$. For $s \in {}^{<\omega}2$, let $T_s = \{\beta \in {}^\omega 2 \mid s^\frown\beta \in T\}$. Note that x^n_α depends only on $\alpha \restriction (\omega\backslash n)$. It follows from this and (9.168) that for any $s \in {}^{<\omega}2$, $T_{s^\frown\langle 0\rangle}$ and $T_{s^\frown\langle 1\rangle}$ are complementary subsets of ${}^\omega 2$.

Since T has the Baire property,[9.148.2] there exists $s \in {}^{<\omega}2$ such that T_s is either meager or comeager.[9.166.1] Hence, $T_{s^\frown\langle 0\rangle}$ and $T_{s^\frown\langle 1\rangle}$ are either both meager or both comeager, which is impossible since they are complementary. □[9.167]

The following theorem characterizes the order type of the Lipschitz and Wadge degrees. The proof is fairly straightforward and not very illuminating and will be omitted. The terms *successor*, *limit*, and *cofinality* as applied to l- or w-degrees refers to their ordinal rank in $\leqslant_l$ or $\leqslant_w$, respectively.

(9.169) **Theorem** [ZF + AD + DC]

1. *The structure of $\leqslant_l$ is as follows.*
    1. *The sets $\{0\}$ and $\{{}^\omega\omega\}$ are dual nonselfdual l-degrees and are the least l-degrees.*
    2. *Every successor l-degree is selfdual.*
    3. *Every limit l-degree of cofinality ω is selfdual.*
    4. *Every limit l-degree of uncountable cofinality is nonselfdual.*
2. *The structure of the w-degrees is as follows.*
    1. *The nonselfdual w-degrees are exactly the nonselfdual l-degrees.*
    2. *Every selfdual w-degree is the union of a maximal set of consecutive selfdual l-degrees, which necessarily has order type ω_1.*

Figure 9.3 illustrates $\leqslant_w$. The order type of the l-degrees is obtained by replacing each selfdual w-degree by an ω_1-sequence of l-degrees.[9.169.2.2] The nonselfdual w-degrees are l-degrees.[9.169.2.1]

Definition [ZF + AD] *A* Wadge class *is a continuously closed pointclass in the sense of (5.52).*

As usual in descriptive set theory, the pointspace ${}^\omega\omega$ is of primary interest, and we often treat Wadge classes as though they were confined to subsets of ${}^\omega\omega$.

The first two Wadge classes are $\{0\}$ and $\{{}^\omega\omega\}$. The next Wadge class is $\mathbf{\Delta}^0_1$. Next are $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$. These are followed by the Hausdorff difference hierarchy on $\mathbf{\Pi}^0_1$, which has ω_1 levels and runs through $\mathbf{\Delta}^0_2$. Next are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$. After this, the Wadge hierarchy is a refinement of the Hausdorff hierarchy (which is itself a refinement of the Borel hierarchy).

[52] Although DC is available, there is no need to use it here, as $\mathsf{AC}_\omega(\mathbb{R})$ suffices, and this follows from AD.

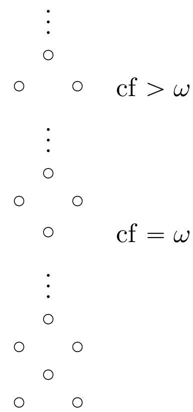


Figure 9.3: The order type of the Wadge degrees

Universal sets for Wadge classes In discussing universal sets for Wadge classes, for our purposes it is sufficient and convenient to restrict our attention to sets $U \subseteq {}^\omega\omega \times {}^\omega\omega$ that are universal for classes $\Gamma \cap \mathcal{P}({}^\omega\omega)$. The following definition just specializes (5.52.2.1) to this setting.

(9.170) **Definition** [ZF]

1. *The following notation is handy.*
   1. *Suppose $A \subseteq X \times Y$ and $x \in X$. Then $A_x \stackrel{\text{def}}{=} \{y \mid \langle x, y\rangle \in A\}$.*
   2. *Recall that $x^1, \dots, x^k \mapsto \langle x^1, \dots, x^k\rangle^{\cdot}$ is a bijection of ${}^k({}^\omega\omega)$ to ${}^\omega({}^k\omega)$. Thus, for example, given $x, y \in {}^\omega\omega$, $\langle x, y\rangle^{\cdot} = \langle\langle x_n, y_n\rangle \mid n \in \omega\rangle$. To define a bijection to ${}^\omega\omega$, for each $k \in \omega$ we let $p^k : {}^k\omega \stackrel{\text{bij}}{\to} \omega$ be a birecursive bijection, and*

$$\langle x^1, \dots, x^k\rangle^{\mathsf{p}} \stackrel{\text{def}}{=} \langle p^k\langle x^1_n, \dots, x^k_n\rangle \mid n \in \omega\rangle.$$

2. *$U \subseteq {}^\omega\omega \times {}^\omega\omega$ is a* universal set *for a Wadge class $\Gamma \stackrel{\text{def}}{\iff} U \in \Gamma$ and $\forall A \in \Gamma \cap \mathcal{P}({}^\omega\omega)\, \exists a \in {}^\omega\omega\ A = U_a$.*

3. *$U \subseteq {}^\omega\omega \times {}^\omega\omega$ is a* good universal set *for $\Gamma \stackrel{\text{def}}{\iff} U$ is universal for Γ and there exists a continuous function $s : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ such that for all $a, x, y \in {}^\omega\omega$*

$$\langle x, y\rangle^{\mathsf{p}} \in U_a \leftrightarrow y \in U_{s\langle a,x\rangle}.$$

As for Σ_1, the existence of a good universal set for any Wadge class implies the recursion theorem for the class.

(9.171) **Theorem** [ZF] *Suppose Γ is a Wadge class, U is a good universal set for Γ, and $A \subseteq {}^\omega\omega \times {}^\omega\omega$ is in Γ. Then there exists $a \in {}^\omega\omega$ such that $A_a = U_a$.*

Proof Let s witness the goodness of U.[9.170.3] As in the proof of (4.80), $\{\langle b, c\rangle^{\mathsf{p}} \mid \langle s\langle b, b\rangle, c\rangle \in A\}$, being a continuous preimage of A, is in Γ, so there exists $d \in {}^\omega\omega$ such that for all $b, c \in {}^\omega\omega$

$$\langle d, \langle b, c\rangle^{\mathsf{p}}\rangle \in U \leftrightarrow \langle s\langle b, b\rangle, c\rangle \in A.$$

Let $a = s\langle d,d\rangle$. Then for all $c \in {}^{\omega}\omega$

$$\begin{aligned}\langle a,c\rangle \in U &\leftrightarrow \langle s\langle d,d\rangle, c\rangle \in U \leftrightarrow \langle d, \langle d,c\rangle^{\mathsf{p}}\rangle \in U\\ &\leftrightarrow \langle s\langle d,d\rangle, c\rangle \in A\\ &\leftrightarrow \langle a,c\rangle \in A,\end{aligned}$$

i.e., $A_a = U_a$. $\square$[9.171]

(9.172) **Theorem** [$\mathsf{ZF} + \mathsf{AD}$] *Suppose Γ is a nonselfdual Wadge class. Then there exists a good universal set for Γ.*

Proof Let A be in $\Gamma\backslash\breve{\Gamma}$. Let t be a birecursive bijection of ${}^{\omega}\omega$ with the set of II-strategies for games on ω. Let

$$U = \{\langle x,y\rangle \mid \vec{tx}y \in A\}.$$

Note that $\langle x,y\rangle \mapsto \vec{tx}y$ is continuous, so $U \in \Gamma$. Suppose $B \in \Gamma$. Let $x \in {}^{\omega}\omega$ be such that tx is a winning II-strategy in $G_l(B,A)$. Then $B = \{y \in {}^{\omega}\omega \mid \langle x,y\rangle \in U\}$. Thus, U is a universal set for Γ.

To show that U is good, let f be a recursive function such that for any $y \in {}^{\omega}\omega$ and any II-strategy τ for games on ω, $f\langle \tau, y\rangle$ is a II-strategy such that

$$\forall z \in {}^{\omega}\omega\ \overrightarrow{f\langle \tau,y\rangle}z = \vec{\tau}\langle y,z\rangle^{\mathsf{p}}.$$

Let $s : {}^{\omega}\omega \times {}^{\omega}\omega \to {}^{\omega}\omega$ be the (recursive) function such that $\forall x,y \in {}^{\omega}\omega\ t(s\langle x,y\rangle) = f\langle tx,y\rangle$. Then for any $x,y,z \in {}^{\omega}\omega$

$$\begin{aligned}\langle y,z\rangle^{\mathsf{p}} \in U_x &\leftrightarrow \vec{tx}\langle y,z\rangle^{\mathsf{p}} \in A \leftrightarrow \overrightarrow{f\langle tx,y\rangle}z \in A \leftrightarrow \overrightarrow{t(s\langle x,y\rangle)}z \in A\\ &\leftrightarrow z \in U_{s\langle x,y\rangle}.\end{aligned}$$

$\square$[9.172]

9.7.3.1 Structural properties of Wadge classes

We now turn our attention to structural properties of Wadge classes under the hypothesis of AD. We will prove just one result of this type, viz., Theorem 9.176 of Steel and Van Wesep. By way of preparation we define another structural property of pointclasses closely related to the separation property and establish its relationship to the separation property.

Definition [ZF] *Suppose Γ is a nonselfdual Wadge class. Γ has the* second separation property $\overset{\text{def}}{\Longleftrightarrow} \mathrm{Sep}_{\mathrm{II}}\,\Gamma \overset{\text{def}}{\Longleftrightarrow}$

$$\forall A,B \in \Gamma\, \exists A',B' \in \breve{\Gamma}\, (A\backslash B \subseteq A' \wedge B\backslash A \subseteq B' \wedge A' \cap B' = 0).$$

Note that since $A\backslash B = \neg B \backslash \neg A$, $\mathrm{Sep}_{\mathrm{II}}\,\Gamma$ iff

$$\forall A,B \in \breve{\Gamma}\, \exists A',B' \in \breve{\Gamma}\, (A\backslash B \subseteq A' \wedge B\backslash A \subseteq B' \wedge A' \cap B' = 0). \tag{9.173}$$

By way of distinction, the separation property[5.111.1] *per se* is also referred to as the *first separation property*: $\mathrm{Sep}_{\mathrm{I}}$.

(9.174) **Theorem** [$\mathsf{ZF} + \mathsf{AD}$] *Suppose Γ is a nonselfdual Wadge class. Then* $\mathrm{Sep}_{\mathrm{II}}\,\breve{\Gamma} \leftrightarrow \neg\, \mathrm{Sep}_{\mathrm{I}}\,\Gamma$.

Proof $\rightarrow$ Let A be in $\Gamma\backslash\breve{\Gamma}$. Let s be a birecursive bijection of ${}^\omega\omega$ with the set of 2-sequences of II-strategies for games on ω, and—to simplify the notation—let $(\cdot,\cdot)$ be a birecursive bijection of ${}^\omega\omega \times {}^\omega\omega$ with ${}^\omega\omega$ (e.g., $(x,y) = \langle x,y\rangle^{\mathsf{p}}$). Let

$$\begin{aligned} A_0 &= \{(x,y) \mid \overrightarrow{(sx)_0}y \in A\} \\ A_1 &= \{(x,y) \mid \overrightarrow{(sx)_1}y \in A\}. \end{aligned}$$

Note that $(x,y) \mapsto \overrightarrow{(sx)_i}y$ is continuous for $i \in 2$, so $A_0, A_1 \in \Gamma$. Also, if we let $x \in {}^\omega\omega$ be such that $(sx)_0$ and $(sx)_1$ are both the copying II-strategy, so that $\forall i \in 2\, \forall y \in {}^\omega\omega\ \overrightarrow{(sx)_i}y = y$, then

$$A = \{y \in {}^\omega\omega \mid (x,y) \in A_0\} = \{y \in {}^\omega\omega \mid (x,y) \in A_1\},$$

so $A_0, A_1 \notin \breve{\Gamma}$, i.e., A_0 and A_1 are both complete in Γ.

Suppose $B_0, B_1 \in \Gamma$. Let $x \in {}^\omega\omega$ be such that for each $i \in 2$, $(sx)_i$ is a winning II-strategy in $G_l(B_i, A_i)$. Then for each $i \in 2$, $B_i = \{y \in {}^\omega\omega \mid (x,y) \in A_i\}$. In this sense, $\langle A_0, A_1\rangle$ is universal for pairs of sets in Γ. Note that for this argument, a universal set is a subset of ${}^\omega\omega$, not ${}^\omega\omega \times {}^\omega\omega$.

(9.175) $\langle \neg A_0, \neg A_1\rangle$ *is likewise universal for pairs of sets in* $\breve{\Gamma}$.

Suppose $\mathrm{Sep}_{\mathrm{II}}\,\breve{\Gamma}$. We will show $\neg\,\mathrm{Sep}_{\mathrm{I}}\,\Gamma$. To this end let[9.173] $B_0, B_1 \in \Gamma$ be disjoint such that $A_0\backslash A_1 \subseteq B_0$ and $A_1\backslash A_0 \subseteq B_1$. For $i \in 2$, let $C_i = \{x \mid (x,x) \in B_i\}$. C_0, C_1 are disjoint sets in Γ. We will show that they are not separable by a set $D \in \Delta = \Gamma \cap \breve{\Gamma}$. For suppose $C_0 \subseteq D$ and $D \cap C_1 = 0$. Let[9.175] z be such that $D = \{x \mid (z,x) \in \neg A_0\}$ and $\neg D = \{x \mid (z,x) \in \neg A_1\}$. Then

$$\begin{aligned} z \in D &\rightarrow (z,z) \in A_1\backslash A_0 \rightarrow (z,z) \in B_1 \rightarrow z \in C_1 \rightarrow z \notin D \\ \text{and } z \notin D &\rightarrow (z,z) \in A_0\backslash A_1 \rightarrow (z,z) \in B_0 \rightarrow z \in C_0 \rightarrow z \in D; \end{aligned}$$

contradiction.

$\leftarrow$ Conversely, suppose $\neg\,\mathrm{Sep}_{\mathrm{I}}\,\Gamma$. Let $C, D \in \Gamma$ be disjoint and not separable by a set in Δ. Suppose $A, B \in \breve{\Gamma}$. To demonstrate $\mathrm{Sep}_{\mathrm{II}}\,\breve{\Gamma}$, we must find disjoint sets $A', B' \in \Gamma$ such that $A\backslash B \subseteq A'$ and $B\backslash A \subseteq B'$. Consider the following game.

I and II produce $x, y \in {}^\omega\omega$, respectively, and I wins iff

$$\begin{aligned} y \in C &\rightarrow x \in A\backslash B \\ y \in D &\rightarrow x \in B\backslash A \\ \text{and } x &\in A \triangle B = (A\backslash B) \cup (B\backslash A). \end{aligned}$$

Suppose σ is a winning I-strategy. Let $E = \vec{\sigma}^{\leftarrow}A$. $\vec{\sigma}$ is continuous, and $\operatorname{im}\vec{\sigma} \subseteq A \triangle B$, so $\vec{\sigma}^{\leftarrow}A = \neg\,\vec{\sigma}^{\leftarrow}B$, and E is therefore in Δ. But $C \subseteq E$ and $E \cap D = 0$, contradicting the inseparability of C, D. There is thus no winning I-strategy.

Therefore let τ be a winning II-strategy. Let $A' = \vec{\tau}^{\leftarrow}D$ and $B' = \vec{\tau}^{\leftarrow}C$. Then A', B' are as desired. $\square$[9.173]

(9.176) **Theorem** $[\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}]$ *Suppose* Γ *is a nonselfdual Wadge class. Then either* Γ *or* $\breve{\Gamma}$ *has the (first) separation property, but not both. Equivalently,*[9.174] *either* Γ *or* $\breve{\Gamma}$ *has the second separation property, but not both.*

Proof We first show that either Γ or $\breve{\Gamma}$ has the second separation property. Let $\langle A_0, A_1\rangle$ be a complete pair of sets in Γ as in the proof of (9.174), i.e., for any $A'_0, A'_1 \in \Gamma$ there exists a continuous $f : {}^\omega\omega \to {}^\omega\omega$ such that $A'_i = f^{\leftarrow} A_i$ for each $i \in 2$. We will show that if $\langle A_0, A_1\rangle$ is $\mathrm{Sep_{II}}$-separable then $\mathrm{Sep_{II}}$ holds for Γ. Thus, suppose $B_0, B_1 \in \breve{\Gamma}$ are disjoint such that $A_0\backslash A_1 \subseteq B_0$ and $A_1\backslash A_0 \subseteq B_1$. Suppose $A'_0, A'_1 \in \Gamma$. Let $f : {}^\omega\omega \to {}^\omega\omega$ be continuous such that $A'_i = f^{\leftarrow} A_i$ for each $i \in 2$. Let $B'_i = f^{\leftarrow} B_i$ for $i \in 2$. Then B'_0 and B'_1 are disjoint sets in $\breve{\Gamma}$ such that $A'_0\backslash A'_1 \subseteq B'_0$ and $A'_1\backslash A'_0 \subseteq B'_1$.

Thus, if $\neg\,\mathrm{Sep_{II}}\,\Gamma$ then there do not exist disjoint $B_0, B_1 \in \breve{\Gamma}$ such that $A_0\backslash A_1 \subseteq B_0$ and $A_1\backslash A_0 \subseteq B_1$. Similarly, if $\neg\,\mathrm{Sep_{II}}\,\breve{\Gamma}$[9.173] then there do not exist disjoint $B_0, B_1 \in \Gamma$ such that $A_0\backslash A_1 \subseteq B_0$ and $A_1\backslash A_0 \subseteq B_1$. We will derive a contradiction from these hypotheses.

Consider the following two games in which I and II play $x, y \in {}^\omega\omega$, respectively.

G_0: II wins iff

$$\begin{aligned} &x \in A_0\backslash A_1 \to y \in A_1\backslash A_0,\\ &x \in A_1\backslash A_0 \to y \in A_0\backslash A_1,\\ \text{and }&y \notin A_0 \cap A_1. \end{aligned}$$

G_1: II wins iff

$$\begin{aligned} &x \in A_0\backslash A_1 \to y \in A_0\backslash A_1,\\ &x \in A_1\backslash A_0 \to y \in A_1\backslash A_0,\\ \text{and }&y \in A_0 \cup A_1. \end{aligned}$$

Suppose τ is a winning II-strategy in G_0 or G_1. Let $f = \vec{\tau}$. In the former case, let $B_i = f^{\leftarrow} A_{1-i}$, and in the latter case let $B_i = f^{\leftarrow}(\neg\, A_{1-i})$. In either case, $B_0 \cap B_1 = 0$, $A_0\backslash A_1 \subseteq B_0$, and $A_1\backslash A_0 \subseteq B_1$, which contradicts our hypothesis.

Hence, there exist winning I-strategies σ_0, σ_1 in G_0, G_1, respectively. Let f_0, f_1 be the corresponding functions from ${}^\omega\omega$ to ${}^\omega\omega$. Then

$$\begin{aligned} &f_0{}^{\rightarrow}(A_0\backslash A_1) \subseteq A_0\backslash A_1,\\ &f_0{}^{\rightarrow}(A_1\backslash A_0) \subseteq A_1\backslash A_0,\\ \text{and }&f_0{}^{\rightarrow}(\neg\, A_0 \cap \neg\, A_1) \subseteq (A_0\backslash A_1) \cup (A_1\backslash A_0). \end{aligned}$$

Similarly,

$$\begin{aligned} &f_1{}^{\rightarrow}(A_0\backslash A_1) \subseteq A_1\backslash A_0,\\ &f_1{}^{\rightarrow}(A_1\backslash A_0) \subseteq A_0\backslash A_1,\\ \text{and }&f_1{}^{\rightarrow}(A_0 \cap A_1) \subseteq (A_0\backslash A_1) \cup (A_1\backslash A_0). \end{aligned}$$

As in the proof of (9.167), given $\alpha \in {}^\omega 2$, let $x^n_\alpha \in {}^\omega\omega$, $n \in \omega$, be such that for each $n \in \omega$

$$x^n_\alpha = f_{\alpha(n)} x^{n+1}_\alpha.$$

This is possible, of course, because for any $y \in {}^\omega\omega$, $(f_i y)m$ is uniquely determined by $y \restriction m$.

(9.177) **Claim** $\{\alpha \in {}^\omega 2 \mid x^0_\alpha \in A_0 \bigtriangleup A_1\}$ *is comeager.*

Proof If not, then since by AD every set of reals has the Baire property, for some $s \in {}^{<\omega}\omega$, $\{\alpha \in {}^\omega 2 \mid x^0_{s^\frown\alpha} \in A_0 \bigtriangleup A_1\}$ is meager. Recall that $A_0 \bigtriangleup A_1 = (A_0\backslash A_1) \cup$

$(A_1 \backslash A_0)$, and note that $f_i{}^{\to}(A_0 \triangle A_1) \subseteq A_0 \triangle A_1$ for each $i \in 2$. Hence, $\{\alpha \in {}^{\omega}2 \mid x^0_\alpha \in A_0 \triangle A_1\}$ is meager. It follows that either $\{\alpha \in {}^{\omega}2 \mid x^0_\alpha \in A_0 \cap A_1\}$ or $\{\alpha \in {}^{\omega}2 \mid x^0_\alpha \in \neg A_0 \cap \neg A_1\}$ is nonmeager. Without loss of generality, suppose the former. For any α in this set, $(\langle 1\rangle ^\frown \alpha)^0 \in A_0 \triangle A_1$, so $\{\alpha \in {}^{\omega}2 \mid x^0_\alpha \in A_0 \triangle A_1\}$ is nonmeager; contradiction. $\square$[9.177]

Let $X = \{\alpha \in {}^{\omega}2 \mid x^0_\alpha \in A_0 \triangle A_1\}$, and let $X_i = \{\alpha \in X \mid x^0_\alpha \in A_i \backslash A_{1-i}\}$. Then X is the disjoint union of X_0 and X_1, and for any $i \in 2$ and $\alpha \in X_i$,

1. $\langle 0\rangle ^\frown \alpha \in X_i$, and
2. $\langle 1\rangle ^\frown \alpha \in X_{1-i}$.

We now derive a contradiction as in the proof of (9.167). Since X is comeager,[9.177] either X_0 or X_1 is nonmeager. Suppose X_i is nonmeager. Let $s \in {}^{<\omega}\omega$ be such that $\{\alpha \in {}^{\omega}\omega \mid s ^\frown \alpha \in X_i\}$ is comeager. Again using the fact that X is comeager, letting $j = i$ or $1-i$ according on whether s has an even or odd number of 0s, X_j is comeager, so X_{1-j} is meager. But $X_{1-j} \supseteq \{\langle 1\rangle ^\frown \alpha \mid \alpha \in X_j\}$, which is nonmeager; contradiction.

This completes the proof that either Γ or $\breve{\Gamma}$ has the second separation property; equivalently, Γ and $\breve{\Gamma}$ do not both have the first separation property.

It remains to be shown that either Γ or $\breve{\Gamma}$ has the first separation property. Let us call a function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ I-*Lipschitz* $\overset{\text{def}}{\iff} f = \vec{\sigma}$ for some I-strategy σ.

(9.178) **Claim** *Suppose $\langle A_0, A_1\rangle$ is an inseparable pair of sets in Γ, and suppose $\langle B_0, B_1\rangle$ is a disjoint pair of sets in Γ or in $\breve{\Gamma}$. Then there exists a* I-*Lipschitz f such that $f^{\to}B_0 \subseteq A_0$ and $f^{\to}B_1 \subseteq A_1$.*

Proof Consider the game in which I and II play respective reals x and y, and I wins iff

$$y \in B_0 \to x \in A_0 \text{ and } y \in B_1 \to x \in A_1.$$

A winning II-strategy in this game would yield a continuous $g : {}^{\omega}\omega \to {}^{\omega}\omega$ such that

$$\begin{aligned} g^{\to}A_0 &\subseteq B_1, \\ g^{\to}A_1 &\subseteq B_0, \\ \text{and } \operatorname{im} g &\subseteq B_0 \cup B_1. \end{aligned}$$

Then $D = g^{\leftarrow}B_1 = g^{\leftarrow}(\neg B_0)$ is in $\Gamma \cap \breve{\Gamma}$ and separates A_0 from A_1. Since A_0 and A_1 are inseparable, this cannot be. Therefore let σ be a winning I-strategy and let $f = \vec{\sigma}$. $\square$[9.178]

Now suppose toward a contradiction that $\langle A_0, A_1\rangle$ is an inseparable pair of Γ sets, and $\langle C_0, C_1\rangle$ is an inseparable pair of $\breve{\Gamma}$ sets. Let f be I-Lipschitz such that $f^{\to}A_0 \subseteq C_0$ and $f^{\to}A_1 \subseteq C_1$, using (9.178) (with $\breve{\Gamma}$ for Γ). Let $B_0 = f^{\leftarrow}C_0$ and $B_1 = f^{\leftarrow}C_1$. Note that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, and $B_0 \cap B_1 = 0$. Since $\langle A_0, A_1\rangle$ is inseparable, so are $\langle A_0, \neg B_0\rangle$ and $\langle A_1, \neg B_1\rangle$.

Now use (9.178) with various disjoint and inseparable pairs to obtain I-Lipschitz functions $f_0, f_1, f_2 : {}^{\omega}\omega \to {}^{\omega}\omega$ such that

$$\begin{aligned} f_0{}^{\to}A_0 \subseteq A_1 &\wedge f_0{}^{\to}A_1 \subseteq A_0, \\ f_1{}^{\to}A_0 \subseteq A_0 &\wedge f_1{}^{\to}(\neg B_0) \subseteq A_1, \\ \text{and } f_2{}^{\to}A_1 \subseteq A_1 &\wedge f_2{}^{\to}(\neg B_1) \subseteq A_0. \end{aligned}$$

As before, given $\alpha \in {}^\omega 3$, let x^n_α, $n \in \omega$, be such that for each $n \in \omega$,

$$x^n_\alpha = f_{\alpha(n)} x^{n+1}_\alpha.$$

(9.179) **Claim** $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \in A_0 \cup A_1\}$ *is comeager.*

Proof Suppose not. Then there exists $s \in {}^{<\omega}3$ such that $\{\alpha \in {}^\omega 3 \mid x^0_{s^\frown\alpha} \notin A_0 \cup A_1\}$ is comeager. But for any $\alpha \in {}^\omega 3$, $x^0_{s^\frown\alpha} \notin A_0 \cup A_1 \rightarrow x^0_\alpha \notin A_0 \cup A_1$, so $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \notin A_0 \cup A_1\}$ is comeager.

Since B_0 and B_1 are disjoint, there exists $i \in 2$ such that $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \notin B_i\}$ is nonmeager. Since $f_{i+1}{}^{\rightarrow}(\neg B_i) \subseteq A_0 \cup A_1$, it follows that $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \in A_0 \cup A_1\}$ is nonmeager; contradiction. $\square$9.179

Now let9.179 $i \in 2$ be such that $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \in A_i\}$ is nonmeager, and let $s \in {}^{<\omega}3$ be such that $\{\alpha \in {}^\omega 3 \mid x^0_{s^\frown\alpha} \in A_i\}$ is comeager. Since $f_0{}^{\rightarrow}A_i \subseteq A_{1-i}$, 0 must occur an even number of times in s, and $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \in A_{1-i}\}$ must be meager; however, for any $\alpha \in {}^\omega 3$, if $x^0_\alpha \in A_i$ then $x^0_{\langle 0\rangle^\frown\alpha} \in A_{1-i}$, so $\{\alpha \in {}^\omega 3 \mid x^0_\alpha \in A_{1-i}\}$ is nonmeager.

Thus, either Γ or $\breve{\Gamma}$ has the first separation property. $\square$9.176

The ostensibly stronger reduction and prewellordering properties are indeed stronger than the separation property, and it has been shown that there are non-selfdual Wadge classes Γ such that neither Γ nor $\breve{\Gamma}$ has the reduction property. On the other hand, certain closure properties of pointclasses have been shown to imply that reduction or prewellordering holds for one or the other of a dual pair. For example, Steel has shown that if Γ is nonselfdual and continuously closed, and $\Gamma \cap \breve{\Gamma}$ is closed under (finite) unions, then either Γ or $\breve{\Gamma}$ has the reduction property.

9.7.4 AD

Up to this point most of the consequences of determinacy we have proved for a pointclass Γ follow from the hypothesis of the determinacy for sets in a suitable continuously closed pointclass Γ' that does not in general contain every subset of ${}^\omega\omega$. In this section we will explore consequences of AD *per se*, i.e., the assumption that all subsets of ${}^\omega\omega$ are determinate. AD is, of course, inconsistent with AC, but this does not render AD useless in the setting of ZFC. For example, as we have seen, $\mathsf{ZF} \vdash \mathsf{ZF}^{L(\mathbb{R})}$, and clearly $\mathsf{ZF} + \mathsf{AD} \vdash \mathsf{AD}^{L(\mathbb{R})}$. We will also show that $\mathsf{ZF} + \mathsf{DC} \vdash \mathsf{DC}^{L(\mathbb{R})}$. This suggests the possibility of $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})}$ as a comprehensive theory of the countably infinitary, within which context $L(\mathbb{R})$ is an inner model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$

The acceptability this theory as an extension of ZFC, however, is contingent on its consistency—which is an absolute requirement—and also on its plausibility: Do we have any reason to believe that it is true, or, contrarily, do we have any reason to believe that it is false? In this section we will address these issues, which we will find to be very precisely related to certain large cardinal hypotheses.

We begin with several theorems that derive large cardinal conclusions from AD. We follow this with several theorems that derive determinacy conclusions from large cardinal hypotheses. Applying the aesthetic criterion these results may be taken as strong evidence in favor of both determinacy and large cardinal hypotheses.

By way of preparation, we discuss two choice axioms.

(9.180) **Theorem** [ZF + AD] $\mathsf{AC}_\omega(\mathbb{R})$.

Proof Suppose for each $n \in \omega$, A_n is a nonempty subset of ${}^\omega\omega$. We must show that there exists $f : \omega \to {}^\omega\omega$ such that $\forall n \in \omega\ fn \in A_n$. Consider the game in which I and II play x and y in ${}^\omega\omega$, and II wins iff $y \in A_{x_0}$. Clearly, I does not have a winning strategy in this game, so II does, say τ. For each $n \in \omega$ let fn be II's response to $\langle n, 0, 0, \dots \rangle$ according to τ. f is a choice function for $\langle A_n \mid n \in \omega \rangle$. $\square$[9.180]

(9.181) **Theorem** [$\mathsf{ZF} + \mathsf{DC}$] $\mathsf{DC}^{L(\mathbb{R})}$.

Proof We work in $\mathsf{GB} + \mathsf{DC}$ to facilitate the use of proper classes. Using the recursive definition of $L(\mathbb{R})$ we may define a function $\Phi : \mathrm{Ord} \times {}^\omega\omega \overset{\text{sur}}{\to} L(\mathbb{R})$ such that for every $\alpha \in \mathrm{Ord}$, $\Phi \restriction (\alpha \times {}^\omega\omega) \in L(\mathbb{R})$. Suppose $X, R \in L(\mathbb{R})$, R is a binary relation on X, and

$$\forall x \in X\ \exists y\ \langle x, y \rangle \in R.$$

Suppose $x_0 \in X$. Then by DC there exists $\langle x_n \mid n \in \omega \rangle$ such $\forall n \in \omega\ \langle x_n, x_{n+1} \rangle \in R$. We will show that there exists such a sequence in $L(\mathbb{R})$.

Since $L(\mathbb{R})$ is transitive, $X \subseteq L(\mathbb{R})$. Let $\lambda \in \mathrm{Ord}$ be such that $\{x_n \mid n \in \omega\} \subseteq \Phi^{\to}(\lambda \times {}^\omega\omega)$. Using AC_ω, which is a special case of DC, let $\big\langle \langle \alpha_n, a_n \rangle \mid n \in \omega \big\rangle$ be such that $\forall n \in \omega \big(\alpha_n < \lambda \wedge \Phi\langle \alpha_n, a_n \rangle = x_n\big)$. Note that $\langle a_n \mid n \in \omega \rangle \in L(\mathbb{R})$ because it can be coded by a single real.

Let $\prec$ be the binary relation on $\lambda \times \omega$ defined by the condition:

$$\langle \alpha, i \rangle \prec \langle \beta, j \rangle \leftrightarrow i = j + 1 \wedge \big\langle \Phi\langle \beta, a_j \rangle, \Phi\langle \alpha, a_i \rangle \big\rangle \in R.$$

Then $\prec \in L(\mathbb{R})$, and $\big\langle \langle \alpha_n, n \rangle \mid n \in \omega \big\rangle$ is a descending $\prec$-sequence, so $\prec$ is illfounded below $\langle \alpha_0, 0 \rangle$. Hence, $L(\mathbb{R}) \models \ulcorner [\prec]$ is illfounded below $[\langle \alpha_0, 0 \rangle] \urcorner$.[53] Working in $L(\mathbb{R})$, define by recursion $\langle \beta_n \mid n \in \omega \rangle$ such that $\beta_0 = \alpha_0$ and for every $n \in \omega$, β_{n+1} is the least β such that $\langle \beta, n+1 \rangle \prec \langle \beta_n, n \rangle$ and $\prec$ is illfounded below $\langle \beta, n+1 \rangle$. This is a legitimate definition in ZF.

Let $y_n = \Phi\langle \beta_n, a_n \rangle$. Then $\langle y_n \mid n \in \omega \rangle \in L(\mathbb{R})$, $y_0 = x_0$ and $\forall n\ \langle y_n, y_{n+1} \rangle \in R$, as desired. $\square$[9.181]

9.7.5 Large cardinals from determinacy

Recall[4.102] that $\mathcal{D} = \mathcal{D}_T$ is the set of Turing degrees, and $\leqslant_T$ is Turing reducibility, i.e., the relation of relative recursiveness. For $x \in {}^\omega\omega$, $[x]$ is the Turing degree of x.

Definition [ZF]

1. *The* cone *of Turing degrees* with base $d \overset{\text{def}}{=} C_d \overset{\text{def}}{=} \{d' \in \mathcal{D} \mid d \leqslant_T d'\}$.
2. *The* cone filter *over* $\mathcal{D} \overset{\text{def}}{=} \{S \subseteq \mathcal{D} \mid \exists d \in \mathcal{D}\, C_d \subseteq S\}$.

It is straightforward to show that what we have defined as the cone filter is indeed a filter.

(9.182) **Theorem (Martin)** [$\mathsf{ZF} + \mathsf{AD}$] *The cone filter over $\mathcal{D}$ is a nonprincipal countably complete ultrafilter.*

[53] By the usual argument: If $L(\mathbb{R})$ thinks $\prec$ is wellfounded then there is a rank function for $\prec$ in $L(\mathbb{R})$, which is a rank function for $\prec$ in V, contradicting the illfoundedness of $\prec$.

Proof Let F be the cone filter. We first show that F is nonprincipal, i.e., for any $d \in \mathcal{D}$, there exists $d' \in \mathcal{D}$ such that $d' \not\leqslant_T d$. This follows from the fact that $\{x' \in {}^\omega\omega \mid x' \leqslant_T x\}$ is countable for any $x \in {}^\omega\omega$, whereas ${}^\omega\omega$ is not countable; or simply let d' be the Turing jump of d.

Next we show that F is countably complete. Suppose $\{D_n \mid n \in \omega\} \subseteq F$. For each $n \in \omega$, let $A_n = \{x \in {}^\omega\omega \mid C_{[x]} \subseteq D_n\}$. Let[9.180] f be a choice function for $\langle A_n \mid n \in \omega\rangle$. Thus, $f : \omega \to {}^\omega\omega$, and $\forall n \in \omega\ C_{[fn]} \subseteq D_n$. Let $j : {}^\omega\omega \overset{\text{bij}}{\to} {}^\omega({}^\omega\omega)$ be the homeomorphism defined in (5.87) from a recursive pairing function $p : {}^2\omega \overset{\text{bij}}{\to} \omega$. Then j and j^{-1} are recursive. Let $x = j^{-1}f$. Then $C_{[x]} \subseteq D_n$ for all $n \in \omega$, so $\bigcap_{n\in\omega} D_n \in F$.

Next we show that F is an ultrafilter. Suppose $D \subseteq \mathcal{D}$. Let $\mathcal{G}$ be the game in which I and II play x and y, respectively, and I wins iff $[x * y] \in D$. Suppose σ is a winning I-strategy in $\mathcal{G}$. Note that $\sigma \subseteq V_\omega$ is a type-1 object to which the notion of Turing degree applies. Suppose $y \in {}^\omega\omega$ and $[\sigma] \leqslant_T y$. Clearly, $\sigma * y \equiv_T y$, i.e., $[\sigma * y] = [y]$. Since σ is a winning I-strategy in $\mathcal{G}$, $[\sigma * y] \in D$, so $[y] \in D$. In other words, $C_{[\sigma]} \subseteq D$, so $D \in F$.

On the other hand, if τ is a winning II-strategy in $\mathcal{G}$, then $C_{[\tau]} \subseteq (\mathcal{D}\backslash D)$, so $\mathcal{D}\backslash D \in F$. F is therefore an ultrafilter. $\square$[9.182]

As above in this chapter, we will use 'measure' interchangeably with 'ultrafilter' for countably complete ultrafilters.

Definition [ZF + AD] *The* Martin measure *over* $\mathcal{D} \overset{\text{def}}{=}$ *the cone filter.*

$\mathcal{D}$ is not a cardinal, and our working theory does not contain AC (but rather the incompatible AD), so we cannot simply quote the theory of measurable cardinals developed above. Nevertheless, the existence of the Martin measure is a very strong consequence of ZF + AD, as the following two theorems demonstrate.

(9.183) **Theorem** [ZF + AD] ω_1 *is measurable.*

Proof Let F be the cone (ultra)filter over $\mathcal{D}$. It suffices to show the existence of a function $f : \mathcal{D} \to \omega_1$ such that for every $\alpha \in \omega_1$, $f^{\leftarrow}\{\alpha\} \notin F$; for then $\{X \subseteq \omega_1 \mid f^{\leftarrow}X \in F\}$ is a nonprincipal countably complete ultrafilter over ω_1.

A simple way to define f is to let fd be the least ordinal α such that α is not the order type of a wellordering R of a subset of ω such that $[R] \leqslant_T d$. Clearly, for any $\alpha \in \omega_1$, letting R be a wellordering of a subset of ω with order type α, $f^{\leftarrow}\{\alpha\} \cap C_{[R]} = 0$. $\square$[9.183]

It would appear from (9.183) that, absent the axiom of choice, a measurable cardinal need not be large. The following theorem shows that ZF + AD nevertheless does imply that ω_1 is large in a certain sense. We formulate the following theorem in GB so that we can easily talk about inner models.

(9.184) **Theorem** [GB + AD] HOD $\models$ $\ulcorner[\omega_1]$ *is measurable*$\urcorner$.

Remark In other words, letting $\kappa = \omega_1$, HOD $\models$ $\ulcorner[\kappa]$ is measurable$\urcorner$. This is not to say that HOD $\models$ $\ulcorner\omega_1$ is measurable$\urcorner$, which is manifestly false, since HOD $\models$ ZFC.

Proof Clearly the Martin measure is ordinal-definable (OD)—it is in fact definable from no parameters at all. Let U be the ultrafilter over ω_1 defined in the proof of (9.183). Then U is OD. Let $U' = U \cap \text{HOD}$. Since HOD is definable, U' is OD and therefore HOD (since it is included in HOD). Since all ordinals are OD, a subset of ω_1 is HOD iff it is OD. U' therefore consists of all OD subsets of ω_1 that are in U. Clearly HOD $\models$ ⌜$[U']$ is an ultrafilter⌝. The intersection of any OD ω-sequence of members of U' is clearly OD and is in U, so it is in U', and therefore HOD $\models$ ⌜$[U']$ is countably complete⌝. Since U is nonprincipal, so is U'. Hence, HOD $\models$ ⌜$[U']$ is a nonprincipal countably complete ultrafilter over $[\omega_1]$⌝. □[9.184]

(9.184) gives us the following relative consistency result.

(9.185) **Theorem** [S] *If* ZF + AD *is consistent then* ZFC+ ⌜*there exists a measurable cardinal*⌝ *is consistent.*

Proof Any proof of inconsistency in the latter theory yields a proof of inconsistency in GB + AD via (9.184), hence a GB-proof of ¬ AD. Since GB is a conservative extension of ZF, ZF ⊢ ¬ AD, so ZF + AD is inconsistent. □[9.185]

With a little more work we obtain the following.

(9.186) **Theorem** [ZF + AD] *The closed unbounded filter over* ω_1 *is an ultrafilter.*

Proof We will make use of a birecursive coding of members of ${}^\omega({}^\omega\omega)$ by members of ${}^\omega\omega$, as in the proof of (9.182). Thus, let $j : {}^\omega\omega \overset{\text{bij}}{\to} {}^\omega({}^\omega\omega)$ be a homeomorphism as defined in (5.87).[54] As a notational convenience, for $x \in {}^\omega\omega$ and $n \in \omega$, let $x^n = (jx)n$.

Recall[5.61] the definition of WO $\subseteq {}^\omega\omega$ as the set of codes of countable ordinals via the map $x \mapsto R_x$, where R_x is a binary relation on V_ω that is a wellorder iff $x \in$ WO. For $x \in$ WO, let $\|x\|$ be the order type of R_x.

Suppose $X \subseteq \omega_1$ and consider the game $\mathcal{G}_X$ in which I and II play $x, y \in {}^\omega\omega$, which are regarded as coding ω-sequences $\langle x^n \mid n \in \omega\rangle$ and $\langle y^n \mid n \in \omega\rangle$, respectively, as above. The winner is defined as follows.

1. If for some $n \in \omega$, $x^0, y^0, x^1, y^1, \ldots, x^n, y^n$ are not all in WO, let n be least with this property. I wins iff $x^n \in$ WO.

2. If $x^n, y^n \in$ WO for all $n \in \omega$, and for some $n \in \omega$, $\langle \|x^0\|, \|y^0\|, \|x^1\|, \|y^1\|, \ldots, \|x^n\|, \|y^n\|\rangle$ is not strictly increasing, let n be least with this property. I wins iff $\langle \|x^0\|, \|y^0\|, \|x^1\|, \|y^1\|, \ldots, \|x^n\|\rangle$ is strictly increasing.

3. If $\langle x^0, y^0, x^1, y^1, \ldots\rangle$ is a strictly increasing sequence of ordinal codes, then I wins iff $\sup_{n\in\omega} \|x^n\| \in X$.

(9.187) **Claim** *If* I *has a winning strategy in* $\mathcal{G}_X$ *then* X *includes a closed unbounded subset of* ω_1. *If* II *has a winning strategy in* $\mathcal{G}_X$ *then* $\omega_1 \backslash X$ *includes a closed unbounded subset of* ω_1.

Proof Suppose σ is a winning I-strategy in $\mathcal{G}_X$. For each $\alpha \in \omega_1$, let

$$X_\alpha = \{((\sigma * y)^{\mathrm{I}})^n \mid y \in {}^\omega\omega \wedge n \in \omega \wedge \forall m < n\,(y^m \in \mathrm{WO} \wedge \|y^m\| < \alpha)\}.$$

Since σ is a winning I-strategy in $\mathcal{G}$, $X_\alpha \subseteq$ WO. Let $a \in$ WO be such that $\|a\| = \alpha$. Then $\{b \in \mathrm{WO} \mid \|b\| < \alpha\}$ is $\Sigma^1_1(a)$. Hence, X_α is $\Sigma^1_1(\sigma, a)$. Thus[5.118] there exists $\beta \in \omega_1$ such that $\forall x \in X_\alpha\ \|x\| < \beta$.

[54] Unlike the proof of (9.182), the present application does not require recursiveness.

Let $f : \omega_1 \to \omega_1$ be defined by the condition that for each $\alpha \in \omega_1$, $f\alpha$ is the least ordinal β such that $\forall b \in X_\alpha\ \|b\| < \beta$. Let $C = \{\gamma \in \omega_1 \mid \operatorname{Lim}\gamma \wedge \forall \alpha < \gamma\ f\alpha < \gamma\}$. C is closed unbounded in ω_1. Let C' be the set of limit points of C. C' is closed unbounded in ω_1.

(9.188) **Claim** *$C' \subseteq X$.*

Proof Suppose $\alpha \in C'$. Let $\langle \alpha_n \mid n \in \omega\rangle$ be a strictly increasing sequence of ordinals in C with limit α. Let $y \in {}^\omega\omega$ be such that for each $n \in \omega$, $y^n \in \mathrm{WO}$ and $\|y^n\| = \alpha_n$ (invoking $\mathsf{AC}_\omega(\mathbb{R})$). Let $x = (\sigma * y)^{\mathrm{I}}$. Since σ is a winning I-strategy, $x^n \in \mathrm{WO}$ for all $n \in \omega$. Suppose $n \in \omega$. Since C consists of limit ordinals, there exists $\beta < \alpha_n$ such that $\alpha_m < \beta$ for all $m < n$. Since $\|y_m\| = \alpha_m$ for all $m < n$, $x^n \in X_\beta$, so $\|x^n\| < f\beta < \alpha_n$, since $\alpha_n \in C$.

Thus, $\|x^n\| < \|y^n\|$ for all $n \in \omega$. Since σ is a winning I-strategy, $\|y^n\| < \|x^{n+1}\|$ for all $n \in \omega$. Hence, $\langle x^0, y^0, x^1, y^1, \dots\rangle$ is a strictly increasing sequence of ordinal codes. Since $\|y_n\| = \alpha_n$, $\sup_{n\in\omega}\|x^n\| = \sup_{n\in\omega}\|y^n\| = \sup_{n\in\omega}\alpha_n = \alpha$. Since σ is a winning I-strategy, $\alpha \in X$. $\square$[9.188]

The analogous argument shows that if II has a winning strategy in $\mathcal{G}_X$ then $\omega_1 \setminus X$ includes a closed unbounded subset of ω_1. $\square$[9.187]

The closed unbounded filter is therefore an ultrafilter. $\square$[9.186]

(9.189) **Theorem** [$\mathsf{ZF} + \mathsf{AD}$] *The closed unbounded filter over ω_1 is a nonprincipal countably complete ultrafilter.*

Proof Let F be the closed unbounded filter over ω_1. We have just shown[9.188] that F is an ultrafilter. F is obviously nonprincipal, so it only remains to be shown that F is countably complete. Suppose $\langle X_n \mid n \in \omega\rangle$ is an ω-sequence of members of F. Let $X = \bigcap_{n\in\omega} X_n$. We claim that $X \in F$. If we had the axiom of choice we could let $C_n \subseteq X_n$ be closed unbounded in ω_1 for each $n \in \omega$, and let $C = \bigcap_{n\in\omega} C_n$. It is easy to show that C is closed unbounded, and clearly $C \subseteq X$.

But we don't have AC. We have $\mathsf{AC}_\omega(\mathbb{R})$,[9.180] but this is not enough for the preceding argument, because for each $n \in \omega$ we have to choose from the set of closed unbounded subsets of X_n, which is a set of subsets of ω_1, not of ${}^\omega\omega$. The proof of (9.186), however, shows us a way around this difficulty. A strategy for a game on ω is a subset of V_ω, i.e., an element of $V_{\omega+1}$, so we may use (9.180)—with the trivial substitution of $V_{\omega+1}$ for ${}^\omega\omega$—to conclude that there exists $\langle \sigma_n \mid n \in \omega\rangle$ such that for each $n \in \omega$, σ_n is a winning I-strategy in the game $\mathcal{G}_{X_n}$ as defined in the proof of (9.186). (II cannot have a winning strategy because then ${}^\omega\omega\setminus X_n$ would include a closed unbounded set;[9.187] hence, by AD, I must have a winning strategy.)

A review of the proof of (9.188) shows that the closed unbounded set C' was defined from the strategy σ. We now define C'_n from σ_n the same way for each $n \in \omega$, so that each C'_n is closed unbounded in ω_1 and $C'_n \subseteq X_n$. Let $C = \bigcap_{n\in\omega} C'_n$. Then C is closed unbounded in ω_1, and $C \subseteq X$. $\square$[9.189]

Another way to show that the closed unbounded filter over ω_1 is countably complete is to make use of the remarkable fact that—assuming $\mathsf{ZF} + \mathsf{AD}$—*any* ultrafilter over *any* set is countably complete.

Definition [ZF] *Suppose $X \subset {}^\omega M$ for some set M. X is a* tail set $\overset{\text{def}}{\iff}$ *for all $x, y \in {}^\omega M$, if $\exists m\, \forall n > m\ x_n = y_n$ (i.e., x and y have a common "tail") then $x \in X \leftrightarrow y \in X$.*

(9.190) **Theorem: the 0-1 law** [ZF] *Suppose $X \subseteq {}^{\omega}2$ is a tail set.*

1. *If X has the Baire property then X is meager or comeager.*
2. *If X is measurable then $\mu X = 0$ or $\mu X = 1$.*

Proof Suppose I and J are basic intervals of the same size, i.e., $I = I_s$ and $J = I_t$ where $|s| = |t|$. Then $X \cap I$ and $X \cap J$ are homeomorphic. Thus, if they have the Baire property and either is meager or comeager, so is the other; and if they are measurable, they have equal measure.

Suppose X has the Baire property. If X is not meager then[9.166.1] it is comeager on a basic interval; hence, it is comeager on every basic interval of the same size, so it is comeager.

By a similar argument, if X is measurable and not null then[9.166.2] for any $p < 1$, $\mu X > p$, so $\mu X = 1$. □[9.190]

(9.191) **Theorem** [ZF] *A nonprincipal ultrafilter over ω, construed as a subset of ${}^{\omega}2$, does not have the Baire property and is not measurable. Therefore, if* AD *then every ultrafilter over ω is principal.*

Proof It is easy to see that a nonprincipal ultrafilter U, construed as a subset of ${}^{\omega}2$, is a tail set. The map $X \mapsto \omega\backslash X$ is a homeomorphism of U with its complement, which is also a tail set. They cannot both be meager, and they cannot both be comeager, so they cannot have the Baire property. Similarly, they cannot both have measure 0 and they cannot both have measure 1, so they cannot be measurable. □[9.191]

(9.192) **Theorem** [ZF + AD] *Every ultrafilter is countably complete.*

Proof Suppose U is an ultrafilter over some set M, and suppose toward a contradiction that $\langle X_n \mid n \in \omega\rangle$ is a sequence of subsets of M such that $\forall n \in \omega\ X_n \in U$ and $\bigcap_{n\in\omega} X_n \notin U$. Let $Y_0 = \bigcap_{n\in\omega} X_n$, and for each $n \in \omega$, let $Y_{n+1} = \left(\bigcap_{m<n} X_m\right)\backslash X_n$, where $\bigcap 0$ is understood in this case to be M, so $Y_1 = M\backslash X_0$. $\langle Y_n \mid n \in \omega\rangle$ is a partition of M into sets not in U.

Let $U' = \left\{A \subseteq \omega \,\middle|\, \left(\bigcup_{n\in A} Y_n\right) \in U\right\}$. U' is a nonprincipal ultrafilter over ω, the existence of which contradicts (9.191). □[9.192]

(9.193) **Theorem** [ZF + AD] *For all $x \in {}^{\omega}\omega$, $x^\sharp$ exists.*

Proof Working in GB for convenience, we note that for any $x \in {}^{\omega}\omega$, $\mathrm{HOD}(\{x\}) \models \ulcorner[\omega_1]$ is measurable$\urcorner$.[9.184] Hence, in $\mathrm{HOD}(\{x\})$ there exists $I \subseteq \omega_1$ of order type ω_1, such that I is a set of indiscernibles for $L[x]$. It follows that I is in fact an uncountable set of indiscernibles for $L[x]$, so $x^\sharp$ (the theory of $(L[x]; \in, i)_{i\in I'}$, where I' is any subset of I of order type ω) exists. □[9.193]

The following theorem of Solovay is an early example of the use of reals (i.e., subsets of ω) to code subsets of an uncountable cardinal, prefiguring the general coding lemma of Moschovakis (9.207).

(9.194) **Theorem (Solovay)** [ZF+AD] *Suppose $X \subseteq \omega_1$. Then $\{x \in \mathrm{WO} \mid \|x\| \in X\}$ is $\mathbf{\Pi}^1_1$.*

Proof We will use the recursive bijection $x \mapsto \langle x^n \mid n \in \omega \rangle$ of ${}^\omega\omega$ with ${}^\omega({}^\omega\omega)$, as described in the proof of (9.186). Consider the following game.

I and II respectively produce a and b in ${}^\omega\omega$ in the usual way. II wins iff

1. $a \notin \mathrm{WO}$; or
2. $a \in \mathrm{WO}$ and
   1. $\forall n \in \omega\ b^n \in \mathrm{WO}$, and
   2. $X \cap (\|a\| + 1) \subseteq \{\|b^n\| \mid n \in \omega\} \subseteq X$.

Thus, in order to win, I must code a countable ordinal, say α; and if I does this then, in order to win, II must code a countable subset of X including $X \cap (\alpha + 1)$.

We first observe that I cannot win this game. For suppose σ is a winning I-strategy. Let $S = \operatorname{im} \vec{\sigma}$. S is Σ^1_1 relative to σ, and $S \subseteq \mathrm{WO}$. Thus there exists $\alpha \in \omega_1$ such that $\forall x \in S\ \|x\| < \alpha$. Let $b \in {}^\omega\omega$ be such that $X \cap \alpha \subseteq \langle b^n \mid n \in \omega \rangle$. Obviously, σ does not win against b.

Hence, there exists a winning II-strategy τ. Suppose $x \in \mathrm{WO}$. If $\|x\| \in X$ then $\exists n \in \omega\ \|(\vec{\tau}x)^n\| = \|x\|$. Conversely, if $\exists n \in \omega\ \|(\vec{\tau}x)^n\| = \|x\|$ then (since $\|(\vec{\tau}x)^n\| \in X$ for all $n \in \omega$) $\|x\| \in X$. Thus, for all $x \in \mathrm{WO}$,

$$\|x\| \in X \leftrightarrow \exists n \in \omega\ \|(\vec{\tau}x)^n\| = \|x\|. \tag{9.195}$$

As we have seen in the discussion of the prewellordering property of Π^1_1, there is a Π^1_1 set $D \subseteq {}^\omega\omega \times {}^\omega\omega$ such that for any $y \in \mathrm{WO}$ and any $z \in {}^\omega\omega$

$$z \in \mathrm{WO} \wedge \|z\| \leqslant \|y\| \leftrightarrow \langle z, y \rangle \in D.$$

Since $x \in \mathrm{WO} \to \forall n \in \omega\ (\vec{\tau})^n \in \mathrm{WO}$, and WO is Π^1_1, it follows that X is $\Pi^1_1(\tau)$. $\square$ [9.194]

The following two of theorems are good examples of early work in the theory of AD.

(9.196) **Theorem (Solovay)** [ZF + AD] *Suppose $X \subseteq \omega_1$. Then X is constructible from a real, i.e., $X \in L[z]$ for some $z \in {}^\omega\omega$.*

Proof See Note 10.33. $\square$

(9.197) **Theorem (Solovay)** [ZF + DC + AD] *ω_2 is measurable.*

Proof See Note 10.34. $\square$

Actually, (9.196) follows from (9.194) by a general theorem of ZF.

(9.198) **Theorem** [ZF] *Suppose $X \subseteq \omega_1$ and $A = \{x \in {}^\omega\omega \mid x \in \mathrm{WO} \wedge \|x\| \in X\}$ is $\Sigma^1_2(z)$ for some $z \in {}^\omega\omega$. Then $X \in L[z]$.*

Proof The proof is an instructive elaboration on the forcing constructions used in the proofs of the preceding two theorems, but it is a bit of a digression and is relegated to Note 10.35. $\square$

9.7.5.1 The size of the continuum

As we have seen, a wellordering of the continuum may be used to define a game on ω that is not determined, so under the assumption of AD, ${}^{\omega}2$ cannot be wellordered. In fact, assuming AD, there does not exist any uncountable wellordered subset of ${}^{\omega}2$.

(9.199) **Theorem** [$\mathsf{ZF} + \mathsf{AD}$] *There does not exist an injection of ω_1 into ${}^{\omega}2$.*

Proof Suppose $f : \omega_1 \overset{\text{inj}}{\to} {}^{\omega}2$. By (9.148.1) since $\operatorname{im} f$ is uncountable, it has a perfect subset S. Any perfect set is equipollent with ${}^{\omega}2$, so let $g : S \overset{\text{bij}}{\to} {}^{\omega}2$. Then $g \circ f$ is a bijection of a subset of ω_1 with ${}^{\omega}2$, which gives a wellordering of ${}^{\omega}2$. As noted above, this is impossible. $\square$[9.199]

Thus, in the presence of AD, $\omega_1 \not\preccurlyeq 2^{\omega}$;[3.126.1] and for all $\kappa \in \mathrm{Ord}$ $2^{\omega} \not\preccurlyeq \kappa$. Injections are therefore of little value in assessing the size of 2^{ω}. Surjections, on the other hand, are quite useful. Of course, any surjection of an ordinal κ to ${}^{\omega}2$ leads to a bijection of a subset of κ with ${}^{\omega}2$, which cannot exist; so it is surjections $f : \mathbb{R} \to \alpha$ that interest us, where '$\mathbb{R}$' stands generically for ${}^{\omega}2$, ${}^{\omega}\omega$, etc.

Definition [ZF] $\Theta \overset{\text{def}}{=}$ *the least ordinal α such that there does not exist $\varphi : {}^{\omega}\omega \overset{\text{sur}}{\to} \alpha$.*

Note that the relevant functions here are those we have previously termed 'norms',[5.107] and we may define Θ as the supremum of the lengths of norms on subsets of ${}^{\omega}\omega$. We have seen that a complete Π^1_1 set has a natural norm of length ω_1, so $\Theta > \omega_1$. Clearly, Θ is a cardinal, so $\Theta \geqslant \omega_2$. In ZFC, of course, $\Theta = (2^{\omega})^+$.

Just as AD imposes limitations on the complexity of subsets of ${}^{\omega}2$ relative to subsets of ω (games on ω being essentially subsets of ${}^{\omega}2$, and strategies being essentially subsets of ω), it also imposes limitations on the complexity of subsets of ordinals below Θ relative to subsets of ω (of which (9.194) and (9.196) are early examples). In particular, it imposes limitations on the lengths of wellorderings of ordinals below Θ, and in this way it implies that there are many cardinals below Θ.

The following theorem of Moschovakis provides considerable information about Θ.

(9.200) **Theorem** [$\mathsf{ZF} + \mathsf{AD}$] *Suppose $\alpha \in \mathrm{Ord}$ and there exists $f : {}^{\omega}\omega \overset{\text{sur}}{\to} \alpha$. Then there exists $g : {}^{\omega}\omega \overset{\text{sur}}{\to} \mathcal{P}\alpha$.*

Proof We will use a fixed coding s of strategies for games on ω by reals. Thus, for every $x \in {}^{\omega}\omega$, sx is a I- or II-strategy, and every strategy is sx for some $x \in {}^{\omega}\omega$. We now define $\langle g_\beta \mid \beta \leqslant \alpha \rangle$ such that $\forall \beta \leqslant \alpha\ g_\beta : {}^{\omega}\omega \overset{\text{sur}}{\to} \mathcal{P}\beta$. For finite β we do this directly, say by letting $g_\beta x = \{\gamma < \beta \mid x_\gamma = 0\}$. For infinite $\beta \leqslant \alpha$ we proceed by recursion on β. Given $\beta \geqslant \omega$, let $h_\beta : \beta \overset{\text{bij}}{\to} \beta + 1$ be any definite bijection, say,

1. $h_\beta 0 = \beta$;
2. $h_\beta(n+1) = n$ for $n \in \omega$; and
3. $h_\beta \gamma = \gamma$ for $\omega \leqslant \gamma < \beta$.

Given g_β, let $g_{\beta+1}$ be defined by the condition that $g_{\beta+1}x = h^{\to}(g_\beta x)$.

To define g_β for limit β we proceed as follows. For the nonce, if $x, y \in {}^\omega\omega$ and $z = \langle x, y\rangle^{\mathsf{p}}$ then let

$$\zeta_z = fx$$
$$S_z = g_{\zeta_z} y.$$

If z is not $\langle x, y\rangle^{\mathsf{p}}$ for any $x, y \in {}^\omega\omega$ then let $\zeta_z = S_z = 0$. Given $X \subseteq \beta$, let $\mathcal{G}^X$ be game on ω in which the players I and II produce x and y, respectively, and II wins iff

$$S_x = X \cap \zeta_x \rightarrow (\zeta_y > \zeta_x \wedge S_y = X \cap \zeta_y).$$

Suppose σ is a winning I-strategy for $\mathcal{G}^X$. Then

$$\forall y \in {}^\omega\omega \; S_{\vec{\sigma} y} = X \cap \zeta_{\vec{\sigma} y}, \tag{9.201}$$

and X is the only subset of β with this property. For suppose $Y \subseteq \beta$ and $\delta \in X \triangle Y$. Let y be such that $\zeta_y = \delta + 1$ and $S_y = X \cap (\delta + 1)$. Let $x = \vec{\sigma} y$. Then $\zeta_x \geqslant \delta + 1$ and $S_x = X \cap \zeta_x \neq Y \cap \zeta_x$.

Now suppose τ is a winning II-strategy in $\mathcal{G}^X$. Then

$$\forall x \in {}^\omega\omega \left(S_x = X \cap \zeta_x \rightarrow (\zeta_{\vec{\tau} x} > \zeta_x \wedge S_{\vec{\tau} x} = X \cap \zeta_{\vec{\tau} x})\right), \tag{9.202}$$

and X is the only subset of β with this property. For suppose $Y \subseteq \beta$, and $Y \neq X$. Let δ be the least member of $X \triangle Y$, and let x be such that $\zeta_x = \delta$ and $S_x = X \cap \delta = Y \cap \delta$. Then $\zeta_{\vec{\tau} x} > \delta$ and $S_{\vec{\tau} x} = X \cap \zeta_{\vec{\tau} x} \neq Y \cap \zeta_{\vec{\tau} x}$.

We now define $g_\beta x$ for $x \in {}^\omega\omega$ as follows. Recall that at the outset we defined the surjection s from ${}^\omega\omega$ to the set of strategies for games on ω. Suppose first that sx is a I-strategy σ, there exists $X \subseteq \beta$ satisfying (9.201) with $\sigma = sx$, and σ is a winning I-strategy in $\mathcal{G}^X$. As we have shown, there is at most one $X \subseteq \beta$ with this property *vis-à-vis* σ. Let $g_\beta x = X$. Similarly, suppose sx is a II-strategy τ, there exists $X \subseteq \beta$ satisfying (9.202) with $\tau = sx$, and τ is a winning II-strategy in $\mathcal{G}^X$. Again, there is at most one $X \subseteq \beta$ with this property, and we let $g_\beta x = X$. If neither of these two scenarios is applicable, let $g_\beta x = 0$.

The preceding arguments have shown that for every $X \subseteq \beta$ one of the first two scenarios is applicable for some $x \in {}^\omega\omega$, so $g_\beta : {}^\omega\omega \overset{\text{sur}}{\to} \mathcal{P}\beta$, as specified.

This completes the definition of $\langle g_\beta \mid \beta \leqslant \alpha\rangle$. Let $g = g_\alpha$. $\square$[9.200]

(9.203) **Theorem (Friedman)** [$\mathsf{ZF} + \mathsf{AD}$] Θ *is a limit cardinal.*

Proof Suppose $\lambda < \Theta$ is a cardinal. We will show that $\lambda^+ < \Theta$. By (9.200) there exists $g : {}^\omega\omega \overset{\text{sur}}{\to} \mathcal{P}\lambda$. It is easy to map $\mathcal{P}\lambda$ onto λ^+, using first a pairing function from λ to $\lambda \times \lambda$ to map $\mathcal{P}\lambda$ onto the set of wellorderings of λ, and then applying the function taking a wellordering to its order type. Thus, we obtain $h : {}^\omega\omega \overset{\text{sur}}{\to} \lambda^+$. $\square$[9.203]

(9.204) **Theorem (Solovay)** [$\mathsf{ZF} + \mathsf{AD}$] $\Theta = \omega_\Theta$.

Proof The arguments for (9.200) and (9.203) show that there exists a function F such that for every cardinal $\lambda < \Theta$ and $f : {}^\omega\omega \overset{\text{sur}}{\to} \lambda$, $F(f) : {}^\omega\omega \overset{\text{sur}}{\to} \lambda^+$. Suppose $\alpha < \Theta$. Let $f : {}^\omega\omega \overset{\text{sur}}{\to} \alpha$. It is a simple matter, using F, to define from f a sequence $\langle g_\beta \mid \beta \leqslant \alpha\rangle$ such that for each $\beta < \alpha$, $g_\beta : {}^\omega\omega \overset{\text{sur}}{\to} \omega_\beta$. It follows that $\omega_\alpha < \Theta$.

It follows that $\Theta = \omega_\Theta$. $\square$[9.204]

It follows from (9.204) that Θ is pretty big: If we let $\kappa_0 = \omega$ and $\kappa_{n+1} = \omega_{\kappa_n}$, then $\Theta > \kappa_n$ for all $n \in \omega$.

It is easy to show that

$$\mathsf{ZF} + \mathsf{AC}_\omega(\mathcal{P}\,\mathbb{R}) \vdash \operatorname{cf}\Theta > \omega, \tag{9.205}$$ [55]

but this does not follow from $\mathsf{ZF} + \mathsf{AD}$, or even from $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$. In fact, Solovay showed that

$$\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \operatorname{cf}\Theta > \omega \vdash \operatorname{Con}(\mathsf{ZF} + \mathsf{AD}_\mathbb{R}).$$

Thus, it follows from Gödel's second incompleteness theorem that $\operatorname{Con}(\mathsf{ZF}+\mathsf{AD}_\mathbb{R}) \to \operatorname{Con}(\mathsf{ZF}+\mathsf{AD}_\mathbb{R} + \operatorname{cf}\Theta = \omega)$. Hence also[9.205] $\operatorname{Con}(\mathsf{ZF} + \mathsf{AD}_\mathbb{R}) \to \operatorname{Con}(\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \neg\,\mathsf{AC}_\omega(\mathcal{P}\,\mathbb{R}))$.

Things are a good deal more regular in the model $L(\mathbb{R})$.

(9.206) **Theorem**

1. $[\mathsf{ZF} + \ulcorner V = L(\mathbb{R})\urcorner]$ Θ *is regular.*
2. $[\mathsf{ZF} + \mathsf{AD} + \ulcorner V = L(\mathbb{R})\urcorner]$ Θ *is weakly inaccessible.*[56]

Proof 1 As in the proof of (9.181) we define $\Phi : \mathrm{Ord} \times {}^\omega\omega \overset{\text{sur}}{\to} V$. Suppose $0 < \alpha < \Theta$. Define $g_\alpha : {}^\omega\omega \overset{\text{sur}}{\to} \alpha$ as follows. Let β be the least ordinal such that there exists $x \in {}^\omega\omega$ such that $\Phi\langle\beta, x\rangle : {}^\omega\omega \overset{\text{sur}}{\to} \alpha$. Given $z \in {}^\omega\omega$, if $z = \langle x, y\rangle^{\mathsf{p}}$ for some $x, y \in {}^\omega\omega$ and $\Phi\langle\beta, x\rangle : {}^\omega\omega \overset{\text{sur}}{\to} \alpha$, let $g_\alpha z = (\Phi\langle\beta, x\rangle)y$. Otherwise, let $g_\alpha z = 0$. Clearly, $g_\alpha : {}^\omega\omega \overset{\text{sur}}{\to} \alpha$.

Now it is easy to use $\langle g_\alpha \mid 0 < \alpha < \Theta\rangle$ to derive a contradiction from the supposition that there exists $f : \alpha \to \Theta$ such that $\alpha < \Theta$ and $\operatorname{im} f$ is cofinal in Θ. Hence Θ is regular.

2 This follows directly from (9.203) and (9.206.1). $\square$[9.206]

Recall that AD implies $\mathsf{AD}^{L(\mathbb{R})}$, so $\mathsf{ZF}+\mathsf{AD} \vdash \ulcorner\Theta$ is weakly inaccessible$\urcorner^{L(\mathbb{R})}$.[9.206.2]

9.7.5.2 The coding lemma

The remarkable coding lemma of Moschovakis improves on the already remarkable (9.200) by relating the complexity of codes for subsets of an ordinal $\lambda < \Theta$ to the complexity of a wellfounded relation on a subset of ${}^\omega\omega$ with rank λ. The use of the recursion theorem is striking.

(9.207) **Theorem: Moschovakis's coding lemma** $[\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}]$ *Suppose* Γ *is a nonselfdual Wadge class including* $\boldsymbol{\Sigma}^1_1$ *and closed under* $\vee$, $\wedge$, $\exists^0$, $\forall^0$, *and* $\exists^1$.[57] *Suppose* $\lambda \in \mathrm{Ord}$, *and* $\prec\, \in \Gamma$ *is an irreflexive wellfounded relation with rank* λ *on a set* $D \subseteq {}^\omega\omega$. *(Note that* $D \in \Gamma$.*) Suppose* $R \subseteq D \times {}^\omega\omega$ *is such that* $\forall x \in D\, \exists y\, \langle x, y\rangle \in R$. *Then there exists* $A \in \Gamma$ *such that* $A \subseteq R$ *and*

$$\forall \alpha < \lambda\, \exists x \in D\, \exists y \in {}^\omega\omega\, (\operatorname{rk}^{\prec} x = \alpha \wedge \langle x, y\rangle \in A).$$

[55] Remember that $\mathsf{AC}_\omega(\mathbb{R})$ follows from AD. $\mathsf{AC}_\omega(\mathcal{P}\,\mathbb{R})$ is of course a special case of AC_ω, which is a special case of DC, which is an axiom frequently adopted in the setting of AD.

[56] Inaccessibility *per se*—i.e., strong inaccessibility—is of course out of the question.

[57] In the language of descriptive set theory, closure under the logical operations $\vee$ and $\wedge$ is equivalent to closure under (finite) union and intersection, respectively; and—given the existence of universal sets for nonselfdual Wadge classes, together with $\mathsf{AC}_\omega(\mathbb{R})$, both of which follow from AD—closure under $\exists^0$ and $\forall^0$ is equivalent to closure under countable union and intersection, respectively. Closure under $\exists^1$ is the same as closure under projection (along a type-1 coordinate axis, such as ${}^\omega\omega$).

Proof Given $\prec$ and R as above, we will use the following definition for $A \subseteq {}^\omega\omega \times {}^\omega\omega$ and $\alpha \in \mathrm{Ord}$.

1. A is a *choice set below* $\alpha \overset{\text{def}}{\iff} A \subseteq R$ and $\forall \beta < \alpha\, \exists x \in D\, \exists y \in {}^\omega\omega\, (\mathrm{rk}^\prec x = \beta \wedge \langle x, y\rangle \in A)$. (Note that A is a choice set below 0 iff $A \subseteq R$.)

2. A is a *choice set at* $\alpha \overset{\text{def}}{\iff} A$ is a choice set below $\alpha + 1$.

Note that a choice set below α is a choice set below any $\beta < \alpha$, and a union of choice sets below α is a choice set below α.

Suppose toward a contradiction that the theorem fails for Γ, and let λ be the least ordinal for which it fails. It is easy to see that λ is a limit ordinal. Let $\prec$ and R be such that $\mathrm{rk}\prec = \lambda$ and there does not exist a choice set in Γ below λ (*vis-à-vis* $\prec$ and R). By virtue of the minimality of λ, for every $\alpha < \lambda$ there exists a choice set in Γ below α.

Let $U \subseteq {}^\omega\omega \times {}^\omega\omega$ be a good universal set for Γ,[9.172] and let $s : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ be recursive such that for all $a, b, c \in {}^\omega\omega$,

$$\langle a, \langle b, c\rangle^{\cdot}\rangle \in U \leftrightarrow \langle s\langle a, b\rangle, c\rangle \in U.$$

As usual, for any $A \subseteq {}^\omega\omega \times {}^\omega\omega$ and $a \in {}^\omega\omega$, let $A_a = \{b \mid \langle a, b\rangle \in A\}$. Thus, for all $a, b, c \in {}^\omega\omega$

$$\langle b, c\rangle^{\cdot} \in U_a \leftrightarrow c \in U_{s\langle a, b\rangle}. \tag{9.208}$$

We will say that $a \in {}^\omega\omega$ is *good below* or *at* $\alpha \overset{\text{def}}{\iff} \{\langle x, y\rangle \mid \langle x, y\rangle^{\cdot} \in U_a\}$ is a choice set below or at α, respectively. By hypothesis, no $a \in {}^\omega\omega$ is good below λ.

Let $\mathcal{G}$ be the following game:

> I and II play a and b in ${}^\omega\omega$, respectively. II wins iff for every $\alpha < \lambda$, if a is good below α then b is good at α. In other words, if a is good then b is better. Note that I wins iff a is good below 0 and for every $\alpha < \lambda$, if b is good below α then a is good below α.

Suppose σ is a winning I-strategy. Then for any $b \in {}^\omega\omega$, $\vec{\sigma} b$ is good below some α, and for every $\alpha < \lambda$, if b is good below α then $\vec{\sigma} b$ is good below α. By hypothesis, for every $\alpha < \lambda$ there exists $b \in {}^\omega\omega$ such that b is good below α. Let $C = \{\vec{\sigma} b \mid b \in {}^\omega\omega\}$. $C \in \boldsymbol{\Sigma}^1_1 \subseteq \Gamma$. Let $A = \{\langle x, y\rangle \mid \exists a \in {}^\omega\omega\, (a \in C \wedge \langle x, y\rangle^{\cdot} \in U_a)\}$. Then $A \in \Gamma$ and A is a choice set below λ; contradiction.

By AD, there is a II-winning strategy τ. Let $B \subseteq {}^\omega\omega \times {}^\omega\omega$ be such that for all $a \in {}^\omega\omega$

$$B_a = \{\langle x, z\rangle^{\cdot} \mid x \in D \wedge z \in {}^\omega\omega \wedge \exists x' \prec x\ z \in U_{\vec{\tau}(s\langle a, x'\rangle)}\}.$$

Then $B \in \Gamma$. By the recursion theorem,[9.171] let a_0 be such that $B_{a_0} = U_{a_0}$. Then[9.208] for any $x, z \in {}^\omega\omega$

$$\begin{aligned} z \in U_{s\langle a_0, x\rangle} &\leftrightarrow \langle x, z\rangle^{\cdot} \in U_{a_0} \\ &\leftrightarrow x \in D \wedge \exists x' \prec x\ z \in U_{\vec{\tau}(s\langle a_0, x'\rangle)}. \end{aligned}$$

For $x \in {}^\omega\omega$, let $gx = s\langle a_0, x\rangle$. Then for all $x \in D$

$$U_{gx} = \bigcup_{x' \prec x} U_{\vec{\tau}(gx')}.$$

Note that if x is $\prec$-minimal then $U_{gx} = 0$, so gx is good below $0 = \mathrm{rk}^{\prec} x$. In general, suppose $x \in D$ and for all $x' \prec x$, gx' is good below $\mathrm{rk}^{\prec} x'$. Then for all $x' \prec x$, $\vec{\tau}(gx')$ is good at $\mathrm{rk}^{\prec} x'$, so gx is good below $\sup_{x' \prec x} \mathrm{rk}^{\prec} x' = \mathrm{rk}^{\prec} x$. It follows by $\prec$-induction that for every $x \in D$, U_{gx} is good below $\mathrm{rk}^{\prec} x$. Let $A = \{\langle x', y'\rangle \mid \exists x \in D\ \langle x', y'\rangle^{\cdot} \in U_{gx}\}$. Then A is a choice set below λ; contradiction.

Having obtained a contradiction from the existence of a winning strategy for either player in $\mathcal{G}$, we conclude that the the theorem holds for Γ. □[9.207]

In the typical application of the coding lemma $R \subseteq D \times {}^{\omega}\omega$ is derived from a function $f : \lambda \to \mathcal{P}({}^{\omega}\omega)$, with $\langle x, y\rangle \in R \leftrightarrow y \in f(\mathrm{rk}^{\prec} x)$. In particular, f may be two-valued, acting as a characteristic function for a subset of λ. For example, we might specify that $f\alpha$ be either $\bar{0} = \langle 0, 0, \ldots, \rangle$ or $\bar{1} = \langle 1, 1, \ldots \rangle$, so that f represents the set $\{\alpha < \lambda \mid f\alpha = \bar{1}\}$. More generally, we may apply this to functions from products of ordinals.

Similarly, we may apply the coding lemma to obtain sets of reals corresponding to sets of ordinals according to the following scheme.

Definition [ZF] *Suppose $n \in \omega$ and for each $m \in n$, $\preccurlyeq_m$ is a prewellordering of a set D_m (typically ${}^{\omega}\omega$ in our applications). Let $\lambda_m = \mathrm{rk} \preccurlyeq_m$. Then for any $A \subseteq \lambda_0 \times \cdots \times \lambda_{n^-}$,* $\mathrm{Code}(A; \preccurlyeq_0, \ldots, \preccurlyeq_{n^-}) \overset{\text{def}}{=}$

$$\big\{\langle x_0, \ldots, x_{n^-}\rangle \in D_0 \times \cdots \times D_{n^-} \,\big|\, \langle \mathrm{rk}^{\preccurlyeq_0} x_0, \ldots, \mathrm{rk}^{\preccurlyeq_{n^-}} x_{n^-}\rangle \in A\big\}.$$

(9.209) **Theorem** [ZF + AD] *Suppose Γ is a nonselfdual Wadge class including $\mathbf{\Sigma}^1_1$ and closed under $\vee$, $\wedge$, $\exists^0$, $\forall^0$, and $\exists^1$. Let $\Delta = \Gamma \cap \breve{\Gamma}$.*

1. *Suppose $\preccurlyeq \in \Delta$ is a prewellordering of ${}^{\omega}\omega$. Let $\lambda = \mathrm{rk} \preccurlyeq$. Then for any $A \subseteq \lambda$,* $\mathrm{Code}(A; \preccurlyeq) \in \Delta$.
2. *More generally, suppose $n \in \omega$ and for each $m \in n$, $\preccurlyeq_m \in \Delta$ is a prewellordering of ${}^{\omega}\omega$. For each $m \in n$ let $\lambda_m = \mathrm{rk} \preccurlyeq_m$. Then for any $A \subseteq \lambda_0 \times \cdots \times \lambda_{n^-}$,* $\mathrm{Code}(A; \preccurlyeq_0, \ldots, \preccurlyeq_{n^-}) \in \Delta$.

Proof 1 Note that $\prec$ and $\equiv$ are in Δ, where $x \equiv y \leftrightarrow x \preccurlyeq y \wedge y \preccurlyeq x$. Suppose $A \subseteq \lambda$. Let

$$R = \{\langle x, \bar{1}\rangle \mid \mathrm{rk}^{\prec} x \in A\} \cup \{\langle x, \bar{0}\rangle \mid x \in D \wedge \mathrm{rk}^{\prec} x \notin A\}$$

Let $B \in \Gamma$ be a choice set for R. Then for any $x \in {}^{\omega}\omega$,

$$\begin{aligned} x \in \mathrm{Code}(A; \preccurlyeq) &\leftrightarrow \exists x'\, (x' \equiv x \wedge \langle x', \bar{1}\rangle \in R) \\ &\leftrightarrow \neg\, \exists x'\, (x' \equiv x \wedge \langle x', \bar{0}\rangle \in R). \end{aligned}$$

Thus, $\mathrm{Code}(A; \preccurlyeq)$ is in both Γ and $\breve{\Gamma}$, so it is in Δ.

2 Let $\prec'$ be the (strict) lexicographic ordering of ${}^{n}({}^{\omega}\omega)$ defined from $\preccurlyeq_0, \ldots, \preccurlyeq_{n^-}$, i.e.,

$$\begin{aligned} &\langle x_0, \ldots, x_{n^-}\rangle \prec' \langle x'_0, \ldots, x'_{n^-}\rangle \\ &\qquad \leftrightarrow \quad x_0 \prec_0 x'_0 \\ &\qquad\qquad \vee\, x_0 \equiv_0 x'_0 \wedge x_1 \prec_1 x'_1 \\ &\qquad\qquad \vee \cdots \\ &\qquad\qquad \vee\, x_0 \equiv_0 x'_0 \wedge \cdots \wedge x_{n-2} \equiv_{n-2} x'_{n-2} \wedge x_{n^-} \prec_{n^-} x'_{n^-}. \end{aligned}$$

$\prec'$ is in Δ and is an irreflexive wellfounded relation (a prewellordering, in fact) Let

$$R = \{\langle\langle x_0, \dots, x_{n^-}\rangle, y\rangle \mid$$

By means of a recursive bijection of ${}^n({}^\omega\omega)$ with ${}^\omega\omega$, such as $\langle x_0, \dots, x_{n^-}\rangle \mapsto \langle x_0, \dots, x_{n^-}\rangle^{\mathsf{p}}$,[9.170.1.2] we may reduce this to the previous case. $\square$[9.209]

(9.210) **Theorem** [ZF + AD] *Suppose Γ is a nonselfdual Wadge class including $\mathbf{\Sigma}^1_1$ and closed under $\vee$, $\wedge$, $\exists^0$, $\forall^0$, and $\exists^1$. Let $\Delta = \Gamma \cap \breve{\Gamma}$. Let*

$$\boldsymbol{\delta} = \sup\{\text{rk}\preccurlyeq \mid \preccurlyeq \in \Delta \textit{ is a reflexive prewellordering of } {}^\omega\omega\}$$

$$\boldsymbol{\sigma} = \sup\{\text{rk}\prec \mid \prec \in \Gamma \textit{ is an irreflexive wellfounded relation on some } D \subseteq {}^\omega\omega\}.$$

1. *$\boldsymbol{\delta}$ is a cardinal.*
2. *$\boldsymbol{\sigma}$ is a regular cardinal.*

Proof 1 It is easy to see that there is no longest Δ prewellordering of ${}^\omega\omega$, so $\boldsymbol{\delta}$ is a limit ordinal, and there is no Δ prewellordering of ${}^\omega\omega$ of length $\boldsymbol{\delta}$.

Suppose toward a contradiction that $\lambda < \boldsymbol{\delta}$ and $f : \lambda \overset{\text{bij}}{\to} \boldsymbol{\delta}$. Let $\preccurlyeq \in \Delta$ be a prewellordering of ${}^\omega\omega$ of length λ. Let $A = \{\langle\alpha, \beta\rangle \in \lambda \times \lambda \mid f\alpha \preccurlyeq f\beta\}$. A is a reflexive wellordering of length $\boldsymbol{\delta}$. Let $C = \text{Code}(A; \preccurlyeq, \preccurlyeq)$. Then $C \in \Delta$,[9.209.2] and C is a reflexive prewellordering of ${}^\omega\omega$ of length $\boldsymbol{\delta}$.

2 Since $\exists^1\Gamma = \Gamma$, if $\prec \in \Gamma$ is a relation on $D \subseteq {}^\omega\omega$, then $D \in \Gamma$, and if $\prec$ is wellfounded, then there is a wellfounded $\prec' \in \Gamma$ with greater rank than $\prec$. Hence, $\boldsymbol{\sigma}$ is a limit ordinal and there does not exist an irreflexive wellfounded relation $\prec \in \Gamma$ on any $D \subseteq {}^\omega\omega$ such that $\text{rk}\prec = \boldsymbol{\sigma}$.

Suppose toward a contradiction that $\lambda < \boldsymbol{\sigma}$ and $f : \lambda \to \boldsymbol{\sigma}$ is cofinal in $\boldsymbol{\sigma}$. Let $\prec \in \Gamma$ be a prewellordering of $D \subseteq {}^\omega\omega$ of length λ. Let U be universal for Γ, and for the purpose of this argument, let $U_a = \{\langle x, y\rangle \mid \langle a, \langle x, y\rangle^{\mathsf{p}}\rangle \in U\}$. Let $R \subseteq D \times {}^\omega\omega$ be the set of $\langle x, a\rangle$ such that U_a is an irreflexive wellfounded relation of rank $f(\text{rk}^{\prec} x)$. Let $A \in \Gamma$ be a choice set for R,[9.207] and let $\prec'$ consist of all pairs $\big\langle\langle x, a, y\rangle^{\mathsf{p}}, \langle x', a', y'\rangle^{\mathsf{p}}\big\rangle$ such that $\langle x, a\rangle = \langle x', a'\rangle \in A$ and

$$\langle y, y'\rangle \in U_a.$$

Then $\prec'$ is an irreflexive wellfounded relation in Γ with rank $\boldsymbol{\sigma}$. $\square$[9.210]

AD provides much additional information about the cardinals associated with pointclasses as in (9.210). We begin with a brief survey of the cardinals associated with the projective hierarchy, omitting proofs.

Definition [ZF] *We apply the naming convention of (9.210) to the projective pointclasses as follows.*

$$\boldsymbol{\delta}^1_n \overset{\text{def}}{=} \sup\{\text{rk}\preccurlyeq \mid \preccurlyeq \in \mathbf{\Delta}^1_n \textit{ is a reflexive prewellordering of } {}^\omega\omega\}$$

$$\boldsymbol{\sigma}^1_n \overset{\text{def}}{=} \sup\{\text{rk}\prec \mid \prec \in \mathbf{\Sigma}^1_n \textit{ is an irreflexive wellfounded relation on some } D \subseteq {}^\omega\omega\}.$$

Several people contributed importantly to the following theorem, including Martin, Kunen, Moschovakis, Kechris, and Jackson.

Theorem [$\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$]

1. *For each* $n > 0$, $\boldsymbol{\delta}^1_n = \boldsymbol{\sigma}^1_n$, *and* $\boldsymbol{\delta}^1_n$ *is a regular cardinal.*
2. $\omega_1 = \boldsymbol{\delta}^1_1 < \boldsymbol{\delta}^1_2 < \cdots$.
3. *For each odd* n, $\boldsymbol{\delta}^1_{n+1} = (\boldsymbol{\delta}^1_n)^+$,
4. *For each odd* n, $\boldsymbol{\delta}^1_n = \omega_{\omega(n-2)+1}$, *where* $\omega(0) = 1$ *and* $\forall n \in \omega\ \omega(n+1) = \omega^{\omega(n)}$. *Thus,* $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, $\boldsymbol{\delta}^1_5 = \omega_{\omega^{\omega^\omega}+1}$, $\ldots$.

As we have previously noted, absent AC, successor cardinals may be singular, and assuming AD, this is demonstrably so for many successor cardinals. In particular, $\operatorname{cf}\omega_n = \omega_2$ for every $2 < n < \omega$ (Martin), and for every $n \in \omega$ there are exactly $2^{n+1} - 1$ regular cardinals less than $\boldsymbol{\delta}^1_{2n+1}$ (Jackson). The first nine regular cardinals (assuming $\mathsf{AD} + \mathsf{DC}$) are

$$\begin{aligned}
&\omega\\
\omega_1 &= \boldsymbol{\delta}^1_1\\
\omega_2 &= \boldsymbol{\delta}^1_2\\
\omega_{\omega+1} &= \boldsymbol{\delta}^1_3\\
\omega_{\omega+2} &= \boldsymbol{\delta}^1_4\\
&\omega_{\omega\cdot 2+1}\\
&\omega_{\omega^\omega+1}\\
\omega_{\omega^{\omega^\omega}+1} &= \boldsymbol{\delta}^1_5\\
\omega_{\omega^{\omega^\omega}+2} &= \boldsymbol{\delta}^1_6.
\end{aligned}$$

AD also has large cardinal implications for cardinals below Θ. We have already mentioned Solovay's proofs that ω_1 and ω_2 are measurable. Jackson showed (assuming $\mathsf{AD} + \mathsf{DC}$) that every uncountable regular cardinal less than $\sup_{n\in\omega} \boldsymbol{\delta}^1_n$ is measurable.

Theorem (Steel, Woodin) [$\mathsf{ZF} + \mathsf{AD} + \ulcorner V = L(\mathbb{R})\urcorner$] *Every uncountable regular cardinal below* Θ *is measurable.*

Much more than measurability is achievable:

(9.211) **Theorem** [$\mathsf{ZF} + \mathsf{AD} + \ulcorner V = L(\mathbb{R})\urcorner$] $\ulcorner[\Theta]$ *is Woodin*$\urcorner^{\mathrm{HOD}}$.

The inner model construction of (9.211) may also be used in an ultrapower construction to obtain an inner model with infinitely many Woodin cardinals.

9.7.6 Determinacy from large cardinals

Martin's proof of $\boldsymbol{\Pi}^1_1$-determinacy from the existence of a measurable cardinal was the first use of a large cardinal hypothesis to prove determinacy. We will present the proof in the context of homogeneous systems of ultrafilters, in which format it is a paradigm of proofs of determinacy from large cardinals.

In this final section of the book we deviate somewhat from our pedagogical principles in that we present a rather elaborate machinery of which we make rather little use in the form of theorems actually proved. The development of homogeneity systems may be justified as an abstraction of the essence of Martin's proof of

$\mathbf{\Pi}^1_1$-determinacy, which does not unduly complicate the presentation of that argument. The discussion of weak homogeneity is not similarly justified by application. Although we do indicate rather strongly how it figures in Martin's proof[9.235] that $\mathbf{\Sigma}^1_3$ sets are ω_2-Suslin, our principal purpose in its presentation is to permit a comprehensible statement of the Martin-Steel theorem[9.236] on the propagation of homogeneity through the projective hierarchy, with projective determinacy as an immediate consequence, which was a stunning breakthrough in the derivation of determinacy from large cardinal hypotheses.

9.7.6.1 Homogeneity systems

We use the terminology of ultrafilters and measures interchangeably. Thus, a measure over a set Z is in this discussion a 2-valued measure on $\mathcal{P}Z$, and it is the characteristic function (relative to $\mathcal{P}Z$) of the corresponding ultrafilter over Z. Additivity of measures is the same as completeness of ultrafilters.

Definition [ZF]

1. *Suppose κ is a cardinal.* $\mathrm{ms}_\kappa Z \overset{\text{def}}{=}$ *the set of κ-complete ultrafilters (κ-additive measures) over ${}^{<\omega}Z$. Note that $\kappa \geqslant \lambda \to \mathrm{ms}_\kappa Z \subseteq \mathrm{ms}_\lambda Z$.*
2. $\mathrm{ms}\, Z \overset{\text{def}}{=} \mathrm{ms}_{\omega_1} Z$. *We will be exclusively interested in the case that $\kappa > \omega$, in which case $\mathrm{ms}_\kappa Z \subseteq \mathrm{ms}\, Z$.*
3. *Suppose $U \in \mathrm{ms}\, Z$. The* dimension of $U \overset{\text{def}}{=} \dim U \overset{\text{def}}{=}$ *the (unique) $n \in \omega$ such that ${}^nZ \in U$.*

(9.212) **Definition** [ZF]

1. *Suppose $U, U' \in \mathrm{ms}\, Z$ and $n = \dim U \geqslant \dim U' = m$. Then U* projects to U' $\overset{\text{def}}{\iff}$ *for all $A \subseteq {}^mZ$, $A \in U' \leftrightarrow \{s \in {}^nZ \mid s \restriction m \in A\} \in U$. (Note that since U' is an ultrafilter and U is a filter, the forward implication implies the reverse.)*
2. *A* tower of ultrafilters over ${}^{<\omega}Z \overset{\text{def}}{=}$ *a sequence $\langle U_n \mid n < k\rangle$, where $k \leqslant \omega$, such that*
   1. *for each $n < k$, $U_n \in \mathrm{ms}\, Z$ and $\dim U_n = n$; and*
   2. *for each $m \leqslant n < k$, U_n projects to U_m.*
3. *An infinite tower $\langle U_n \mid n < \omega\rangle$ of ultrafilters over ${}^{<\omega}Z$ is* countably complete $\overset{\text{def}}{\iff}$ *for every $\langle Z_n \mid n \in \omega\rangle$ such that $\forall n \in \omega\ Z_n \in U_n$, there exists $z \in {}^\omega Z$ such that $\forall n \in \omega\ z \restriction n \in Z_n$.*

(9.213) **Definition** [ZF]

1. *$\bar{U}$ is a* homogeneity system over X with support Z $\overset{\text{def}}{\iff}$
   1. $\bar{U} : {}^{<\omega}X \to \mathrm{ms}\, Z$;
   2. $\forall s \in {}^{<\omega}X\ \ \dim \bar{U}_s = |s|$; *and*
   3. $\forall s, t \in {}^{<\omega}X\ (s \subseteq t \to \bar{U}_t$ *projects to* $\bar{U}_s)$.
2. *A homogeneity system is κ-*complete $\overset{\text{def}}{\iff}$ *all of its ultrafilters are κ-complete.*
3. *Suppose $\bar{U}$ is a homogeneity system over X with support Z.*

1. *Suppose* $x \in {}^{\omega}X$. *Then* $\bar{U}^x \overset{\text{def}}{=} \langle \bar{U}_{x \restriction n} \mid n \in \omega \rangle$, *which is a tower of ultrafilters over* ${}^{<\omega}Z$.
2. $S_{\bar{U}} \overset{\text{def}}{=} \{x \in {}^{\omega}X \mid \bar{U}^x$ *is countably complete*$\}$.

4. *Suppose* $A \subseteq {}^{\omega}X$. A *is* (κ-)homogeneous (with support Z) $\overset{\text{def}}{\iff}$ *for some* (κ-*complete*) *homogeneity system* $\bar{U}$ *over* X (*with support* Z), $A = S_{\bar{U}}$.

The above notion of a homogeneous subset of ${}^{\omega}X$ is similar to that of a Suslin set. We may say that $A \subseteq {}^{\omega}X$ is Z-Suslin[5.120] iff there is a sequence tree T on $X \times Z$ such that $A = \mathfrak{p}^{\cdot}[T] = \{x \in {}^{\omega}X \mid \exists z \in {}^{\omega}Z \; \langle x, z\rangle^{\cdot} \in [T]\}$. (Recall the definitions[5.58] of $\mathfrak{p}^{\cdot}$ and of T_s, $T_{[s]}$, and $T_{[x]}$.) If instead of a homogeneity system we use the function $\bar{U} : {}^{<\omega}X \to \mathcal{P}({}^{<\omega}Z)$ defined by the condition that $\bar{U}_s = \{T_s\}$, then for any $x \in {}^{\omega}X$, $\bar{U}^x = \langle \{T_{x \restriction n}\} \mid n \in \omega\rangle$, which is countably complete just in case there exists $z \in {}^{\omega}Z$ such that $\forall n \in \omega \; z \restriction n \in T_{x \restriction n}$, i.e., $\langle x, z\rangle^{\cdot} \in [T]$, so $S_{\bar{U}} = \mathfrak{p}^{\cdot}[T]$. Thus, '$S$' here stands for 'Suslin'.

The relationship of homogeneity *per se* to Suslin properties is illuminated by the following theorem of Woodin.

(9.214) **Theorem** [ZFC] *Suppose* $\bar{U}$ *is a* $|X|^+$-*complete homogeneity system over* X *with support* Z. *Then there is a tree* T *on* $X \times Z$ *such that*

1. $\forall s \in {}^{<\omega}X \; T_s \in \bar{U}_s$; *and*
2. $S_{\bar{U}} = \mathfrak{p}^{\cdot}[T]$.

Proof For each $x \in {}^{\omega}X \backslash S_{\bar{U}}$ let $\langle Z^x_n \mid n \in \omega\rangle$ witness that $\bar{U}^x$ is not countably complete. Thus, $\forall n \in \omega \; Z^x_n \in \bar{U}_{x \restriction n}$, but $\neg \exists z \in {}^{\omega}Z \, \forall n \in \omega \; z \restriction n \in Z^x_n$. We may suppose that

$$\forall n < \omega \, \forall u \in Z^x_n \, \forall m < n \; u \restriction m \in Z^x_m; \tag{9.215}$$

otherwise, replace Z^x_n by $\{u \in Z^x_n \mid \forall m < n \; u \restriction m \in Z^x_m\}$, which can be done by virtue of the fact that $\bar{U}_{x \restriction n}$ projects to $\bar{U}_{x \restriction m}$.

Let T be the tree on $X \times Z$ such that for every $n \in \omega$ and $s \in {}^{n}X$,

$$T_s = {}^{n}Z \cap \bigcap \{Z^x_n \mid x \in {}^{\omega}X \backslash S_{\bar{U}} \wedge s \subseteq x\}.$$

T is a tree by virtue of (9.215). $\bar{U}$ is assumed to be $|X|^+$-complete, and it is by definition[9.213.1.1] ω_1-complete. The completeness of any nonprincipal ω_1-complete ultrafilter is measurable,[9.15.1] and any principal ultrafilter is λ-complete for all λ, so $\bar{U}$ is $|{}^{\omega}X|^+$-complete. Hence, $T_s \in \bar{U}_s$ for each $s \in {}^{<\omega}X$, i.e., (9.214.1) holds.

Suppose $x \in {}^{\omega}X \backslash S_{\bar{U}}$, and suppose $z \in {}^{\omega}Z$. Then there exists $n \in \omega$ such that $z \restriction n \notin Z^x_n$, so $z \restriction n \notin T_{x \restriction n}$; hence, $z \notin T_{[x]}$. Thus, if $x \notin S_{\bar{U}}$ then $x \notin \mathfrak{p}^{\cdot}[T]$. Inversely, suppose $x \in S_{\bar{U}}$. Then $\bar{U}^x$ is countably complete, so[9.214.1] there exists $z \in {}^{\omega}Z$ such that $\forall n \in \omega \; z \restriction n \in T_{x \restriction n}$; hence, $z \in T_{[x]}$, so $x \in \mathfrak{p}^{\cdot}[T]$. □[9.214]

Definition [ZFC]

1. *Suppose* T *is a sequence tree on* $X \times Z$. T *is* κ-homogeneous $\overset{\text{def}}{\iff}$ *there is a* κ-*complete homogeneity system* $\bar{U}$ *on* X *with support* Z *such that*
    1. $\forall s \in {}^{<\omega}X \; T_s \in \bar{U}_s$; *and*
    2. $\mathfrak{p}^{\cdot}[T] = S_{\bar{U}}$.

$\bar{U}$ *is a* homogeneity system for T *in this case.*

2. *Suppose* $A \subseteq {}^{\omega}X$. *A is* κ-homogeneously (Z-)Suslin $\overset{\text{def}}{\iff} A = \mathfrak{p}^{\cdot}[T]$ *for some* κ*-homogeneous tree* T *(on* $X \times Z$*).*

(9.216) *Clearly, if* $A \subseteq {}^{\omega}X$ *is* κ*-homogeneously* Z*-Suslin then* A *is* κ*-homogeneous with support* Z*, and by virtue of (9.214), if* A *is* κ*-homogeneous with support* Z *and* $\kappa > |X|$ *then* A *is* κ*-homogeneously* Z*-Suslin.*

Before continuing with the general theory, we illustrate the notion of homogeneity with the urexample of $\mathbf{\Pi}^1_1$.

(9.217) **Theorem** [ZFC] *Suppose* $A \subseteq {}^{\omega}\omega$ *is* $\mathbf{\Pi}^1_1$*, and suppose* κ *is a measurable cardinal. Then* A *is* κ*-homogeneously Suslin.*

Proof Let[5.60] $T \subseteq {}^{<\omega}(\omega \times \omega)$ be a sequence tree such that $A = \neg\,\mathfrak{p}^{\cdot}[T] = \{x \in {}^{\omega}\omega \mid T_{[x]} \text{ is wellfounded}\}$. Let $\langle \boldsymbol{s}_n \mid n \in \omega \rangle$ be an enumeration of ${}^{<\omega}\omega$ such that $\forall m, n \in \omega\, (m < n \rightarrow \boldsymbol{s}_n \nsubseteq \boldsymbol{s}_m)$.

Given $n \leqslant \omega$ and $s \in {}^{n}\omega$, let $<_s$ be the total order on n such that for all $m, m' < n$, $m <_s m'$ iff

1. $\boldsymbol{s}_m, \boldsymbol{s}_{m'} \in T_{[s]} \wedge \boldsymbol{s}_m <_{BK} \boldsymbol{s}_{m'}$, or
2. $\boldsymbol{s}_m \in T_{[s]} \wedge \boldsymbol{s}_{m'} \notin T_{[s]}$, or
3. $\boldsymbol{s}_m, \boldsymbol{s}_{m'} \notin T_{[s]} \wedge m < m'$,

where $<_{BK}$ is the Brouwer-Kleene ordering of ${}^{<\omega}\omega$.[5.62] Note that for $s, t \in {}^{\leqslant\omega}\omega$,

$$s \subseteq t \rightarrow <_s \subseteq <_t,$$

and we have allowed for the possibility that $n = \omega$. If $x \in {}^{\omega}\omega$ then $<_x$ is a total order of ω that arranges those $n \in \omega$ such that $\boldsymbol{s}_n \in T_{[x]}$ according to the Brouwer-Kleene ordering of $T_{[x]}$, and arranges the remaining elements of ω above them in (increasing) numerical order. Thus,

(9.218) $<_x$ *is wellordered iff* $T_{[x]}$ *is wellfounded.*

Let $T^* \subseteq {}^{<\omega}(\omega \times \kappa)$ *be the sequence tree such that for every* $n \in \omega$*,* $s \in {}^{n}\omega$ *and* $\theta \in {}^{n}\kappa$*,* $\langle s, \theta \rangle^{\cdot} \in T^*$ *iff* $\forall m, m' \in n\, (\theta_m < \theta_{m'} \leftrightarrow m <_s m')$*.*

Suppose $n \in \omega$ and $s \in {}^{n}\omega$.

(9.219) *Then*

1. T^*_s *consists of injective elements of* ${}^{n}\kappa$*; and*
2. *for every* $a \in [\kappa]^n$ *there is a unique* $\theta \in T^*_s$ *such that* $\operatorname{im}\theta = a$*.*

(9.220) **Claim** *For all* $x \in {}^{\omega}\omega$*,* $x \in \mathfrak{p}^{\cdot}[T^*] \leftrightarrow x \notin \mathfrak{p}^{\cdot}[T]$*.*

Proof Suppose $x \in \mathfrak{p}^{\cdot}[T^*]$. Let $\theta \in {}^{\omega}\kappa$ be such that $\langle x, \theta \rangle^{\cdot} \in [T^*]$. Then $\{(\boldsymbol{s}_n, \theta n) \mid n \in \omega\}$ is an order-preserving function from $(\omega; <_x)$ to $(\kappa; <)$. Hence $<_x$ is wellordered, so[9.218] $T_{[x]}$ is wellfounded, so $x \notin \mathfrak{p}^{\cdot}[T]$.

Conversely, if $x \notin \mathfrak{p}^{\cdot}[T]$ then $T_{[x]}$ is wellfounded, so $<_x$ is a wellordering of ω. Since κ is uncountable there exists $\theta : \omega \rightarrow \kappa$ that is order-preserving. Clearly, $\langle x, \theta \rangle^{\cdot} \in [T^*]$, so $x \in \mathfrak{p}^{\cdot}[T^*]$. $\square$[9.220]

Suppose U is a normal ultrafilter over κ. For each $n \in \omega$, let U_n be the corresponding ultrafilter over $[\kappa]^n$.[9.42] By virtue of the preceding remarks, there is a natural map $\bar{U} : {}^{<\omega}\omega \rightarrow \operatorname{ms}_\kappa \kappa$ such that for each $s \in {}^{n}\omega$

$\bar{U}_s = \{X \subseteq {}^{<\omega}\kappa \,|\, \{\operatorname{im}\theta \mid \theta \in X \cap T^*_s\} \in U_n\}$.

Note that[9.219.2]

$$T^*_s \in \bar{U}_s. \tag{9.221}$$

It follows from the proof of (9.66) that for any $Y \in U_n$ there exists $Y' \in U$ such that $[Y']^n \subseteq Y$. Conversely, for any $Z \in U$, $[Z]^n \in U_n$; otherwise, $[\kappa]^n \backslash [Z]^n \in U_n$, so for some $Z' \in U$, $[Z']^n \cap [Z]^n = 0$, which is impossible, since $Z \cap Z' \in U$, so $|Z \cap Z'| \geqslant n$. Thus, for all $X \subseteq T^*_s$

$$X \in \bar{U}_s \leftrightarrow \exists Z \in U\, \forall \theta \in T^*_s\, (\operatorname{im}\theta \subseteq Z \rightarrow \theta \in X). \tag{9.222}$$

(9.223) **Claim** *Suppose $m \leqslant n < \omega$, $t \in {}^m\omega$, and $t \subseteq s \in {}^n\omega$. Then $\bar{U}_s$ projects to $\bar{U}_t$.*[9.212.1] *Hence, $\bar{U}$ is a homogeneity system over ω with support κ.*[9.213.1]

Proof Suppose $X \subseteq {}^m\kappa$. If $X \in \bar{U}_t$, let[9.222] $Z \in U$ be such that $\forall \theta \in T^*_t\, (\operatorname{im}\theta \subseteq Z \rightarrow \theta \in X)$. Then $\forall \theta \in T^*_s\, (\operatorname{im}\theta \subseteq Z \rightarrow \theta \restriction m \in X)$, so[9.222] $\{\theta \in {}^n\kappa \mid \theta \restriction m \in X\} \in \bar{U}_s$. □[9.223]

Thus, if $x \in {}^\omega\omega$ then $\bar{U}^x = \langle \bar{U}_{x \restriction n} \mid n \in \omega \rangle$[9.213.3.1] is a tower of ultrafilters over ${}^{<\omega}\omega$.

(9.224) **Claim** *Suppose $x \in {}^\omega\omega$. Then $x \in \mathfrak{p}``[T^*]$ iff $\bar{U}^x$ is countably complete.*

Proof Suppose $x \in \mathfrak{p}``[T^*]$. Then[9.220] $x \notin \mathfrak{p}``[T]$, i.e., $T_{[x]}$ is wellfounded, so[9.218] $(\omega; <_x)$ is wellordered. Suppose for each $n \in \omega$, $X_n \in \bar{U}_{x \restriction n}$. For each $n \in \omega$, let[9.222] $Z_n \in U$ be such that for all $\theta \in T^*_{x \restriction n}$, $\operatorname{im}\theta \subseteq Z_n \rightarrow \theta \in X_n$. Let $Z = \bigcap_{n \in \omega} Z_n$. Then $Z \in U$, so Z is uncountable. Let $\theta : (\omega; <_x) \rightarrow (Z; <)$ be order-preserving. Then for each $n \in \omega$, $\theta \restriction n \in T^*_{x \restriction n}$ and $\operatorname{im}(\theta \restriction n) \subseteq Z \subseteq Z_n$, so $\theta \restriction n \in X_n$.

Conversely, suppose $\langle \bar{U}_{x \restriction n} \mid n \in \omega \rangle$ is countably complete. Then there exists $\theta \in {}^\omega\kappa$ such that $\forall n \in \omega\ \theta \restriction n \in T^*_{x \restriction n}$.[9.221] $\langle x, \theta \rangle^\cdot \in [T^*]$, so $x \in \mathfrak{p}``[T^*]$. □[9.224]

Thus, $A = \neg\, \mathfrak{p}``[T] = \mathfrak{p}``[T^*] = S_{\bar{U}}$, so A is κ-homogeneously Suslin. □[9.217]

Homogeneous sets have various regularity properties, of which the most important is determinacy.

(9.225) **Theorem** [ZFC] *Suppose $A \subseteq {}^\omega X$ is $|X|^+$-homogeneous. Then A is determinate.*

Proof Let $\bar{U}$ be an $|X|^+$-complete homogeneity system over X with support Z such that $A = S_{\bar{U}}$, and let T be a sequence tree on $X \times Z$ such that[9.214]

1. $\forall s \in {}^{<\omega}X\ T_s \in \bar{U}_s$; and
2. $S_{\bar{U}} = \mathfrak{p}``[T]$.

Let $\mathcal{G}^T$ be the following game:

1. I plays pairs $\langle k, l \rangle$, where $k \in X$ and $l \in Z$.
2. II plays elements of X.
3. Let $x^{\mathrm{I}} \in {}^\omega X$ and $y \in {}^\omega Z$ be such that $\langle x^{\mathrm{I}}, y \rangle^\cdot$ is I's sequence of moves, and let $x^{\mathrm{II}} \in {}^\omega X$ be II's sequence of moves. Let $x = x^{\mathrm{I}} * x^{\mathrm{II}}$. I wins iff $\langle x, y \rangle^\cdot \in [T]$.

Let us compare $\mathcal{G}^T$ to the game $\mathcal{G}_A$, where $A = \mathfrak{p}^{\cdot}[T]$. In both cases I is trying to get $x = x^{\mathrm{I}} * x^{\mathrm{II}}$ in $\mathfrak{p}^{\cdot}[T]$; however, in the former game, I is required to provide a witness y, i.e., $y \in {}^{\omega}Z$ such that $\langle x, y\rangle^{\cdot} \in [T]$.

$\mathcal{G}^T$ is a closed game, i.e., I wins iff the successive values of $\langle x \restriction n, y \restriction n\rangle^{\cdot}$, as they become known, are always in the tree T.[58] $\mathcal{G}^T$ is therefore determined. We will use this to show that $\mathcal{G}_A$ is determined.

It is straightforward to show that if I has a winning strategy σ in $\mathcal{G}^T$ then I has a winning strategy σ' in the easier game $\mathcal{G}_A$ by playing according to σ and discarding the auxiliary information about the companion sequence y, i.e., letting $\sigma' s = (\sigma s)_0$.

Conversely, suppose II has a winning strategy τ in $\mathcal{G}^T$. Let τ' be the following strategy for II in $\mathcal{G}_A$.

> Given $s^{\mathrm{I}} \in {}^{n+1}X$ and $s^{\mathrm{II}} \in {}^{n}X$, let $s = s^{\mathrm{I}} * s^{\mathrm{II}}$, and let $s' = s \restriction (n+1)$. Let $\tau'(s)$ be that (unique) $k \in X$ such that $\{t \in T_{s'} \mid \tau(\langle s^{\mathrm{I}}, t\rangle^{\cdot} * s^{\mathrm{II}}) = k\} \in \bar{U}_{s'}$. ($k$ exists because $T_{s'} \in \bar{U}_{s'}$ and $\bar{U}_{s'}$ is $|X|^{+}$-complete.) For future reference, let $X_{n+1} = \{t \in T_{s'} \mid \tau(\langle s^{\mathrm{I}}, t\rangle^{\cdot} * s^{\mathrm{II}}) = \tau'(s)\}$; and let $X_0 = \{0\}$, which is the only member of U_0.

Suppose toward a contradiction that τ' is not a winning strategy for II.

Suppose $x^{\mathrm{I}} \in {}^{\omega}X$ is such that $x = x^{\mathrm{I}} * \tau' \in A = \mathfrak{p}^{\cdot}[T] = S_{\bar{U}}$, i.e., $\bar{U}^x = \langle \bar{U}_{x \restriction n} \mid n \in \omega\rangle$ is countably complete. Let $\langle X_n \mid n < \omega\rangle$ be the sequence of sets $X_n \in \bar{U}_{x \restriction n}$ used by II in playing according to τ' against x^{I}; and let $y \in {}^{\omega}Z$ be such that $\forall n \in \omega\, (y \restriction n \in X_n)$. Then $\langle x^{\mathrm{I}}, y\rangle^{\cdot} * x^{\mathrm{II}} = \langle x^{\mathrm{I}}, y\rangle^{\cdot} * \tau$.

Then, on the one hand, $y \restriction n \in T_{x \restriction n}$ for each $n \in \omega$, so $\langle x, y\rangle^{\cdot} \in [T]$; whereas, on the other hand, since τ is a winning II-strategy in $\mathcal{G}^T$, $\langle x, y\rangle^{\cdot} \notin [T]$. This contradiction establishes that τ' is a winning II-strategy in $\mathcal{G}_A$. $\square_{9.225}$

Putting (9.217) together with (9.225) we have Martin's theorem:

(9.226) **Theorem** [ZFC] *Suppose there exists a measurable cardinal. Then* Det $\boldsymbol{\Pi}^1_1$.

Measurability is much more than is needed for $\boldsymbol{\Pi}^1_1$-determinacy. For example, it is sufficient that there exist a cardinal κ such that $\kappa \to (\omega_1)^{<\omega}_{\omega}$.[59] A proof from this hypothesis is fairly easily constructed along the lines of the proof given above (with homogeneous sets implied by the partition relation in place of those derived from a homogeneity system), but for our purposes this refinement is unnecessary, and we are actually interested in the generalization to more complex sets, for which the above treatment in terms of homogeneity systems is appropriate.

9.7.6.2 Weak homogeneity

Suppose $B \subseteq {}^{\omega}(X \times \omega)$ is Z-Suslin, say $B = \mathfrak{p}^{\cdot}[T]$, where T is a sequence tree on $(X \times \omega) \times Z$; and suppose $A = \mathfrak{p}^{\cdot} B = \{x \in {}^{\omega}X \mid \exists y \in {}^{\omega}\omega\ \langle x, y\rangle^{\cdot} \in B\}$. Then $A = \mathfrak{p}^{\cdot}[T']$, where $T' = \{\langle s, \langle t, u\rangle^{\cdot}\rangle^{\cdot} \mid \langle\langle s, t\rangle^{\cdot}, u\rangle^{\cdot} \in T\}$. T' is a sequence tree on $X \times (\omega \times Z)$, so A is $(\omega \times Z)$-Suslin.

Note that if $X = \omega$ and B is in a pointclass Γ then $A \in \exists^1\Gamma$, the class whose members are obtained from sets in Γ by existential quantification over a real variable (equivalently, projection along a coordinate of type 1). $\boldsymbol{\Pi}^1_n$ and $\boldsymbol{\Sigma}^1_{n+1}$, for example, are related in this way.

[58] It doesn't matter that by the time we know $y \restriction n$ we already know $x \restriction (2n-1)$. We still only check that $\langle x \restriction n, y \restriction n\rangle^{\cdot} \in T^{*}$.

[59] In fact, it suffices that sharps exist.

Suppose now that T is homogeneously Suslin, via a homogeneity system $\bar{U} : {}^{<\omega}(X \times \omega) \to \mathrm{ms}_\kappa Z$, so that $\langle x, y\rangle^{\cdot} \in B$ iff the tower $\bar{U}^{\langle x,y\rangle^{\cdot}}$ of ultrafilters over ${}^{<\omega}Z$ is countably complete. Then $x \in A$ iff there exists $y \in {}^{\omega}\omega$ such that $\bar{U}^{\langle x,y\rangle^{\cdot}}$ is countably complete. A is *weakly homogeneous* in the following sense.

(9.227) **Definition** [ZF]

1. *Suppose* $A \subseteq {}^{\omega}X$. A *is* (κ-)weakly homogeneous (with support Z) $\overset{\text{def}}{\Longleftrightarrow}$ *there is a (κ-complete) homogeneity system* $\bar{U}$ *over* $X \times \omega$ *(with support* Z*) such that* $A = \{x \in {}^{\omega}X \mid \exists y \in {}^{\omega}\omega\ \langle x, y\rangle^{\cdot} \in S_{\bar{U}}\} = \mathfrak{p}^{\cdot}\, S_{\bar{U}}$.[9.213.3.2]

2. *Suppose* T *is a sequence tree on* $X \times Z$. T *is* (κ-)weakly homogeneous $\overset{\text{def}}{\Longleftrightarrow}$ *there is a (κ-complete) homogeneity system* $\bar{U}$ *over* $X \times \omega$ *with support* Z *such that*

    1. $\forall \langle s, t\rangle^{\cdot} \in {}^{<\omega}(X \times \omega)\ T_s \in \bar{U}_{\langle s,t\rangle^{\cdot}}$*; and*
    2. $\mathfrak{p}^{\cdot}[T] = \mathfrak{p}^{\cdot}\, S_{\bar{U}}$.

3. $A \subseteq {}^{\omega}X$ *is* (κ-)weakly homogeneously (Z-)Suslin $\overset{\text{def}}{\Longleftrightarrow}$ $A = \mathfrak{p}^{\cdot}[T]$ *for some (κ-)weakly homogeneous tree* T *(on* $X \times Z$*).*

It turns out that for sufficiently large cardinals κ, κ-weak homogeneity is itself a powerful regularity property.

Note that in the scenario described just preceding this definition, the tree T' is not rendered weakly homogeneous by $\bar{U}$, because the ultrafilters $\bar{U}_{\langle s,t\rangle^{\cdot}}$ are over ${}^{<\omega}Z$, not ${}^{<\omega}(\omega \times Z)$, but this is easily remedied.

(9.228) **Theorem** [GBC] *Suppose* X *is a set,* κ *is an uncountable cardinal.*

1. *Suppose* $B \subseteq {}^{\omega}(X \times \omega)$ *is* κ*-homogeneous with support* Z*. Then* $\mathfrak{p}^{\cdot}\, B$ *is* κ*-weakly homogeneous with support* Z*.*

2. *Suppose* Z *is infinite*[60] *and* $B \subseteq {}^{\omega}(X \times \omega)$ *is* κ*-homogeneously* Z*-Suslin. Then* $\mathfrak{p}^{\cdot}\, B$ *is* κ*-weakly homogeneously* Z*-Suslin.*

Proof See Note 10.36. $\square$[9.228]

With reference to the urexample of homogeneity of $\mathbf{\Pi}^1_1$ sets,[9.217] we have:

(9.229) **Theorem** [ZFC] *Suppose* A *is* $\mathbf{\Sigma}^1_2$ *and* κ *is measurable. Then* A *is* κ*-weakly homogeneously Suslin.*[9.227.1]

For countable sets X, weak homogeneity has the following equivalent definition.

(9.230) **Theorem** [ZFC] *Suppose* X *is countable,* T *is a tree on* $X \times Z$*, and* $\kappa > \omega$*. Then* T *is* κ*-weakly homogeneous iff there exists a countable set* $\mathcal{U} \subseteq \mathrm{ms}_\kappa Z$ *such that for all* $x \in {}^{\omega}X$*, if* $x \in \mathfrak{p}^{\cdot}[T]$ *then there is a countably complete tower* $\langle U_n \mid n \in \omega\rangle \in {}^{\omega}\mathcal{U}$ *such that* $\forall n \in \omega\ T_{x \restriction n} \in U_n$*.*

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[60] We assume Z is infinite so that $|\omega \times Z| = |Z|$, but this is also the only interesting case. If Z is finite and $T \subseteq {}^{<\omega}(X \times Z)$ is a tree, let $S = \{s \in {}^{<\omega}X \mid T_s \neq 0\}$. Then S is a tree, and for all $x \in {}^{\omega}X$, $x \in [S]$ iff $T_{[x]}$ is infinite iff $T_{[x]}$ has an infinite branch, by König's lemma.[9.56] Hence, $\mathfrak{p}^{\cdot}[T] = [S]$, and every Z-Suslin set is closed. If Z is finite (or even countable) then all countably complete ultrafilters over ${}^{<\omega}Z$ are principal, so all towers are countably complete, and $\mathfrak{p}^{\cdot}\, \bar{U} = {}^{\omega}X$ for any homogeneity system $\bar{U}$ over X with support Z. (The reason *homogeneously* Z*-Suslin* is more restrictive than Z*-Suslin* in the finite case is that it excludes the possibility that $T_s = 0$ by the requirement that T_s be in $\bar{U}_s$.)

Proof See Note 10.37. □[9.230]

It is often useful to view towers of ultrafilters in terms of elementary embeddings of their ultrapowers. Suppose $\langle U_n \mid n \in \omega\rangle$ is a tower of ultrafilters over ${}^{<\omega}Z$. Recall that by definition, for each $n \in \omega$, $\dim U_n = n$, so U_n is—in effect—an ultrafilter over ${}^{n}Z$. Since U_n is ω_1-complete, ${}^{{}^nZ}V/U_n$ is wellfounded, and we let $\pi_n : {}^{{}^nZ}V/U_n \to \mathrm{Ult}_{U_n}(V)$ be the transitive collapse. Note that ${}^0Z = \{0\} = 1$, so $U_0 = \{1\}$ and ${}^{{}^0Z}V/U_0 \cong {}^1V \cong V$, and $\mathrm{Ult}_{U_0}(V) = V$.

Suppose $m \leqslant n < \omega$. Given $f : {}^mZ \to V$, let $\bar{f} : {}^nZ \to V$ be such that $\forall u \in {}^nZ\ \bar{f}(u) = f(u \restriction m)$. Then $f \equiv^{U_m} g \leftrightarrow \bar{f} \equiv^{U_n} \bar{g}$,[9.212.2.2] so $[f]_{U_m} \mapsto [\bar{f}]_{U_n}$ is a well defined injection of ${}^{{}^mZ}V/U_m$ into ${}^{{}^nZ}V/U_n$, and it is elementary. Define $i_{mn} : \mathrm{Ult}_{U_m}(V) \to \mathrm{Ult}_{U_n}(V)$ so that for all $f : {}^mZ \to V$,

$$i_{mn}\pi_m[f]_{U_m} = \pi_n[\bar{f}]_{U_n}.$$

For $n \in \omega$ let $M_n = \mathrm{Ult}_{U_n}(V)$ and let $\mathfrak{M}_n = (M_n; \in)$. $\big[[i_{mn} \mid m \leqslant n < \omega], [\mathfrak{M}_n \mid n \in \omega]\big]$ is a directed system of elementary embeddings, and its direct limit is defined in terms of its continuation to a directed elementary system $\big[[i_{mn} \mid m \leqslant n \leqslant \omega], [\mathfrak{M}_n \mid n \leqslant \omega]\big]$ of length $\omega+1$, such that $|\mathfrak{M}_\omega| = \bigcup_{n<\omega} i_{n\omega}{}^{\to}M_n$. $\mathfrak{M}_\omega$ and $i_{n\omega}$ are uniquely defined up to isomorphism, and we refer to $\mathfrak{M}_\omega$ as the direct limit of the system, with the understanding that the embeddings $i_{n\omega}$ are intrinsic to its status as such. Let $\mathfrak{M}_\omega = (M_\omega; E_\omega)$.

(9.231) *Concretely, we may take M_ω to consist of pairs $\langle n, a\rangle$, where $n \in \omega$, $a \in M_n$, and there do not exist $n' < n$ and $a' \in M_{n'}$ such that $i_{n'n}a' = a$. Given $\langle n_0, a_0\rangle$ and $\langle n_1, a_1\rangle$ in M_ω, let $n = \max\{n_0, n_1\}$. Then $\langle n_0, a_0\rangle\, E_\omega\, \langle n_1, a_1\rangle$ iff $i_{n_0n}a_0 \in i_{n_1n}a_1$.*

(9.232) **Theorem** [GBC] *Suppose $\langle U_n \mid n \in \omega\rangle$ is a tower of ultrafilters over ${}^{<\omega}Z$. Then it is countably complete iff the direct limit of the corresponding directed system of embeddings is wellfounded.*

Proof Suppose $\langle U_n \mid n \in \omega\rangle$ is countably complete. Using the above terminology, suppose toward a contradiction that $\mathfrak{M}_\omega$ is not wellfounded. Then there is a descending ω-sequence $\big\langle\langle n_k, \alpha_k\rangle \,\big|\, k \in \omega\big\rangle$[9.231] in the ordinals of $\mathfrak{M}_\omega$. Since each $\mathfrak{M}_n$ is wellfounded, $\langle n_k \mid k \in \omega\rangle$ is unbounded, and by taking a subsequence we may suppose that $\langle n_k \mid k \in \omega\rangle$ is strictly increasing. For each $k \in \omega$ let $f_k : {}^{n_k}Z \to \mathrm{Ord}$ represent α_k in $M_{n_k} = \mathrm{Ult}_{U_{n_k}}(V)$. Let $Z_k = \{u \in {}^{n_{k+1}}Z \mid f_{n_k}(u \restriction n_k) > f_{n_{k+1}}(u)\} \in U_{n_{k+1}}$. By hypothesis there exists $z \in {}^{\omega}Z$ such that for each $k \in \omega$, $z \restriction n_{k+1} \in Z_k$. Then $\langle f_{n_k}(z \restriction n_k) \mid k \in \omega\rangle$ is a descending ω-sequence in Ord.

Inversely, suppose $\langle U_n \mid n \in \omega\rangle$ is not countably complete, and let Z_n $(n \in \omega)$ be such that $Z_n \in U_n$ and there does not exist $z \in {}^{\omega}Z$ such that $\forall n \in \omega\ z \restriction n \in Z_n$. By virtue of (9.212.2.2) we may suppose that for all $m < n < \omega$, if $u \in Z_n$ then $u \restriction m \in Z_m$. Let $T = \bigcup_{n\in\omega} Z_n$. Then T is a sequence tree on Z, which by hypothesis has no infinite branch, so it is wellfounded. Let $\rho : T \to \mathrm{Ord}$ be its rank function. For each $n \in \omega$, let $f_n : {}^nZ \to \mathrm{Ord}$ be such that for all $u \in {}^nZ$,

$$f_n u = \begin{cases} \rho u & \text{if } u \in Z_n \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \omega$, let $a_n = \pi_n[f_n]_{U_n}$. For each $u \in Z_{n+1}$, $u \restriction n \in Z_n$ and $f_n(u \restriction n) = \rho(u \restriction n) > \rho(u) = f_{n+1}(u)$, so $i_{n(n+1)}a_n > a_{n+1}$, and $i_{(n+1)\omega}a_{n+1}\, E_\omega\, i_{n\omega}a_n$. It follows that $\mathfrak{M}_\omega = (M_\omega; E_\omega)$ is not wellfounded. □[9.232]

By virtue of (9.232), we use 'wellfounded' synonymously with 'countably complete' in reference to towers of ultrafilters.

The above point of view is quite useful in the consideration of homogeneity systems. Suppose $\bar{U} : {}^{<\omega}X \to \operatorname{ms} Z$ is a homogeneity system. For each $n \in \omega$ and $s \in {}^{n}X$, ${}^{{}^nZ}V/\bar{U}_s$ is wellfounded, and we let $\pi_s : {}^{{}^nZ}V/\bar{U}_s \to \operatorname{Ult}_{\bar{U}_s}(V)$ be the transitive collapse. Suppose $s, t \in {}^{<\omega}X$ and $s \subseteq t$. Let $m = |s|$ and $n = |t|$. Given $f : {}^{m}Z \to V$, let $\bar{f} : {}^{n}Z \to V$ be such that $\bar{f}(u) = f(u \restriction m)$. Then $[f]_{\bar{U}_s} \mapsto [\bar{f}]_{\bar{U}_t}$ is a well defined injection of ${}^{{}^mZ}V/\bar{U}_s$ into ${}^{{}^nZ}V/\bar{U}_t$, and we define the elementary embedding $i_{st} : \operatorname{Ult}_{\bar{U}_s}(V) \to \operatorname{Ult}_{\bar{U}_t}(V)$ so that for all $f : {}^{m}Z \to V$,

$$i_{st}\pi_s[f]_{\bar{U}_s} = \pi_t[\bar{f}]_{\bar{U}_t}.$$

For $s \in {}^{<\omega}X$ let $M_s = \operatorname{Ult}_{\bar{U}_s}(V)$ and let $\mathfrak{M}_s = (M_s; \in)$.

Suppose $x \in {}^{\omega}X$. Then $\bar{U}^x = \langle \bar{U}_{x \restriction n} \mid n \in \omega \rangle$ is a tower of ultrafilters over ${}^{<\omega}Z$. For $m \leqslant n \in \omega$, let $\mathfrak{M}^x = [\mathfrak{M}_{x \restriction n} \mid n < \omega]$, and let $i^x = [i_{(x \restriction m)(x \restriction n)} \mid m \leqslant n < \omega]$. Then $[i^x, \mathfrak{M}^x]$ is a directed system of elementary embeddings. Let $\mathfrak{M}^x_\omega$ be its direct limit. By virtue of (9.232), $S_{\bar{U}}$ is the set of $x \in {}^{\omega}X$ such that $\mathfrak{M}^x_\omega$ is wellfounded.

As noted previously, the above proof of $\operatorname{Det} \mathbf{\Pi}^1_1$ is a model for proofs of determinacy from large cardinal hypotheses. The following are several key elements of the proof.

1. *If $A \subseteq {}^{\omega}\omega$ is $\mathbf{\Sigma}^1_1$ then A is the projection of a closed subset of ${}^{\omega}\omega \times {}^{\omega}\omega$. In other words, there is a sequence tree T on $\omega \times \omega$ such that $A = \mathfrak{p}^{\cdot}[T]$.*
2. *If $A \subseteq {}^{\omega}\omega$ is the projection of a closed subset of ${}^{\omega}\omega \times {}^{\omega}\omega$ then $\neg A$ is κ-Suslin for any uncountable cardinal κ, i.e., there is a sequence tree T^* on $\omega \times \kappa$ such that $\neg A = \mathfrak{p}^{\cdot}[T^*]$.*
3. *If κ is measurable then the tree T^* may be designed to be homogeneous, i.e., for each $s \in {}^{<\omega}\omega$ there is an ω_1-complete ultrafilter U_s over T^*_s such that for any $x \in {}^{\omega}\omega$, $\langle U_{x \restriction n} \mid n \in \omega \rangle$ is a tower, which is countably complete iff $x \in \mathfrak{p}^{\cdot}[T^*]$.*
4. *If T^* is homogeneous in this sense then $\mathfrak{p}^{\cdot}[T^*]$ is determinate.*

A key construct relates trees T on $\omega \times Z$ and T' on $\omega \times Z'$ so that an infinite branch in $T'_{[x]}$ witnesses the wellfoundedness of $T_{[x]}$ and hence the nonexistence of an infinite branch in $T_{[x]}$. In the case of $\mathbf{\Sigma}^1_1/\mathbf{\Pi}^1_1$, Z may be taken to be ω, which makes it quite easy to define T', with Z' being any cardinal $\kappa \geqslant \omega_1$, and a branch of $T'_{[x]}$ amounting to an order-preserving tagging of $T_{[x]}$ by ordinals $< \kappa$. By taking κ measurable, we can make T' homogeneous and derive regularity consequences for $\mathfrak{p}^{\cdot}[T]$. When Z is not countable a more sophisticated construction of T' is necessary so that the wellfoundedness of $T_{[x]}$ may be witnessed by a branch of $T'_{[x]}$, with homogeneity systems playing a critical role. Specifically, a κ-complete weak homogeneity system for T is used to define T', along with a λ-complete homogeneity system for T', with $\lambda < \kappa$ typically.

(9.233) **Definition** [GBC] *Suppose T is a κ-weakly homogeneous tree on X via the homogeneity system $\bar{U} : {}^{<\omega}(X \times \omega) \to \operatorname{ms}_\kappa Z$. Using the terminology elaborated following Theorem 9.232, for any $\lambda \in \operatorname{Ord}$, the* Martin-Solovay tree $\stackrel{\text{def}}{=} \tilde{T}^\lambda \stackrel{\text{def}}{=}$ *the sequence tree on $X \times \lambda$ such that for all $n \in \omega$ and $\langle s, v \rangle^{\cdot} \in {}^{n}(X \times \lambda)$, $\langle s, v \rangle^{\cdot} \in \tilde{T}$ iff*

$$\forall i, j < n \left(\boldsymbol{s}_i \subsetneq \boldsymbol{s}_j \to v_j < i_{(\langle s \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i \rangle^{\cdot})(\langle s \restriction |\boldsymbol{s}_j|, \boldsymbol{s}_j \rangle^{\cdot})}(v_i) \right),$$

where $\boldsymbol{s}$ is the fixed enumeration of ${}^{<\omega}\omega$ posited at the beginning of the proof of (9.217).

(9.234) **Theorem** [GBC] *Suppose T, $\bar U$, and $\tilde T^\lambda$ are as in (9.233). If $\lambda > 2^{|Z|}$ then $\mathfrak{p}^\cdot[\tilde T^\lambda] = {}^\omega X \setminus \mathfrak{p}^\cdot[T]$.*

Proof Given $x \in {}^\omega X$ and $i \in \omega$, let

$$U_i^x = \bar U_{\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot}.$$

Analogously, let $\pi_i^x = \pi_{\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot}$ and $M_i^x = M_{\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot}$; and let $i_{ij}^x = i_{(\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot)(\langle x \restriction |\boldsymbol{s}_j|, \boldsymbol{s}_j\rangle^\cdot)}$ whenever $\boldsymbol{s}_i \subseteq \boldsymbol{s}_j$.

Let $\tilde T = \tilde T^\lambda$, and suppose $x \in \mathfrak{p}^\cdot[\tilde T]$. Let $f \in {}^\omega\lambda$ be such that $\langle x, f\rangle^\cdot \in [\tilde T]$. Given $i, j \in \omega$ such that $\boldsymbol{s}_i \subsetneqq \boldsymbol{s}_j$, it follows from the definition of $\tilde T$[9.233] that $f_j < i_{ij}^x f_i$. Note that if $y \in {}^\omega\omega$ is such that $\boldsymbol{s}_i \subsetneqq \boldsymbol{s}_j \subseteq y$ then $i_{ij}^x = i^{\langle x,y\rangle^\cdot}_{|\boldsymbol{s}_i||\boldsymbol{s}_j|}$, so $f_j < i^{\langle x,y\rangle^\cdot}_{|\boldsymbol{s}_i||\boldsymbol{s}_j|} f_i$, and—more to the point—

$$i^{\langle x,y\rangle^\cdot}_{|\boldsymbol{s}_j|\omega} f_j < i^{\langle x,y\rangle^\cdot}_{|\boldsymbol{s}_i|\omega} f_i.$$

Now suppose toward a contradiction that $x \in \mathfrak{p}^\cdot[T]$. Let $y \in {}^\omega\omega$ be such that $\bar U^{\langle x,y\rangle^\cdot}$ is countably complete, so that $\mathfrak{M}_\omega^{\langle x,y\rangle^\cdot}$ is wellfounded. Let $\langle i_n \mid n \in \omega\rangle$ be such that $\langle \boldsymbol{s}_{i_n} \mid n \in \omega\rangle$ is a strictly increasing sequence of initial segments of y. Then $\left\langle i^{\langle x,y\rangle^\cdot}_{|\boldsymbol{s}_{i_n}|\omega} f_{i_n} \,\middle|\, n \in \omega\right\rangle$ is a strictly decreasing sequence of ordinals in $\mathfrak{M}_\omega^{\langle x,y\rangle^\cdot}$, which is impossible, as it is wellfounded.

Conversely, suppose $x \notin \mathfrak{p}^\cdot[T]$. Then $T_{[x]}$ is wellfounded. Let $\rho : T_{[x]} \overset{\text{sur}}{\to} \eta$ be its rank function, where η is the height of $T_{[x]}$. Note that $|\eta| \leqslant |Z|$.

For $i \in \omega$ let

$$f_i = \pi_i^x[\rho \restriction T_{x \restriction |\boldsymbol{s}_i|}]_{U_i^x},$$

regarding $U_i^x = \bar U_{\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot}$ as an ultrafilter over $T_{x \restriction |\boldsymbol{s}_i|}$ (or, equivalently, extending ρ arbitrarily to a total function on ${}^{<\omega}Z$).

Note that Z cannot be finite (or even countable), because in that case all countably complete ultrafilters over ${}^{<\omega}Z$ are principal, so all towers are countably complete, and $\mathfrak{p}^\cdot[T] = {}^\omega X$, contradicting the choice of x. Given that Z is infinite, $|f_i| \leqslant |{}^Z\eta| \leqslant |{}^Z Z| = 2^{|Z|}$. Hence, $f_i < \lambda$.

Suppose $\boldsymbol{s}_i \subsetneqq \boldsymbol{s}_j$. Then for any $u \in T_{x \restriction |\boldsymbol{s}_j|}$, $u \restriction |\boldsymbol{s}_i| \in T_{x \restriction |\boldsymbol{s}_i|}$, and $\rho(u) < \rho(u \restriction |\boldsymbol{s}_i|)$. Since U_j^x projects to U_i^x,

$$f_j < i_{ij}^x f_i = i_{(\langle x \restriction |\boldsymbol{s}_i|, \boldsymbol{s}_i\rangle^\cdot)(\langle x \restriction |\boldsymbol{s}_j|, \boldsymbol{s}_j\rangle^\cdot)} f_i.$$

Thus, $\langle x, f\rangle^\cdot \in [\tilde T]$, so $x \in \mathfrak{p}^\cdot[\tilde T]$. □[9.234]

The above construction of $\tilde T$ was originally carried out by Martin and Solovay, starting from a κ-weakly homogeneous Suslin tree T for an arbitrary $\boldsymbol{\Sigma}^1_2$ set $A \subseteq {}^\omega\omega \times {}^\omega\omega$, where κ is measurable.[9.229] In this case $\tilde T$ is a λ-Suslin tree for the arbitrary $\boldsymbol{\Pi}^1_2$ set $B = (\omega \times \omega)\setminus A$, where λ is a cardinal $> 2^\kappa$. It follows that the arbitrary $\boldsymbol{\Sigma}^1_3$ set $C = \mathfrak{p}\, B \subset {}^\omega\omega$ is also λ-Suslin. As noted following (9.226), a measurable cardinal is more than is needed for this. Assuming the existence of sharps, Martin was able to define a $\tilde T$-like sequence tree on $\omega \times u_\omega$, where u_ω is the ωth uniform indiscernible. It can be shown that $u_\omega < \omega_3$, which yields the following theorem.

(9.235) **Theorem** [ZFC] *Suppose $\forall x \in {}^\omega\omega\, (x^\sharp$ exists). Then every $\boldsymbol{\Sigma}^1_3$ set is ω_2-Suslin.*

Note that $\mathbf{\Pi}^1_2$ sets are not shown by this construction to be homogeneously Suslin, nor are $\mathbf{\Sigma}^1_3$ sets shown to be weakly homogeneously Suslin; they are simply shown to be Suslin. Nevertheless, the Martin-Solovay construction holds the key to propagating homogeneous and weakly homogeneous Suslin representations up the projective hierarchy, and the following theorem of Martin and Steel unlocks the door.

(9.236) **Theorem (Martin, Steel)** [ZFC] *Suppose λ is a Woodin cardinal and T is a λ^+-weakly homogeneous tree. Then for every $\mu < \lambda$, for every sufficiently large ν, the tree $\tilde{T}^\nu$ defined as in (9.233) (from a given homogeneity system) is μ-homogeneous.*

As we have seen, if κ is measurable then any $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin, so any $\mathbf{\Sigma}^1_2$ set is κ-weakly homogeneously Suslin. If $\lambda < \kappa$ is Woodin then any $\mathbf{\Sigma}^1_2$ set is λ^+-weakly homogeneously Suslin, so any $\mathbf{\Pi}^1_2$ set is μ-homogeneously Suslin for every $\mu < \lambda$,[9.236] so any $\mathbf{\Sigma}^1_3$ set is μ-weakly homogeneously Suslin for every $\mu < \lambda$. If there is a Woodin cardinal $\lambda' < \lambda$, then by the same reasoning any $\mathbf{\Pi}^1_3$ set is μ-homogeneously Suslin for every $\mu < \lambda'$,[9.236] so any $\mathbf{\Sigma}^1_4$ set is μ-weakly homogeneously Suslin for every $\mu < \lambda'$.

This yields the following corollary of (9.236).

Theorem [ZFC] *Suppose $n \in \omega$ and there are n Woodin cardinals with a measurable cardinal above them. Then* $\operatorname{Det} \mathbf{\Pi}^1_{n+1}$.

This of course implies that if there are infinitely many Woodin cardinals with a measurable cardinal above them then all projective sets are determinate. This result was soon improved by Woodin as follows.

Theorem [GBC] *Suppose there exist infinitely many Woodin cardinals $\langle \lambda_n \mid n \in \omega \rangle$ and a measurable cardinal above them. Then every set of reals in $L(\mathbb{R})$ is λ-weakly homogeneously Suslin for every $\lambda < \sup_{n\in\omega} \lambda_n$. Hence* $\mathsf{AD}^{L(\mathbb{R})}$.

At the level of consistency, the measurable above the Woodin cardinals is unnecessary, and its elimination allows the converse to be proved.

Theorem (Woodin) [S] *The following theories are equiconsistent, i.e., the consistency of each implies the consistency of the other:*

1. ZFC+ ⌜*there exist infinitely many Woodin cardinals*⌝.
2. ZF + AD.

We have previously established relationships between large cardinals and ideals over relatively small cardinals. Such relationships also hold for determinacy. Recall Solovay's result[9.186] that ZF + AD ⊢ ⌜the closed unbounded filter over ω_1 is an ultrafilter⌝, which is to say ⌜NS_{ω_1} is 2-saturated⌝, where $\mathrm{NS}_\lambda \overset{\text{def}}{=}$ the nonstationary ideal over a cardinal λ. In the context of ZFC, of course, $\operatorname{sat} \omega_1 \geqslant \omega_2$.[9.101.5.2] Steel and Van Wesep, using a forcing construction of Steel, showed that if $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}+\mathsf{AC}_\mathbb{R}$ is consistent then so is $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})} +$ ⌜NS_{ω_1} is ω_2-saturated⌝, where $\mathsf{AC}_\mathbb{R}$ is the axiom of choice for functions with domain $\mathbb{R}$. Shelah derived the consistency of ⌜NS_{ω_1} is ω_2-saturated⌝ from the existence of a Woodin cardinal.

Definition [ZFC] *Suppose I is an ideal over a cardinal κ, and λ is a cardinal. I is* λ-dense $\overset{\text{def}}{\iff}$ *there a dense set in $\mathcal{P}\,\kappa/I$ of size λ.*

Clearly the existence of a dense set in a boolean algebra of size λ precludes the existence of an antichain of size $> \lambda$, so if I is λ-dense and $\mu > \lambda$, then I is μ-saturated.

The following theorem summarizes several relevant results, all due to Woodin.

Theorem [ZFC]

1. *If there is an ω_1-dense ideal over ω_1 then* $\mathsf{AD}^{L(\mathbb{R})}$.
2. *If* $\mathsf{AD}^{L(\mathbb{R})}$ *then there is a generic extension of $L(\mathbb{R})$ satisfying* ZFC+ ⌜NS_{ω_1} *is ω_1-dense*⌝.

It follows that the theories

1. ZFC+ ⌜there exist infinitely many Woodin cardinals⌝,
2. ZF + AD, and
3. ZFC+ ⌜NS_{ω_1} is ω_1-dense⌝

are equiconsistent, with associated closely related outright implications.

9.8 Summary

As Chapters 7 and 8 have already suggested, ZF is better viewed as a canon of core principles than as a comprehensive theory of membership, because so many reasonable questions about fairly simple objects, such as projective sets of real numbers, are not settled by ZF, even with the addition of AC. A complete theory of membership does not, of course, exist, as Gödel has taught us, but the possibility of broadly explanatory extensions of ZFC is not ruled out, and the definition and elaboration of such extensions has been a vital field of research in which remarkable progress has been made, as described in this chapter.

One can easily imagine a proliferation of "schools" of set theory, arising by the adoption of mutually inconsistent new "axioms", and it is of interest that this has not happened (to date). This is not to say that mutually inconsistent propositions—such as AC and AD—are not entertained, but in any instance at most one is treated as true *per se*, i.e., in $\mathfrak{V} = (V; \in)$, with the rest being true in other models, often related to $\mathfrak{V}$ by generic extension and/or inner model constructions: all being simultaneously regarded as expressing aspects of a single reality.

One is tempted to attribute this to some underlying principle that—while it may not be expressible as an axiom—may usefully expand our intuition as to the nature of membership that already gives rise to ZF(C). This has been done with some success for large cardinal axioms, which are justified as strong statements of the principle that the cumulative hierarchy goes on forever and ever. The existence of inaccessibles (or perhaps more naturally the assertion that there are arbitrarily large inaccessibles) is an obvious—albeit relatively weak—example of this. Vopěnka's principle stands out as a natural—and, as it happens, extremely powerful—statement of this intuition, and for that reason would actually appear to have a reasonable claim to the status of "axiom". On the other hand, determinacy does not appear to derive from any prior intrinsically plausible principle.

Perhaps it should suffice that the theory described in this chapter is very beautiful. Why it works is an interesting but inessential question; that it works is undeniable, and how it works irresistibly fascinating.

We begin the chapter with a discussion of large cardinals, noting first that Infinity is the paradigm and (relatively) the most powerful of large cardinal hypotheses. We have also noted that Power is a large cardinal axiom in the context of Infinity. The weakest large cardinal properties beyond the reach of ZF are weak and strong inaccessibility.[9.1] The derivative notion of α-inaccessibility is introduced, based on the notion of κ being "large$^+$" if the set of "large" cardinals below κ is cofinal in κ. We then introduce Mahlo properties, based on the qualitatively important notion of κ being "large$^+$" if the set of "large" cardinals below κ is stationary in κ.

Next we revisit the concept of measure, introduced by Lebesgue as a useful way of assigning sizes in $[0, \infty)$ to complicated sets of real numbers, and discussed by us in Section 5.7 with an emphasis on delimiting the class of sets of reals to which a Lebesgue measure can be assigned. Lebesgue's *measure problem* is whether a nontrivial translation-invariant measure can apply to all sets of reals. Vitali showed that this is inconsistent with ZFC; but the problem is still interesting in the setting of ZF + DC. In the setting of ZFC, Banach's *measure problem*, which drops the requirement of translation invariance, is quite productive and leads to the notion of a *real-valued measurable* cardinal.[9.9]

Without the requirement of translation invariance, the concept of measure becomes essentially combinatorial, and further investigation leads to the important case of 2-valued (as opposed to more generally real-valued) measures, and thereby to the notion of a *measurable cardinal*, which is where our discussion of large cardinals begins in earnest. In the presence of AC, a countably complete ultrafilter (i.e., a countably additive 2-valued measure) over a cardinal κ determines an elementary embedding $j : V \to M$, where M is transitive and $\operatorname{crit} j = \kappa$. Conversely, such an elementary embedding provides a simple definition of a nontrivial (i.e., κ-complete nonprincipal) ultrafilter over κ, in fact, a normal ultrafilter.

The notion of a large cardinal as the critical point of an elementary embedding of transitive classes with certain closure properties is enormously productive. The ultimate example, that of a nontrivial elementary embedding of V into V, is excluded by Kunen's theorem.[9.131] It is perhaps important that Kunen's proof and all other known proofs of this theorem use Choice, so it remains open whether the existence of a nontrivial elementary embedding of V into V is consistent with ZF.

One criterion of the usefulness of a new hypothesis is whether it settles a preexisting question of interest. The question of the size of powersets was raised very early in the development of the theory of membership—as Cantor's *continuum hypothesis* and the *generalized continuum hypothesis*—and the behavior of the continuum function $\kappa \mapsto 2^\kappa$, or more generally $\kappa \to \kappa^{\operatorname{cf} \kappa}$, remains a fundamental issue. We have seen that large cardinal hypotheses bear importantly on this issue, but the basic question of whether $2^\omega = \omega_1$ is still unsettled; indeed, based on the intuition that has been gained through years of research into this topic, it is reasonable to suggest that this question concerning sets of reals may be of a sort that is intrinsically incapable of satisfactory resolution. That would, of course, in itself be an important insight into the nature of infinitarity.

On the other hand, many other questions about sets of reals have received satisfactory answers by the methods described in this chapter. The web of implications relating the theory $\mathsf{ZF} + \mathsf{AC} + \mathsf{AD}^{L(\mathbb{R})}$ to hypotheses concerning large cardinals, as sketched in this chapter, convey an impression of inevitability hardly imaginable when the extraordinary *axiom of determinacy* was proposed as a *deus ex machina* by Mycielski and Steinhaus—somewhat as Planck proposed the quantization of the electromagnetic field as a solution to the black-body problem. Its intimate relationship to the properties of relatively small objects, like the nonstationary ideal on ω_1, only strengthen this impression. In the other direction, the program of extending ZFC by large-cardinal hypotheses derives credibility from its connection—at least at the level of Woodin cardinals—to this elegant and powerfully explanatory theory.

As discussed in the introduction to this chapter, it may be a vain hope that mathematical principles such as these will ever receive confirmation as conclusive as that accorded Planck's quantization principle, for example; but it may not be too much to hope for a comparable degree of affirmation.

Chapter 10

Notes

10.1 Multisorted signatures and structures

[Refer to p. 29.]

(10.1) **Definition** [C^0] *A* multisorted signature *is a 4-indexed family* $[\Delta, \Pi, \Phi, T]$ *with the following properties:*

1. Π *and* Φ *are disjoint classes.*
2. T *is a function with domain* $\Pi \cup \Phi$.
3. *For each* $P \in \Pi$, T_P *is a nonempty subset of* ${}^n\Delta$ *for some* $n \in \omega$. *We define* $\mathrm{ar}^\rho(P)$, *the* ρ-arity *of* P, *to be the unique* n *for which this is true. If* $0 \in \Pi$ *then* $T_0 = {}^2\Delta$.
4. *For each* $F \in \Phi$, T_F *is a nonempty function with* $\operatorname{dom} T_F \subseteq {}^n\Delta$ *for some* $n \in \omega$, *and* $\operatorname{im} T_F \subseteq \Delta$. *We define* $\mathrm{ar}^\rho(F)$, *the* (ρ-)arity *of* F, *to be the unique* n *for which this is true.*

Definition [C^0] *For* $\rho = [\Delta, \Pi, \Phi, T]$ *as in (10.1),*

1. $\Delta^\rho \overset{\mathrm{def}}{=} \Delta$, $\Pi^\rho \overset{\mathrm{def}}{=} \Pi$, $\Phi^\rho \overset{\mathrm{def}}{=} \Phi$, *and* $T^\rho \overset{\mathrm{def}}{=} T$.

We refer to the members of Δ as *sort* or *domain indices*, to the members of Π as *predicate* or *relation indices*, and to the members of Φ as *function* or *operation indices*. Given an index $X \in \Pi \cup \Phi$, we call $T_{[X]}$ the *type* of X; it specifies the domain sequences from which it is legitimate to form expressions using X, and if X is an operation index it also specifies the sort of the resulting term. 0 is reserved to index the identity predicate, if present, so the last sentence in (10.1.4) stipulates that it shall make sense to ask whether two individuals are identical regardless of what sorts they respectively are.

We suppose that for each $D \in \Delta$ we have an infinite class $\mathcal{V}_D$ of variables such that $D \neq D' \to \mathcal{V}_D \cap \mathcal{V}_{D'} = 0$. Thus, each variable is of a specific sort.

(10.2) **Definition** [C^0] *Given a multisorted signature* ρ, *we define the* sort *of a* ρ*-term* $\tau \overset{\mathrm{def}}{=} \operatorname{srt} \tau$ *by recursion on complexity as follows:*

1. *For each* $D \in \Delta$ *and* $v \in \mathcal{V}_D$, $\operatorname{srt} \bar{v} = D$.

2. *Suppose F is n-ary operation index. An n-sequence $\langle \tau_0, \ldots, \tau_{n^-} \rangle$ of terms is in* $\operatorname{dom} \tilde{F}$ *iff* $\langle \operatorname{srt} \tau_0, \ldots, \operatorname{srt} \tau_{n^-} \rangle \in \operatorname{dom} T$, *and in this case* $\operatorname{srt} \big(\tilde{F} \langle \tau_0, \ldots, \tau_{n^-} \rangle \big) = T \langle \operatorname{srt} \tau_0, \ldots, \operatorname{srt} \tau_{n^-} \rangle$.

(10.3) **Definition** [C^0] *Given a signature* $\rho = [\Delta, \Pi, \Phi, T]$, *a* ρ-structure $\mathfrak{S}$ *is a 4-indexed family* $[\rho, \delta, \pi, \phi]$ *with the following properties:*

1. δ, π, *and* ϕ *are prefunctions with* $\operatorname{dom} \delta \subseteq \Delta$, $\operatorname{dom} \pi \subseteq \Pi$, *and* $\operatorname{dom} \phi \subseteq \Phi$. *Let* $|\mathfrak{S}| \stackrel{\text{def}}{=} \bigcup_{D \in \Delta} \delta_{[D]}$.
2. *For each* $P \in \Pi$,
$$\pi_{[P]} \subseteq \bigcup_{\boldsymbol{D} \in T_P} \bigtimes_{m \in \operatorname{dom} \boldsymbol{D}} \delta_{[\boldsymbol{D}_m]}.$$
If $0 \in \Pi$ *then* $\pi_{[0]} = \{ \langle x, x \rangle \mid x \in |\mathfrak{S}| \}$.
3. *For each* $F \in \Phi$, $\phi_{[F]}$ *is a function with domain*
$$\bigcup_{\boldsymbol{D} \in \operatorname{dom} T_F} \bigtimes_{m \in \operatorname{dom} \boldsymbol{D}} \delta_{[\boldsymbol{D}_m]}$$
such that for all $\boldsymbol{D} \in \operatorname{dom} T_{[F]}$,
$$\phi_{[F]}{}^{\rightarrow} \Big(\bigtimes_{m \in \operatorname{dom} \boldsymbol{D}} \delta_{[D_m]} \Big) \subseteq T_F \, \boldsymbol{D}.$$

Definition [C^0] *Given a multisorted signature* $\rho = [\Delta, \Pi, \Phi, T]$ *and a* ρ*-structure* $\mathfrak{S} = [\rho, \delta, \pi, \phi]$,

$$\forall D \in \Delta \;\; D^{\mathfrak{S}} \stackrel{\text{def}}{=} \delta_{[D]}$$
$$\forall P \in \Pi \;\; P^{\mathfrak{S}} \stackrel{\text{def}}{=} \pi_{[P]}$$
$$\forall F \in \Phi \;\; F^{\mathfrak{S}} \stackrel{\text{def}}{=} \phi_{[F]}.$$

The valuation of ρ-terms and ρ-formulas is defined in the same way as for unisorted languages and structures. Note that for any ρ-term τ, $\tau^{\mathfrak{S}} \in (\operatorname{srt} \tau)^{\mathfrak{S}}$, i.e., the value in $\mathfrak{S}$ of a term of sort D is a member of the D-domain of $\mathfrak{S}$. Note that, while every ρ-term has a unique sort, this is not necessarily the case for the individuals of a ρ-structure, i.e., a given member of $|\mathfrak{S}|$ may be denoted by terms of distinct sorts.

10.1.1 Example: Vector spaces over ordered fields

A concrete example will help to solidify these ideas. We will define a typographical signature ρ for vector spaces over ordered fields. We use either the underline or the single-quote convention to produce metalanguage names for object-language expressions. Let the character $\underline{\mathbb{F}}$ (= '$\mathbb{F}$') index the domain of scalars; let $\underline{\mathsf{V}}$ index the domain of vectors; let $\underline{<}$ index the order relation on the field of scalars;[1] let $\underline{+}$ index the addition operation (on scalars or vectors); and let $\underline{\cdot}$ index the multiplication

[1] '$\underline{<}$' is a name for the symbol '$<$', meaning 'less than'; it is not the symbol '$\leqslant$', meaning 'less than or equal to'.

operation (applicable to two scalars or to a scalar and a vector). Remember that by convention 0 indexes the identity relation. Thus $\rho = [\Delta, \Pi, \Phi, T]$, where

$$\begin{aligned} \Delta &= \{\underline{\mathbb{F}}, \underline{\mathsf{V}}\}, \\ \Pi &= \{0, \underline{\leq}\}, \\ \Phi &= \{\underline{+}, \underline{\cdot}\}, \end{aligned}$$

and T is given by

$$\begin{aligned} T_{[0]} &= \{\langle\underline{\mathbb{F}}, \underline{\mathbb{F}}\rangle, \langle\underline{\mathbb{F}}, \underline{\mathsf{V}}\rangle, \langle\underline{\mathsf{V}}, \underline{\mathbb{F}}\rangle, \langle\underline{\mathsf{V}}, \underline{\mathsf{V}}\rangle\} \\ &= \{\underline{\mathbb{F}}, \underline{\mathsf{V}}\} \times \{\underline{\mathbb{F}}, \underline{\mathsf{V}}\}, \\ T_{[\underline{\leq}]} &= \{\langle\underline{\mathbb{F}}, \underline{\mathbb{F}}\rangle\}, \\ T_{[\underline{+}]} &= \{(\langle\underline{\mathbb{F}}, \underline{\mathbb{F}}\rangle, \underline{\mathbb{F}}), (\langle\underline{\mathsf{V}}, \underline{\mathsf{V}}\rangle, \underline{\mathsf{V}})\}, \\ T_{[\underline{\cdot}]} &= \{(\langle\underline{\mathbb{F}}, \underline{\mathbb{F}}\rangle, \underline{\mathbb{F}}), (\langle\underline{\mathbb{F}}, \underline{\mathsf{V}}\rangle, \underline{\mathsf{V}})\}. \end{aligned}$$

10.1.2 Example: Theory of membership with sets and classes

The theory of membership with sets and classes is also naturally treated as multisorted. The domains are those of sets and classes. An expression $\tau \in \tau'$ is well formed iff τ is of the set sort; τ' may be of set or class sort. Let $\underline{S}$ and $\underline{C}$ index the domains of sets and classes; and let $\underline{\in}$ index the membership predicate. Remember that by convention 0 indexes the identity predicate. Thus $\rho = [\Delta, \Pi, \Phi, T]$, where

$$\begin{aligned} \Delta &= \{\underline{S}, \underline{C}\}, \\ \Pi &= \{0, \underline{\in}\}, \\ \Phi &= 0, \end{aligned}$$

and T is given by

$$\begin{aligned} T_{[0]} &= \{\underline{S}, \underline{C}\} \times \{\underline{S}, \underline{C}\}, \\ T_{[\underline{\in}]} &= \{\langle\underline{S}, \underline{S}\rangle, \langle\underline{S}, \underline{C}\rangle\}. \end{aligned}$$

As noted above, domains with distinct indices need not have disjoint interpretations in a structure. The theory of membership is a case in point, as $\underline{S}^{\mathfrak{S}} \subseteq \underline{C}^{\mathfrak{S}}$ for any structure $\mathfrak{S}$ interpreting the theory of membership, i.e., every set is a class.

10.2 Proof of (1.69)

[Refer to p. 56.]

(10.4) **Theorem** [C^0] *Suppose $\mathfrak{S}$ is a (weakly) satisfactory ρ-structure and ρ' is an expansion of ρ. Then there is a (weakly) satisfactory ρ'-structure $\mathfrak{S}'$ that is an expansion of $\mathfrak{S}$.*

Remark We will call a ρ'-index *new* iff it is not a ρ-index. To expand $\mathfrak{S}$ to a ρ'-structure one simply assigns relations and operations on $|\mathfrak{S}|$ to the new indices X of ρ'. We have to show that the structure $\mathfrak{S}'$ so defined is (weakly) satisfactory. If $|\mathfrak{S}|$ is a proper class this does not follow for an arbitrary choice of denotations, but by choosing carefully we can arrange that the Φ'-satisfaction relation for $\mathfrak{S}'$ for any $\Phi' \subseteq \mathcal{F}^{\rho'}$ is definable from the Φ-satisfaction relation for $\mathfrak{S}$ for an appropriate $\Phi \subseteq \mathcal{F}^{\rho}$.

Proof For every new ρ'-predicate index R let $R^{\mathfrak{S}'} = 0$, i.e., R is universally false in $\mathfrak{S}'$. Let a_0 be a fixed member of $|\mathfrak{S}|$,[2] and for every new (n-ary) ρ'-operation index F let $F^{\mathfrak{S}'}$ be $\{(s, a_0) \mid s \in {}^n|\mathfrak{S}|\}$. Let $\mathfrak{S}'$ be the resulting ρ'-structure.

Expand ρ to a signature $\dot\rho$ by the addition of a single nulary predicate index, P (a constant predicate index). Let $\mathrm{T} = \bar{P}$ $(= \tilde{P}0)$, and let $\mathrm{F} = \neg\, T$. Expand $\mathfrak{S}$ to a $\dot\rho$-structure $\dot{\mathfrak{S}}$ by stipulating that $P^{\dot{\mathfrak{S}}} = 1$ (so that T is true in $\dot{\mathfrak{S}}$, and F is false).

We now define by recursion on complexity a function H that assigns to each ρ'-expression a $\dot\rho$-expression, which we will use to define partial satisfaction relations for $\mathfrak{S}'$. Without loss of generality, we will restrict our attention to the minimal set $\{\neg, \rightarrow, \exists\}$ of formula-building operations.[1.18]

(10.5) *Suppose ϵ is a ρ'-expression.*

1. *If $\epsilon = \bar{\mathrm{v}}_n$ for some $n \in \omega$, then $H\epsilon = \bar{\mathrm{v}}_{n+1}$.*
2. *If $\epsilon = \tilde{F}\langle \tau_0, \ldots, \tau_{n^\cdot} \rangle$ for some $F \in \Phi^{\rho'}$ then*

$$H\epsilon = \begin{cases} \tilde{F}\langle H\tau_0, \ldots, H\tau_{n^\cdot} \rangle & \textit{if } F \in \Phi^{\rho} \\ \bar{\mathrm{v}}_0 & \textit{otherwise.} \end{cases}$$

3. *If $\epsilon = \tilde{R}\langle \tau_0, \ldots, \tau_{n^\cdot} \rangle$ for some $R \in \Pi^{\rho'}$ then*

$$H\epsilon = \begin{cases} \tilde{R}\langle H\tau_0, \ldots, H\tau_{n^\cdot} \rangle & \textit{if } R \in \Phi^{\rho} \\ \mathrm{F} & \textit{otherwise.} \end{cases}$$

4. *If $\epsilon = \neg\, \phi$ then*
   1. *if $H\phi = \mathrm{T}$ then $H\epsilon = \mathrm{F}$ (i.e., $\neg\, \mathrm{T}$);*
   2. *if $H\phi = \mathrm{F}$ then $H\epsilon = \mathrm{T}$ (which is not $\neg\, \mathrm{F}$);*
   3. *otherwise $H\epsilon = \neg\, H\phi$.*
5. *If $\epsilon = \phi \rightarrow \psi$ then*
   1. *if $H\phi = \mathrm{T}$ then $H\epsilon = H\psi$;*
   2. *if $H\phi = \mathrm{F}$ then $H\epsilon = \mathrm{T}$;*
   3. *otherwise,*
      1. *if $H\psi = \mathrm{T}$ then $H\epsilon = \mathrm{T}$;*
      2. *if $H\psi = \mathrm{F}$ then $H\epsilon = \neg\, H\phi$;*
      3. *otherwise, $H\epsilon = H\phi \rightarrow H\psi$.*
6. *If $\epsilon = \exists \mathrm{v}_n\, \phi$ then*
   1. *if $H\phi = \mathrm{T}$ then $H\epsilon = \mathrm{T}$;*
   2. *if $H\phi = \mathrm{F}$ then $H\epsilon = \mathrm{F}$;*
   3. *otherwise, $H\epsilon = \exists \mathrm{v}_{n+1}\, H\phi$.*

Note that H replaces every occurrence of v_n by v_{n+1}, both in terms via (10.5.1) and in quantifier phrases via (10.5.6). This makes v_0 available to be used to represent the common value of the new operations in ρ', which we have specified as a_0. We let H^* be the corresponding transformation of assignments:

$$H^*A = \{(\mathrm{v}_0, a_0)\} \cup \{(\mathrm{v}_{n+1}, A\mathrm{v}_n) \mid \mathrm{v}_n \in \operatorname{dom} A\}.$$

[2] Recall that a structure is by definition nonempty.

(10.6) **Claim** *For any ρ'-term τ and any $\mathfrak{S}'$-assignment A (which is also an $\mathfrak{S}$-assignment, since $|\mathfrak{S}'| = |\mathfrak{S}|$)*

$$\mathrm{Val}^{\mathfrak{S}'}\, \tau[A] = \mathrm{Val}^{\mathfrak{S}}\, H\tau[H^* A].$$

Proof Straightforward induction on complexity of terms.[10.5.1, 2] $\square$[10.6]

Suppose $\Phi' \subseteq \mathcal{F}^{\rho'}$. Let $\dot{\Phi} = \{H\epsilon \mid \epsilon \in \overline{\Phi'}\}$. Let $\Phi = \dot{\Phi} \cap \mathcal{F}^{\rho}$. Note that Φ' is a class of ρ'-formulas, $\overline{\Phi'}$ is a class of ρ'-expressions (generally containing terms as well as formulas), $\dot{\Phi}$ is a class of $\dot{\rho}$-expressions, and Φ is a class of ρ-formulas. It is easy to show by induction that $\forall \phi \in \mathcal{F}^{\rho'}\ H\phi \in \mathcal{F}^{\rho} \cup \{\mathrm{T}, \mathrm{F}\}$, i.e., neither T nor F is ever incorporated into a complex formula during the construction of H; hence, the only formulas in $\dot{\Phi}$ that are not in Φ are (potentially) T and F. Note that if Φ' is finite then Φ is finite.

Suppose S is the Φ-satisfaction relation for $\mathfrak{S}$, which is also the Φ-satisfaction relation for $\dot{\mathfrak{S}}$. Let

$$\dot{S} = S \cup \{\langle \mathrm{T}, A\rangle \mid A \text{ is an } \mathfrak{S}\text{-assignment}\},$$

which is the $\dot{\Phi}$-satisfaction relation for $\dot{\mathfrak{S}}$, and let

$$S' = \{\langle \epsilon, A\rangle \mid \epsilon \in \overline{\Phi'} \wedge \langle H\epsilon, H^* A\rangle \in \dot{S}\}.$$

Note that we have not defined S' by recursion, but rather directly from $\dot{S}$.

(10.7) **Claim** *S' is the Φ'-satisfaction relation for $\mathfrak{S}'$.*

Proof We just need to check the various clauses. Suppose $\epsilon \in \overline{\Phi'}$ and A is an $\mathfrak{S}'$-assignment for ϵ.

Suppose $\epsilon = \tilde{R}\langle \tau_0, \ldots, \tau_{n^-}\rangle$, where $\tau_0, \ldots, \tau_{n^-}$ are ρ'-terms. If $R \in \Pi^{\rho}$ then $R^{\mathfrak{S}'} = R^{\mathfrak{S}}$, and

$$\begin{aligned}\langle \epsilon, A\rangle \in S' &\leftrightarrow \langle H\epsilon, H^* A\rangle \in \dot{S} \leftrightarrow \langle \tilde{R}\langle H\tau_0, \ldots, H\tau_{n^-}\rangle, H^* A\rangle \in S\\ &\leftrightarrow \langle \mathrm{Val}^{\mathfrak{S}}\, H\tau_0[H^* A], \ldots, \mathrm{Val}^{\mathfrak{S}}\, H\tau_{n^-}[H^* A]\rangle \in R^{\mathfrak{S}}\\ &\leftrightarrow \langle \mathrm{Val}^{\mathfrak{S}'}\, \tau_0[A], \ldots, \mathrm{Val}^{\mathfrak{S}'}\, \tau_{n^-}[A]\rangle \in R^{\mathfrak{S}'},\end{aligned}$$

as required. On the other hand, if $R \notin \Pi^{\rho}$, then

1. $R^{\mathfrak{S}'} = 0$, so $\langle \mathrm{Val}^{\mathfrak{S}'}\, \tau_0[A], \ldots, \mathrm{Val}^{\mathfrak{S}'}\, \tau_{n^-}[A]\rangle \notin R^{\mathfrak{S}'}$, and[10.5.3]
2. $H\epsilon = \mathrm{F}$, so $(H\epsilon, H^* A\rangle \notin \dot{S}$,

whence $(H\epsilon, H^* A\rangle \in \dot{S} \leftrightarrow \langle \mathrm{Val}^{\mathfrak{S}'}\, \tau_0[A], \ldots, \mathrm{Val}^{\mathfrak{S}'}\, \tau_{n^-}[A]\rangle \in R^{\mathfrak{S}'}$, and therefore

$$\langle \epsilon, A\rangle \in S' \leftrightarrow \langle \mathrm{Val}^{\mathfrak{S}'}\, \tau_0[A], \ldots, \mathrm{Val}^{\mathfrak{S}'}\, \tau_{n^-}[A]\rangle \in R^{\mathfrak{S}'}.$$

Suppose $\epsilon = \neg\, \phi$. Then[10.5.4]

$$\langle H\epsilon, H^* A\rangle \in \dot{S} \leftrightarrow \langle H\phi, H^* A\rangle \notin \dot{S},$$

so

$$\langle \epsilon, A\rangle \in S' \leftrightarrow \langle \phi, A\rangle \notin S'.$$

Suppose $\epsilon = \phi \to \psi$. Then[10.5.5]

$$\langle H\epsilon, H^*A\rangle \in \dot{S} \leftrightarrow \big(\langle H\phi, H^*A\rangle \in \dot{S} \to \langle H\psi, H^*A\rangle \in \dot{S}\big),$$

so

$$\langle \epsilon, A\rangle \in S' \leftrightarrow \big(\langle \phi, A\rangle \in S' \to \langle \psi, A\rangle \in S'\big).$$

Finally, suppose $\epsilon = \exists \mathrm{v}_n\, \phi$. Then[10.5.6]

$$\langle H\epsilon, H^*A\rangle \in \dot{S} \leftrightarrow \exists a \in |\mathfrak{S}|\, \big\langle H\phi, (H^*A)\big\langle{}^{\mathrm{v}_{n+1}}_{\;a}\big\rangle\big\rangle \in \dot{S}$$

so

$$\begin{aligned}\langle \epsilon, A\rangle \in S' &\leftrightarrow \exists a \in |\mathfrak{S}|\, \big\langle H\phi, (H^*A)\big\langle{}^{\mathrm{v}_{n+1}}_{\;a}\big\rangle\big\rangle \in \dot{S}\\ &\leftrightarrow \exists a \in |\mathfrak{S}|\, \big\langle H\phi, H^*\big(A\big\langle{}^{\mathrm{v}_{n}}_{\,a}\big\rangle\big)\big\rangle \in \dot{S}\\ &\leftrightarrow \exists a \in |\mathfrak{S}'|\, \big\langle \phi, A\big\langle{}^{\mathrm{v}_{n}}_{\,a}\big\rangle\big\rangle \in S'.\end{aligned}$$

□[10.7]

It follows from (10.7) that if $\mathfrak{S}$ is (weakly) satisfactory then $\mathfrak{S}'$ is (weakly) satisfactory. Since $|\mathfrak{S}'| = |\mathfrak{S}|$ and $X^{\mathfrak{S}'} = X^{\mathfrak{S}}$ for every index X of ρ, $\mathfrak{S}'$ is an expansion of $\mathfrak{S}$. □[10.4]

10.3 Proof of (2.59)

[Refer to p. 90.]

(10.8) **Theorem** [S^0] *A theory Θ is inconsistent iff there exist a finite $\Sigma \subseteq \Theta$, finite witness sequence W for Σ, and finite instance set I such that $\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent.*

Proof $\leftarrow$ Suppose $\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent, i.e., $\Sigma \cup \operatorname{im} W \cup I \vdash^{\mathsf{P}} \mathsf{F}$.[2.58] *A fortiori*, $\Sigma \cup \operatorname{im} W \cup I \vdash \mathsf{F}$, and we will work entirely in **ND** from this point on. Suppose $\big(\psi\binom{v}{\bar{c}} \to \exists v\, \psi\big) \in I$. Let I' be I with this sentence omitted. Since $\{\exists v\, \psi\} \vdash \psi\binom{v}{\bar{c}} \to \exists v\, \psi$,

$$\Sigma \cup \operatorname{im} W \cup I' \cup \{\exists v\, \psi\} \vdash \mathsf{F}.$$

One application of Rule 7 will extend a proof of $\Sigma \cup \operatorname{im} W \cup I' \cup \{\exists v\, \psi\} \Rightarrow \mathsf{F}$ to a proof of $\Sigma \cup \operatorname{im} W \cup I' \cup \big\{\psi\binom{v}{\bar{c}}\big\} \Rightarrow \mathsf{F}$, so

$$\Sigma \cup \operatorname{im} W \cup I' \cup \big\{\psi\tbinom{v}{\bar{c}}\big\} \vdash \mathsf{F}.$$

Since[2.43.6] $\neg\,\psi\binom{v}{\bar{c}} \vdash \psi\binom{v}{\bar{c}} \to \exists v\, \psi$,

$$\Sigma \cup \operatorname{im} W \cup I' \cup \big\{\neg\,\psi\tbinom{v}{\bar{c}}\big\} \vdash \mathsf{F}.$$

Hence,[2.44.1.3]

$$\Sigma \cup \operatorname{im} W \cup I' \vdash \mathsf{F}.$$

Proceeding in this way we eliminate every element of I, so

$$\Sigma \cup \operatorname{im} W \vdash \mathsf{F}.$$

We now similarly eliminate the members of $\operatorname{im} W$ as premises, but in this case we must proceed in a specific order. Let $W = \left\langle \exists v_n\, \psi_n \to \psi_n\binom{v_n}{\bar{c}_n} \,\middle|\, n < N \right\rangle$. Let $v = v_{N-1}$, $c = c_{N-1}$, and $\psi = \psi_{N-1}$, and let W' be W with the last item removed, so $W = W' ^\frown \left\langle \exists v\, \psi \to \psi\binom{v}{\bar{c}} \right\rangle$. Since $\psi\binom{v}{\bar{c}} \vdash \exists v\, \psi \to \psi\binom{v}{\bar{c}}$,

$$\Sigma \cup \operatorname{im} W' \cup \left\{ \psi\binom{v}{\bar{c}} \right\} \vdash \mathsf{F}.$$

Since W is a witness sequence for Σ, c does not occur in $\Sigma \cup \operatorname{im} W' \cup \{\exists v\, \psi\} \cup \{\mathsf{F}\}$. Hence, we may apply Rule 4 to extend a proof of $\Sigma \cup \operatorname{im} W' \cup \left\{ \psi\binom{v}{\bar{c}} \right\} \Rightarrow \mathsf{F}$ to a proof of $\Sigma \cup \operatorname{im} W' \cup \{\exists v\, \psi\} \Rightarrow \mathsf{F}$, whence,

$$\Sigma \cup \operatorname{im} W' \cup \{\exists v\, \psi\} \vdash \mathsf{F}.$$

As before, we also have

$$\Sigma \cup \operatorname{im} W' \cup \{\neg \exists v\, \psi\} \vdash \mathsf{F},$$

so

$$\Sigma \cup \operatorname{im} W' \vdash \mathsf{F}.$$

Proceeding in this fashion, we progressively reduce W to the empty sequence, and conclude that

$$\Sigma \vdash \mathsf{F}.$$

$\to$ This direction requires the greater detail provided by proof trees. Thus, we suppose π_0 is an **ND**-proof tree with the root sequent

$$\Sigma \Rightarrow \mathsf{F},$$

for some finite $\Sigma \subseteq \Theta$. Let N be the number of instances of Rule 4 in π_0. By recursion on $n \leqslant N$ we will construct a witness sequence $W = \left\langle \exists v_n\, \psi_n \to \psi_n\binom{v_n}{\bar{c}_n} \,\middle|\, n < N \right\rangle$ for Σ, and for each $n \leqslant N$ an **ND** proof tree π_n with root sequent

$$\Sigma \cup \operatorname{im}(W \upharpoonright n) \Rightarrow \mathsf{F},$$

such that π_n contains $N - n$ instances of Rule 4.

To this end, given $n < N$, let S be any element of π_n of the form[3]

$$\begin{array}{c} \Sigma' \cup \left\{ \psi\binom{v}{\bar{c}} \right\} \Rightarrow \sigma \\ \hline \Sigma' \cup \{\exists v\, \psi\} \Rightarrow \sigma \\ \vdots \\ \Sigma \cup \operatorname{im}(W \upharpoonright n) \Rightarrow \mathsf{F} \end{array}$$

where the top inference is justified by Rule 4. Let c_n be a constant that does not occur in π_n, and let π' be the result of replacing $\bar{c}$ by $\bar{c}_n$ in the last sequent of S

[3]Recall that a proof tree is a set of finite sequences of sequents, each of which progresses from the root of the tree up to a node.

and everywhere in π_n above that point. By the definition of RULE 4, c does not occur in Σ', $\exists v\,\psi$, or σ, so S becomes

$$\begin{array}{c}\dfrac{\Sigma' \cup \left\{\psi\binom{v}{\bar{c}_n}\right\} \Rightarrow \sigma}{\Sigma' \cup \{\exists v\,\psi\} \Rightarrow \sigma}\\ \vdots \\ \Sigma \cup \operatorname{im}(W \restriction n) \Rightarrow \mathsf{F}\end{array}$$

and π' is a proof tree. Let $v_n = v$ and $\psi_n = \psi$; let $W(n) = \exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}$; and let π'' be the result of adding $W(n)$ to the antecedent of every sequent in π'. π'' is again a proof tree, and S has now become

$$\begin{array}{c}\dfrac{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \psi_n\binom{v_n}{\bar{c}_n}\right\} \Rightarrow \sigma}{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \exists v_n\,\psi_n\right\} \Rightarrow \sigma}\\ \vdots \\ \Sigma \cup \operatorname{im}(W \restriction (n+1) \Rightarrow \mathsf{F}\end{array}$$

Finally, let π_{n+1} be obtained from π'' by replacing the fragment

$$\begin{array}{c}\vdots\\ \dfrac{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \psi_n\binom{v_n}{\bar{c}_n}\right\} \Rightarrow \sigma}{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \exists v_n\,\psi_n\right\} \Rightarrow \sigma}\\ \vdots\end{array}$$

which uses RULE 4, by the fragment

$$\begin{array}{c}\vdots\\ \dfrac{\dfrac{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \psi_n\binom{v_n}{\bar{c}_n}\right\} \Rightarrow \sigma}{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \psi_n\binom{v_n}{\bar{c}_n}, \exists v_n\,\psi_n\right\} \Rightarrow \sigma} \qquad \overline{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \exists v_n\,\psi_n\right\} \Rightarrow \psi_n\binom{v_n}{\bar{c}_n}}}{\Sigma' \cup \left\{\exists v_n\,\psi_n \rightarrow \psi_n\binom{v_n}{\bar{c}_n}, \exists v_n\,\psi_n\right\} \Rightarrow \sigma}\\ \vdots\end{array}$$

which instead uses RULES 0, 5 (as modified[2.45.1.2]), and 3.

After N steps, we have a proof π_N with the root sequent $\Sigma \cup \operatorname{im} W \Rightarrow \mathsf{F}$ in which there are no uses of RULE 4. Let I be the set of all sentences

$$\psi\binom{v}{\bar{c}} \rightarrow \exists v\,\psi$$

such that there exists in π_N a use

$$\begin{array}{c}\vdots\\ \dfrac{\Sigma' \cup \{\exists v\,\psi\} \Rightarrow \sigma}{\Sigma' \cup \left\{\psi\binom{v}{\bar{c}}\right\} \Rightarrow \sigma}\\ \vdots\end{array} \tag{10.9}$$

of RULE 7. Let π' be the result of adding I to the antecedent of every sequent in π_N. Then π' is a proof tree with root sequent $\Sigma \cup \operatorname{im} W \cup I \Rightarrow \mathsf{F}$. Note that (10.9) has become

$$\tag{10.10} \begin{array}{c} \vdots \\ \dfrac{\Sigma'' \cup \{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \exists v\,\psi\} \Rightarrow \sigma}{\Sigma'' \cup \left\{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \psi\binom{v}{\bar{c}}\right\} \Rightarrow \sigma} \\ \vdots \end{array}$$

where $\Sigma'' = \Sigma' \cup \left(I \backslash \left\{\psi\binom{v}{\bar{c}} \to \exists v\,\psi\right\}\right)$. We now eliminate each use of RULE 7 as we did for RULE 4, i.e., replace (10.10) by

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots \\ \Sigma'' \cup \left\{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \exists v\,\psi\right\} \Rightarrow \sigma\end{array}}{\Sigma'' \cup \left\{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \exists v\,\psi, \psi\binom{v}{\bar{c}}\right\} \Rightarrow \sigma} \qquad \Sigma'' \cup \left\{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \psi\binom{v}{\bar{c}}\right\} \Rightarrow \exists v\,\psi}{\Sigma'' \cup \{\psi\binom{v}{\bar{c}} \to \exists v\,\psi, \psi\binom{v}{\bar{c}}\} \Rightarrow \sigma} \\ \vdots \end{array}$$

to obtain a proof tree π'' with root sequent $\Sigma \cup \operatorname{im} W \cup I \Rightarrow \mathsf{F}$ that has no use of RULE 4 or 7. π'' is an **NDP**-proof tree, so $\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent. $\square^{10.8}$

10.4 Proof of (2.85)

[REFER TO P. 101.]

(10.11) **Theorem** $[\mathsf{S}^0]$ *Every formula is propositionally equivalent to a formula in disjunctive normal form and to a formula in conjunctive normal form.*

Proof We first note the duality of the two forms as mediated by negation. Suppose I is a nonempty finite set and for each $i \in I$, J_i is a nonempty finite set. Suppose for each $i \in I$ and $j \in J_i$, $\eta_{i,j}$ is a formula. Although we are specifically interested in the case that $\eta_{i,j}$ is a prime formula or the negation of a prime formula, the following claim is generally true.

(10.12) **Claim**

1. $\neg \bigwedge_{i\in I} \bigvee_{j\in J_i} \eta_{i,j}$ *is propositionally equivalent to* $\bigvee_{i\in I} \bigwedge_{j\in J_i} \neg\,\eta_{i,j}$.
2. $\neg \bigvee_{i\in I} \bigwedge_{j\in J_i} \eta_{i,j}$ *is propositionally equivalent to* $\bigwedge_{i\in I} \bigvee_{j\in J_i} \neg\,\eta_{i,j}$.

Proof Straightforward. $\square^{10.12}$

Next we note that there is a canonical way to convert from one form to the other. Let Γ be the set of choice functions for $\langle J_i \mid i \in I\rangle$, i.e., the set of functions f such that $\operatorname{dom} f = I$ and $\forall i \in I\ f(i) \in J_i$.

(10.13) **Claim**

1. $\bigwedge_{i\in I}\bigvee_{j\in J_i}\eta_{i,j}$ *is propositionally equivalent to* $\bigvee_{f\in\Gamma}\bigwedge_{i\in I}\eta_{i,f(i)}$.
2. $\bigvee_{i\in I}\bigwedge_{j\in J_i}\eta_{i,j}$ *is propositionally equivalent to* $\bigwedge_{f\in\Gamma}\bigvee_{i\in I}\eta_{i,f(i)}$.

Proof Both of these are generalizations of the distributive properties (of disjunction over conjunction and *vice versa*), and the straightforward proof is left to the reader. Note that either may be derived from the other using (10.12). $\square^{10.13}$

It is worth noting that although $\bigwedge_{i\in I}\bigvee_{j\in J_i}\eta_{i,j}$ and $\bigvee_{f\in\Gamma}\bigwedge_{i\in I}\eta_{i,f(i)}$ are formed from the same set $\{\eta_{i,j} \mid i\in I\wedge j\in J_i\}$ of formulas, the latter typically is much lengthier. In the case that the index set I is a (finite) ordinal and for some finite ordinal J, for each $i\in I$, $J_i = J$, the first expression has $I\times J$ "elements", whereas the latter has $J^I\times I$ elements. Note that if we first apply (10.13.1) to a conjunctive form and then apply (10.13.2) to the resulting disjunctive form, we obtain another conjunctive form—again, typically much lengthier than the original: $I^{J^I}\times J^I$ *vs* $I\times J$, in our example. It is a mildly entertaining exercise to show directly that the final form is equivalent to the original, i.e., that

$$\bigwedge_{l\in{}^{({}^{I}J)}I}\bigvee_{k\in{}^{I}J}\eta_{l(k),k(l(k))}$$

is propositionally equivalent to

$$\bigwedge_{i\in I}\bigvee_{j\in J}\eta_{i,j}.$$

[Hint: For any $l : {}^{I}J\to I$ there exists $i\in I$ such that for all $j\in J$ there exists $k : I\to J$ such that $l(k) = i$ and $k(i) = j$.]

We now note that any propositional expression is equivalent to one involving only negation, disjunction, and conjunction, as $\phi\to\psi$ is propositionally equivalent to $\neg\phi\vee\psi$.

Finally we observe that prime formulas are already in both normal forms. It is now straightforward to show by induction on complexity that any formula is propositionally equivalent to a formula in conjunctive and to a formula in disjunctive normal form. $\square^{10.11}$

10.5 Proof of (2.93)

[Refer to p. 104.]

(10.14) **Theorem** [S^0] *Suppose ρ is a signature with at least one constant, and σ is a purely universal ρ-sentence.*[2.88] *Let $\sigma = \forall v_0\cdots\forall v_{N^-}\mu$, where μ is quantifier-free. Then $\{\sigma\}$ is inconsistent iff there exists $M\in\omega$ and variable-free ρ-terms τ^m_n, $m\in M$, $n\in N$, such that*

$$\left\{\mu\begin{pmatrix} v_0 & \cdots & v_{N^-}\\ \tau^m_0 & \cdots & \tau^m_{N^-}\end{pmatrix}\,\middle|\, m\in M\right\}$$

is propositionally inconsistent.

Proof As a matter of convenience, we will work in a language with the universal but not the existential quantifier. Let ρ' be an expansion of ρ by the addition of countably infinitely many constants.[4] Unless otherwise specified, all expressions are ρ'-expressions. We make the following definitions specific to this proof.

[4] We are working in S^0, which does not have the $\mathsf{Infinity}$ axiom, so we do not presume the actual existence of ρ'; we merely allude to it as a concept in the formation of expressions like 'ρ'-expression'.

(10.15)

1. *A* witness sequence $\overset{\text{def}}{=}$ *a finite sequence W of ρ'-sentences of the form*

$$\left\langle \phi\binom{v_i}{\bar{c}_i} \to \forall v\, \phi \,\middle|\, i \in n \right\rangle,$$

where for each $i \in n$, c_i is a ρ'-constant that is not an index of the original signature ρ and does not occur previously in W.

2. *An instance set is a finite set of ρ'-sentences of the form $\forall v\, \phi \to \phi\binom{v}{\tau}$.*

3. *The* quantifier depth *of a formula ϕ $\overset{\text{def}}{=}$ $\operatorname{qd}\phi$ $\overset{\text{def}}{=}$ the number of quantifiers in it.*

4. *The* quantifier depth *of a finite sequence W of formulas $\overset{\text{def}}{=}$ $\operatorname{qd} W$ $\overset{\text{def}}{=}$ is the maximum quantifier depth of its items if $W \neq 0$; otherwise, $\operatorname{qd} W = 0$.*

5. *The* type *of a finite sequence W of formulas $\overset{\text{def}}{=}$ $\operatorname{tp} W$ $\overset{\text{def}}{=}$ the pair $\langle D, L\rangle$, where $D = \operatorname{qd} W$ and L is the number of items in W with quantifier depth D.*

6. *We order types so that*

$$\langle D, L\rangle \prec \langle D', L'\rangle \leftrightarrow \big(D < D' \vee (D = D' \wedge L < L')\big).$$

We will deal primarily with pairs of witness sequences and instance sets satisfying the following conditions.

(10.16)

1. *For every sentence $\forall v\, \phi \to \phi\binom{v}{\tau} \in I$, there is a sentence $\phi\binom{v}{\bar{c}} \to \forall v\, \phi \in \operatorname{im} W$, i.e., any sentence that occurs as an antecedent in I occurs as a consequent in W.*

2. *For $i, i' \in |W|$, if $i \neq i'$, $W(i) = \phi\binom{v}{\bar{c}} \to \forall v\, \phi$, and $W(i') = \phi'\binom{v'}{\bar{c}'} \to \forall v'\, \phi'$, then $\forall v\, \phi \neq \forall v'\, \phi'$, i.e., no sentence $\forall v\, \phi$ is witnessed more than once in W.*

(10.17) *For the nonce we make the following definitions.*

1. $\langle \Sigma, W, I\rangle$ *is* good $\overset{\text{def}}{\iff}$
   1. *Σ is a finite set of constant instances of μ (i.e., sentences of the form $\mu\binom{v_0\ \cdots\ v_{N\text{-}}}{\tau_0\ \cdots\ \tau_{N\text{-}}}$, where $\tau_0, \ldots, \tau_{N\text{-}}$ are variable-free ρ'-terms);*
   2. *W is a witness sequence;*
   3. *I is an instance set;*
   4. *$\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent; and*
   5. *(10.16.1) is satisfied.*

2. $\langle \Sigma, W, I\rangle$ *is* fine $\overset{\text{def}}{\iff}$ *it is good and (10.16.2) is satisfied.*

Note that the constants that occur as witnesses in W may also occur in Σ, i.e., we do not specify that W is a witness sequence for Σ.

(10.18) **Claim** *Suppose $\langle \Sigma, W, I\rangle$ is good. Then there exists a fine $\langle \Sigma', W', I'\rangle$ such that $\operatorname{qd} W' \preccurlyeq \operatorname{qd} W$.*

Proof Since W is finite, $|W|$ is a finite ordinal. Let $m = |W| + 1$, also a finite ordinal, so $(m; \in)$ is by definition a wellorder. Let $n \in m$ be least such that there exists a good $\langle \Sigma_0, W_0, I_0 \rangle$ such that $\operatorname{qd} W_0 \preccurlyeq \operatorname{qd} W$ and $|W_0| = n$, and let $\langle \Sigma_0, W_0, I_0 \rangle$ be good such that $\operatorname{qd} W_0 \preccurlyeq \operatorname{qd} W$ and $|W_0| = n$. We will show that $\langle \Sigma_0, W_0, I_0 \rangle$ is fine. To this end, suppose toward a contradiction that $i < i'$, $W_0(i) = \phi\binom{v}{\bar{c}} \to \forall v\, \phi$, and $W_0(i') = \phi\binom{v}{\bar{c}'} \to \forall v\, \phi$. For any formula θ, let θ' be the result of substituting $\bar{c}$ for every occurrence of $\bar{c}'$. Let Σ', W', and I' be the corresponding transformations of Σ_0, W_0, and I_0. By the definition[10.15.1] of *witness sequence*, c' does not occur in μ, so Σ' is a set of constant instances of μ.

It is easy to show that $\Sigma' \cup \operatorname{im} W' \cup I'$ is propositionally inconsistent. For suppose $\mathfrak{I}'$ is a propositional interpretation such that $\mathfrak{I}' \models \Sigma' \cup \operatorname{im} W' \cup I'$. Let $\mathfrak{I}$ be the propositional interpretation given by $\mathfrak{I}\theta = \mathfrak{I}'\theta'$ for every prime sentence θ such that $\theta' \in \operatorname{dom} \mathfrak{I}'$. Then $\mathfrak{I} \models \Sigma_0 \cup \operatorname{im} W_0 \cup I_0$, contrary to hypothesis. (10.16.1) is still satisfied, so $\langle \Sigma', W', I' \rangle$ is good. Also, $\operatorname{qd} W' = \operatorname{qd} W_0 \preccurlyeq \operatorname{qd} W$.

By design, $W'(i') = W'(i)$. Let W'' be the sequence that results from deleting the i'th item from W', which is now entirely superfluous. Then $\langle \Sigma', W'', I' \rangle$ is good, $\operatorname{qd} W'' = \operatorname{qd} W' \preccurlyeq \operatorname{qd} W$, and $|W''| < n$; contradiction. □[10.18]

We now turn to the proof of the theorem. The 'if' direction is trivial. We will prove the 'only if' direction. Suppose therefore that $\{\sigma\}$ is inconsistent. Thus there exist a finite witness sequence W for $\{\sigma\}$ in the original sense of Theorem 2.59, and a finite instance set I, both in the signature ρ', such that

$$\{\sigma\} \cup \operatorname{im} W \cup I \tag{10.19}$$

is propositionally inconsistent. We may clearly arrange that W be a witness sequence in the sense of (10.15.1).

Let $d_0, \ldots, d_{N^-}$ be ρ'-constants that do not occur in μ, W, or I, and let

$$W' = \left\langle \forall v_{n+1} \cdots \forall v_{N^-}\, \mu\binom{v_0 \cdots v_n}{\bar{d}_0 \cdots \bar{d}_n} \to \forall v_n \cdots \forall v_{N^-}\, \mu\binom{v_0 \cdots v_{n^-}}{\bar{d}_0 \cdots \bar{d}_{n^-}} \,\middle|\, n \in N \right\rangle {}^\frown W.$$

Then

$$\operatorname{im} W' \cup \left\{\mu\binom{v_0 \cdots v_{N^-}}{\bar{d}_0 \cdots \bar{d}_{N^-}}\right\} \vdash^{\mathsf{P}} \sigma,$$

so

$$\left\{\mu\binom{v_0 \cdots v_{N^-}}{\bar{d}_0 \cdots \bar{d}_{N^-}}\right\} \cup \operatorname{im} W' \cup I$$

is propositionally inconsistent.

For each sentence $\forall v\, \phi \to \phi\binom{v}{\tau} \in I$, if there is not already a sentence $\phi\binom{v}{\bar{c}} \to \forall v\, \phi \in \operatorname{im} W'$ then we add one at the end with a new constant. We do this sequentially until (10.16.1) is satisfied. Let W'' be the resulting witness sequence. Then

(10.20) $\left\langle \left\{\mu\binom{v_0 \cdots v_{N^-}}{\bar{d}_0 \cdots \bar{d}_{N^-}}\right\}, W'', I \right\rangle$ *is good.*

(10.21) **Claim** *There exists a finite set Σ of constant ρ'-instances of μ such that $\langle \Sigma, 0, 0 \rangle$ is good. Hence Σ is propositionally inconsistent.*

Proof Let $\langle D_0, L_0 \rangle$ be the $\preccurlyeq$-least type $\langle D, L \rangle$ such that there exists a good

$\langle \Sigma, W, I\rangle$ such that $\operatorname{tp} W = \langle D, L\rangle$.[5] It suffices to show that $\langle D_0, L_0\rangle = \langle 0, 0\rangle$.[6] Thus, suppose toward a contradiction that $\langle D_0, L_0\rangle \neq \langle 0, 0\rangle$.

By (10.18) there exists a fine $\langle \Sigma_0, W_0, I_0\rangle$ with $\operatorname{tp} W_0 = \langle D_0, L_0\rangle$. Let

$$W_0 = \left\langle \phi_i\tbinom{v_i}{\bar{c}_i} \to \forall v_i\, \phi_i \,\middle|\, i \in |W_0| \right\rangle,$$

and

(10.22) *let $i \in |W_0|$ be such that $\operatorname{qd}(\forall v_i\, \phi_i)$ is D_0 and $\operatorname{qd}(\forall v_{i'}\, \phi_{i'}) < D_0$ for every $i' \in |W_0|$ such that $i' > i$.*

Let W' be W_0 with the ith item deleted. Let T be the set of terms τ such that $\forall v_i\, \phi_i \to \phi_i\tbinom{v_i}{\tau} \in I_0$, let I' be I_0 with all those sentences deleted, and let

$$J = \left\{ \phi_i\tbinom{v_i}{\bar{c}_i} \to \phi\tbinom{v_i}{\tau} \,\middle|\, \tau \in T \right\}. \tag{10.23}$$

(10.24) **Claim** $\Sigma_0 \cup \operatorname{im} W' \cup I' \cup J$ *is propositionally inconsistent.*

Proof Suppose toward a contradiction that $\mathfrak{I}$ is a propositional interpretation and $\mathfrak{I} \models \Sigma_0 \cup \operatorname{im} W' \cup I' \cup J$. Since $\langle \Sigma_0, W_0, I_0\rangle$ is fine and $\operatorname{qd}(\forall v_i\, \phi_i)$ is maximal in $\operatorname{im} W_0$, $\forall v_i\, \phi_i$ does not occur in W'. Also, no member of I_0 has quantifier depth greater than D_0, so any occurrence of $\forall v_i\, \phi_i$ in I_0 is in a sentence of the form $\forall v_i\, \phi_i \to \phi_i\tbinom{v_i}{\tau}$, so $\forall v_i\, \phi_i$ does not occur in I' or J. Σ_0 contains only quantifier-free sentences, so $\forall v_i\, \phi_i$ does not occur in Σ_0, either. We may therefore suppose that $\forall v_i\, \phi_i \notin \operatorname{dom} \mathfrak{I}$. Since $\forall v_i\, \phi_i$ is prime[2.47.2] we may extend $\mathfrak{I}$ to an interpretation $\mathfrak{I}'$ such that $\forall v_i\, \phi_i \in \operatorname{dom} \mathfrak{I}$ and

$$\mathfrak{I}'\big(\forall v_i\, \phi_i\big) = \mathfrak{I}\big(\phi_i\tbinom{v_i}{\bar{c}_i}\big). \tag{10.25}$$

Obviously $\mathfrak{I}' \models \phi_i\tbinom{v_i}{\bar{c}_i} \to \forall v_i\, \phi_i$, so $\mathfrak{I}' \models \operatorname{im} W_0$.

Suppose $\tau \in T$. Then $\mathfrak{I} \models \phi_i\tbinom{v_i}{\bar{c}_i} \to \phi\tbinom{v_i}{\tau}$.[10.23] If $\mathfrak{I}' \models \forall v_i\, \phi_i$ then[10.25] $\mathfrak{I} \models \phi_i\tbinom{v_i}{\bar{c}_i}$, so[10.23] $\mathfrak{I} \models \phi_i\tbinom{v_i}{\tau}$, so $\mathfrak{I}' \models \phi_i\tbinom{v_i}{\tau}$, so $\mathfrak{I}' \models \forall v_i\, \phi_i \to \phi_i\tbinom{v_i}{\tau}$. Also, if $\mathfrak{I}' \not\models \forall v_i\, \phi_i$ then $\mathfrak{I}' \models \forall v_i\, \phi_i \to \phi_i\tbinom{v_i}{\tau}$. Thus, for any $\tau \in T$, $\mathfrak{I}' \models \forall v_i\, \phi_i \to \phi_i\tbinom{v_i}{\tau}$. Since these are just the sentences deleted from I_0 to make I', $\mathfrak{I}' \models I_0$.

Finally, $\mathfrak{I} \models \Sigma_0$, so $\mathfrak{I}' \models \Sigma_0$. Thus, $\mathfrak{I}' \models \Sigma_0 \cup \operatorname{im} W_0 \cup I_0$, contradicting the assumption that $\langle \Sigma_0, W_0, I_0\rangle$ is good. $\square$[10.24]

It follows from (10.24) that

$$\Sigma_0 \cup \operatorname{im} W' \cup I' \cup \left\{ \phi\tbinom{v_i}{\tau} \,\middle|\, \tau \in T \right\} \tag{10.26}$$

---

[5] That there exists a $\preccurlyeq$-least such type follows from the fact that $\preccurlyeq$ is a wellorder. In C^0 we would express this by saying that any nonempty class of types has a $\preccurlyeq$-least member. As we are working in S^0, we have to do a little more work. Let $D' = \operatorname{qd} W'' + 1$. D' is a finite ordinal and is therefore wellordered by $\in$. The set of $D \in D'$ such that there exists a good $\langle \Sigma_0, W_0, I_0\rangle$ such that $\operatorname{qd} W_0 = D$ is nonempty[10.20] and therefore has a least element, say D_0. Let $\langle \Sigma_0, W_0, I_0\rangle$ be good such that $\operatorname{qd} W_0 = D_0$. Similarly, let $L' = |W_0| + 1$, and let L_0 be the least $L \in L'$ such that there exists a good $\langle \Sigma_1, W_1, I_1\rangle$ such that $\operatorname{qd} W_1 = D_0$ and $|W_1| = L$. $\langle D_0, L_0\rangle$ is as desired.

[6] Since every item in a witness sequence has quantifier depth at least 1, if $\operatorname{qd} W = 0$ then $W = 0$; and if $\langle \Sigma, 0, I\rangle$ is good then[10.16.1] $I = 0$.

$$\Sigma_0 \cup \operatorname{im} W' \cup I' \cup \left\{ \neg\, \phi_i \binom{v_i}{\bar{c}_i} \right\} \tag{10.27}$$

are both propositionally inconsistent.

Let $W'_0 = W_0 \restriction i$ ($= W' \restriction i$), and let W'_1 be such that $W' = W'_0 {}^\frown W'_1$ (so $W'_1(k) = W'(i+k) = W_0(i+k+1)$). Let c^τ_j be a new constant for each $\tau \in T$ and $j \in |W_0|$ with $j > i$. For any sentence θ let

$$\theta^\tau = \theta\begin{pmatrix} \bar{c}_i & \bar{c}_{i+1} & \cdots & \bar{c}_{|W_0|^-} \\ \tau & \bar{c}^\tau_{i+1} & \cdots & \bar{c}^\tau_{|W_0|^-} \end{pmatrix}.$$

Let

$$\begin{aligned} \Sigma^\tau &= \{\theta^\tau \mid \theta \in \Sigma_0\} \\ I^\tau &= \{\theta^\tau \mid \theta \in I'\} \\ W^\tau &= \langle W'_1(k)^\tau \mid k \in |W'_1| \rangle. \end{aligned}$$

Note that for $j \geqslant i$, c_j does not occur in W'_0, so W'_0 is unaltered by these transformations, and

$$\{\theta^\tau \mid \theta \in \operatorname{im} W'\} = \operatorname{im} W'_0 \cup \operatorname{im} W^\tau.$$

By the definition of *witness sequence*, none of the constants c_i, $i \in |W_0|$, is in ρ, so Σ^τ is a finite set of constant instances of μ.

Clearly,[10.27] for each $\tau \in T$,

$$\Sigma^\tau \cup \operatorname{im} W'_0 \cup \operatorname{im} W^\tau \cup I^\tau \cup \left\{ \neg\, \phi_i \binom{v_i}{\tau} \right\}$$

is propositionally inconsistent. It follows[10.26] that

$$\Sigma_0 \cup \operatorname{im} W' \cup I' \cup \bigcup_{\tau \in T} \left(\Sigma^\tau \cup \operatorname{im} W^\tau \cup I^\tau \right)$$

is propositionally inconsistent.

Let

$$\begin{aligned} \Sigma_1 &= \Sigma_0 \cup \bigcup_{\tau \in T} \Sigma^\tau \\ I_1 &= I' \cup \bigcup_{\tau \in T} I^\tau. \end{aligned}$$

Let $\langle \tau_m \mid m \in M \rangle$ enumerate T, and let

$$W_1 = W' {}^\frown W^{\tau_0} {}^\frown \ldots {}^\frown W^{\tau_{M^-}}$$

Then

$$\Sigma_1 \cup \operatorname{im} W_1 \cup I_1$$

is propositionally inconsistent, and it is easy to check that $\langle \Sigma_1, W_1, I_1 \rangle$ is good. Let $\langle D_1, L_1 \rangle$ be the type of W_1. By our choice[10.22] of i every sentence in W^τ has quantifier depth $< D_0$. Since we have eliminated $\forall v_i\, \phi_i$ from W_0 to obtain W', either $D_1 = D_0 \wedge L_1 < L_0$ or $D_1 < D_0$. Either way, $\langle D_1, L_1 \rangle \prec \langle D_0, L_0 \rangle$; contradiction. □[10.21]

Let[10.21] Σ be a propositionally inconsistent finite set of constant ρ'-instances of μ. Let[10.14] c be a ρ-constant, and let Σ' be obtained from Σ by substituting $\bar{c}$ for every occurrence in Σ of a ρ'-constant that is not in ρ. Then Σ is a propositionally inconsistent finite set of constant ρ-instances of μ. □[10.14]

10.6 Proof of (2.99)

[Refer to p. 107.]

(10.28) **Theorem** [S^0] *Suppose Θ is a set of sentences, ψ is a formula, $\langle v, v_0, \ldots, v_{n^-}\rangle$ is an enumeration of* Free ψ*, and $\bar{v}_m$ is free for v in ψ for all $m \in n$. Let F be an n-ary operation index that does not appear in Θ or ψ. Suppose Θ is consistent. Then*

$$\Theta \cup \left\{\forall v_0 \cdots \forall v_{n^-} \left(\exists v\, \psi \to \psi\left(\substack{v \\ \tilde{F}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}\rangle}\right)\right)\right\}$$

is consistent.

By way of preparation, we prove the following lemma, which is central to the finitary treatment of skolemization.

(10.29) **Theorem** [S^0] *Suppose Θ is a consistent set of prenex sentences. Let Θ' be a standard skolemization*[2.97.4] *of Θ. Then Θ' is consistent.*

Proof It clearly suffices to prove this for finite Θ. For each $\theta \in \Theta$, let θ' be the skolemization of θ, so that $\Theta' = \{\theta' \mid \theta \in \Theta\}$. We may suppose without loss of generality that there exist variables v_m, u_m ($m \in \omega$) such that for each $\theta \in \Theta$, v_m is the mth universally quantified and u_m the mth existentially quantified variable in θ. (Here and throughout we start the numbering at 0, so the 0th is the first, the 1th is the second, etc.)

1. Let E^θ be the number of existential quantifiers in θ.

2. For each $e < E^\theta$, let A^θ_e be the number of universal quantifiers preceding the eth existential quantifier, and let $A^\theta_{E^\theta}$ be the total number of universal quantifiers. Let $A^\theta = A^\theta_{E^\theta}$.

3. Let $Q^\theta = E^\theta + A^\theta$, the total number of quantifiers in θ.

4. For $q < Q^\theta$, let w^θ_q be the qth quantified variable in θ. Thus $w^\theta_q = u_e$ iff $q = A^\theta_e + e$; otherwise, $w^\theta_q = v_a$ for some $a < A^\theta$.

5. For $q < Q^\theta$, let $\mathsf{Q}^\theta_q = \exists$ or $\forall$ according as w_q is u_e or v_a.

Let μ^θ be the matrix of θ. Let F^θ_e be the operation index skolemizing the eth existential quantifier in the formation of θ', so F^θ_e is A^θ_e-ary, and the matrix $\mu^{\theta'}$ of θ' is obtained from μ^θ by substituting

$$\tilde{F}^\theta_e\langle \bar{v}_0, \ldots, \bar{v}_{A^\theta_e{}^-}\rangle$$

for u_e, for each $e < E^\theta$. Note that, letting $A = A^\theta$, $Q = Q^\theta$, and $w_q = w^\theta_q$,

$$\theta = \mathsf{Q}^\theta_0 w_0 \cdots \mathsf{Q}^\theta_{Q^-} w_{Q^-}\ \mu^\theta$$
$$\theta' = \forall v_0 \cdots \forall v_{A^-}\ \mu^{\theta'}.$$

(10.30) *Suppose toward a contradiction that Θ' is inconsistent.*

Let ρ' be $\rho^{\Theta'}$ expanded by the addition of as many new constants (nulary operation indices) as required by the following construction. In particular, we suppose there is at least one ρ'-constant. By Herbrand's theorem[2.94] there exists a finite set $S = \bigcup_{\theta\in\Theta} S^\theta$ of ρ'-sentences, where each S^θ is finite set of constant instances of $\mu^{\theta'}$, such that S is propositionally inconsistent. Suppose $\sigma \in S^\theta$. Then

$$\sigma = \mu'\Bigl(\begin{smallmatrix} v_0 & \cdots & v_{A^-} \\ \tau_0^\sigma & \cdots & \tau_{A^-}^\sigma \end{smallmatrix}\Bigr) = \mu\Bigl(\begin{smallmatrix} w_0 & \cdots & w_{Q^-} \\ \eta_0^\sigma & \cdots & \eta_{Q^-}^\sigma \end{smallmatrix}\Bigr),$$

where

1. $\mu' = \mu^{\theta'}$, $\mu = \mu^\theta$;
2. $A = A^\theta$, $E = E^\theta$, $Q = Q^\theta$;
3. $\tau_0^\sigma, \ldots, \tau_{A^-}^\sigma$ are ρ'-terms; and
4. for each $q < Q$,
    1. if $w_q^\theta = u_e$ then $\eta_q^\sigma = \tilde{F}_e^\theta\langle\tau_0^\sigma, \ldots, \tau_{A_e^-}^\sigma\rangle$;
    2. if $w_q^\theta = v_a$ then $\eta_q^\sigma = \tau_a^\sigma$.

For each $q \leqslant Q^\theta$, let

$$\sigma_q = \mathsf{Q}_q^\theta w_q^\theta \cdots \mathsf{Q}_{Q^{\theta-}}^\theta w_{Q^{\theta-}}^\theta\ \mu\bigl(\begin{smallmatrix} w_0^\theta & \cdots & w_{q^-}^\theta \\ \eta_0^\sigma & \cdots & \eta_{q^-}^\sigma \end{smallmatrix}\bigr). \tag{10.31}$$

Note that $\sigma_0 = \theta$, $\sigma_{Q^\theta} = \sigma$, and

$$\{\theta\} \cup \{\sigma_q \to \sigma_{q+1} \mid q < Q^\theta\} \vdash^{\mathsf{P}} \sigma.$$

Hence,

$$\Theta \cup \bigcup_{\theta\in\Theta} \bigcup_{\sigma\in S^\theta} \{\sigma_q \to \sigma_{q+1} \mid q < Q^\theta\} \tag{10.32}$$

is propositionally inconsistent.

We will call $\sigma_q \to \sigma_{q+1}$ an *instantiation*. If $w_q^\theta = u_e$ for some $e < E^\theta$ then $\sigma_q \to \sigma_{q+1}$ is an *existential instantiation*, and it has the form of a witness sentence in the sense of (2.57.1), where the witness is the term $\eta_q^\sigma = \tilde{F}_e^\theta\langle\tau_0^\sigma, \ldots, \tau_{A_e^-}^\sigma\rangle$; if $w_q^\theta = v_a$ for some $a < A^\theta$ then $\sigma_q \to \sigma_{q+1}$ is a *universal instantiation*, and it is an instance sentence in the sense of (2.57.2), where the instantiation is again to η_q^σ, which in this case is τ_a^σ.

We now wish to convert the above set of sentences[10.32] to a proof

$$\Theta \cup \operatorname{im} W \cup I$$

of the inconsistency of Θ in the sense of (2.60), where W is a witness sequence for Θ and I is an instance set, both in the signature ρ^Θ extended by the addition of constants. Thus, we must eliminate all the Skolem indices F_e^θ, and we must order the existential instantiations so as to form a witness sequence.[2.57.1]

Suppose τ is the witness for an existential instantiation[10.32] $\sigma_q \to \sigma_{q+1}$, i.e., letting $\theta \in \Theta$ and $e < E^\theta$ be such that $\sigma \in S^\theta$ and $w_q^\theta = u_e$,

$$\tau = \eta_q^\sigma = \tilde{F}_e^\theta\langle\tau_0^\sigma, \ldots, \tau_{A_e^{\theta-}}^\sigma\rangle.$$

Note that θ and e, and therefore also q, are uniquely determined by τ, as are $\tau^\sigma_0, \dots, \tau^\sigma_{A_e\text{-}}$.

Thus, if τ is also the witness for $\sigma'_{q'} \to \sigma'_{q'+1}$, then $\sigma' \in S^\theta$, $q' = q$, and

$$\langle \tau^{\sigma'}_0, \dots, \tau^{\sigma'}_{A^\theta_e\text{-}} \rangle = \langle \tau^\sigma_0, \dots, \tau^\sigma_{A^\theta_e\text{-}} \rangle.$$

For each $\bar{e} < e$, letting $\bar{q}$ be such that $w^\theta_{\bar{q}} = u_{\bar{e}}$,

$$\eta^{\sigma'}_{\bar{q}} = \tilde{F}^\theta_{\bar{e}} \langle \tau^\sigma_0, \dots, \tau^\sigma_{A^\theta_{\bar{e}}\text{-}} \rangle = \eta^\sigma_{\bar{q}}.$$

Hence

$$\langle \eta^{\sigma'}_0, \dots, \eta^{\sigma'}_q \rangle = \langle \eta^\sigma_0, \dots, \eta^\sigma_q \rangle,$$

so[10.31] $\sigma'_{\bar{q}} = \sigma_{\bar{q}}$ for all $\bar{q} \leqslant q+1$. In particular, since $q' = q$,

$$\sigma'_{q'} \to \sigma'_{q'+1} = \sigma_q \to \sigma_{q+1}.$$

Thus, each Skolem witness term occurs in just one existential instantiation, although that instantiation may listed more than once in (10.32).

We use this fact to put the existential instantiations in a satisfactory order. Define the *Skolem rank* of a ρ'-expression $\epsilon \stackrel{\text{def}}{=} \operatorname{skr} \epsilon$ recursively by stipulating that $\operatorname{skr} \epsilon$ is the supremum of the Skolem ranks of the proper subexpressions of ϵ unless $\epsilon = \tilde{F}\langle \tau_0, \dots, \tau_{n\text{-}} \rangle$, where F is a Skolem operation index, in which case

$$\operatorname{skr} \tilde{F}\langle \tau_0, \dots, \tau_{n\text{-}} \rangle = \sup\{\operatorname{skr} \tau_0, \dots, \operatorname{skr} \tau_{n\text{-}}\} + 1.$$

In particular, if v is a variable then $\operatorname{skr} \bar{v} = 0$, since it has no proper subexpressions.

Thus, for any $\sigma \in S^\theta$ and $e < E^\theta$, letting q be such that $w^\theta_q = u_e$,

$$\operatorname{skr} \eta^\sigma_q = \sup\{\operatorname{skr} \tau^\sigma_a \mid a < A^\theta_e\} + 1.$$

so for all $q' < q$,

1. if $\exists e' < e \; w^\theta_{q'} = u_{e'}$ then $\operatorname{skr} \eta^\sigma_{q'} \leqslant \operatorname{skr} \eta^\sigma_q$;
2. otherwise, $\operatorname{skr} \eta^\sigma_{q'} < \operatorname{skr} \eta^\sigma_q$.

Let $\prec$ be a linear ordering of the Skolem witnesses η^σ_q such that $\eta^{\sigma'}_{q'} \prec \eta^\sigma_q$ if either

(10.33)

1. $\operatorname{skr} \eta^{\sigma'}_{q'} < \operatorname{skr} \eta^\sigma_q$, *or*
2. $\operatorname{skr} \eta^{\sigma'}_{q'} = \operatorname{skr} \eta^\sigma_q$ *and* $q' < q$.

It is not necessary to specify $\prec$ beyond this. (10.33.2) is unambiguous because, as noted above, if $\eta^{\sigma'}_{q'} = \eta^\sigma_q$ then $q' = q$.

We may therefore define W^0 to be the enumeration of the set

$$\bigcup_{\theta \in \Theta} \bigcup_{\sigma \in S^\theta} \{\sigma_q \to \sigma_{q+1} \mid \exists e < E^\theta \; w^\theta_q = u_e\}$$

of existential instantiations according to the $\prec$-order of the Skolem terms η^σ_q.

By the preceding remarks, the Skolem rank of any $\sigma_q \to \sigma_{q+1} \in \operatorname{im} W^0$ is $\operatorname{skr} \eta^\sigma_q$, and apart from terms $\eta^\sigma_{q'}$ with $q' < q$, which may have the same Skolem rank as

η_q^σ, every other Skolem term occurring as a subexpression of $\sigma_q \rightarrow \sigma_{q+1}$ has lower Skolem rank. Thus, the first occurrence of η_q^σ as a subexpression of an item in W^0 is as the witness for the existential instantiation $\sigma_q \rightarrow \sigma_{q+1}$.

Let I^0 be the set

$$\bigcup_{\theta \in \Theta} \bigcup_{\sigma \in S^\theta} \{\sigma_q \rightarrow \sigma_{q+1} \mid \exists a < A^\theta \; w_q^\theta = v_a\}$$

of universal instantiations.

We now proceed to the elimination of Skolem operations. For each Skolem witness τ let c_τ be a distinct new constant. Let σ^1 and σ_q^1 derive from σ and σ_q by the substitution of c_τ for every occurrence of τ (as a subexpression) for every Skolem witness τ of maximum Skolem rank. Let W^1 and I^1 be the corresponding transformations of W^0 and I^0. Since no Skolem witness of maximum Skolem rank is a subterm of any other Skolem witness, this is a well defined substitution. Propositional relationships are not affected by this substitution, so

$$\Theta \cup \operatorname{im} W^1 \cup I^1$$

is propositionally inconsistent.

By the preceding remarks, for each constant c_τ introduced at this stage, the first occurrence of c_τ in W^1 is as the Skolem witness in a sentence $\sigma_q^1 \rightarrow \sigma_{q+1}^1$, which is

$$\exists u_e \, \psi \rightarrow \psi\binom{u_e}{c_\tau},$$

where $w_q^\theta = u_e$,

$$\psi = \mathsf{Q}_{q+1}^\theta w_{q+1}^\theta \cdots \mathsf{Q}_{Q^{\theta}\text{-}}^\theta w_{Q^{\theta}\text{-}}^\theta \; \mu\begin{pmatrix} w_0^\theta & \cdots & w_{q\text{-}}^\theta \\ \eta_0^\sigma & \cdots & \eta_{q\text{-}}^\sigma \end{pmatrix},$$

and $\tau = \eta_q^\sigma$.

We now define σ^2, σ_q^2, W^2, and I^2 by substitution of c_τ for τ for all remaining Skolem terms τ of highest rank. For each constant c_τ introduced so far, the first occurrence of c_τ in W^2 is in a sentence of the form

$$\exists u_e \, \psi \rightarrow \psi\binom{u_e}{c_\tau},$$

for appropriate e and ψ.

Continue in this fashion until all Skolem witnesses have been replaced by new constants. Let W^n and I^n be the final versions of W^0 and I^0. $\operatorname{im} W^n$ consists entirely of sentences of the form

$$\exists u_e \, \psi \rightarrow \psi\binom{u_e}{c_\tau},$$

for appropriate e and ψ, and c_τ does not occur in $\exists u_e \, \psi$ or in any item that occurs earlier in W^n. Thus, W^n is a witness sequence for Θ, I^n is an instance set, and

$$\Theta \cup \operatorname{im} W^n \cup I^n$$

is propositionally inconsistent.

Now eliminate any remaining occurrences of Skolem operation indices by successive substitutions of new constants for terms involving these indices, transforming W^n to a sequence W and I^n to a set I. Clearly, W is a witness sequence for Θ, I is an instance set, and

$$\Theta \cup \operatorname{im} W \cup I$$

is propositionally inconsistent. Hence,[2.59] Θ is inconsistent; contradiction. $\square$[10.29]

(10.34) **Theorem** [S^0] *Suppose* Θ *is a set of sentences,* ψ *is a formula,* $\langle v, v_0, \dots, v_{n^-}\rangle$ *is an enumeration of* $\operatorname{Free}\psi$*, and* $\bar{v}_m$ *is free for* v *in* ψ *for all* $m \in n$*. Let* F *be an* n*-ary operation index that does not appear in* Θ *or* ψ*. If*

$$\Theta \cup \{\forall v_0 \cdots \forall v_{n^-}\, \exists v\, \psi\}$$

is consistent then

$$\Theta \cup \left\{\forall v_0 \cdots \forall v_{n^-}\, \psi\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right)\right\} \tag{10.35}$$

is consistent.

Proof Let ψ' be a prenexification of ψ, and let Θ' be a prenexification of Θ. It is enough to show that if

$$\Theta' \cup \{\forall v_0 \cdots \forall v_{n^-}\, \exists v\, \psi'\}$$

is consistent then

$$\Theta' \cup \left\{\forall v_0 \cdots \forall v_{n^-}\, \psi'\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right)\right\}$$

is consistent.

Suppose

$$\Theta' \cup \{\forall v_0 \cdots \forall v_{n^-}\, \exists v\, \psi'\}$$

is consistent. Let σ be a standard skolemization of

$$\forall v_0 \cdots \forall v_{n^-}\, \psi'\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right),$$

and let Θ'' be a standard skolemization of Θ'. Then σ is a standard skolemization of $\forall v_0 \cdots \forall v_{n^-}\, \exists v\, \psi'$, so[10.29]

$$\Theta'' \cup \{\sigma\}$$

is consistent. Trivially,

$$\Theta'' \vdash \Theta'$$

and

$$\{\sigma\} \vdash \forall v_0 \cdots \forall v_{n^-}\, \psi'\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right),$$

so

$$\Theta' \cup \left\{\forall v_0 \cdots \forall v_{n^-}\, \psi'\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right)\right\}$$

is consistent, as claimed. □[10.34]

Proof of (10.28) Suppose Θ is consistent. Let w be a new variable, and let

$$\phi = \exists v\, \psi \rightarrow \psi\left(\begin{smallmatrix} v \\ w\end{smallmatrix}\right).$$

$\forall v_0 \cdots \forall v_{n^-}\, \exists w\, \phi$ is a validity, so

$$\Theta \cup \{\forall v_0 \cdots \forall v_{n^-}\, \exists w\, \phi\}$$

is consistent. Hence,[10.34]

$$\Theta \cup \left\{\forall v_0 \cdots \forall v_{n^-}\, \phi\left(\begin{smallmatrix} w \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right)\right\}$$

is consistent. Since

$$\phi\left(\begin{smallmatrix} w \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right) = \exists v\, \psi \rightarrow \psi\left(\begin{smallmatrix} v \\ \tilde{F}\langle \bar{v}_0, \dots, \bar{v}_{n^-}\rangle\end{smallmatrix}\right),$$

the point is proved. □[10.28]

10.7 Proof of (2.106)

[Refer to p. 109.]

(10.36) **Theorem** [S^0] *If Θ is a ρ-theory and Θ is consistent then $\Theta \cup \Theta^{\rho,=}$ is consistent.*

Proof Suppose Θ is a ρ-theory and $\Theta \cup \Theta^{\rho,=}$ is inconsistent. We will show that Θ is inconsistent.

(10.37) *Since consistency is preserved under expansions and contractions of signature, we may suppose that ρ has at least one constant.*

Using compactness, let $\Sigma \subseteq \Theta$ and $\Sigma^= \subseteq \Theta^{\rho,=}$ be finite such that $\Sigma \cup \Sigma^=$ is inconsistent. Let $\sigma = \bigwedge \Sigma$, and let σ' be a skolemization[2.97] of σ with operation indices that are not in ρ. Let ρ' be the expansion of ρ with these new indices, and let $\rho'^=$ be ρ' with the addition of 0 as a binary predicate index. Then σ' is universal, $\vdash \sigma' \rightarrow \sigma$, and $\{\sigma'\} \cup \Sigma^=$ is inconsistent.[2.100] Let

$$\sigma' = \forall u_0 \cdots \forall u_{J^-} \mu,$$

where μ is quantifier-free.

Note that $\Sigma^=$ also consists of universal sentences. By Herbrand's theorem[2.94]there are finite sets $\mathcal{M}$, $\mathcal{K}$, and $\mathcal{L}$, of variable-free sentences such that

(10.38)

1. *if $\theta \in \mathcal{M}$ then θ is an instance of μ;*
2. *if $\theta \in \mathcal{K}$ then*[2.105.1]

$$\theta = \left(\bigwedge_{m\in n} \tau_m = \tau'_m \rightarrow \tilde{F}\langle \tau_0, \ldots, \tau_{n^-}\rangle = \tilde{F}\langle \tau'_0, \ldots, \tau'_{n^-}\rangle\right),$$

   for some $n \in \omega$, n-ary ρ'-operation index F, and variable-free terms $\tau_0, \ldots, \tau_{n^-}, \tau'_0, \ldots \tau'_{n^-}$;
3. *if $\theta \in \mathcal{L}$ then*[2.105.2]

$$\theta = \left(\bigwedge_{m\in n} \tau_m = \tau'_m \rightarrow (\tilde{P}\langle \tau_0, \ldots, \tau_{n^-}\rangle \leftrightarrow \tilde{P}\langle \tau'_0, \ldots, \tau'_{n^-}\rangle)\right),$$

   for some $n \in \omega$, n-ary $\rho^=$-predicate index P, and variable-free terms $\tau_0, \ldots, \tau_{n^-}, \tau'_0, \ldots \tau'_{n^-}$; and
4. *$\mathcal{M} \cup \mathcal{K} \cup \mathcal{L}$ is propositionally inconsistent.*

(10.39) **Claim** *$\mathcal{M}$ is propositionally inconsistent.*

Proof Suppose toward a contradiction that $\mathfrak{I}$ is an interpretation whose domain is the set of prime expressions of $\mathcal{M}$, such that $\mathfrak{I} \models \mathcal{M}$. We wish to extend $\mathfrak{I}$ to an interpretation $\mathfrak{I}'$ such that $\mathfrak{I}' \models \mathcal{M} \cup \mathcal{K} \cup \mathcal{L}$. To this end we first extend $\mathfrak{I}$ if necessary to cover all the ρ'-sentences that occur in in $\mathcal{K} \cup \mathcal{L}$[2.48] by assigning arbitrary truth values to any new prime sentences $\tilde{P}\langle \tau_0, \ldots, \tau_{n^-}\rangle$. We then cover all the $\rho'^{=}$ sentences in $\mathcal{K} \cup \mathcal{L}$ by setting $\mathfrak{I}'(\tau = \tau') = 1$ iff $\tau = \tau'$.[7] It is easy to check that $\mathfrak{I}' \models \mathcal{K} \cup \mathcal{L}$. $\square$[10.39]

It follows from (10.39) that $\{\sigma'\}$ is inconsistent, so $\{\sigma\}$ is inconsistent,[2.100] so Σ is inconsistent, so Θ is inconsistent, as claimed. $\square$[10.36]

10.8 Proof of (2.125)

[Refer to p. 124.]

(10.40) **Theorem** [PG^2] $\mathsf{DP} \to \mathsf{DP}^*$.

Proof Suppose A, B, C, a, b, c and A', B', C', a', b', c' are triangles perspective from a line d. Let A'', B'', C'' be points on d such that A'' is on a and a', B'' is on b and b', and C'' is on c and c'.

We first deal with degenerate cases. Suppose $A = A'$. Then the triangles are perspective from any point collinear with B, B' and with C, C' (of which there is at least one). Thus,

we assume that $A \neq A'$, and likewise $B \neq B'$ and $C \neq C'$.

Now suppose $a = a'$. Then B, B', C, C' are on a, so the triangles are perspective from any point on (A, A') and a. Thus,

we assume that $a \neq a'$, and likewise $b \neq b'$ and $c \neq c'$.

(10.41) *Suppose B'' is on (A, A').*

If B'' is neither A nor A' then $(A, B'') = (A', B'') = (A, A')$. Since B'' is on (A, C) and (A', C'), C and C' are on (A, A'), so the triangles are perspective from any point on (A, A') and (B, B'). Thus, we may suppose that B'' is either A or A', and

we assume without loss of generality that $B'' = A'$.

Since B'' is on (A, C), C is on (A, A'). By hypothesis, B'' is on the axis of perspectivity d, so A' is on d. A'' cannot be A' because by hypothesis A'' is collinear with B', C', so $d = (A', A'')$. Also C'' cannot be A', because C'' is on (A, B), so in that case, A' would be on (A, B) as well as on (A, C), but A is the only point on both (A, B) and (A, C). Since C'' is on (A', B'), $d = (A', B')$. Hence, A'' is on (A', B'). A'' is also on (B', C') by hypothesis. Since A', B', C' are not collinear, $A'' = B'$. Thus, B' is on (B, C). Hence, the triangles are perspective from C.

We therefore assume B'' is not on (A, A'),[10.41] *and we likewise assume A'' is not on (B, B').*

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[7]The sentences (10.38.2) would be satisfied by a variety of interpretations, e.g., making $\mathfrak{I}'(\tau = \tau')$ always 1 or always 0. The particular definition we have made is mandated by the necessity of satisfying the sentences[10.38.3]

$$(\tau_0 = \tau_0' \wedge \tau_1 = \tau_1') \to (\tau_0 = \tau_1 \leftrightarrow \tau_0' = \tau_1'),$$

which it clearly does.

Thus, $AA'B''$ and $BB'A''$ are triangles, which are perspective from C'', since C'' is collinear with A, B, with A', B', and with A'', B''. By DP, they are perspective from a line d'. C is collinear with B'', A and with A'', B, and it is the only such point; otherwise, C would be collinear with A and B. Similarly, C' is the only point collinear with B'', A' and A'', B'. Hence, $d' = (C, C')$. Thus, there exists a point D on (C, C') that is on (A, A') and (B, B'). ABC and $A'B'C'$ are perspective from D. $\square^{10.40}$

10.9 Proof of (2.147)

[Refer to p. 144.]

(10.42) **Theorem** [C^0] *Suppose $I = (\Gamma \Rightarrow \Delta)$ is a sequent in a signature ρ without identity that is not $\mathbf{LK}^-$-provable. Then there exists a ρ-structure $\mathfrak{S}$, an $\mathfrak{S}$-assignment A to the free variables of I, and a subvaluation S for $\mathfrak{S}$ such that for each $\phi \in \Gamma \cup \Delta$,*

1. $\langle \phi, A \rangle \in \operatorname{dom} S$;
2. $\phi \in \Gamma \rightarrow S\langle \phi, A \rangle = 1$; *and*
3. $\phi \in \Delta \rightarrow S\langle \phi, A \rangle = 0$.

Proof We will construct a sequence $\langle I_n \mid n \in \omega \rangle$, where $I_n = (\Gamma_n \Rightarrow \Delta_n)$ with the following properties.

(10.43)

1. $I_0 = I$.
2. *For all* $n \in \omega$, $\Gamma_n \subseteq \Gamma_{n+1}$ *and* $\Delta_n \subseteq \Delta_{n+1}$.
3. *For all* $n \in \omega$, I_n *is not* $\mathbf{LK}^-$*-provable. Note that this implies that* $\Gamma_n \cap \Delta_n = 0$.

Ultimately, we will use $\bigcup_{n \in \omega} \Gamma_n$ and $\bigcup_{n \in \omega} \Delta_n$ to define a structure $\mathfrak{S}$, assignment A, and subvaluation function S as required by the theorem. The resemblance to the Henkin construction will be obvious.

We will work with a fixed signature ρ without identity.

(10.44)

1. *As a convenience, we suppose that no variable occurs both free and bound in* $\Gamma \cup \Delta$. *This can be achieved without loss of generality by a change of variables.*
2. *Let V be a countably infinite class of variables that do not occur in a quantifier phrase in $\Gamma \cup \Delta$, and let $<^V$ be an enumeration of V.*
3. *Let T be the class of terms τ such that* $\operatorname{Free} \tau \subseteq V$, *and let $<^T$ be an enumeration of T.*

We will maintain a "to do" list, which is, for each $n \in \omega$, a subset L_n of $\Gamma_n \cup \Delta_n$ together with a linear ordering $<^L_n$ of L_n. At the outset, $L_0 = \Gamma_0 \cup \Delta_0$ and $<^L_0$ is an arbitrary linear ordering of L_0.

Now suppose we have $\Gamma_n \Rightarrow \Delta_n$, which is an $\mathbf{LK}^-$-unprovable sequent. Let ϕ be the $<^L_n$-first formula in the list L_n.

(10.45) *The next step depends on the structure of ϕ.*

1. *Suppose $\phi = \neg\,\psi$.*
   1. *If $\phi \in \Gamma_n$ let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{\psi\}$. Note that $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable.*[2.143.3]
   2. *If $\phi \in \Delta_n$ let $\Gamma_{n+1} = \Gamma_n \cup \{\psi\}$ and $\Delta_{n+1} = \Delta_n$. Note that $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable.*[2.143.4]
   3. *Let $L_{n+1} = L_n \backslash \{\phi\} \cup \{\psi\}$, and define $<^L_{n+1}$ by removing ϕ from the head of the list $<^L_n$ and adding ψ at the tail.*
2. *Suppose $\phi = \psi_0 \rightarrow \psi_1$.*
   1. *Suppose $\phi \in \Gamma_n$. Then either $\Gamma_n \Rightarrow \Delta_n \cup \{\psi_0\}$ is unprovable or $\Gamma_n \cup \{\psi_1\} \Rightarrow \Delta_n$ is unprovable.*[2.143.5] *If the former, let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{\psi_0\}$; otherwise let $\Gamma_{n+1} = \Gamma_n \cup \{\psi_1\}$ and $\Delta_{n+1} = \Delta_n$. In either case $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable. To obtain L_{n+1} from L_n, remove ϕ and add ψ_0 or ψ_1, respectively, and define $<^L_{n+1}$ from $<^L_n$ by removing ϕ from the head of the list and adding ψ_0 or ψ_1, respectively, at the tail.*
   2. *If $\phi \in \Delta_n$ let $\Gamma_{n+1} = \Gamma_n \cup \{\psi_0\}$ and $\Delta_{n+1} = \Delta_n \cup \{\psi_1\}$. Note that $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable.*[2.143.6] *To obtain L_{n+1} from L_n, remove ϕ and add ψ_0 and ψ_1, and define $<^L_{n+1}$ from $<^L_n$ by removing ϕ from the head of the list and adding ψ_0, ψ_1 at the tail (in either order).*
3. *Suppose $\phi = \exists v\,\psi$.*
   1. *If $\phi \in \Gamma_n$ let u be the $<^V$-first variable in V that does not occur free in $\Gamma_n \cup \Delta_n$, and let $\Gamma_{n+1} = \Gamma_n \cup \left\{\psi\binom{v}{\bar{u}}\right\}$ and $\Delta_{n+1} = \Delta_n$. By virtue of (10.44.2) u does not occur in $\Gamma_n \Rightarrow \Delta_n$, and $\bar{u}$ is free for v in ψ; hence, $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable.*[2.143.7] *To obtain L_{n+1} from L_n, remove ϕ and add $\psi\binom{v}{\bar{u}}$, and define $<^L_{n+1}$ from $<^L_n$ by removing ϕ from the head of the list and adding $\psi\binom{v}{\bar{u}}$ at the tail.*
   2. *If $\phi \in \Delta_n$ let τ be the $<^T$-first term in T for which $\psi\binom{v}{\tau} \notin \Gamma_n \cup \Delta_n$, and let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \left\{\psi\binom{v}{\tau}\right\}$. τ is free for v in ψ,*[10.44.2,3] *so $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is unprovable.*[2.143.8] *To obtain L_{n+1} from L_n, add $\psi\binom{v}{\tau}$,* but do not remove ϕ, *and define $<^L_{n+1}$ from $<^L_n$ by moving ϕ from the head to the tail and adding $\psi\binom{v}{\tau}$ after it.*

Let $\Gamma_\omega = \bigcup_{n\in\omega} \Gamma_n$ and $\Delta_\omega = \bigcup_{n\in\omega} \Delta_n$. Define the structure $\mathfrak{S}$ as follows.

1. $|\mathfrak{S}| = T$.[10.44.3]
2. Suppose X is an n-ary ρ-operation index. Let
$$X^{\mathfrak{S}} = \{(\langle \tau_0, \ldots, \tau_{n^-}\rangle, \tilde{X}\langle \tau_0, \ldots, \tau_{n^-}\rangle) \mid \tau_0, \ldots, \tau_{n^-} \in T\}.$$
3. Suppose X is an n-ary ρ-predicate index. The only conditions we must impose are that for any $\tau_0, \ldots, \tau_{n^-} \in T$,
   1. $\tilde{X}\langle \tau_0, \ldots, \tau_{n^-}\rangle \in \Gamma_\omega \rightarrow \langle \tau_0, \ldots, \tau_{n^-}\rangle \in X^{\mathfrak{S}}$; and
   2. $\tilde{X}\langle \tau_0, \ldots, \tau_{n^-}\rangle \in \Delta_\omega \rightarrow \langle \tau_0, \ldots, \tau_{n^-}\rangle \notin X^{\mathfrak{S}}$.

These are consistent by virtue of (10.43.3). To complete the definition we arbitrarily declare that

$$X^{\mathfrak{S}} = \{\langle \tau_0, \dots, \tau_{n^-} \rangle \in {}^nT \mid \tilde{X}\langle \tau_0, \dots, \tau_{n^-} \rangle \in \Gamma_\omega\}.$$

We now define a subvaluation function S for $\mathfrak{S}$ in the natural way.

1. For any ρ-term τ and $\mathfrak{S}$-assignment A for τ, $\langle \tau, A \rangle \in \operatorname{dom} S$ and $S\langle \tau, A \rangle = \tau(A)$. Recall that $|\mathfrak{S}| = T$, the class of ρ-terms derived from the class V of variables, so A is an assignment of terms in T to the free variables of τ, and $\tau(A)$ is the term obtained by indicated substitution, which is in $T = |\mathfrak{S}|$, so this definition is well made.

2. For a ρ-formula ϕ and $\mathfrak{S}$-assignment A for ϕ, $\langle \phi, A \rangle \in \operatorname{dom} S$ iff $\phi(A) \in \Gamma_\omega \cup \Delta_\omega$, in which case, $S\langle \phi, A \rangle = 1$ if $\phi(A) \in \Gamma_\omega$, and $S\langle \phi, A \rangle = 0$ if $\phi(A) \in \Delta_\omega$.

Conditions 1–3 of the definition[2.146] of *subvaluation* are clearly met by S. To check Condition 4, suppose $\langle \neg\, \psi, A \rangle \in \operatorname{dom} S$. Then $\neg\, \psi(A) \in \Gamma_\omega \cup \Delta_\omega$. Let $n \in \omega$ be the least such that $\neg\, \psi(A) \in \Gamma_n \cup \Delta_n$. Then $\neg\, \psi(A) \in L_n$ and for some $m \geqslant n$, $\neg\, \psi(A)$ is the $<^L_m$-first member of L. At this stage in the construction, if $\neg\, \psi(A)$ was in Γ_m then $\psi(A)$ was put into Δ_{m+1},[10.45.1.1] and *vice versa*.[10.45.1.2] In either case $\langle \psi, A \rangle \in \operatorname{dom} S$ and $S\langle \neg\, \psi, A \rangle = 1 - S\langle \psi, A \rangle$, as required.

A similar argument applies to Condition 7. (Note that Conditions 5, 6, 8, and 10 are automatically satisfied since we have excluded such formulas from our construction.) Specifically, if $\langle \psi_0 \rightarrow \psi_1, A \rangle \in \operatorname{dom} S$, then for some $m \in \omega$, $\psi_0(A) \rightarrow \psi_1(A)$ was $<^L_m$-first in L_m. If $S\langle \psi_0 \rightarrow \psi_1, A \rangle = 1$, $\psi_0(A) \rightarrow \psi_1(A)$ was in Γ_m, and we either put $\psi_0(A)$ in Δ_{m+1} or $\psi_1(A)$ in Γ_{m+1},[10.45.2.1] making $S\langle \psi_0, A \rangle = 0$ or $S\langle \psi_1, A \rangle = 1$, respectively. On the other hand, if $S\langle \psi_0 \rightarrow \psi_1, A \rangle = 0$, $\psi_0(A) \rightarrow \psi_1(A)$ was in Δ_m, and we put $\psi_0(A)$ in Γ_{m+1} and $\psi_1(A)$ in Δ_{m+1},[10.45.2.2] making $S\langle \psi_0, A \rangle = 1$ and $S\langle \psi_1, A \rangle = 0$.

To verify Condition 9, suppose $\langle \exists v\, \psi, A \rangle \in \operatorname{dom} S$. Suppose first that $(\exists v\, \psi)(A) \in \Gamma_\omega$, and let $m \in \omega$ be such that $(\exists v\, \psi)(A)$ is the $<^L_m$-first member of Γ_m. The for some variable $u \in V$, $\psi\big(A\big\langle \begin{smallmatrix} v \\ \bar{u} \end{smallmatrix} \big\rangle\big) \in \Gamma_{m+1}$,[10.45.3.1] so $S\big\langle \psi, A\big\langle \begin{smallmatrix} v \\ \bar{u} \end{smallmatrix} \big\rangle\big\rangle = 1$. Since $\bar{u} \in T = |\mathfrak{S}|$, this suffices.

Finally, suppose $(\exists v\, \psi)(A) \in \Delta_\omega$. By construction, for infinitely many $m \in \omega$, $(\exists v\, \psi)(A)$ is the $<^L_m$-first member of L_m. Each time this happens we have let $\Delta_{m+1} = \Delta_m \cup \big\{\psi\big(A\big\langle \begin{smallmatrix} v \\ \tau \end{smallmatrix} \big\rangle\big)\big\}$, where τ is the $<^T$-first term in T for which $\psi\big(A\big\langle \begin{smallmatrix} v \\ \tau \end{smallmatrix} \big\rangle\big) \notin \Gamma_n \cup \Delta_n$.[10.45.3.2] Consequently, $\psi\big(A\big\langle \begin{smallmatrix} v \\ \tau \end{smallmatrix} \big\rangle\big) \in \Delta_\omega$ for every $\tau \in T$ $\big($since $\psi\big(A\big\langle \begin{smallmatrix} v \\ \tau \end{smallmatrix} \big\rangle\big) \notin \Gamma_n$ for any $n\big)$. Again, since $|\mathfrak{S}| = T$, this suffices.

Thus, S is a subvaluation function for $\mathfrak{S}$. Let A be the assignment of $\bar{v}$ to v for each variable v that is free in $\Gamma \cup \Delta$. Then for each $\phi \in \Gamma \cup \Delta$, $\phi(A) = \phi$. Since $\Gamma = \Gamma_0 \subseteq \Gamma_\omega$ and $\Delta = \Delta_0 \subseteq \Delta_\omega$, $\forall \phi \in \Gamma\ S\langle \phi, A \rangle = 1$ and $\forall \phi \in \Delta\ S\langle \phi, A \rangle = 0$, as required. □[10.42]

Note that S is not necessarily even a partial valuation function for $\mathfrak{S}$, much less a full valuation function, so (2.147) is not the completeness theorem for $\mathbf{LK}^-$. If we assume Infinity, $\mathfrak{S}$ is a set, so it has a full satisfaction relation. Thus, an $\mathbf{LK}^-$-unprovable sequent is false in a satisfactory structure; equivalently, an $\mathbf{LK}^-$-consistent theory has a satisfactory model, i.e., $\mathbf{LK}^-$ is a complete system of deduction. Obviously, this implies the completeness of $\mathbf{LK}$. Thus, a sequent is $\mathbf{LK}^-$-

provable iff it is semantically valid iff it is **LK**-provable. We therefore have a proof of the cut-elimination theorem in ZF.

Alternatively, working finitarily, we can obtain the completeness theorem for **LK** by a modification of the construction used in the proof of (2.147). We do this in Note 10.10.

Of course, this does not give us a finitary proof of cut-elimination, and we provide such a proof in Note 10.11. With this in hand, we have a finitary proof that a sequent is $\mathbf{LK}^-$-provable iff it is **LK**-provable iff it is semantically valid. We therefore have a finitary proof of the completeness of $\mathbf{LK}^-$.

10.10 Proof of (2.148)

[REFER TO P. 144.]

(10.46) **Theorem** [C^0] *Suppose $I = (\Gamma \Rightarrow \Delta)$ is a sequent in a signature ρ without identity that is not* **LK***-provable. Then there exists a satisfactory ρ-structure $\mathfrak{S}$ and an $\mathfrak{S}$-assignment A to the free variables of I such that for each $\phi \in \Gamma \cup \Delta$,*

1. *if $\phi \in \Gamma$ then $\mathfrak{S} \models \phi[A]$; and*
2. *if $\phi \in \Delta$ then $\mathfrak{S} \models \neg\,\phi[A]$.*

Proof Let V' be a countably infinite class of variables that contains all variables that occur in quantifier phrases in $\Gamma \cup \Delta$, and let V be a countably infinite class of variables not in V'. Let F be the class of ρ-formulas all of whose bound variables are in V'. Note that $\Gamma, \Delta \subseteq F$. As before, let T be the class of terms τ such that $\operatorname{Free}\tau \subseteq V$.

Assume $\Gamma \Rightarrow \Delta$ is not **LK**-provable. Construct **LK**-unprovable sequents $\Gamma_n \Rightarrow \Delta_n$ as before, but alternate steps of the type (10.45) with steps of the following type. Let ϕ be the first formula in F (in some fixed enumeration $<^F$) that is not in $\Gamma_n \cup \Delta_n$. Since $\Gamma_n \Rightarrow \Delta_n$ is by construction not **LK**-provable, either $\Gamma_n \Rightarrow \Delta_n \cup \{\phi\}$ or $\Gamma_n \cup \{\phi\} \Rightarrow \Delta_n$ is not **LK**-provable.[2.143.9] If the former, let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{\phi\}$; otherwise let $\Gamma_{n+1} = \Gamma_n \cup \{\phi\}$ and $\Delta_{n+1} = \Delta_n$. In either case $\Gamma_{n+1} \Rightarrow \Delta_{n+1}$ is **LK**-unprovable.

Now define Γ_ω, Δ_ω, and $\mathfrak{S}$ as before. We will define a (full) valuation function for $\mathfrak{A}$. Given a ρ-formula ϕ, let ϕ' be any formula in F obtained from ϕ by a change of bound variables. Given an $|\mathfrak{S}|$-assignment A for ϕ, A is also an $|\mathfrak{S}|$-assignment for ϕ', and $\phi'(A) \in \Gamma_\omega \cup \Delta_\omega$. If ϕ'' is any other formula in F obtained from ϕ by a change of bound variables, then ϕ' and ϕ'' are related by a change of bound variables. Hence, $\phi'(A)$ and $\phi''(A)$ are related by a change of bound variables, and it is not hard to show that $\{\phi'(A)\} \Rightarrow \{\phi''(A)\}$ is **LK**-provable.[8] Thus, $\phi'(A)$ and

[8] We show by induction on logical complexity that for any ϕ, ϕ' related by a change of bound variables, $\{\phi\} \Rightarrow \{\phi'\}$ is **LK**-provable. Suppose $\exists v\, \phi$ and $\exists v'\, \phi'$ are related by a change of bound variables. Let u be a variable that does not occur in either formula. Then $\phi\binom{v}{\bar u}$ and $\phi'\binom{v'}{\bar u}$ are related by a change of bound variables. Suppose by induction hypothesis that $\phi\binom{v}{\bar u} \Rightarrow \phi'\binom{v'}{\bar u}$ is **LK**-provable. We can append the proof-segment $\dfrac{\dfrac{\phi\binom{v}{\bar u} \Rightarrow \phi'\binom{v'}{\bar u}}{\phi\binom{v}{\bar u} \Rightarrow \exists v'\, \phi'}}{\exists v\, \phi \Rightarrow \exists v'\, \phi'}$ to obtain an **LK**-proof of $\{\phi\} \Rightarrow \{\phi'\}$. (Clearly, we have proved the result for $\mathbf{LK}^-$, but that is inconsequential here.)

$\phi''(A)$ cannot occur on opposite sides of any sequent $\Gamma_n \Rightarrow \Delta_n$, otherwise it would be **LK**-provable. We therefore unambiguously define $S\langle\phi, A\rangle$ to be 1 or 0 according as $\phi'(A)$ is in Γ_ω or Δ_ω for any (or all) $\phi' \in F$ obtained from ϕ by a change of bound variables. $\square^{2.148}$

10.11 Proof of (2.149)

[Refer to p. 144.]

We will need the following theorem.

(10.47) **Theorem** [C^0] *Suppose a sequent $\Gamma \Rightarrow \Delta$ is* **LK**$^-$*-provable, v is a variable, τ is a term, and τ is free for v in every formula in $\Gamma \cup \Delta$. Let $\Gamma' \Rightarrow \Delta'$ be the result of substituting τ for v in every formula in $\Gamma \Rightarrow \Delta$. Then $\Gamma' \Rightarrow \Delta'$ is* **LK**$^-$*-provable.*

Proof There is a direct syntactical argument involving changes of variables and induction on the length of proofs, but there is a simpler semantical argument. Suppose $\Gamma' \Rightarrow \Delta'$ is not **LK**$^-$-provable. By (2.147) there is a structure $\mathfrak{S}$, an $\mathfrak{S}$-assignment A' to the free variables of $\Gamma' \Rightarrow \Delta'$, and a subvaluation S' for $\mathfrak{S}$ such that for all $\phi \in \Gamma' \cup \Delta'$,

1. $\langle\phi, A'\rangle \in \operatorname{dom} S'$;
2. $\phi \in \Gamma' \to S'\langle\phi, A'\rangle = 1$; and
3. $\phi \in \Delta' \to S'\langle\phi, A'\rangle = 0$.

Let $A = A'\left\langle {}^{\ v}_{\mathrm{Val}^{\mathfrak{S}}\,\tau[A']} \right\rangle$. It is a simple matter to modify S' to obtain a subvaluation S for $\mathfrak{S}$ such that for all $\phi \in \Gamma \cup \Delta$,

1. $\langle\phi, A\rangle \in \operatorname{dom} S$;
2. $\phi \in \Gamma \to S\langle\phi, A\rangle = 1$; and
3. $\phi \in \Delta \to S\langle\phi, A\rangle = 0$.

$\square^{10.47}$

(10.48) **Theorem** [C^0] *Suppose $I = (\Gamma \Rightarrow \Delta)$ is a sequent in a signature ρ without identity. If I is* **LK***-provable then I is* **LK**$^-$*-provable.*

Proof By induction on the number of applications of the cut rule in an **LK**-proof. The following claim clearly suffices.

(10.49) **Claim** *Suppose π is an* **LK***-proof of a sequent I with one application of the cut rule, occurring as the last inference. Then I is* **LK**$^-$*-provable.*

Proof For technical reasons we will consider proofs with the following modification of the cut rule:

(10.50)
$$\frac{\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\} \qquad \Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

Any application of this rule may be replaced by the following proof segment

$$\dfrac{\dfrac{\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1 \cup \{\phi\}} \qquad \dfrac{\Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1}{\Gamma_0 \cup \Gamma_1 \cup \{\phi\} \Rightarrow \Delta_0 \cup \Delta_1}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where the dotted lines represent possible insertions of weakening inferences (2.143.1,2), so despite its greater generality, (10.50) does not increase the strength of the deductive system.[9] For the nonce, we will call a proof π that ends as in (10.50) and is otherwise cut-free, a *ϕ-proof*. We say that π is *essential* iff its final sequent does not have a cut-free proof. The proof of Claim 10.49 proceeds by induction on the complexity of proofs. The primary measure of complexity of a ϕ-proof is the *logical height* of ϕ, defined as follows: the logical height of a term is 0, the logical height of an atomic formula is 1, and the logical height of a complex formula is the least $n \in \omega$ that exceeds the logical height(s) of its immediate subformula(s). Since there do not exist formulas with logical height 0, the following claim suffices.

(10.51) **Claim** *Suppose ϕ is a ρ-formula, and for every formula ψ of lower logical height, no ψ-proof is essential. Then no ϕ-proof is essential.*

Proof Suppose for every formula ψ of lower logical height than ϕ, no ψ-proof is essential. We seek to show that no ϕ-proof is essential.

(10.52) **Claim** *Suppose π is an essential ϕ-proof that terminates thus:*

(10.53)
$$\dfrac{\begin{matrix}\vdots\, \pi_0 \\ \Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}\end{matrix} \qquad \begin{matrix}\vdots\, \pi_1 \\ \Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1\end{matrix}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where π_0 and π_1 are the subproofs terminating with $\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}$ and $\Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1$, respectively. (In general, a label next to a vertical ellipsis indicates the proof that ends with the following sequent.) Then ϕ occurs in either a succedent in π_0 or an antecedent in π_1 immediately above the final sequent (of π_0 or π_1, respectively) and is not the principal formula of the final inference.

Proof Suppose the contrary. Suppose $\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}$ is an axiom. Then $\Gamma_0 = \{\phi\}$, so $\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1$ is obtainable by weakening $\Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1$, and a cut is not required. Hence, $\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}$ is not an axiom. Likewise, $\Gamma_1 \cup \{\phi\} \Rightarrow \Delta_1$ is not an axiom.

Suppose $\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}$ in π_0 is obtained by weakening. Since weakening rules do not have principal formulas, ϕ does not occur as a succedent in the penultimate sequent of π_0, so π_0 is

$$\dfrac{\begin{matrix}\vdots\pi_2 \\ \Gamma_0 \Rightarrow \Delta'\end{matrix}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\phi\}}$$

where $\phi \notin \Delta'$ and $\Delta' \subseteq \Delta_0$, and

$$\dfrac{\begin{matrix}\vdots\pi_2 \\ \Gamma_0 \Rightarrow \Delta'\end{matrix}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

[9] Any of the rules with two upper sequents may be generalized in this way, and they are often so stated.

is therefore a cut-free proof of $\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1$. Hence, π_0 does not end with a weakening inference; nor does π_1, by a similar argument.

It follows that ϕ must be the principal formula of the final inferences of both π_0 and π_1. There are three cases, depending on the primary structure of ϕ. Suppose first that $\phi = \neg\,\psi$. Then π ends as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \cup \{\psi\} \Rightarrow \Delta_0\end{array}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\neg\,\psi\}} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \Rightarrow \Delta_1 \cup \{\psi\}\end{array}}{\Gamma_1 \cup \{\neg\,\psi\} \Rightarrow \Delta_1}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

We may replace this segment in π by

$$\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \cup \{\psi\} \Rightarrow \Delta_0\end{array} \qquad \begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \Rightarrow \Delta_1 \cup \{\psi\}\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

which yields a proof that ends with a cut, with the cut formula ψ, and is otherwise cut-free, i.e., a ψ-proof. By the induction hypothesis (of Claim 10.51) there is therefore a cut-free proof of $\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1$.

Next, suppose $\phi = \psi \rightarrow \psi'$. Then π ends as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \cup \{\psi\} \Rightarrow \Delta_0 \cup \{\psi'\}\end{array}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\psi \rightarrow \psi'\}} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \Rightarrow \Delta_1 \cup \{\psi\}\end{array} \qquad \begin{array}{c}\vdots\,\pi_4\\ \Gamma_1 \cup \{\psi'\} \Rightarrow \Delta_1\end{array}}{\Gamma_1 \cup \{\psi \rightarrow \psi'\} \Rightarrow \Delta_1}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

We may replace this segment in π by

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \cup \{\psi\} \Rightarrow \Delta_0 \cup \{\psi'\}\end{array} \qquad \begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \Rightarrow \Delta_1 \cup \{\psi\}\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1 \cup \{\psi'\}} \qquad \begin{array}{c}\vdots\,\pi_4\\ \Gamma_1 \cup \{\psi'\} \Rightarrow \Delta_1\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

Using the induction hypothesis twice, first for ψ and then for ψ', there is therefore a cut-free proof of $\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1$.

Finally, suppose $\phi = \exists v\, \psi$. Then π ends as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \Rightarrow \Delta_0 \cup \left\{\psi\binom{v}{\tau}\right\}\end{array}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\exists v\, \psi\}} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \cup \left\{\psi\binom{v}{\bar{u}}\right\} \Rightarrow \Delta_1\end{array}}{\Gamma_1 \cup \{\exists v\, \psi\} \Rightarrow \Delta_1}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where τ and $\bar{u}$ are free for v in ψ, and u does not occur free in $\Gamma_1 \cup \{\exists v\, \psi\} \Rightarrow \Delta_1$.

τ is free for u in $\psi\binom{v}{\bar{u}}$, and u does not occur free in $\Gamma_1 \cup \Delta_1$, so by Theorem 10.47 there is a cut-free proof, say π_4, of $\Gamma_1 \cup \left\{\psi\binom{v}{\tau}\right\} \Rightarrow \Delta_1$. Thus we have a $\psi\binom{v}{\tau}$-proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \Rightarrow \Delta_0 \cup \left\{\psi\binom{v}{\tau}\right\}\end{array} \qquad \begin{array}{c}\vdots\,\pi_4\\ \Gamma_1 \cup \left\{\psi\binom{v}{\tau}\right\} \Rightarrow \Delta_1\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

By induction hypothesis, there is therefore a cut-free proof of $\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1$. $\square$[10.52]

Continuing with the proof of Claim 10.51, suppose toward a contradiction that $\pi =$

$$\dfrac{\begin{array}{c}\vdots\, \pi_0 \\ \Gamma_0 \Rightarrow \Delta_0 \cup \{\eta\}\end{array} \qquad \begin{array}{c}\vdots\, \pi_1 \\ \Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

is an essential η-proof of minimum length, where the length of a proof is defined as the number of inferences in it.[10]

We first note that

$$\eta \notin \Gamma_0 \cup \Delta_0 \cup \Gamma_1 \cup \Delta_1; \tag{10.54}$$

otherwise the final sequent is obtainable by a weakening either of $\Gamma_0 \Rightarrow \Delta_0 \cup \{\eta\}$ if $\eta \in \Delta_0 \cup \Delta_1$, or of $\Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1$ if $\eta \in \Gamma_0 \cup \Gamma_1$.

By (10.52), η occurs in either a succedent in π_0 or an antecedent in π_1 immediately above the final sequent and is not the principal formula.

(10.55) *Suppose the former.*

Then, except in the case of Rule 5, π terminates thus:

$$\dfrac{i\ \dfrac{\begin{array}{c}\vdots\, \pi_2 \\ \Gamma_2 \Rightarrow \Delta_2 \cup \{\eta\}\end{array}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\eta\}} \qquad \begin{array}{c}\vdots\, \pi_1 \\ \Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where inference i is either a weakening or an instance of a unary inference rule with principal formula other than η.

Obviously, we may suppose $\eta \notin \Delta_2$. Remember that we have shown above[10.54] that $\eta \notin \Delta_0$.

If i is an instance of Rule 1 or 2[2.143] then

$$\frac{\Gamma_2 \Rightarrow \Delta_2}{\Gamma_0 \Rightarrow \Delta_0} \tag{10.56}$$

is an instance of the same rule.

In Rules 3, 4, 6, 7, 8 as listed in (2.143), any formula η that occurs in the succedent of the lower sequent and is not the principal formula, occurs in Δ. Thus, if i is an instance of Rule 4 or 7 then (10.56) is an instance of the same rule.

If i is an instance of Rule 3, 6, or 8, then either (10.56) or

$$\frac{\Gamma_2 \Rightarrow \Delta_2 \cup \{\eta\}}{\Gamma_0 \Rightarrow \Delta_0} \tag{10.57}$$

is an instance of the same rule, depending on whether η is involved in the formation of the principal formula. Specifically, (10.56) is an instance of the same rule as i, unless i is an instance of

(10.58)

[10] We use 'η' rather than 'ϕ' to facilitate comparison with the inference rules as listed in (2.143).

1. *Rule 3 with principal formula* $\neg\,\eta$*;*
2. *Rule 6 with principal formula* $\phi \to \eta$*; or*
3. *Rule 8 with principal formula* $\exists\,\phi$*, where* $\eta = \phi\binom{v}{\tau}$*;*

in which case (10.57) is an instance of the same rule.

Since

$$\dfrac{\begin{matrix}\vdots\,\pi_2 \\ \Gamma_2 \Rightarrow \Delta_2 \cup \{\eta\}\end{matrix} \qquad \begin{matrix}\vdots\,\pi_1 \\ \Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1\end{matrix}}{\Gamma_2 \cup \Gamma_1 \Rightarrow \Delta_2 \cup \Delta_1}$$

is an η-proof shorter than π, it is replaceable by a cut-free proof. Thus we have the cut-free proof

$$i'\ \dfrac{\dfrac{\begin{matrix}\vdots\,\pi_4 \\ \Gamma_2 \cup \Gamma_1 \Rightarrow \Delta_2 \cup \Delta_1\end{matrix}}{\Gamma_2 \cup \Gamma_1 \Rightarrow \Delta_2 \cup \Delta_1 \cup \Pi}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where the inference i' is an instance of the same rule as inference i, with the same principal formula, if any, and $\Pi = 0$, unless (10.58) applies, in which case $\Pi = \{\eta\}$.

In the case of Rule 5, π is of the form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_2 \\ \Gamma_2 \Rightarrow \Delta_2 \cup \{\psi_0\} \cup \Pi_0\end{matrix} \qquad \begin{matrix}\vdots\,\pi_3 \\ \Gamma_3 \cup \{\psi_1\} \Rightarrow \Delta_3 \cup \Pi_1\end{matrix}}{\Gamma_0 \Rightarrow \Delta_0 \cup \{\eta\}} \qquad \begin{matrix}\vdots\,\pi_1 \\ \Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1\end{matrix}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

where $\eta \notin \Delta_2 \cup \Delta_3$; Π_0 and Π_1 are individually either 0 or $\{\eta\}$, and at least one of them is $\{\eta\}$; and

$$\begin{aligned} \Gamma_0 &= \Gamma_2 \cup \Gamma_3 \cup \{\psi_0 \to \psi_1\} \\ \Delta_0 &= \Delta_2 \cup \Delta_3. \end{aligned} \tag{10.59}$$

We will now use the minimality of π to conclude that there are cut-free proofs

$$\begin{matrix}\vdots\,\pi_5 \\ \Gamma_1 \cup \Gamma_2 \Rightarrow \Delta_1 \cup \Delta_2 \cup \{\psi_0\}\end{matrix}$$

and

$$\begin{matrix}\vdots\,\pi_6 \\ \Gamma_1 \cup \Gamma_3 \cup \{\psi_1\} \Rightarrow \Delta_1 \cup \Delta_3\end{matrix}$$

For the first, if ψ_0 is η or Π_0 is empty, the final sequent is just a weakening of $\Gamma_2 \Rightarrow \Delta_2 \cup \{\psi_0\} \cup \Pi_0$; if ψ_0 is not η and $\Pi_0 = \{\eta\}$, we use the fact that the final cut may be eliminated in

$$\dfrac{\begin{matrix}\vdots\,\pi_2 \\ \Gamma_2 \Rightarrow \Delta_2 \cup \{\psi_0\} \cup \{\eta\}\end{matrix} \qquad \begin{matrix}\vdots\,\pi_1 \\ \Gamma_1 \cup \{\eta\} \Rightarrow \Delta_1\end{matrix}}{\Gamma_2 \cup \Gamma_1 \Rightarrow \Delta_2 \cup \Delta_1 \cup \{\psi_0\}}$$

because this is a shorter η-proof than π.

For the second, if $\Pi_1 = 0$ then the final sequent is a weakening of $\Gamma_3 \cup \{\psi_1\} \Rightarrow \Delta_3 \cup \Pi_1$; otherwise we use cut-elimination as above.

We now have the cut-free proof

$$\dfrac{\begin{array}{c}\vdots\ \pi_5\\ \Gamma_1 \cup \Gamma_2 \Rightarrow \Delta_1 \cup \Delta_2 \cup \{\psi_0\}\end{array} \qquad \begin{array}{c}\vdots\ \pi_6\\ \Gamma_1 \cup \Gamma_3 \cup \{\psi_1\} \Rightarrow \Delta_1 \cup \Delta_3\end{array}}{\Gamma_0 \cup \Gamma_1 \Rightarrow \Delta_0 \cup \Delta_1}$$

using Rule 5 with principal formula $\psi_0 \to \psi_1$ and the identities (10.59).

The preceding argument supposed that η occurs as a succedent in π_0.[10.55] The other possibility is that η occurs as an antecedent in π_1. The proof of this case is handled similarly. □[10.51] □[10.49] □[10.48]

10.12 Proof of (2.183)

[Refer to p. 160.]

(10.60) **Theorem** [S^0]

1. C^0 *is a conservative extension of* S^0.
2. C *is a conservative extension of* S.

Proof The following list of axioms of C^1 is similar to (3.16) with the following modifications:

1. The Extension axiom C1 has been rewritten to be meaningful in the absence of identity.
2. The Collection axiom C5 has been replaced by the schema $\mathsf{C}^1$5. This is done for two reasons. First, since we do not have identity we cannot introduce operations by definition, so the ordered pair operation used in C5 is not available. Second, $\mathsf{C}^1$5 incorporates the Collection schema of pure set theory when class variables are absent.

For each number n, let v_n^S and v_n^C be the nth variable of set and class sort, respectively.

(10.61) **Axioms of** C^1

$\mathsf{C}^1$1. Extension

$$\forall \mathrm{v}_0^C\, \forall \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\, (\forall \mathrm{v}_2^S\, (\bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_0^S \leftrightarrow \bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_1^S) \to (\bar{\mathrm{v}}_0^S \in \bar{\mathrm{v}}_0^C \leftrightarrow \bar{\mathrm{v}}_1^S \in \bar{\mathrm{v}}_0^C))$$

$\mathsf{C}^1$2a. Comprehension

$$\forall v_0 \cdots \forall v_{n^-}\, \exists_C v\, \forall_S w\, (\bar{w} \in \bar{v} \leftrightarrow \phi),$$

where ϕ is any c^1-formula with only set quantification, and

$$v_0, \ldots, v_{n^-}, v, w$$

are distinct variables such that $\operatorname{Free}\phi \subseteq \{v_0, \ldots, v_{n^-}, w\}$.

$\mathsf{C}^1$2b. Separation

$$\forall \mathrm{v}_0^C\, \forall \mathrm{v}_0^S\, \exists \mathrm{v}_1^S\, \forall \mathrm{v}_2^S\, (\mathrm{v}_2^S \in \mathrm{v}_1^S \leftrightarrow \mathrm{v}_2^S \in \mathrm{v}_0^S \wedge \mathrm{v}_2^S \in \mathrm{v}_0^C)$$

$\mathsf{C}^1$3. Existence

$$\exists \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\ \bar{\mathrm{v}}_1^S \notin \bar{\mathrm{v}}_0^S$$

$\mathsf{C}^1$4. Pair

$$\forall \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\, \exists \mathrm{v}_2^S\, (\bar{\mathrm{v}}_0^S \in \bar{\mathrm{v}}_2^S \wedge \bar{\mathrm{v}}_1^S \in \bar{\mathrm{v}}_2^S)$$

$\mathsf{C}^1$5. Collection

$$\forall v_0 \cdots \forall v_{n^-}\, \forall_S u\, \big(\forall_S v \in \bar{u}\, \exists_S w\, \forall_S a\, (\phi \rightarrow \bar{a} \in \bar{w}) \\ \rightarrow \exists_S w\, \forall_S v \in \bar{u}\, \forall_S a\, (\phi \rightarrow \bar{a} \in \bar{w})\big),$$

where ϕ is any c^1-formula with only set quantification, and

$$v_0, \ldots, v_{n^-}, a, u, v, w$$

are distinct variables such that $\mathrm{Free}\, \phi \subseteq \{v_0, \ldots, v_{n^-}, a, v\}$.

We will show that C^1 is a conservative extension of the c^1-theory S^1, defined here.[10.62] Note that no class variables occur in axioms of S^1, and it is easily seen that S^1 is equi-interpretable with S^0.

(10.62) **Axioms of** S^1

$\mathsf{S}^1$1. Extension

$$\forall \mathrm{v}_3^S\, \forall \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\, (\forall \mathrm{v}_2^S\, (\bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_0^S \leftrightarrow \bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_1^S) \rightarrow (\bar{\mathrm{v}}_0^S \in \bar{\mathrm{v}}_3^S \leftrightarrow \bar{\mathrm{v}}_1^S \in \bar{\mathrm{v}}_3^S))$$

$\mathsf{S}^1$2. Comprehension

$$\forall v_0 \cdots \forall v_{n^-}\, \forall u\, \exists w\, \forall v\, \big(\bar{v} \in \bar{w} \leftrightarrow (\bar{v} \in \bar{u} \wedge \phi)\big),$$

where ϕ is any c^1-formula with only set quantification, and $u, v, w, v_0, \ldots, v_{n^-}$ are distinct set variables such $\mathrm{Free}\, \phi \subseteq \{v, v_0, \ldots, v_{n^-}\}$.

$\mathsf{S}^1$3. Existence

$$\exists \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\ \bar{\mathrm{v}}_1^S \notin \bar{\mathrm{v}}_0^S$$

$\mathsf{S}^1$4. Pair

$$\forall \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\, \exists \mathrm{v}_2^S\, (\bar{\mathrm{v}}_0^S \in \bar{\mathrm{v}}_2^S \wedge \bar{\mathrm{v}}_1^S \in \bar{\mathrm{v}}_2^S)$$

$\mathsf{S}^1$5. Collection

$$\forall v_0, \ldots, v_{n^-}\, \forall u\, \big(\forall v \in u\, \exists w\, \forall a\, (\phi \rightarrow a \in w) \rightarrow \exists w\, \forall v \in u\, \forall a\, (\phi \rightarrow a \in w)\big),$$

where ϕ is any c^1-formula with only set quantification, and $a, u, v, w, v_0, \ldots, v_{n^-}$ are distinct set variables such that $\mathrm{Free}\, \phi \subseteq \{a, v, v_0, \ldots, v_{n^-}\}$.

Note that

$$\forall \mathrm{v}_3^S\, \forall \mathrm{v}_0^S\, \forall \mathrm{v}_1^S\, \big(\forall \mathrm{v}_2^S\, (\bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_0^S \leftrightarrow \bar{\mathrm{v}}_2^S \in \bar{\mathrm{v}}_1^S) \rightarrow (\bar{\mathrm{v}}_3^S \in \bar{\mathrm{v}}_0^S \leftrightarrow \bar{\mathrm{v}}_3^S \in \bar{\mathrm{v}}_1^S)\big)$$

is a logical validity. It follows by induction on logical complexity that

$$\mathsf{S}^1 \vdash \forall v_0 \cdots \forall v_{n^-}\, \forall u_0\, \forall u_1 \left(\forall w\, (\bar{w} \in \bar{u}_0 \leftrightarrow \bar{w} \in \bar{u}_1) \rightarrow \big(\phi\big(\tbinom{u}{\bar{u}_0}\big) \leftrightarrow \phi\big(\tbinom{u}{\bar{u}_1}\big)\big)\right), \tag{10.63}$$

for any c^1-formula ϕ with only set quantification, and distinct set variables $u, u_0, u_1, w, v_0, \dots, v_{n^-}$, such that $\operatorname{Free}\phi \subseteq \{u, v_0, \dots, v_{n^-}\}$, and $\bar{u}_0$ and $\bar{u}_1$ are free for u in ϕ.

The first part of the proof is modeled on the proof of Herbrand's theorem.[10.14] To begin, we will work in a language with the universal but not the existential quantifier. '$\exists$' is understood to mean '$\neg\forall\neg$'. The following definition adapts (10.15) to the present purpose. For the first part of the construction, to align notation with that of (10.15), we let ρ be c^1 and let ρ' be ρ extended by new constants (nulary operation indices) of both set and class sort. We will not introduce any operations other than constants, so all terms are constants or variables. Note that substitution of a term for a variable requires that they be of the same sort.

(10.64)

1. *A* witness sequence $\overset{\text{def}}{=}$ *a finite sequence*

$$W = \left\langle \phi_i\big(\tbinom{v_i}{\bar{c}_i}\big) \rightarrow \forall v_i\, \phi_i \,\middle|\, i \in n \right\rangle$$

of ρ'-sentences, where for each $i \in n$, c_i is a ρ'-constant that does not occur previously in W.

2. *An instance set is a finite set of ρ'-sentences of the form $\forall v\, \phi \rightarrow \phi\big(\tbinom{v}{\bar{c}}\big)$.*

We will define the *quantifier depth* $\operatorname{qd}\phi$ of a formula ϕ as a 2-sequence $\langle Q, q\rangle$ of numbers, and we will order these lexicographically, so that

$$\langle Q, q\rangle < \langle Q', q'\rangle \leftrightarrow \big(Q < Q' \vee (Q = Q' \wedge q < q')\big).$$

The definition is by recursion on complexity:

1. The quantifier depth of a quantifier-free formula is $\langle 0, 0\rangle$.
2. The quantifier depth of a propositional combination of formulas is the maximum quantifier depth of the constituent formulas.
3. Suppose $\operatorname{qd}\phi = \langle Q, q\rangle$.
   1. $\operatorname{qd}\exists_S u\,\phi = \operatorname{qd}\forall_S u\,\phi = \langle Q, q+1\rangle$.
   2. $\operatorname{qd}\exists_C u\,\phi = \operatorname{qd}\forall_C u\,\phi = \langle Q+1, 0\rangle$.

The *quantifier depth* of a finite sequence W of formulas $\overset{\text{def}}{=} \operatorname{qd} W \overset{\text{def}}{=}$ is the maximum quantifier depth of its items if $W \neq 0$; otherwise, $\operatorname{qd} W = \langle 0, 0\rangle$. The *type* of a finite sequence W of formulas $\overset{\text{def}}{=} \operatorname{tp} W \overset{\text{def}}{=} \langle Q, q, L\rangle$, where $\langle Q, q\rangle = \operatorname{qd} W$ and L is the number of items in W with quantifier depth $\langle Q, q\rangle$. We order types lexicographically, so that

$$\begin{aligned} &\langle Q, q, L\rangle < \langle Q', q', L'\rangle \\ &\qquad \leftrightarrow \big(Q < Q' \vee (Q = Q' \wedge q < q') \vee (Q = Q' \wedge q < q' \wedge L < L')\big). \end{aligned}$$

We will deal primarily with witness sequences and instance sets satisfying conditions (10.16), which we repeat here for convenience.

(10.65)

1. *For every sentence* $\forall v\,\phi \to \phi\binom{v}{\bar c} \in I$, *there is a sentence* $\phi\binom{v}{\bar c'} \to \forall v\,\phi \in \operatorname{im} W$, *i.e., any sentence that occurs as an antecedent in* I *occurs as a consequent in* W.
2. *For* $i, i' \in |W|$, *if* $i \neq i'$, $W(i) = \phi\binom{v}{\bar c} \to \forall v\,\phi$, *and* $W(i') = \phi'\binom{v'}{\bar c'} \to \forall v'\,\phi'$, *then* $\forall v\,\phi \neq \forall v'\,\phi'$, *i.e., no sentence* $\forall v\,\phi$ *is witnessed more than once in* W.

Suppose $\mathsf{C}^1 \vdash \sigma$, where σ is a c^1-sentence with no class variables. Let Σ_0 be a finite set of c^1-sentences including $\neg\,\sigma$ that is inconsistent. Recall that we have defined ρ to be c^1 and ρ' to be ρ with additional constants. Let W_0 and I be respectively a ρ'-witness sequence for Σ_0 and a ρ' instance set such that

$$\Sigma_0 \cup \operatorname{im} W_0 \cup I$$

is propositionally inconsistent.

Note that every axiom of Σ_0 is of the form

$$\forall v_0 \cdots \forall v_{n^-}\,\psi,$$

where ψ either has no class quantification or is of the form $\exists_C v\ \psi'$, where ψ' has only set quantification; n may be 0. The latter form is specific to $\mathsf{C}^1\mathsf{2a}$.

(10.66) *We stipulate without loss of generality that in the initial universal quantifier strings just mentioned, all class quantifications precede all set quantifications.*

In the case of $\mathsf{C}^1\mathsf{1}$, $\mathsf{C}^1\mathsf{2b}$, $\mathsf{C}^1\mathsf{3}$, and $\mathsf{C}^1\mathsf{4}$, this is the form in which they have been presented. In the case of $\mathsf{C}^1\mathsf{2a}$ and $\mathsf{C}^1\mathsf{5}$, we have

We define for the nonce the *matrix* μ^θ of a formula θ to be the formula obtained by removing any initial universal quantifications. Let $M = \{\mu^\theta \mid \theta \in \Sigma_0\}$.

For each $\theta \in \Sigma_0$, let $\forall v_0^\theta \cdots \forall v_{N^{\theta-}}^\theta$ be its initial universal quantification sequence (which may be null). Let d_n^θ ($\theta \in \Sigma_0$, $n < N^\theta$) be distinct ρ'-class constants that do not occur in W_0 or I such that d_n^θ is of the same sort as v_n^θ. Let $\langle \theta_0, \ldots, \theta_{K^-} \rangle$ be an enumeration of Σ_0 in an arbitrary order. For each $k < K$, let $\mu_k = \mu^{\theta_k}$, $N^k = N^{\theta_k}$, $v_n^k = v_n^{\theta_k}$, and $d_n^k = d_n^{\theta_k}$; let
(10.67)

$$W^k = \Big\langle \forall v_{n+1}^k \cdots \forall v_{N^{k-}}^k\, \mu_k\left(\begin{smallmatrix} v_0^k & \cdots & v_n^k \\ \bar d_0^k & \cdots & \bar d_n^k \end{smallmatrix}\right) \to \forall v_n^k \cdots \forall v_{N^{k-}}^k\, \mu_k\left(\begin{smallmatrix} v_0^k & \cdots & v_{n^-}^k \\ \bar d_0^k & \cdots & \bar d_{n^-}^k \end{smallmatrix}\right) \Bigm| n \in N^k \Big\rangle,$$

and let

$$W = W^0 \frown W^1 \frown \cdots \frown W^{K^-} \frown W_0.$$

Let

$$\Sigma = \Big\{ \mu_k\left(\begin{smallmatrix} v_0^k & \cdots & v_{N^{k-}}^k \\ \bar d_0^k & \cdots & \bar d_{N^{k-}}^k \end{smallmatrix}\right) \Bigm| k < K \Big\}.$$

Then

$$\Sigma \cup \operatorname{im}\big(W^0 \frown \cdots \frown W^{K^-}\big) \vdash^{\mathsf{P}} \Sigma_0,$$

so

(10.68) $$\Sigma \cup \operatorname{im} W \cup I$$

is propositionally inconsistent.

We now proceed as in the proof of Herbrand's theorem to remove elements from W and I, while adding elements to Σ. For this purpose we modify (10.17) to refer to sets Σ of constant instances of M.

(10.69)

1. $\langle \Sigma, W, I \rangle$ *is* good $\overset{\text{def}}{\iff}$
   1. Σ *is a finite set of constant* ρ'*-instances of* M*;*
   2. W *is a witness sequence;*
   3. I *is an instance set;*
   4. $\Sigma \cup \operatorname{im} W \cup I$ *is propositionally inconsistent; and*
   5. *(10.65.1) is satisfied.*
2. $\langle \Sigma, W, I \rangle$ *is* fine $\overset{\text{def}}{\iff}$ *it is good and (10.65.2) is satisfied.*

Note that the constants that occur as witnesses in W may also occur in Σ, i.e., we do not specify that W is a witness sequence for Σ. (10.18), which we repeat here, remains valid, with the same proof.

(10.70) *Suppose* $\langle \Sigma, W, I \rangle$ *is good. Then there exists a fine* $\langle \Sigma', W', I' \rangle$ *such that* $\operatorname{qd} W' \preccurlyeq \operatorname{qd} W$.

The following claim corresponds to (10.21) and has a similar proof.

(10.71) **Claim** *There exists a fine* $\langle \Sigma, W, I \rangle$ *such that the only items of* W *involving class quantification are of the form*

$$\phi\binom{v}{\bar{c}} \to \forall_C v\, \phi.$$

where $\phi = \neg\, \psi$*, and* $\neg \forall_C v\, \phi \in \Sigma$ *(having been obtained from* $\exists_C v\, \psi$ *by the substitution of* $\neg \forall v \neg$ *for* $\exists$*).*

Proof We first observe that in (10.68), since we may append witness sentences to W until (10.65.1) is satisfied, so there exists a good $\langle \Sigma, W, I \rangle$.

(10.72) **Claim** *There exists a fine* $\langle \Sigma, W, I \rangle$ *such that* $\operatorname{qd} W \preccurlyeq \langle 1, 0 \rangle$.

Proof We follow the proof of (10.21) quite closely. Suppose $\langle \Sigma, W, I \rangle$ is fine, where W has the least possible type. Suppose toward a contradiction that $\operatorname{qd} W \succ \langle 1, 0 \rangle$. Let i be such that $W_i = \phi_i\binom{v_i}{\bar{c}_i} \to \forall v_i\, \phi_i$ has the highest quantifier depth in W and is the last item of W with this depth. We let W' be W with W_i omitted; we let T be the set of constants c' such that $\forall v_i\, \phi_i \to \phi_i\binom{v_i}{\bar{c}'} \in I$; we let I' be I with all these instances omitted; and we let[10.23]

$$J = \left\{ \phi_i\binom{v_i}{\bar{c}_i} \to \phi\binom{v_i}{\bar{c}'} \,\middle|\, c' \in T \right\}.$$

As in the proof of (10.24), we show that $\forall v_i\, \phi_i$ does not occur in W', I', or J. Since $\operatorname{qd}(\forall v_i\, \phi_i) = \operatorname{qd}\left(\phi_i\binom{v_i}{\bar{c}_i} \to \forall v_i\, \phi_i\right)$, and no sentence in Σ has quantifier depth greater than $\langle 1, 0 \rangle$, $\forall v_i\, \phi_i$ also does not occur in Σ. It follows as before[10.24] that $\Sigma \cup \operatorname{im} W' \cup I' \cup J$ is propositionally inconsistent.

The rest of the proof of (10.21) goes through without essential modification to show that there exists a good and therefore[10.70] a fine $\langle \Sigma'', W'', I'' \rangle$ with $\operatorname{tp} W'' \prec \operatorname{tp} W$. □[10.72]

Now let $\langle \Sigma, W, I \rangle$ be fine such that $\operatorname{qd} W \preccurlyeq \langle 1, 0 \rangle$ and W has the least possible number of items involving class quantification not of the form specified in (10.71).

Suppose toward a contradiction that the number of such items in W is not zero. Let $W_i = \phi_i\binom{v_i}{\bar{c}_i} \to \forall v_i\, \phi_i$ be the last such item of W. Then v_i is necessarily of class sort and ϕ_i has no class quantifiers. Define W', T, I', and J as before. Since $\langle \Sigma, W, I\rangle$ is fine, $\forall v_i\, \phi_i$ does not occur in W' or I'. Since qd $J \prec \langle 1, 0\rangle$, $\forall v_i\, \phi_i$ does not occur in J. The only class quantifiers in Σ are in sentences of the form $\neg\forall v\, \neg\psi$, so the only way $\forall v_i\, \phi_i$ could occur in Σ would be if $\phi = \neg\psi$ and $\neg\forall v\, \phi \in \Sigma$, which is excluded by hypothesis. It therefore follows as before that $\Sigma \cup \operatorname{im} W' \cup I' \cup J$ is propositionally inconsistent, and the rest of the proof of (10.21) goes through without essential modification to complete the proof of the claim. $\square^{10.71}$

Let $\langle \Sigma, W, I\rangle$ be as specified in (10.71). The only occurrences of class quantification in W, and therefore also in I, are in expressions of the form $\forall_C V\, \psi$, where $\neg\forall_C V\, \psi \in \Sigma$. ψ is therefore of the form $\neg\forall_S v\, (\bar{v} \in \bar{V} \leftrightarrow \phi)$, where ϕ has no class quantification, so $\neg\forall_C V\, \psi$ is the instance

$$\exists_C V\, \forall_S v\, (\bar{v} \in \bar{V} \leftrightarrow \phi)$$

of the Comprehension schema (with $\neg\forall_C V\, \neg$ standing in for $\exists_C V$). We now wish to arrange that any occurrence of class quantification occurring in Σ also occurs in W. Let T be the set of sentences θ of the form $\forall_C V\, \psi$ such that $\neg\theta \in \Sigma$ but there is no sentence $\psi\binom{V}{\bar{C}} \to \forall_C V\, \psi$ in W. Let $\Sigma^0 \setminus \{\neg\theta \mid \theta \in T\}$. Then no member of T occurs in Σ^0, W, or I. Thus, if $\Sigma^0 \cup \operatorname{im} W \cup I$ is propositionally consistent then there exists an interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models \Sigma^0 \cup \operatorname{im} W \cup I$ and $\theta \notin \operatorname{dom} \mathfrak{I}$. We may therefore extend $\mathfrak{I}$ to an interpretation $\mathfrak{I}'$ such that $\mathfrak{I}' \models \Sigma \cup \operatorname{im} W \cup I$ by letting $\mathfrak{I}'(\theta) = 0$ for all $\theta \in T$. Since $\Sigma \cup \operatorname{im} W \cup I$ is propositionally inconsistent by hypothesis, $\Sigma^0 \cup \operatorname{im} W \cup I$ is propositionally inconsistent.

In the interest of notational uniformity, let $W^0 = W$ and $I^0 = I$. Then $\langle \Sigma^0, W^0, I^0\rangle$ is fine, and class quantification occurs in $\Sigma^0 \cup \operatorname{im} W^0 \cup I^0$ only in expressions of the form $\forall_C V\, \psi$, where ψ is of the form $\neg\forall_S v\, (\bar{v} \in \bar{V} \leftrightarrow \phi)$ and ϕ is class-quantification-free. Each such expression occurs once in Σ^0 and once in W; it may occur any number of times in I.

We will now eliminate all these remaining occurrences of class quantification. In the interest of notational uniformity, let ρ^0 be ρ', the 2-sorted signature c^1 expanded with constants. ρ^0 has one predicate index, which is binary, and no operation indices other than constants. Thus, Σ^0 is a finite set of constant instances of M, W^0 is a witness sequence, and I^0 is an instance set, all in the signature ρ^0; and $\Sigma^0 \cup \operatorname{im} W^0 \cup I^0$ is propositionally inconsistent.

(10.73) *For each set constant c in $\Sigma^0 \cup \operatorname{im} W^0 \cup I^0$ let v^c be a distinct new set variable.*

Recall that the only remaining occurrences of class quantification are related to instances of the class comprehension schema:

$$\neg\forall_C V\, \neg\forall_S v\, (\bar{v} \in \bar{V} \leftrightarrow \phi),$$

where ϕ has no class quantification and only the free variable v (along with set and class constants). Let

$$\langle \neg\forall v_k\, (\bar{v}_k \in \bar{C}_k \leftrightarrow \phi_k) \to \forall V_k\, \neg\forall v_k\, (\bar{v}_k \in \bar{V}_k \leftrightarrow \phi_k) \mid k \in K\rangle \tag{10.74}$$

enumerate the corresponding items of W^0 in the order in which they occur there.

Let $\langle c_0, \dots, c_{n^-}\rangle$ enumerate the set constants that occur in ϕ_0. Let P_0 be an $(n+1)$-ary predicate index, and, using the new set variables introduced above,[10.73] let

$$\eta_0 = \forall v_0 \left(\tilde{P}_0\langle \bar{v}^{c_0}, \dots, \bar{v}^{c_{n^-}}, \bar{v}_0\rangle \leftrightarrow \phi_0\left\{\begin{smallmatrix}\bar{c}_0 & \cdots & \bar{c}_{n^-}\\ \bar{v}^{c_0} & \cdots & \bar{v}^{c_{n^-}}\end{smallmatrix}\right\}\right), \tag{10.75}$$

and let

$$\delta_0 = \forall v^{c_0} \cdots \forall v^{c_{n^-}}\, \eta_0, \tag{10.76}$$

which is a ρ^0-definition of P_0 as a predicate index that takes $n+1$ set arguments.

Let ρ^1 be the expansion of ρ_0 by the addition of the predicate index P_0. Given an expression θ, let θ^1 be the result of replacing all occurrences of expressions of the form $\tau \in \bar{C}_0$ by $\tilde{P}_0\langle \bar{c}_0, \dots, \bar{c}_{n^-}, \tau\rangle$. Let Σ', W', and I' be the result of making this replacement in all sentences in Σ^0, W^0 and I^0, respectively. Then

$$\Sigma' \cup \operatorname{im} W' \cup I'$$

is propositionally inconsistent. Since W^0 is a witness sequence, C_0 does not occur in ϕ_0, and the first sentence in the sequence (10.74) has become

$$\neg \forall v_0\, (\tilde{P}_0\langle \bar{c}_0, \dots, \bar{c}_{n^-}, \bar{v}_0\rangle \leftrightarrow \phi_0) \rightarrow \forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0). \tag{10.77}$$

1. Let Σ^1 be obtained from Σ' by removing $\neg \forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0)$ and adding δ_0.[10.76]
2. Let W^1 be obtained from W' by removing

   $$\neg \forall v_0\, (\tilde{P}_0\langle \bar{c}_0, \dots, \bar{c}_{n^-}, \bar{v}_0\rangle \leftrightarrow \phi_0) \rightarrow \forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0).$$[10.77]

   W^1 is a witness sequence.
3. Let I^1 be obtained from I' by removing all sentences of the form

   $$\forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0) \rightarrow \neg \forall v_0\, (\bar{v}_0 \in \bar{C} \leftrightarrow \phi_0)$$

   and adding the instance sentences

   $$\forall v^{c_m} \cdots \forall v^{c_{n^-}}\, \forall v_0\, \eta_0\left(\begin{smallmatrix}v^{c_0} & \cdots & v^{c_{m^-}}\\ \bar{c}_0 & \cdots & \bar{c}_{m^-}\end{smallmatrix}\right) \rightarrow \forall v^{c_{m+1}} \cdots \forall v^{c_{n^-}}\, \forall v_0\, \eta_0\left(\begin{smallmatrix}v^{c_0} & \cdots & v^{c_m}\\ \bar{c}_0 & \cdots & \bar{c}_m\end{smallmatrix}\right), \tag{10.78}$$

   for all $m \in n$.

We will now show that $\Sigma^1 \cup \operatorname{im} W^1 \cup I^1$ is propositionally inconsistent. Suppose toward a contradiction that $\mathfrak{I} \models \Sigma^1 \cup \operatorname{im} W^1 \cup I^1$. Then $\mathfrak{I} \models \delta_0$ and $\mathfrak{I} \models$ (10.78) for all $m \in n$, so $\mathfrak{I} \models \forall v_0\, \eta_0\left(\begin{smallmatrix}v^{c_0} & \cdots & v^{c_{n^-}}\\ \bar{c}_0 & \cdots & \bar{c}_{n^-}\end{smallmatrix}\right)$, which is $\neg \forall v_0\, (\tilde{P}_0\langle \bar{c}_0, \dots, \bar{c}_{n^-}, \bar{v}_0\rangle \leftrightarrow \phi_0)$. Hence, $\mathfrak{I} \models$ (10.77), so $\mathfrak{I} \models W'$.

Since $\forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0)$ does not occur in $\Sigma^1 \cup \operatorname{im} W^1 \cup I^1$, we may assume that it is not $\operatorname{dom} \mathfrak{I}$. Extend $\mathfrak{I}$ to an interpretation $\mathfrak{I}'$ by setting

$$\mathfrak{I}'\big(\forall V_0\ \neg \forall v_0\, (\bar{v}_0 \in \bar{V}_0 \leftrightarrow \phi_0)\big) = 0.$$

Then $\mathfrak{I}' \models \Sigma'$ and $\mathfrak{I}' \models I'$. Hence, $\mathfrak{I}' \models \Sigma' \cup \operatorname{im} W' \cup I'$; contradiction.

Now eliminate all occurrences of $\forall V_1 \ \neg \forall v_1 \, (\bar{v}_1 \in \bar{V}_1 \leftrightarrow \phi_1^1)$[10.74] from Σ^1, W^1, and I^1 in the same way (ϕ_1^1 being the sentence that has resulted from the replacement of each occurrence of an expression $\tau \in \bar{C}_0$ by $\tilde{P}_0\langle \bar{c}_0, \ldots, \bar{c}_{n^-}, \tau\rangle$ in ϕ_1 as part of the previous elimination). Let δ_1 be the corresponding definition of P_1. For each sentence θ^1 arising from the previous transformation, let θ^2 be the result of replacing all expressions of the form $\tau \in \bar{C}_1$ by $\tilde{P}_2\langle \ldots, \tau\rangle$. Note that since W^1 is a witness sequence, C_1 does not occur in ϕ_0, so δ_0 is unaffected by this transformation, and δ_0 therefore remains a ρ^0-definition of P_0.

Let Σ^2, W^2, and I^2 be derived from Σ^1, W^1 and I^1 as before, by making the above replacement, eliminating all sentences containing $\forall V_1 \ \neg \forall v_1 \, (\bar{v}_1 \in \bar{V}_1 \leftrightarrow \phi_1)$, and adding δ_1 to Σ^1 and the appropriate instance sentences to I^1. δ_1 is a ρ^1-definition of P_1.

Proceed in this fashion to eliminate all class quantification. For each $k \leqslant K$, ρ^k is the expansion of ρ^0 by the addition of the predicate indices $P_0, \ldots, P_{k-1}$; and for each $k < K$, δ_k is a ρ^k-definition of P_k. Σ^K consists of the sentences θ^K derived from class-quantifier-free members θ of Σ^0, together with the definitions δ_k ($k < K$). W^K is the residual witness sequence. I^K is the final instance set.

For each remaining class constant C, let P^C be a new unary predicate; and let

$$\delta^C = \forall v \, (\tilde{P}^C\langle \bar{v}\rangle \leftrightarrow \phi^C),$$

where ϕ^C is an arbitrary c^1-formula with one free set variable v and only set quantification, in the original signature, specifically, without any constants. Let $\bar{\rho}$ be the expansion of ρ by the addition of these predicate indices, and for any ρ^K-expression θ let $\bar{\theta}$ be the result of replacing all occurrences of expressions of the form $\tau \in \bar{C}$ by $\tilde{P}^C\langle\tau\rangle$.[11] Let $\bar{\Sigma}^K$ be $\{\bar{\theta} \mid \theta \in \Sigma^K\}$ together with the definitions δ^C; and let $\bar{W}^K$ and $\bar{I}^K$ be the transformed versions of W^K and I^K. Then $\bar{\Sigma}^K \cup \operatorname{im} \bar{W}^K \cup \bar{I}^K$ is propositionally inconsistent.

$\bar{\Sigma}^K$ consists of $\bar{\rho}^K$-sentences derived from three sources:

1. Constant instances of the matrices obtained by removing all initial universal quantifications from C^1-axioms other than Comprehension.

2. Definitions δ_k.

3. Definitions δ^C.

All class constants have been eliminated in favor of defined predicates. The remaining constants in $\bar{\Sigma}^K$ are set constants in sentences of the first type. We transfer these from Σ to I as we have done previously. Thus, suppose $\theta \in \bar{\Sigma}^K$. Let $\langle c_0, \ldots, c_{n^-}\rangle$ enumerate the (set) constants that occur in θ. Let $\zeta = \theta\left\{\begin{smallmatrix} \bar{c}_0 & \cdots & \bar{c}_{n^-} \\ \bar{v}^{c_0} & \cdots & \bar{v}^{c_{n^-}} \end{smallmatrix}\right\}$. Replace θ in $\bar{\Sigma}^K$ by $\forall v^{c_0} \cdots \forall v^{c_{n^-}} \zeta$, and add the sentences

$$\forall v^{c_m} \cdots \forall v^{c_{n^-}} \forall v_0 \, \zeta\left(\begin{smallmatrix} v^{c_0} & \cdots & v^{c_{m^-}} \\ \bar{c}_0 & \cdots & \bar{c}_{m^-} \end{smallmatrix}\right) \rightarrow \forall v^{c_{m+1}} \cdots \forall v^{c_{n^-}} \forall v_0 \, \zeta\left(\begin{smallmatrix} v^{c_0} & \cdots & v^{c_m} \\ \bar{c}_0 & \cdots & \bar{c}_m \end{smallmatrix}\right)$$

to $\bar{I}^K$ for all $m \in n$. Since θ derives propositionally from ζ and the above instantiations, the resulting theory remains propositionally inconsistent.

[11] If our original proof that $\mathsf{C}^1 \vdash \sigma$ was efficient, there are no remaining class constants; nevertheless, we must eliminate them if present. It is not surprising that this may be done entirely arbitrarily. If we wished, however, we could choose the same formula for each ϕ^C, and we could use a single new predicate, rather than one for each remaining class constant. We could further simplify the transformation by choosing ϕ so that $\phi\binom{v}{\tau}$ is tautologically true (or false) for every τ.

Let Σ^* and I^* be the result of eliminating all constants from Σ in this way. Let $W^* = \bar{W}^K$. Then $\Sigma^* \cup \operatorname{im} W^* \cup I^*$ is propositionaly inconsistent. Since W^* is a witness sequence and no constant occurs in Σ^*, W^* is a witness sequence for Σ^*; hence, Σ^* is inconsistent.

Note that $\neg\,\sigma$ has been unaffected by all of the above transformations, and the only element of Σ^* derived from $\neg\,\sigma$ is $\neg\,\sigma$ itself. Let S^K be Σ^* with $\neg\,\sigma$ removed. Then

$$S \vdash \sigma.$$

(10.79) *S consists of $\bar{\rho}^K$ sentences of the following kinds.*

1. *Sentences derived from* C^1*-axioms, which are of the following forms, where P is a defined predicate, and all variables are of set sort.*

   $\mathsf{C}^1 1$:

$$\forall v_0 \cdots \forall v_{n^-}\, \forall u\, \forall v\, (\forall w\, (\bar{w} \in \bar{u} \leftrightarrow \bar{w} \in \bar{v}) \\ \to (\tilde{P}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}, \bar{u}\rangle \leftrightarrow \tilde{P}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}, \bar{v}\rangle)\big).$$

   $\mathsf{C}^1 2\mathsf{b}$: $\forall v_0 \cdots \forall v_{n^-}\, \forall u\, \exists v\, \forall w\, (\bar{w} \in \bar{v} \leftrightarrow \bar{w} \in \bar{u} \wedge \tilde{P}\langle \bar{v}_0, \ldots, \bar{v}_{n^-}, \bar{w}\rangle)$.

   $\mathsf{C}^1 3$: $\exists u\, \forall v\;\, \bar{v} \notin \bar{u}$.

   $\mathsf{C}^1 4$: $\forall u\, \forall v\, \exists w\, (\bar{u} \in \bar{w} \wedge \bar{v} \in \bar{w})$.

   $\mathsf{C}^1 5$:

$$\forall v_0 \cdots \forall v_{n^-}\, \forall u\, \big(\forall v \in \bar{u}\, \exists w\, \forall a\, (\phi \to \bar{a} \in \bar{w}) \\ \to \exists w\, \forall v \in \bar{u}\, \forall a\, (\phi \to \bar{a} \in \bar{w})\big),$$

   where ϕ is any $\bar{\rho}^K$-formula with only set quantification, and

$$v_0, \ldots, v_{n^-}, a, u, v, w$$

   are distinct variables such that $\operatorname{Free} \phi \subseteq \{v_0, \ldots, v_{n^-}, a, v\}$.

2. *Sentences derived from the definitions δ_k, $k = 0, \ldots, K-1$.*

3. *The definitions δ^C.*

Note that there are no elements of S derived from $\mathsf{C}^1 2\mathsf{a}$, as these have all been eliminated.

Using the definitions $\delta_{K^-}, \ldots, \delta_0$ in that order, and then the definitions δ^C in any order, we may replace each $\theta \in S$ by a sentence θ^* that is equivalent to θ over the theory D consisting of the definitions δ_k and δ^C. Let $S^* = \{\theta^* \mid \theta \in S \backslash D\}$. Since the definitions δ^C do not mention any added predicates, and—as noted above—δ_k only mentions predicates already defined by $\delta_{k'}$ $(k' < k)$, S is an extension-by-definition of S^*. Since extension-by-definition is conservative,[2.108.1]

$$S^* \vdash \sigma.$$

We now show that $\mathsf{S}^1 \vdash S^*$. First note that any element of S^* derived from $\mathsf{C}^1 2\mathsf{b}$, $\mathsf{C}^1 3$, $\mathsf{C}^1 4$, or $\mathsf{C}^1 5$ is logically equivalent to the correpsonding S^1-axiom: $\mathsf{S}^1 2$, $\mathsf{S}^1 3$, $\mathsf{S}^1 4$ or $\mathsf{S}^1 5$, respectively. Next note that any element of S^* derived from $\mathsf{C}^1 1$ is an S^1-theorem.[10.63]

It follows that $\mathsf{S}^1 \vdash \sigma$. $\square^{10.60.1}$

To show that C is a conservative extension of S we could adapt the preceding proof by including the $\mathsf{Foundation}$ axiom of C in the initial set Σ_0 of c^1-sentences assumed to be inconsistent. This leads to instances

$$\exists v \in \bar{C} \rightarrow \exists v \in \bar{C}\, \forall w \in \bar{v}\ \bar{w} \notin \bar{C},$$

for various class terms C, which get replaced by predicates P which are subsequently written out in terms of their definitions, yielding instances of the $\mathsf{Foundation}$ schema of S:

$$\forall v_0, \ldots, v_{n^-} \left(\exists v\, \phi \rightarrow \exists v \left(\phi \wedge \forall u \in v\ \neg\, \phi\tbinom{v}{u}\right)\right).$$

It is simpler, however, to use Theorem 3.116, which states that there are two instances, θ_0[3.111] and θ_1,[3.112] of the $\mathsf{Foundation}$ schema of S, from which the entire schema follows (in S^0). The proof of (3.116) is easily adapted to show that the $\mathsf{Foundation}$ axiom of C also follows (in C^0) from θ_0 and θ_1 (appropriately written with all quantification restricted to sets). Suppose σ is a sentence of pure set theory and $\mathsf{C} \vdash \sigma$. Then

$$\mathsf{C}^0 \vdash (\theta_0 \wedge \theta_1) \rightarrow \sigma.$$

Hence,

$$\mathsf{S}^0 \vdash (\theta_0 \wedge \theta_1) \rightarrow \sigma,$$

so $\mathsf{S} \vdash \sigma$. $\square^{10.60}$

10.13 Proof of (2.186)

[Refer to p. 163.]

(10.80) **Theorem** [S^0] S^0 *and* PA *are equi-interpretable.*

We make use of a system for using certain numbers to code certain finite sequences of numbers. Specifically, we define special numbers that code sequences $0, 1, 2, \ldots, k$ and $2^0, 2^1, 2^2, \ldots, 2^k$ in such a way that we can state arithmetically that two numbers occur in corresponding respective positions in these sequences.

We begin with some definitions and theorems that show that the Peano axioms permit the development of arithmetic in the familiar way. It would be excessively tedious to provide an exhaustive list of principles that we have understood since childhood, and the intent is only to develop an intuition for what it means to operate within the constraints of first-order predicate logic in the context of arithmetic.

(10.81) **Theorem** [PA]

1. $\forall n\, (0 + n = n + 0 = n)$.
2. $\forall m, n \left(\mathrm{S}\, m + n = m + \mathrm{S}\, n = \mathrm{S}(m + n)\right)$.
3. $\forall m, n\, (m + n = n + m)$.
4. $\forall k, m, n \left(k + (m + n) = (k + m) + n\right)$.
5. $\forall n\, (0 \cdot n = n \cdot 0 = 0)$.
6. $\forall m, n\, (m \cdot n = n \cdot m)$.

7. $\forall k, m, n\, \big(k \cdot (m \cdot n) = (k \cdot m) \cdot n\big)$.
8. $\forall k, m, n\, \big(k \cdot (m + n) = k \cdot m + k \cdot n\big)$.
9. $\forall m, m', n\, (n + m = n + m' \rightarrow m = m')$.
10. $\forall m, m'\, \forall n \neq 0\, (n \cdot m = n \cdot m' \rightarrow m = m')$.

Proof See Note 10.13.1. □

Definition [PA]

1. $1 \stackrel{\text{def}}{=} \mathrm{S}\,0$, $2 \stackrel{\text{def}}{=} \mathrm{S}\,1$.
2. $m \leqslant n \stackrel{\text{def}}{\iff} \exists k\, (m + k = n)$.
3. $m < n \stackrel{\text{def}}{\iff} m \leqslant n \wedge m \neq n$.
4. $n - m \stackrel{\text{def}}{=}$ *the unique*[10.81.9] k *such that* $m + k = n$ *if* $m \leqslant n$*; otherwise 0.*
5. m divides $n \stackrel{\text{def}}{\iff} m$ is a factor of $n \stackrel{\text{def}}{\iff} m|n \stackrel{\text{def}}{\iff} \exists k\ m \cdot k = n$.[12]
6. n is prime $\stackrel{\text{def}}{\iff} \forall m\, (m|n \rightarrow m = 1 \vee m = n)$.
7. *Suppose* p *is prime.* n *is a* p-power $\stackrel{\text{def}}{\iff} \forall m \neq 1\, (m|n \rightarrow p|m)$.

(10.82) **Theorem** [PA]

1. $\forall n\, (n \neq 0 \rightarrow \exists m\ n = \mathrm{S}\,m)$.
2. $\forall n\ 0 \leqslant n$.
3. $\forall n, k, l\, (n + k = n + l \rightarrow k = l)$.
4. $\forall m, n\, (m + n = 0 \rightarrow m = n = 0)$.
5. $\forall m, n\, (m \leqslant n \vee n \leqslant m)$.
6. $\forall m, n\, (m \leqslant n \wedge n \leqslant m \rightarrow m = n)$.
7. $\forall n\, \forall m \neq 0\, \exists! q\, \exists! r < m\, (n = m \cdot q + r)$.

Proof See Note 10.13.2. □

There are a number of other identities that we will use from time to time without proof. These are all easily derived using techniques similar to the preceding. Some examples are:

1. $n + 1 = \mathrm{S}\,n$.
2. $n \cdot 1 = n$.
3. If $n > 0$ and $m > 1$ then $n \cdot m > n$.
4. $m \leqslant n \leftrightarrow m + k \leqslant n + k$.
5. If $k \neq 0$ then $m \leqslant n \leftrightarrow m \cdot k \leqslant n \cdot k$.

Definition [PA] *Given* n, m*, if* $m \neq 0$*, then* $\mathrm{Rem}(n, m)$ *and* n/m *are by definition the unique numbers such that* $\mathrm{Rem}(n, m) < m$ *and* $n = (n/m) \cdot m + \mathrm{Rem}(n, m)$*. If* $m = 0$ *then both are* 0 *(for completeness).*

[12] Note that everything divides 0, 1 divides everything, and 0 divides only 0.

(10.83) **Theorem** [S^0] *Suppose ϕ is an* a*-formula and u is a variable not in ϕ. Then*

$$\mathsf{PA} \vdash \big(\exists v\ \phi \to \exists v\ \big(\phi \wedge \forall u < \bar{v}\ \neg\, \phi\tbinom{v}{\bar{u}}\big)\big).$$

Remark Remember that universal closure (i.e., universal quantification over all free variables) is understood in statements of provability.

Proof This is another form of the induction schema PA4. To prove it for a given formula ϕ, let

$$\phi' = \exists v < \bar{w}\ \phi \to \exists v < \bar{w}\ \big(\phi \wedge \forall u < \bar{v}\ \neg\, \phi\tbinom{v}{\bar{u}}\big),$$

where w is a variable other than v, u that is not in ϕ. Clearly, $\mathsf{PA} \vdash \phi'\binom{w}{\mathbf{0}}$ and $\mathsf{PA} \vdash \big(\phi' \to \phi'\binom{w}{\mathbf{S}\,\bar{w}}\big)$, so by PA2, $\mathsf{PA} \vdash \phi'$, i.e.,

$$\mathsf{PA} \vdash \forall w\ \big(\exists v < \bar{w}\ \phi \to \exists v < \bar{w}\ \big(\phi \wedge \forall u < \bar{v}\ \neg\, \phi\tbinom{v}{\bar{u}}\big)\big).$$

It follows directly that

$$\mathsf{PA} \vdash \big(\exists v\ \phi \to \exists v\ \big(\phi \wedge \forall u < \bar{v}\ \neg\, \phi\tbinom{v}{\bar{u}}\big)\big),$$

as claimed. $\square$ 10.83

We often use (10.83) in the form of the *least counterexample* principle: Letting $\phi = \neg\,\psi$, we use (10.83) to infer that if there is a counterexample to ψ, there is a least counterexample. We use 'proof by induction' to refer to invocations of (10.83) generally, not just to direct applications of PA4.

(10.84) **Theorem** [PA]

1. *Suppose p is prime and $p|(m \cdot n)$. Then $p|m$ or $p|n$.*
2. *Every $n > 1$ has a prime factor.*
3. *Suppose p is prime. n is a p-power iff n has no prime factor other than p.*

Proof 1 Suppose p does not divide m. We claim that for every n

$$p|(m \cdot n) \to p|n. \tag{10.85}$$

Suppose toward a contradiction that this is not universally true, and let n be the least counterexample, so $p|(m \cdot n)$ and $p \nmid n$. Note that n is not 0, 1, or p. If $n > p$ then $p|(m \cdot (n - p))$, and $p \nmid (n - p)$, so $n - p$ is a smaller counterexample. Hence, $n < p$ (since $n \neq p$). Let q, r be such that $p = n \cdot q + r$, with $r < n$. Then $m \cdot p = m \cdot n \cdot q + m \cdot r$, so $p|m \cdot r$. Since p is prime and n is not 1 or p, $r \neq 0$, and $r < n < p$, so $p \nmid r$. Thus, r is a smaller counterexample than n to (10.85). Contradiction.

2 Suppose toward a contradiction that for some $n > 1$, n has no prime factor. Let n be the least such. Then n is not prime. Let $m, k \neq 1$ be such that $m \cdot k = n$. Then $m, k < n$, so m, for example, has a prime factor $p|m$. Then $p|(m \cdot k) = n$. Contradiction.

3 Left to the reader. □[10.84]

(10.86) **Theorem** [PA] *Suppose p is prime.*

1. *1 is the least p-power.*
2. *Suppose n is a p-power. If $n > 1$ then $p|n$ and n/p is a p-power.*
3. *Suppose n is a p-power. Then for all m, if $n < m < p \cdot n$ then m is not a p-power; and $p \cdot n$ is a p-power. In other words, $p \cdot n$ is the next p-power.*
4. *Suppose m, n are p-powers and $m \leqslant n$. Then $m|n$ and n/m is a p-power.*

Proof 1 Trivial.

2 Suppose n is a p-power and $n > 1$. Then[10.84.2] n has a prime factor, which must be p, so $p|n$. Every divisor of n/p is a divisor of n and is therefore divisible by p, so n/p is a p-power.

3 By induction. Suppose n is a p-power, and for all $n' < n$, if n' is a p-power then $p \cdot n'$ is the next p-power. Suppose $n < m < p \cdot n$. If m is a p-power, let $m' = m/p$ and $n' = n/p$. Then n' is a p-power[10.86.3] and $n' < m' < p \cdot n'$, so by the induction hypothesis, m' is not a p-power. Hence,[10.86.3] m is not a p-power.

To show that $p \cdot n$ is a p-power, we observe that if it were not then[10.84.3] for some prime $q \neq p$, $q|(p \cdot n)$, whence[10.84.1] $q|n$, contrary to supposition.

4 By induction on $n \geqslant m$ for a given a p-power m. The result is trivial for $n = m$. Suppose $n > m$ is a p-power. Then $n \geqslant p \cdot m$.[10.86.4] Let $n' = n/p$. Then n' is a p-power[10.86.3] and $n' \geqslant m$, so by the induction hypothesis, $m|n'$ and n'/m is a p-power. Hence, $m|n$, and $n/m = p \cdot (n'/m)$ is a p-power.[10.86.3] □[10.86]

Definition [PA]

1. *n is* even $\overset{\text{def}}{\iff} 2|n$.
2. *n is* odd $\overset{\text{def}}{\iff}$ *n is not even.*
3. *m and n* have the same parity $\overset{\text{def}}{\iff}$ *both are even or both are odd; otherwise they* have opposite parity.

Definition [PA] m participates in n *or* is a participant in n $\overset{\text{def}}{\iff} m \,\mathrm{P}\, n \overset{\text{def}}{\iff} m$ *is a 2-power and n/m is odd.*

The following theorem shows how to use a number as a representation of the set of its participants, with numerical addition corresponding to set-theoretic union in the limited sense that if n_0 and n_1 have no participants in common then $n_0 + n_1$ represents the union of the sets represented by n_0 and n_1.

(10.87) **Theorem** [PA]

1. *$m \cdot n$ is odd iff either m and n are both odd.*
2. *$m + n$ is even iff m and n have the same parity (are either both even or both odd). m and $m + n$ have the same parity iff n is even.*

3. *Suppose m is a 2-power. Then m is the sole participant in m.*
4. *Suppose m is a 2-power.*
   1. *Suppose $m \not\mathrm{P}\, n$. Then*
   $$\forall k\,(k\,\mathrm{P}(n+m) \leftrightarrow k\,\mathrm{P}\,n \vee k = m).$$
   2. *Suppose $m\,\mathrm{P}\,n$. Then $n \geqslant m$ and*
   $$\forall k\,(k\,\mathrm{P}(n-m) \leftrightarrow k\,\mathrm{P}\,n \wedge k \neq m).$$
5. *Suppose n_0 and n_1 have no participants in common. Let $n = n_0 + n_1$. Then*
$$\forall m\,(m\,\mathrm{P}\,n \leftrightarrow m\,\mathrm{P}\,n_0 \vee m\,\mathrm{P}\,n_1).$$

Proof See Note 10.13.3. □

Definition [PA] *Suppose $k < l$ are 2-powers. Then $I(m,k,l) \stackrel{\text{def}}{=} (\mathrm{Rem}(m,l) - \mathrm{Rem}(m,k))/k$.*

(10.88) **Definition** [PA]

1. *a is an A-number* $\stackrel{\text{def}}{\Longleftrightarrow}$ $A(a)$ $\stackrel{\text{def}}{\Longleftrightarrow}$
   1. $1\,\mathrm{P}\,a$*;*
   2. $2\,\mathrm{P}\,a$*; and*
   3. *if $m_0 < m_1 < m_2$ are consecutive participants in a then $m_2/m_1 = (m_1/m_0) \cdot 2$.*
2. a and b code 2-exponentiation $\stackrel{\text{def}}{\Longleftrightarrow}$ $B(a,b)$ $\stackrel{\text{def}}{\Longleftrightarrow}$
   1. $A(a)$*;*
   2. *b is the smallest number such that*
      1. $I(b,1,2) = 1$*; and*
      2. *if $m_0 < m_1 < m_2$ are consecutive participants in a then $I(b,m_1,m_2) = \mathrm{S}(I(b,m_0,m_1))$.*

(10.89) **Theorem** [PA]

1. *$1+2$ is the least A-number, and $B(1+2,1)$. Note that every A-number has at least two participants, e.g., 1 and 2.*
2. *Suppose $B(a,b)$. Note that a is an A-number.*[10.88.2.1]
   1. *The A-numbers less than a are the numbers $\mathrm{Rem}(a,m)$, where $m > 1+2$ is a participant in a. Note that the A-numbers less than or equal to a are the numbers $\mathrm{Rem}(a,m)+m$, where $m > 1$ is a participant in a.*
   2. *Suppose $m > 1$ and $m\,\mathrm{P}\,a$. Let $a' = \mathrm{Rem}(a,m)+m$ and let $b' = \mathrm{Rem}(b,m)$. Then $B(a',b')$. Note that if m is the largest participant in a then $a' = a$ and $b' = b$; note that $b < m$.*
3. *Suppose $m_0 < m_1$ are the largest two participants in a. Let*
$$\begin{aligned} n &= I(b,m_0,m_1) \\ m_2 &= m_1 \cdot (m_1/m_0) \cdot 2 \\ a' &= a + m_2 \\ b' &= b + m_1 \cdot \mathrm{S}\,n. \end{aligned}$$

   Then

1. *a' is next A-number after a;*
2. *$m_1 < m_2$ are the largest participants in a';*
3. *$I(b', m_1, m_2) = \mathrm{S}\, n$; and*
4. *$B(a', b')$.*

4. *For each $n \geqslant 1$ there are unique a, b such that*
   1. *$B(a, b)$;*
   2. *$I(b, m_0, m_1) = n$, where $m_0 < m_1$ are the largest two participants in a.*

Proof 1 Clearly $1 + 2$ is an A-number, 1 and 2 are its only participants, and $I(1, 1, 2) = (\mathrm{Rem}(1, 2) - \mathrm{Rem}(1, 1))/1 = 1 - 0 = 1$, whereas $I(0, 1, 2) = 0$.

2 Straightforward.

3 Suppose $a, b, m_0, m_1, \ldots$ are as stated in (10.89.3).

3.1, 3.2 The participants of a' are those of a with the addition of m_2, which is the largest participant in a'. It is straightforward to check that a' is the next A-number after a.

3.3 In general, for 2-powers $k < l$, for any n, since $\mathrm{Rem}(n, l) < l$ and $k|(\mathrm{Rem}(n, l) - \mathrm{Rem}(n, k))$, $I(n, k, l) < l/k$. Hence $n < m_1/m_0$. It follows that $\mathrm{S}\, n = n + 1 < (m_1/m_0) + 1 \leqslant (m_1/m_0) \cdot 2 = m_2/m_1$. Since $b < m_1$,[10.89.3.2] $I(b', m_1, m_2) = \mathrm{S}\, n$.

3.4 It is straightforward to check that b' satisfies (10.88.2.1–2) *vis-à-vis* a' and that it is the least number that does, so $B(a', b')$.

4 Straightforward induction on n. $\square$[10.89]

Definition [PA]

1. $\mathrm{Exp}\, n \overset{\text{def}}{=} 1$ *if $n = 0$; otherwise m_1/m_0, where $m_0 < m_1$ are the largest two participants in an A-number a for which there exists b such that $B(a, b)$ and $I(b, m_0, m_1) = n$.*
2. $m \,\mathrm{E}\, n \overset{\text{def}}{\iff} \mathrm{Exp}\, m$ *participates in n.*

(10.90) **Theorem** [PA]

1. $\mathrm{Exp}\, 0 = 1$.
2. $\forall n\ \mathrm{Exp}(\mathrm{S}\, n) = 2 \cdot \mathrm{Exp}\, n$.
3. *Suppose n is a 2-power. Then $\exists! m\ n = \mathrm{Exp}\, m$.*
4. $n = 0 \leftrightarrow \forall m\ \neg\, m \,\mathrm{E}\, n$.
5. $\forall m, n\ \mathrm{Exp}(m + n) = \mathrm{Exp}\, m \cdot \mathrm{Exp}\, n$.

Proof The proofs of (1–3) are straightforward given (10.89). For (4), if $n = 0$ then all 2-powers divide it evenly, so $\forall m \neg m \,\mathrm{P}\, n$; whereas if $n > 0$ then $m \,\mathrm{P}\, n$, where m is the largest 2-power less than or equal to n. (5) follows from (1) by induction on n for any given m. $\square^{6.10}$

The following theorem shows that S^0 is interpretable in PA. a^+ is the expansion of the basic signature a of arithmetic by the addition of the various predicate and operation indices introduced in the course of this discussion. In particular, E is a predicate of $\mathcal{L}^{\mathsf{a}^+}$; in fact, we really only have to consider expressions that use 'E' and '=' exclusively, as these correspond to the membership and identity predicates of set theory.

(10.91) **Theorem** [S^0]

1. $\mathsf{PA} \vdash (\forall \mathrm{v}_2\, (\mathrm{v}_2 \,\mathbf{E}\, \mathrm{v}_0 \leftrightarrow \mathrm{v}_2 \,\mathbf{E}\, \mathrm{v}_1) \to \mathrm{v}_0 = \mathrm{v}_1)$.[13]
2. *Suppose ϕ is an a^+-formula, and u, v are distinct variables with $u \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \big(\exists v\, \phi \to \exists v\, \big(\phi \wedge \forall u \,\mathbf{E}\, v\; \neg \phi\tbinom{v}{u}\big)\big).$$
3. *Suppose ϕ is an a^+-formula, and u, v, w are distinct variables with $u, w \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \exists w\, \forall v\, (v \,\mathbf{E}\, w \leftrightarrow v \,\mathbf{E}\, u \wedge \phi).$$
4. $\mathsf{PA} \vdash \exists \mathrm{v}_0\, \forall \mathrm{v}_1\; \mathrm{v}_1 \not\mathbf{E}\, \mathrm{v}_0$.
5. $\mathsf{PA} \vdash \exists \mathrm{v}_2\, (\mathrm{v}_0 \,\mathbf{E}\, \mathrm{v}_2 \wedge \mathrm{v}_1 \,\mathbf{E}\, \mathrm{v}_2)$.
6. $\mathsf{PA} \vdash \exists \mathrm{v}_1\, \forall \mathrm{v}_2, \mathrm{v}_3\, (\mathrm{v}_2 \,\mathbf{E}\, \mathrm{v}_3 \,\mathbf{E}\, \mathrm{v}_0 \to \mathrm{v}_2 \,\mathbf{E}\, \mathrm{v}_1)$.
7. $\mathsf{PA} \vdash \exists \mathrm{v}_1\, \forall \mathrm{v}_2\, \big(\forall \mathrm{v}_1\, (\mathrm{v}_1 \,\mathbf{E}\, \mathrm{v}_2 \to \mathrm{v}_1 \,\mathbf{E}\, \mathrm{v}_0) \to \mathrm{v}_2 \,\mathbf{E}\, \mathrm{v}_1\big)$.
8. *Suppose ϕ is an a-formula, and a, u, v, w are distinct variables with $u, w \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \big(\forall v \,\mathbf{E}\, u\, \exists w\, \forall a\, (\phi \to a \,\mathbf{E}\, w) \to \exists w\, \forall v \,\mathbf{E}\, u\, \forall a\, (\phi \to a \,\mathbf{E}\, w)\big).$$

Proof See Note 10.13.4. $\square$

We are now in a position to define the requisite interpretations. To interpret[2.115] S^0 in PA we extend PA by definition to incorporate the predicate E, and we interpret $\in$ as E. The domain-defining formula ψ may be taken to be $\ulcorner \mathrm{v}_0 = \mathrm{v}_0 \urcorner$ or any other formula with one free variable v such that $\mathsf{PA} \vdash \forall v\ \psi$. Theorem 10.91 shows that this works.

To interpret PA in S^0, using an appropriate extension-by-definition of S^0 with ordinal operations, we may take the domain-defining formula ψ to be $\ulcorner \mathrm{v}_0$ is a finite ordinal$\urcorner$, and interpret the zero, successor, addition, and multiplication operations of PA as the corresponding operations on ordinals as given by (3.120) (the successor operation being $\alpha \mapsto \alpha + 1$). $\square^{10.80}$

Letting ι be the interpretation just defined of S^0 in PA, PA easily shows that if $x \,\mathrm{E}\, y$ then $x < y$, so given any instance θ of the Foundation schema of S, $\iota\theta$ is provable in PA using the related instance of the Induction schema of PA. Hence, ι interprets S in PA. Also, the set-theoretical axiom of finiteness, $\neg$ Infinity, becomes a PA-theorem (easily proved by induction). Thus, ι interprets F in PA. As an

[13] Remember that $\Theta \vdash \psi \overset{\text{def}}{\Longleftrightarrow} \Theta \vdash \bar{\forall}\psi$, where $\bar{\forall}\psi$ is the universal closure of ψ.

interpretation of F, ι is conservative in the sense that any set-theoretical sentence whose arithmetical interpretation is a PA-theorem is an F-theorem.

The interpretation given above of PA in S^0 is, of course, also an interpretation in the stronger theory F, and as such it is also conservative in this sense.

10.13.1 Proof of (10.81)

[REFER TO P. 740.]

Theorem [PA]

1. $\forall n\ 0+n=n+0=n$.
2. $\forall m,n\ \mathrm{S}\,m+n=m+\mathrm{S}\,n=\mathrm{S}(m+n)$.
3. $\forall m,n\,m+n=n+m$.
4. $\forall k,m,n\,(k+(m+n))=((k+m)+n)$.
5. $\forall n\ 0\cdot n=n\cdot 0=0$.
6. $\forall m,n\,m\cdot n=n\cdot m$.
7. $\forall k,m,n\,(k\cdot(m\cdot n))=((k\cdot m)\cdot n)$.
8. $\forall k,m,n\ k\cdot(m+n)=k\cdot m+k\cdot n$.
9. $\forall m,m',n\,(n+m=n+m'\rightarrow m=m')$.
10. $\forall m,m'\,\forall n\neq 0\,(n\cdot m=n\cdot m'\rightarrow m=m')$.

Proof 1 The second equality is in PA2. We prove the first equality by induction on n. This is trivial for $n=0$. Now suppose $0+n=n+0$. By PA2, $n+0=n$. Using PA2 twice again, $0+\mathrm{S}\,n=\mathrm{S}(0+n)=\mathrm{S}(n+0)=\mathrm{S}\,n=\mathrm{S}\,n+0$. By Induction, therefore, $\forall n\ 0+n=n+0$.

2 The second equality is in PA2. We prove the first equality by induction on n. For $n=0$ we have $\mathrm{S}\,m+0=\mathrm{S}\,m=\mathrm{S}(m+0)=m+\mathrm{S}\,0$. Suppose for a given n that for all m, $\mathrm{S}\,m+n=m+\mathrm{S}\,n$. Then for any m, $\mathrm{S}\,m+\mathrm{S}\,n=\mathrm{S}(\mathrm{S}\,m+n)=\mathrm{S}(m+\mathrm{S}\,n)=m+\mathrm{S}(\mathrm{S}\,n)$. By Induction, therefore, $\forall n,m\ \mathrm{S}\,m+n=m+\mathrm{S}\,n$.

3 By induction on m. For $m=0$ this is (10.81.1). Now suppose, for a given m, that $\forall n\ m+n=n+m$. Then for any n, $n+\mathrm{S}\,m=\mathrm{S}(n+m)=\mathrm{S}(m+n)=\mathrm{S}\,m+n$.[10.81.2] By Induction, therefore, $\forall m,n\ m+n=n+m$.

4 By induction on k. We start with $0+(m+n)=m+n=(0+m)+n$. Then assume $\forall m,n\,(k+(m+n))=((k+m)+n)$ and infer that $\mathrm{S}\,k+(m+n)=\mathrm{S}(k+(m+n))=\mathrm{S}((k+m)+n)=\mathrm{S}(k+m)+n=(\mathrm{S}\,k+m)+n$.

5 The second equality is in PA3. We now show by induction on n that $0\cdot n=0$. For $n=0$ this again is in PA3. Suppose $0\cdot n=0$. Then by PA3, $0\cdot\mathrm{S}\,n=0\cdot n+0=0\cdot n=0$. By Induction, therefore, $\forall n\ 0\cdot n=n\cdot 0=0$.

6 We first show by induction on n that

$$\forall m \ \mathrm{S}\, m \cdot n = (m \cdot n) + n. \tag{10.92}$$

For $n = 0$ we have $\mathrm{S}\, m \cdot 0 = 0 = (m \cdot 0) + 0$. Now suppose (10.92) holds for a given n. Then for any m, $\mathrm{S}\, m \cdot \mathrm{S}\, n = (\mathrm{S}\, m \cdot n) + \mathrm{S}\, m = ((m \cdot n) + n) + \mathrm{S}\, m = (m \cdot n) + (n + \mathrm{S}\, m) = (m \cdot n) + (\mathrm{S}\, n + m) = (m \cdot n) + (m + \mathrm{S}\, n) = ((m \cdot n) + m) + \mathrm{S}\, n = (m \cdot \mathrm{S}\, n) + \mathrm{S}\, n$. By Induction, therefore, $\forall n, m \ \mathrm{S}\, m \cdot n = (m \cdot n) + n$.

Now we show by induction on m that

$$\forall n\, m \cdot n = n \cdot m. \tag{10.93}$$

For $m = 0$ this is (10.81.5). Now suppose (10.93) holds for a given m. Then for any n, $\mathrm{S}\, m \cdot n = (m \cdot n) + n = (n \cdot m) + n = n \cdot \mathrm{S}\, m$. By Induction, therefore, $\forall m, n \ m \cdot n = n \cdot m$.

7 Left to the reader.

8 By induction on k. For $k = 0$ the result is trivial. Suppose it holds for k. Then $\mathrm{S}\, k \cdot (m + n) = k \cdot (m + n) + (m + n) = (k \cdot m + k \cdot n) + (m + n) = (k \cdot m + m) + (k \cdot n + n) = \mathrm{S}\, k \cdot m + \mathrm{S}\, k \cdot n$.

9, 10 Left to reader. $\square$[10.81]

10.13.2 Proof of (10.82)

[Refer to p. 741.]

Theorem [PA]

1. $\forall n\, (n \neq 0 \rightarrow \exists m \ n = \mathrm{S}\, m)$.
2. $\forall n \ 0 \leqslant n$.
3. $\forall n, k, l\, (n + k = n + l \rightarrow k = l)$.
4. $\forall m, n\, (m + n = 0 \rightarrow m = n = 0)$.
5. $\forall m, n\, (m \leqslant n \vee n \leqslant m)$.
6. $\forall m, n\, (m \leqslant n \wedge n \leqslant m \rightarrow m = n)$.
7. $\forall n\, \forall m \neq 0\, \exists! q\, \exists! r < m \ n = m \cdot q + r$.

Proof 1 By induction on n. This is trivially true for $n = 0$, and $\mathrm{S}\, n = \mathrm{S}\, n$, so it is true for $\mathrm{S}\, n$ (even without the induction hypothesis).

2 $n = 0 + n$.

3 This is trivial for $n = 0$. If it is true for n and $\mathrm{S}\, n + k = \mathrm{S}\, n + l$ then $n + \mathrm{S}\, k = n + \mathrm{S}\, l$, so $\mathrm{S}\, k = \mathrm{S}\, l$, so $k = l$ by PA1.

4 Suppose $m + n = 0$ and suppose toward a contradiction and without loss of generality that $m \neq 0$. Let l be such that $m = \mathrm{S}\, l$. Then $0 = m + n = \mathrm{S}\, l + n = \mathrm{S}(l + n)$, which contradicts PA1.

5 Let m be given. We will show by induction on n that

$$m \leqslant n \vee n \leqslant m.$$

For $n = 0$, $n \leqslant m$.[10.82.2] To handle the induction step, suppose first that $m \leqslant n$, say $n = m + k$. Then $\mathrm{S}\, n = m + \mathrm{S}\, k$, so $m \leqslant \mathrm{S}\, n$. Now suppose $n \leqslant m$, say $m = n + k$. If $k = 0$ then $n = m$, so $\mathrm{S}\, n = \mathrm{S}\, m = \mathrm{S}(m + 0) = m + \mathrm{S}\, 0$, so $m \leqslant \mathrm{S}\, n$; whereas if $k \neq 0$ then[10.82.1] for some l, $k = \mathrm{S}\, l$, so $m = n + \mathrm{S}\, l = \mathrm{S}\, n + l$, so $\mathrm{S}\, n \leqslant m$.

6 Suppose $m \leqslant n$ and $n \leqslant m$. Let k, l be such that $n = m + k$ and $m = n + l$. Then $n + 0 = n = (n + l) + k = n + (l + k)$, so[10.82.3] $l + k = 0$, so[10.82.4] $l = k = 0$, so $m = n$.

7 To show uniqueness, suppose $m \cdot q + r = m \cdot q' + r'$, where $r, r' < m$. If $q \neq q'$, suppose without loss of generality that $q < q'$, and let $k \neq 0$ be such that $q' = q + k$. Then $m \cdot q + r = m \cdot (q + a) + r' = m \cdot q + m \cdot k + r'$. Hence, $r = m \cdot k + r'$. Let l be such that $k = \mathrm{S}\, l$. Then $r = m \cdot l + m + r'$, so $r \geqslant m$, contrary to supposition.

We show existence by induction on n, given m. If $n = 0$ then $r = q = 0$ will do. Suppose $n = m \cdot q + r$ with $r < m$. Then $\mathrm{S}\, n = m \cdot q + \mathrm{S}\, r$. If $\mathrm{S}\, r < m$ we are finished. Otherwise, $\mathrm{S}\, r \geqslant m$. In fact, $\mathrm{S}\, r = m$, because if $\mathrm{S}\, r > m$ then for some $k \neq 0$, $\mathrm{S}\, r = m + k$; letting l be such that $k = \mathrm{S}\, l$, $\mathrm{S}\, r = m + \mathrm{S}\, l$, so $r = m + l$, so $r \geqslant m$, contrary to supposition. Thus, $\mathrm{S}\, n = m \cdot q + m = m \cdot \mathrm{S}\, q$. □[10.82]

10.13.3 Proof of (10.87)

[Refer to p. 743.]

Theorem [PA]

1. *$m \cdot n$ is odd iff either m and n are both odd.*
2. *$m + n$ is even iff m and n have the same parity (are either both even or both odd). m and $m + n$ have the same parity iff n is even.*
3. *Suppose m is a 2-power. Then m is the sole participant in m.*
4. *Suppose m is a 2-power.*
    1. *Suppose $m \not\mathrm{P}\, n$. Then*

    $$\forall k\, (k\, \mathrm{P}(n + m) \leftrightarrow k\, \mathrm{P}\, n \vee k = m).$$

    2. *Suppose $m\, \mathrm{P}\, n$. Then $n \geqslant m$ and*

    $$\forall k\, (k\, \mathrm{P}(n - m) \leftrightarrow k\, \mathrm{P}\, n \wedge k \neq m).$$

5. *Suppose n_0 and n_1 have no participants in common. Let $n = n_0 + n_1$. Then*

$$\forall m\, (m\, \mathrm{P}\, n \leftrightarrow m\, \mathrm{P}\, n_0 \vee m\, \mathrm{P}\, n_1).$$

Proof 1, 2 Straightforward.

3 $m/m = 1$. If $k < m$ is a 2-power then m/k is a 2-power > 1, hence even; and if $k > m$ then $m/k = 0$, also even.

4.1 Suppose $m \not\mathrm{P}\, n$. Let $r < m$ and q be such that

$$n = r + q \cdot m.$$

Then $q = n/m$, so q is even, since $m \not\mathrm{P}\, n$. Let $n' = n + m = r + (q + 1) \cdot m$. $q + 1$ is odd, so $m \,\mathrm{P}\, n'$.

If $k < m$ is a 2-power, then $\mathrm{Rem}(m, k) = 0$ so $n'/k = n/k + m/k$. Since m/k is even, n/k and n'/k are both even or both odd, so $k \,\mathrm{P}\, n \leftrightarrow k \,\mathrm{P}\, n'$.

If $k > m$ is a 2-power then $k \geqslant m \cdot 2$ and $(m \cdot 2)|k$. Let $k' = k/(m \cdot 2)$. As we have noted, q is even. Let $q' = q/2$. Then

$$n = r + q' \cdot (m \cdot 2).$$

Let $r' < k'$ and q'' be such that

$$q' = r' + q'' \cdot k'.$$

Then

$$\begin{aligned} n &= r + (r' + q'' \cdot k') \cdot (m \cdot 2) \\ &= r + r' \cdot m \cdot 2 + q'' \cdot k \\ \text{and } n' &= r + m + r' \cdot m \cdot 2 + q'' \cdot k. \end{aligned}$$

Since $r < m$, $r + m < m + m = m \cdot 2$. Since $r' < k'$, $r' + 1 \leqslant k'$. Hence

$$\begin{aligned} r + m + r' \cdot m \cdot 2 &< m \cdot 2 + r' \cdot m \cdot 2 \\ &= (r' + 1) \cdot m \cdot 2 \\ &\leqslant k' \cdot m \cdot 2 \\ &= k. \end{aligned}$$

It follows that $n/k = q'' = n'/k$, so $k \,\mathrm{P}\, n \leftrightarrow k \,\mathrm{P}\, n'$.

4.2 Similar to the preceding.

5 Suppose the assertion is false. Let n_0 be least such that it fails for some n_1, and let n_1 be least such that it fails for n_0, n_1. Let $n = n_0 + n_1$. Then n_0 and n_1 have no participants in common, and for some m either

1. $m \,\mathrm{P}\, n_0 \vee m \,\mathrm{P}\, n_1$, but $m \not\mathrm{P}\, n$; or
2. $m \,\mathrm{P}\, n$, but $m \not\mathrm{P}\, n_0$ and $m \not\mathrm{P}\, n_1$.

(10.94) *Let m be the least such.*

In case (1), suppose $m \,\mathrm{P}\, n_0$. Then $m \not\mathrm{P}\, n_1$. Note that $m \leqslant n_0 \leqslant n$. The participants in $n_0 - m$ are those in n_0 except for m,[10.874.2] so $n_0 - m$ and n_1 have no participants in common, and m participates in neither of them, so by the minimality of n_0, $m \not\mathrm{P}(n - m)$. But by (10.87.4.), $m \,\mathrm{P}(n - m)$; contradiction. If $m \,\mathrm{P}\, n_1$ we obtain a contradiction similarly.

In case (2), by arguments similar to those presented for (10.87.4), there exist $r, r_0, r_1 < m$ and q, q_0, q_1 such that

$$\begin{aligned} n &= r + m + q \cdot m \cdot 2 \\ n_0 &= r_0 + q_0 \cdot m \cdot 2 \\ n_1 &= r_1 + q_1 \cdot m \cdot 2. \end{aligned}$$

Note that

$$n = n_0 + n_1 = r_0 + r_1 + (q_0 + q_1) \cdot m \cdot 2.$$

Since $r + m < m \cdot 2$ and $r_0 + r_1 < m \cdot 2$,

$$r + m = \mathrm{Rem}(n, m \cdot 2) = r_0 + r_1.$$

If $m = 1$ then $r = r_0 = r_1 = 0$; contradiction. Hence $m > 1$. As m is a 2-power, $2|m$. Since $r + m \geqslant m$, either $r_0 \geqslant m/2$ or $r_1 \geqslant m/2$.

Suppose $r_0 \geqslant m/2$. Then $n_0 = r'_0 + m/2 + q_0 \cdot m \cdot 2$, with $r'_0 < m/2$, so $(m/2)\,\mathrm{P}\,n_0$. By the assumed[10.94] minimality of m, $(m/2)\,\mathrm{P}\,n$, from which it follows that $r \geqslant m/2$. Since n_0 and n_1 have no participants in common, $(m/2) \not\mathrm{P}\, n_1$, so $r_1 < m/2$. Hence

$$r_0 + r_1 < m + m/2 \leqslant m + r;$$

contradiction.

Similarly, if we suppose $r_1 \geqslant m/2$, we arrive at a contradiction. $\square$[10.87]

10.13.4 Proof of (10.91)

[Refer to p. 746.]

Theorem $[\mathsf{S}^0]$

1. $\mathsf{PA} \vdash (\forall \mathrm{v}_2\,(\mathrm{v}_2\,\mathbf{E}\,\mathrm{v}_0 \leftrightarrow \mathrm{v}_2\,\mathbf{E}\,\mathrm{v}_1) \rightarrow \mathrm{v}_0 = \mathrm{v}_1)$.[14]
2. *Suppose ϕ is an a^+-formula, and u, v are distinct variables with $u \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \big(\exists v\,\phi \rightarrow \exists v\,\big(\phi \wedge \forall u\,\mathbf{E}\,v\ \neg\,\phi\big(\tbinom{v}{u}\big)\big)\big).$$
3. *Suppose ϕ is an a^+-formula, and u, v, w are distinct variables with $u, w \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \exists w\,\forall v\,(v\,\mathbf{E}\,w \leftrightarrow v\,\mathbf{E}\,u \wedge \phi).$$
4. $\mathsf{PA} \vdash \exists \mathrm{v}_0\,\forall \mathrm{v}_1\ \mathrm{v}_1 \not\mathbf{E}\,\mathrm{v}_0$.
5. $\mathsf{PA} \vdash \exists \mathrm{v}_2\,(\mathrm{v}_0\,\mathbf{E}\,\mathrm{v}_2 \wedge \mathrm{v}_1\,\mathbf{E}\,\mathrm{v}_2)$.
6. $\mathsf{PA} \vdash \exists \mathrm{v}_1\,\forall \mathrm{v}_2, \mathrm{v}_3\,(\mathrm{v}_2\,\mathbf{E}\,\mathrm{v}_3\,\mathbf{E}\,\mathrm{v}_0 \rightarrow \mathrm{v}_2\,\mathbf{E}\,\mathrm{v}_1)$.
7. $\mathsf{PA} \vdash \exists \mathrm{v}_1\,\forall \mathrm{v}_2\,\big(\forall \mathrm{v}_1\,(\mathrm{v}_1\,\mathbf{E}\,\mathrm{v}_2 \rightarrow \mathrm{v}_1\,\mathbf{E}\,\mathrm{v}_0) \rightarrow \mathrm{v}_2\,\mathbf{E}\,\mathrm{v}_1\big)$.
8. *Suppose ϕ is an a-formula, and a, u, v, w are distinct variables with $u, w \notin \mathrm{Free}\,\phi$. Then*
$$\mathsf{PA} \vdash \big(\forall v\,\mathbf{E}\,u\,\exists w\,\forall a\,(\phi \rightarrow a\,\mathbf{E}\,w) \rightarrow \exists w\,\forall v\,\mathbf{E}\,u\,\forall a\,(\phi \rightarrow a\,\mathbf{E}\,w)\big).$$

Proof 1 Reason as follows in PA:[15]

⌜Suppose toward a contradiction that for some n_0 and n_1

$$\forall n_2\,(n_2\,\mathrm{E}\,n_0 \leftrightarrow n_2\,\mathrm{E}\,n_1) \wedge n_0 \neq n_1.$$

Let n_0 be least for which such n_1 exists, and let n_1 be such that $\forall n_2\,(n_2\,\mathrm{E}\,n_0 \leftrightarrow n_2\,\mathrm{E}\,n_1)$, but $n_0 \neq n_1$. If $n_0 = 0$ then $\forall n_2\ \neg\,n_2\,\mathrm{E}\,n_0$, so $\forall n_2\ \neg\,n_2\,\mathrm{E}\,n_1$, from which it follows that $n_1 = 0$.

Hence, $n_0 > 0$. Let[10.90.4] m be such that $m\,\mathrm{E}\,n$. Let $n'_0 = n_0 - \mathrm{Exp}\,m$ and $n'_1 = n_1 - \mathrm{Exp}\,m$. Then $\forall n_2\,(n_2\,\mathrm{E}\,n'_0 \leftrightarrow n_2\,\mathrm{E}\,n'_1)$, so by the posited minimality of n_0, $n'_0 = n'_1$. Hence $n_0 = n_1$, contrary to hypothesis.⌝

[14] Remember that $\Theta \vdash \psi \overset{\text{def}}{\iff} \Theta \vdash \bar{\forall}\psi$, where $\bar{\forall}\psi$ is the universal closure of ψ.

[15] As always, we suppose that variables indicated by distinct names are distinct.

2 Reason as follows in PA:

$\ulcorner$Suppose that there exists a number (v) such that (ϕ), and let (v) be the least such. Then, since $(u)\,\mathrm{E}\,(v) \to (u) < (v)$, $(\forall u\,\mathbf{E}\,v\ \neg\,\phi\binom{v}{u})$.$\urcorner$

In other words, the wellorderedness of universe of arithmetic—embodied in the induction principle—implies the wellfoundedness of E.

3 Reason as follows in PA:

$\ulcorner$Suppose toward a contradiction that for some (u) it is not the case that

$$(\exists w\,\forall v\,(v\,\mathbf{E}\,w \leftrightarrow (v\,\mathbf{E}\,u \wedge \phi))),$$

and suppose (u) is the least such number. If $(u) = 0$ then, letting $(w) = 0$, trivially, $(\forall v\,(v\,\mathbf{E}\ w \leftrightarrow (v\,\mathbf{E}\,u \wedge \phi)))$. Hence $(u) \neq 0$. Let (r) be such that $(r\,\mathbf{E}\,u)$. Then $\mathrm{Exp}\,(r)\,\mathrm{P}\,(u)$. Let $(u') = (u) - \mathrm{Exp}\,(r)$. Then $(r\,\not\mathbf{E}\,u')$ and $\forall l\,(l\,\mathrm{E}\,(u) \leftrightarrow l\,\mathrm{E}\,(u') \vee l = (r))$. Since $(u') < (u)$, for some (w'),

$$(\forall v\,(v\,\mathbf{E}\,w' \leftrightarrow (v\,\mathbf{E}\,u' \wedge \phi))).$$

If $(\phi\binom{v}{r})$, let $(w) = (w)' + \mathrm{Exp}\,(r)$; otherwise, let $(w) = (w')$. Then clearly

$$(\forall v\,(v\,\mathbf{E}\,w \leftrightarrow (v\,\mathbf{E}\,u \wedge \phi))),$$

contradicting the choice of (u).$\urcorner$

4 Reason as follows in PA:

$\ulcorner$As we have previously noted, $\forall n\ n \not\mathrm{E}\, 0$.$\urcorner$

Note that all we really need for the Existence axiom of set theory is the assertion that something exists, and in PA the mere presence of the 0-ary operation symbol $\ulcorner 0 \urcorner$ suffices, as $\exists v\ v = \bar{C}$ is a theorem of pure logic in any signature containing a 0-ary operation symbol C.

5 Reason as follows in PA:

$\ulcorner$Let

$$n_2 = \begin{cases} \mathrm{Exp}\,n_0 + \mathrm{Exp}\,n_1 & \text{if } n_0 \neq n_1 \\ \mathrm{Exp}\,n_0 & \text{if } n_0 = n_1. \end{cases}$$

Then $n_0\,\mathrm{E}\,n_2$ and $n_1\,\mathrm{E}\,n_2$.$\urcorner$

6 Reason as follows in PA:

$\ulcorner$Suppose toward a contradiction that for some n_0 it is not the case that

$$\exists n_1\,\forall n_2, n_3\,(n_2\,\mathrm{E}\,n_3\,\mathrm{E}\,n_0 \to n_2\,\mathrm{E}\,n_1),$$

and suppose n_0 is the least such. Then $n_0 \neq 0$, because in that case we could take n_1 to be any number. Let n_3' be such that $n_3'\,\mathrm{E}\,n_0$, i.e., $(\mathrm{Exp}\,n_3')\,\mathrm{P}\,n_0$, and let $n_0' = n_0 - \mathrm{Exp}\,n_3'$. Since $n_0' < n_0$, there exists n_1' such that

$$\forall n_2, n_3\,(n_2\,\mathrm{E}\,n_3\,\mathrm{E}\,n_0' \to n_2\,\mathrm{E}\,n_1').\urcorner$$

Here we insert a PA-proof of

$$\ulcorner\exists k\,\forall n_2\,(n_2\,\mathrm{E}\,k \leftrightarrow n_2\,\mathrm{E}\,n_3' \wedge n_2\,\not\mathrm{E}\,n_1')\urcorner,$$

which we know exists by virtue of (10.91.3). Now we continue to reason as follows in PA:

⌜Let k be such that

$$\forall n_2 \, (n_2 \, \mathrm{E} \, k \leftrightarrow n_2 \, \mathrm{E} \, n_3' \wedge n_2 \, \not\mathrm{E} \, n_1').$$

Thus, for any m, $m \, \mathrm{P} \, k$ iff $m \, \mathrm{P} \, n_3' \wedge m \, \not\mathrm{P} \, n_1'$. k and n_1' have no participants in common, so,[10.87.5] letting $n_1 = n_1' + k$, for all m

$$\begin{aligned} m \, \mathrm{P} \, n_1 &\leftrightarrow m \, \mathrm{P} \, n_1' \vee m \, \mathrm{P} \, k \\ &\leftrightarrow m \, \mathrm{P} \, n_1' \vee m \, \mathrm{P} \, n_3'. \end{aligned}$$

Hence, for all n_2

$$n_2 \, \mathrm{E} \, n_1 \leftrightarrow n_2 \, \mathrm{E} \, n_1' \vee n_2 \, \mathrm{E} \, n_3'.$$

Also,[10.87.3, .4.2] since $(\mathrm{Exp} \, n_3') \, \mathrm{P} \, n_0$ and $n_0' = n_0 - \mathrm{Exp} \, n_3'$, for all m

$$m \, \mathrm{P} \, n_0 \leftrightarrow m \, \mathrm{P} \, n_0' \vee m = \mathrm{Exp} \, n_3',$$

so for all n_3,

$$n_3 \, \mathrm{E} \, n_0 \leftrightarrow n_3 \, \mathrm{E} \, n_0' \vee n_3 = n_3'.$$

Thus, for all n_2, n_3

$$\begin{aligned} n_2 \, \mathrm{E} \, n_3 \, \mathrm{E} \, n_0 &\leftrightarrow n_2 \, \mathrm{E} \, n_3 \wedge (n_3 \, \mathrm{E} \, n_0' \vee n_3 = n_3') \\ &\rightarrow n_2 \, \mathrm{E} \, n_1' \vee n_2 \, \mathrm{E} \, n_3' \\ &\rightarrow n_2 \, \mathrm{E} \, n_1, \end{aligned}$$

contradicting our assumption about n_0.

7 Reason as follows in PA:

Begin with a proof of the PA-version of Extension.[10.91.1] Then continue with ⌜

(10.95) **Claim** *Suppose $n < n'$. Then $\exists m \, (m \, \mathrm{E} \, n' \wedge m \, \not\mathrm{E} \, n)$.*

Proof Suppose toward a contradiction that for some n

$$\exists n' > n \, \forall m \, (m \, \mathrm{E} \, n' \rightarrow m \, \mathrm{E} \, n),$$

and suppose n is the least such. Let $n' > n$ be such that $\forall m \, (m \, \mathrm{E} \, n' \rightarrow m \, \mathrm{E} \, n)$. Since $n \neq n'$, by Extension there exists $m_0 \, \mathrm{E} \, n$ such that $m_0 \, \not\mathrm{E} \, n'$. Let $k = n - \mathrm{Exp} \, m_0$. Then for any $m \neq m_0$, $m \, \mathrm{E} \, n \rightarrow m \, \mathrm{E} \, k$, so $\forall m \, (m \, \mathrm{E} \, n' \rightarrow m \, \mathrm{E} \, k)$, and $k < n'$, Since $k < n$, this contradicts the minimality of n. □[10.95]

(10.96) **Claim** *Suppose $m < n$. Then $m \, \mathrm{E} (\mathrm{Exp} \, n - 1)$.*

Proof $\mathrm{Exp} \, n / \mathrm{Exp} \, m$ is a 2-power > 1 and is therefore even. $\mathrm{Exp} \, n - 1 = \mathrm{Exp} \, m - 1 + \mathrm{Exp} \, m + ((\mathrm{Exp} \, m \cdot (\mathrm{Exp} \, n / \mathrm{Exp} \, m) - (\mathrm{Exp} \, m) \cdot 2)$. Hence $(\mathrm{Exp} \, n - 1)/ \mathrm{Exp} \, m = 1 + (\mathrm{Exp} \, n / \mathrm{Exp} \, m - 2)$, which is odd, so $m \, \mathrm{E} (\mathrm{Exp} \, n - 1)$. □[10.96]

Using these two claims, it is clear that for any n_0 and n_2, if

$$\forall n_1 (n_1 \, \mathrm{E} \, n_2 \rightarrow n_1 \, \mathrm{E} \, n_0)$$

then $n_2 \leqslant n_0$, so $n_2 \, \mathrm{E} (\mathrm{Exp}(n_0 + 1) - 1)$. Hence

$$\forall n_0 \, \exists n_1 \, \forall n_2 \, (\forall n_1 \, (n_1 \, \mathrm{E} \, n_2 \rightarrow n_1 \, \mathrm{E} \, n_0) \rightarrow n_2 \, \mathrm{E} \, n_1).$$

⌝

8 Left to the reader. Essentially, one translates the following argument into PA. If one has a w that works in the consequent clause for u and u' is obtained by adding one element to u, then let w' be something that works for u' in the antecedent clause and let $w'' = w \cup w'$. Then w'' works for u' in the consequent clause. $\square$[10.91]

10.14 Proof of (4.101)

[REFER TO P. 306.]

[C^+] **First incompleteness theorem, Rosser's improvement** *If* S *is consistent, then* S' *neither proves nor disproves* ρ*, so* S *is syntactically incomplete.*

Proof Suppose S is consistent; hence S' is consistent.

(10.97) **Claim** $\mathsf{S}' \nvdash \rho$.

Proof Suppose toward a contradiction that

(10.98) $\mathsf{S}' \vdash \rho$.

Then[4.100]

$$\mathsf{S}' \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_2 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_2 & \tau \end{smallmatrix}\right) \rightarrow \exists_{\mathrm{HF}}\, \mathrm{v}_3 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \tau' \end{smallmatrix}\right) \wedge \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \bar{\mathrm{v}}_2 \end{smallmatrix}\right) \right)\right). \tag{10.99}$$

Let π be an S'-proof of ρ, and let $\boldsymbol{\pi} = \mathrm{Nm}\, \pi$. Then

$$\mathsf{S}' \vdash \mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \boldsymbol{\pi} & \tau \end{smallmatrix}\right) \tag{10.100}$$

because Prf' is Σ_1' and $\mathrm{Sat}_1^{\Sigma'}\, \mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \boldsymbol{\pi} & \tau \end{smallmatrix}\right)$, so[10.99]

$$\mathsf{S}' \vdash \exists_{\mathrm{HF}}\, \mathrm{v}_3 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \tau' \end{smallmatrix}\right) \wedge \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \boldsymbol{\pi} \end{smallmatrix}\right) \right). \tag{10.101}$$

Since S' is supposed consistent, and we have supposed $\mathsf{S}' \vdash \rho$,[10.98] there does not exist an S'-proof of $\neg\,\rho$. In S' we can simply enumerate all proofs shorter than π and verify that none of them is a proof of $\neg\,\rho$, which we name by the term τ'. Hence

$$\mathsf{S}' \vdash \neg\, \exists_{\mathrm{HF}}\, \mathrm{v}_3 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \tau' \end{smallmatrix}\right) \wedge \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \boldsymbol{\pi} \end{smallmatrix}\right) \right),$$

so[10.101] S' is inconsistent, contrary to hypothesis. $\square$[10.97]

(10.102) **Claim** $\mathsf{S}' \nvdash \neg\,\rho$.

Proof Suppose toward a contradiction that

(10.103) $\mathsf{S}' \vdash \neg\,\rho$.

Let π be a proof of $\neg\,\rho$, and let $\boldsymbol{\pi} = \mathrm{Nm}\, \pi$. Then, as before,[10.100]

$$\mathsf{S}' \vdash \mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \boldsymbol{\pi} & \tau' \end{smallmatrix}\right). \tag{10.104}$$

Since we have assumed that S' is consistent and $\mathsf{S}' \vdash \neg\, \rho$,[10.103] there is no S'-proof of ρ. In S' we can therefore enumerate all proofs with length no greater than that of π and verify that none is a proof of ρ. So

$$\mathsf{S}' \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_2 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_2 & \tau \end{smallmatrix}\right) \rightarrow \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \boldsymbol{\pi} & \bar{\mathrm{v}}_2 \end{smallmatrix}\right)\right).$$

Therefore[10.104]

$$\mathsf{S}' \vdash \forall_{\mathrm{HF}}\, \mathrm{v}_2 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_2 & \tau \end{smallmatrix}\right) \rightarrow \exists_{\mathrm{HF}}\, \mathrm{v}_3 \left(\mathrm{Prf}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \tau' \end{smallmatrix}\right) \wedge \mathrm{Sh}' \left(\begin{smallmatrix} \mathrm{v}_0 & \mathrm{v}_1 \\ \bar{\mathrm{v}}_3 & \bar{\mathrm{v}}_2 \end{smallmatrix}\right)\right)\right),$$

so $\mathsf{S}' \vdash \rho$,[4.100] which contradicts (10.103). $\square$[10.102] $\square$[4.101]

10.15 Proof of (5.77)

[Refer to p. 352.]

Theorem [ZF]

1. ${}^{\omega}2$ *is homeomorphic to the Cantor set in* $\mathbb{R}$.
2. ${}^{\omega}\omega$ *is homeomorphic to* $\mathbb{R}\backslash\mathbb{Q}$.

Proof

1 It is straightforward to show that the map ι given by

$$\iota f = \sum_{n=0}^{\infty} 2 \cdot f(n) \cdot 3^{-n^{-}}$$

is a homeomorphism of ${}^{\omega}2$ with the Cantor set, where we have taken the liberty of identifying the ordinals 0 and 1 with the real numbers of the same name.

2 We will define a homeomorphism ι of ${}^{\omega}(\omega\backslash\{0\})$ with $(1,\infty)\backslash\mathbb{Q}$. We can then compose this with

1. a homeomorphism of ${}^{\omega}\omega$ with ${}^{\omega}(\omega\backslash\{0\})$ and
2. a homeomorphism of $(1,\infty)\backslash\mathbb{Q}$ with $\mathbb{R}\backslash\mathbb{Q}$

to obtain the desired result. For the former, it is enough to note that ω and $\omega\backslash\{0\}$ are equipollent; any bijection of ω with $\omega\backslash\{0\}$ can be use to define a homeomorphism of ${}^{\omega}\omega$ with ${}^{\omega}(\omega\backslash\{0\})$. For the latter, let $\iota' : (1,\infty) \xrightarrow{\text{sur}} \mathbb{R}$ be given by

$$\iota' x = \begin{cases} x - 3 & \text{if } x \in [2,\infty) \\ 1/(1-x) & \text{if } x \in (1,2). \end{cases}$$

It is easily checked that ι' is continuous and strictly increasing (in fact, differentiable with derivative $\geqslant 1$) and is therefore bicontinuous; hence, a homeomorphism. Note also that ι' preserves (ir)rationality. Let $\iota'' = \iota' \restriction ((1,\infty)\backslash\mathbb{Q})$.

To define ι, given $f \in {}^{\omega}(\omega\backslash\{0\})$, let

$$\begin{aligned}\iota f &= f(0) + \cfrac{1}{f(1) + \cfrac{1}{f(2) + \cdots}} \\ &\stackrel{\text{def}}{=} \lim_{n\to\infty} f(0) + \cfrac{1}{f(1) + \ddots + \cfrac{1}{f(n-1) + \cfrac{1}{f(n)}}}.\end{aligned}$$

Clearly, ι is a continuous injection of ${}^{\omega}(\omega\backslash\{0\})$ into $(1,\infty)$.

On the other hand, given $x \in (1,\infty)\backslash\mathbb{Q}$, define $g : \omega \to (1,\infty)$ by letting $g(0) = x$, $g(n+1) = 1/(g(n) - [g(n)])$, where $[y] \stackrel{\text{def}}{=}$ the greatest integer $\leqslant y$. Note that since x is irrational, $g(n)$ is irrational for all n, so $g(n) - [g(n)] \in (0,1)$, and $g_{n+1} \in (1,\infty)$. Define $f(n) = [g(n)]$, where we have again taken the liberty of identifying finite ordinals with integers in $\mathbb{R}$. Let $\eta x = f$. Clearly, η is a continuous injection of $(1,\infty)$ into ${}^{\omega}(\omega\backslash\{0\})$.

It is easy to see that for any $f \in {}^{\omega}(\omega\backslash\{0\})$, $\eta(\iota f) = f$. In particular, ιf is irrational; otherwise the above construction would terminate and not yield an infinite sequence. Thus, $\operatorname{im}\iota \subseteq (1,\infty)\backslash\mathbb{Q}$. Clearly, also, $\forall x \in (1,\infty)\backslash\mathbb{Q}\ \iota(\eta x) = x$, which shows that ι is a surjection to $(1,\infty)\backslash\mathbb{Q}$, hence a bicontinuous bijection, i.e., a homeomorphism. $\square^{5.77}$

10.16 Proof of (5.154)

[Refer to p. 393.]

(10.105) **Theorem** [ZF] *Suppose $\mathfrak{S}$ is a semiring on Ω and $\mathsf{AC}_\omega({}^{\omega}\mathfrak{S})$. Let $\mathfrak{R}$ be the set of finite disjoint unions of members of $\mathfrak{S}$. Let $\mathfrak{C}$ be the set of countable disjoint unions of members of $\mathfrak{S}$. Suppose μ is a measure on $\mathfrak{S}$.*

1. *$\mathfrak{R}$ is the smallest ring that includes $\mathfrak{S}$, i.e., the ring* generated by *$\mathfrak{S}$.*
2. *μ extends uniquely to a measure μ_1 on $\mathfrak{R}$.*
3. *$\mathfrak{C}$ is the smallest set that includes $\mathfrak{S}$ and is closed under countable union.*
4. *$\mathfrak{C}$ is closed under (finite) intersection.*[16]
5. *μ extends uniquely to a measure μ_2 on $\mathfrak{C}$. μ_2 has the following properties.*

    1. (Monotonicity) *For any $C, D \in \mathfrak{C}$, if $C \subseteq D$ then $\mu_2 C \leqslant \mu_2 D$.*
    2. (Subadditivity) *For any $C \in \mathfrak{C}$, if $C = \bigcup_{n\in\omega} C_n$ with $C_n \in \mathfrak{C}$ for each $n \in \omega$, then $\mu_2 C \leqslant \sum_{n\in\omega} \mu_2 C_n$.*
    3. (Downward continuity) *Suppose $C, C_0, C_1, \ldots$ are members of $\mathfrak{C}$, $\forall n \in \omega\ C_n \supseteq C_{n+1}$, $\mu_2 C_0 < \infty$, and $\bigcap_{n\in\omega} C_n \subseteq C$. Then $\lim_{n\to\infty} \mu_2 C_n \leqslant \mu_2 C$.*

[16] Note that $\mathfrak{C}$ is not in general closed under difference or countable intersection.

Proof 1 To show that $\mathfrak{R}$ is closed under difference it suffices to show that if $A \in \mathfrak{R}$ and $B \in \mathfrak{S}$ then $A \backslash B \in \mathfrak{R}$; and to show that $\mathfrak{R}$ is closed under union it suffices to show that if $A \in \mathfrak{R}$ and $B \in \mathfrak{S}$ then $A \cup B \in \mathfrak{R}$. To this end, suppose $A_0, \ldots, A_{n^-}, B \in \mathfrak{S}$ and $A = \bigsqcup_{m \in n} A_m$. Then $A \backslash B = \bigsqcup_{m \in n}(A_m \backslash B)$. Since $\mathfrak{S}$ is a semiring, each $A_m \backslash B$ is a finite disjoint union of members of $\mathfrak{S}$, so $A \backslash B$ is as well. Since $A \cup B = (A \backslash B) \sqcup B$, $A \cup B$ is also a finite disjoint union of members of $\mathfrak{S}$.

Hence, $\mathfrak{R}$ is a ring. Clearly, any ring that includes $\mathfrak{S}$ must include $\mathfrak{R}$, so $\mathfrak{R}$ is the smallest such ring. $\square$[10.105.1]

2 Suppose $A \in \mathfrak{R}$ and $A = \bigsqcup_{m \in n} B_m = \bigsqcup_{m \in n'} B'_m$, where $B_m, B'_{m'} \in \mathfrak{S}$ for all $m \in n$ and $m' \in n'$.

(10.106) **Claim** $\sum_{m \in n} \mu B_m = \sum_{m \in n'} \mu B'_m$.

Proof For $m \in n$ and $m' \in n'$, let $C_{m,m'} = B_m \cap B'_{m'}$. Note that $C_{m,m'} \in \mathfrak{S}$. Clearly, for each $m \in n$, $B_m = \bigsqcup_{m' \in n'} C_{m,m'}$, so $\mu B_m = \sum_{m' \in n'} \mu C_{m,m'}$. Similarly, for each $m' \in n'$, $B_{m'} = \bigsqcup_{m \in n} C_{m,m'}$, so $\mu B_{m'} = \sum_{m \in n} \mu C_{m,m'}$. Hence $\sum_{m \in n} \mu B_m = \sum_{m \in n, m' \in n'} \mu C_{m,m'} = \sum_{m \in n'} \mu B'_m$. $\square$[10.106]

In view of the claim we may define $\mu_1 : \mathfrak{R} \to [0, \infty]$ by the condition that for any $A = \bigsqcup_{m \in n} B_m$, with $B_m \in \mathfrak{S}$ for each $m \in n$, $\mu_1 A = \sum_{m \in n} \mu B_m$. It is straightforward to show that μ_1 is a measure on $\mathfrak{R}$, and it is clearly the only extension of μ to $\mathfrak{R}$. $\square$[10.105.2]

3

(10.107) **Claim** *Suppose* $C = \bigcup_{n \in \omega} S_n$, *where* $S_n \in \mathfrak{S}$ *for each* $n \in \omega$. *Then* $C \in \mathfrak{C}$.

Proof For each $n \in \omega$ let $R_n = S_n \backslash \bigcup_{m \in n} S_m$. (Remember that the union of the empty set is empty, so $R_0 = S_0$.) The R_ns are pairwise disjoint and $\bigsqcup_{n \in \omega} R_n = \bigcup_{n \in \omega} S_n = C$. Note that each R_n must be in any ring that includes $\mathfrak{S}$, so $R_n \in \mathfrak{R}$,[10.105.1] and R_n is therefore a finite disjoint union of members of $\mathfrak{S}$. Invoking $\mathsf{AC}_\omega({}^{\omega}\mathfrak{S})$, there is a function f such that for each $n \in \omega$, fn is a finite sequence of disjoint members of $\mathfrak{S}$ whose union is R_n, and this gives us an ω-sequence of disjoint members of $\mathfrak{S}$ whose union is C. $\square$[10.107]

Now suppose $C = \bigcup_{n \in \omega} C_n$, where $C_n \in \mathfrak{C}$ for each $n \in \omega$. Using $\mathsf{AC}_\omega({}^{\omega}\mathfrak{S})$, let $\langle S_n^m \mid m, n \in \omega \rangle$ be such that $S_n^m \in \mathfrak{S}$, and $\forall n \in \omega\ C_n = \bigcup_{m \in \omega} S_n^m$. Then $C = \bigcup_{m,n \in \omega} S_n^m$, so $C \in \mathfrak{C}$.[10.107]

Thus $\mathfrak{C}$ is closed under countable union. It is obviously the smallest such set including $\mathfrak{S}$. $\square$[10.105.3]

4 $\bigsqcup_{m \in \omega} S_m \cap \bigsqcup_{m \in \omega} T_m = \bigsqcup_{m,n \in \omega}(S_m \cap T_n)$. $\square$[10.105.4]

5 Suppose $A \in \mathfrak{C}$ and $A = \bigsqcup_{m \in \omega} B_m = \bigsqcup_{m \in \omega} B'_m$, where $B_m, B'_m \in \mathfrak{S}$ for all $m \in \omega$. Then $\sum_{m \in \omega} \mu B_m = \sum_{m \in \omega} \mu B'_m$ by the same argument used to prove (10.106). Hence we may define $\mu_2 : \mathfrak{C} \to [0, \infty]$ by the condition that for any $A = \bigsqcup_{m \in \omega} B_m$, with $B_m \in \mathfrak{S}$ for each $m \in \omega$, $\mu_2 A = \sum_{m \in \omega} \mu B_m$. It is straightforward to show that μ_2 is a measure on $\mathfrak{C}$, and it is clearly the only extension of μ to $\mathfrak{C}$.

5.1 Suppose $C, D \in \mathfrak{C}$ and $C \subseteq D$. Suppose $C = \bigsqcup_{n\in\omega} S_n$ and $D = \bigsqcup_{n\in\omega} T_n$, with $S_n, T_n \in \mathfrak{S}$ for all $n \in \omega$. Then $S_m = \bigsqcup_{n\in\omega}(S_m \cap T_n)$ and $T_n \supseteq \bigsqcup_{m\in\omega}(S_m \cap T_n)$. Suppose $n \in \omega$. For each $M \in \omega$, $\bigsqcup_{m\in M}(S_m \cap T_n)$ is in $\mathfrak{R}$, which is closed under difference, so letting $U = \bigsqcup_{m\in M}(S_m \cap T_n)$,

$$\mu_1 T_n = \mu_1 U + \mu_1(T_n \backslash U) \geqslant \mu_1 U = \sum_{m\in M} \mu_1(S_m \cap T_n).$$

Since μ_2 extends μ_1, $\mu_2 T_n \geqslant \sum_{m\in M} \mu_2(S_m \cap T_n)$ for all $M \in \omega$, so $\mu_2 T_n \geqslant \sum_{m\in\omega} \mu_2(S_m \cap T_n)$. Hence

$$\mu_2 D = \sum_{n\in\omega} \mu_2 T_n \geqslant \sum_{m,n\in\omega} \mu_2(S_m \cap T_n) = \sum_{m\in\omega} \mu_2 S_m = \mu_2 C.$$

5.2 We first prove the following special case.

(10.108) **Claim** *For any $C \in \mathfrak{C}$, if $C = \bigcup_{n\in\omega} S_n$ with $S_n \in \mathfrak{S}$ for each $n \in \omega$, then $\mu_2 C \leqslant \sum_{n\in\omega} \mu_2 S_n$*

Proof Note that in the disjointing process used for $C = \bigcup_{n\in\omega} S_n$ in the proof of (10.105.3), $R_n \subseteq S_n$ for each $n \in \omega$, so $\mu_2 S_n = \mu_1 S_n = \mu_1 R_n + \mu_1(S_n \backslash R_n) \geqslant \mu_1 R_n = \mu_2 R_n$, so $\mu_2 C = \sum_{n\in\omega} \mu_2 R_n \leqslant \sum_{n\in\omega} \mu_2 S_n$. $\square$[10.108]

To treat the general case, suppose $C = \bigcup_{n\in\omega} C_n$ with $C_n \in \mathfrak{S}$ for each $n \in \omega$. Let $S_n^m \in \mathfrak{S}$ $(m, n \in \omega)$ be such that for each $n \in \omega$, $C_n = \bigsqcup_{m\in\omega} S_n^m$ $(\mathsf{AC}_\omega({}^\omega\mathfrak{S}))$. Then

$$\mu_2 C \leqslant \sum_{m,n\in\omega} \mu_2 S_n^m = \sum_{n\in\omega}\sum_{m\in\omega} \mu_2 S_n^m = \sum_{n\in\omega} \mu_2 C_n.$$

5.3 The essence of the proof is an approximation of the C_ns by members of $\mathfrak{R}$, which is closed under difference. Note that $\forall n \in \omega\ \mu_2 C_n \geqslant \mu_2 C_{n+1}$.[10.105.5.1] Suppose toward a contradiction that $\lim_{n\to\infty} \mu_2 C_n > \mu_2 C$. Let $\varepsilon = \frac{1}{2}\big((\lim_{n\to\infty} \mu_2 C_n) - \mu_2 C\big) > 0$.

Let $T_n^m \in \mathfrak{S}$ $(m, n \in \omega)$ be such that $C_n = \bigsqcup_{m\in\omega} T_n^m$ (using $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$).

(10.109) *Construct $C'_n \in \mathfrak{C}$, $R_n^m \in \mathfrak{R}$, $M_n \in \omega$, and $R_n, R'_n \in \mathfrak{R}$ for $m, n \in \omega$, with the following properties:*

1. $R_0^m = T_0^m$.
2. *For any $m \in \omega$, $R_n^m \subseteq T_n^m$.*
3. $C'_n = \bigsqcup_{m\in\omega} R_n^m$.
4. $R'_n = \bigsqcup_{m\geqslant M_n} R_n^m$ *and* $\mu_2 R'_n < 2^{-n-1}\varepsilon$.
5. $R_n = C'_n \backslash R'_n = \bigsqcup_{m<M_n} R_n^m$.
6. $C'_{n+1} = R_n \cap C_{n+1}$.

(10.109.1) begins the construction. For each $n \in \omega$, supposing R_n^m $(m \in \omega)$ to have been defined satisfying (10.109.2),

$$\sum_{m\in\infty} \mu_2 R_n^m \leqslant \sum_{m\in\infty} \mu_2 T_n^m = \mu_2 C_n \leqslant \mu_2 C_0 < \infty.$$

Let M_n be least such that $\sum_{m\geqslant M_n} \mu_2 R_n^m < 2^{-n-1}\varepsilon$, and define C'_n, R'_n and R_n as in (10.109.3,4,5). Let $R_{n+1}^m = T_{n+1}^m \cap R_n$.

Note that by this construction, for each $n \in \omega$,

1. $R_n \subseteq C'_n \subseteq C_n$;
2. $R_{n+1} \subseteq C'_{n+1} \subseteq R_n$;
3. $C_n \backslash C'_n \subseteq \bigcup_{n'<n} R'_{n'}$;
4. $C_n \backslash R_n \subseteq \bigcup_{n' \leqslant n} R'_{n'}$; and
5. $\mu_2 C_n - \mu_2 R_n < (1 - 2^{-n-1})\varepsilon$.

The first two assertions are immediate. The third and fourth are proved by simultaneous induction. For the last assertion we note that $C_n \subseteq R_n \cup \bigcup_{n' \leqslant n} R'_{n'}$, so[10.105.5.2] $\mu_2 C_n \leqslant \mu_2 R_n + \sum_{n' \leqslant n} 2^{-n'-1}\varepsilon = \mu_2 R_n + (1 - 2^{-n-1})\varepsilon$.

Note that $\bigcap_{n \in \omega} R_n \subseteq \bigcap_{n \in \omega} C_n \subseteq C$, and $\lim_{n \to \infty} \mu_2 R_n \geqslant \lim_{n \to \infty} \mu_2 C_n - \varepsilon > \mu_2 C$. However, $R_0 = \bigsqcup_{m<n} (R_m \backslash R_{m+1}) \sqcup R_n$, so $\mu_2 R_n = \mu_2 R_0 - \sum_{m<n} \mu_2(R_m \backslash R_{m+1})$, and $\lim_{n \to \infty} \mu_2 R_n = \mu_2 R_0 - \sum_{m \in \omega} \mu_2(R_m \backslash R_{m+1})$. Hence $\mu_2 C < \mu_2 R_0 - \sum_{m \in \omega} \mu_2(R_m \backslash R_{m+1})$, which contradicts the fact that $C \cup \bigcup_{m \in \omega} (R_m \backslash R_{m+1}) \supseteq R_0$.[10.105.5.1] $\square$[10.105]

10.17 Proof of (5.156)

[Refer to p. 394.]

(10.110) **Theorem** [ZF] *Suppose $\mathfrak{S}$ is a semiring on Ω and $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$. Suppose μ is a σ-finite measure on $\mathfrak{S}$. Then*

1. *$\mathfrak{S} \subseteq \mathfrak{M}^\mu$;*
2. *$\mathfrak{M}^\mu$ is a σ-algebra; and*
3. *$\bar{\mu}$ is the unique extension of μ to a measure on $\mathfrak{M}^\mu$.*

Proof 1 Suppose $S \in \mathfrak{S}$. Since μ is σ-finite, let $S_n \in \mathfrak{S}$, $n \in \omega$, be such that $\Omega = \bigcup_{n \in \omega} S_n$. For each $n \in \omega$, let $R_n = S_n \backslash S$. Then for each $n \in \omega$, R_n is a finite union of members of $\mathfrak{S}$, and $\Omega \backslash S = \bigcup_{n \in \omega} R_n$, so by $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$ there exist $T_n \in \mathfrak{S}$, $n \in \omega$, such that $\Omega \backslash S = \bigcup_{n \in \omega} T_n$, so S is μ-measurable. $\square$[10.110.1]

2 Let $\mathfrak{R}$ be the set of finite disjoint unions of members of $\mathfrak{S}$, and let $\mathfrak{C}$ be the set of countable disjoint unions of members of $\mathfrak{S}$. Recall[5.154] that $\mathfrak{R}$ is the ring generated by $\mathfrak{S}$, $\mathfrak{C}$ is the set of countable unions of members of $\mathfrak{S}$, and we may suppose that μ is defined on $\mathfrak{C}$. Keep in mind that $\mathfrak{R}$ is closed under finite union, finite intersection, and difference; $\mathfrak{C}$ is closed under countable union and finite intersection, but not necessarily difference or countable intersection.

(10.111) *By virtue of (5.154) a set $A \subseteq \Omega$ is μ-measurable iff for all $\varepsilon > 0$ there exists $C, D \in \mathfrak{C}$ such that*

1. *$A \subseteq C$;*
2. *$\Omega \backslash A \subseteq D$; and*
3. *$\mu(C \cap D) < \varepsilon$.*

Obviously, $\mathfrak{M}^\mu$ is closed under complementation (relative to Ω). It suffices, therefore to show that it is closed under countable union. Suppose $M_n \in \mathfrak{M}^\mu$ for each $n \in \omega$. Let $M = \bigcup_{n\in\omega} M_n$. We wish to show that M is μ-measurable.

Let $S \in \mathfrak{S}$ with $\mu S < \infty$, and $\varepsilon > 0$ be fixed for the moment.

(10.112) *Invoking* $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$*, let* $C_n, D_n \in \mathfrak{C}$ $(n \in \omega)$ *be such that*

1. $M_n \subseteq C_n$*;*
2. $\Omega\backslash M_n \subseteq D_n$*; and*
3. $\mu(C_n \cap D_n) < 2^{-n-1}\varepsilon$*.*

Let $C = \bigcup_{n\in\omega} C_n$, $D = \bigcap_{n\in\omega} D_n$, and $E_m = \bigcap_{n\in m} D_n$ for $m \in \omega$. Then

1. $M \subseteq C$;
2. $\Omega\backslash M \subseteq D$;
3. $C \cap D \subseteq \bigcup_{n\in\omega}(C_n \cap D_n)$; and
4. $\langle C \cap E_m \mid m \in \omega\rangle$ is a decreasing sequence of members of $\mathfrak{C}$ with $\bigcap_{m\in\omega}(C \cap E_m) = C \cap D \subseteq \bigcup_{n\in\omega}(C_n \cap D_n)$.

Restricting to S, $\langle C \cap E_m \cap S \mid m \in \omega\rangle$ is a decreasing sequence of members of $\mathfrak{C}$ with $\bigcap_{m\in\omega}(C \cap E_m \cap S) = C \cap D \cap S \subseteq \bigcup_{n\in\omega}(C_n \cap D_n \cap S)$.

Since $\mu S < \infty$, (5.154.5.3) applies, so

$$\lim_{m\to\infty} \mu_2(C \cap E_m \cap S) \leqslant \mu_2 \bigcup_{n\in\omega}(C_n \cap D_n \cap S).$$

By (5.154.5.2)

$$\mu_2 \bigcup_{n\in\omega}(C_n \cap D_n \cap S) \leqslant \sum_{n\in\omega} \mu(C_n \cap D_n \cap S) < \sum_{n\in\omega} 2^{-n-1}\varepsilon = \varepsilon.^{10.112.3}$$

Let $m \in \omega$ be such that $\mu_2(C \cap E_m \cap S) < \varepsilon$. Let $C' = C \cap S$ and $D' = E_m \cap S$. Then

1. $C', D' \in \mathfrak{C}$;
2. $C', D' \subseteq S$;
3. $M \cap S \subseteq C'$;
4. $S\backslash M \subseteq D'$; and
5. $\mu_2(C' \cap D') < \varepsilon$.

Since μ is σ-finite there exist $S_i \in \mathfrak{S}$ $(i \in \omega)$ such that $\forall i \in \omega\ \mu S_i < \infty$ and $\Omega = \bigcup_{i\in\omega} S_i$. By replacing S_i with $S_i\backslash\bigcup_{j<i} S_j$, we may assume that the S_is are disjoint. Thus, $\Omega = \bigsqcup_{i\in\omega} S_i$.

Now suppose $\varepsilon > 0$. For each $i \in \omega$, by the above argument with S_i for S and $2^{-i-1}\varepsilon$ for ϵ, there exist $C_i', D_i' \in \mathfrak{C}$ be such that

1. $C_i', D_i' \subseteq S_i$;
2. $M \cap S_i \subseteq C_i'$;

3. $S_i \backslash M \subseteq D'_i$; and

4. $\mu_2(C'_i \cap D'_i) < 2^{-i-1}\varepsilon$.

Use $\mathsf{AC}_\omega({}^\omega\mathfrak{S})$ to show that there is a sequence $\langle\langle C_i, D_i\rangle \mid i \in \omega\rangle$ of such pairs, and let $C' = \bigcup_{i\in\omega} C'_i$ and $D' = \bigcup_{i\in\omega} D'_i$. Then

1. $C', D' \in \mathfrak{C}$;

2. $M \subseteq C'$;

3. $\Omega\backslash M \subseteq D'$; and

4. $\mu_2(C' \cap D') \leqslant \sum_{i\in\omega} \mu_2(C'_i \cap D'_i) < \sum_{i\in\omega} 2^{-i-1}\varepsilon = \varepsilon$.[5.154.5.2]

Hence M is μ-measurable. □[10.110.2]

3 Using the characterization (10.111) of $\mathfrak{M}^\mu$, it is straightforward to show that $\bar{\mu}$, which is by definition μ^* restricted to $\mathfrak{M}^\mu$, is a measure. It is clearly unique. □[10.110.3] □[10.110]

10.18 Proof of (5.158)

[Refer to p. 395.]

(10.113) **Theorem** [$\mathsf{ZF} + \mathsf{AC}_\omega$] *Suppose, for $i = 0, 1$, that μ_i is a σ-finite measure on a σ-algebra $\mathfrak{M}_i$ on a set Ω_i. Let $\mu = \mu_0 \times \mu_1$ be the product measure on the product algebra $\mathfrak{M} = \mathfrak{M}_0 \times \mathfrak{M}_1$ on $\Omega = \Omega_0 \times \Omega_1$.*[5.157] *Suppose $A \in \mathfrak{M}$. For $x \in \Omega_0$, let $A_x = \{y \in \Omega_1 \mid \langle x, y\rangle \in A\}$. Let $E = \{x \in \Omega_0 \mid \mu_1 A_x > 0\}$. Then $\mu A = 0$ iff $\mu_0 E = 0$.*

Remark We call A_x the *section of A at x*. E is the *exceptional set of A*. Thus the theorem states that A is null iff almost every section of A is null, i.e., iff the members of the exceptional set E truly are exceptional.

Proof Let $\mathfrak{S}$ be the semiring of rectangles $A_0 \times A_1$, where $A_0 \in \mathfrak{M}_0$ and $A_1 \in \mathfrak{M}_1$.

($\rightarrow$) Suppose N is some index set, which will often be ω or a subset of ω. Suppose $m \in \omega$. We will say that $\langle X_n \mid n \in N\rangle$

1. *covers Y* $\overset{\text{def}}{\iff}$ for every $y \in Y$, $y \in X_n$ for some $n \in N$;

2. *covers Y m times* $\overset{\text{def}}{\iff}$ for every $y \in Y$, $|\{n \in \omega \mid y \in X_n\}| = m$;

3. *covers Y at least m times* $\overset{\text{def}}{\iff}$ for every $y \in Y$, $|\{n \in \omega \mid y \in X_n\}| \geqslant m$; and

4. *covers Y infinitely often* $\overset{\text{def}}{\iff}$ it covers Y at least m times for every $m \in \omega$.

By definition, for any measure μ on a semiring $\mathfrak{S}$, $Y \in \mathfrak{M}^\mu$ is null iff for any $\varepsilon > 0$ there exists $\langle X_n \mid n \in \omega \rangle$, with $X_n \in \mathfrak{S}$, that covers Y such that $\sum_{n\in\omega} \mu X_n < \varepsilon$. It is easy to see that if Y is null then for any $\varepsilon > 0$ there exists a sequence $\langle X_n \mid n \in \omega \rangle$, with $X_n \in \mathfrak{S}$, that covers Y infinitely often such that $\sum_{n\in\omega} \mu X_n < \varepsilon$. (For each $m \in \omega$, let $\langle X_n^m \mid n \in \omega \rangle$ be a sequence that covers Y, such that $\sum_{n\in\omega} \mu X_n^m < \varepsilon/2^{m+1}$. Merge these sequences into a single ω-sequence.)

(10.114) **Claim** *On the other hand, if Y is not null then for any $\langle X_n \mid n \in N \rangle$, with $X_n \in \mathfrak{S}$, that covers Y infinitely often, $\sum_{n\in N} \mu X_n$ diverges.*

Proof It suffices to prove this for $N = \omega$. We will show by induction on $m \geqslant 1$ that if $\langle X_n \mid n \in \omega \rangle$ covers Y at least m times then $\sum_{n\in\omega} \mu X_n \geqslant m \cdot \mu Y$. This is trivial for $m = 1$. Suppose it is true for m, and suppose $\langle X_n \mid n \in \omega \rangle$ covers Y at least $m + 1$ times. For each $n \in \omega$ let $Y_n = X_n \cap \bigcup_{n'<n} X_{n'}$, and let $Z_n = X_n \backslash Y_n$. Note that $X_n = Y_n \sqcup Z_n$. $\bigcup_{n\in\omega} Z_n = \bigcup_{n\in\omega} X_n$, so $\langle Z_n \mid n \in \omega \rangle$ covers Y. Since the Z_ns are pairwise disjoint, $\langle Z_n \mid n \in \omega \rangle$ covers Y 1 time. It follows that $\langle Y_n \mid n \in \omega \rangle$ covers Y at least m times. Since $\mathfrak{S}$ is a semiring and we have AC_ω, there exists a sequence of finite decompositions of the Y_ns into members of $\mathfrak{S}$, which can be arranged as a single ω-sequence $\langle Y_n' \mid n \in \omega \rangle$, so that $\sum_{n\in\omega} \mu Y_n' = \sum_{n\in\omega} \mu Y_n$, and $\langle Y_n' \mid n \in \omega \rangle$ covers Y at least m times. By induction hypothesis, therefore,

$$\sum_{n\in\omega} \mu Y_n = \sum_{n\in\omega} \mu Y_n' \geqslant m \cdot \mu Y.$$

By a similar sequence of decompositions we obtain a sequence $\langle Z_n' \mid n \in \omega \rangle$ of disjoint members of $\mathfrak{S}$ such that $\sum_{n\in\omega} \mu Z_n' = \sum_{n\in\omega} \mu Z_n$, and $\langle Z_n' \mid n \in \omega \rangle$ covers Y, so

$$\sum_{n\in\omega} \mu Z_n = \sum_{n\in\omega} \mu Z_n' \geqslant \mu Y.$$

Again invoking the fact that $X_n = Y_n \sqcup Z_n$, $\mu X_n = \mu Y_n + \mu Z_n$, from which it follows that $\sum_{n\in\omega} \mu X_n \geqslant (m + 1) \cdot \mu Y$.

If $\langle X_n \mid n \in \omega \rangle$ covers Y infinitely often then it covers Y at least m times for every $m \in \omega$, so $\sum_{n\in N} \mu X_n \geqslant m \cdot \mu Y$ for all $m \in \omega$. Since $\mu Y > 0$ by assumption, $\sum_{n\in N} \mu X_n$ diverges. □[10.114]

Suppose $A \in \mathfrak{M}$ and $\mu A = 0$. Suppose $\varepsilon > 0$. Let[10.114] $\langle X_n \times Y_n \mid n \in \omega \rangle$ be an ω-sequence of rectangles in $\mathfrak{S}$ that covers A infinitely often, such that

$$\sum_{n\in\omega} \mu_0 X_n \cdot \mu_1 Y_n < \varepsilon. \tag{10.115}$$

(10.116) *By partitioning the X_ns as necessary, we may arrange that for each $m < n < \omega$, either $X_n \subseteq X_m$ or $X_n \cap X_m = 0$. The new X_ns are still in $\mathfrak{M}_0$.*

For $x \in \Omega_0$, let $N_x = \{n \in \omega \mid x \in X_n\}$. Note that for each $x \in \Omega_0$, $\langle Y_n \mid n \in N_x \rangle$ covers A_x infinitely often. Thus, for every $x \in E$, $\sum_{n\in N_x} \mu_1 Y_n$ diverges.

For $n \in \omega$, let $\phi_n : \Omega_0 \to \mathbb{R}$ be defined by the condition that

$$\phi_n x = \sum \{\mu_1 Y_m \mid m < n \wedge x \in X_m\},$$

with the convention that the sum of the empty set is 0, so $\phi_0 x = 0$ for all x. Thus,

$$\phi_{n+1} x - \phi_n x = \begin{cases} \mu_1 Y_n & \text{if } x \in X_n \\ 0 & \text{otherwise.} \end{cases}$$

(10.117) *By virtue of (10.116) the region under the graph of ϕ_n is the disjoint union of the rectangles $X_m \times [a, b)$, $m < n$, where $\phi_m x = a$ for all $x \in X_m$, and $b = a + \mu_1 Y_m$.*

Note that for each $x \in \Omega_0$, $\langle \phi_n x \mid n \in \omega \rangle$ is nondecreasing, and for any $x \in E$, $\lim_{n\to\infty} \phi_n x = \infty$. Let N be the set of $n \in \omega$ such that for some x, $\phi_n x < 1 \leqslant \phi_{n+1} x$. Then for any $x \in E$, $x \in X_n$ for some $n \in N$, i.e., $\langle X_n \mid n \in N \rangle$ covers E.

Using a partitioning argument based on the above tiling,[10.117] it is easy to show that $\sum_{n\in N} \mu_0 X_n \leqslant \sum_{n\in\omega} \mu_0 X_n \cdot \mu_1 Y_n$. Thus, $\sum_{n\in N} \mu_0 X_n < \varepsilon$.[10.115]

Since ε was an arbitrary positive real number, this shows that $\mu_0 E = 0$.

($\leftarrow$) Conversely, suppose $\mu_0 E = 0$, and suppose toward a contradiction that $\mu A = c > 0$. Let[5.155.3] $S_n, T_n \in \mathfrak{S}$, $n \in \omega$, be such that

1. $A \subseteq \bigcup_{n\in\omega} S_n$;
2. $\Omega \backslash A \subseteq \bigcup_{n\in\omega} T_n$; and
3. $\sum_{m,n\in\omega} \mu(S_m \cap T_n) < c/2$.

(10.118) *By partitioning, if necessary, we may arrange that*[17]

1. *the S_ns and T_ns are finite rectangles;*
2. *the T_ns are pairwise disjoint; and*
3. *letting $T_n = T_n^0 \times T_n^1$, if $m < n < \omega$ then either $T_n^0 \subseteq T_m^0$ or $T_n^0 \cap T_m^0 = 0$.*

By construction,

$$\mu\big(\bigcup_{m\in\omega} S_m \backslash \bigcup_{m,n\in\omega} (S_m \cap T_n)\big) > c/2,$$

so there exists $k \in \omega$ such that

$$\mu\big(S_k \backslash \bigcup_{n\in\omega} (S_k \cap T_n)\big) > 0. \tag{10.119}$$

Since

$$S_k \backslash \bigcup_{n\in\omega} (S_k \cap T_n) \subseteq A,$$

for any $x \in \Omega_0 \backslash E$, $\big(S_k \backslash \bigcup_{n\in\omega} (S_k \cap T_n)\big)_x$ is null.

In the terminology of (10.118.3), let $X = S_k^0$, $Y = S_k^1$, $X_n = S_k^0 \cap T_n^0$, and $Y_n = S_k^1 \cap T_n^1$. Thus,

1. $S_k = X \times Y$; and
2. for each $n \in \omega$, $S_k \cap T_n = X_n \times Y_n$.

Then

$$\forall x \in X \backslash E \ \big(Y \backslash \bigcup \{Y_n \mid x \in X_n\} \text{ is null}\big). \tag{10.120}$$

The rectangles $X_n \times Y_n$ are pairwise disjoint,[10.118.2] so[10.119]

$$\mu(X \times Y) - \textstyle\sum_{n\in\omega} \mu(X_n \times Y_n) > 0, \tag{10.121}$$

[17] We could arrange that (10.118.2) apply to S_ns as well as the T_ns, and that (10.118.3) apply to the T_n^1s, and to the S_n^0s and S_n^1s similarly defined, but the listed properties are all we will need.

but[10.120]

$$\forall x \in X \backslash E \ \sum\{\mu Y_n \mid x \in X_n\} = \mu Y. \tag{10.122}$$

We have arranged[10.118.3] that for each $m < n < \omega$, either $X_n \subseteq X_m$ or $X_n \cap X_m = 0$. As before, define $\phi_n : X \to \mathbb{R}$ for $n \in \omega$ by the condition that

$$\phi_n x = \sum\{\mu_1 Y_m \mid m < n \wedge x \in X_m\}.$$

Then

$$\phi_{n+1} x - \phi_n x = \begin{cases} \mu_1 Y_n & \text{if } x \in X_n \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $0 < \varepsilon < \mu_1 Y$. Let N be the set of $n \in \omega$ such that for some $x \in X$, $\phi_n x < \mu_1 Y - \varepsilon \leqslant \phi_{n+1} x$. For any $x \in X \backslash E$, $\lim_{n\to\infty} \phi_n x = \mu_1 Y$,[10.122] so $x \in X_n$ for some $n \in N$, i.e., $\langle X_n \mid n \in N\rangle$ covers $X \backslash E$. Since E is assumed null, there exists $n \in \omega$ such that $\sum\{\mu_0 X_m \mid m < n \wedge m \in N\} > \mu_0 X - \varepsilon$.

Using a tiling argument as before, we see that

$$\sum_{m<n} \mu(X_m \times Y_m) > (\mu_0 X - \varepsilon)(\mu_1 Y - \varepsilon).$$

Since ε was an arbitrary positive number,

$$\sum_{m\in\omega} \mu(X_m \times Y_m) \geqslant (\mu_0 X)(\mu_1(Y) = \mu(X \times Y),$$

which contradicts (10.121). $\square$[10.113]

10.19 Proof of (5.177): Borel determinacy

[Refer to p. 406.]

In proofs of determinacy, *auxiliary games* are often useful. Suppose $G = \langle T, X\rangle$ is a game. Informally, an auxiliary game for G is a game $G' = \langle T', X'\rangle$ with the property that a strategy σ' in G' determines a strategy σ in G (for the same or the opposite player), such that if σ' is a winning strategy in G' then σ is a winning strategy in G. Therefore, if G' is determined, so is G. For the proof of Borel determinacy, a particular sort of auxiliary game is used, viz., a *covering*. This differs from the usual case in that the auxiliary games for $\langle T, X\rangle$ and $\langle T, [T]\backslash X\rangle$ are the same, i.e., in the definition of a covering we are indifferent as to which outcome each player is imagined to be trying to achieve.

(10.123) **Definition** [ZF] *Suppose T is a nonempty good tree.*

1. $\langle T', \pi, \varphi\rangle$ *is a* covering of T $\overset{\text{def}}{\iff}$
   1. *T' is a nonempty good tree;*
   2. *$\pi : T' \to T$, where*
      1. *π is monotone, i.e., $s \subseteq t \to \pi\, s \subseteq \pi\, t$; and*
      2. *π is length-preserving, i.e., $|\pi\, s| = |s|$.*

      We extend π to $[T']$ by letting $\pi\, x = \bigcup_{n\in\omega} \pi(x \restriction n)$, so $\pi \restriction [T'] : [T'] \to [T]$.

3. $\varphi = \varphi_{\mathrm{I}} \cup \varphi_{\mathrm{II}}$, *where, letting* P *be* I *or* II,
   1. $\varphi_{\mathrm{P}} : \bar{\mathcal{S}}^{T'}_{\mathrm{P}} \to \bar{\mathcal{S}}^{T}_{\mathrm{P}}$;[5.166.9.4]
   2. $\varphi_{\mathrm{P}}{}^{\to}\mathcal{S}^{T'|n}_{\mathrm{P}} \subseteq \mathcal{S}^{T|n}_{\mathrm{P}}$; *and*
   3. *for all* $\sigma \in \bar{\mathcal{S}}^{T'}_{\mathrm{P}}$ *and* $n \in \omega$, $(\varphi_{\mathrm{P}}\,\sigma)|n = \varphi_{\mathrm{P}}(\sigma|n)$.

   We extend φ_{P} *to* $\mathcal{S}^{T'}_{\mathrm{P}}$ *by letting* $\varphi_{\mathrm{P}}\,\sigma = \bigcup_{n\in\omega} \varphi_{\mathrm{P}}(\sigma|n)$.
4. *for every* $\sigma \in \mathcal{S}^{T'}$, $\pi^{\to}[\sigma] \supseteq [\varphi\sigma]$.

2. *Suppose* $k \in \omega$. $\langle T', \pi, \varphi\rangle$ *is a* k-covering of T $\overset{\text{def}}{\iff}$
   1. $\langle T', \pi, \varphi\rangle$ *is a covering of* T;
   2. $T'|2k = T|2k$; *and*
   3. $\pi \restriction (T'|2k)$ *is the identity function.*

Definition [ZF] *Suppose* $\langle T', \pi, \varphi\rangle$ *is a covering of a nonempty good tree* T *and* $X \subseteq [T]$. $\langle T', \pi, \varphi\rangle$ unravels X $\overset{\text{def}}{\iff}$ $\pi^{\leftarrow}X$ *is clopen. Note that* $\langle T', \pi, \varphi\rangle$ *unravels* X *iff* $\langle T', \pi, \varphi\rangle$ *unravels* $[T]\backslash X$.

(10.124) **Theorem** [ZFC] *Suppose* $G = \langle T, X\rangle$ *is a game and* $\langle T', \pi, \varphi\rangle$ *is a covering of* T *that unravels* X. *Then* G *is determined.*

Proof Let $X' = \pi^{\leftarrow}X$, and let $G' = \langle T', X'\rangle$. By the Gale-Stewart theorem, there is a P-strategy σ' in T' that is winning in $\langle T', X'\rangle$. Let $\sigma = \varphi\sigma'$ be the corresponding strategy in T. Suppose $x \in [\sigma]$. Let $x' \in [\sigma']$ be such that $\pi x' = x$. Then

$$x' \in \begin{cases} X' & \text{if P is I} \\ [T']\backslash X' & \text{if P is II,} \end{cases}$$

and, accordingly,

$$x \in \begin{cases} X & \text{if P is I} \\ [T]\backslash X & \text{if P is II,} \end{cases}$$

so σ is a winning P-strategy in G. □[10.124]

(10.125) **Theorem** [ZFC] *Suppose* T *is a nonempty good tree and* $X \subseteq [T]$ *is closed. Then for any* $k \in \omega$ *there is a* k*-covering of* T *that unravels* X *(and hence also* $[T]\backslash X$*).*

Proof Let $T^X = \{p \in T \mid \exists x \in X\ p \subseteq x\}$. Since X is closed, $X = [T^X]$.

(10.126) *Let* $\tilde{T}$ *be the good tree such that* $[\tilde{T}]$ *consists of all* ω*-sequences*

$$\langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, r\rangle, x_{2k+2}, \ldots\rangle,$$

where, letting $p' = \langle x_0, \ldots, x_{2k}\rangle$, $p'' = \langle x_0, \ldots, x_{2k}, x_{2k+1}\rangle$, *and* $x = \langle x_0, x_1, \ldots\rangle$,

1. $x \in T$;
2. Σ^{I} *is a* I*-imposed subtree of* $T_{(p')}$; *and*
3. *either*
   1. $r = \langle 1, u\rangle$, *where* u *is a sequence of even length extending* p'' *such that*
      1. $u \in \Sigma^{\mathrm{I}}\backslash T^X$; *and*

2. $x \in [T_{(u)}]$; *or*

2. $r = \langle 2, \Sigma^{\mathrm{II}} \rangle$, *where* Σ^{II} *is a* II*-imposed subtree of* $\Sigma^{\mathrm{I}}_{(p'')}$ *such that*

   1. $\Sigma^{\mathrm{II}} \subseteq T^X$; *and*
   2. $x \in [\Sigma^{\mathrm{II}}]$.

Note that $\langle x_{2k}, \Sigma^{\mathrm{I}} \rangle$ is a play by I in the auxiliary game on $\tilde{T}$, which consists of a play x_{2k} by I in the original game on T, together with a restriction Σ^{I} on the rest of the T-game imposable by I. The option (10.126.3.1) corresponds to II observing that the restriction Σ^{I} allows for the possibility of a continuation of play to a position $u \in \Sigma^{\mathrm{I}}$ that is not in T^X, and hence is a losing position for I.

Option (10.126.3.2) corresponds to II observing that it may restrict Σ^{I} so as to preclude the possibility of option 1.

If option 1 is not possible after II's play of x_{2k+1} then option 2 is possible—indeed, Σ^{II} may be taken to be $\Sigma^{\mathrm{I}}_{(p'')}$. Note that either option may constrain the subsequent play: option 1 requires that $x_m = u_m$ for $2k+2 \leqslant m < |u|$; option 2 requires that $\langle x_{2k+2}, \dots \rangle \in \Sigma^{\mathrm{II}}_{p''}$.

Let π be given by:

$$\pi \langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}} \rangle, \langle x_{2k+1}, r \rangle, x_{2k+2}, \dots \rangle = \langle x_0, \dots \rangle. \tag{10.127}$$

Let $\tilde{X} = \pi^{\leftarrow} X$, and let $\tilde{G} = \langle \tilde{T}, \tilde{X} \rangle$. Since X is closed, $\tilde{X}$ is closed.

Note also that $\tilde{x} \in \tilde{X}$ iff $\tilde{x}_{2k+1}$ is of the form $\langle x_{2k+1}, \langle 2, \Sigma^{\mathrm{II}} \rangle \rangle$, i.e.,

$$((\tilde{x}_{2k+1})_1)_0 = 2.$$

Hence $\tilde{X} = \bigcup \{ \tilde{T}_{(\tilde{p})} \mid |\tilde{p}| = 2k+2 \wedge ((\tilde{p}_{2k+1})_1)_0 = 2 \}$, so $\tilde{X}$ is a union of basic open sets in $[\tilde{T}]$ and is therefore open.

Thus $\tilde{X}$ is clopen. To complete the proof we must define the mapping $\varphi : \mathcal{S}^{\tilde{T}} \to \mathcal{S}^T$. Recall that $\varphi = \varphi_{\mathrm{I}} \cup \varphi_{\mathrm{II}}$, with φ_{I} and φ_{II} applying to I- and II-strategies, respectively.

Case I Suppose $\tilde{\sigma} \subseteq \tilde{T}$ is a I-strategy. We will define the I-strategy $\sigma = \varphi\, \tilde{\sigma} \subseteq T$.

Let $\sigma|2k = \tilde{\sigma}|2k$. Suppose $p = \langle x_0, \dots, x_{2k-1} \rangle \in \sigma$. We must define the unique immediate extension of p in σ, which is I's play according to σ at position p in T. Since p is in $\tilde{\sigma}$, it has a unique immediate extension $p^\frown \langle \langle x_{2k}, \Sigma^{\mathrm{I}} \rangle \rangle$ in $\tilde{\sigma}$. Let $p' = p^\frown \langle x_{2k} \rangle$ be the unique extension of p in σ.

Being a I-strategy, σ does not restrict the immediate extensions of p' in σ, so we must define $\sigma_{(p'')}$ for all $p'' = p'^\frown \langle x_{2k+1} \rangle \in T$. To do this we consider the game

$$G' = \langle \Sigma^{\mathrm{I}}_{(p'')}, [\Sigma^{\mathrm{I}}_{(p'')}] \backslash X \rangle.$$

Note that $X \cap [\Sigma^{\mathrm{I}}_{(p'')}] = [T^X \cap \Sigma^{\mathrm{I}}_{(p'')}]$. There are two cases to consider.

Case 1 Suppose there is a winning I-strategy in G'. Let σ' be the first such strategy in some fixed wellordering of strategies. Then $\forall x \in [\sigma']\, \exists n \in \omega\ x \restriction n \notin T^X$. Let U consist of the $\subseteq$-minimal members of the set of $u \in \sigma' \backslash T^X$ of even length. Then $\sigma' = \bigcup_{u \in U} \sigma'_{(u)}$, i.e., every play according to σ' contains a (unique) position $u \in U$. For each $u \in U$, let

$$\tilde{u} = \langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}} \rangle, \langle x_{2k+1}, \langle 1, u \rangle \rangle, u_{2k+2}, \dots, u_{|u|-1} \rangle,$$

and note that $\tilde{u} \in \tilde{\sigma}$. Let

$$\sigma_{p''} = \bigcup_{u \in U} \{u ^\frown s \mid s \in \tilde{\sigma}_{\tilde{u}}\}$$

In other words, from position p'', I plays according to σ' until a position u of even length is reached such that $u \notin T_X$, which must happen since σ' is a winning I-strategy in G'. As soon as this happens, I imagines that II has played $\langle x_{2k+1}, \langle 1, u\rangle\rangle$ after p' in $\tilde{G}$ and that both players in that game have played—as they must—the remainder of u. From this point on in G, I plays as instructed by $\tilde{\sigma}$ in $\tilde{G}$.

Case 2 Now suppose there is no winning I-strategy in G'. Let Σ^{II} be the II-nonlosing subtree of $\Sigma^{\mathrm{I}}_{(p'')}$ for G', i.e., the set of positions $s \in \Sigma^{\mathrm{I}}_{(p'')}$ such that there is no winning I-strategy in $\langle \Sigma^{\mathrm{I}}_{(s)}, [\Sigma^{\mathrm{I}}_{(s)}] \backslash X$. Clearly, Σ^{II} is a nonempty good subtree of $\Sigma^{\mathrm{I}}_{(p'')}$ and $[\Sigma^{\mathrm{II}}] \subseteq T^X$, so

$$\tilde{p}'' = \langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, \langle 2, \Sigma^{\mathrm{II}}\rangle\rangle \in \tilde{\sigma}.$$

We will complete the description of σ in game-theoretic terminology, leaving a strictly formal definition to the reader. I plays according to $\tilde{\sigma}$ as long as II plays on Σ^{II}. Note that in $\tilde{G}$, after $\tilde{p}''$, both players are required to play on Σ^{II}, and $\tilde{\sigma}$ therefore never instructs I to deviate from Σ^{II}. In G, however, II may deviate from Σ^{II}. If and when it does, I is confronted by a position $s \in \Sigma^{\mathrm{I}} \backslash \Sigma^{\mathrm{II}}$.[18] By the definition of Σ^{II}, there is a winning I-strategy in $G'' = \langle \Sigma^{\mathrm{I}}_{(s)}, [\Sigma^{\mathrm{I}}_{(s)}] \backslash X\rangle$. As in Case 1, we let σ' be the first such strategy in the fixed wellordering mentioned above. I follows σ' until a position $u \in \sigma' \backslash T^X$ of even length is reached. I now imagines that II has played $\langle x_{2k+1}, \langle 1, u\rangle\rangle$ after p' in $\tilde{G}$ and that both players in that game have played—as they must—the remainder of u. From this point on in G, I plays as instructed by $\tilde{\sigma}$ in $\tilde{G}$.

Case II Suppose $\tilde{\sigma} \subseteq \tilde{T}$ is a II-strategy. We will define the II-strategy $\sigma = \varphi\,\tilde{\sigma} \subseteq T$.

Let $\sigma|2k = \tilde{\sigma}|2k$. Suppose $p = \langle x_0, \ldots, x_{2k-1}\rangle \in \sigma$. We must define II's reply to any move x_{2k} of I, i.e., any x_{2k} such that $p ^\frown \langle x_{2k}\rangle$ is in T. Therefore suppose $p' = p ^\frown \langle x_{2k}\rangle$. Let

$$U = \{((\tilde{s}_{2k+1})_1)_1 \mid \tilde{s} \in \tilde{\sigma}_{(p)} \wedge |\tilde{s}| = 2k+2 \wedge (\tilde{s}_{2k})_0 = x_{2k} \wedge ((\tilde{s}_{2k+1})_1)_0 = 1\}.$$

In other words, U consists of all the sequences $u \in T_{(p')}$ of even length that occur in a sequence in $\tilde{\sigma}$ of the form

$$\langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma\rangle, \langle x_{2k+1}, \langle 1, u\rangle\rangle\rangle.$$

Let $Y = \{y \in [T] \mid \forall u \in U\ u \nsubseteq y\}$, which is closed in $[T_{(p')}]$. Let T^Y be the corresponding tree, and let $G' = \langle T_{(p')}, Y\rangle$. We again consider two cases.

Case 1 Suppose there is a winning II-strategy in G'. Let σ' be the first such strategy in the fixed wellordering we have posited. Let σ instruct II to play according to σ' until a position $u \in U$ is reached, as must happen since σ' is

[18] Since Σ^{I} is a I-imposed subtree of $T_{(p')}$, II cannot be the first to deviate from it.

a winning II-strategy in G'. At this point let Σ^{I} be the first Σ in some fixed wellordering such that

$$\langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma\rangle, \langle x_{2k+1}, \langle 1, u\rangle\rangle\rangle \in \tilde{\sigma},$$

where, of course, $x_{2k+1} = u_{2k+1}$ (just as $x_m = u_m$ for all $m < 2k+1$). Let σ instruct II to play according to $\tilde{\sigma}$ for the rest of the game.

Case 2 Now suppose there is no winning II-strategy in G'. Let Σ^{I} be the I-nonlosing subtree of $T_{(p')}$ for G', i.e., the set of positions $s \in T_{(p')}$ such that there is no winning II-strategy in $\langle T_{(s)}, Y\rangle$. Σ^{I} is a I-imposed subtree of $T_{(p')}$, and $\langle x_{2k}, \Sigma^{\mathrm{I}}\rangle$ is a legal move by I after p. Let

$$\tilde{p}' = \langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle\rangle.$$

Clearly, $[\Sigma^{\mathrm{I}}] \subseteq T^Y$, so the response of $\tilde{\sigma}$ to $\tilde{p}'$ cannot be of the form $\langle x_{2k+1}, \langle 1, u\rangle\rangle$, because then $u \in U$ by definition, so $u \notin T^Y$ by definition, whereas $u \in \Sigma^{\mathrm{I}}$ by the definition of $\tilde{T}$. Therefore let x_{2k+1} and Σ^{II} be such that

$$\tilde{p}'' = \langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, \Sigma^{\mathrm{II}}\rangle\rangle \in \tilde{\sigma}.$$

Let σ instruct II to play x_{2k+1} and then play as instructed by $\tilde{\sigma}$ from $\tilde{p}''$ as long as the position remains in Σ^{II}. $\tilde{\sigma}$ never instructs II to leave Σ^{II}, and since Σ^{II} is a II-imposed subtree of Σ^{I}, I cannot make a move that leaves Σ^{II} without leaving Σ^{I}, so if a position s arises that is not in Σ^{II}, it is also not in Σ^{I}. By the definition of Σ^{I}, there is a winning II-strategy in $G'' = \langle T_{(s)}, Y\rangle$, and we let σ' be the first such strategy in the fixed wellordering mentioned above. II follows σ' until a position $u \in U$ is reached, as must eventually happen; otherwise, the ultimate result would be $x \in Y$, which is a win for I in G''. Let Σ be the first I-imposed subtree of $T_{(p')}$ such that

$$\langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma\rangle, \langle x_{2k+1}, \langle 1, u\rangle\rangle\rangle \in \tilde{\sigma}.$$

Then $u \restriction (2k+2) = x \restriction (2k+2)$, and

$$\langle u_0, \ldots, u_{2k-1}, \langle u_{2k}, \Sigma\rangle, \langle u_{2k+1}, \langle 1, u\rangle\rangle, u_{2k+2}, \ldots, u_{|u|-1}\rangle \in \tilde{\sigma},$$

so II may now play in G as instructed by $\tilde{\sigma}$ after this position in $\tilde{G}$.

It is easy to check that the maps π defined by (10.127) and $\varphi : \mathcal{S}^{\tilde{T}} \to \mathcal{S}^T$ as just defined satisfy (10.123), so $\langle \tilde{T}, \pi, \varphi\rangle$ is a k-covering of $\langle T, X\rangle$. Given

$$\tilde{x} = \langle x_0, \ldots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, r\rangle, \ldots\rangle \in [\tilde{T}],$$

and

$$x = \langle x_0, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}, \ldots\rangle = \pi\, x,$$

clearly $x \in X$ iff $r_0 = 2$, i.e., r is of the form $\langle 2, \Sigma^{\mathrm{II}}\rangle$ with $[\Sigma^{\mathrm{II}}] \subseteq X$, as opposed to $\langle 1, u\rangle$ with $u \notin T^X$. Hence $\tilde{X} = \pi^{\leftarrow} X$ is clopen; in fact, the winner of $\langle \tilde{T}, \tilde{X}\rangle$ is known after the first $2k+2$ moves. □10.125

For convenience in the proof of (10.125) we have defined the auxiliary tree $\tilde{T}$ in terms of subtrees Σ^{I} of $T_{(p')}$ and $T_{(p'')}$. Since all the sequences in these trees begin with p, the same information is contained in the the trees Σ^{I}_p and Σ^{II}_p consisting of the sequences obtained by removing this initial segment, i.e. $\Sigma^{\mathrm{I}} = p^\frown \Sigma^{\mathrm{I}}_p$ and

$\Sigma^{\mathrm{II}} = p^\frown\Sigma^{\mathrm{II}}_p$. The following definition takes advantage of this opportunity for efficiency,[19] which will become important when we compute the complexity of the trees that arise during iteration of the operation of forming coverings.

(10.128) **Definition** [ZFC] *Suppose T is a nonempty good tree and $X \subseteq [T]$ is closed. The* standard k-covering of $\langle T, X\rangle \overset{\text{def}}{=} \langle \tilde{T}^{T,X}, \pi^{T,X}, \varphi^{T,X}\rangle$*, where $\tilde{T}^{T,X}$ is the good tree such that*

1. $[\tilde{T}^{T,X}]$ *consists of all ω-sequences*

$$\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, r\rangle, x_{2k+2}, \dots\rangle,$$

where, letting $p' = \langle x_0, \dots, x_{2k}\rangle$, $p'' = \langle x_0, \dots, x_{2k}, x_{2k+1}\rangle$, *and* $x = \langle x_0, x_1, \dots\rangle$,

   1. $x \in T$;
   2. $p^\frown\Sigma^{\mathrm{I}}$ *is a* I*-imposed subtree of* $T_{(p')}$; *and*
   3. *either*
      1. $r = \langle 1, u\rangle$, *where u is a sequence of even length extending p'' such that*
         1. $u \in (p^\frown\Sigma^{\mathrm{I}})\backslash T^X$; *and*
         2. $x \in [T_{(u)}]$; *or*
      2. $r = \langle 2, \Sigma^{\mathrm{II}}\rangle$, *where $p^\frown\Sigma^{\mathrm{II}}$ is a* II*-imposed subtree of $p^\frown\Sigma^{\mathrm{I}}_{(p'')}$ such that*
         1. $p^\frown\Sigma^{\mathrm{II}} \subseteq T^X$; *and*
         2. $x \in [p^\frown\Sigma^{\mathrm{II}}]$;
2. $\pi\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma^{\mathrm{I}}\rangle, \langle x_{2k+1}, r\rangle, x_{2k+2}, \dots\rangle = \langle x_0, \dots\rangle$;[10.125] *and*
3. $\varphi^{T,X}$ *is defined* mutatis mutandis *as in the proof of (10.125).*

We now define the induction step in Martin's proof of Borel determinacy.

(10.129) **Definition** [ZFC] *Suppose $k \in \omega$.*

1. $\langle\langle T_i \mid i \in \omega\rangle, \langle \pi_i \mid i \in \omega\rangle, \langle \varphi_i \mid i \in \omega\rangle\rangle$ *is a* $\langle k, \omega\rangle$-covering system $\overset{\text{def}}{\iff}$ T_0 *is a nonempty good tree, and for each $i \in \omega$, $\langle T_{i+1}, \pi_i, \varphi_i\rangle$ is a $(k+i)$-covering of T_i.*
2. *Suppose $\langle\langle T_i \mid i \in \omega\rangle, \langle \pi_i \mid i \in \omega\rangle, \langle \varphi_i \mid i \in \omega\rangle\rangle$ is a $\langle k, \omega\rangle$-covering system. The* standard k-limit *of this system is the unique $\langle \hat{T}, \langle \hat{\pi}_i \mid i \in \omega\rangle, \langle \hat{\varphi}_i \mid i \in \omega\rangle\rangle$ such that*
   1. $\hat{T} = \bigcup_{i\in\omega} T_i|2(k+i)$;
   2. *for all $i \in \omega$*
      1. $\hat{\pi}_i \restriction (\hat{T}|2(k+i))$ *is the identity; and*
      2. $\hat{\pi}_i = \pi_i \circ \hat{\pi}_{i+1}$ $(= \pi_i \circ \pi_{i+1} \circ \hat{\pi}_{i+2}$, *etc.); and*
   3. *for all $i \in \omega$*
      1. $\hat{\varphi}_i \restriction \mathcal{S}^{\hat{T}|2(k+i)}$ *is the identity; and*
      2. $\hat{\varphi}_i = \varphi_i \circ \hat{\varphi}_{i+1}$ $(= \varphi_i \circ \varphi_{i+1} \circ \hat{\varphi}_{i+2}$, *etc.).*

[19] In fact, we could use $\Sigma^{\mathrm{I}}_{p'}$ and $\Sigma^{\mathrm{II}}_{p''}$, but we will not need this degree of efficiency, and in the interest of uniformity we will forgo this adjustment.

(10.130) **Theorem** [ZFC] *Suppose* $\langle\langle T_i \mid i \in \omega\rangle, \langle \pi_i \mid i \in \omega\rangle, \langle \varphi_i \mid i \in \omega\rangle\rangle$ *is a* $\langle k, \omega\rangle$*-covering system. Then (10.129.2) uniquely determines the standard* k*-limit* $\langle \hat{T}, \langle \hat{\pi}_i \mid i \in \omega\rangle, \langle \hat{\varphi}_i \mid i \in \omega\rangle\rangle$*, and for each* $i \in \omega$*,* $\langle \hat{T}, \hat{\pi}_i, \hat{\varphi}_i\rangle$ *is a* $(k+i)$*-covering of* T_i*.*

Proof By (10.123.2.2), for all $i \in \omega$, $T_i|2(k+i) = T_{i+1}|2(k+i)$, so for all $j > i$, $T_i|2(k+i) = T_j|2(k+i)$. It follows that $\hat{T}$[10.129.2.1] is a nonempty good tree with $\hat{T}|2(k+i) = T_i|2(k+i)$ for each $i \in \omega$, thus satisfying (10.123.2.2).

Given $s \in \hat{T}$, let j be least such that $|s| \leqslant 2(k+j)$. For any $i \geqslant j$ let $\hat{\pi}_i\, s = s$. In particular, $\hat{\pi}_j\, s = s$. Now let $\hat{\pi}_{j-1}\, s = \pi_{j-1}\, s$, $\hat{\pi}_{j-2}\, s = \pi_{j-2}(\pi_{j-1}\, s)$, etc. In other words, for any $i < j$ let

$$\hat{\pi}_i\, s = \pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{j-1}\, s.$$

This uniquely determines $\hat{\pi}_i\, s$ for all $s \in \hat{T}$ and $i \in \omega$ so as to satisfy (10.129.2.2). It is easy to check that (10.123.1.2) and (10.123.2.3) are satisfied.

The maps $\hat{\varphi}_i$ are defined in identical fashion. Thus, given $\sigma \in \mathcal{S}^{\hat{T}|n}$, let j be least such that $n \leqslant 2(k+j)$. For any $i \geqslant j$ let $\hat{\varphi}_i\, \sigma = \sigma$. For $i < j$, let

$$\hat{\varphi}_i\, \sigma = \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{j-1}\, \sigma.$$

This uniquely determines $\hat{\varphi}_i\, \sigma$ for all $\sigma \in \bar{\mathcal{S}}^{\hat{T}}$ and $i \in \omega$ so as to satisfy (10.129.2.3) and (10.123.1.3).

We extend $\hat{\pi}$ to $[\hat{T}]$ and $\hat{\varphi}$ to $\mathcal{S}^{\hat{T}}$ in the usual way, as in (10.123.1.2) and (10.123.1.3). For each $j \in \omega$ and $\sigma \in \mathcal{S}^{T_{j+1}}$,[10.123.1.4]

$$\pi_j{}^{\to}[\sigma] \supseteq [\varphi_j\, \sigma]. \tag{10.131}$$

To verify (10.123.1.4) for $\hat{\pi}_i$ and $\hat{\varphi}_i$, suppose $\sigma \in \mathcal{S}^{\hat{T}}$ and $x_i \in [\hat{\varphi}_i\, \sigma]$. Using some fixed wellordering of a suitable set, define x_j for $j > i$ by recursion on j so that

(10.132) *for* $j \geqslant i$*,*

1. $x_j \in [\hat{\varphi}_j\, \sigma]$*;*
2. $\pi_j\, x_{j+1} = x_j$*,*

which is possible[10.131] because $\hat{\varphi}_j\, \sigma \in \mathcal{S}^{T_j}$ and $\hat{\varphi}_j\, \sigma = \varphi_j \circ \hat{\varphi}_{j+1}\, \sigma$. Note that for any $j \geqslant i$,[10.129.2.3.1]

$$x_j \restriction 2(k+j) \in \hat{\varphi}_j(\sigma)|2(k+j) = \hat{\varphi}_j(\sigma|2(k+j)) = \sigma|2(k+j). \tag{10.133}$$

Let $x = \bigcup_{j \geqslant i} x_j \restriction 2(k+j)$. Then[10.123.2.3] x is an ω-sequence, and $x \restriction 2(k+j) = x_j \restriction 2(k+j)$ for all $j \geqslant i$, so[10.133] $x \in [\sigma]$.

It only remains to be shown that $\hat{\pi}_i\, x = x_i$. By construction,[10.132] for all $j \geqslant i$, $\pi_j\, x_{j+1} = x_j$, which is to say, for all $n \in \omega$, $\pi_j\,(x_{j+1} \restriction n) = x_j \restriction n$. Hence, for all $n \in \omega$ and $j \geqslant i$,

$$x_i \restriction n = \pi_i(x_{i+1} \restriction n) = \cdots = \pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{j-1}(x_j \restriction n).$$

For all $j \geqslant i$,[10.129.2.2.1]

$$\hat{\pi}_j\,(x \restriction 2(k+j)) = \hat{\pi}_j\,(x_j \restriction 2(k+j)) = x_j \restriction 2(k+j),$$

so

$$\begin{aligned} x_i \restriction 2(k+j) &= \pi_i \circ \cdots \circ \pi_{j-1}(x_j \restriction 2(k+j)) \\ &= \pi_i \circ \cdots \circ \pi_{j-1} \circ \hat{\pi}_j\,(x \restriction 2(k+j)) \\ &= \hat{\pi}_i(x \restriction 2(k+j)), \end{aligned}$$

so $\hat{\pi}_i\, x = x_i$. $\square^{10.130}$

We now state and prove the theorem (see (5.177) in the main text).

Theorem [ZFC] *Suppose T is a good tree and $X \subseteq [T]$ is Borel. Then $\langle T, X\rangle$ is determined.*

Proof It suffices to show that there is a covering $\langle \tilde{T}, \pi, \varphi\rangle$ of T that unravels X.[10.124] The following claim provides the framework for a proof of this by induction on the complexity of X.

(10.134) **Claim** *Suppose $0 < \alpha < \omega_1$. Then for any nonempty good tree T, $X \in \mathbf{\Sigma}^0_\alpha([T])$, and $k \in \omega$ there is a k-covering of $\langle T, X\rangle$ that unravels X.*

Proof By (10.125) this is true for $\alpha = 1$. Keep in mind that a covering that unravels $\langle T, X\rangle$ also unravels $\langle T, [T]\backslash X\rangle$.

Suppose $0 < \alpha < \omega_1$, and suppose the claim holds for every $\beta < \alpha$. Suppose $X \in \mathbf{\Sigma}^0_\alpha([T])$. Specifically, suppose $X = \bigcup_{i\in\omega} X_i$, where $X_i \in \mathbf{\Pi}^0_{\beta_i}([T])$, where $\beta_i < \alpha$ for each $i \in \omega$. Let $T_0 = T$ and let $\langle T_1, \pi_0, \varphi_0\rangle$ be a k-covering of T_0 that unravels X_0. Then $\pi_0{}^{\leftarrow}X_0 \in \mathbf{\Delta}^0_1([T_1])$, and since $\pi_0 : [T_1] \to [T_0]$ is continuous, $\pi_0{}^{\leftarrow}X_i \in \mathbf{\Pi}^0_{\beta_i}$ for each $i > 0$.

Now let $\langle T_2, \pi_1, \varphi_1\rangle$ be a $(k+1)$-covering of T_1 that unravels $\pi_0{}^{\leftarrow}X_1$. Then $\pi_1{}^{\leftarrow}\pi_0{}^{\leftarrow}X_i \in \mathbf{\Delta}^0_1([T_2])$ for $i = 0, 1$, while $\pi_1{}^{\leftarrow}\pi_0{}^{\leftarrow}X_i \in \mathbf{\Pi}^0_{\beta_i}([T_2])$ for $i > 1$.

Continue in this fashion to obtain a $\langle k, \omega\rangle$-covering system $\langle\langle T_i \mid i \in \omega\rangle, \langle \pi_i \mid i \in \omega\rangle, \langle \varphi_i \mid i \in \omega\rangle\rangle$, and let $\langle \hat{T}, \langle \hat{\pi}_i \mid i \in \omega\rangle, \langle \hat{\varphi}_i \mid i \in \omega\rangle\rangle$ be its standard k-limit. Then $\langle \hat{T}, \hat{\pi}_0, \hat{\varphi}_0\rangle$ unravels X_i, i.e., $\hat{\pi}_0{}^{\leftarrow}X_i \in \mathbf{\Delta}^0_1([\hat{T}])$, for each $i \in \omega$. Thus,

$$\hat{\pi}_0{}^{\leftarrow}X = \hat{\pi}_0{}^{\leftarrow}\bigcup_{i\in\omega} X_i = \bigcup_{i\in\omega} \hat{\pi}_0{}^{\leftarrow}X_i \in \mathbf{\Sigma}^0_1([\hat{T}]).$$

Now let $\langle \tilde{T}, \pi, \varphi\rangle$ be a k-covering of $\hat{T}$ that unravels $\hat{\pi}_0{}^{\leftarrow}X$. Then $\langle \tilde{T}, \hat{\pi}_0 \circ \pi, \hat{\varphi}_0 \circ \varphi\rangle$ is a k-covering of T that unravels X. $\square^{10.134}$ $\square^{5.177}$

10.20 Iterations of the powerset operation in Martin's proof of Borel determinacy

[Refer to p. 407.]

To analyze the use of the powerset operation in Martin's proof it is convenient to use the following modification $\mathcal{Q}$ of $\mathcal{P}$, and its iterates $\mathcal{Q}^\alpha$, $\alpha \in \mathrm{Ord}$.

(10.135) **Definition** [ZF] *Suppose M is a set.*

1. $\mathcal{Q}M \overset{\text{def}}{=} \mathcal{F}M \cup \mathcal{F}\mathcal{P}\mathcal{F}M$, *where $\mathcal{F}$ is the operation of finitary closure, i.e., closure under the formation of finite subsets.*[20]

[20] Note that for any set A, $\mathcal{F}A$ includes HF and is closed under the formation of finite sequences.

2. $\mathcal{Q}^0 M \stackrel{\text{def}}{=} M$;
3. $\mathcal{Q}^{\alpha+1} \stackrel{\text{def}}{=} \mathcal{Q}\mathcal{Q}^\alpha M$; *and*
4. $\mathcal{Q}^\alpha M = \bigcup_{\beta<\alpha} \mathcal{Q}^\beta M$, *if α is a limit ordinal.*

In the context of ZF^-, of course, $\mathcal{Q}M$ may not exist, or $\mathcal{Q}^\alpha M$ may exist, while $\mathcal{Q}^{\alpha+1}M$ does not exist. Clearly:

(10.136) *if α is a limit ordinal and $\mathcal{Q}^\beta M$ exists for all $\beta < \alpha$ then $\mathcal{Q}^\alpha$ exists.*

We first observe that to demonstrate the determinacy of a Borel set $X \subseteq [T]$ it is not necessary to completely unravel X. For example, suppose $X \subseteq {}^\omega M$ is $\mathbf{\Sigma}^0_4$, say

$$X = \bigcup_{k\in\omega}\bigcap_{l\in\omega}\bigcup_{m\in\omega} X_{\langle k,l,m\rangle},$$

where each $X_{\langle k,l,m\rangle}$ is $\mathbf{\Pi}^0_1$. Let $\langle s_i \mid i \in \omega\rangle$ enumerate ${}^3\omega$. As in the proof of (10.134) let $T_0 = T$ and let $\langle T_1, \pi_0, \varphi_0\rangle$ be the standard 0-covering[10.128] of T_0 that unravels X_{s_0}. Then $\pi_0{}^{\leftarrow}X_{s_0} \in \mathbf{\Delta}^0_1([T_1])$, and since $\pi_0 : [T_1] \to [T_0]$ is continuous, $\pi_0{}^{\leftarrow}X_{s_i} \in \mathbf{\Pi}^0_1$ for each $i > 0$.

Now let $\langle T_2, \pi_1, \varphi_1\rangle$ be the standard 1-covering of T_1 that unravels $\pi_0{}^{\leftarrow}X_{s_1}$. Then $\pi_1{}^{\leftarrow}\pi_0{}^{\leftarrow}X_{s_i} \in \mathbf{\Delta}^0_1([T_2])$ for $i = 0, 1$, while $\pi_1{}^{\leftarrow}\pi_0{}^{\leftarrow}X_{s_i} \in \mathbf{\Pi}^0_1([T_2])$ for $i > 1$.

Continue in this fashion to obtain a $\langle 0,\omega\rangle$-covering system $\langle\langle T_i \mid i \in \omega\rangle, \langle \pi_i \mid i \in \omega\rangle, \langle \varphi_i \mid i \in \omega\rangle\rangle$, and let $\langle \hat{T}, \langle \hat{\pi}_i \mid i \in \omega\rangle, \langle \hat{\varphi}_i \mid i \in \omega\rangle\rangle$ be its standard 0-limit. Then $\langle \hat{T}, \hat{\pi}_0, \hat{\varphi}_0\rangle$ unravels X_{s_i}, i.e., $\hat{\pi}_0{}^{\leftarrow}X_{s_i} \in \mathbf{\Delta}^0_1([\hat{T}])$, for each $i \in \omega$. Thus,

$$\hat{\pi}_0{}^{\leftarrow}X = \hat{\pi}_0{}^{\leftarrow}\bigcup_{k\in\omega}\bigcap_{l\in\omega}\bigcup_{m\in\omega} X_{\langle k,l,m\rangle} = \bigcup_{k\in\omega}\bigcap_{l\in\omega}\bigcup_{m\in\omega} \hat{\pi}_0{}^{\leftarrow}X_{\langle k,l,m\rangle} \in \mathbf{\Sigma}^0_3([\hat{T}]).$$

By (5.176) $\hat{\pi}_0{}^{\leftarrow}X$ is determined, so X is determined.

Note that we specified that the standard k-covering[10.128] be used at each step in the above construction of the coverings $\langle T_{i+1}, \pi_i, \varphi_i\rangle$.

1. The nodes of T_0 are of the form $\langle s_0, s_1, s_2, s_3, \ldots\rangle$, where $s_n \in \mathcal{F}M$ for every $n \in \omega$.

2. The nodes of T_1 are of the form $\langle S_0, S_1, s_2, s_3, \ldots\rangle$, where $S_0, S_1 \in \mathcal{Q}M$, and $s_n \in \mathcal{F}M$ for every $n \geqslant 2$.

3. Similarly, the nodes of T_2 are of the form $\langle S_0, S_1, S_2, S_3, s_4, \ldots\rangle$. Note that for any $q \in T_1$ with $|q| \geqslant 2$, $(T_1)_q \subseteq \mathcal{F}M$, so $S_2, S_3 \in \mathcal{Q}M$,[10.126.1] as are S_0, S_1, which are inherited from T_1; and $s_n \in \mathcal{F}M$ for every $n \geqslant 4$. This is the reason we use the standard coverings instead of coverings as described in (10.126).

4. Continuing in this way, we see that $T_i \subseteq \mathcal{Q}M$ for every $i \in \omega$.

5. Hence, $\hat{T} = \bigcup_{i\in\omega} T_i|2(k+i) \subseteq \mathcal{Q}M$,[10.129.2.1] where $\hat{T}$ is the tree of the standard 0-limit of this system.

(5.176) is a theorem of ZF^-, so the only "powerset" that needs to be posited to prove $\mathbf{\Sigma}^0_4$-determinacy for games on a set M is $\mathcal{Q}M$.

To prove $\mathbf{\Sigma}^0_5$-determinacy for games on a set M, we apply the same reasoning to reduce it to $\mathbf{\Sigma}^0_4$-determinacy for games on $\mathcal{Q}M$, reduce this to $\mathbf{\Sigma}^0_3$-determinacy for games on $\mathcal{Q}^2M$, and then apply (5.176). In general,

(10.137) $$\mathsf{ZF}^- \vdash \ulcorner\forall M\, \forall n \in \omega\, (\mathcal{Q}^n M \text{ exists} \to \mathbf{\Sigma}^0_{n+3}({}^\omega M)\text{-determinacy})\urcorner.$$

To see what happens at limit stages, take ω as an example. Suppose X is $\mathbf{\Sigma}^0_{\omega+2}$, say

$$X = \bigcup_{k\in\omega}\bigcap_{l\in\omega}\bigcup_{m\in\omega} X_{\langle k,l,m\rangle},$$

where each $X_{\langle k,l,m\rangle}$ is $\mathbf{\Sigma}^0_n$ for some $n<\omega$. If we have $\mathcal{Q}^n M$ for every $n<\omega$ (i.e., we have $\mathcal{Q}^\omega M$[10.136]), then we have coverings that unravel each $X_{\langle k,l,m\rangle}$, and we can put these together to form a single covering that unravels them all. This covering reduces X to a $\mathbf{\Sigma}^0_3$ set, and (5.176) applies.

To prove $\mathbf{\Sigma}^0_{\omega+3}$-determinacy for games on a set M, we apply this reasoning to reduce it to $\mathbf{\Sigma}^0_4$-determinacy for games on $\mathcal{Q}^\omega M$, reduce this to $\mathbf{\Sigma}^0_3$-determinacy for games on $\mathcal{Q}^{\omega+1}M$, and then apply (5.176). In general, given the existence of $\mathcal{Q}^{\omega+n}M$, we can prove $\mathbf{\Sigma}^0_{\omega+n+2}$-determinacy, and this works for any limit ordinal in place of ω. This leads to the following conclusion, which is stated so as to incorporate the previous result[10.137] for finite ordinals,

$$\mathsf{ZF}^- \vdash \ulcorner \forall M\,\forall\rho<\omega_1\,(\mathcal{Q}^\rho M \text{ exists} \rightarrow \mathbf{\Sigma}^0_{1+\rho+2}({}^\omega M)\text{-determinacy})\urcorner.$$

10.21 Proof of (8.36)

[Refer to p. 487.]

Theorem [GB] *For any* $\mathsf{s}^{M,\mathbb{P}}$*-sentence* σ*,* $\{p\in|\mathbb{P}| \mid p\Vdash^{M,\mathbb{P}}\sigma\}$ *is regular.*

Proof If there is no $\{\sigma\}^{M,\mathbb{P}}$-forcing relation then $\{p\in|\mathbb{P}| \mid p\Vdash^{M,\mathbb{P}}\sigma\} = |\mathbb{P}|$, which is regular. Suppose therefore that $\Vdash$ is a $\{\sigma\}^{M,\mathbb{P}}$-forcing relation. We proceed by induction on complexity of $\theta\in\{\sigma\}^{M,\mathbb{P}}$ to show that $\{p\in|\mathbb{P}| \mid p\Vdash\theta\}$ is regular. Recall[8.10.2] that regular sets are open, and if $X\subseteq|\mathbb{P}|$ is open then X is regular iff for any $p\in|\mathbb{P}|$, if X is dense below p then $p\in X$.

Suppose, therefore, that $\theta\in\{\sigma\}^{M,\mathbb{P}}$, and suppose first that θ is atomic, i.e., $\theta = \tau\in\tau'$, $\tau=\tau'$, or $\mathsf{V}\tau$. Let $X=\{p\in|\mathbb{P}| \mid p\Vdash\theta\}$. It is easy to see from (8.27.1, 2) and (8.29.3) that X is open. To show X is regular, we consider each of the three cases separately.

$(\theta=\tau\in\tau')$ Suppose X is dense below p, i.e.,

$$\forall q\leqslant p\,\exists r\leqslant q\,(r\Vdash\tau\in\tau'). \tag{10.138}$$

We must show that $p\in X$, i.e.,[8.27.1]

$$\forall q\leqslant p\,\exists r\leqslant q\;\exists\langle\tau_0,r'\rangle\in\tau'\,(r\leqslant r'\wedge r\Vdash\tau_0=\tau).$$

To this end, suppose $q\leqslant p$. Let[10.138] $r\leqslant q$ be such that $r\Vdash\tau\in\tau'$. Then there exists $\langle\tau_0,r'\rangle\in\tau'$, and s extending both r and r', such that $s\Vdash\tau_0=\tau$. Since $s\leqslant q$, this justifies the claim.

$(\theta=\tau=\tau')$ Suppose X is dense below p. We must show that $p\in X$, i.e.,[8.27.2]

$$\forall q\leqslant p\,\forall\langle\tau_0,r'\rangle\in\tau\,(q\leqslant r'\rightarrow q\Vdash\tau_0\in\tau'),$$

and likewise with τ and τ' switched. To this end, suppose $\langle\tau_0,r'\rangle\in\tau$ and $q\leqslant p,r'$. By hypothesis, $\forall r\leqslant q\,\exists s\leqslant r\; s\Vdash\tau=\tau'$. Since $q\leqslant r'$, it follows by definition that $\forall r\leqslant q\,\exists s\leqslant r\; s\Vdash\tau_0\in\tau'$, i.e., $\{s \mid s\Vdash\tau_0\in\tau'\}$ is dense below q. By induction hypothesis, therefore, $q\Vdash\tau_0\in\tau'$. This justifies the first half of the claim, and the second half (with τ and τ' switched) is justified by the corresponding argument.

($\theta = \mathsf{V}(\tau)$) Suppose X is dense below p. We must show that $p \in X$, i.e.,[8.29.3]

$$\forall q \leqslant p\, \exists r \leqslant q\, \exists x \in M\, (r \Vdash \tau = \check{x}).$$

To this end, suppose $q \leqslant p$. Since X is dense below p, there exists $r \leqslant q$ such that $r \in X$, so there exist $s \leqslant r$ and $x \in M$ such that $s \Vdash \tau = \check{x}$. Since $s \leqslant q$ this justifies the claim.

Now suppose θ is complex, and suppose that for all $\theta' \in \{\theta\}^{M,\mathbb{P}}$ of lower logical complexity, $\{p \in |\mathbb{P}| \mid p \Vdash \theta'\}$ is regular. It is easy to see from Definition 8.29.4–10 that X is open. To show X is regular, we

(10.139) *suppose X is dense below p,*

and we show that $p \in X$.

($\theta = \neg\,\phi$) Suppose $q \leqslant p$. Let[10.139] $r \leqslant q$ be such that $r \Vdash \theta$. Then $r \nVdash \phi$,[8.29.4] so $q \nVdash \phi$, since $\{p \in |\mathbb{P}| \mid p \Vdash \phi\}$ is open by induction hypothesis. Hence, $p \Vdash \neg\,\phi$.[8.29.4]

($\theta = \phi \wedge \psi$) For any r, if $r \Vdash \theta$ then $r \Vdash \phi$ and $r \Vdash \psi$, so[10.139] $\forall q \leqslant p\, \exists r \leqslant q\, (r \Vdash \phi)$ and $\forall q \leqslant p\, \exists r \leqslant q\, (r \Vdash \psi)$, so by the induction hypothesis, $p \Vdash \phi$ and $p \Vdash \psi$, so $p \Vdash \phi \wedge \psi$.

($\theta = \phi \vee \psi$) Suppose $q \leqslant p$. Let[10.139] $r \leqslant q$ be such that $r \Vdash \phi \vee \psi$. Let $s \leqslant r$ be such that $s \Vdash \phi$ or $s \Vdash \psi$. Then $s \leqslant q$ and $s \Vdash \phi \vee s \Vdash \psi$. Hence $p \Vdash \phi \vee \psi$.

($\theta = \phi \rightarrow \psi$) Suppose $q \leqslant p$ and $q \Vdash \phi$. By hypothesis,[10.139] $\forall q' \leqslant q\, \exists r \leqslant q'\ r \Vdash (\phi \rightarrow \psi)$, so $\forall q' \leqslant q\, \exists r \leqslant q'\, (r \Vdash \phi \rightarrow r \Vdash \psi)$. Since $q \Vdash \phi$, $\forall q' \leqslant q\, \exists r \leqslant q'\ r \Vdash \psi$, so by induction hypothesis, $q \Vdash \psi$.

($\theta = \phi \leftrightarrow \psi$) Essentially immediate.

($\theta = \forall v\, \phi$) Suppose $\tau \in M^{\mathbb{P}}$. Then[10.139] $\forall q \leqslant p\, \exists r \leqslant q\, r \Vdash \phi(\tau)$, so by induction hypothesis, $p \Vdash \phi(\tau)$. Hence $p \Vdash \theta$.

($\theta = \exists v\, \phi$) Suppose $q \leqslant p$. There exists[10.139] $r \leqslant q$ such that $r \Vdash \exists v\, \phi$, and by definition, there exist $s \leqslant r$ and $\tau \in M^{\mathbb{P}}$ such that $s \Vdash \phi(\tau)$. $s \leqslant q$, so $\forall q \leqslant p\, \exists s \leqslant q\, \exists \tau \in M^{\mathbb{P}}\ s \Vdash \phi(\tau)$. Hence $p \Vdash \exists v\, \phi$. $\square$[8.36]

10.22 Proof of (8.44)

[REFER TO P. 489.]

Theorem [GB] *Suppose M is a transitive model of* ZF, *$\mathbb{P} \in M$ is a partial order, and G is an M-generic filter on $\mathbb{P}$. For all $\tau, \tau' \in M^{\mathbb{P}}$,*

$$\begin{aligned} \tau^G \in \tau'^G &\leftrightarrow \exists p \in G\ p \Vdash \tau \in \tau' \\ \tau^G = \tau'^G &\leftrightarrow \exists p \in G\ p \Vdash \tau = \tau'. \end{aligned} \tag{10.140}$$

Remark Note that we do not assume that for every $p \in |\mathbb{P}|$ there exists an M-generic filter G with $p \in G$, so Theorem 8.21 does not apply.

Proof We prove (10.140) by induction using the ordering (8.22) of ordinal pairs as in the proof of (8.21). We will use (8.10.6, 7) repeatedly, usually without reference. We will also use the definability of the forcing relation as discussed in Section 8.2.4, specifically (8.43), often without reference.

Suppose $p \in G$ and $p \Vdash \tau \in \tau'$. Then[8.27.1] the set $\{r \in |\mathbb{P}| \mid \exists \langle \tau_0, r' \rangle \in \tau' \, (r \leqslant r' \wedge r \Vdash \tau_0 = \tau)\}$ is dense below p and is in M,[8.43] so G meets this set,[8.10.7] say at r. Let $\langle \tau_0, r' \rangle \in \tau'$ be such that $r \leqslant r'$ and $r \Vdash \tau_0 = \tau$. Then $r' \in G$, so $\tau_0^G \in \tau'^G$, and by induction hypothesis $\tau_0^G = \tau^G$, so $\tau^G \in \tau'^G$.

Conversely, suppose $\tau^G \in \tau'^G$. Then there exist $r' \in G$ and $\langle \tau_0, r' \rangle \in \tau'$ such that $\tau_0^G = \tau^G$. By induction hypothesis, there exists $s \in G$ such that $s \Vdash \tau_0 = \tau$. Let p be a common extension of r' and s in G. Then for all $q \leqslant p$, there exists $r \leqslant q$ (e.g., q itself) such that $r \leqslant r'$ and $r \Vdash \tau_0 = \tau$, so $p \Vdash \tau \in \tau'$.[8.27.1]

Now suppose $p \in G$ and $p \Vdash \tau = \tau'$. Suppose $x \in \tau^G$. Then there exists $\langle \tau_0, r' \rangle \in \tau$ such that $r' \in G$ and $\tau_0^G = x$. Let q be a common extension of p and r' in G. Then[8.27.2] $q \Vdash \tau_0 \in \tau'$, so $x = \tau_0^G \in \tau'^G$ by induction hypothesis. Similarly, for any $x \in \tau'^G$, $x \in \tau^G$. Hence, $\tau^G = \tau'^G$.

Conversely, suppose $\tau^G = \tau'^G$. Let

$$\begin{aligned} S_0 &= \{p \mid \forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \rightarrow \exists r \leqslant q \; r \Vdash \tau_0 \in \tau')\} \\ S_1 &= \{q \mid \exists \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \wedge \forall r \leqslant q \; r \nVdash \tau_0 \in \tau')\}. \end{aligned}$$

Note that S_0 and S_1 are open. Suppose $p \in |\mathbb{P}|$. Then either $p \in S_0$ or there exists $q \leqslant p$ such that $q \in S_1$. Thus, $S_0 \cup S_1$ is dense, so there exists $p \in G$ such that $p \in S_0$ or $p \in S_1$. Suppose $p \in S_1$. Let $\langle \tau_0, r' \rangle \in \tau$ be such that $p \leqslant r'$ and for all $q \leqslant p$, $q \nVdash \tau_0 \in \tau'$. Since $p \leqslant r'$, $\tau_0^G \in \tau^G$, so $\tau_0^G \in \tau'^G$. Let $s \in G$ be such that $s \Vdash \tau_0 \in \tau'$. Let q be a common extension of p and s. Then $q \nVdash \tau_0 \in \tau'$ and $q \Vdash \tau_0 \in \tau'$; contradiction.

Thus, $p \notin S_1$, so $p \in S_0$. Now suppose $q \leqslant p$, $\langle \tau_0, r' \rangle \in \tau$, and $q \leqslant r'$. Since $q \in S_0$ and $q \leqslant r'$, $\forall q' \leqslant q \, \exists r \leqslant q' \; r \Vdash \tau_0 \in \tau'$, so[8.36] $q \Vdash \tau_0 \in \tau'$. Thus,

$$\forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau').$$

Similarly, there exists $p \in G$ such that

$$\forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau).$$

Thus, there exists $p \in G$ satisfying both of these open conditions, i.e.,

$$\begin{aligned} &\forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau') \\ \wedge &\forall q \leqslant p \, \forall \langle \tau_0, r' \rangle \in \tau' \, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau), \end{aligned}$$

which is to say, $p \Vdash \tau = \tau'$.

□[8.44]

10.23 Proof of (8.45)

[Refer to p. 490.]

(10.141) **Theorem** [GB] *Suppose M is a transitive model of* ZF*, $\mathbb{P} \in M$ is a partial order, G is an M-generic filter on $\mathbb{P}$, and σ is an $\mathsf{s}^{M,\mathbb{P}}$-sentence. Suppose there is a $\{\sigma\}^{M,\mathbb{P}}$-forcing relation. Then there is a $\{\sigma\}^{M[G]}$-satisfaction relation, and*

$$M[G] \models \sigma \leftrightarrow \exists p \in G\, p \Vdash \sigma.$$

Proof Let $\Vdash$ be the $\{\sigma\}^{M,\mathbb{P}}$-forcing relation. Recall that $M[G] = \{\tau^G \mid \tau \in M^{\mathbb{P}}\}$. Thus, given an $\mathsf{s}^{M,\mathbb{P}}$-formula θ, every $M[G]$-assignment for θ is of the form T^G, where $T = \left\langle\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right\rangle$ is an $M^{\mathbb{P}}$-substitution for $\operatorname{Free}\theta$, and

$$T^G \stackrel{\text{def}}{=} \left\langle\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau_0^G & \cdots & \tau_{n^-}^G\end{smallmatrix}\right\rangle.$$

Let S^G be the class of $\langle \theta, T^G\rangle$ such that $\theta \in \overline{\{\sigma\}}$, T is an $M^{\mathbb{P}}$-substitution for $\operatorname{Free}\theta$*, and*

$$\exists p \in G\ p \Vdash \theta(T).$$

Note that if A is an $M[G]$-assignment for θ then $\langle \theta, A\rangle \in S^G$ iff there exists an $M^{\mathbb{P}}$-substitution T for $\operatorname{Free}\theta$ such that $A = T^G$ and $\exists p \in G\ p \Vdash \theta(T)$. *A priori* the possibility exists that for some substitution T, $\langle \theta, T^G\rangle \in S^G$ by virtue of the fact that $T^G = T'^G$ for some substitution T' such that $\exists p \in G\ p \Vdash \theta(T')$, while it is not the case that $\exists p \in G\ p \Vdash \theta(T)$. The first statement of the following claim is that this does not happen.[21]

(10.142) **Claim** *Suppose $\theta \in \overline{\{\sigma\}}$.*

1. *Suppose T, T' are $M^{\mathbb{P}}$-substitutions for* $\operatorname{Free}\theta$ *and $T^G = T'^G$. Then*

$$\exists p \in G\ p \Vdash \theta(T) \leftrightarrow \exists p \in G\ p \Vdash \theta(T').$$

2. *S^G includes the $\{\theta\}^{\mathfrak{M}[G]}$-satisfaction relation.*

Proof By induction on the complexity of $\theta \in \overline{\{\sigma\}}$. Note that (10.142.1) implies that for any $M^{\mathbb{P}}$-substitution T for $\operatorname{Free}\theta$,

$$\langle \theta, T^G\rangle \in S^G \leftrightarrow \exists p \in G\ p \Vdash \theta(T).$$

The proof consists therefore of showing that the relation defined by $\ulcorner \exists p \in G\ p \Vdash \theta(T)\urcorner$ satisfies the recursive definition of satisfaction for $\mathfrak{M}[G]$.

$(\theta = t_0 \in t_1)$ Here t_0, t_1 are terms of the forcing language, which may be variables or constant terms, i.e., elements of $M^{\mathbb{P}}$. In any case, if T is an $M^{\mathbb{P}}$-substitution for $\operatorname{Free}\theta$, then $\theta(T) = \tau_0 \in \tau_1$ for some $\tau_0, \tau_1 \in M^{\mathbb{P}}$. By Theorem 8.44

$$\tau_0^G \in \tau_1^G \leftrightarrow \exists p \in G\ p \Vdash \tau_0 \in \tau_1,$$

and the claim follows at once.

$(\theta = t_0 = t_1)$ Similar to the previous case.

[21]Since

$$\bigwedge_{m \in n} \tau_m = \tau'_m \to \left(\theta\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau_0 & \cdots & \tau_{n^-}\end{smallmatrix}\right) \leftrightarrow \theta\left(\begin{smallmatrix} u_0 & \cdots & u_{n^-} \\ \tau'_0 & \cdots & \tau'_{n^-}\end{smallmatrix}\right)\right)$$

is a logical validity, we could derive (10.142.1) from Theorem 8.82, but we don't have this theorem yet.

$(\theta = \mathsf{V}(t))$ Whether t is a variable or a constant term, the proof boils down to showing that for any $\tau \in M^{\mathbb{P}}$

$$\tau^G \in M \leftrightarrow \exists p \in G\ p \Vdash \mathsf{V}(\tau).$$

Suppose $\tau^G = x \in M$. Then $\tau^G = \check{x}^G$, where $\check{x}$ is the standard name for x. Thus for some $p \in G$, $p \Vdash \tau = \check{x}$, so[8.29.3] $p \Vdash \mathsf{V}(\tau)$. Conversely, suppose $p \in G$ and $p \Vdash \mathsf{V}(\tau)$. Then[8.29.3] the set of $r \in |\mathbb{P}|$ such that $r \Vdash \tau = \check{x}$ for some $x \in M$ is dense below p. Hence for some $r \in G$ and $x \in M$, $r \Vdash \tau = \check{x}$, so $\tau^G = \check{x}^G = x \in M$.

$(\theta = \neg\,\phi)$ As for the preceding cases, the essence of the argument concerns the sentences that arise from substitution of constant terms for all free variables, and we will henceforth confine our remarks to this case. We must show that

$$\exists p \in G\ p \Vdash \neg\,\phi \leftrightarrow \neg\,\exists p \in G\ p \Vdash \phi.$$

Suppose $p \in G$ and $p \Vdash \neg\,\phi$. Then[8.29.4] no extension of p forces ϕ. It follows that there does not exist $p' \in G$ such that $p' \Vdash \phi$; otherwise there is $q \in G$ extending both p and p', and q is an extension of p forcing ϕ. Conversely, suppose $\neg\,\exists p \in G\ p \Vdash \phi$. Since[8.38] $\{p \mid p \mid \phi\}$ is dense (and is in M), and no $p \in G$ forces ϕ, for some $p \in G$, $p \Vdash \neg\,\phi$.

$(\theta = \phi \wedge \psi)$ We must show that

$$\exists p \in G\ p \Vdash (\phi \wedge \psi) \leftrightarrow \exists p \in G\ p \Vdash \phi \wedge \exists p \in G\ p \Vdash \psi.$$

Suppose $p \in G$ and $p \Vdash (\phi \wedge \psi)$. Then $p \Vdash \phi$ and $p \Vdash \psi$. Conversely, suppose $p, p' \in G$ are such that $p \Vdash \phi$ and $p' \Vdash \psi$. Let $p'' \in G$ extend both p and p'. Then $p'' \Vdash \phi$ and $p'' \Vdash \psi$, so $p'' \Vdash (\phi \wedge \psi)$.

$(\theta = \phi \vee \psi)$ We must show that

$$\exists p \in G\ p \Vdash (\phi \vee \psi) \leftrightarrow \exists p \in G\ p \Vdash \phi \vee \exists p \in G\ p \Vdash \psi.$$

Suppose $p \in G$ and $p \Vdash (\phi \vee \psi)$. Then the conditions forcing either ϕ or ψ are dense below p, and therefore there exists $r \in G$ such that $r \Vdash \phi$ or $r \Vdash \psi$. Conversely, suppose $\exists p \in G\ p \Vdash \phi$ or $\exists p \in G\ p \Vdash \psi$. Without loss of generality, let $p \in G$ be such that $p \Vdash \phi$. Then for any $q \leqslant p$ there exists $r \leqslant q$—e.g., q itself—such that $r \Vdash \phi$, so $p \Vdash \phi \vee \psi$.

$(\theta = \phi \rightarrow \psi)$ We must show that

$$\exists p \in G\ p \Vdash (\phi \rightarrow \psi) \leftrightarrow \big((\exists p \in G\ p \Vdash \phi) \rightarrow (\exists p \in G\ p \Vdash \psi)\big).$$

Suppose $p \in G$ and $p \Vdash (\phi \rightarrow \psi)$, i.e.,[8.29.7] $\forall q \leqslant p\,(q \Vdash \phi \rightarrow q \Vdash \psi)$. Suppose there exists $p' \in G$ such that $p' \Vdash \phi$. Let $q \in G$ be a common extension of p and p'. Then $q \leqslant p$ and $q \Vdash \phi$, so $q \Vdash \psi$. Conversely, suppose $(\exists p \in G\ p \Vdash \phi) \rightarrow (\exists p \in G\ p \Vdash \psi)$. Then either $\forall p \in G\ p \nVdash \phi$ or $\exists p \in G\ p \Vdash \psi$. In the former instance, since the set of conditions deciding ϕ is dense, there exists $p \in G$ such that $p \Vdash \neg\,\phi$, i.e., $\forall q \leqslant p\ q \nVdash \phi$, so[8.29.7] $p \Vdash (\phi \rightarrow \psi)$. In the latter instance, let $p \in G$ be such that $p \Vdash \psi$. Then $p \Vdash (\phi \rightarrow \psi)$.

$(\theta = \phi \leftrightarrow \psi)$ Straightforward.

$(\theta = \forall v\, \phi)$ Remember that every member of $M[G]$ is τ^G for some $\tau \in M^{\mathbb{P}}$, so we must show that

$$\exists p \in G\ p \Vdash \forall v\, \phi \leftrightarrow \forall \tau \in M^{\mathbb{P}}\, \exists p \in G\ p \Vdash \phi(\tau),$$

where $\phi(\tau) = \phi\binom{v}{\tau}$, v being the sole free variable of ϕ, θ being presumed to be a sentence. Suppose $p \in G$ and $p \Vdash \forall v\, \phi$. Then $\forall \tau \in M^{\mathbb{P}}\ p \Vdash \phi(\tau)$.

Inversely, suppose $\forall p \in G\, p \not\Vdash \forall v\, \phi$. Note that for every $p \in |\mathbb{P}|$, if $p \not\Vdash \forall v\, \phi$ then for some $\tau \in M^{\mathbb{P}}$, $p \not\Vdash \phi(\tau)$. Hence, $\{q \mid q \Vdash \phi(\tau)\}$ is not dense below p, so for some $q \leqslant p$, $\forall r \leqslant q\ r \not\Vdash \phi(\tau)$, whence $q \Vdash \neg\, \phi(\tau)$. Hence $\{p \in |\mathbb{P}| \mid (p \Vdash \forall v\, \phi) \vee \exists \tau \in M^{\mathbb{P}}\, (p \Vdash \neg\, \phi(\tau))\}$ is dense, so G meets this set. Since we have supposed $\forall p \in G\, p \not\Vdash \forall v\, \phi$, there exists $\tau \in M^{\mathbb{P}}$ and $p \in G$ such that $p \Vdash \neg\, \phi(\tau)$ so $\neg\, \exists p \in G\ p \Vdash \phi(\tau)$.

$(\theta = \exists v\, \phi)$ We must show that

$$\exists p \in G\ p \Vdash \exists v\, \phi \leftrightarrow \exists \tau \in M^{\mathbb{P}}\, \exists p \in G\ p \Vdash \phi(\tau).$$

Suppose $p \in G$ and $p \Vdash \exists v\, \phi$. Then the conditions r such that $\exists \tau \in M^{\mathbb{P}}\ r \Vdash \phi(\tau)$ are dense below p, so $\exists \tau \in M^{\mathbb{P}}\, \exists r \in G\ r \Vdash \phi(\tau)$.

Conversely, suppose $\tau \in M^{\mathbb{P}}$, $p \in G$, and $p \Vdash \phi(\tau)$. Then $p \Vdash \exists v\ \phi$. □10.142

S^G is therefore the $\{\sigma\}^{M,\mathbb{P}}$-satisfaction relation for $\mathfrak{M}[G]$, and $\mathfrak{M}[G] \models \sigma \leftrightarrow \exists p \in G\ p \Vdash \sigma$. □10.141

10.24 Proof of (8.66)

[Refer to p. 497.]

(10.143) **Theorem** [GB]

1. $[\![x = y]\!] \wedge [\![y = z]\!] \leqslant [\![x = z]\!]$.
2. $[\![x \in y]\!] \wedge [\![x = z]\!] \leqslant [\![z \in y]\!]$.
3. $[\![y \in x]\!] \wedge [\![x = z]\!] \leqslant [\![y \in z]\!]$.

Proof By induction on $\langle \rho x, \rho y, \rho z\rangle$ with the following order $\preccurlyeq$:[8.22]

Given a 3-sequence s of ordinals, let $\bar{s}$ be the (unique) 3-sequence of the form $s \circ \pi$, where $\pi : 3 \xrightarrow{\text{bij}} 3$ is a permutation of $\{0, 1, 2\}$, and $\bar{s}_0 \geqslant \bar{s}_1 \geqslant \bar{s}_2$. $s \preccurlyeq s' \overset{\text{def}}{\iff} \bar{s}$ precedes $\bar{s}'$ lexicographically.

Note that $\preccurlyeq$ is a prewellorder, sequences related by a permutation of their domain (viz., 3) are at the same level of $\preccurlyeq$, and if s and s' differ at one coordinate, then whichever is lower at that coordinate is lower in $\preccurlyeq$.

For convenience we let

$$[\![x \subseteq y]\!] \overset{\text{def}}{=} \bigwedge_{z \in \operatorname{dom} x} \big(x(z) \to [\![z \in y]\!]\big),$$

so $[\![x = y]\!] = [\![x \subseteq y]\!] \wedge [\![y \subseteq x]\!]$. Suppose $x, y, z \in M^{\mathbb{P}}$ and the theorem holds for all $x', y', z' \in M^{\mathbb{P}}$ such that $\langle \rho x', \rho y', \rho z'\rangle \prec \langle \rho x, \rho y, \rho z\rangle$.

1 We first show that

$$[\![x \subseteq y]\!] \wedge [\![y = z]\!] \leqslant [\![x \subseteq z]\!], \tag{10.144}$$

for which it suffices to show that for any $w \in \operatorname{dom} x$,

$$\big(x(w) \to [\![w \in y]\!]\big) \wedge [\![y = z]\!] \leqslant \big(x(w) \to [\![w \in z]\!]\big).$$

This follows from the instance $[\![w \in y]\!] \wedge [\![y = z]\!] \leqslant [\![w \in z]\!]$ of the induction hypothesis. (Since $\rho w < \rho x$, $\langle \rho y, \rho w, \rho z\rangle \prec \langle \rho x, \rho y, \rho z\rangle$.)

Next we show that

$$[\![y \subseteq x]\!] \wedge [\![y = z]\!] \leqslant [\![z \subseteq x]\!], \tag{10.145}$$

for which it suffices to show that $[\![y \subseteq x]\!] \wedge [\![z \subseteq y]\!] \leqslant [\![z \subseteq x]\!]$, for which it suffices to show that for any $w \in \operatorname{dom} z$,

$$\big(z(w) \to [\![w \in y]\!]\big) \wedge [\![y \subseteq x]\!] \leqslant \big(z(w) \to [\![w \in x]\!]\big),$$

for which it suffices to show that $[\![w \in y]\!] \wedge [\![y \subseteq x]\!] \leqslant [\![w \in x]\!]$, i.e.,

$$\bigvee_{w' \in \operatorname{dom} y} \big(y(w') \wedge [\![w' = w]\!]\big) \wedge \bigwedge_{w' \in \operatorname{dom} y} \big(y(w') \to [\![w' \in x]\!]\big) \leqslant [\![w \in x]\!],$$

for which it suffices to show that for each $w' \in \operatorname{dom} y$,

$$y(w') \wedge [\![w' = w]\!] \wedge [\![w' \in x]\!] \leqslant [\![w \in x]\!]$$

This follows from the instance $[\![w' \in x]\!] \wedge [\![w' = w]\!] \leqslant [\![w \in x]\!]$ of the induction hypothesis. $(\langle \rho w', \rho x, \rho w\rangle \prec \langle \rho x, \rho y, \rho z\rangle$.)

Conjoining corresponding sides of (10.144) and (10.145), we have (10.143.1).

2 Suppose $w \in \operatorname{dom} y$. By induction hypothesis, $[\![x = z]\!] \wedge [\![x = w]\!] \leqslant [\![z = w]\!]$, so

$$[\![x = z]\!] \wedge [\![x = w]\!] \wedge y(w) \leqslant [\![z = w]\!] \wedge y(w).$$

Disjoining over all $w \in \operatorname{dom} y$,

$$[\![x = z]\!] \wedge [\![x \in y]\!] \leqslant [\![z \in y]\!].$$

3 Suppose $w \in \operatorname{dom} x$. Then

$$x(w) \wedge [\![x = z]\!] \leqslant x(w) \wedge \big(x(w) \to [\![w \in z]\!]\big) \leqslant [\![w \in z]\!],$$

so

$$[\![y = w]\!] \wedge x(w) \wedge [\![x = z]\!] \leqslant [\![y = w]\!] \wedge [\![w \in z]\!].$$

By induction hypothesis, $[\![y = w]\!] \wedge [\![w \in z]\!] \leqslant [\![y \in z]\!]$, so

$$[\![y = w]\!] \wedge x(w) \wedge [\![x = z]\!] \leqslant [\![y \in z]\!],$$

so

$$\bigvee_{w \in \operatorname{dom} x} \big([\![y = w]\!] \wedge x(w)\big) \wedge [\![x = z]\!] \leqslant [\![y \in z]\!],$$

i.e., $[\![y \in x]\!] \wedge [\![x = z]\!] \leqslant [\![y \in z]\!]$, as claimed. $\square$10.143

10.25 Proof of (8.74)

[Refer to p. 499.]

(10.146) **Theorem** [GB] *If ϕ is a formula with the single free variable u, then*

1. $[\![x = y]\!] \wedge [\![\phi(x)]\!] \leqslant [\![\phi(y)]\!]$.
2. $[\![\exists u \in x\, \phi]\!] = \bigvee_{y \in \operatorname{dom} x} \big(x(y) \wedge [\![\phi(y)]\!]\big)$.
3. $[\![\forall u \in x\, \phi]\!] = \bigwedge_{y \in \operatorname{dom} x} \big(x(y) \rightarrow [\![\phi(y)]\!]\big)$.

Proof If the $\{\phi\}$-valuation function does not exist all boolean values in the statement of the theorem are **1**, so it is trivially true. We therefore assume that the $\{\phi\}$-valuation function exists and proceed accordingly.

1 We proceed by induction on logical complexity of subformulas ψ of ϕ. The case that ψ is atomic is easily handled by (8.66) and (8.65).

Now suppose $\psi = \neg\, \psi'$. By induction hypothesis, $[\![x = y]\!] \wedge [\![\psi'(y)]\!] \leqslant [\![\psi'(x)]\!]$ (reversing the roles of x and y and using (8.65)). Thus

$$[\![x = y]\!] \wedge [\![\psi(x)]\!] = [\![x = y]\!] \wedge \neg\, [\![\psi'(x)]\!] \leqslant \neg\, [\![\psi'(y)]\!] = [\![\psi(y)]\!].$$

The remaining cases are left to the reader.

2, 3 These are dual to each other. We'll prove the former.

$$\begin{aligned}
[\![\exists u \in x\, \phi]\!] &= \bigvee_{z \in M^{\mathfrak{A}}} ([\![z \in x]\!] \wedge [\![\phi(z)]\!]) \\
&= \bigvee_{z \in M^{\mathfrak{A}}} \bigvee_{y \in \operatorname{dom} x} (x(y) \wedge [\![z = y]\!]) \wedge [\![\phi(z)]\!]) \\
&= \bigvee_{y \in \operatorname{dom} x} \bigvee_{z \in M^{\mathfrak{A}}} (x(y) \wedge [\![z = y]\!] \wedge [\![\phi(z)]\!]) \\
&= \bigvee_{y \in \operatorname{dom} x} \Big(x(y) \wedge \bigvee_{z \in M^{\mathfrak{A}}} ([\![z = y]\!] \wedge [\![\phi(z)]\!])\Big) \\
&= \bigvee_{y \in \operatorname{dom} x} (x(y) \wedge [\![\phi(y)]\!]),
\end{aligned}$$

where the last line follows from the fact that

$$\begin{aligned}
[\![\phi(y)]\!] &= [\![y = y]\!] \wedge [\![\phi(y)]\!] \\
&\leqslant \bigvee_{z \in M^{\mathfrak{A}}} ([\![z = y]\!] \wedge [\![\phi(z)]\!]) \\
&\leqslant \bigvee_{z \in M^{\mathfrak{A}}} [\![\phi(y)]\!] \\
&= [\![\phi(y)]\!],
\end{aligned}$$

using (10.146.1). $\square$[10.146]

10.26 Proof of (8.82)

[REFER TO P. 504.]

(10.147) **Theorem** [GB] *Suppose M is a transitive model of* ZF *and $\mathfrak{A} \in M$ is an M-complete boolean algebra.*

1. *Suppose an $\mathsf{s}^{M,\mathfrak{A}}$-sentence σ is a logical validity, i.e., $\vdash \sigma$, then $[\![\sigma]\!]^{M,\mathfrak{A}} = \mathbf{1}$.*
2. *Hence, if $\mathsf{s}^{M,\mathfrak{A}}$-formulas σ and θ are logically equivalent, i.e., σ and θ have the same free variables and $\vdash \sigma \leftrightarrow \theta$, then for any substitution T of forcing terms for the free variables, $[\![\sigma(T)]\!] = [\![\theta(T)]\!]$.*

Proof 1 If there is no $\{\sigma\}^{M,\mathfrak{A}}$-valuation then $[\![\sigma]\!]^{M,\mathfrak{A}} = \mathbf{1}$ automatically, so suppose S is a $\{\sigma\}^{M,\mathfrak{A}}$-valuation. Let[8.32] S' be an extension of S to a $(\Sigma \cup \{\sigma\})^{M,\mathfrak{A}}$-valuation, where Σ consists of the following axioms of identity.[2.79,2.177]

1. $\forall \mathrm{v}_0\ \mathrm{v}_0 = \mathrm{v}_0$.
2. $\forall \mathrm{v}_0, \mathrm{v}_1\, (\mathrm{v}_0 = \mathrm{v}_1 \rightarrow \mathrm{v}_1 = \mathrm{v}_0)$.
3. $\forall \mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\, (\mathrm{v}_0 = \mathrm{v}_1 \wedge \mathrm{v}_1 = \mathrm{v}_2 \rightarrow \mathrm{v}_0 = \mathrm{v}_2)$.
4. $\forall \mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\, \big(\mathrm{v}_0 = \mathrm{v}_1 \wedge \mathrm{v}_2 = \mathrm{v}_3 \rightarrow (\mathrm{v}_0 \in \mathrm{v}_2 \leftrightarrow \mathrm{v}_1 \in \mathrm{v}_3)\big)$.

For the remainder of the proof, '$[\![\ ]\!]$' refers to $[\![\ ]\!]^{S'}$.

Suppose $\vdash \sigma$, and let π be an $\mathbf{LK}^-$-proof of $\Sigma \Rightarrow \{\sigma\}$.

(10.148) **Claim** *Suppose $\Gamma \Rightarrow \Delta$ occurs in π and T is an $M^{\mathfrak{A}}$-substitution for the free variables of $\Gamma \cup \Delta$. Then*

$$\bigwedge_{\gamma \in \Gamma} [\![\gamma(T)]\!] \leqslant \bigvee_{\delta \in \Delta} [\![\delta(T)]\!].$$

Proof The proof is by induction, starting at the topmost sequents of the tree π, and working downward. The topmost sequents are axioms $\{\phi\} \Rightarrow \{\phi\}$, for which the claim is immediate. We now deal with each of the inference rules (2.143.1–8) in turn.

1, 2 Immediate.

3, 4, 5, 6 Suppose T is an $M^{\mathfrak{A}}$-substitution for the lower sequent. Let $g = \bigwedge_{\gamma \in \Gamma} [\![\gamma(T)]\!]$ and $d = \bigvee_{\delta \in \Delta} [\![\delta(T)]\!]$. Let $a = [\![\phi(T)]\!]$ and $b = [\![\psi(T)]\!]$. For (3), by induction hypothesis, $g \leqslant d \vee a$, so

$$\begin{aligned} g \wedge \neg a &\leqslant (d \vee a) \wedge \neg a = (d \wedge \neg a) \vee (a \wedge \neg a) = d \wedge \neg a \\ &\leqslant d, \end{aligned}$$

as claimed. (4), (5), and (6) follow by similar arguments.

7 Suppose T is an $M^{\mathfrak{A}}$-substitution for $\Gamma \cup \{\exists v\, \phi\} \cup \Delta$, and let g, d be as before. We claim that $g \wedge [\![(\exists v\, \phi)(T)]\!] \leqslant d$.

Let $T' = T \restriction \mathrm{Free}(\exists v\, \phi)$. Note that

$$(\exists v\, \phi)(T) = \exists v\, (\phi(T')). \tag{10.149}$$

Given $x \in M^{\mathfrak{A}}$, let $T'' = T\left\langle {u \atop x} \right\rangle$. The conditions imposed on u in (2.143.7) guarantee that $\Gamma(T'') = \Gamma(T)$, $\Delta(T'') = \Delta(T)$, and $\phi\binom{v}{u}(T'') = \phi(T')\binom{v}{x}$. By induction hypothesis, therefore, $g \wedge \left[\!\!\left[\phi(T')\binom{v}{x}\right]\!\!\right] \leqslant d$. By definition,

$$[\![\exists v\, (\phi(T'))]\!] = \bigvee_{x \in M^{\mathfrak{A}}} \left[\!\!\left[\phi(T')\binom{v}{x}\right]\!\!\right],$$

so[10.149] $g \wedge [\![(\exists v\, \phi)(T)]\!] \leqslant d$, as claimed.

8 Suppose T is an $M^{\mathfrak{A}}$-substitution for $\Gamma \cup \Delta \cup \{\exists v\, \phi\}$, and let g, d be as before. We claim that $g \leqslant d \vee [\![(\exists v\, \phi)(T)]\!]$.

As before, let $T' = T \restriction \mathrm{Free}(\exists v\, \phi)$, and note that (10.149) holds. The term τ appearing in the inference (2.143.8) is either a variable u or a constant $x \in M^{\mathfrak{A}}$.

1. If τ is a variable u then
   1. if $u \in \operatorname{dom} T$ then let $x = T(u)$ and let $T'' = T$; and
   2. if $u \notin \operatorname{dom} T$ then let $x \in M^{\mathfrak{A}}$ be arbitrary and let $T'' = T\left\langle {u \atop x} \right\rangle$.
2. If τ is a constant $x \in M^{\mathfrak{A}}$ then let $T'' = T$.

In either case, $\Gamma(T'') = \Gamma(T)$, $\Delta(T'') = \Delta(T)$, and $\phi\binom{v}{\tau}(T'') = \phi(T')\binom{v}{x}$. By induction hypothesis, therefore, $g \leqslant d \vee \left[\!\!\left[\phi(T')\binom{v}{x}\right]\!\!\right]$.

We conclude as in the previous case. $\square$[10.148]

Thus, $\bigwedge_{\gamma \in \Sigma} [\![\gamma]\!] \leqslant [\![\sigma]\!]$. By virtue of (8.65) and (8.66), for every $\gamma \in \Sigma$, $[\![\gamma]\!] = \mathbf{1}$, so $[\![\sigma]\!] = \mathbf{1}$, as claimed. $\square$[10.147.1]

2 This follows from (10.147.1) since in any boolean algebra, $a \leftrightarrow b = \mathbf{1} \leftrightarrow a = b$. $\square$[10.147.2] $\square$[10.147]

10.27 Proof of (8.108)

[Refer to p. 517.]

(10.150) **Theorem** [S] *Θ' is a conservative extension of Θ in the sense that for any* s*-*sentence σ, if $\Theta' \vdash \sigma$ then $\Theta \vdash \sigma$.*

Remark We will first give an infinitary proof of this result—in ZF, for example—and then show how to modify it to a finitary proof, i.e., a proof in C, from which a proof in S may be derived. We could be satisfied with a proof of this theorem in ZF, as the theory of forcing is not of interest without Infinity; however, since the theorem is a finitary statement, it is reasonable to give a finitary proof.

Proof Suppose $\Theta' \vdash \sigma$ and suppose toward a contradiction that $\Theta \nvdash \sigma$. Let $\mathfrak{M} = (M; \in^{\mathfrak{M}}, M_0, \mathbb{P}, G)$ be a satisfactory s^*-structure such that $\mathfrak{M} \models \Theta \cup \{\neg\, \sigma\}$, where $M_0 = \mathsf{V}^{\mathfrak{M}}$, $\mathbb{P} = \mathsf{P}^{\mathfrak{M}}$ and $G = \mathsf{G}^{\mathfrak{M}}$. Let T be the (full) satisfaction relation for $\mathfrak{M}$. Thus, $\models^T \Theta \cup \{\neg\, \sigma\}$. Let s^{V} be the expansion of the signature s by the addition of the unary predicate symbol V (without the constant symbols P and G of s^*). Extend $\mathfrak{M}$ to a c^*-structure $\mathfrak{M}'$ by adding (as proper classes) all subsets of M definable over $\mathfrak{M}$ from a parameter in M. This is the standard method of obtaining a model of GB from a model of ZF, and it is straightforward to show that it is indeed a model of GB.

(10.151) **Claim** $\mathfrak{M}' \models \Theta'$.

Proof It is straightforward to check that $\mathfrak{M}' \models \theta$ for all $\theta \in \Theta'$ other than $\ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$, so the following claim completes the proof.

(10.152) **Claim** $\mathfrak{M}' \models \ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$.

Proof

(10.153) *Suppose toward a contradiction that* $\mathfrak{M}' \models \ulcorner \mathsf{V} \not\models \mathsf{ZF} \urcorner$*, so there exist* $\theta \in M$ *and* $S \in M'$ *such that* $\mathfrak{M} \models \ulcorner [\theta] \in \mathsf{ZF} \urcorner$ *and* $\mathfrak{M}' \models \ulcorner [S]$ *is the* $\{[\theta]\}$*-satisfaction relation for* V*, and* $\not\models^{[S]} [\theta] \urcorner$.

$\in^{\mathfrak{M}}$ is not necessarily wellfounded, but the ordinals of $\mathfrak{M}$ are linearly ordered and have a wellordered initial segment of length at least ω. Accordingly, $\mathrm{HF}^{\mathfrak{M}}$ has a wellfounded "intial segment", which is necessarily isomorphic to HF. Let H be this *standard part* of $\mathrm{HF}^{\mathfrak{M}}$, which is also the standard part of $\mathrm{HF}^{\mathfrak{M}_0}$, where $\mathfrak{M}_0$ is the substructure of $\mathfrak{M}$ corresponding to M_0. Let $x \mapsto \bar{x}$ be the (unique) isomorphism of H with HF. The standard part of $(\mathcal{L}^{\mathsf{s}^*})^{\mathfrak{M}}$ is included in H and is isomorphic to $\mathcal{L}^{\mathsf{s}^*}$. To simplify the notation we suppose that this isomorphism is the identity. Thus, any actual s^*-expression ϵ is in $(\mathcal{L}^{\mathsf{s}^*})^{\mathfrak{M}}$, and $\bar{\epsilon} = \epsilon$.

It is easy to show that if θ is in H then $\mathfrak{M}' \models \ulcorner \models^{[S]} [\theta] \urcorner$ iff $\mathfrak{M} \models \theta^{\mathsf{V}}$, so $\mathfrak{M}' \models \ulcorner \models^{[S]} [\theta] \urcorner$, since $\mathfrak{M} \models \mathsf{ZF}^{\mathsf{V}}$ by hypothesis. Thus, θ is in the nonstandard part of $\mathsf{ZF}^{\mathfrak{M}}$, which means that it is an instance of one of the axiom schemas for a nonstandard formula.

The schemas are $\mathsf{Collection}$ and $\mathsf{Comprehension}$.[22] A sufficiently general version of $\mathsf{Collection}$ is

$$\ulcorner \forall y\, \forall x\, \exists_{\mathrm{Ord}} \alpha\, \forall z \in x\, (\exists_{\mathrm{Ord}} \beta\ [\zeta](z, \beta, y) \rightarrow \exists \beta < \alpha\ [\zeta](z, \beta, y)) \urcorner,$$

where ζ is an s-formula with three free variables. Suppose θ is this above instance of $\mathsf{Collection}$. We will derive a contradiction by showing that $\mathfrak{M}' \models \ulcorner \models^{[S]} [\theta] \urcorner$.

Since $\mathfrak{M}' \models \ulcorner [S]$ is the $\{[\theta]\}$-satisfaction relation for $\mathsf{V} \urcorner$ and $\mathfrak{M} \models \ulcorner [\zeta]$ is a subformula of $[\theta] \urcorner$, it suffices to show that

$$\mathfrak{M}' \models \ulcorner \forall y, x \in \mathsf{V}\, \exists_{\mathrm{Ord}} \alpha\, \forall z \in x\, \big(\exists_{\mathrm{Ord}} \beta\ \models^{[S]} [\zeta][z, \beta, y] \\ \rightarrow \exists_{\mathrm{Ord}} \beta < \alpha\ \models^{[S]} [\zeta][z, \beta, y]\big) \urcorner.$$

This is a theorem of GB (following from the $\mathsf{Collection}$ axiom and an appropriate instance of the $\mathsf{Comprehension}$ schema), so it holds in $\mathfrak{M}'$.

[22] Although in S we must formulate $\mathsf{Foundation}$ as a schema, in ZF, the single $\mathsf{Foundation}$ axiom for sets suffices.

Now suppose θ is the following instance of the Comprehension schema:

$$\ulcorner \forall y\, \forall x\, \exists x'\, \forall z\, (z \in x' \leftrightarrow z \in x \wedge (\zeta)(z, y))\urcorner,$$

where ζ is an s-formula with two free variables. Given $y, x \in M_0$, we must show that there exists $x' \in M_0$ such that for all $z \in M_0$, $z \in^{\mathfrak{M}} x'$ iff $z \in^{\mathfrak{M}} x$ and $\mathfrak{M}' \models \ulcorner \models^{[S]} [\zeta][[z, y]]\urcorner$.

By construction, the $\in^{\mathfrak{M}'}$-extension of S is a subclass of M definable over $\mathfrak{M}$ by an s^{V}-formula ϕ from a parameter in M, which is a^G for some $a \in M_0$, so

$$\mathfrak{M}' \models \ulcorner \models^{[S]} [\zeta][[A]]\urcorner \leftrightarrow \langle \zeta, A\rangle \in^{\mathfrak{M}'} S \leftrightarrow \mathfrak{M} \models \phi[\zeta, A, a^G].$$

(10.154) *Thus, we must show that, given $y, x \in M_0$, there exists $x' \in M_0$ such that for all $z \in M_0$, $z \in^{\mathfrak{M}} x'$ iff $z \in^{\mathfrak{M}} x$ and $\models^T \phi[\zeta, A, a^G]$, where A is the assignment of z and y to the free variables of ζ.*

Let u, v be new variables, and let ϕ' be the s^{V}-formula with free variables u, v, obtained from $\ulcorner S$ is the $\{(u)\}$-satisfaction relation for $\mathsf{V}\urcorner$ by replacing each subformula of the form $\ulcorner\langle \psi, A\rangle \in S\urcorner$ by $\phi(\psi, A, v)$. Without belaboring the issue, suffice it to say that ϕ' is a conjunction of formulas such as

1. $\ulcorner(u)$ is an s^*-formula$\urcorner$;
2. $\ulcorner$for any subformulas ψ and ψ' of (u) and V-assignment A for ψ, if $\psi = \neg\,\psi'$, then $(\phi)(\psi, A, v)$ iff $\neg\,(\phi)(\psi', A, v)\urcorner$ (with similar formulas for the other propositional connectives); and
3. $\ulcorner$for any subformulas ψ and ψ' of (u), variable w, and V-assignment A for ψ, if $\psi = \exists w\, \psi'$, then $(\phi)(\psi, A, v)$ iff for some x such that $\mathsf{V}(x)$, $(\phi)\left(\psi', A\left\langle {w \atop x}\right\rangle, v\right)\urcorner$ (with a similar formula for the universal quantifier).

We now have

$$\mathfrak{M} \models \phi'[\theta, a^G].$$

Since $\mathfrak{M}$ is satisfactory and $\mathfrak{M} \models \Theta$, any deduction from Θ holds in $\mathfrak{M}$. We now argue in Θ as follows.[23]

$\ulcorner$Suppose $(\phi'(\theta, a^{\mathsf{G}}))$. Let $p \in \mathsf{G}$ be such that

(10.155) $$(\phi'^{\Vdash})(\mathsf{P}, p, \check{\theta}, a).$$

(10.156) **Claim** *For every subformula ψ of θ and every V-assignment A for ψ,*

$$(\phi^{\Vdash})(\mathsf{P}, p, \check{\psi}, \check{A}, a) \vee ((\neg\,\phi)^{\Vdash})(\mathsf{P}, p, \check{\psi}, \check{A}, a).$$

Proof By induction on the complexity of ψ. By way of illustration, suppose $\psi = \exists w\, \psi'$, and suppose A is a V-assignment for ψ. For any $x \in \mathsf{V}$, let $A^x = A\left\langle {w \atop x}\right\rangle$. By induction hypothesis, for any $x \in \mathsf{V}$,

(10.157) $$(\phi^{\Vdash})(\mathsf{P}, p, \check{\psi}', \check{A}^x, a) \vee ((\neg\,\phi)^{\Vdash})(\mathsf{P}, p, \check{\psi}', \check{A}^x, a).$$

[23] For the purpose of this proof we will suppose that the formulas $\psi^{\Vdash}$ are defined as relativized to V, so we do not have to write '$\psi^{\Vdash^{\mathsf{V}}}$' to refer to forcing over the ground model.

Suppose $\neg\,(\phi^{\Vdash})(\mathsf{P}, p, \check{\psi}, \check{A}, a)$. By virtue of (10.155)—which says that p forces that ϕ defines the $\{\theta\}$-satisfaction relation for V from the parameter a^{G}—for all $x \in \mathsf{V}$, $\neg\,(\phi^{\Vdash})(\mathsf{P}, p, \check{\psi}', \check{A}^x, a)$. By virtue of (10.157), for all $x \in \mathsf{V}$, $((\neg\,\phi)^{\Vdash})(\mathsf{P}, p, \check{\psi}', \check{A}^x, a)$. Hence, using (10.155) again, we conclude that

$$((\neg\,\phi)^{\Vdash})(\mathsf{P}, p, \check{\psi}, \check{A}, a).$$

The other recursive clauses in the definition of satisfaction are handled similarly, and the atomic formulas are easily dealt with. $\square^{10.156}$

Let x' be the set of $z \in x$ such that $(\phi^{\Vdash})(\mathsf{P}, p, \check{\zeta}, \check{A}, a)$, where A is the assignment of z and y to the free variables of ζ. Then $x' \in \mathsf{V}$ by virtue of $\mathsf{Comprehension}^{\mathsf{V}}$. Given $z \in \mathsf{V}$, let A be the assignment of z and y to the free variables of ζ.

If $z \in x'$ then $(\phi^{\Vdash})(\mathsf{P}, p, \check{\zeta}, \check{A}, a)$, so $(\phi)(\zeta, A, a^{\mathsf{G}})$, since $p \in \mathsf{G}$. On the other hand, if $z \notin x'$ then $\neg\,(\phi^{\Vdash})(\mathsf{P}, p, \check{\zeta}, \check{A}, a)$, so by Claim 10.156

$$((\neg\,\phi)^{\Vdash})(\mathsf{P}, p, \check{\zeta}, \check{A}, a),$$

whence $(\neg\,\phi)(\zeta, A, a^{\mathsf{G}})$, since $p \in \mathsf{G}$, so $\neg\,(\phi)(\zeta, A, a^{\mathsf{G}})$.⌝

As noted above, the existence of this argument in Θ shows that there exists $x' \in M_0$ as required by (10.154), and this completes the proof that $\mathfrak{M}' \models \ulcorner \models^{[S]} [\theta] \urcorner$ for the case that θ is an instance of $\mathsf{Collection}$. Thus, (10.153) is untenable. $\square^{10.152}$

$\square^{10.151}$

We now know that $\mathfrak{M}' \models \Theta'$, $\Theta' \vdash \sigma$, and $\mathfrak{M}' \models \neg\,\sigma$. Since we are working infinitarily, e.g., in ZF, $\mathfrak{M}'$ is satisfactory, so this is an immediate contradiction.

This concludes the infinitary proof of the theorem. To achieve a proof in C, we again suppose toward a contradiction that σ is an s*-sentence such that $\Theta' \vdash \sigma$ and $\Theta \nvdash \sigma$, and we let $\mathfrak{M} = (M; \in^{\mathfrak{M}}, M_0, \mathbb{P}, G)$ be a satisfactory structure such that $\mathfrak{M} \models \Theta \cup \{\neg\,\sigma\}$. At this point in the infinitary proof, we expanded $\mathfrak{M}$ to a model $\mathfrak{M}'$ of GB, as in the infinitary proof of Theorem 2.183, stating the conservativity of GB over ZF. To achieve the result finitarily, we now proceed as in the finitary proof of Theorem 2.183, showing that any proof π of σ from Θ' may be replaced by a proof of σ from premises that are true in $\mathfrak{M}$.

Let $\theta' = \ulcorner \mathsf{V} \models \mathsf{ZF} \urcorner$, i.e.,

(10.158) ⌜*for all S, for all $x \in \mathsf{ZF}$, if S is an $\{x\}$-satisfaction relation for V, then $\langle x, 0 \rangle \in S$*⌝ *($0$ being the empty assignment, appropriate for sentences).*

We systematically eliminate class variables in favor of defined new predicates to obtain a proof of σ from a theory which is derived from Θ' as S is derived from C^1 in the finitary proof of (2.183).[10.79] Eliminating the new predicates in favor of their definitions, we arrive at a proof π' of σ whose premises are

1. axioms of ZF^{V};

2. the "fixed premises" (8.101.2, 4, 5, 6), which are the same in Θ and Θ'; and

3. sentences θ^{ϕ} obtained from θ' by omitting the universal quantifications of S and x in (10.158); replacing each expression of the form $v \in S$ by $\phi(v_0, \ldots, v_{n^-}, v)$, where ϕ is an s-formula and $v_0, \ldots, v_{n^-}$ are new variables; and universally quantifying over x and $v_0, \ldots, v_{n^-}$.

Premises of the first two sorts are axioms of Θ and therefore hold in $\mathfrak{M}$ by virtue of the fact that $\mathfrak{M} \models \Theta$. The proof of Claim 10.152 is, in effect, a proof that $\mathfrak{M} \models \overline{\theta^{\phi}}$ for any ϕ as above. Thus, π' is an s*-proof of σ from premises that are true in $\mathfrak{M}$, so $\mathfrak{M} \models \sigma$, contradicting our assumption that $\mathfrak{M} \models \neg\,\sigma$. $\square^{10.150}$

10.28 Distributivity properties of boolean algebras

[Refer to p. 528.]

Definition [ZFC] *Suppose $\mathfrak{A}$ is a boolean algebra, and κ and λ are cardinals. $\mathfrak{A}$ is* (κ, λ)-distributive $\overset{\text{def}}{\iff}$ *for every $\langle a_{\alpha,\beta} \mid \alpha < \kappa \wedge \beta < \lambda\rangle$*

(10.159)
$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} a_{\alpha,\beta} = \bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} a_{\alpha,f(\alpha)}.$$

By duality, this is equivalent to

(10.160)
$$\bigvee_{\alpha<\kappa} \bigwedge_{\beta<\lambda} a_{\alpha,\beta} = \bigwedge_{f:\kappa\to\lambda} \bigvee_{\alpha<\kappa} a_{\alpha,f(\alpha)}.$$

For $\kappa = \lambda = 2$, (10.159) expresses the distributivity of $\wedge$ over $\vee$, i.e.,

$$(a_0 \vee a_1) \wedge (b_0 \vee b_1) = (a_0 \wedge b_0) \vee (a_0 \wedge b_1) \vee (a_1 \wedge b_0) \vee (a_1 \wedge b_1),$$

and (10.160) expresses the distributivity of $\vee$ over $\wedge$. Every boolean algebra is (m, n)-distributive for all $m, n \in \omega$.

Definition [ZFC] *Suppose $\mathfrak{A}$ is a boolean algebra, and κ is a cardinal. $\mathfrak{A}$ is* κ-distributive $\overset{\text{def}}{\iff}$ *$\mathfrak{A}$ is (κ, λ)-distributive for all cardinals λ.*

It is easy to show that this definition of κ-distributive is equivalent to (8.130.2).

The following theorem enlarges on (8.132). The proof is fairly straightforward and is left to the reader.

Theorem [ZFC] *Suppose $\mathfrak{A}$ is a complete boolean algebra, and κ and λ are cardinals. $\mathfrak{A}$ is (κ, λ)-distributive iff $[\![$every function $f : \kappa \to \lambda$ is in $\mathsf{V}]\!]^{\mathfrak{A}} = \mathbf{1}$.*

10.29 Proof of (8.159)

[Refer to p. 544.]

(10.161) **Theorem** [GB] *There exists a Π_1 s^{P}-formula θ with two free variables such that for any transitive model $(M; \in, \mathbb{P})$ of* ZFP*, if we let*

$$p \Vdash^{M,\mathbb{P}} \phi \overset{\text{def}}{\iff} \theta^{(M;\in,\mathbb{P})}(p, \phi)$$

for any $p \in |\mathbb{P}|$ and $\phi \in A^{M,\mathbb{P}}$, then for all $\tau, \tau' \in M^{\mathbb{P}}$

1. $p \Vdash^{M,\mathbb{P}} \tau \in \tau' \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle \tau_0, r' \rangle \in \tau'\, (r \leqslant r' \wedge r \Vdash^{M,\mathbb{P}} \tau_0 = \tau)$,

and

2.
$$p \Vdash^{M,\mathbb{P}} \tau = \tau' \leftrightarrow \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (q \leqslant r' \rightarrow q \Vdash^{M,\mathbb{P}} \tau_0 \in \tau') \\ \wedge \forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash^{M,\mathbb{P}} \tau_0 \in \tau).$$

Remark The proof may be simplified somewhat if we restrict our attention to the case that M is a set, in which case (8.159.1, 2) constitutes a legitimate recursive definition of $\Vdash^{M,\mathbb{P}}$, and we only have to show that this relation satisfies (10.165) and is therefore given by a Π_1-definition over $(M; \in, \mathbb{P})$. We have, however, stated the theorem without this restriction. M may, for example, be V. We will therefore work in the theory ZFP, in effect placing ourselves inside an arbitrary transitive model $(M; \in, \mathbb{P})$, so that $V = M$.

Proof By convention, we may omit references to M when $M = V$. Thus, in particular, $A^{\mathbb{P}} \overset{\text{def}}{=} A^{V,\mathbb{P}}$.[8.158] We also frequently omit reference to $\mathbb{P}$. Thus, for the purpose of this proof, $A = A^{\mathbb{P}}$.

For $\alpha \in \text{Ord}$, let $A_\alpha = A \cap V_\alpha$ and let $\mathbb{P}_\alpha = (|\mathbb{P}_\alpha|; \leqslant_\alpha)$ be the suborder of $\mathbb{P}$ such that $|\mathbb{P}_\alpha| = |\mathbb{P}| \cap V_\alpha$. Let $\prec$ be the wellordering of A used previously in the definition of forcing for atomic sentences.[8.22] Note that each A_α is an initial segment of $\prec$.

We will define a $\subseteq$-increasing sequence $\langle F_\alpha \mid \alpha \in \text{Ord} \rangle$ of functions such that for each $\alpha \in \text{Ord}$,

1. $\operatorname{dom} F_\alpha$ is a set of pairs $\langle p, \phi \rangle$, where $p \in |\mathbb{P}_\alpha|$ and $\phi \in A_\alpha$; and

2. for each $\langle p, \phi \rangle \in \operatorname{dom} F_\alpha$, $F_\alpha \langle p, \phi \rangle = \langle i, d \rangle$, where $i \in 2$ and d is a nonempty set of conditions in $|\mathbb{P}_\alpha|$ extending p (i.e., $\forall q \in d\ q \leqslant_\alpha p$).

For ease of reference, for $i \in 2$, let $F^i_\alpha = \{(x, d) \mid (x, \langle i, d \rangle) \in F_\alpha\}$.

Ultimately, if $F_\alpha \langle p, \phi \rangle = \langle i, d \rangle$, then if $i = 1$ then $\forall q \in d\ q \Vdash \phi$, and if $i = 0$ then $\forall q \in d\ q \Vdash \neg\phi$. This foreknowledge will help to explain the construction, but it is not used in the construction, as we have not yet defined $\Vdash$.

We will define $F = \bigcup_{\alpha \in \text{Ord}} F_\alpha$. The pretameness of $\mathbb{P}$ will allow us to show that F is total, and we will ultimately define $p \Vdash \phi$ iff for all $q \leqslant p$ $F\langle q, \phi \rangle$ is of the form $\langle 1, d \rangle$ (rather than $\langle 0, d \rangle$). This foreknowledge will also help to explain the construction, but of course it is not used in the construction.

Necessarily, $F_0 = 0$. For limit ordinals α, let $F_\alpha = \bigcup_{\beta < \alpha} F_\beta$. Given F_α we define $F_{\alpha+1}\langle p, \phi \rangle$ by $\prec$-recursion on $\phi \in A_{\alpha+1}$ as follows.[24]

(10.162)

1. Suppose $\phi = \tau \in \tau'$ and $p \in |\mathbb{P}_{\alpha+1}|$. If $\langle p, \phi \rangle \in \operatorname{dom} F_\alpha$ then let $F_{\alpha+1}\langle p, \phi \rangle = F_\alpha \langle p, \phi \rangle$. Otherwise, proceed as follows.
    1. Suppose there exists d such that there exist $\langle \tau_0, r' \rangle \in \tau'$ and $q \leqslant_{\alpha+1}$ both p and r', such that $d = F^1_{\alpha+1}\langle q, \tau_0 = \tau \rangle$. Let e be the union of all such d, and let $F_{\alpha+1}\langle p, \phi \rangle = \langle 1, e \rangle$.
    2. If there is no such d then suppose there exists $q \leqslant_{\alpha+1} p$ and a function d mapping τ' into $\mathcal{P}\,|\mathbb{P}_{\alpha+1}|$ such that for each $\langle \tau_0, r' \rangle \in \tau'$

[24] What we are actually defining by recursion is the function $G_{\alpha+1}$ with domain $A_{\alpha+1}$ such that for each $\phi \in A_{\alpha+1}$, $G_{\alpha+1}\phi$ is $F_{\alpha+1} \restriction \{\langle p, \phi \rangle \mid p \in |\mathbb{P}_{\alpha+1}|\}$.

1. $d_{\langle \tau_0, r' \rangle} \subseteq D^\alpha_{\langle \tau_0, r' \rangle}$, where
$$D^\alpha_{\langle \tau_0, r' \rangle} = \{r'\}^\perp \cup \bigcup \{F^0_{\alpha+1}\langle q, \tau_0 = \tau \rangle \mid q \leqslant r'\},$$
2. and $d_{\langle \tau_0, r' \rangle}$ is predense[25] below q.

Let e be the set of all $q \leqslant_{\alpha+1} p$ for which such a function d exists, and let $F_{\alpha+1}\langle p, \phi \rangle = \langle 0, e \rangle$.

3. If neither of the above suppositions holds, then $\langle p, \phi \rangle \notin \operatorname{dom} F_{\alpha+1}$.

2. Now suppose $\phi = \tau = \tau'$ and $p \in |\mathbb{P}_{\alpha+1}|$. If $\langle p, \phi \rangle \in \operatorname{dom} F_\alpha$ then let $F_{\alpha+1}\langle p, \phi \rangle = F_\alpha \langle p, \phi \rangle$. Otherwise, proceed as follows.

1. Suppose there exists d' such that there exist $\langle \tau_0, r \rangle \in \tau \cup \tau'$, $q \leqslant_{\alpha+1}$ both p and r, $i \in 2$, and $q' \in F^i_{\alpha+1}\langle q, \tau_0 \in \tau \rangle$, such that $d' = F^{1-i}_{\alpha+1}\langle q', \tau_0 \in \tau' \rangle$. Let e be the union of all such d', and let $F_{\alpha+1}\langle p, \phi \rangle = \langle 0, e \rangle$.
2. If there is no such d' then suppose there exist $q \leqslant_{\alpha+1} p$ and a function d mapping $\tau \cup \tau'$ into $\mathcal{P}|\mathbb{P}_{\alpha+1}|$ such that for each $\langle \tau_0, r \rangle \in \tau \cup \tau'$

1. $d_{\langle \tau_0, r \rangle} \subseteq D^\alpha_{\langle \tau_0, r \rangle}$, where
$$\begin{aligned} D^\alpha_{\langle \tau_0, r \rangle} &= \{r\}^\perp \\ &\quad \cup \bigcup \{F^i_{\alpha+1}\langle q', \tau_0 \in \tau' \rangle \mid i \in 2 \wedge \exists q \leqslant r \; q' \in F^i_{\alpha+1}\langle q, \tau_0 \in \tau \rangle\}, \end{aligned}$$
2. and $d_{\langle \tau_0, r \rangle}$ is predense below q.

Let e be the set of all $q \leqslant_{\alpha+1} p$ for which such a function d exists, and let $F_{\alpha+1}\langle p, \phi \rangle = \langle 1, e \rangle$.

3. If neither of the above suppositions holds, then $\langle p, \phi \rangle \notin \operatorname{dom} F_{\alpha+1}$.

(10.163) **Claim** *F is total, i.e.,* $\operatorname{dom} F^0 \cup \operatorname{dom} F^1 = |\mathbb{P}| \times A$.

Remark Note also that, as an increasing union of functions, F is a function, i.e., $\operatorname{dom} F^0 \cap \operatorname{dom} F^1 = 0$.

Proof By $\prec$-recursion. Thus, we suppose for all $\phi' \prec \phi \, \forall p \in |\mathbb{P}| \, (\langle p, \phi' \rangle \in \operatorname{dom} F^0 \cup \operatorname{dom} F^1)$.

$\phi = \tau \in \tau'$ Suppose $p \in |\mathbb{P}|$, and suppose $\langle p, \phi \rangle \notin \operatorname{dom} F^1$. Then it is not the case that there exist $\langle \tau_0, r' \rangle \in \tau'$ and q extending both p and r' such that $\langle q, \tau_0 = \tau \rangle \in \operatorname{dom} F^1$, because if there were, then for some $\alpha \in \mathrm{Ord}$, (10.162.1.1) would have applied and assigned a value to $F^1_{\alpha+1}\langle p, \phi \rangle$. Thus, by induction hypothesis, for all $\langle \tau_0, r' \rangle \in \tau'$ and q extending both p and r', $\langle q, \tau_0 = \tau \rangle \in \operatorname{dom} F^0$.

We will show that in this case each class $D_{\langle \tau_0, r' \rangle}$ is dense below p, where $D_{\langle \tau_0, r' \rangle}$ is defined as in (10.162.1.2.1) with F^0 in place of $F^0_{\alpha+1}$. To this end, suppose $r \leqslant p$. Then either $r \perp r'$, in which case $r \in D_{\langle \tau_0, r' \rangle}$; or there exists q extending both r and r', in which case q extends both p and r', so $\langle q, \tau_0 = \tau \rangle \in \operatorname{dom} F^0$. Any member of $F^0 \langle q, \tau_0 = \tau \rangle$ is in $D_{\langle \tau_0, r' \rangle}$ and extends r.

Thus, $[D_{[x]} \mid x \in \tau']$ is a set-indexed family of subclasses of $\mathbb{P}$ dense below p. Since $\mathbb{P}$ is pretame, there exist $q \leqslant p$ and d as described in (10.162.1.2) for some $\alpha \in \mathrm{Ord}$, so $\langle p, \phi \rangle \in \operatorname{dom} F^0_{\alpha+1}$. Thus, $\langle p, \phi \rangle \in \operatorname{dom} F^0$.

[25] Note that references to compatibility, denseness, etc., are with reference to $\mathbb{P}$, not $\mathbb{P}_{\alpha+1}$.

$\phi = \tau = \tau'$ Suppose $p \in |\mathbb{P}|$, and suppose $\langle p, \phi\rangle \notin \operatorname{dom} F^0$. Then it is not the case that there exist $\langle \tau_0, r\rangle \in \tau \cup \tau'$, q extending both p and r, $i \in 2$, and $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, such that $\langle q', \tau_0 \in \tau'\rangle \in \operatorname{dom} F^{1-i}$, because if there were, then for some $\alpha \in \mathrm{Ord}$, (10.162.2.1) would have applied and assigned a value to $F^0_{\alpha+1}\langle p, \phi\rangle$. Thus, by induction hypothesis, for all $\langle \tau_0, r\rangle \in \tau \cup \tau'$, q extending both p and r, $i \in 2$, and $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, $\langle q', \tau_0 \in \tau'\rangle \in \operatorname{dom} F^i$.

We will show that in this case each class $D_{\langle \tau_0, r\rangle}$ is dense below p, where $D_{\langle \tau_0, r\rangle}$ is defined as in (10.162.2.2.1) with F^i in place of $F^i_{\alpha+1}$. To this end, suppose $r' \leqslant p$. Then either $r' \perp r$, in which case $r' \in D_{\langle \tau_0, r\rangle}$; or there exists q extending both r' and r, in which case q extends both p and r, so for each $i \in 2$ and $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, $\langle q', \tau_0 \in \tau'\rangle \in \operatorname{dom} F^i$. By induction hypothesis, there exists (a unique) $i \in 2$ such that $\langle q, \tau_0 \in \tau\rangle \in \operatorname{dom} F^i$, and letting q' be any member of $F^i\langle q, \tau_0 \in \tau\rangle$, we see that any member of $F^i\langle q', \tau_0 \in \tau'\rangle$ is in $D_{\langle \tau_0, r\rangle}$ and extends r'.

Thus, $[D_{[x]} \mid x \in \tau \cup \tau']$ is a set-indexed family of subclasses of $\mathbb{P}$ dense below p. Since $\mathbb{P}$ is pretame, there exist $q \leqslant p$ and d as described in (10.162.2.2) for some $\alpha \in \mathrm{Ord}$, so $\langle p, \phi\rangle \in \operatorname{dom} F^1_{\alpha+1}$. Thus, $\langle p, \phi\rangle \in \operatorname{dom} F^1$. □[10.163]

The following claim summarizes the properties of F *vis-à-vis* $\mathbb{P}$ corresponding to the clauses in the definition of F_α *vis-à-vis* $\mathbb{P}_\alpha$.

(10.164) **Claim**

1. *For all $p' \in F^1\langle p, \tau \in \tau'\rangle$ there exist $\langle \tau_0, r\rangle \in \tau'$ and q extending both p and r, such that $p' \in F^1\langle q, \tau_0 = \tau\rangle$.*
2. *For all $p' \in F^0\langle p, \tau \in \tau'\rangle$ and $\langle \tau_0, r\rangle \in \tau'$, $D^\tau_{\langle \tau_0, r\rangle}$ is predense below p', where*
$$D^\tau_{\langle \tau_0, r\rangle} = \{r\}^\perp \cup \bigcup\{F^0\langle q, \tau_0 = \tau\rangle \mid q \leqslant r\}.$$
3. *For all $p' \in F^1\langle p, \tau = \tau'\rangle$ and $\langle \tau_0, r\rangle \in \tau \cup \tau'$, $D^{\tau,\tau'}_{\langle \tau_0, r\rangle}$ is predense below p', where*
$$D^{\tau,\tau'}_{\langle \tau_0, r\rangle} = \{r\}^\perp \cup \bigcup\{F^i\langle q', \tau_0 \in \tau'\rangle \mid i \in 2 \wedge \exists q \leqslant r\ q' \in F^i\langle q, \tau_0 \in \tau\rangle\}.$$
4. *For all $p' \in F^0\langle p, \tau = \tau'\rangle$ there exist $\langle \tau_0, r\rangle \in \tau \cup \tau'$, q extending both p and r, $i \in 2$, and $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, such that $p' \in F^{1-i}\langle q', \tau_0 \in \tau'\rangle$.*

Proof These follow directly from the definition[10.162] and the fact that $\alpha < \beta \rightarrow F^i_\alpha \subseteq F^i_\beta$. □[10.164]

Recall[8.158] that A ($= A^{\mathbb{P}} = A^{V,\mathbb{P}}$) is the class of sentences of $\mathcal{L}^{\mathbb{P}}$ of the form $\tau \in \tau'$ or $\tau = \tau'$. Let A^- be the class of negations of these.

(10.165) *For $p \in |\mathbb{P}|$ and $\phi \in A$, let*

1. $p \Vdash^{\mathbb{P}} \phi \overset{\text{def}}{\iff} \forall q \leqslant p\ \langle q, \phi\rangle \in \operatorname{dom} F^1$, *and*
2. $p \Vdash^{\mathbb{P}} \neg\phi \overset{\text{def}}{\iff} \forall q \leqslant p\ q \nVdash \phi$, *i.e.,* $\forall q \leqslant p\, \exists r \leqslant q\ \langle r, \phi\rangle \in \operatorname{dom} F^0$.

(10.166) **Claim**

1. *Suppose $\phi \in A \cup A^-$. If $p \Vdash \phi$ and $q \leqslant p$ then $q \Vdash \phi$.*

2. *Suppose* $\phi \in A$. *Then it is not the case that* $p \Vdash \phi$ *and* $p \Vdash \neg\,\phi$.

Proof Immediate from the definition. $\square$[10.166]

(10.167) **Claim** *Suppose* $p \in |\mathbb{P}|$ *and* $\phi \in A$.

1. $q \in F^1\langle p, \phi\rangle \rightarrow q \Vdash \phi$.
2. $q \in F^0\langle p, \phi\rangle \rightarrow q \Vdash \neg\,\phi$.

Proof We prove both claims together by $\prec$-induction.

1 Suppose $q \in F^1\langle p, \phi\rangle$. We will show that $q \Vdash \phi$. To this end, suppose $r \leqslant q$. We must show that $\langle r, \phi\rangle \in \operatorname{dom} F^1$. Suppose toward a contradiction that $\langle r, \phi\rangle \in \operatorname{dom} F^0$, and suppose $p' \in F^0\langle r, \phi\rangle$.

$\phi = \tau \mathrel{\varepsilon} \tau'$ Since $q \in F^1\langle p, \phi\rangle$,[10.164.1] there exist $\langle \tau_0, r'\rangle \in \tau'$ and q' extending both p and r' such that $q \in F^1\langle q', \tau_0 = \tau\rangle$. By induction hypothesis, $q \Vdash \tau_0 = \tau$. Since $p' \leqslant r \leqslant q$, $p' \Vdash \tau_0 = \tau$.[10.166.1] But[10.164.2]

$$\{r'\}^\perp \cup \bigcup\{F^0\langle s, \tau_0 = \tau\rangle \mid s \leqslant r'\}$$

is predense below p'. Note that $p' \leqslant r \leqslant q \leqslant q' \leqslant r'$. Thus, there exist $s \leqslant r'$, $s' \in F^0\langle s, \tau_0 = \tau\rangle$, and p'' extending both p' and s'. Since $p'' \leqslant p'$, $p'' \Vdash \tau_0 = \tau$, and since $p'' \leqslant s' \in F^0\langle s, \tau_0 = \tau\rangle$, $p'' \Vdash \tau_0 \neq \tau$ (by induction hypothesis and 10.166.1); contradiction.

$\phi = \tau = \tau'$ There exist $\langle \tau_0, r'\rangle \in \tau \cup \tau'$, q' extending both r and r', $i \in 2$, and $q'' \in F^i\langle q', \tau_0 \mathrel{\varepsilon} \tau\rangle$, such that $p' \in F^{1-i}\langle q'', \tau_0 \mathrel{\varepsilon} \tau'\rangle$. Since $q \in F^1\langle p, \phi\rangle$,

$$\{r'\}^\perp \cup \bigcup\{F^j\langle s', \tau_0 \mathrel{\varepsilon} \tau'\rangle \mid j \in 2 \wedge \exists s \leqslant p\ s' \in F^j\langle s, \tau_0 \mathrel{\varepsilon} \tau\rangle\}$$

is predense below q. Since p' extends both q and r', there exist $s \leqslant p$, $j \in 2$, $s' \in F^j\langle s, \tau_0 \mathrel{\varepsilon} \tau\rangle$, $s'' \in F^j\langle s', \tau_0 \mathrel{\varepsilon} \tau'\rangle$, and p'' extending both p' and s''.

Since $q'' \in F^i\langle q', \tau_0 \mathrel{\varepsilon} \tau\rangle$, $s' \in F^j\langle s, \tau_0 \mathrel{\varepsilon} \tau\rangle$, and p'' extends both q'' and s', $i = j$ (using the induction hypothesis and 10.166.2). Similarly, since $p' \in F^{1-i}\langle q'', \tau_0 \mathrel{\varepsilon} \tau'\rangle$, $s'' \in F^j\langle s', \tau_0 \mathrel{\varepsilon} \tau'\rangle$, and p'' extends both p' and s'', $1 - i = j$; contradiction.

2 Suppose $q \in F^0\langle p, \phi\rangle$. We claim that $q \Vdash \neg\,\phi$, i.e., for every $r \leqslant q$, $r \not\Vdash \phi$, i.e., $\exists s \leqslant r\ \langle s, \phi\rangle \in \operatorname{dom} F^0$. In fact, we will show that $\langle r, \phi\rangle \in \operatorname{dom} F^0$. To this end, suppose toward a contradiction that $\langle r, \phi\rangle \in \operatorname{dom} F^1$, and suppose $p' \in F^1\langle r, \phi\rangle$.

$\phi = \tau \mathrel{\varepsilon} \tau'$ Since $p' \in F^1\langle r, \phi\rangle$, there exist $\langle \tau_0, r'\rangle \in \tau'$ and q' extending both r and r', such that $p' \in F^1\langle q', \tau_0 = \tau\rangle$. Since $q \in F^0\langle p, \phi\rangle$,

$$\{r'\}^\perp \cup \bigcup\{F^0\langle q', \tau_0 = \tau\rangle \mid q' \leqslant r'\}$$

is predense below q. Since $p' \leqslant q$ and $p' \leqslant r'$, there exist $q'' \leqslant r'$, $s \in F^0\langle q'', \tau_0 = \tau\rangle$, and p'' extending both p' and s. By induction hypothesis, $p'' \Vdash \tau_0 = \tau$ and $p'' \Vdash \tau_0 \neq \tau$; contradiction.

$\phi = \tau = \tau'$ Since $q \in F^0\langle p, \phi\rangle$, there exist $\langle \tau_0, r'\rangle \in \tau \cup \tau'$, q' extending both p and r', $i \in 2$, and $q'' \in F^i\langle q', \tau_0 \in \tau\rangle$, such that $q \in F^{1-i}\langle q'', \tau_0 \in \tau'\rangle$. Since $p' \in F^1\langle r, \phi\rangle$,

$$\{r'\}^\perp \cup \bigcup\{F^j\langle s', \tau_0 \in \tau'\rangle \mid j \in 2 \wedge \exists s \leqslant r'\ s' \in F^j\langle s, \tau_0 \in \tau\rangle\}$$

is predense below p'. Note that $p' \leqslant r \leqslant q \leqslant q'' \leqslant q' \leqslant r'$. Thus, there exist $j \in 2$, $s \leqslant r'$, $s' \in F^j\langle s, \tau_0 \in \tau\rangle$, $s'' \in F^j\langle s', \tau_0 \in \tau'\rangle$, and p'' extending both p' and s''. Using the induction hypothesis, since $q'' \in F^i\langle q', \tau_0 \in \tau\rangle$ and $s' \in F^j\langle s, \tau_0 \in \tau\rangle$, and p'' extends both q'' and s', $i = j$. On the other hand, since $q \in F^{1-i}\langle q'', \tau_0 \in \tau'\rangle$ and $s'' \in F^j\langle s', \tau_0 \in \tau'\rangle$, and p'' extends both q and s'', $1 - i = j$; contradiction. □[10.167]

(10.168) **Claim** *Suppose $p \in |\mathbb{P}|$ and $\phi \in A$.*

1. *Suppose $\phi = \tau = \tau$. Then $p \Vdash \phi$.*
2. *Suppose $\phi = \tau \in \tau'$ and $\langle \tau, p\rangle \in \tau'$. Then $p \Vdash \phi$.*

Proof 1 We claim that $\langle p', \phi\rangle \in \operatorname{dom} F^1$ for any $p' \in |\mathbb{P}|$. If not, then $\langle p', \phi\rangle \in \operatorname{dom} F^0$, and[10.164.4] for any $p'' \in F^0\langle p', \phi\rangle$ there exist $\langle \tau_0, r\rangle \in \tau$, q extending both p' and r, $i \in 2$, and $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, such that $p'' \in F^{1-i}\langle q', \tau_0 \in \tau\rangle$. If $i = 1$ then[10.167.1] $q' \Vdash \tau_0 \in \tau$, and[10.167.2] $p'' \Vdash \tau_0 \notin \tau$, a contradiction, since $p'' \leqslant q'$. Similarly, if $i = 0$, then $q' \Vdash \tau_0 \notin \tau$, and $p'' \Vdash \tau_0 \in \tau$.

2 Suppose $p' \leqslant p$. We claim that $\langle p', \phi\rangle \in \operatorname{dom} F^1$. If not, then $\langle p', \phi\rangle \in \operatorname{dom} F^0$, so[10.164.2] there exists $p'' \leqslant p'$ such that

$$\{p\}^\perp \cup \bigcup\{F^0\langle q, \tau = \tau\rangle \mid q \leqslant p\}$$

is predense below p''. Since no condition forces $\tau \neq \tau$,[10.168.1] $\langle q, \tau = \tau\rangle$ is not in $\operatorname{dom} F^0$ for any q, so $\{p\}^\perp$ is predense below p'', which is impossible, since $p'' \leqslant p' \leqslant p$. □[10.168]

(10.169) **Claim** *Suppose $p \in |\mathbb{P}|$ and $\phi \in A$.*

1. *Suppose $\phi = \tau \in \tau'$. Then*

$$p \Vdash \phi \leftrightarrow \forall q \leqslant p\, \exists r \leqslant q\, \exists \langle \tau_0, r'\rangle \in \tau'\, (r \leqslant r' \wedge r \Vdash \tau_0 = \tau).$$

2. *Suppose $\phi = \tau = \tau'$. Then*

$$\begin{aligned} p \Vdash \phi \leftrightarrow &\forall q \leqslant p\, \forall \langle \tau_0, r'\rangle \in \tau\, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau') \\ &\wedge \forall q \leqslant p\, \forall \langle \tau_0, r'\rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau). \end{aligned}$$

Proof 1 Suppose $\phi = \tau \in \tau'$.

$\rightarrow$ Suppose $p \Vdash \phi$, and suppose $q \leqslant p$. Then[10.165] $\langle q, \phi\rangle \in \operatorname{dom} F^1$, so[10.164.1] there exist $\langle \tau_0, r'\rangle \in \tau'$ and r extending both q and r' such that $\langle r, \tau_0 = \tau\rangle \in \operatorname{dom} F^1$. Suppose $s \in F^1\langle r, \tau_0 = \tau\rangle$. Then s extends both q and r', and $s \Vdash \tau_0 = \tau$.[10.167.1]

$\leftarrow$ Suppose $\forall q \leqslant p\, \exists \langle \tau_0, r' \rangle \in \tau'\, \exists r \leqslant q, r' (r \Vdash \tau_0 = \tau)$. Suppose $p' \leqslant p$. We wish to show that $\langle p', \phi \rangle \in F^1$. Suppose not; then $\langle p', \phi \rangle \in F^0$, so there exists $p'' \leqslant p'$ such that for every $\langle \tau_0, r' \rangle \in \tau'$, the class

$$D^{\tau}_{\langle \tau_0, r' \rangle} = \{r'\}^{\perp} \cup \bigcup \{F^0 \langle q, \tau_0 = \tau \rangle \mid q \leqslant r'\}$$

is predense below p''. Since $p'' \leqslant p$, by assumption there exist $\langle \tau_0, r' \rangle \in \tau'$ and r extending both p'' and r', such that $r \Vdash \tau_0 = \tau$. Let $s \leqslant r$ be such that s extends a member of $D^{\tau}_{\langle \tau_0, r' \rangle}$. $s \leqslant r'$, so it does not extend a member of $\{r'\}^{\perp}$, so there exists $q \leqslant r'$ and $q' \in F^0 \langle q, \tau_0 = \tau \rangle$, such that $s \leqslant q'$. Now $q' \Vdash \tau_0 \neq \tau$ and $r \Vdash \tau_0 = \tau$ and $s \leqslant r, q'$; contradiction.

2 Suppose $\phi = \tau = \tau'$.

$\rightarrow$ Suppose $p \Vdash \phi$.

(10.170) **Claim**

$$\begin{aligned} &\forall p' \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (p' \leqslant r' \rightarrow \exists r \leqslant p'\, r \Vdash \tau_0 \in \tau') \\ \wedge &\forall p' \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (p' \leqslant r' \rightarrow \exists r \leqslant p'\, r \Vdash \tau_0 \in \tau). \end{aligned}$$

Proof Suppose $p' \leqslant p$. By definition, $\langle p', \phi \rangle \in \operatorname{dom} F^1$, so[10.164.3] there exists $p'' \leqslant p'$ such that for every $\forall \langle \tau_0, r' \rangle \in \tau \cup \tau'$

$$\begin{aligned} D^{\tau,\tau'}_{\langle \tau_0, r' \rangle} = \{r'\}^{\perp} & \\ &\cup \bigcup \{F^i \langle q', \tau_0 \in \tau' \rangle \mid i \in 2 \wedge \exists q \leqslant r'\ q' \in F^i \langle q, \tau_0 \in \tau \rangle\} \end{aligned}$$

is predense below p''.

Suppose first that $\langle \tau_0, r' \rangle \in \tau$ and $p' \leqslant r'$. Since $D^{\tau,\tau'}_{\langle \tau_0, r' \rangle}$ is predense below p'', there exists $r \leqslant p''$ such that r extends a member of $D^{\tau,\tau'}_{\langle \tau_0, r' \rangle}$. $r \leqslant p'' \leqslant p' \leqslant r'$, so r does not extend a member of $\{r'\}^{\perp}$, so there exist $i \in 2$, $q \leqslant r'$, $q' \in F^i \langle q, \tau_0 \in \tau \rangle$, and $q'' \in F^i \langle q', \tau_0 \in \tau' \rangle$, such that $r \leqslant q''$. Since $\langle \tau_0, r' \rangle \in \tau$, $r' \Vdash \tau_0 \in \tau$.[10.168.2] Since $r \leqslant r'$, $r \Vdash \tau_0 \in \tau$. Since $r \leqslant q'' \leqslant q'$, $q' \nVdash \tau_0 \notin \tau$, so $i = 1$. Hence, $q'' \Vdash \tau_0 \in \tau'$, so $r \Vdash \tau_0 \in \tau'$.

Now suppose that $\langle \tau_0, r' \rangle \in \tau'$ and $p' \leqslant r'$. We argue as before that there exist $r \leqslant p''$, $i \in 2$, $q \leqslant r'$, $q' \in F^i \langle q, \tau_0 \in \tau \rangle$, and $q'' \in F^i \langle q', \tau_0 \in \tau' \rangle$, such that $r \leqslant q''$. Since $\langle \tau_0, r' \rangle \in \tau'$, $r' \Vdash \tau_0 \in \tau'$. Since $r \leqslant r'$, $r \Vdash \tau_0 \in \tau'$. Since $r \leqslant q''$, $q'' \nVdash \tau_0 \notin \tau'$, so $i = 1$. Hence, $q' \Vdash \tau_0 \in \tau$, so $r \Vdash \tau_0 \in \tau$. $\square$[10.170]

Suppose $q \leqslant p$, $\langle \tau_0, r' \rangle \in \tau$, and $q \leqslant r'$. We will show that $q \Vdash \tau_0 \in \tau'$. To this end, suppose toward a contradiction that $s \leqslant q$ and $\langle s, \tau_0 \in \tau' \rangle \notin \operatorname{dom} F^1$.[10.165.1] Then[10.163] $\langle s, \tau_0 \in \tau' \rangle \in \operatorname{dom} F^0$. Let t be a member of $F^0 \langle s, \tau_0 \in \tau' \rangle$. Then[10.167] $t \Vdash \tau_0 \notin \tau'$. But $t \leqslant p$ and $t \leqslant r'$, so[10.170] there exists $r \leqslant t$ such that $r \Vdash \tau_0 \in \tau'$. Since[10.166.1] $r \Vdash \tau_0 \notin \tau'$, this contradicts (10.166.2).

Similarly, $\forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau)$.

$\leftarrow$ Suppose

$$\begin{aligned} &\forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau\, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau') \\ \wedge &\forall q \leqslant p\, \forall \langle \tau_0, r' \rangle \in \tau'\, (q \leqslant r' \rightarrow q \Vdash \tau_0 \in \tau). \end{aligned}$$

Suppose $p' \leqslant p$. We wish to show that $\langle p', \phi\rangle \in F^1$. Suppose not; then $\langle p', \phi\rangle \in F^0$. Thus, there exist $\langle \tau_0, r'\rangle \in \tau \cup \tau'$, q extending both p' and r', $i \in 2$, $q' \in F^i\langle q, \tau_0 \in \tau\rangle$, and $q'' \in F^{1-i}\langle q', \tau_0 \in \tau'\rangle$.

Suppose first that $\langle \tau_0, r'\rangle \in \tau$. By hypothesis, since $q'' \leqslant p$ and $q'' \leqslant r'$, $q'' \Vdash \tau_0 \in \tau'$. Hence, $i = 0$ (otherwise $q'' \Vdash \tau_0 \notin \tau'$), so $q' \Vdash \tau_0 \notin \tau$, which is impossible, since $\langle \tau_0, r'\rangle \in \tau$, so $r' \Vdash \tau_0 \in \tau$, and $q' \leqslant r'$.[10.166]

The possibility that $\langle \tau_0, r'\rangle \in \tau'$ is excluded similarly. □[10.169]

Note that for any limit ordinal α, $\langle F_\beta \mid \beta < \alpha\rangle$ is included in V_α and is defined over V_α by (10.162) relativized to V_α. Thus the preceding argument in ZFP contains a Σ_1 definition of F, viz., $\ulcorner(x,y) \in F$ iff there is a limit ordinal α and a sequence $\langle F_\beta \mid \beta \in \alpha\rangle$ satisfying (10.162) relativized to V_α, such that for some $\beta \in \alpha$, $(x,y) \in F_\beta\urcorner$. For $\phi \in A^{\mathbb{P}}$ and $p \in |\mathbb{P}|$,

$$\begin{aligned} p \Vdash \phi &\leftrightarrow \forall q \leqslant p\ \langle q, \phi\rangle \in \operatorname{dom} F^1 \\ &\leftrightarrow \forall q \leqslant p\ \langle q, \phi\rangle \notin \operatorname{dom} F^0 \\ &\leftrightarrow \forall q \leqslant p\, \forall d\, (\langle q, \phi\rangle, \langle 0, d\rangle) \notin F, \end{aligned}$$

so $\{\langle p, \phi\rangle \mid \phi \in A^{\mathbb{P}} \wedge p \Vdash^{\mathbb{P}} \phi\}$ is Π_1. □[10.161]

10.30 Proof of (8.162)

[Refer to p. 545.]

(10.171) **Theorem** [GB] *Suppose $(M; \in, \mathbb{P})$ is a transitive model of* ZFP. *Then every axiom of* $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$ *with the possible exception of* Power *is a $M^{\mathbb{P}}$-validity, where* $\mathsf{ZF}^{\mathsf{s}^{\mathsf{V}}}$ *is* ZF *with the axiom schemas extended to all* s^{V}*-formulas.*

Proof Remember that for any $\mathsf{s}^{M,\mathbb{P}}$-sentence σ, if the $\{\sigma\}^{M,\mathbb{P}}$-forcing relation does not exist then $\Vdash \sigma$, so we will assume throughout this proof that the forcing relation exists for the axiom in question.

Extension We will show that

$$\forall u\, \forall v\, \big((\forall w \in u\ w \in v \wedge \forall w \in v\ w \in u) \rightarrow u = v\big),$$

is a validity. Suppose $x, y \in M^{\mathbb{P}}$. We must show that

$$\Vdash \big((\forall w \in x\ w \in y \wedge \forall w \in y\ w \in x) \rightarrow x = y\big),$$

i.e.,[8.29.7]

$$\forall p\, \big(p \Vdash (\forall w \in x\ w \in y \wedge \forall w \in y\ w \in x) \rightarrow p \Vdash x = y\big).$$

Suppose $p \in |\mathbb{P}|$ and $p \Vdash (\forall w \in x\ w \in y \wedge \forall w \in y\ w \in x)$. Then[8.161.4.3]

$$\begin{aligned} &\forall q \leqslant p\, \forall \langle z, r'\rangle \in x\, (q \leqslant r' \rightarrow q \Vdash z \in y) \\ \wedge &\forall q \leqslant p\, \forall \langle z, r'\rangle \in y\, (q \leqslant r' \rightarrow q \Vdash z \in x). \end{aligned}$$

Hence,[8.159.2] $p \Vdash x = y$.

Comprehension We must show that for any $x \in M^{\mathbb{P}}$ and $\mathsf{s}^{M,\mathbb{P}}$-formula ϕ with one free variable,[26]

$$\Vdash \exists w\, \forall v\, \big(v \in w \leftrightarrow (v \in x \wedge \phi(v))\big),$$

i.e.,

$$\forall p \in |\mathbb{P}|\, \exists q \leqslant p\, \exists z \in M^{\mathbb{P}}\, q \Vdash \forall v\, \big(v \in z \leftrightarrow (v \in x \wedge \phi(v))\big).$$

Suppose $p \in |\mathbb{P}|$. Let $D \subseteq M$ be the $(\operatorname{dom} x)$-indexed family such that for each $y \in \operatorname{dom} x$,

$$D_{[y]} = \big\{ r \in |\mathbb{P}| \,\big|\, r \,|\, (y \in x \wedge \phi(y)) \big\}. \tag{10.172}$$

(Recall that $r \,|\, \theta$ iff r decides θ iff either $r \Vdash \theta$ or $r \Vdash \neg\theta$.) Note that each $D_{[y]}$ is dense,[8.160: 8.38] *a fortiori* predense below p. Let $q \leqslant p$ and $d \in M$ be a $(\operatorname{dom} x)$-indexed family such that for all $y \in \operatorname{dom} x$, $d_{[y]}$ is included in $D_{[y]}$ and is predense below q.

Let

$$z = \{\langle y, r\rangle \mid r \in d_{[y]} \wedge r \Vdash (y \in x \wedge \phi(y))\}.$$

We will show that $q \Vdash \forall v\, \big(v \in z \leftrightarrow (v \in x \wedge \phi(v))\big)$, i.e., for all $y \in M^{\mathbb{P}}$

$$q \Vdash \big(y \in z \leftrightarrow (y \in x \wedge \phi(y))\big).$$

To this end, suppose[8.29.8] $s \leqslant q$; we will show that

$$s \Vdash y \in z \leftrightarrow s \Vdash (y \in x \wedge \phi(y)).$$

Suppose first that $s \Vdash y \in z$. Then

$$\forall s' \leqslant s\, \exists s'' \leqslant s'\, \exists \langle y', r\rangle \in z\, (s'' \leqslant r \wedge s'' \Vdash y = y').$$

It suffices[8.160: 8.36] to show that

$$\forall s' \leqslant s\, \exists s'' \leqslant s'\;\; s'' \Vdash (y \in x \wedge \phi(y)).$$

To this end, suppose $s' \leqslant s$ and let $s'' \leqslant s'$ and $\langle y', r\rangle \in z$ be such that $s'' \leqslant r \wedge s'' \Vdash y = y'$. Then $r \Vdash (y' \in x \wedge \phi(y'))$, so $s'' \Vdash (y' \in x \wedge y = y' \wedge \phi(y'))$, so $s'' \Vdash (y \in x \wedge \phi(y))$.[8.161]

Conversely, suppose $s \Vdash (y \in x \wedge \phi(y))$. We must show that $s \Vdash y \in z$. It suffices to show that

$$\forall s' \leqslant s\, \exists s'' \leqslant s'\;\; s'' \Vdash y \in z.$$

To this end, suppose $s' \leqslant s$. Since $d_{[y]}$ is predense below q and $s' \leqslant q$, there exists $s'' \leqslant s'$ and $r \in d_{[y]}$ such that $s'' \leqslant r$. Since $r \,|\, (y \in x \wedge \phi(y))$ and $s'' \Vdash (y \in x \wedge \phi(y))$, $r \Vdash (y \in x \wedge \phi(y))$. Hence, $\langle y, r\rangle \in z$, so $r \Vdash y \in z$, so $s'' \Vdash y \in z$.

Existence We show that $\Vdash \exists u\, \forall v\;\, v \notin u$. This follows from the fact that 0 is in $M^{\mathbb{P}}$ and denotes the empty set in the sense that for any $y \in M^{\mathbb{P}}$, $\Vdash y \notin 0$, as is easily shown.

Pair Given $x, y \in M^{\mathbb{P}}$, let $z = \{\langle x, \mathbf{1}\rangle, \langle y, \mathbf{1}\rangle\}$. Then for any $w \in M^{\mathbb{P}}$, $\Vdash (w \in z \leftrightarrow w = x \vee w = y)$.

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[26] As a convenience, the parameters that ordinarily occur in the axiom schema are in ϕ as constant terms, rather than variables.

Collection We must show that for any $x \in M^{\mathbb{P}}$, distinct variables a, v, w, and $\mathsf{s}^{M,\mathbb{P}}$-formula ϕ with free variables in $\{a, v\}$,

$$\Vdash \forall v \in x \, \exists w \, \forall a \, (\phi(a, v) \rightarrow a \in w) \rightarrow \exists w \, \forall v \in x \, \forall a \, (\phi(a, v) \rightarrow a \in w),$$

i.e., for all $p \in |\mathbb{P}|$,

$$p \Vdash \forall v \in x \, \exists w \, \forall a \, (\phi(a, v) \rightarrow a \in w) \rightarrow p \Vdash \exists w \, \forall v \in x \, \forall a \, (\phi(a, v) \rightarrow a \in w).$$

To this end, suppose $p \Vdash \forall v \in x \, \exists w \, \forall a \, (\phi(a, v) \rightarrow a \in w)$, i.e.,[8.161.4.3]

$$\forall \langle y, s\rangle \in x \, \forall q \leqslant p \, \big(q \leqslant s \rightarrow q \Vdash \exists w \, \forall a \, (\phi(a, y) \rightarrow a \in w)\big).$$

We must show that

$$p \Vdash \exists w \, \forall v \in x \, \forall a \, (\phi(a, v) \rightarrow a \in w),$$

i.e.,

$$\forall q \leqslant p \, \exists r \leqslant q \, \exists z \in M^{\mathbb{P}} \; r \Vdash \forall v \in x \, \forall a \, (\phi(a, v) \rightarrow a \in z). \tag{10.173}$$

Let D be the x-indexed family such that for each $\langle y, s\rangle \in x$

$$D_{[\langle y,s\rangle]} = \{r \in |\mathbb{P}| \mid r \perp s \vee \exists z \in M^{\mathbb{P}} \, r \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)\}.$$

Then

(10.174) *$D_{[\langle y,s\rangle]}$ is dense below p.*

For suppose $q \leqslant p$. Then either $q \perp s$ or there exists $q' \leqslant q$ such that $q' \leqslant s$, in which case $q' \Vdash \exists w \, \forall a \, (\phi(a, y) \rightarrow a \in w)$, so there exist $z \in M^{\mathbb{P}}$ and $r \leqslant q'$ such that $r \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)$.

To prove (10.173), suppose $q \leqslant p$. Based on (10.174), let $r \leqslant q$ and $d \in M$ be such that d is an x-indexed family, $\forall \langle y, s\rangle \in x \; d_{[\langle y,s\rangle]} \subseteq D_{[\langle y,s\rangle]}$, and each $d_{[\langle y,s\rangle]}$ is predense below r. Since $\mathfrak{M} \models$ Collection, there exists $X \in M$ such that

$$\forall \langle y, s\rangle \in x \, \forall t \in d_{[\langle y,s\rangle]} \, \big(t \parallel s \rightarrow \exists z \in X \; t \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)\big). \tag{10.175}$$

Let $Y = \bigcup_{z \in X} \operatorname{dom} z$. Let $z = Y \times \{\mathbf{1}\}$.

We will show that $r \Vdash \forall v \in x \, \forall a \, (\phi(a, v) \rightarrow a \in z)$, i.e.,

$$\forall t \leqslant r \, \forall \langle y, s\rangle \in x \, (t \leqslant s \rightarrow t \Vdash \forall a \, (\phi(a, v) \rightarrow a \in z). \tag{10.176}$$

Suppose $t \leqslant r$, $\langle y, s\rangle \in x$, and $t \leqslant s$. Since $d_{[\langle y,s\rangle]}$ is predense below r, for every $r' \leqslant t$ there exist $r'' \leqslant r'$ and $t' \in d_{[\langle y,s\rangle]}$ such that $r'' \leqslant t'$. t' is compatible with s, so[10.175] there exists $z' \in X$ such that $t' \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z')$. Note that $\forall \langle y, s\rangle \in z' \; \langle y, \mathbf{1}\rangle \in z$, from which it follows easily that $t' \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)$. Hence, $r'' \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)$. Thus the set of conditions forcing $\forall a \, (\phi(a, y) \rightarrow a \in z)$ is dense below t, so $t \Vdash \forall a \, (\phi(a, y) \rightarrow a \in z)$, as claimed.[10.176]

Infinity We will show that

$$\Vdash \exists u \, (\exists v \; v \in u \wedge \forall v \in u \, \exists w \in u \; v \in w).$$

To this end, let $x_0 = 0$ and for each $n \in \omega$, let $x_{n+1} = \{\langle x_n, \mathbf{1}\rangle\}$. Let $x = \{\langle x_n, \mathbf{1}\rangle \mid n \in \omega\}$. Since Infinity^M, $x \in M^{\mathbb{P}}$. It suffices to show that

$$\Vdash \exists v \; v \in x \wedge \forall v \in x \, \exists w \in x \; v \in w.$$

Note that for all $n \in \omega$,

$$\Vdash x_n \in x \text{ and } \Vdash x_n \in x_{n+1}.$$

Thus, $\Vdash x_0 \in x$, so $\Vdash \exists v\, v \in x$.

To show that $p \Vdash \forall v \in x\, \exists w \in x\, v \in w$, we must show that $\forall q \leqslant p\, \forall \langle y, s\rangle \in x\, \exists r \leqslant q\, (r \leqslant s \wedge r \Vdash \exists w \in x\, y \in w)$, i.e., $\forall q \leqslant p\, \forall n \in \omega\, \exists r \leqslant q\, r \Vdash \exists w \in x\, x_n \in w$. This follows from the fact that $\Vdash x_{n+1} \in x$ and $\Vdash x_n \in x_{n+1}$.

Foundation The proof is left to the reader. $\square$[10.171]

10.31 Proof of (8.225)

[Refer to p. 572.]

The purpose of this note is to fill out the sketch of the proof of (8.225) given in the main text.

10.31.1 Properties of the Levy collapse

The following theorem states that for any infinite cardinal κ, $\mathbb{C}(\omega, \kappa)$ is essentially the only way to collapse κ to ω with a partial order of cardinality κ.[27]

(10.177) **Theorem** [ZFC] *Suppose $\mathbb{P}$ is a partial order, $\kappa = ||\mathbb{P}|| > \omega$, and $\Vdash^{\mathbb{P}} \ulcorner(\check{\kappa})$ is countable$\urcorner$. Then $\mathfrak{R}\,\mathbb{P} \cong \mathfrak{C}(\omega, \kappa)$.*

Remark Note that if we let $\kappa = \omega$, and we require that $\mathfrak{R}\,\mathbb{P}$ be atomless (i.e., that $\mathbb{P}$ have no have no minimal element) then this is just the familiar fact that there is—up to isomorphic equivalence—only one atomless complete boolean algebra with a countable dense subset, viz., the Cohen algebra $\mathfrak{C}(\omega, 2)$, which is isomorphic to $\mathfrak{C}(\omega, \omega)$.

Proof Since $\mathfrak{R}\,\mathbb{P} = \mathfrak{R}\,\overline{\mathbb{P}}$, where $\overline{\mathbb{P}}$ is the canonical separative quotient of $\mathbb{P}$,[8.59] we may assume without loss of generality that $\mathbb{P}$ is separative. Recall that conditions in $\mathbb{C}(\omega, \kappa)$ are functions whose domains are finite subsets of ω. Let $Q = \{p \in |\mathbb{C}(\omega, \kappa)| \mid \operatorname{dom} p \in \omega\}$, and let $\mathbb{Q}$ be the corresponding partial order. Since Q is dense in $\mathbb{C}(\omega, \kappa)$, $\mathfrak{R}\,\mathbb{Q} = \mathfrak{R}\,\mathbb{C}(\omega, \kappa) = \mathfrak{C}(\omega, \kappa)$. We will show that $\mathbb{Q}$ is isomorphic to a dense subset of $\mathbb{P}$, from which it follows that $\mathfrak{C}(\omega, \kappa) = \mathfrak{R}\,\mathbb{C} \cong \mathfrak{R}\,\mathbb{P}$.

Let $\mathsf{G} = \mathsf{G}^{\mathbb{P}}$ be the canonical term for the generic filter on $\mathbb{P}$, as usual. Let[8.109] $\dot{f} \in M^{\mathbb{P}}$ be such that $\Vdash^{\mathbb{P}} \dot{f} : \check{\omega} \overset{\text{sur}}{\to} \mathsf{G}$. Let $P = |\mathbb{P}|$ and let $\leqslant = \leqslant^{\mathbb{P}}$. We will define (from a choice function adequate to the purpose) an isomorphism $\iota : \mathbb{Q} \overset{\text{inj}}{\to} \mathbb{P}$ by $\subseteq$-recursion on Q. Specifically, we will show how to define $\iota(q ^\frown \langle\alpha\rangle)$ for each $\alpha \in \kappa$, given $\iota(q)$.

Suppose $|q| = n$. Since $\Vdash^{\mathbb{P}} \ulcorner(\check{\kappa})$ is countable$\urcorner$, there exists an antichain R in $\mathbb{P}$ below $\iota(q)$ such that $|R| = \kappa$. We can arrange that R is a maximal antichain, and we can also arrange that for every $r \in R$, r *decides* the value of $\dot{f}$, i.e., $\exists p \in P\; r \Vdash^{\mathbb{P}} \dot{f}(\check{n}) = \check{p}$. Let $\langle r_\alpha \mid \alpha \in \kappa\rangle$ enumerate R. For each $\alpha \in \kappa$, let $\iota(q ^\frown \langle\alpha\rangle) = r_\alpha$.

ι is clearly injective and order-preserving, so it is an isomorphism. We will show that $\{\iota(q) \mid q \in Q\}$ is dense in $\mathbb{P}$. Suppose $p \in P$. Then $p \Vdash \check{p} \in \mathsf{G}$, so

[27] Any partial order of cardinality $< \kappa$ satisfies the κ-chain condition, so it cannot render κ countable.

$p \Vdash \exists n \in \check{\omega}\ \dot{f}(n) = \check{p}$. Hence there exists $r \leqslant p$ and $n \in \omega$ such that $r \Vdash \dot{f}(\check{n}) = \check{p}$. By the maximality of the antichains R used in the definition of ι, for each $m \in \omega$ there exists $q \in Q$ of length m such that $\iota(q)$ is compatible with r. In particular, there exists such a q of length $n+1$. By the definition of ι, $\iota(q)$ decides the value of $\dot{f}(\check{n})$. Since $r \Vdash \dot{f}(\check{n}) = \check{p}$, necessarily $\iota(q) \Vdash \dot{f}(\check{n}) = \check{p}$. Since $\Vdash \dot{f} : \check{\omega} \overset{\text{sur}}{\to} \mathsf{G}$, $\iota(q) \Vdash \check{p} \in \mathsf{G}$.

Since $\mathbb{P}$ is separative, if $\iota(q) \nleqslant p$ there exists $s \leqslant \iota(q)$ such that s is incompatible with p, in which case $s \Vdash \check{p} \notin \mathsf{G}$. Hence $\iota(q) \leqslant p$. $\square$[10.177]

We have as a corollary the following theorem of Kripke.

(10.178) **Theorem** [ZFC] *Suppose κ is an infinite cardinal, $\mathfrak{A}$ is a complete boolean algebra, and $||\mathfrak{A}|| \leqslant \kappa$. Then $\mathfrak{A}$ is isomorphic to a complete subalgebra of $\mathfrak{C}(\omega, \kappa)$.*

Proof Let $\mathfrak{A}^+$ be the partial order of nonzero elements of $\mathfrak{A}$, and let $\mathbb{P} = \mathfrak{A}^+ \times \mathbb{C}(\omega, \kappa)$. Then $\mathbb{P}$ has cardinality κ and collapses κ to ω, so $\mathfrak{R}\,\mathbb{P} \cong \mathfrak{C}(\omega, \kappa)$.[10.177] Let $\iota : |\mathfrak{A}| \to |\,\mathfrak{R}\,\mathbb{P}|$ be such that

$$\iota(a) = \{\langle a', c\rangle \mid a' \leqslant a \wedge c \in |\mathbb{C}(\omega, \kappa)|\}.$$

Clearly, ι is an isomorphism of $\mathfrak{A}$ with a complete subalgebra of $\mathfrak{R}\,\mathbb{P}$, so $\mathfrak{A}$ isomorphic to a complete subalgebra of $\mathfrak{C}(\omega, \kappa)$. $\square$[10.178]

(10.179) **Theorem** [ZFC] *Suppose κ is an infinite cardinal, $\mathfrak{A}$ is a complete boolean algebra, and $||\mathfrak{A}|| = \kappa$. Suppose $\mathfrak{B}$ is a complete subalgebra of $\mathfrak{A}$, $||\mathfrak{B}|| < \kappa$, and ι is a complete embedding of $\mathfrak{B}$ in $\mathfrak{C}(\omega, \kappa)$. Then there exists an extension ι' of ι to a complete embedding of $\mathfrak{A}$ in $\mathfrak{C}(\omega, \kappa)$.*

Proof Let $\dot{\mathfrak{D}} = \mathfrak{A} : \mathfrak{B}$, i.e., a term denoting the quotient of $\check{\mathfrak{A}}$ by the canonical generic filter G on $\check{\mathfrak{B}}$ in $V^{\mathfrak{B}}$. Let $\mathfrak{C} = \mathfrak{C}(\omega, \kappa)$, let $\mathfrak{B}'$ be the image of $\mathfrak{B}$ under ι, and let $\dot{\mathfrak{D}}' = \mathfrak{C} : \mathfrak{B}'$. Let $\dot{\mathfrak{C}}^{\mathfrak{B}}$ and $\dot{\mathfrak{C}}^{\mathfrak{B}'}$ be terms for the collapsing algebra $\mathfrak{C}(\omega, \check{\kappa})$ in $V^{\mathfrak{B}}$ and $V^{\mathfrak{B}'}$, respectively.

It is easy to check that $\ulcorner(\check{\kappa})$ is a cardinal$\urcorner$ and $\ulcorner||(\dot{\mathfrak{D}})|| \leqslant (\check{\kappa})\urcorner$ are $\mathfrak{B}$-valid. Thus,[10.178, 8.109] there exists $\dot{k} \in V^{\mathfrak{B}}$ such that $\ulcorner(\dot{k})$ is a complete embedding of $(\dot{\mathfrak{D}})$ in $(\dot{\mathfrak{C}}^{\mathfrak{B}})\urcorner$ is $\mathfrak{B}$-valid.

We now define a complete embedding j of $\mathfrak{A}$ in $\mathfrak{B} * \dot{\mathfrak{C}}^{\mathfrak{B}}$. Suppose $a \in |\mathfrak{A}|$. Let $\tilde{a} \in V^{\mathfrak{B}}$ be the canonical name for the element of $\dot{\mathfrak{D}}$ corresponding to a (i.e., the equivalence class in $\check{\mathfrak{A}}/\mathsf{G}^{\mathfrak{B}}$ containing $\check{a}$), as defined in the proof of (8.184). Let $\dot{c} \in V^{\mathfrak{B}}$ be such that $\dot{c} = \dot{k}(\tilde{a})$ is $\mathfrak{B}$-valid. Let $j(a)$ be the element of $\mathfrak{B} * \dot{\mathfrak{C}}^{\mathfrak{B}}$ corresponding to $\dot{c}$, i.e., the canonical element $\dot{c}'$ of $V^{\mathfrak{B}}$ such that $[\![\dot{c}' = \dot{c}]\!]^{\mathfrak{B}} = \mathbf{1}$. Note that if $a \in |\mathfrak{B}|$ then

$$[\![\tilde{a} = \mathbf{1}^{\dot{\mathfrak{D}}}]\!]^{\mathfrak{B}} = a \text{ and } [\![\tilde{a} = \mathbf{0}^{\dot{\mathfrak{D}}}]\!]^{\mathfrak{B}} = \neg\, a,$$

so

$$[\![\dot{c} = \mathbf{1}^{\dot{\mathfrak{C}}^{\mathfrak{B}}}]\!]^{\mathfrak{B}} = a \text{ and } [\![\dot{c} = \mathbf{0}^{\dot{\mathfrak{C}}^{\mathfrak{B}}}]\!]^{\mathfrak{B}} = \neg\, a.$$

Hence, $j(a)$ is the image of a under the canonical (complete) embedding of $\mathfrak{B}$ in $\mathfrak{B} * \dot{\mathfrak{C}}^{\mathfrak{B}}$.[8.182]

Working briefly in $V[G']$, where G' is a hypothetical V-generic filter on $\mathfrak{B}'$, and letting $\mathfrak{D}' = \dot{\mathfrak{D}}'^{G'}$, we find that κ is an uncountable cardinal (because $||\mathfrak{B}'|| < \kappa$), and $\mathfrak{D}'$ collapses κ to ω (since $\mathfrak{B}' * \dot{\mathfrak{D}}' \cong \mathfrak{C} = \mathfrak{C}(\omega, \kappa)$). Let C be the canonical image of $\mathbb{C}(\omega,\kappa)$ in $\mathfrak{C}$ ($= \mathfrak{R}\,\mathbb{C}(\omega,\kappa)$), i.e., $C = \{[p] \mid p \in \mathbb{C}(\omega,\kappa)\}$. C is dense in $\mathfrak{C}$. Let $E = \{\tilde{c}^{G'} \mid c \in C\}$, the projection[8.193.2] of C in $\mathfrak{D}'$ (as the quotient of $\mathfrak{C}$ by G'). Let $\mathbb{E} = \left(E\backslash\{\mathbf{0}\}; \leqslant^{\mathfrak{D}'}\right)$ be the corresponding partial order. Then $\mathbb{E}$ is dense in $\mathfrak{D}'$,[8.194] so $\mathfrak{R}\,\mathbb{E} \cong \mathfrak{D}'$, so $\mathbb{E}$ collapses κ to ω. It follows that $\kappa \leqslant ||\mathbb{E}|| \leqslant ||\mathbb{C}|| = \kappa$, so $||\mathbb{E}|| = \kappa$, and[10.177] $\mathfrak{D}' \cong \mathfrak{C}$.

It follows from the existence of this argument in $V[G']$ that $\ulcorner(\dot{\mathfrak{D}}') \cong (\dot{\mathfrak{C}}^{\mathfrak{B}'})\urcorner$ is $\mathfrak{B}'$-valid. Let $\dot{k}' \in V^{\mathfrak{B}'}$ be such that $\ulcorner(\dot{k}')$ is an isomorphism of $(\dot{\mathfrak{D}}')$ with $(\dot{\mathfrak{C}}^{\mathfrak{B}'})\urcorner$ is $\mathfrak{B}'$-valid.[8.109]

We now proceed as above to define an isomorphism j' of $\mathfrak{C}$ with $\mathfrak{B}' * \dot{\mathfrak{C}}^{\mathfrak{B}'}$ such that for any $c \in |\mathfrak{B}'|$, $j'(c)$ is the image of c under the canonical (complete) embedding of $\mathfrak{B}'$ in $\mathfrak{B}' * \dot{\mathfrak{C}}^{\mathfrak{B}'}$.

Since $\iota : \mathfrak{B} \cong \mathfrak{B}'$, there exists $\pi : \mathfrak{B} * \dot{\mathfrak{C}}^{\mathfrak{B}} \cong \mathfrak{B}' * \dot{\mathfrak{C}}^{\mathfrak{B}'}$ such that for all $b \in |\mathfrak{B}|$, π maps the canonical image of b in $\mathfrak{B} * \dot{\mathfrak{C}}^{\mathfrak{B}}$ to the canonical image of $\iota(b)$ in $\mathfrak{B}' * \dot{\mathfrak{C}}^{\mathfrak{B}'}$. Now, given $a \in |\mathfrak{A}|$, let $\iota'(a) = j'^{-1}(\pi(j(a)))$. Then ι' is the desired extension of ι to a complete embedding of $\mathfrak{A}$ in $\mathfrak{C}$. $\square$[10.179]

(10.180) **Theorem** [ZF] *Suppose κ is an inaccessible cardinal. Suppose $\mathfrak{A}$ and $\mathfrak{A}'$ are complete subalgebras of $\mathfrak{C}(\omega,{<}\kappa)$ of size $< \kappa$, and $\iota : \mathfrak{A} \cong \mathfrak{A}'$. Then ι may be extended to an automorphism of $\mathfrak{C}(\omega,{<}\kappa)$.*

Proof Let $\mathbb{C} = \mathbb{C}(\omega,{<}\kappa)$ and let $\mathfrak{C} = \mathfrak{C}(\omega,{<}\kappa) = \mathfrak{R}\,\mathbb{C}$. For each $\nu < \kappa$, let[8.220]

$$\mathbb{C}_\nu = \mathbb{C}(\omega,{<}\nu)$$
$$\mathbb{C}^\nu = \mathbb{C}(\omega,[\nu,\kappa)).$$

Note that $\mathbb{C}$ is canonically isomorphic to $\mathbb{C}_\nu \times \mathbb{C}^\nu$. Let $\mathfrak{C}_\nu$ be the subalgebra of $\mathfrak{C}$ corresponding to $\mathfrak{C}(\omega,{<}\nu)$. Thus, $X \in \mathfrak{C}_\nu$ iff $X = \{p \in \mathbb{C} \mid p{\restriction}(\nu \times \omega) \in Y\}$ for some $Y \in \mathfrak{C}(\omega,{<}\nu)$. $\mathfrak{C}_\nu$ is a complete subalgebra of $\mathfrak{C}$.[8.146] Since κ is inaccessible, $||\mathfrak{C}_\nu|| < \kappa$ for every $\nu < \kappa$.

Suppose $X \in |\mathfrak{C}|$. Then X is a regular subset of $\mathbb{C}$. Let $Y \subseteq X$ be a maximal antichain in X. Then $X = \overline{Y}$. Since $\mathbb{C}$ has the κ-chain condition,[8.223] $Y \subseteq |\mathfrak{C}_\nu|$ for some $\nu < \kappa$, so $X \in \mathfrak{C}_\nu$. Hence $|\mathfrak{C}| = \bigcup_{\nu<\kappa} |\mathfrak{C}_\nu|$.

Suppose ν is a cardinal $< \kappa$. Then $\mathbb{C}_{\nu+1}$ collapses ν to ω, and $||\mathbb{C}_{\nu+1}|| = \nu$, so[10.177]

$$\mathfrak{C}_{\nu+1} \cong \mathfrak{C}(\omega,\nu). \tag{10.181}$$

Let $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}'_0 = \mathfrak{A}'$, and $\iota_0 = \iota$. Let $\nu_0, \nu' < \kappa$ be infinite cardinals such that $|\mathfrak{A}_0| \subseteq |\mathfrak{C}_{\nu_0}|$, $||\mathfrak{C}_{\nu_0}|| = \nu'$, $||\mathfrak{A}_0|| < \nu'$, and $|\mathfrak{A}'_0| \subseteq |\mathfrak{C}_{\nu'+1}|$. By (10.179) with $\mathfrak{C}_{\nu_0}$ for $\mathfrak{A}$, ν' for κ, $\mathfrak{A}_0$ for $\mathfrak{B}$, ι_0 for ι, and $\mathfrak{C}_{\nu'+1}$ for $\mathfrak{C}(\omega,\nu')$,[10.181] there exists a complete embedding ι_1 of $\mathfrak{C}_{\nu_0}$ in $\mathfrak{C}_{\nu'+1}$ that extends ι_0.

Let $\mathfrak{A}_1 = \mathfrak{C}_{\nu_0}$ and $\mathfrak{A}'_1 = \iota_1{}^{\rightarrow}\mathfrak{A}_1$. Since $\mathfrak{C}_\mu$ is a complete subalgebra of $\mathfrak{C}$ for any $\mu < \kappa$, $\mathfrak{A}_1$ and $\mathfrak{A}'_1$ are isomorphic complete subalgebras of $\mathfrak{C}$ of size $< \kappa$, and we can carry out the above construction in the other direction to extend ι_1 to an isomorphism ι_2 of a complete subalgebra $\mathfrak{A}_2$ of $\mathfrak{C}$ with $\mathfrak{A}'_2 = \mathfrak{C}_{\nu_2}$ for some cardinal $\nu_2 < \kappa$.

We proceed in this alternating fashion for ω steps. Let $\nu = \bigcup_{n\in\omega} \nu_n$ and $\iota' = \bigcup_{n\in\omega} \iota_n$. Then $\iota \subseteq \iota'$ and $\iota' : \mathfrak{C}_\nu \cong \mathfrak{C}_\nu$. It is easy to extend ι' to an automorphism of $\mathfrak{C}$. For example, let j be the automorphism $\mathbb{C}_\nu$ corresponding to ι'. Then $\langle p_\nu, p^\nu\rangle \mapsto \langle j(p_\nu), p^\nu\rangle$ is an automorphism of $\mathbb{P}_\nu \times \mathbb{P}^\nu$, and by using natural correspondences, we can define from this an extension of ι' to an automorphism of $\mathfrak{C}$. $\square$[10.180]

The next two theorems establish an important factorization property of collapsing extensions, in particular the Levy collapse.

(10.182) **Theorem** [GBC] *Suppose M is a transitive model of* ZFC *and κ is an infinite cardinal in M. Suppose G is an M-generic filter on $\mathbb{C}(\omega,\kappa)$ and $X \in M[G]$ is a set of ordinals. Then either $M[X] = M[G]$*[28] *or there exists an $M[X]$-generic filter H on $\mathbb{C}(\omega,\kappa)$ such that $M[X][H] = M[G]$.*

Proof Let $\mathfrak{C} = \mathfrak{C}(\omega,\kappa)$, let $\dot{X}$ be such that $\dot{X}^G = X$, and let $\mathfrak{A}$ be the complete subalgebra of $\mathfrak{C}$ generated by elements of $|\mathfrak{C}|$ of the form $[\![\check{x} \in \dot{X}]\!]$ for $x \in M$, as in the proof of (8.196). Let $G' = G \cap |\mathfrak{A}|$, so that G' is an M-generic filter on $\mathfrak{A}$ and $M[X] = M[G']$. Let $\dot{\mathfrak{B}} = \mathfrak{C} : \mathfrak{A}$, and let $\mathfrak{B} = \dot{\mathfrak{B}}^{G'}$. Then $\mathfrak{C} \cong \mathfrak{A} * \dot{\mathfrak{B}}$,[8.184] and $M[G]$ is a $\mathfrak{B}$-generic extension of $M[G']$.

As before (in the proof of (10.179)), let C be the canonical image of $\mathbb{C}(\omega,\kappa)$ in its regular algebra, i.e., $C = \{[p] \mid p \in \mathbb{C}(\omega,\kappa)\}$. Let $B = \{\tilde{c}^{G'} \mid c \in C\}$.[8.193.2] Then[8.194]

(10.183) $\ulcorner [B] \setminus \{\mathbf{0}\}$ *is dense in* $[\mathfrak{B}]\urcorner^{M[G']}$.

We now consider two cases. Suppose first that κ is uncountable in $M[G']$, and let $\lambda = \ulcorner \mathrm{Card}\,[\kappa]\urcorner^{M[G']}$. Working in $M[G']$, we observe that $\mathfrak{B}$ is a forcing algebra in $M[G']$ that collapses λ to ω. Let $\mathbb{P}$ be the partial order consisting of $B \setminus \{\mathbf{0}\}$ with the order inherited from $\mathfrak{B}$. $\mathbb{P}$ is dense in $\mathfrak{B}$,[10.183] so $\mathfrak{R}\,\mathbb{P} \cong \mathfrak{B}$, so $\mathbb{P}$ collapses λ to ω. $||\mathbb{P}|| = |\kappa| = \lambda$, so[10.177] $\mathfrak{B} \cong \mathfrak{C}(\omega,\lambda)$. Since $\lambda \sim \kappa$, $\mathfrak{C}(\omega,\lambda) \cong \mathfrak{C}(\omega,\kappa)$. Hence, $M[G]$ is a $\mathfrak{C}(\omega,\kappa)$-generic extension of $M[G']$.

Now suppose κ is countable in $M[G']$. If $M[G'] = M[G]$ then we are finished, since $M[X] = M[G']$. Suppose therefore that $M[G'] \neq M[G]$. In this case, $\mathfrak{C} \cong \mathfrak{C}(\omega,\omega)$ in $M[G']$, and the dense subset B of $\mathfrak{B}$ is countable in $M[G']$. To show that $\mathfrak{B} \cong \mathfrak{C}$ in $M[G']$ it therefore suffices to show that $\mathfrak{B}$ is atomless, as there is only one atomless countably generated complete boolean algebra up to isomorphic equivalence.

Suppose toward a contradiction that b is an atom in $\mathfrak{B}$, i.e., $b \neq \mathbf{0}$ and $\forall b' \leqslant b\,(b' = b \vee b' = \mathbf{0})$. By (10.183) there exists $b' \in B$ such that $\mathbf{0} < b' \leqslant b$. Thus, $b = b' \in B$. Let $c_0 \in \mathbb{C}(\omega,\kappa)$ be such that $\tilde{c}_0^{G'} = b$. Again using the fact that b is an atom, for any finite $D \subseteq \omega$ such that $\mathrm{dom}\, c_0 \subseteq D$, there exists a unique extension c' of c_0 with domain D such that $\tilde{c}'^{G'} \neq \mathbf{0}$ (and, in fact, $\tilde{c}'^{G'} = b$). Let C' be the set of all such extensions of c_0. Clearly,[8.188.3] $C' \subseteq G$, and, in fact, for any $c \in |\mathbb{C}|$, $c \in G \leftrightarrow \exists c' \in C'\ c' \leqslant c$. Thus, $G \in M[G']$, so $M[G'] = M[G]$, contradicting our assumption that $M[G'] \neq M[G]$.

In this case also, therefore, $\mathfrak{B} \cong \mathfrak{C}(\omega,\kappa)$, so $M[G]$ is a $\mathfrak{C}(\omega,\kappa)$-generic extension of $M[G']$. Since $M[X] = M[G']$, this completes the proof. □[10.182]

Finally, we have Solovay's *factor lemma.*

(10.184) **Theorem** [ZFC] *Suppose M is a transitive model of* ZFC*, κ is an inaccessible cardinal in M, G is an M-generic filter on $\mathbb{C} = \mathbb{C}(\omega,{<}\kappa)$, and s is an ω-sequence of ordinals in $M[G]$. Then κ is inaccessible in $M[s]$, and there exists an $M[s]$-generic filter H on $\mathbb{C}$ such that $M[G] = M[s][H]$.*

Proof For any $\nu < \kappa$, $\mathbb{C}$ is naturally isomorphic to $\mathbb{C}_\nu \times \mathbb{C}^\nu$, where $\mathbb{C}_\nu = \mathbb{C}(\omega,{<}\nu)$ and $\mathbb{C}^\nu = \mathbb{C}(\omega,[\nu,\kappa))$. Let $\mathfrak{C}, \mathfrak{C}_\nu, \mathfrak{C}^\nu$ be the corresponding regular algebras. Note

[28]See (8.196) for the definition of $M[X]$. Note that $M[G]$ is a generic extension of $M[X]$.

that if ν is an infinite cardinal in M, $\mathbb{C}_{\nu+1}$ has cardinality ν and collapses ν to ω, so $\mathfrak{C}_{\nu+1} \cong \mathfrak{C}(\omega, \nu)$ (by virtue of (10.177) if ν is uncountable in M, and by direct evaluation if $\nu = \omega$; we'll only need the uncountable case).

Let $p \in G$ be such that $p \Vdash \ulcorner(\dot{s}) : \omega \to \mathrm{Ord}\urcorner$. Working in M, for each $n \in \omega$ let X_n be a maximal antichain in $\mathbb{C}$ below p such that $\forall q \in X_n\, \exists_{\mathrm{Ord}}\alpha\ q \Vdash \dot{s}(\check{n}) = \check{\alpha}$. Since $\mathbb{C}$ satisfies[8.223] the κ-chain condition, $|X_n| < \kappa$ for each $n \in \omega$, and there exists $\nu < \kappa$ such that $\{p\} \cup \bigcup_{n\in\omega} X_n \subseteq \mathbb{C}_\nu$.

Hence (working in $M[G]$), $s \in M[G_\nu]$, where $G_\nu = G \cap |\mathbb{C}_\nu|$. Since $||\mathbb{C}_\nu|| < \kappa$ in M, κ is inaccessible in $M[G_\nu]$, hence in $M[s]$. Since $\mathbb{C} \cong \mathbb{C}_{\nu+1} \times \mathbb{C}^{\nu+1}$, $M[G] = M[G_{\nu+1}][G^{\nu+1}]$, where $G^{\nu+1} = G \cap |\mathbb{C}^{\nu+1}|$. Since $M[s] \neq M[G_{\nu+1}]$, by (10.182) and the preceding paragraph, there is an $M[s]$-generic filter G' on $\mathbb{C}_{\nu+1}$ such that $M[s][G'] = M[G_{\nu+1}]$. It follows that $M[G] = M[s][G'][G^{\nu+1}]$. Let $H = G' \times G^{\nu+1}$. Then H is an $M[s]$-generic filter on $\mathbb{C}$ and $M[G] = M[s][H]$. $\square$[10.184]

10.31.2 The Solovay model

We presume a fixed enumeration $\langle \phi_n \mid n \in \omega \rangle$ of s-formulas.

Definition [ZF] $\Delta(s, x) \overset{\text{def}}{\iff}$

1. $s : \omega \to \mathrm{Ord}$*;*
2. $s_0 \in \omega$ *and if we let* $\phi = \phi_{s_0}$ *then* ϕ *is an* s*-formula with exactly two free variables;*
3. *if we let* $\alpha = s_1$ *and* $s_{>1} = \langle s_2, s_3, \dots \rangle$ *then* $s_{>1}, x \in V_\alpha$*; and*
4. $V_\alpha \models \phi[s_{>1}, x]\}$.

Note that $\{x \mid \Delta(s, x)\}$ is 0 if it is not the case that $s_0 \in \omega$ and ϕ_{s_0} has exactly two free variables, or if $s_{>1} \notin V_{s_1}$; and in any case, $\{x \mid \Delta(s,x)\} \subseteq V_{s_1}$.

(10.185) [GBC] *Suppose M is a transitive model of* ZFC*, κ is an inaccessible cardinal in M, and G is an M-generic filter on $\mathbb{C}(\omega, {<}\kappa)$ (or $\mathfrak{C}(\omega, {<}\kappa)$ as the spirit moves us). Let S be the class of ω-sequences of ordinals in $M[G]$.*

1. *Let N_1 be the class of $x \in M[G]$ such that there exists $s \in S$ such that $x = \{y \mid \Delta(s, y)\}$, i.e., N_1 is $\mathrm{OD}(S)$ in the sense of $M[G]$.*
2. *Let N be the class of x such that $\mathrm{tc}\{x\} \subseteq N_1$. I.e., N is $\mathrm{HOD}(S)$ in the sense of $M[G]$.*

(10.186) **Theorem** [GBC] *Suppose M, N, etc. are as in (10.185).*

1. *Suppose $f \in M[G]$ and $f : \omega \to N$. Then $f \in N$.*
2. *$N \models$ DC.*

Proof 1 For each $n \in \omega$ let s^n be as in (10.185.2) such that $f(n) = \{x \mid \Delta^{M[G]}(s^n, x)\}$. Let $t : \omega \to \mathrm{Ord}^M$ code the sequence $\langle s^n \mid n \in \omega \rangle$ in some simple way (by dovetailing perhaps), and let $\alpha \in \mathrm{Ord}^M$ be such that $t \in M[G]_\alpha$ and f is definable over $M[G]_\alpha$ from t by some formula ϕ, i.e., $f = \{x \mid M[G]_\alpha \models \phi[t, x]\}$. Let $n \in \omega$ be such that $\phi = \phi_n$, and let $s = \langle n, \alpha \rangle ^\frown t$. Then $f = \{x \mid \Delta^{M[G]}(s, x)\}$, so $f \in N_1$. Since $f(n) \in N$ for each $n \in \omega$, it is easy to see that $\mathrm{tc}\, f \subseteq N_1$. Hence, $f \in N$.

2 Suppose $R \in N$ is a binary relation on a nonempty set $X \in N$ such that $\forall x \in X\, \exists y \in X\; x\,R\,y$. Since $M[G]$ satisfies AC, for each $x_0 \in X$ there exists $f \in M[G]$ such that $f : \omega \to X$, $f(0) = x_0$, and $\forall n \in \omega\; f(n)\,R\,f(n+1)$. It follows from (10.186.1) that $f \in N$. □[10.186]

10.31.3 Cohen and random reals

We will be working with submodels $M[s]$ of N, where $s : \omega \to \mathrm{Ord}$, and $M[s]$ is defined as in the remark following the statement of Theorem 8.196. Thus, $M[s]$ models ZFC, *a fortiori*, $\mathsf{ZF} + \mathsf{DC}$. In the following discussion, we will use 'M' and 'N' generally to denote transitive models of $\mathsf{ZF} + \mathsf{DC}$ with $M \subseteq N$. Note that $\mathsf{ZF} + \mathsf{DC}$ is sufficient to develop most of descriptive set theory, including the theory of category and Lebesgue measure on $\mathbb{R}$ (for which $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$ actually suffices). That part of the discussion that does not deal explicitly with models of $\mathsf{ZF} + \mathsf{DC}$ may be regarded as taking place in $\mathsf{ZF} + \mathsf{DC}$, so that it is applicable within these models.

Let $\mathbf{Borel} = \mathbf{Borel}^{\mathbb{R}}$ be the boolean algebra of Borel subsets of $\mathbb{R}$, where $\mathbb{R}$ is formally defined as the set of Dedekind cuts[5.71] in the rational number line $\mathbb{Q}$. As we have previously shown, all members of **Borel** have the Baire property[5.147] and are Lebesgue measurable.[5.160] Within **Borel** let $\mathfrak{m}$ and $\mathfrak{n}$ be respectively the ideals of meager[5.143] and null[5.155] sets. Let $\mathfrak{B} = \mathbf{Borel}/\mathfrak{m}$ and let $\mathfrak{L} = \mathbf{Borel}/\mathfrak{n}$. As shown previously,[5.148, 5.161] $\mathfrak{B}$ and $\mathfrak{L}$ are complete boolean algebras satisfying the countable chain condition.[29]

As we did in the introduction,[8.7] we now exploit the correspondence between generic filters on $\mathfrak{B}$ or $\mathfrak{L}$ and real numbers.

(10.187) **Definition** [GB] *Suppose M is a transitive model of $\mathsf{ZF} + \mathsf{DC}$ and G is an M-generic filter on $\mathfrak{B}^M$ ($\mathfrak{L}^M$). $x_G \overset{\text{def}}{=} \{r \in \mathbb{Q} \mid [(r, \infty)] \in G\}$.*

(10.188) **Theorem** [GB] *Under the conditions of (10.187), x_G is a Dedekind cut in $\mathbb{Q}$.*

Proof x_G is clearly closed downward. To see that it is bounded above, we note that any nonzero element of $\mathfrak{B}$ ($\mathfrak{L}$) has a nonzero extension of the form $[X]$ for some Borel set X such that $X \subseteq (-\infty, s)$ for some rational s. It follows that G contains such an element, so $[(-\infty, s)] \in G$ for some $s \in \mathbb{Q}$, whence it follows that $[(r, \infty)] \notin G$ for any $r \geqslant s$. To see that x_G has no greatest element (and is therefore a cut in the strict sense), a similar density argument applies: given $r \in \mathbb{Q}$, any nonzero element of $\mathfrak{B}$ ($\mathfrak{L}$) has a nonzero extension of the form $[X]$ for some Borel set X such that $X \subseteq (-\infty, s)$ for some rational $s < r$ or $X \subseteq (s, \infty)$ for some rational $s > r$, from which it follows that either $r \notin x_G$, or $s \in x_G$ for some $s > r$. □[10.188]

Definition [GB] *Suppose M is a transitive model of $\mathsf{ZF} + \mathsf{DC}$ and $x \in \mathbb{R}$. x is* Cohen (random) over M $\overset{\text{def}}{\iff}$ *there exists an M-generic filter on $\mathfrak{B}^M(\mathfrak{L}^M)$ such that $x = x_G$.*

[29] Since we are only concerned with $\mathfrak{B}$ and $\mathfrak{L}$ as defined in M, and the models that serve in this role in the subsequent proof all satisfy AC, it would suffice to have established these facts in ZFC, but we have shown that they follow from $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$.

The rationale for the use of ‘Cohen’ here is the fact noted previously that since $\mathfrak{B}$ and the Cohen algebra are both atomless with countable dense sets, they are isomorphic.

Note that for any $r \in \mathbb{Q}$,

$$x_G \in (r, \infty) \leftrightarrow [(r, \infty)] \in G, \tag{10.189}$$

where ‘(r, ∞)’ on the left refers to (r, ∞) in the sense of N, and on the right to the corresponding interval in the sense of M. Note that $\mathbb{Q}^M = \mathbb{Q}^N$. Using the M-genericity of G and the boolean operations of complementation and intersection, (10.189) generalizes to arbitrary intervals, e.g.,

$$\begin{gathered} x_G \in [r, \infty) \leftrightarrow [[r, \infty)] \in G \\ x_G \in (-\infty, r) \leftrightarrow [(-\infty, r)] \in G \\ x_G \in (r, s) \leftrightarrow [(r, s)] \in G, \end{gathered}$$

for any rationals r, s, where again we understand the intervals on the right in the sense of M, and those on the left in the sense of N.

We can generalize these identities to arbitrary Borel sets in the sense of M if we appropriately define the corresponding sets in N. For this purpose we employ a propositional language $\mathcal{B}$, which has a *prime proposition* P_r for each $r \in \mathbb{Q}$, along with a negation (complementation) relation and countable disjunction (join) and conjunction (meet) operations. We call the expressions of $\mathcal{B}$, *boolean expressions*, or more specifically *Borel expressions* in recognition of the countability condition. An *interpretation* of $\mathcal{B}$ is a map $\mathfrak{I}$ of the $\mathcal{B}$-expressions into a countably complete boolean algebra $\mathfrak{A}$ that follows the obvious rules:

$$\begin{gathered} \mathfrak{I}(\neg\, \epsilon) = \neg\, \mathfrak{I}\epsilon \\ \mathfrak{I}\Big(\bigvee_{n \in \omega} \epsilon_n\Big) = \bigvee_{n \in \omega} \mathfrak{I}\epsilon_n. \end{gathered}$$

Clearly, any interpretation is uniquely determined by its values on the prime propositions. A natural interpretation B of $\mathcal{B}$ in **Borel** is obtained by letting $B(P_r) = (r, \infty)$. In this context, we may regard $\mathcal{B}$-expressions as Borel codes, which we have previously defined[5.89] somewhat differently but equivalently.

Now suppose, as above, that $M \subseteq N$ are transitive models of $\mathsf{ZF} + \mathsf{DC}$ and $G \in N$ is an M-generic filter on $\mathfrak{B}^M$. In the present terminology, (10.189) states that

$$x_G \in B(P_r)^N \leftrightarrow \big[B(P_r)^M\big] \in G. \tag{10.190}$$

As we have indicated, the purpose of $\mathcal{B}$ is to allow us to generalize (10.190) to $\mathbf{Borel}^N$ on the left and $\mathbf{Borel}^M$ on the right, which it does:

(10.191) **Theorem** [GB] *In the above context, suppose ϵ is a $\mathcal{B}^M$-expression. Then*

$$x_G \in (B\epsilon)^N \leftrightarrow \big[(B\epsilon)^M\big] \in G. \tag{10.192}$$

Proof By induction on complexity. Prime propositions are covered by (10.190). Suppose the theorem holds for ϵ. Then

$$\begin{aligned} x_G \in (B(\neg\, \epsilon))^N \leftrightarrow x_G \notin (B\epsilon)^N \leftrightarrow \big[(B\epsilon)^M\big] \notin G \leftrightarrow \neg \big[(B\epsilon)^M\big] \in G \\ \leftrightarrow \big[(B(\neg\, \epsilon))^M\big] \in G, \end{aligned}$$

because G is an ultrafilter.

Next suppose E is a countable set of $\mathcal{B}$-expressions in M and the theorem holds for each $\epsilon \in E$. Let $\eta = \bigvee E$. Then

$$\begin{aligned} x_G \in (B\eta)^N &\leftrightarrow \exists \epsilon \in E\, \big(x_G \in (B\epsilon)^N\big) \\ &\leftrightarrow \exists \epsilon \in E\, \big(\big[(B\epsilon)^M\big] \in G\big) \\ &\leftrightarrow \Big(\bigvee_{\epsilon \in E} \big[(B\epsilon)^M\big]\Big) \in G \\ &\leftrightarrow \big[(B\eta)^M\big] \in G, \end{aligned}$$

because, firstly, G is M-generic and, secondly, E is countable in M. □[10.191]

(10.192) shows how to recover G from x_G. Theorem 10.195 provides a useful condition on a real x that guarantees that (10.192), with x for x_G, defines an M-generic filter G, from which it follows that $x = x_G$, since (10.192) includes the definition[10.187] of x_G as the special case in which ϵ is prime.

Note that although the expression ϵ in (10.195) is required to be in M, it is $(B\epsilon)^N$, not $(B\epsilon)^M$, that is evaluated as to whether it is meager (null). Thus, preparatory to proving the theorem, we establish some absoluteness properties of category and measure.

(10.193) **Theorem** [GB] *Suppose $M \subseteq N$ are transitive models of* $\mathsf{ZF} + \mathsf{DC}$, *and ϵ is a $\mathcal{B}^M$-expression.*

1. $(B\epsilon)^M = (B\epsilon)^N \cap \mathbb{R}^M$.
2. $(B\epsilon)^M$ *is empty iff* $(B\epsilon)^N$ *is empty.*

Proof 1 Straightforward induction on the complexity of ϵ.

2 We have seen[5.97] that any Borel set is $\mathbf{\Sigma}^1_1$. It is in fact not hard to define a Σ^1_1 formula ϕ with free variables u, v such that for any $\mathcal{B}^M$-expression ϵ, which we may suppose by a suitable coding scheme to be in $\mathbb{R}^M$, and any $x \in \mathbb{R}$, for any transitive model N of $\mathsf{ZF} + \mathsf{DC}$ that includes M and contains x,

$$x \in (B\epsilon)^N \leftrightarrow N \models \phi\big[\begin{smallmatrix} u & v \\ x & \epsilon \end{smallmatrix}\big].$$

($B\epsilon$ is uniformly $\Pi^1_1(\epsilon)$ in the same sense.) Let $\psi = \exists u\, \phi$. ψ is Σ^1_1. $(B\epsilon)^M$ is nonempty iff $M \models \psi[\epsilon]$; and $(B\epsilon)^N$ is nonempty iff $N \models \psi[\epsilon]$. By the Σ^1_1-absoluteness theorem,[6.7] $M \models \psi[\epsilon]$ iff $N \models \psi[\epsilon]$, which proves the claim. □[10.193]

(10.194) **Theorem** [GB] *Suppose $M \subseteq N$ are transitive models of* $\mathsf{ZF} + \mathsf{DC}$, *and ϵ is a $\mathcal{B}^M$-expression.*

1. $M \models \ulcorner B[\epsilon]$ *is meager*$\urcorner \leftrightarrow N \models \ulcorner B[\epsilon]$ *is meager*$\urcorner$.
2. $M \models \ulcorner B[\epsilon]$ *is null*$\urcorner \leftrightarrow N \models \ulcorner B[\epsilon]$ *is null*$\urcorner$.

Proof 1 Suppose $M \models \ulcorner B[\epsilon]$ is meager$\urcorner$. Let $\langle \epsilon_n \mid n \in \omega \rangle \in M$ be a sequence of codes for open sets such that $M \models \ulcorner B[\epsilon_n]$ is open and dense$\urcorner$ for each $n \in \omega$, and $M \models \ulcorner B[\epsilon] \cap \bigcap_{n\in\omega} B[\epsilon_n] = 0\urcorner$, i.e., $M \models \ulcorner B[\epsilon']$ is empty$\urcorner$, where $\epsilon' = \epsilon \wedge \bigwedge_{n\in\omega} \epsilon_n$. Using (10.193), we see that $N \models \ulcorner B[\epsilon_n]$ is open and dense$\urcorner$ for each $n \in \omega$, and $N \models \ulcorner B[\epsilon']$ is empty$\urcorner$, so $N \models \ulcorner B[\epsilon] \cap \bigcap_{n\in\omega} B[\epsilon_n] = 0\urcorner$. Hence, $N \models \ulcorner B[\epsilon]$ is meager$\urcorner$.

Now suppose $M \models \ulcorner B[\epsilon]$ is not meager$\urcorner$. Then[5.147.4] there is are rationals $r < s$ such that $M \models \ulcorner B[\epsilon]$ is comeager on the open interval $([r], [s])$, i.e., $([r], [s]) \setminus B[\epsilon]$ is meager$\urcorner$. It follows by the preceding argument that N agrees, so $N \models \ulcorner B[\epsilon]$ is not meager$\urcorner$.

2 It is straightforward to show that M and N compute the same measure for open sets coded in M as countable unions of rational open intervals, and for closed sets coded in M as countable intersections of complements of rational open intervals. If $M \models \ulcorner B[\epsilon]$ is null$\urcorner$ then for any $n > 0$, there is an open set $G \in M$ with measure $< 1/n$ such that $M \models \ulcorner B[\epsilon] \subseteq [G]$, so—again using (10.193)—$N \models \ulcorner B[\epsilon] \subseteq [G]$. Hence $M \models \ulcorner B[\epsilon]$ is null$\urcorner$.

Inversely, if $M \models \ulcorner B[\epsilon]$ is not null$\urcorner$ then there is a closed set of positive measure included in $B[\epsilon]$ in the sense of M and therefore also in the sense of N, so $N \models \ulcorner B[\epsilon]$ is not null$\urcorner$. $\square$[10.194]

(10.195) **Theorem** [GB] *Suppose $M \subseteq N$ are transitive models of* ZF + DC, $\mathbb{R}^M \in N$, *and* $x \in \mathbb{R}^N$.

1. *x is Cohen over M iff x is not in any meager Borel set in N that is coded in M, i.e., for all $\mathcal{B}^M$-expressions ϵ, if $(B\epsilon)^N$ is meager then $x \notin B\epsilon$, where we take '$(B\epsilon)^N$ is meager' to mean that $N \models \ulcorner B[\epsilon]$ is meager$\urcorner$.*
2. *x is random over M iff x is not in any null Borel set in N that is coded in M.*

Proof We will give one proof for both cases, giving case-specific details as necessary. For the forward direction, suppose $G \in N$ is M-generic on $\mathfrak{B}^M(\mathfrak{L}^M)$, and suppose $\epsilon \in \mathcal{B}^M$ is such that $(B\epsilon)^N$ is meager (null). Then[10.194] $(B\epsilon)^M$ is meager (null), so $[(B\epsilon)^M] = \mathbf{0}$. Hence $[B(\neg\,\epsilon)^M] = \mathbf{1} \in G$, from which it follows[10.191] that $x_G \in B(\neg\,\epsilon)^N$, so $x_G \notin (B\epsilon)^N$.

For the reverse direction, suppose x is not in any meager (null) Borel set in N that is coded in M. Let $G = \{[(B\epsilon)^M] \mid \epsilon \in \mathcal{B}^M \wedge x \in (B\epsilon)^N\}$. Since $M \subseteq N$ and $\mathbb{R}^M \in N$, $\mathcal{P}(\mathbb{R})^M$, $\mathbf{Borel}^M$, $\mathfrak{B}^M$, $\mathfrak{L}^M$, etc. are in N, so category- and measure-theoretic notions as defined in M are also definable in N. Thus, in particular, $G \in N$. Since x is not in any meager (null) Borel set in N that is coded in M, G does not contain $\mathbf{0}$, and G is clearly closed upward and closed under (finite) meet, so G is a filter.

To show that G is M-generic, suppose $A \in M$ is dense in $\mathfrak{B}^M(\mathfrak{L}^M)$. Then $\bigvee A = \mathbf{1}$.

(10.196) **Claim** *There exists $B \subseteq A$, such that $B \in M$ and B is countable in M, such that $\bigvee B = \mathbf{1}$.*

Remark With AC this is easy to show. As we have noted above, for the present purpose we could be satisfied with this, as M may be assumed to model ZFC. Nevertheless, we will give a proof in ZF + DC, in fact, in ZF + $\mathsf{AC}_\omega(\mathbb{R})$.

Proof We argue in M. First consider the case of $\mathfrak{B}$. Let $\{a_n \mid m \in \omega\}$ be a countable dense set in $\mathfrak{B}$. Using $\mathsf{AC}_\omega(\mathbb{R})$ to choose Borel codes, for each $n \in \omega$, if there exists $b \in A$ such that $b \geqslant a_n$, let b_n be such an element. Let $B \subseteq A$ be the set of all such elements b_n. Then B is countable. We claim that $b = \bigvee B = \mathbf{1}$. For if not, let $a \in A$ be such that $a \leqslant \neg b$, and let $n \in \omega$ be such that $a_n \leqslant a$. Then a_n is nonzero and disjoint from b. Since $a_n \leqslant a$, b_n is defined, and $b_n \in B$, so $b_n \leqslant b$. But $a_n \leqslant b_n$, which is a contradiction.

Next we consider the case of $\mathfrak{L}$. Suppose toward a contradiction that for every countable $B \subseteq A$, $\bigvee B < \mathbf{1}$. We will use methods similar to those used in the proof of (5.161.2). For each $m \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, let $i_m = [(m, m+1)]$, the equivalence class in $\mathfrak{L}$ of the interval $(m, m+1)$. Using $\mathsf{AC}_\omega(\mathbb{R})$ with Borel codes, one can show that there exists $m \in \mathbb{Z}$ such that for every countable $B \subseteq A$, $\bigvee_{b \in B}(b \wedge i_m) < i_m$. It follows that for every countable $B \subseteq A$, $\mu\big(\bigvee_{b \in B}(b \wedge i_m)\big) < \mu(i_m) = 1$. Let μ_0 be the supremum over all countable $B \subseteq A$ of $\mu\big(\bigvee_{b \in B}(b \wedge i_m)\big)$. Using $\mathsf{AC}_\omega(\mathbb{R})$ again, it follows that $\mu_0 < 1$ (since countable sets of Borel sets can be coded by reals). Using $\mathsf{AC}_\omega(\mathbb{R})$ again, we show that there exists a countable $B \subseteq A$ such that $\mu\big(\bigvee_{b \in B}(b \wedge i_m)\big) = \mu_0$. Let $b_0 = \bigvee_{b \in B}(b \wedge i_m)$. Then $\mathbf{0} < b_0 < i_m$. Since A is dense, there exists $a \in A$ be such that $a \leqslant i_m \cap (\neg b_0)$. Let $B' = B \cup \{a\}$. Then B' is a countable subset of A and $\mu\big(\bigvee_{b \in B'}(b \wedge i_m)\big) = \mu_0 + \mu(a) > \mu_0$; contradiction. □[10.196]

Still arguing in M as in the proof of the claim, and using the claim, let B be a countable subset of A such that $\bigvee B = \mathbf{1}$. Using $\mathsf{AC}_\omega(\mathbb{R})$, for each $b \in B$ let ϵ_b be a $\mathcal{B}$-expression such that $[B(\epsilon_b)] = b$, and let $\epsilon = \bigvee_{b \in B} \epsilon_b$. Note that ϵ is a $\mathcal{B}$-expression, since B is countable, and $[B(\epsilon)] = \mathbf{1}$.

Now moving to N, let $X = \mathbb{R}^N \backslash B(\epsilon)^N$. X is meager (null), so $x \notin X$. Hence, $x \in B(\epsilon)^N = \bigcup_{b \in B} B(\epsilon_b)^N$, so there exists $b \in B$ such that $x \in B(\epsilon_b)^N$, whence $\big[B(\epsilon_b)^M\big] \in G$, so G meets A.

We have shown that G is M-generic. It is clear from the definition of G that $x = x_G$. □[10.195]

10.31.4 Solovay's theorem

(10.197) **Theorem** [GBC] *Under the conditions of (10.185), $N \models \ulcorner$for every $X \subseteq \mathbb{R}$*

1. *X has the Baire property;*
2. *X is Lebesgue measurable; and*
3. *X has the perfect set property.$\urcorner$*

Proof We will make use of the homeomorphism[5.77.2] of $\mathbb{R}\backslash\mathbb{Q}$ with ${}^\omega\omega$. Thus, reals are effectively ω-sequences of ordinals, and $\mathbb{R}^N = \mathbb{R}^{M[G]}$. Suppose $X \subseteq \mathbb{R}$ and $X \in N$. Then

$$X = \{x \in \mathbb{R}^{M[G]} \mid M[G] \models \Delta[s, x]\},$$

for some $s \in S$ (i.e., $s : \omega \to \mathrm{Ord}^M$).

Let $M' = M[s]$. Since[10.184] κ is inaccessible in M', there exist $\lambda < \kappa$ and $f \in M' \subseteq N$ such that $f : \lambda \overset{\text{sur}}{\to} ({}^\omega 2)^{M'}$. In $M[G]$ there exists $g : \omega \overset{\text{sur}}{\to} \lambda$. Note that $f \circ g : \omega \overset{\text{sur}}{\to} ({}^\omega 2)^{M'}$. By (10.186.1) $g \in N$, so $f \circ g \in N$.

(10.198) *Hence, $\mathbb{R}^{M'}$ is countable in N.*[30]

Let $\mathbb{C} = \mathbb{C}(\omega, {<}\kappa)$, and let '$\mathfrak{C}$' abbreviate '$\mathfrak{C}(\omega, {<}\kappa)$', so that for any transitive model M' of ZF containing $\mathbb{C}$, $\mathfrak{C}^{M'}$ is the regular algebra obtained from $\mathbb{C}$ in M'. (Since the elements of $\mathbb{C}$ are finite, $\mathbb{C}^{M'}$ is the same for any M' that contains κ.)

Suppose $x \in \mathbb{R}^N$. Since s and x may be coded together as a single ω-sequence of ordinals, the argument of the preceding paragraph applies to show that $M'[x] \neq N$, and (10.184) applies to show that there exists an $M'[x]$-generic filter H on $\mathbb{C}$ such that $M[G] = M'[x][H]$. Let

$$c = [\![\Delta(\check{s}, \check{x})]\!]^{\mathfrak{C}^{M'[x]}}.$$

$\check{s}$ and $\check{x}$ are invariant under all automorphisms of $\mathfrak{C}^{M'[x]}$ (since all that is necessary is that $\mathbf{1}$ be fixed), so c is invariant under automorphisms of $\mathfrak{C}^{M'[x]}$ in $M'[x]$.

(10.199) **Claim** $c = \mathbf{0}$ *or* $c = \mathbf{1}$.

Proof Suppose not. Let c' be any element of $\mathfrak{C}^{M'[x]}$ other than c, $\mathbf{0}$, or $\mathbf{1}$—e.g., $\neg\, c$. Then $\{\mathbf{0}, c, \neg\, c, \mathbf{1}\}$ and $\{\mathbf{0}, c', \neg\, c', \mathbf{1}\}$ are complete subalgebras of $\mathfrak{C}^{M'[x]}$, with an isomorphism π that takes c to c', so by (10.180) there is an automorphism of $\mathfrak{C}^{M'[x]}$ in $M'[x]$ that takes c to c', whence

$$c = [\![\Delta(\check{s}, \check{x})]\!]^{\mathfrak{C}^{M'[x]}} = c';$$

contradiction. $\square$10.199

Thus

$$\begin{aligned} x \in X &\leftrightarrow M[G] \models \Delta[s, x] \\ &\leftrightarrow M'[x][H] \models \Delta[s, x] \\ &\leftrightarrow c \in H \\ &\leftrightarrow c = \mathbf{1}. \end{aligned}$$

Equivalently,

(10.200) $$x \in X \leftrightarrow M'[x] \models \Delta^{\Vdash}[\mathbb{C}, 0, \check{s}, \check{x}],$$

where $\Delta^{\Vdash}$ is the s-formula that expresses the forcing relation for sentences derived by substitution of forcing terms for the (two) free variables of Δ, 0 is the empty condition in $\mathbb{C}$ (which is its maximum element), and $\check{s}, \check{x}$ are the canonical terms for s, x in $M'[x]^{\mathbb{C}}$.

(10.201) **Definition** [ZF] $D(\mathbb{P}, a, b) \overset{\text{def}}{\Longleftrightarrow} \Delta^{\Vdash}(\mathbb{P}, 0, \check{a}, \check{b})$.

Then

(10.202) $$x \in X \leftrightarrow M'[x] \models D[\mathbb{C}, s, x].$$

[30] Recall that $\mathbb{R}$, ${}^{\omega}2$, $\mathcal{P}\omega$, etc. are all equinumerous and for many purposes interchangeable interpretations of 'the reals'.

1 Let $\mathfrak{B} = \mathfrak{B}^{M'}$. Let $\dot{x} \in M'^{\mathfrak{B}}$ denote x_{G},[10.187] where G is the canonical term for the generic filter on $\mathfrak{B}$. Thus, for any M'-generic filter F on $\mathfrak{B}$, $\dot{x}^F = x_F$. Let

$$b = [\![D(\check{\mathbb{C}}, \check{s}, \dot{x})]\!]^{\mathfrak{B}}, \tag{10.203}$$

and let $\epsilon \in M'$ be a Borel code such that $(B\epsilon)^{M'} \in b$.

Now suppose F is an M'-generic filter on $\mathfrak{B}$ and let $x = x_F$. Then $\dot{x}^F = x$ and $M'[F] = M'[x]$, so

$$\begin{aligned} x \in X &\leftrightarrow M'[x] \models D(\mathbb{C}, s, x) \\ &\leftrightarrow M'[F] \models D(\mathbb{C}, s, x) \\ &\leftrightarrow b \in F \\ &\leftrightarrow x \in (B\epsilon)^N, \end{aligned}$$

where we have used successively (10.202), (10.203), and (10.192).

Borel expressions may be coded as reals, so the set of Borel codes in M' is countable in N.[10.198] Let A be defined in N as the set of reals that are in some meager Borel set coded in M'. Then $N \models \ulcorner [A]$ is meager, and[10.195] every real not in $[A]$ is Cohen over $M[[s]]\urcorner$. Let $B = (B\epsilon)^N$. Then $X \triangle B \subseteq A$, so $N \models \ulcorner[X]$ has the Baire property$\urcorner$.

2 The proof is entirely parallel to the preceding.

3 Given the background provided in this Note, the proof of the perfect set property given in the main text is adequately detailed. □[10.197]

10.32 The Hausdorff difference hierarchies

[Refer to p. 663.]

Definition [ZF] *Suppose α is a positive ordinal, and A is a decreasing α-sequence of sets, i.e., if $\beta \leqslant \gamma < \alpha$ then $A_\beta \supseteq A_\gamma$. $D_\alpha A \overset{\text{def}}{=}$ the set of x such that the least β such that either $x \notin A_\beta$ or $\beta = \alpha$ is odd, where an ordinal β is even or odd according as $\beta = 2 \cdot \gamma$ or $\beta = 2 \cdot \gamma + 1$, respectively.*

Thus

$$\begin{aligned} D_1\langle A_0\rangle &= A_0 \\ D_2\langle A_0, A_1\rangle &= A_0 \backslash A_1 \\ D_3\langle A_0, A_1, A_2\rangle &= (A_0 \backslash A_1) \cup A_2 \\ D_4\langle A_0, A_1, A_2, A_3\rangle &= (A_0 \backslash A_1) \cup (A_2 \backslash A_3) \\ &\vdots \\ D_{\omega+1}\langle A_0, \ldots, A_\omega\rangle &= \bigcup_{\gamma<\omega} (A_{2\cdot\gamma} \backslash A_{2\cdot\gamma+1}) \cup A_\omega \\ &\vdots \end{aligned}$$

In the following discussion we presume a fixed Polish topology $\mathcal{T}$ on a set $X = \bigcup \mathcal{T}$.

Given a class Γ of sets and $\alpha > 0$, $\alpha\text{-}\Gamma \overset{\text{def}}{=}$ the class of sets obtained by the application of D_α to decreasing α-sequences from Γ. Recall that for any class Γ of subsets of X, $\breve{\Gamma} = \neg\,\Gamma = \{X \backslash Y \mid Y \in \Gamma\}$ is the class dual to Γ. For convenience, we let $\breve{\alpha}\text{-}\Gamma \overset{\text{def}}{=} \alpha\text{-}\breve{\Gamma}$.

Definition [ZF] *Suppose* $0 < \xi < \omega_1$. *The* Hausdorff difference hierarchy on $\mathbf{\Pi}^0_\xi$ *consists of the classes* $\alpha\text{-}\mathbf{\Pi}^0_\xi$ *and* $\breve{\alpha}\text{-}\mathbf{\Pi}^0_\xi$, $0 < \alpha < \omega_1$.

Clearly, $\alpha\text{-}\mathbf{\Pi}^0_\xi$ is continuously closed.

In general, the difference hierarchy on Γ is not simply related to the one on $\neg\,\Gamma$, and our choice of $\mathbf{\Pi}^0_\xi$ over $\mathbf{\Sigma}^0_\xi$ as the base for the difference hierarchy is significant. In particular, the closure of $\mathbf{\Pi}^0_\xi$ under countable intersections is useful, as the following argument demonstrates.

Suppose $0 < \xi, \alpha < \omega_1$. First note that we can replace decreasing α-sequences of $\mathbf{\Pi}^0_\xi$ sets by arbitrary α-sequences as arguments of D_α by the following device. Given $A \in {}^\alpha(\mathbf{\Pi}^0_\xi)$, define $\tilde{A}$ by the condition that $\tilde{A}_\beta = \bigcap_{\gamma \leqslant \beta} A_\gamma$. Note that $\tilde{A} \in {}^\alpha(\mathbf{\Pi}^0_\xi)$, $\tilde{A}$ is decreasing, and if A is decreasing then $\tilde{A} = A$. Extend the definition of D_α by letting

$$D_\alpha A \overset{\text{def}}{=} D_\alpha \tilde{A} \tag{10.204}$$

for any $A \in {}^\alpha(\mathbf{\Pi}^0_\xi)$.

Let π be a fixed bijection of $\omega \times \alpha$ with ω, and for $x \in {}^\omega\omega$ and $\beta \in \alpha$, let $x_{(\beta)} \in {}^\omega\omega$ be defined by the condition: $\forall n \in \omega\,\big(x_{(\beta)}(n) = x(\pi\langle n, \beta\rangle)\big)$. Let $S \subseteq {}^\omega\omega \times {}^\omega\omega$ be a universal set for $\mathbf{\Pi}^0_\xi$, i.e., S is $\mathbf{\Pi}^0_\xi$, and for any $\mathbf{\Pi}^0_\xi$ set $X \subseteq {}^\omega\omega$, there exists $a \in {}^\omega\omega$ such that $X = S_a \overset{\text{def}}{=} \{x \in {}^\omega\omega \mid \langle a, x\rangle \in S\}$. Let $T = \big\{\langle a, x\rangle \,\big|\, a \in {}^\omega\omega \wedge x \in D_\alpha\langle S_{a_{(\beta)}} \mid \beta \in \alpha\rangle\big\}$, using the generalized difference operation (10.204).

T is clearly universal for $\alpha\text{-}\mathbf{\Pi}^0_\xi$. It follows that $\alpha\text{-}\mathbf{\Pi}^0_\xi \neq \breve{\alpha}\text{-}\mathbf{\Pi}^0_\xi$, by the usual argument. (Otherwise, let $a \in {}^\omega\omega$ be such that $T_a = \{x \in {}^\omega\omega \mid \langle x, x\rangle \notin T\}$. Then $\langle a, a\rangle \in T \leftrightarrow \langle a, a\rangle \notin T$.)

Next we show that

1. $\alpha\text{-}\mathbf{\Pi}^0_\xi \subseteq (\alpha+1)\text{-}\mathbf{\Pi}^0_\xi$; and
2. $\breve{\alpha}\text{-}\mathbf{\Pi}^0_\xi \subseteq (\alpha+1)\text{-}\mathbf{\Pi}^0_\xi$.

For the first claim, suppose $A \in {}^\alpha(\mathbf{\Pi}^0_\xi)$ is decreasing. Let $A' = A^\frown\langle 0\rangle$. Then $D_{\alpha+1} A' = D_\alpha A$.

To prove the second claim, suppose $A \in {}^\alpha(\mathbf{\Pi}^0_\xi)$ is decreasing. Define $A' \in {}^{\alpha+1}(\mathbf{\Pi}^0_\xi)$ as follows:

1. $A'_0 = X$ (the entire space under consideration).
2. For each successor $\beta \leqslant \alpha$, $A'_\beta = A_{\beta\text{-}}$.
3. For each limit $\beta \leqslant \alpha$, $A'_\beta = \bigcap_{\gamma < \beta} A_\gamma$.

(Note that we have again used the closure of $\mathbf{\Pi}^0_\xi$ under countable intersections. Then $D_{\alpha+1} A' = X \backslash D_\alpha A$.

Thus, the Hausdorff difference hierarchy over $\mathbf{\Pi}^0_\xi$ is a hierarchy in the familiar sense. The following theorem of Hausdorff and Kuratowski is analogous to Suslin's theorem 5.106, inasmuch as it states that the difference operations generate $\mathbf{\Delta}^0_{\xi+1}$ from $\mathbf{\Pi}^0_\xi$, as the operations of countable union and complementation generate $\mathbf{\Delta}^1_1$ from $\mathbf{\Pi}^0_1$. In the proof of the theorem we will have occasion to consider multiple topologies on a given set X, so we refer to the topology explicitly in the statement of the theorem.

(10.205) **Theorem** [ZF] *Suppose $\mathcal{T}$ is a Polish topology and $0 < \xi < \omega_1$. $\mathbf{\Delta}^0_{\xi+1}(\mathcal{T}) = \bigcup_{0<\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^0_\xi(\mathcal{T})$.*

Proof Given $0 < \alpha < \omega_1$ and $A \in {}^\alpha(\mathbf{\Pi}^0_\xi)$, $D_\alpha A$ is a countable union of differences of $\mathbf{\Pi}^0_\xi$ sets, each of which is $\mathbf{\Sigma}^0_{\xi+1}$, so $D_\alpha A$ is $\mathbf{\Sigma}^0_{\xi+1}$. By the same reasoning, $\breve{D}_\alpha A$ is $\mathbf{\Sigma}^0_{\xi+1}$, so $D_\alpha A$ is $\mathbf{\Pi}^0_{\xi+1}$. Hence, $\bigcup_{0<\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$. It remains to be shown that $\mathbf{\Delta}^0_{\xi+1} \subseteq \bigcup_{0<\alpha<\omega_1} \alpha\text{-}\mathbf{\Pi}^0_\xi$.

We will prove the theorem for $\xi = 1$. The general case may be derived from this case as described in[15, §22.E]. Briefly, if $A \in \mathbf{\Delta}^0_{\xi+1}(\mathcal{T})$, then there is a Polish topology $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathbf{\Sigma}^0_\xi(\mathcal{T})$ such that $A \in \mathbf{\Delta}^0_2(\mathcal{T}')$. Applying the theorem for $\xi = 1$ to $\mathcal{T}'$, for some $0 < \alpha < \omega_1$, $A \in \alpha\text{-}\mathbf{\Pi}^0_1(\mathcal{T}') \subseteq \alpha\text{-}\mathbf{\Pi}^0_\xi(\mathcal{T})$, since $\mathbf{\Pi}^0_1(\mathcal{T}') \subseteq \mathbf{\Pi}^0_\xi(\mathcal{T})$.

We now deal with a fixed Polish topology on a set X. Suppose $F \subseteq X$ is closed. We define the *boundary relative to* F of a set $A \subseteq X$ to be

$$\partial_F A \stackrel{\text{def}}{=} \overline{F \cap A} \cap \overline{F \backslash A}.$$

Note that $\partial_F A$ is closed and included in F.

Suppose $A \in \mathbf{\Delta}^0_2$. Define $\langle F_\gamma \mid \gamma \in \mathrm{Ord}\rangle$ recursively as follows:

1. $F_0 = X$.
2. $F_{\gamma+1} = \partial_{F_\gamma} A$.
3. $F_\gamma = \bigcap_{\delta<\gamma} F_\delta$, if $\gamma \in \mathrm{Lim}$.

Let η be least such that $F_{\eta+1} = F_\eta$. Since the topology is Polish, it has a countable base, and any strictly decreasing sequence of closed sets is countable. Hence $\eta < \omega_1$.

(10.206) **Claim** $F_\eta = 0$.

Proof Suppose toward a contradiction that $F_\eta \neq 0$. Then F_η with the relative topology is a Polish space, and $F_\eta = \overline{F_\eta \cap A} = \overline{F_\eta \backslash A}$. Thus, $F_\eta \cap A$ and $F_\eta \backslash A$ are disjoint G_δ subsets of F_η, both of which are dense in F_η, and both of which are therefore comeager, which is impossible. □10.206

Let $\alpha = 2 \cdot \eta$, and define a decreasing α-sequence $C = \langle C_\beta \mid \beta < \alpha\rangle$ of closed sets as follows. For $\gamma < \eta$,

1. $C_{2\cdot\gamma} = F_\gamma$; and
2. $C_{2\cdot\gamma+1} = \overline{F_\gamma \backslash A}$.

Then

$$D_\alpha C = \bigcup_{\gamma<\eta} (C_{2\cdot\gamma} \backslash C_{2\cdot\gamma+1}) = \bigcup_{\gamma<\eta} (F_\gamma \backslash \overline{F_\gamma \backslash A}).$$

We claim that this is A. To prove it, suppose $x \in A$. Let γ be such that $x \in F_\gamma \backslash F_{\gamma+1}$, which must exist, since $F_0 = X$, $F_\eta = 0$, and $\langle F_\gamma \mid \gamma \leqslant \eta\rangle$ is continuous at limits.

Then $x \in F_\gamma \cap A$ and $x \notin F_{\gamma+1} = \partial_{F_\gamma} A = \overline{F_\gamma \cap A} \cap \overline{F_\gamma \backslash A}$, so $x \notin \overline{F_\gamma \backslash A}$. Hence, $x \in C_{2\cdot\gamma} \backslash C_{2\cdot\gamma+1}$, so $x \in D_\alpha C$.

Conversely, suppose $x \in D_\alpha C$, say $x \in F_\gamma \backslash \overline{F_\gamma \backslash A}$. Then $x \in A$; otherwise, $x \in F_\gamma \backslash A$, so $x \notin F_\gamma \backslash \overline{F_\gamma \backslash A}$. $\square$[10.205]

10.33 Proof of (9.196)

[Refer to p. 679.]

(10.207) **Theorem** [ZF + AD] *Suppose $X \subseteq \omega_1$. Then X is constructible from a real, i.e., $X \in L[z]$ for some $z \in {}^\omega\omega$.*

Proof Let τ be a winning II-strategy in the game defined for X in the proof of (9.194). We can obviously use (9.195) to show that $X \cap \omega_1^{L[\tau]} \in L[\tau]$. Since $\omega_1^{L[\tau]} < \omega_1$, this is not enough, but we can achieve the desired result by the consideration of generic extensions of $L[\tau]$.

For each $\alpha \in \omega_1$ let $\mathbb{P}_\alpha$ be the partial order that collapses α to ω with finite conditions; specifically, let $|\mathbb{P}_\alpha| = {}^{<\omega}\alpha$, with $q \leqslant p \leftrightarrow p \subseteq q$. Note that $\mathbb{P}_\alpha \in L$, and if G is an L-generic filter on $\mathbb{P}_\alpha$ then $\bigcup G : \omega \overset{\text{sur}}{\to} \alpha$. Clearly, we may use this to define a canonical $\mathbb{P}_\alpha$-term $\dot{x}_\alpha$ such that if G is an L-generic filter on $\mathbb{P}_\alpha$ then $\dot{x}_\alpha^G \in \mathrm{WO}$ and $\|\dot{x}_\alpha^G\| = \alpha$.[31] Suppose G is an $L[\tau]$-generic filter on $\mathbb{P}_\alpha$. Let $x = \dot{x}_\alpha^G$. Then $x \in \mathrm{WO}$ and $\|x\| = \alpha$, so[9.195]

$$\begin{aligned}
\alpha \in X &\leftrightarrow \exists n \in \omega\ \|(\vec{\tau} x)^n\| = \|x\| \\
&\leftrightarrow L[\tau][G] \models \ulcorner \exists n \in \omega\ \|(\overrightarrow{[\tau]}[x])^n\| = \|[x]\| \urcorner \\
&\leftrightarrow \exists p \in G\ L[\tau] \models \ulcorner [p] \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner,
\end{aligned} \tag{10.208}$$

where

$$\psi_\alpha = \ulcorner \exists n \in \omega\ \|(\overrightarrow{(\check{\tau})}(\dot{x}_\alpha))^n\| = \|(\dot{x}_\alpha)\| \urcorner.$$

(10.209) **Claim** *For any $\alpha \in \omega_1$,*

$$\alpha \in X \leftrightarrow L[\tau] \models \ulcorner \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner.$$

Proof Since $\tau^\sharp$ exists[9.193] ω_1 is inaccessible in $L[\tau]$. It follows that for any $\alpha < \omega_1$, $\ulcorner \mathcal{P}[|\mathbb{P}_\alpha|] \urcorner^{L[\tau]}$ is countable, so for any $p \in |\mathbb{P}_\alpha|$ there exists an $L[\tau]$-generic filter G on $\mathbb{P}_\alpha$ with $p \in P$.

Suppose $\alpha \in \omega_1$. If $L[\tau] \models \ulcorner \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner$ then let G be any $L[\tau]$-generic filter on $\mathbb{P}_\alpha$. Then $0 \in G$ and $L[\tau] \models \ulcorner [0] \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner$, so[10.208] $\alpha \in X$.

Inversely, suppose $L[\tau] \not\models \ulcorner \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner$. Let $p_0 \in |\mathbb{P}_\alpha|$ be such that $L[\tau] \models \ulcorner [p_0] \Vdash^{\mathbb{P}_{[\alpha]}} \neg \psi_{[\alpha]} \urcorner$, and let G be an $L[\tau]$-generic filter on $\mathbb{P}_\alpha$ such that $p_0 \in G$. Then there is no $p \in G$ such that $L[\tau] \models \ulcorner [p] \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner$, so[10.208] $\alpha \notin X$. $\square$[10.209]

Thus,[10.209] X is definable over $L[\tau]$, so $X \in L[\tau]$. $\square$[10.207]

[31] Given G, let $r_0 = \bigcup G$, and let $r_1 = \{\langle m, n\rangle \in {}^2\omega \mid r_0(m) \leqslant r_0(n)\}$. r_1 is a prewellordering of ω of length α. Let r_2 be the subset of r_1 obtained by taking just the numerically least member of each level of r_1. r_2 is a wellordering of a subset of ω of length α. Let $r_3 : \omega \to 2$ code the characteristic function of r_2 as in (5.61.1), i.e.,

$$r_3(m) = 1 \leftrightarrow \vec{B}\, m \in r_2.$$

Let $\dot{x}_\alpha$ be the natural $\mathbb{P}_\alpha$ term denoting r_3 for any given G.

10.34 Proof of (9.197)

[Refer to p. 679.]

(10.210) **Theorem** [ZF + DC + AD] ω_2 *is measurable.*

Proof For each $x \in {}^\omega\omega$ let $fx = \ulcorner \lceil\omega_1\rceil^{+} \urcorner^{L[x]}$, the successor in the sense of $L[x]$ of (the real) ω_1. Since $x^\sharp$ exists, $fx < \omega_2$.[32] Note that fx depends only on the Turing degree of x (indeed, only on its constructibility degree). Let $\mathcal{F}$ be the cone ultrafilter on Turing degrees, and let $U \subseteq \mathcal{P}\omega_2$ by defined by the condition that $X \in U$ iff

$$\{[x] \mid fx \in X\} \in \mathcal{F},$$

where $[x]$ is the Turing degree of x. U is a countably complete ultrafilter since F is.

To show that U is nonprincipal, suppose $\alpha \in \omega_2$. Let $X \subseteq \omega_1$ code a wellordering of ω_1 of length α. Let[9.196] $a \in {}^\omega\omega$ be such that $X \in L[a]$. Then for any $x \geqslant_T a$, $\ulcorner \lceil\omega_1\rceil^{+} \urcorner^{L[x]} > \alpha$, so $\omega_2 \backslash \{\alpha\} \in U$.

It remains to be shown that U is ω_2-complete. Suppose $\langle X_\alpha \mid \alpha < \omega_1\rangle \in {}^{\omega_1}U$, where we may assume that $\alpha < \beta \rightarrow X_\alpha \supseteq X_\beta$ by virtue of the ω_1-completeness of U. We must show that $\bigcap_{\alpha<\omega_1} X_\alpha \in U$. Consider the following game:

> I and II play reals x and y in ${}^\omega\omega$ in the usual way. II wins iff either $x \notin \mathrm{WO}$ or $\forall z \geqslant_T y\ f(z) \in X_{\|x\|}$.

A $\mathbf{\Sigma}^1_1$-bounding argument shows that I does not have a winning strategy. For suppose σ is a winning I-strategy. Then for all $y \in {}^\omega\omega$, $\vec{\sigma}(y) \in \mathrm{WO}$. Hence, $\{x \in {}^\omega\omega \mid \exists y \in {}^\omega\omega\ x = \vec{\sigma}(y)\}$ is a $\Sigma^1_1(\sigma)$ subset of WO. Let[5.118] $\alpha \in \omega_1$ be such that $\forall y \in {}^\omega\omega\ \|\vec{\sigma}(y)\| < \alpha$. Since $X_\alpha \in U$ there exists $y \in {}^\omega\omega$ such that $\forall z \geqslant_T y\ f(z) \in X_\alpha$. Let $\beta = \|\vec{\sigma}(y)\|$. Then $\beta < \alpha$, so $\forall z \geqslant_T y\ f(z) \in X_\beta$, since $X_\beta \supseteq X_\alpha$. Thus, $\sigma * y$ is a win for II.

Therefore, let τ be a winning II-strategy. We will show that for all $x \in {}^\omega\omega$, if $x \geqslant_T \tau$ then $fx \in \bigcap_{\alpha<\omega_1} X_\alpha$. To this end, suppose $x \geqslant_T \tau$ and $\alpha < \omega_1$. Let $\mathbb{P}_\alpha$ be as in the proof of (9.196), let G be an $L[x]$-generic filter on $\mathbb{P}_\alpha$, and let $x' = \dot{x}^G_\alpha$. Then $x' \in \mathrm{WO}$ and $\|x'\| = \alpha$. G is easily coded by a real, and there exists $y \geqslant_T x$ that efficiently codes both x and G, so $L[y] = L[x][G]$. Since $L[x] \models \ulcorner\lceil\mathbb{P}_\alpha\rceil$ has the $\lceil\alpha\rceil^+$-chain condition$\urcorner$, $\mathbb{P}_\alpha$-forcing over $L[x]$ preserves cardinals (in $L[x]$) above α, so $fx = fy$. Let $y' = \vec{\tau}x'$. Then $y' \leqslant_T y$. Thus, since $\|x'\| = \alpha$ and τ is a winning II-strategy, $fy \in X_\alpha$; hence, $fx \in X_\alpha$.

U is therefore an ω_2-complete nontrivial ultrafilter over ω_2. □[10.210]

10.35 Proof of (9.198)

[Refer to p. 679.]

(10.211) **Theorem** [ZF] *Suppose* $X \subseteq \omega_1$ *and* $A = \{x \in {}^\omega\omega \mid x \in \mathrm{WO} \wedge \|x\| \in X\}$ *is* $\Sigma^1_2(z)$ *for some* $z \in {}^\omega\omega$. *Then* $X \in L[z]$.

[32] Otherwise, $L[x] \models \lceil\omega_2\rceil = \lceil\omega_1\rceil^+$. But then, since all (real) uncountable cardinals are indiscernible in $L[x]$,[9.193] $L[x] \models \lceil\omega_3\rceil = \lceil\omega_1\rceil^+$, which is patently absurd.

Proof For simplicity, suppose $z = 0$. The proof relativizes directly to arbitrary $z \in {}^{\omega}\omega$. Working in GB for convenience, let ϕ be a Σ^1_2 s^1-formula[5.2] with one free variable v_0 such that for all $x \in {}^{\omega}\omega$

$$x \in A \leftrightarrow V \models \phi[x].^{33}$$

If we wished only to show that $X \in L[a]$ for some $a \in {}^{\omega}\omega$, rather than $X \in L$, we could proceed as follows.

Suppose first that

$$\exists a \in {}^{\omega}\omega \; \omega_1^{L[a]} = \omega_1. \tag{10.212}$$

Then for any $\alpha \in \omega_1$

$$\begin{aligned} \alpha \in X &\leftrightarrow \exists x \in \mathrm{WO}^{L[a]} \, (\|x\| = \alpha \wedge V \models \phi[x]) \\ &\leftrightarrow \exists x \in \mathrm{WO}^{L[a]} \, (\|x\| = \alpha \wedge L[a] \models \phi[x]) \\ &\leftrightarrow L[a] \models \ulcorner \exists x \in \mathrm{WO} \, (\|x\| = [\alpha] \wedge (\phi)(x)) \urcorner, \end{aligned}$$

by virtue of the absoluteness of Σ^1_2 formulas between inner models.[6.7.2] Thus, X is definable over $L[a]$, so $X \in L[a]$.

Now suppose

$$\forall a \in {}^{\omega}\omega \; \omega_1^{L[a]} < \omega_1. \tag{10.213}$$

Then for all $a \in {}^{\omega}\omega$, ω_1 is a limit cardinal in $L[a]$. (If ω_1 is the successor of some cardinal κ in the sense of $L[a]$, let $b \in {}^{\omega}\omega$ code κ, so that κ is countable in $L[a, b]$. Let $c \in {}^{\omega}\omega$ encode both a and b. Then $\omega_1^{L[c]} = \omega_1$.)

We now proceed as in the proof of (9.196), using the partial orders $\mathbb{P}_\alpha$ and canonical ordinal codes $\dot{x}_\alpha$. Suppose G is an L-generic filter on $\mathbb{P}_\alpha$. Let $x = \dot{x}_\alpha^G$. Then $x \in \mathrm{WO}$ and $\|x\| = \alpha$, so (again using Σ^1_2-absoluteness)

$$\begin{aligned} \alpha \in X &\leftrightarrow V \models \phi[x] \\ &\leftrightarrow L[G] \models \phi[x] \\ &\leftrightarrow \exists p \in G \; L \models \ulcorner [p] \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner, \end{aligned}$$

where

$$\psi_\alpha = \ulcorner (\phi)\big((\dot{x}_\alpha)\big) \urcorner.$$

The following claim is analogous to (10.209).

(10.214) **Claim** *For any* $\alpha \in \omega_1$

$$\alpha \in X \leftrightarrow L \models \ulcorner \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]}.$$

Proof For any $\alpha \in \omega_1$, letting $a \in {}^{\omega}\omega$ be such that $\alpha < \omega_1^{L[a]}$,

$$L[a] \models \ulcorner |\mathcal{P}(|\mathbb{P}_{[\alpha]}|)| = \omega_1 \urcorner.$$

---

[33] Since the quantifiers in ϕ are restricted to $V_{\omega+1}$ (specifically to ${}^{\omega}\omega$ and ω in the case of type-1 and type-0 quantifiers, respectively, we could replace V by $V_{\omega+1}$ here, but we want to follow the format of Theorem 6.7.

Since $\omega_1^{L[a]} < \omega_1$ by hypothesis, $\mathcal{P}|\mathbb{P}_\alpha| \cap L[a]$ is countable, so $\mathcal{P}|\mathbb{P}_\alpha| \cap L$ is countable, and for every $p \in |\mathbb{P}_\alpha|$ there exists an L-generic filter G on $\mathbb{P}_\alpha$ with $p \in G$.

Suppose $\alpha \in \omega_1$. If $L \models \ulcorner \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \psi_{\lceil \alpha \rceil} \urcorner$, let G be any L-generic filter on $\mathbb{P}_\alpha$. Then $0 \in G$ and $L \models \ulcorner \lceil 0 \rceil \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \psi_{\lceil \alpha \rceil} \urcorner$, so $\alpha \in X$.

Inversely, suppose $L \not\models \ulcorner \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \psi_{\lceil \alpha \rceil} \urcorner$. Let $p_0 \in |\mathbb{P}_\alpha|$ be such that $L \models \ulcorner \lceil p_0 \rceil \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \neg \psi_{\lceil \alpha \rceil} \urcorner$, and let G be an L-generic filter on $\mathbb{P}_\alpha$ such that $p_0 \in G$. Then there is no $p \in G$ such that $L \models \ulcorner \lceil p \rceil \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \psi_{\lceil \alpha \rceil} \urcorner$, so $\alpha \notin X$. □[10.214]

Thus,[10.214] X is definable over L, so in this case[10.215] we have the desired result that $X \in L$. In the first case,[10.216] we have only shown that $X \in L[a]$ for any $a \in {}^\omega\omega$ such that $\omega_1^{L[a]} = \omega_1$. One way to get the optimum result in general is to argue in a generic extension of V in which ω_1 is collapsed, i.e., to show that the following statement is a theorem of GB:

(10.215) $\ulcorner$*Suppose M is an inner model of* ZF, $\mathbb{P} \in M$ *is a partial order, H is a M-generic filter on $\mathbb{P}$, $V = M[H]$, and $\omega_1^M < \omega_1$. Suppose $X \subseteq \omega_1^M$, $X \in M$, and $M \models \ulcorner \{x \in {}^\omega\omega \mid x \in \mathrm{WO} \wedge \|x\| \in \lceil X \rceil\}$ is $\Sigma_2^1 \urcorner$. Then $X \in L$.*$\urcorner$

The fact that this is a theorem of GB allows us to infer as usual that for any partial order $\mathbb{P}$ such that $\Vdash^{\mathbb{P}} \ulcorner \lceil \check{\omega}_1 \rceil < \omega_1 \urcorner$, $\Vdash^{\mathbb{P}} \ulcorner \lceil \check{X} \rceil \in L \urcorner$. Since such partial orders exist, and for any partial order $\mathbb{P}$, if $\Vdash^{\mathbb{P}} \ulcorner \lceil \check{X} \rceil \in L \urcorner$ then $X \in L$, it follows that $X \in L$.

Here is a GB-proof of (10.215):

$\ulcorner$Let $A = \ulcorner \{x \in {}^\omega\omega \mid x \in \mathrm{WO} \wedge \|x\| \in \lceil X \rceil\} \urcorner^M$, and let ϕ be Σ_2^1 such that for all $x \in {}^\omega\omega \cap M$

$$x \in A \leftrightarrow M \models \phi[x].$$

Let

$$\theta = \ulcorner y \in \mathrm{WO} \wedge \exists x \in \mathrm{WO}\, (\lceil \phi \rceil(x) \wedge \|x\| = \|y\|) \urcorner.$$

Note that θ is Σ_2^1 (with one free variable 'y').

Suppose $\alpha < \omega_1^M$. Let $a \in \mathrm{WO} \cap M$ be such that $\|a\| = \alpha$. Then by Σ_2^1 absoluteness

$$\begin{aligned} \alpha \in X &\leftrightarrow M \models \theta[a] \\ &\leftrightarrow V \models \theta[a]. \end{aligned}$$

Let G be an L-generic filter on $\mathbb{P}_\alpha$, and let $b \in \mathrm{WO} \cap L[G]$ be such that $\|b\| = \alpha$ (e.g., $b = \dot{x}_\alpha^G$). Since $\|a\| = \alpha = \|b\|$,

$$V \models \theta[a] \leftrightarrow V \models \theta[b],$$

and by Σ_2^1-absoluteness

$$V \models \theta[b] \leftrightarrow L[G] \models \theta[b].$$

Hence,

$$\begin{aligned} \alpha \in X &\leftrightarrow L[G] \models \theta[\dot{x}_\alpha^G] \\ &\leftrightarrow \exists p \in G\ L \models \ulcorner \lceil p \rceil \Vdash^{\mathbb{P}_{\lceil \alpha \rceil}} \psi_{\lceil \alpha \rceil} \urcorner, \end{aligned}$$

where

$$\psi_\alpha = \ulcorner \lceil \theta \rceil \big(\lceil \dot{x}_\alpha \rceil \big).$$

The following claim is analogous to (10.214)

(10.216) **Claim** *For any $\alpha \in \omega_1^M$*

$$\alpha \in X \leftrightarrow L \models \ulcorner \Vdash^{\mathbb{P}_{[\alpha]}} \psi_{[\alpha]} \urcorner.$$

Proof ω_1^M is a cardinal in L, and for $\alpha < \omega_1^M$, every constructible subset of $|\mathbb{P}_\alpha|$ is constructed before ω_1^M, so $\mathcal{P}\,|\mathbb{P}_\alpha|$ is countable. Hence for any $p \in |\mathbb{P}_\alpha|$, there is an L-generic filter G on $\mathbb{P}_\alpha$ with $p \in G$. This fact permits the proof of the claim to go through exactly as before. $\square_{10.216}$

Hence $X \in L$.$\urcorner$

This concludes the proof of (10.215), from which it follows, as discussed above, that $X \in L$. $\square_{10.211}$

10.36 Proof of (9.228)

[Refer to p. 692.]

(10.217) **Theorem** [GBC] *Suppose X is a set, κ is an uncountable cardinal, and Z is an infinite set.*

1. *Suppose $B \subseteq {}^{\omega}(X \times \omega)$ is κ-homogeneous with support Z. Then $\mathfrak{p}^{\cdot}\, B$ is κ-weakly homogeneous with support Z.*
2. *Suppose $B \subseteq {}^{\omega}(X \times \omega)$ is κ-homogeneously Z-Suslin. Then $\mathfrak{p}^{\cdot}\, B$ is κ-weakly homogeneously Z-Suslin.*

Proof 1 Immediate from the definitions.

2 Let $\bar{U}$ be a κ-complete homogeneity system for a sequence tree T on $(X \times \omega) \times Z$ such that $B = \mathfrak{p}^{\cdot}[T]$. Let $T' = \{\langle s, \langle t, u\rangle^{\cdot}\rangle^{\cdot} \mid \langle\langle s, t\rangle^{\cdot}, u\rangle^{\cdot} \in T\}$, so T' is a sequence tree on $X \times (\omega \times Z)$, and $\mathfrak{p}^{\cdot}\, B = \mathfrak{p}^{\cdot}[T']$. Define $\bar{U}' : {}^{<\omega}(X \times \omega) \to \mathrm{ms}(\omega \times Z)$ so that for any $n \in \omega$, $\langle s, t\rangle^{\cdot} \in {}^{n}(X \times \omega)$, and $W \subseteq {}^{n}(\omega \times Z)$

$$W \in \bar{U}'_{\langle s,t\rangle^{\cdot}} \leftrightarrow \{u \in {}^{n}Z \mid \langle t, u\rangle^{\cdot} \in W\} \in \bar{U}_{\langle s,t\rangle^{\cdot}}. \tag{10.218}$$

(10.219) **Claim**

1. *$\bar{U}'$ is a κ-complete homogeneity system.*
2. *For any $\langle x, y\rangle^{\cdot} \in {}^{\omega}(X \times \omega)$, $\bar{U}^{\langle x,y\rangle^{\cdot}}$ is countably complete iff $\bar{U}'^{\langle x,y\rangle^{\cdot}}$ is countably complete.*

Proof 1 We must verify (9.213.1) with $\bar{U}'$ for $\bar{U}$, $X \times \omega$ for X, and $\omega \times Z$ for Z. The first two clauses are satisfied automatically. To justify Condition 9.213.1.3, suppose $n_0 \leqslant n_1 < \omega$, $\langle s_0, t_0\rangle^{\cdot} \in {}^{n_0}(X \times \omega)$, $\langle s_1, t_1\rangle^{\cdot} \in {}^{n_1}(X \times \omega)$, $\langle s_0, t_0\rangle^{\cdot} \subseteq \langle s_1, t_1\rangle^{\cdot}$, and $W_0 \in \bar{U}'_{\langle s_0,t_0\rangle^{\cdot}}$. Let

$$W = \{\langle t, u\rangle^{\cdot} \in {}^{n_1}(\omega \times Z) \mid \langle t, u\rangle^{\cdot} \restriction n_0 \in W_0\}.$$

We must show that $W \in \bar{U}'_{\langle s_1,t_1\rangle^{\cdot}}$.[9.212.1]

Let $Y_0 = \{u \in {}^{n_0}Z \mid \langle t_0, u\rangle^{\cdot} \in W_0\}$. Then $Y_0 \in \bar{U}_{\langle s_0,t_0\rangle^{\cdot}}$. Let $Y_1 = \{u \in {}^{n_1}Z \mid u \restriction n_0 \in Y_0\}$. Since $\bar{U}$ is a homogeneity system, $\bar{U}_{\langle s_1,t_1\rangle^{\cdot}}$ projects to $\bar{U}_{\langle s_0,t_0\rangle^{\cdot}}$, so $Y_1 \in \bar{U}_{\langle s_1,t_1\rangle^{\cdot}}$. Let $W_1 = \{\langle t_1, u\rangle^{\cdot} \mid u \in Y_1\}$. Then $W_1 \in \bar{U}'_{\langle s_1,t_1\rangle^{\cdot}}$. Clearly, $W_1 \subseteq W$, so $W \in \bar{U}'_{\langle s_1,t_1\rangle^{\cdot}}$. $\square^{10.219.1}$

2 Suppose $\bar{U}^{\langle x,y\rangle^{\cdot}}$ is countably complete, and suppose for each $n \in \omega$, $W_n \in \bar{U}'_{\langle x \restriction n, y \restriction n\rangle^{\cdot}}$. For each $n \in \omega$ let $Z_n = \{u \mid \langle y \restriction n, u\rangle^{\cdot} \in W_n\}$. Then $Z_n \in \bar{U}_{\langle x \restriction n, y \restriction n\rangle^{\cdot}}$, so there exists $z \in {}^{\omega}Z$ such that $\forall n \in \omega\ z \restriction n \in Z_n$. It follows that $\forall n \in \omega\ \langle y, z\rangle^{\cdot} \restriction n \in W_n$. Hence, $\bar{U}'^{\langle x,y\rangle^{\cdot}}$ is countably complete. Conversely, suppose $\bar{U}'^{\langle x,y\rangle^{\cdot}}$ is countably complete, and suppose for each $n \in \omega$, $Z_n \in \bar{U}_{\langle x \restriction n, y \restriction n\rangle^{\cdot}}$. For each $n \in \omega$ let $W_n = \{\langle u, y \restriction n\rangle^{\cdot} \mid u \in Z_n\}$. Then $W_n \in \bar{U}'_{\langle x \restriction n, y \restriction n\rangle^{\cdot}}$, so there exists $\langle y', z\rangle^{\cdot} \in {}^{\omega}(\omega \times Z)$ such that $\forall n \in \omega\ \langle y', z\rangle^{\cdot} \restriction n \in W_n$. Necessarily, $y' = y$, so it follows that $\forall n \in \omega\ z \restriction n \in Z_n$. Hence, $\bar{U}^{\langle x,y\rangle^{\cdot}}$ is countably complete. $\square^{10.219}$

To show that $\bar{U}'$ is a κ-complete weak homogeneity system for T', we must verify (9.227.2) with T' for T and $\bar{U}'$ for $\bar{U}$

Suppose $\langle s, t\rangle^{\cdot} \in {}^{<\omega}(X \times \omega)$. Then

$$\{u \mid \langle t, u\rangle^{\cdot} \in T'_s\} = \{u \mid \langle\langle s,t\rangle^{\cdot}, u\rangle^{\cdot} \in T\} = T_{\langle s,t\rangle^{\cdot}} \in \bar{U}_{\langle s,t\rangle^{\cdot}},$$

so[10.218] $T'_s \in \bar{U}'_{\langle s,t\rangle^{\cdot}}$. In fact, $\{\langle t', u\rangle^{\cdot} \in T'_s \mid t' = t\} \in \bar{U}'_{\langle s,t\rangle^{\cdot}}$. Thus, Condition 9.227.2.1 is satisfied with $\bar{U}'$ for $\bar{U}$ and T' for T.

It only remains to show (9.227.2.2). By the definition of T' and using (10.219.2), we see that for any $x \in {}^{\omega}X$, $x \in \mathfrak{p}^{\cdot}[T']$ iff there exists $y \in {}^{\omega}\omega$ such that $\langle x, y\rangle^{\cdot} \in \mathfrak{p}^{\cdot}[T]$ iff there exists $y \in {}^{\omega}\omega$ such that $\bar{U}^{\langle x,y\rangle^{\cdot}}$ is countably complete iff there exists $y \in {}^{\omega}\omega$ such that $\bar{U}'^{\langle x,y\rangle^{\cdot}}$ is countably complete. Hence, (9.227.2.2) is satisfied with T' for T and $\bar{U}'$ for $\bar{U}$.

Since Z is infinite, $|\omega \times Z| = |Z|$; hence, $\mathfrak{p}^{\cdot} B$ is κ-weakly homogeneously Z-Suslin. $\square^{10.217}$

10.37 Proof of (9.230)

[Refer to p. 692.]

(10.220) **Theorem** [ZFC] *Suppose X is countable, T is a tree on $X \times Z$, and $\kappa > \omega$. Then T is κ-weakly homogeneous iff there exists a countable set $\mathcal{U} \subseteq \mathrm{ms}_\kappa Z$ such that for all $x \in {}^{\omega}X$, if $x \in \mathfrak{p}^{\cdot}[T]$ then there is a countably complete tower $\langle U_n \mid n \in \omega\rangle \in {}^{\omega}\mathcal{U}$ such that $\forall n \in \omega\ T_{x \restriction n} \in U_n$.*

Proof Clearly, if T is κ-weakly homogeneous via a homogeneity system $\bar{U} : {}^{<\omega}(X \times \omega) \to \mathrm{ms}_\kappa Z$, then we may let $\mathcal{U} = \mathrm{im}\,\bar{U}$. Conversely, suppose $\mathcal{U}$ is as specified. We may assume that $\mathcal{U}$ is closed under projection, i.e., for all $U \in \mathcal{U}$, if $\dim U = n$ and $m \leqslant n$ then there exists $U' \in \mathcal{U}$ such that $\dim U' = m$ and U projects to U'. To achieve this, for each $U \in \mathcal{U}$ and each $m < \dim U = n$, we add to $\mathcal{U}$ the ultrafilter $\{W \subseteq {}^{<\omega}Z \mid \exists W' \in U\ \forall u \in W' \cap {}^{n}Z\ u \restriction m \in W\}$, which preserves the countability of $\mathcal{U}$.

We now define a homogeneity system $\bar{U} : {}^{<\omega}(X \times \omega) \to \mathrm{ms}_\kappa Z$ by recursion. $\bar{U}_0 = \bar{U}_{\langle 0,0\rangle^{\cdot}}$ is of course $\{1\}$. Suppose $n \in \omega$ and $\bar{U}_{\langle s,t\rangle^{\cdot}}$ has been defined for all

$\langle s,t\rangle^{\cdot} \in {}^{n}(X\times\omega)$. For each $\langle s,t\rangle^{\cdot} \in {}^{n}(X\times\omega)$ and $i\in\omega$, let $\bar{U}_{\langle s^\frown\langle i\rangle, t^\frown\langle e\rangle\rangle^{\cdot}}$ be chosen (for $e\in\omega$) so that $\langle \bar{U}_{\langle s^\frown\langle i\rangle, t^\frown\langle e\rangle\rangle^{\cdot}} \mid e\in\omega\rangle$ enumerates the ultrafilters $U\in\mathcal{U}$ such that

1. $T_{s^\frown\langle i\rangle}\in U$; and
2. U projects to $\bar{U}_{\langle s,t\rangle^{\cdot}}$.

It is straightforward to show that T is κ-weakly homogeneous via $\bar{U}$. $\square$[10.220]

Notation

$\overset{\text{def}}{=}$	equals, by definition	5
$\overset{\text{def}}{\Longleftrightarrow}$	if and only if, by definition	5
$\frown$	concatenation	6
$\underline{\cdots}$	use of underline to create a name for a typographical expression	7
$\neg, \vee, \wedge, \rightarrow, \leftrightarrow$	propositional connectives	12
$\exists, \forall$	quantifiers	13
$\tilde{X}$	specification operation corresponding to index X	19
n^{-}	$n-1$ if n is a successor ordinal (typically finite)	27
$\lvert\mathfrak{S}\rvert$	universe of structure $\mathfrak{S}$	30
Δ, Π, Φ	classes of domain, predicate, and operation indices of a signature	30
v_n	nth standard variable	34
$\mathcal{V}, \mathcal{P}, \mathcal{O}, \mathcal{C}, \mathcal{Q}$	classes of variables, predicate and operation indices, propositional connectives, and quantifier phrases of a language	34
$\hat{\varsigma}$	expression-forming operation corresponding to grammatical sign ς	34
$\mathcal{E}$	class of expressions of a language	37
$\mathcal{T}, \mathcal{F}$	classes of terms and formulas of a language	39
$\mathcal{L}^{\rho}$	the standard ρ-language	39
$\Delta\,\epsilon$	diagram of expression ϵ	41
$\left\{{p \atop \eta}\right\}$	substitution of expression η at place p	44
$\binom{v}{\tau}$	substitution of term τ for variable v	44
$\ulcorner \ldots \urcorner$	flexible quoting convention	45
(ϵ)	insertion of expression ϵ in quoted text	45
$\left[{v \atop a}\right]$	assignment of element a to variable v	48
$\mathrm{Val}^{\mathfrak{S}}\,\tau[A]$	value of term τ in structure $\mathfrak{S}$ at assignment A	49
$\mathfrak{S} \models \phi[A]$	structure $\mathfrak{S}$ satisfies formula ϕ at assignment A	50
$\mathrm{Th}\,\mathfrak{S}$	theory of (a satisfactory structure) $\mathfrak{S}$	55
$\Theta \vdash \sigma$	theory Θ proves sentence σ	64
$\Sigma \Rightarrow \sigma$	sequent with antecedent Σ and succedent σ	73
$\dfrac{\mathrm{A}\ \ \mathrm{B}\ \ \cdots}{\mathrm{Z}}$	inference rule justifying sequent Z from sequents A B $\cdots$	73
$\bar{\forall}\phi$	universal closure of formula ϕ	94

$\phi \overset{\mathfrak{A}}{\equiv} \psi$	formulas ϕ, ψ are equivalent over structure $\mathfrak{A}$	95
$\phi \overset{\Theta}{\equiv} \psi$	formulas ϕ, ψ are equivalent modulo theory Θ	95
$\bigvee \Phi$ ($\bigwedge \Phi$)	disjunction (conjunction) of the formulas in the finite set Φ	97
ρ^{Θ}	signature of a theory Θ	102
$\overline{\Theta}$	deductive closure theory Θ	103
$\Theta \vert \rho$	restriction of theory Θ to signature ρ	103
ρ^{+}, Θ^{+}	expansion of signature ρ, extension of theory Θ, by definition	111
$\Gamma \Rightarrow \Delta$	(Gentzen) sequent with antecedent Γ and succedent Δ	140
$\mathfrak{A}' \prec^{\Phi} \mathfrak{A}$	$\mathfrak{A}'$ is a Φ-elementary substructure of $\mathfrak{A}$	146
$\prod_{x \in X} \mathfrak{A}_x / U$	ultraproduct of structure-valued function $\langle \mathfrak{A}_x \mid x \in X \rangle$ mod ultrafilter U on $\mathcal{P} X$	152
${}^{X}\mathfrak{A}/U$	ultrapower of structure $\mathfrak{A}$ mod ultrafilter U on $\mathcal{P} X$	152
$\{u \mid \phi\}$	the class of elements u such that ϕ	176, 183
0	the empty set	177
(x, y)	the ordered pair of elements x and y	178
$x \cup y$, $x \cap y$	union, intersection, of classes x and y	184
$x \backslash y$	difference of classes x and y	184
$\bigcup x$, $\bigcap x$	union, intersection, of the members of class x	184
$R \restriction X$	restriction of prefunction or binary relation[3.63] R to class X	185
$R^{\to} X$, $R^{\leftarrow} X$	image, inverse image, of class X by prefunction or binary relation[3.63] R	185
R^{-1}	inverse of prefunction or binary relation[3.63] R	185
$F x$, $F(x)$, F_x	the value of function F at x	185
$F : X \to Y$	F is a function from X to Y	187
$F : X \rightharpoonup Y$	F is a partial function from X to Y	187
$\xrightarrow{\text{inj}}$, $\xrightarrow{\text{sur}}$, $\xrightarrow{\text{bij}}$	injection, surjection, bijection	187
${}^{Y}X$	the class of functions from set Y to class X	187
$\langle \tau \mid \phi \rangle_u$	function $\{(u, \tau(u)) \mid \phi(u)\}$	188
$[\tau \mid \phi]_u$	indexed family A with domain $\{u \mid \phi\}$ such that $A_{[u]} = \tau(u)$	188
ω	the class of finite ordinals	190
$\left\langle \begin{smallmatrix} a_0 & \cdots & a_{n^-} \\ b_0 & \cdots & b_{n^-} \end{smallmatrix} \right\rangle$	function mapping a_0 to b_0, $\ldots$, a_{n^-} to b_{n^-}	193
$\langle x_0, \ldots, x_{n^-} \rangle$	n-sequence	193
$[x_0, \ldots, x_{n^-}]$	n-indexed family	193
$(x_0, \ldots, x_{n^-})$	ordered n-tuple	193
$\times$, $\bigtimes$	cartesian product via finite sequences, or functions generally	195
$\dot{\times}$, $\dot{\bigtimes}$	cartesian product via ordered n-tuples	195
$R(x_0, \ldots, x_{n^-})$	$\langle x_0, \ldots, x_{n^-} \rangle \in R$ (for n-ary relation R)	195
$F(x_0, \ldots, x_{n^-})$	$F\langle x_0, \ldots, x_{n^-} \rangle$ (for n-ary function F)	195
$x \, R \, y$	$\langle x, y \rangle \in R$ (for binary relation R)	196
$[x]_R$	R-equivalence class of x	197

$\equiv^R$	equivalence as far as relation R is concerned	197
X/E	quotient of class, relation, or function X modulo equivalence relation E	197
$=^X$	the identity relation on class X	198
$\alpha+\beta$, $\alpha\cdot\beta$, α^β	ordinal arithmetic	214
$x \sim y$	sets x, y are equipollent (of equal size)	216
$\lvert x\rvert$	cardinality of set x	217
$x \prec y$	set x is smaller than set y	217
$a \prec b$	cardinality a is smaller than cardinality b	217
α^+	least cardinal $> \alpha$	227
ω_α or $\aleph_\alpha$	the αth infinite cardinal	227
$f : x \overset{\text{cof}}{\to} X$	$\operatorname{im} f$ is cofinal in (ordered) set X	227
$\operatorname{cf} R$	cofinality of total order R	227
$a+b$, $a\cdot b$, a^b	cardinal arithmetic	229
$\sum_{i\in I}\kappa_i, \prod_{i\in I}\kappa_i$	cardinal sum, product	229
$x \vee y$, $x \wedge y$	join, meet, of elements x, y of a semilattice	234
$\neg\, x$	complement of element x of a lattice or boolean algebra	235
$\lceil x\rceil$	$\{y \in \lvert\mathfrak{A}\rvert \mid y \leqslant x\}$, for boolean algebra $\mathfrak{A}$, which is the (principal) ideal of x	237
$\lfloor x\rfloor$	$\{y \in \lvert\mathfrak{A}\rvert \mid y \geqslant x\}$, for boolean algebra $\mathfrak{A}$, which is the (principal) filter of x	238
$\Delta_{\alpha\in\gamma}A_\alpha$	the diagonal intersection of γ-sequence of sets $A_\alpha \subseteq \gamma$	238
H^+	complement of ideal H or ideal H^* dual to filter H	240
H^{+*}	dual ideal (filter) of filter (ideal) H	240
T_s, $T_{(s)}$, $T\lvert n$	certain fragments of sequence tree T	243
$[T]$	the set of infinite branches of sequence tree T	243
A^o, A^c, ∂A	the interior, closure, boundary, of pointset A	246
Δ_0^ρ, Σ_0^ρ, Π_0^ρ	the class of bounded ρ-formulas	265
Σ_n^ρ, Π_n^ρ	the nth levels of the Levy hierarchy	265
Σ_n^T, Π_n^T, Δ_n^T	the nth levels of the Levy hierarchy relative to theory T extending S^+	265
$\mathcal{D}$, $\mathcal{R}$	the Turing degrees, r.e. Turing degrees	306
$x \mid n$, $U_\mathfrak{s}^n$	the pointwise restriction of point x, pointspace $U_\mathfrak{s}$ to length n	321
Σ_1^0, Π_1^0, Δ_1^0	the semirecursive, co-semirecursive, recursive, pointclasses	322
$\neg\,\Gamma$, $\vee\,\Gamma$, etc.	the result of applying an operation to the members of (recursively closed) pointclass Γ	325
$\breve{\Gamma}$	$\neg\,\Gamma$, the dual of pointclass Γ	325
Σ_n^i, Π_n^i, Δ_n^i	the Kleene pointclasses	326
$\utilde{\Gamma}$	the (full) relativization of pointclass Γ	335
$F^{\cdot}\langle f_0, \ldots, f_{n^-}\rangle$	the pointwise application of function F to sequence $\langle f_0, \ldots, f_{n^-}\rangle$ of argument-valued functions	340
$\langle f_0, \ldots, f_{n^-}\rangle^{\cdot}$	function of sequences formed from sequence of functions	340

$\sqcup$, $\bigsqcup$	disjoint union	392
$x * y$	composition of plays in a game	399
$\sigma * y$, $x * \tau$	composition of strategy and play in a game	399
$\sigma * \tau$	composition of strategies in a game	399
z^{I} z^{II}	I's, II's, contribution to play z of a game	399
$s^\frown T$	the concatenation of sequence s with set T of sequences	401
$<^L$	the canonical wellordering of L	438
$\lceil X \rceil$, $\lfloor X \rfloor$	downward, upward, closure of subset X of a partial order	476
$p \parallel q$, $p \perp q$	elements p, q of a partial order are compatible, incompatible	476
$X^\perp$, $\overline{X}$	the complement, completion, of a subset X of a partial order	476
$\mathbf{1}$	maximum element of a partial order (for forcing purposes)	478
$M^{\mathbb{P}}_\alpha$	the initial segment of length α of the class $M^{\mathbb{P}}$ of $\mathbb{P}$-forcing terms for transitive model M	478
$\check{x}$	the canonical forcing term to denote element x of the ground model	478, 496
$\Vdash^*$	extrinsically defined forcing relation	479
$p \Vdash^{M,\mathbb{P}} \sigma$	condition $p \in \lvert\mathbb{P}\rvert$ forces sentence $\sigma \in \mathcal{L}^{M,\mathbb{P}}$	485
$p \mid \sigma$	forcing condition p decides sentence σ	487
$\phi^{\Vdash}$	s-formula that expresses $\ulcorner \cdot \Vdash \phi(\cdot) \urcorner$	487
$\overline{\mathbb{P}}$	the canonical separative quotient of partial order $\mathbb{P}$	493
$\mathfrak{A}^+$	the partial order of nonzero elements of boolean algebra $\mathfrak{A}$	494
$M^{\mathfrak{A}}_\alpha$	the initial segment of length α of the $\mathfrak{A}$-valued universe $M^{\mathfrak{A}}$ for transitive model M	496
$[\![\sigma]\!]^{M,\mathfrak{A}}$	the boolean value of sentence $\sigma \in \mathcal{L}^{M,\mathfrak{A}}$	496
$\phi^{[\![\,]\!]}$	s-formula that expresses $\ulcorner [\![\phi(\cdot)]\!] = \cdot \urcorner$	500
$\bar{x}, \hat{x}$	the regularization of a forcing term $x \in M^{\mathbb{P}}$, the corresponding element of $M^{\mathfrak{R}\,\mathbb{P}}$	501
$\mathbb{P} * \dot{\mathbb{Q}}$	forcing iteration with $\mathbb{P}$ followed by $\dot{\mathbb{Q}}$	549
$\mathsf{Q}^U x$	for almost all x in the sense of ultrafilter U	599
$\mathcal{L}^\rho_{\kappa\lambda}$	infinitary language with parameters κ, λ	602
$\mathcal{L}^\rho_\kappa$	infinitary propositional language with parameter κ	603
$[X]^\gamma$	set of subsets of X with size (or order type) γ	608
$\alpha \to (\beta)^\nu_\lambda$	partition relation	609
$x^\sharp$	theory of Silver indiscernibles in $L[x]$	627
$\langle x^1, \ldots, x^k \rangle^{\mathsf{p}}$	$\left\langle p^k \langle x^1_n, \ldots, x^k_n \rangle \mid n \in \omega \right\rangle$	668
$\lVert x \rVert$	order type of $x \in \mathrm{WO}$	676
Θ	least ordinal not surjective image of $\mathbb{R}$	680
$\boldsymbol{\delta}^1_n$, $\boldsymbol{\sigma}^1_n$	ordinals associated with $\boldsymbol{\Delta}^1_n$ and $\boldsymbol{\Sigma}^1_n$	685
$\tilde{T}$	Martin-Solovay tree for T	694

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Index

$\mathcal{A}$, **407**
absoluteness
  of a formula, **418**
  of constructibility, **437**
absorption, **234**
abstraction term, 176, 180, **183**
  functional ∼, **188**
AC_A, **225**
$\mathsf{AC}_A(B)$, **225**
accumulation point (topology), **246**
AD, **654**
Addison, John West, Jr., 660
adequate pointclass, **369**
adjunction (operation on sets), **277**
algebra (of sets), **392**
  σ-∼, **392**
almost
  category, **387**
almost universal (class), **207**
analytic, **362**
Analytic, **362**
Analytical, **326**
antecedent (of a sequent), **72**
antichain, **411**
antisymmetric (relation), **197**
appropriateness
  of a class of prime propositions to a class of propositions, 87
  of a signature to a theory, 77
Aquinas, Thomas, 578
Arithmetical, **326**
Aronszajn tree, **608**
Aronszajn, Nachman (Nathan), 608
assignment (to variables), **48**
asymmetric (relation), **196**
atom
  for a measure, **585**
atom (in a theory of membership), **171**
atomless (measure), **585**
axiom
  in Gentzen system, **141**
  of choice, **216**
  of membership
    Aussonderungs∼, 176
    collection, **175**
    comprehension, **175**
    comprehension (predicative), **179**
    extension, **175**
    foundation, **176**, **179**
    pair, **175**
    power, **185**
    replacement, **186**
    selection, 176
    separation (for class theory), **179**
    union, **184**
axiom of determinacy, 673

$\vec{B}$, **255**
$\overleftarrow{B}$, **255**
Baire, **388**
Baire property (of a topological space), **387**
Baire property (of pointsets), **388**, 573, 655
Baire space, **352**
Baire, René-Louis, 352
ball (metric topology), **353**
Banach, Stefan, 583
base (topology), **246**
Baumgartner, James Earl, 568
BC, **358**
Bendixson, Ivar Otto, 383
Bernstein, Felix, 218
bijective (function), **187**
Bin, **254**
bin, **254**
BK, **342**
Blackwell, David Harold, 659
boldface (pointclass, theory), **335**
Bona, Jerry Lloyd, 223
Boole, George, 236
boolean algebra, **235**
boolean valuation

definability of ∼, **500**
boolean valuation function, **497**
partial ∼, **497**
boolean valuation operation, **499**
boolean-valued structure, **492**
Borel
κ-∼, **372**
Borel, **357**
Borel code, **358**
Borel determinacy, 452
Borel hierarchy, 329, **357**
Borel, Félix Édouard Justin Émile, 357
Boswell, James, 169
bound occurrence, 14
boundary (topology), **246**
bounded
lattice, **235**
bounded (formula), **266**
branch of a tree, **242**
Brouwer, Luitzen Egbertus Jan, 342
Brouwer-Kleene ordering, **342**

C, **179**
c, **179**
C^0, **179**
C^+, **265**
c^+, **265**
C', **277**
Cantor set, **352**
Cantor space, **352**
Cantor, Georg Ferdinand Ludwig Philipp, 218, 232, 352, 383
Card, **217**
cardinal, **217**
regular, **228**
singular, **228**
cardinal arithmetic, **228**
cardinality, **216, 217**
of a set, **217**
Carroll, Lewis, 8
category, 471
first ∼, **387**
second ∼, **387**
Cauchy construction of real numbers, 347
Cauchy sequence, 347, **353**
Cauchy, Augustin-Louis, 347
ccc (countable chain condition), **411**
CH, **232**
chain condition, **411**
change of variables, 15, **46**
Chinese remainder theorem, 163
choice function, **215**
Church, Alonzo, 289
Church-Turing thesis, **289**
class
equivalence ∼, **197**
proper ∼, **171**
class (in a theory of membership), **171**
class forcing, **540**
classical descriptive set theory, 339
clopen, **401**
closed (class of ordinals), **226**
closed (pointset), **245**
closure
deductive ∼, **103**
closure (topology), **246**
club (class of ordinals), **226**
cluster point (topology), **246**
cofinal (function on ordinals), **228**
cofinality, **228**
Cohen algebra, **523**
Cohen real, **573**
Cohen, Paul, 468
comeager, **387**
compact (cardinal)
strongly∼, **602**
weakly∼, **602**
compactness
of first-order predicate logic, 68, **78**
of propositional logic, **88**
compactness (topology), **251**
compatible (elements of a partial order), **476**
complement (of a subset of a partial order), **476**
complemented lattice, **235**
complete
κ-∼ (filter, ideal), **587**
complete (linear order), **349**
complete (object in a pointclass), **329**
Γ-∼, **329**
completely metrizable, **354**
completeness (of a system of deduction), **65**
completeness (of metric space), **354**
completion (of a subset of a partial order), **476**
complexity
of classes, **279**
of satisfaction, **283**

over type 1, **316**
set-theoretic ∼, **266**
composition (of plays of a game), **399**
computable
class, 283
function, 289
partial ∼ function, 289
computable (function), **290**
computation
science of ∼, **263**
Con, **304**
concatenation, 6
cone filter (on Turing degrees), **674**
cone of Turing degrees, **674**
consistency, 25, **65**
ω-∼, 291
propositional ∼, **88**
consistency proofs using forcing, **521**
constant, 11
constructibility, **434**
relative, **444**
continuous function, **248**
from Ord to Ord, **226**
on a pointspace, 338
continuum hypothesis, **232**, 443
continuum problem, 653
convergent sequence, **250**
countability, **225**
countable chain condition, **411**
cover
open, **251**
crit, **591**
criterion
Tarski-Vaught ∼, **147**
Cummings, James, 654
cut rule, **141**

Davis, Morton, 404
DC, **225**
decidable (theory), 263
decision procedure, 263
decomposition (of a play of a game), **399**
Dedalus, Stephen, 578
Dedekind completion, 350
Dedekind construction of real numbers, 347
Dedekind cut, **348**, 350
Dedekind, Richard, 220, 347
Dedekind-finiteness, **220**
deduction, 65
system of ∼, 65
Gentzen ∼, **140**
Hilbert ∼, **139**
natural, **73**
system of ∼ for propositional logic, **87**
definability
relative, **306**
definability of boolean valuation, **500**
definability of forcing, **487**
definition, **110**
extension by ∼, **110**
dense (ideal), **696**
dense (linear order), **349**
dense (pointset), **246**
dense (set in a partial order), **476**
denumerability, **225**
Desargues's principle, **123**
Desargues, Girard (Gérard), 122
Descartes, René, 117, 345
descriptive set theory, **339**
classical ∼, **339**
effective ∼, **339**
determinacy, 397
Borel ∼, 452
determinate (game, set), **400**
determined (game, set), **400**
diagonal argument, **218**
diagonal intersection, **238**
diagram of an expression, **42**
diamond principle, **449**, 567
directed set
elementary ∼, **148**
disjoint union, **392**
distributivity, **235**
Dodgson, Samuel, 8
domain
of prefunction, **185**
of relation, **196**
universal ∼, 16
domain of discourse, 16
DP, **123**
dual (of a pointclass), **325**
duality
in boolean algebras, **236**
duality (in projective geometry), 119
synthetic and analytic, 132

Easton forcing, **537**
Easton, William Bigelow, 537

effective descriptive set theory, **339**
effective procedure, **263**
Ehrenfeucht, Andrzej, 611
element
    proper ∼, **171**
element (in a theory of membership), **171**
elementary
    class, **114**
    methods, **114**
    substructure, 146
elementary directed set, **148**
Elements of Euclid, 113
EM-set, **623**
embedding
    elementary ∼, **146**
entailment, **56**
Epimenides the Cretan, 261
equipollence, **216**
equivalence, 5
    logical ∼, **95**
    modulo a theory, **95**
    over a structure, **95**
equivalence class, **197**
equivalence relation, **197**
Erdös, Paul, 603, 647
Euclid of Alexandria, 113, 314
evaluation
    of formulas, 49
    of terms, 49
eventually (for sequences), **251**
expansion
    of a signature, **29**
    of a structure, **55**
expression, **37**
    standard ∼, **41**
extendible (cardinal), **650**
extension
    by definition, **110**
    conservative ∼, **103**
        logic with *vs* without identity, **109**
    of a theory, **103**
extension of a sequence, **244**
    immediate ∼, **244**
    proper ∼, **244**
extensional relation, **213**

F, **273**
F, **253**
family
    indexed ∼ of classes, **188**
F_σ, **361**
Fermat, Pierre de, 237
field
    of relation, **196**
fine (filter), **605**, 645
finite
    σ-∼, **394**
finite axiomatizability
    non-∼ of ZF, **422**
    of C, **423**
finite sequence, **193**
finiteness, **192**
    Dedekind-∼, **220**
    definitions of ∼, **220**
first countability (topology), **248**
Fodor, Géza, 240
forcing, 474
    class ∼, **540**
    definability of ∼, **487**
    Easton ∼, **537**
    iterated ∼, **549**
    product ∼, **534**
forcing language, **479**
forcing predicate, **487**
forcing relation, 475, **485**
    extrinsically defined ∼, **479**
    partial ∼, **485**
forcing term, **478**
formula, 11, **39**
    atomic ∼, **39**
    complex ∼, **39**
    compound ∼, **39**
    simple ∼, **39**
Fraenkel, Abraham Halevi (Adolf), 172, 428
free for, **14**
free occurrence, 14
frequently (for sequences), **251**
Friedberg, Richard, 308
Friedman, Harvey, 406
function, **185**
    n-ary ∼, **195**
    partial ∼, **187**
    total ∼, **187**

G, **253**
G, **478**, **496**
Gaifman, Haim, 621
Gale, David, 403
Galvin, Frederick William, 233

game, 398
GB, 172
G_δ, **361**
GB$^-$, **222**
GCH, **232**
Geisel, Theodor Seuss, 578
generalized continuum hypothesis, **232**
generic
  arguing in a ∼ extension, **513**
  arguing with ∼ extensions, **490**
  extension, 475, **478**
  filter, 475
  set, 471
generic (filter), **476**
genericity conjecture, **628**
Gentzen system of deduction, **140**
Gentzen, Gerhard, 73
geometrical point, 345
geometry
  analytic *sim*, 115
  euclidean ∼, 113
  projective ∼, 115
  synthetic ∼, 117
Gitik, Moti, 653
Gödel, Kurt, 25, 58, 169, 426, 433
Goldbach's conjecture, 314
Goldbach, Christian, 314
good (tree), **401**
good universal set, **668**
Gulliver, Lemuel, 64

Hajnal, András, 233, 647
halting problem, **297**
Hamlet, Prince of Denmark, 79
Hanf, William Porter, 602, 616
Hausdorff difference hierarchies, 663, **807**
Hausdorff space, **247**
Hausdorff, Felix, 247
Henkin construction, 68, 142, 145
Henkin, Leon, 68
Herbrand, Jacques, 104
hereditary (property of sets), **204**
hereditary ordinal-definability, **426**
HF, 264
hierarchy
  complexity ∼, 663
  cumulative ∼ (in the theory of membership), **173**
  von Neumann ∼(in the theory of membership), **173**
Hilbert system of deduction, **139**
Hilbert, David, 113
HOD, **426**
homeomorphism, **248**
homogeneity system, **687**
homogeneous
  set for a partition, **608**
  subset of $^\omega X$, **688**
  tree, **688**
homogeneously Suslin, **689**
homology, **33**
Horatio, friend to Hamlet, 79
Hrbáček, Karel, 618
ht, **242**
huge, n-huge, almost huge (cardinal), **648**
hyperarithmetical (pointset), 329

I_s, **340**
I0–I3 (large cardinal hypotheses), **649**
ideal
  in a ring, 237
  on a boolean algebra, **237**
  principal ∼ in a ring, 237
idempotence, **234**
identity, 5, 15
im, **185**
image
  by a prefunction, **185**
  by a relation, **196**
  of prefunction, **185**
  of relation, **196**
indescribability (of cardinals), **616**
index
  Turing ∼, **294**
indexed family
  of classes, **188**
indiscernible
  uniform ∼, **627**
  x-∼, **627**
indiscernibles, 621
indiscernibles (for a structure), **611**
induction
  mathematical ∼, **191**
  on ordinals, **191**
  proof by ∈-∼, **211**
inference rule, **72**
  cut rule, **141**
inference rules
  for universal quantification, 91
  logical, **141**

structural, **141**
weakening, **141**
infinitarity, **314**
infinity, **220**
(point, line, . . .) at ∼, 115
injective (function), **187**
inner model, **207**
interior (topology), **246**
interpretation, 2, 12, 47
of s^+
finitary, **272**
standard, **272**
transitive, **272**
of a propositional language, 86
of a theory in a language or theory, **113**
intersection (of classes), **184**
invariance
of a relation *vis-à-vis* an equivalence relation, **197**
inverse
of prefunction, **185**
of relation, **196**
inverse image
by a relation, **196**
by prefunction, **185**
irreflexive (relation), **196**
isolated point, **382**
isomorphism, **32**
iterated forcing, **549**
iterated ultrapower, **597**

Jackson, Stephen Craig, 685, 686
Jensen, Ronald Björn, 449, 549
Johnson, Samuel, 169
Jónsson function, 647
Jónsson, Bjarni, 647
Joyce, James Augustine Aloysius, 578

Keats, John, 578
Kechris, Alekos (Alexander) Sotirios, 685
Keisler, Howard Jerome, 602
Kelly, John Leroy, 172
Kleene pointclass, **326**
Kleene, Stephen Cole, 298, 326, 342
König, Dénes, 231, 606
Kripke, Saul Aaron, 797
Kronecker, Leopold, 314
Kummer, Ernst Eduard, 237
Kunen, Herbert Kenneth (Ken), 628
Kuratowski, Kazimierz, 355, 583

L, **434**
$L[A]$, **444**
$L(A)$, **445**
language, 2
formal ∼, 3
meta∼, 4
object ∼, 4
propositional ∼, 86
standard ρ-∼, **34**
standard ∼, **41**
lattice, **233**
Lebesgue, Henri-Léon, 583
Levy collapse, **571**
Levy hierarchy, **265**, 316
Levy, Azriel, 265, 571
Lim, **191**
limit (of convergent sequence), **251**
limit cardinal, **227**
limit ordinal, **191**
limit point (topology), **246**
linear preorder, **198**
Lipschitz game, **664**
Lipschitz reducibility, degree, **664**
Lipschitz, Rudolf Otto Sigismund, 664
LK, **140**
LK$^-$, **142**
LO, **342**
logic, 3
formal ∼, 3
propositional ∼, 86
Łoś, Jerzy, 152
Lusin, Nikolai Nikolaevich, 342
Lusin-Sierpinski ordering, **342**

$\mathfrak{m}$, **387**
MA, **561**
Mahlo (cardinal), **582**
Mahlo, Paulo, 582
Martin measure (on Turing degrees), **675**
Martin's axiom, **561**
Martin, Donald Anthony, 406, 561, 665
Martin-Solovay tree, **694**
mathematics, 4
maximal (element), **200**
maximum (element), **200**
meager, **387**
measurability property (of pointsets), **396**, 573, 655
measurable (cardinal), **587**
real-valued ∼, **584**

measurable (set), **394**
measure
  Lebesgue ~, **393**
  over a set, 583
  real-valued, **392**
  uniform ~, **393**
metalanguage, 4
metatheory, 5
metric (topology), **352**
metric space (topology), **353**
metrizable, **353**
minimal (element), **200**
minimum (element), **200**
Mitchell, William J., 233, 608
mod(ulo), **197**
model, **51**
Monk, Leonard Gaines, 665
Morse, Anthony Perry, 172
Morse-Kelly theory of membership, **180**
Moschovakis, Yiannis Nicholas, 660, 680, 682
Mostowski collapse, **214**
Mostowski, Andrzej, 214, 419, 428, 611
Moulton plane, **124**
Moulton, Forest Ray, 124
Muchnik, Albert Abramovich, 308
Mycielski, Jan, 654

$\mathbb{N}$, 347
natural deduction, **73**
ND, **73**
neighborhood (topology), **245**
neighborhood base (topology), **246**
neighborhood subbase (topology), **247**
New Foundation (theory of sets), 172
node of a tree, **242**
nontrivial (ideal over a cardinal), **630**
norm (on a pointset), **368**
normal (filter), **241**, 645
normal form
  conjunctive, **101**
  disjunctive, **101**
  prenex, **102**
normed pointclass, **370**
nowhere dense, **385**
Num, **191**
number, **191**

o, **242**
object language, 4
occur, 42
occurrence, 14, 43
OD, **425**
on (vs. over), **583**
open (pointset), **245**
open (subset of a partial order), **476**
open cover, **251**
operation, 10
operation $\mathcal{A}$, **407**
oracle, **306**
Ord, **189**
order (first-~ and higher-~)
  in geometry, 114
  in mathematical theories, 114
ordered pair, **178**
ordinal
  initial ~, **217**
  limit ~, **191**
  successor ~, **191**
ordinal (von Neumann ~), **189**
ordinal arithmetic, **214**
ordinal sequence, **215**
ordinal-definability, **425**
over (vs. on), **583**

$\mathfrak{p}^{\cdot}$, **342**
PA, **162**
pair
  of elements, **177**
  ordered ~, **178**
paradox
  Cantor ~, **262**
  Epimenides ~, **261**
  liar ~, **261**
  Richard ~, **262**
  Russell ~, **172**, 262
  Skolem ~, **80**
Paris, Jeffrey B., 406
partial computable (function), 289
partial order, **198**
partial recursive function, **292**
partition, **608**
Pbl, **300**
Peano arithmetic, 23
perfect set, **382**
perfect set property, **385**, 573, 655
perspective (from a point or line), **122**
perspectivity, 135
place in an expression, **42**
player, 398
point, **321**

point (topology), **245**
pointclass, **321**
pointset, **321**
pointset (topology), **245**
pointspace, **321**
Polish space, **355**
Pope, Alexander, 416
powerclass (of a class), **185**
p.r., **292**
PRA, 158
predense (set in a partial order), **476**
prefunction, **185**
preimage
  by prefunction, **185**
preorder, **198**
presses down, **238**
pretameness, **543**
prewellordering property, **370**, 660
Prikry, Karel Libor, 568
prime formula, **86**
prime proposition, 86
principal
  filter, 587
principal formula, **141**
principal ideal
  in a boolean algebra, **237**
problem
  halting ∼, **297**
procedure
  decision ∼, 263
  effective ∼, 263
  interminable ∼, **288**, 338
  relative ∼, **338**
  terminable ∼, **288**, 338
product forcing, **534**
product measure, **395**
product topology, **249**
$\mathfrak{p}$, **362**
projection, **248**
projective hierarchy, **362**
projectivity, 134
proof, 64, **74**
  tree, **141**
proposition, 86
  prime ∼, 86
propositional (language, logic, etc.), 86
propositional connective, 12
propositional part (of a predicate language), **86**
provability, 64, **74**

$\mathbb{Q}$, 348
quantification, 12
quantifier
  phrase, 13
Quine, Willard Van Orman, 172
quotient
  of a class modulo an equivalence relation, **197**

$\mathbb{R}$, 348
R, **290**
Ramsey cardinal, **612**
Ramsey, Frank Plumpton, 608
random real, **573**
rank (of a set), **211**
r.e., **292**
real, **352**
real number, 345
recursion
  definition by ∼ on wellfounded relation, **201**, 281
recursive
  class, **292**
  function, **292**
    partial ∼, **292**
    total ∼, **292**
  function on pointset, 323
  pointset, **322**
recursive substitution, **324**
recursively closed (pointclass), **324**
recursively closed pointclass, **324**
recursively enumerable, **292**
reducible
  Turing ∼, **306**
reductio ad absurdum, 65
reduction property, **369**
  Π^1_1-∼, 659
  Π^1_1-∼, **344**
reflection (set theory), **421**
reflexive (relation), **196**
$\mathfrak{R}$, **494**
regressive function, **238**
regular (norm on a pointset), **368**
regular (subset of a partial order), **476**
regular algebra, 475
regular algebra (of a partial order), **494**
regular cardinal, **228**
regular completion (of a boolean algebra), **494**
Reinhardt, William Nelson, 647

relation, **195**
  n-ary ∼, **195**
  equivalence ∼, **197**
  on a class, **196**
relational (signature), **29**
relative constructibility, **444**
relative definability, **306**, 335
relative topology, **246**
relativization, **112**
relativization (of a pointclass)
  full ∼, **335**
  to a type-1 object, **335**
relativized (pointclass), **335**
remarkable (EM-set), **626**
restriction
  of a prefunction to a class, **185**
Richard, Jules, 262
ring (of sets), **392**
  σ-∼, **392**
rk, **211**
Rosser, J. Barkley, 26
Rowbottom (filter, cardinal), **637**
Russell paradox∼, **172**
Russell, Bertrand, **172**, 174

S (basic set theory), **175**
σ-finite, **394**
s, **173**
S^0 (basic set theory without Foundation), **176**
S^+, **265**
s^+, **265**
S', **277**
s', **277**
sat, **629**
$\mathrm{Sat}_0, \mathrm{Sat}_1$, **276**
satisfaction, **51**
  relation, **50**
    partial ∼, **50**
satisfactoriness (of structures), **50**
  strong ∼, **50**
  weak ∼, **50**
satisfiability, **56**
  propositional ∼, **87**
saturation
  of ideals, **629**
scale (on a pointset), 378
scale property (of a pointclass), 380, 663
SCH, **233**
Schröder, Ernst, 218
Scott, Dana Stewart, 596, 616
second countability (topology), **248**
semirecursive, **292**
  pointset, **322**
semiring (of sets), **392**
semiscale (on a pointset), 377
separability (topology), **248**
separable (linear order), **349**
separation property, **247**, **369**
separative partial order, **493**
sequence, **193**
  finite ∼, **193**
  ordinal, **215**
sequent, **72**
  Gentzen ∼, **140**
set (in a theory of membership), **171**
setlike relation, **201**
Seuss, Dr., 578
SH (Suslin's hypothesis), **411**
SH, 561
Shakespeare, William, 79
Shanin, Nikolai Alexandrovich, 526
Shelah, Saharon, 233
Shoenfield, Joseph, 419
Sierpiński, Wacław Franciszek, 342, 355
sign, 35
signature, **28**
  appropriate to an expression or class of expressions, **56**
  relational ∼, **29**
  standard ∼, **33**
Silver, Jack Howard, 233, 568, 608, 620, 622
singleton
  of an element, **177**
singular cardinal, **228**
singular cardinals hypothesis, **233**, 653
size of a set, **217**
Skolem
  function, **151**
  operation, **106**
Skolem paradox, **80**
Skolem, Thoralf, 80
skolemization, **106**
Smith, Edgar C., Jr., 633
Solovay, Robert Martin, 561, 619
sort (of individual), 16
soundness (of a system of deduction), 65
s^{P}, **540**
space

topological, **245**
specification, 11
stationary set, **238**
Steel, John Robert, 669, 686
Steinhaus, Władysław Hugo Dionizy, 654
Stewart, F. M., 403
strategy, 398
I-∼, **399**
II-∼, **399**
winning ∼, **400**
strategy (in an infinite game), **399**
strict (pre)order, **199**
strong (pre)order, **199**
structure, 2, 20, **30**
empty ∼, 102
empty ∼, **57**
subbase (topology), **247**
subformula property (of deductive systems), **142**
subspace (topology), **246**
substitution, 11
in diagrams, **44**
in formulas, **14**
in terms, **11**
of equivalents, **100**
substructure, **112**
elementary ∼, **146**
subtree
I-, II-imposed ∼, **401**
non-losing ∼, **402**
subvaluation function, **143**
Suc, **191**
succedent (of a sequent), **72**
successor cardinal, **227**
successor ordinal, **191**
supercompact, γ-supercompact (cardinal), **644**
surjective (function), **187**
Suslin
κ-∼, **373**
Suslin line, **411**
Suslin tree, **447**
Suslin's hypothesis, **411**, 561
Suslin's operation $\mathcal{A}$, **407**
Suslin, Mikhail Yakovlevich, 366, 407
Swift, Jonathan, 64
symmetric (relation), **196**
syntactical element, 35

T, **273**
tail set, **677**
tameness, **545**
Tarski, Alfred, 58, 355, 602
Tarski-Vaught criterion, **147**
tautology, **87**
tc, **212**
Tennenbaum, Stanley, 561, 565
Tennyson, Alfred (Lord), 578
term, 10, **39**
theorem
Σ_1-uniformization, **293**
0-1 law, **678**
analytic boundedness, **371**
analytic separation, **366**
Baire category ∼, **386**
Cantor's ∼ on the size of the continuum, **218**
Cantor-Bendixson ∼, **383**
closure of Δ_1^{T} under recursive definition, **270**
compactness ∼ for first-order predicate logic, **78**
completeness ∼, **77**
for **LK**, **144**
for **ND**, **77**
for uncountable theories, **145**
condensation lemma, **440**
consistency
$\mathsf{ZF} + \boldsymbol{V} \neq \boldsymbol{L}$, **523**
$\mathsf{ZF} + \neg\,\mathsf{AC}$, **532**
$\mathsf{ZFC} + \neg\,\mathsf{CH}$, **530**
consistency of $\mathsf{ZF} + \boldsymbol{V} = \boldsymbol{L}$, **438**
consistency of $\mathsf{ZFA} + \neg\,\mathsf{AC}$, **431**
consistency of ZFC, **428**, **440**
cut-elimination ∼, **144**
Desargues's ∼, **123**
determinacy
Borel-∼, **406**
$\boldsymbol{\Sigma}_1^0$-∼, **403**
$\boldsymbol{\Sigma}_2^0$-∼, **403**
$\boldsymbol{\Sigma}_3^0$-∼, **406**
Easton's ∼, **537**, 547
Ehrenfeucht-Mostowski ∼, **611**
first periodicity ∼, **660**
Fodor's lemma, **240**
Friedberg-Muchnik ∼, 308
Fubini's ∼, 395
Fubini's ∼ for nullsets, **395**
fundamental ∼ of projective geometry, 134

Gale-Stewart ∼, **403**
Gödel's first incompleteness ∼, 25
finitary semantic form, **303**
infinitary semantic form, **257**
Rosser's improvement, **306**
syntactic form, **304**
Gödel's second incompleteness ∼, 25, **305**
Gödel-Tarski undefinability ∼, 58
Herbrand's ∼, **104**
Jensen's covering ∼, **629**
König's lemma, **231**, 606
Kuratowski-Ulam ∼, **390**
Łoś' ∼, **152**
Löwenheim-Skolem ∼, **149**
Π^1_1-reduction ∼, 344
recursion ∼, **298**
recursive definition ∼, **201**, 281
reflection principle, **421**
Σ^1_1-absoluteness, **419**
Σ^1_2-absoluteness, **419**
Schröder-Bernstein ∼, **218**
separation ∼ for Wadge classes, **670**
Silver's singular cardinals ∼, **568**
Skolem-Löwenheim, **149**
Suslin's ∼, **368**
undecidability of first-order predicate logic, **291**
unsolvability of the halting problem, **298**
$V \models \mathsf{S}$, **256**
Wadge's lemma, **664**
theory, 4, **55**
closed ∼, **103**
complete ∼, **102**
maximal ∼, 66
of a (satisfactory) structure, **56**
with witnesses, 68
thesis
Church-Turing, **289**
topology, **245**
of ordinals, **226**
relative, **246**
topology (of a pointspace), **337**
topology of $\mathbb{R}$, 351
total order, **199**
total preorder, **198**
total recursive function, **292**
tower (of ultrafilters), **687**
transitive closure, **212**
transitive collapse, **214**
transitivity
of classes, **189**
of relations, **197**
tree, **242**
on a set, **243**
sequence ∼, **243**
tree property (of cardinals), **606**
truth table, 88
truth value, 12
tuple
ordered n-∼, **193**
Turing index, **294**
Turing, Alan Mathison, 289, 294
type-0, 1, **316**
type (of a point), **321**
type (of a set-theoretic object), **316**

$U_0, U_1, U_{\scriptscriptstyle S}$, **321**
u_α, **627**
Ulam matrix, **584**
Ulam, Stanislaw Marcin, 390, 584
ultrapower, **152**
iterated, **597**
of proper class, **154**
ultraproduct, **152**
unbounded (class of ordinals), **226**
uniformization
property, **293**
Σ_1-∼, **293**
uniformization property (of a pointclass), **370**, 663
uniformize, **293**, **370**
union (of classes), **184**
unique readability, **37**
universal (object in a pointclass)
0-$\imath$-∼, **330**
0-∼, **329**
universal closure, **94**
universal set
good ∼, **668**
urelement, **171**

V (universe of sets), 173
$\mathcal{V}$ (the class of standard variables), **35**
v_n (the nth standard variable), **35**
V, **479**
V_α (level of cumulative hierarchy), 173
Val_0, **275**
validity, **81**
valuation

of formulas, 49
of terms, 49
valuation function, **49**
Δ_0-$\sim$, **273**
partial $\sim$, **50**
value (of a function at an element), **186**
Van Wesep, Robert Alan, 669
variable, 10
Vaught, Robert Lawson, 147, 617
Vitali (construction, set), **390**, 397
Vitali, Giuseppe, 390
von Neumann, John, 173
Vopěnka's principle, **649**
Vopěnka, Petr, 618, 649

Wadge class, 667
Wadge game, **664**
Wadge reducibility, degree, **664**
Wadge, William Wilfred, 663
weak (pre)order, **199**
wellfoundedness, **200**
wellorder, **200**
wellordering principle, **224**
Whitehead, Alfred North, 174
winning strategy, **400**
witness
sequence, **90**
to an existential sentence, **68**
WO, **342**
Wolfe, Philip, 403
Woodin cardinal, **649**
Woodin, William Hugh, 686

$\mathbb{Z}$, 348
Zermelo, Ernst Friedrich Ferdinand, 172
ZF, 172
ZFA, 172
ZFC, **222**
ZF^-, **222**
ZF^P, **540**
ZFP, **543**
Zorn's lemma, **224**
Zorn, Max August, 224