

AN INTRODUCTION TO CLASSICAL REAL ANALYSIS

KARL R. STROMBERG

AMS CHELSEA PUBLISHING
American Mathematical Society • Providence, Rhode Island



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2010 *Mathematics Subject Classification*. Primary 26-01, 28-01.

For additional information and updates on this book, visit
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Library of Congress Cataloging-in-Publication Data

Stromberg, Karl R. (Karl Robert), 1931–1994.

An introduction to classical real analysis / Karl R. Stromberg.
pages cm

Originally published: Belmont, California : Wadsworth, 1981.

“Reprinted with corrections by the American Mathematical Society, 2015”—Galley t.p. verso.

Includes bibliographical references and index.

ISBN 978-1-4704-2544-9 (alk. paper)

1. Mathematical analysis. I. American Mathematical Society. II. Title.

QA300.S89 2015

515—dc23

2015024928

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10 9 8 7 6 5 4 3 2 1 20 19 18 17 16 15

CONTENTS

ABOUT THE AUTHOR

xiii

0

PRELIMINARIES

1

Sets and Subsets

1

Operations on Sets

2

Ordered Pairs and Relations

3

Equivalence Relations

3

Functions

4

Products of Sets

5

1

NUMBERS

7

Axioms for \mathbb{R}

8

The Supremum Principle

12

The Natural Numbers

13

Integers

16

Decimal Representation of Natural Numbers

17

Roots

20

Rational and Irrational Numbers

21

Complex Numbers

22

Some Inequalities

24

Extended Real Numbers

27

Finite and Infinite Sets

28

Newton's Binomial Theorem

33

Exercises

35

2	SEQUENCES AND SERIES	<u>39</u>
	Sequences in \mathbb{C}	<u>39</u>
	Sequences in $\mathbb{R}^{\#}$	<u>43</u>
	Cauchy Sequences	<u>51</u>
	Subsequences	<u>52</u>
	Series of Complex Terms	<u>53</u>
	Series of Nonnegative Terms	<u>59</u>
	Decimal Expansions	<u>65</u>
	The Number e	<u>67</u>
	The Root and Ratio Tests	<u>69</u>
	Power Series	<u>71</u>
	Multiplication of Series	<u>72</u>
	Lebesgue Outer Measure	<u>76</u>
	Cantor Sets	<u>80</u>
	Exercises	<u>84</u>
3	LIMITS AND CONTINUITY	<u>91</u>
	Metric Spaces	<u>91</u>
	Topological Spaces	<u>95</u>
	Compactness	<u>102</u>
	Connectedness	<u>107</u>
	Completeness	<u>107</u>
	Baire Category	<u>109</u>
	Exercises	<u>111</u>
	Limits of Functions at a Point	<u>114</u>
	Exercises	<u>119</u>
	Compactness, Connectedness, and Continuity	<u>122</u>
	Exercises	<u>124</u>
	Simple Discontinuities and Monotone Functions	<u>128</u>
	Exercises	<u>131</u>
	Exp and Log	<u>134</u>
	Powers	<u>136</u>
	Exercises	<u>138</u>
	Uniform Convergence	<u>139</u>
	Exercises	<u>144</u>
	Stone-Weierstrass Theorems	<u>146</u>
	Exercises	<u>156</u>
	Total Variation	<u>159</u>
	Absolute Continuity	<u>162</u>
	Exercises	<u>163</u>
	Equicontinuity	<u>164</u>
	Exercises	<u>168</u>

4	DIFFERENTIATION	<u>170</u>
	Dini Derivates	<u>170</u>
	**A Nowhere Differentiable, Everywhere Continuous, Function	<u>174</u>
	Some Elementary Formulas	<u>175</u>
	Local Extrema	<u>177</u>
	Mean Value Theorems	<u>178</u>
	L'Hospital's Rule	<u>179</u>
	Exercises	<u>182</u>
	Higher Order Derivatives	<u>188</u>
	Taylor Polynomials	<u>192</u>
	Exercises	<u>197</u>
	*Convex Functions	<u>199</u>
	*Exercises	<u>203</u>
	Differentiability Almost Everywhere	<u>206</u>
	Exercises	<u>211</u>
	*Termwise Differentiation of Sequences	<u>213</u>
	*Exercises	<u>215</u>
	*Complex Derivatives	<u>219</u>
	*Exercises	<u>223</u>
 5	 THE ELEMENTARY TRANSCENDENTAL FUNCTIONS	 <u>226</u>
	The Exponential Function	<u>226</u>
	The Trigonometric Functions	<u>227</u>
	The Argument	<u>231</u>
	Exercises	<u>232</u>
	*Complex Logarithms and Powers	<u>236</u>
	*Exercises	<u>240</u>
	** π is Irrational	<u>240</u>
	**Exercises	<u>242</u>
	*Log Series and the Inverse Tangent	<u>242</u>
	**Rational Approximation to π	<u>245</u>
	*Exercises	<u>246</u>
	**The Sine Product and Related Expansions	<u>247</u>
	**Stirling's Formula	<u>253</u>
	**Exercises	<u>255</u>
 6	 INTEGRATION	 <u>257</u>
	Step Functions	<u>257</u>
	The First Extension	<u>263</u>
	Integrable Functions	<u>264</u>

Two Limit Theorems	266
The Riemann Integral	269
Exercises	274
Measureable Functions	285
Complex-Valued Functions	289
Measurable Sets	293
Structure of Measurable Functions	300
Integration Over Measurable Sets	305
Exercises	306
The Fundamental Theorem of Calculus	318
Integration by Parts	323
Integration Substitution	323
Two Mean Value Theorems	327
*Arc Length	329
Exercises	331
Hölder's and Minkowski's Inequalities	339
The L_p Spaces	341
Exercises	343
Integration on \mathbb{R}^n	345
Iteration of Integrals	349
Exercises	356
Some Differential Calculus in Higher Dimensions	364
Exercises	376
Transformations of Integrals on \mathbb{R}^n	385
Exercises	393

7	INFINITE SERIES AND INFINITE PRODUCTS	398
	Series Having Monotone Terms	398
	Limit Comparison Tests	401
	**Two Log Tests	404
	**Other Ratio Tests	406
	*Exercises	409
	**Infinite Products	411
	**Exercises	418
	Some Theorems of Abel	420
	Exercises	426
	**Another Ratio Test and the Binomial Series	434
	**Exercises	440
	Rearrangements and Double Series	443
	Exercises	454
	**The Gamma Function	460

**Exercises	<u>470</u>
Divergent Series	<u>473</u>
Exercises	<u>484</u>
Tauberian Theorems	<u>494</u>
Exercises	<u>500</u>

8 TRIGONOMETRIC SERIES	<u>502</u>
Trigonometric Series and Fourier Series	<u>503</u>
Which Trigonometric Series are Fourier Series?	<u>510</u>
Exercises	<u>518</u>
*Divergent Fourier Series	<u>526</u>
*Exercises	<u>530</u>
Summability of Fourier Series	<u>534</u>
Riemann Localization and Convergence Criteria	<u>541</u>
Growth Rate of Partial Sums	<u>551</u>
Exercises	<u>554</u>

BIBLIOGRAPHY	<u>567</u>
---------------------	-------------------

OTHER WORK BY THE AUTHOR	569
---------------------------------	------------

INDEX	<u>571</u>
--------------	-------------------

*Sections marked with a single asterisk are not actually needed in conjunction with any of the unmarked sections. These sections may be safely omitted if time permits only a short course.

**Sections marked with a double asterisk are not actually needed in conjunction with any other section. They are included as interesting and useful applications of the theory.

PREFACE

This volume has evolved from lectures that I have given at the University of Oregon and at Kansas State University during the past twenty years. The subject is classical analysis. It is “real analysis” in the sense that none of the Cauchy theory of analytic functions is discussed. Complex numbers, however, do appear throughout. Infinite series and products are discussed in the setting of complex numbers. The elementary functions are defined as functions of a complex variable. I do depart from the classical theme in Chapter 3, where limits and continuity are presented in the contexts of abstract topological and metric spaces.

The approach here is to begin with the axioms for a complete ordered field as the definition of the real number system. Based only upon that, an uncompromisingly rigorous Definition-Theorem-Proof style is followed to completely justify all else that is said. For better or for worse, I have scrupulously avoided any presumption at all that the reader has any knowledge of mathematical concepts until they are formally presented here. Thus, for example, the number π is not mentioned until it has been precisely defined in Chapter 5.

I hope that this book will be found useful as a text for the sort of courses in analysis that are normally given nowadays in most American universities to advanced undergraduate and beginning graduate students. I have included every topic that I deem necessary as a preparation for learning complex and abstract analysis. I have also included a selection of optional topics. The table of contents is a brief guide to the topics included and to which ones may be safely omitted without disturbing the logical continuity of the presentation. I also hope that this book will be found useful as a reference tool for mature mathematicians and other scientific workers.

One significant way in which this book differs from other texts at this level is that the integral which we first mention is the Lebesgue integral on the real line. There are at least three good reasons for doing this. First, the F. Riesz approach (after which mine is modelled) is no more difficult to understand than is the traditional theory of the Riemann integral as it currently appears in nearly every calculus book. Second, I feel that students profit from acquiring a thorough understanding

of Lebesgue integration on Euclidean spaces before they enter into a study of abstract measure theory. Third, this is the integral that is most useful to current applied mathematicians and theoretical scientists whether or not they ever study abstract mathematics. Of course, it is clearly shown in Chapter 6 how the Riemann integral is a special case of the Lebesgue integral. Stieltjes integration is presented in a graded sequence of exercises. The proofs of these exercises are easy, but any instructor who wishes to include them in his lectures is obviously free to do so.

I sincerely hope that the exercise sets will prove to be a particularly attractive feature of this book. I spent at least three times as much effort in preparing them as I did on the main text itself. Most of the exercises take the form of simple assertions. The exercise is to prove the assertion. A great many of the exercises are projects of many parts which, when completed in the order given, lead the student by easy stages to important and interesting results. Many of the exercises are supplied with copious hints. I feel that the only way to truly learn mathematics is by just plain hard work. It does not suffice to simply read through a book and agree with the author. I do encourage all serious students to work diligently through the exercises provided here. Thomas Edison's dictum that genius is ten percent inspiration and ninety percent perspiration has never been truer than it is here.

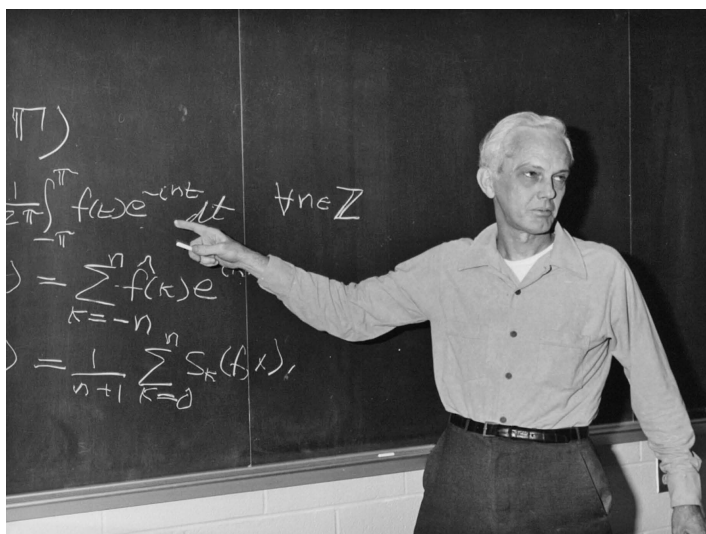
I have found that for a two semester (or three quarter) course, it is easy to cover all the sections in Chapters 1 through 7 that are not marked with asterisks in the Table of Contents. I also find time to include some of the optional sections or part of Chapter 8. In doing this, I make it a practice of assigning a lot of the easier textual material as reading for the students, while I work through many of the harder exercises in class. I see no point in copying the text onto the blackboard.

If it is only possible to spend one semester (about fifteen weeks) on classical real analysis, then one can proceed as follows. Assign all of Chapter 0 and much of Chapter 1 as reading. Omit all sections which bear asterisks in the Table of Contents. Spend only one week on each of Chapters 1, 5, and 7 and only three weeks on each of Chapters 2, 3, 4, and 6 by making the following additional omissions. In Chapter 3, proceed only through "Uniform Convergence," omitting "Baire Category." In Chapter 4 omit "Differentiability Almost Everywhere." In Chapter 6 stop with "The Riemann Integral," but be sure to work through many of the exercises at the end of that section. In Chapter 7, stop with "Some Theorems of Abel."

I take great pleasure in offering my warmest thanks to my good friends Bob Burckel and Louis Pigno who gave me such valuable assistance in preparing this book through their constant encouragement, their proofreading and their many stimulating conversations. I also thank the four women who valiantly typed the technically complicated manuscript. They are Twila Peck, Judy Bernhart, Marie Davis, and Marlyn Logan. Finally, it is a pleasure to thank my publishers and editors John Martindale, Arthur Weber, Paul Prindle, John Kimmel, and David Foss for their excellent help and for their patience and understanding through this seemingly interminable project.

Karl Stromberg
Manhattan, Kansas
July, 1980

About the Author



Courtesy Morse Department of Special Collections, Kansas State University Libraries, Photo Collection—People, “Stromberg, Karl”.

Karl Stromberg (1 December 1931 — 3 July 1994) received his Ph.D. in mathematics from the University of Washington under the direction of Edwin Hewitt in 1958. After postdoctoral years at Yale University and the University of Chicago, he served on the faculty of the University of Oregon from 1960 until 1968. In 1966, he was given the Ersted Award as the outstanding teacher at the University of Oregon. From 1968 on he was Professor of Mathematics at Kansas State University where he received the William L. Stamey Award for exceptional teaching in 1990. He lectured at many universities in the United States and Europe; he spent 1966-7 at Uppsala University (Sweden) and 1974-5 at the University of York (England) as Visiting Professor. He also served on the National Research Council in the United States. In addition to his many research papers in mathematical analysis (see “Other Work by the Author” in the backmatter of this book), he wrote the well-known text *Real and Abstract Analysis* together with Edwin Hewitt. His last major work was a high-level text, *Probability for Analysts*, published in 1994.

The absence of figures and the few typographical imperfections in this present book should be attributed to the fact that for his whole professional life the author was virtually (and, indeed, legally) blind.

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INDEX

- $A(\mathbb{T})$, 524
- a -adic expansions of real numbers, 88
- Abel-Dini theorems, 403–404
- Abel's convergence theorems, 421
- Abel's Limit Theorem, 425
- Abel's Theorem, 57
- Abel summable series, 425, 474
- Abelian theorems, 495
- Absolute continuity, 162, 202, 333, 337
- Absolute convergence of infinite products, 417 of series, 58
- Absolute value, 11, 23
- Absolutely convergent Fourier series, 519–526
- Additive functions, 307
- A. e., 206, 346
- Algebra of functions, 146
- Algebraic number, 37, 187, 241
- Almost everywhere, 206, 346
- Alternating series test, 58
- Approximate units, 338, 358
- Arc length, 329
- Archimedean order, 14
- Arctan, 243
- Arcwise-connected set, 126
- Arg, 232
- Argument function, 232
- Argument of a complex number, 231
- Arithmetic mean, 27, 184
- Arzela-Ascoli Theorem, 167
- Ascoli's Theorem, 167
- Axioms for \mathbb{R} , 8–12
- b -adic expansions, 18, 65, 88
- Baire class 1, 309
- Baire Category Theorem, 110
- Ball in a metric space, 93
- Banach indicatrix, 332
- Banach's fixed point theorem, 113
- Banach-Zarecki Theorem, 333
- Base of a power, 136
- Bernoulli functions, 430
- Bernoulli numbers, 430
- Bernoulli polynomials, 429
- Bernoulli's Inequality, 26, 179
- Bernstein polynomials, 157
- Bernstein's Theorem, 520
- Bertrand's Test, 408
- Bessel's Inequality, 517
- Beta Function, 394, 465
- Binary expansion, 66
- Binomial coefficients, 33, 196, 438
- Binomial series, 197, 438
- Binomial Theorem, 33
- Bonnet's mean value theorems, 334
- Borel sets, 308
- Borel summability, 493
- Boundary of a set, 96
- Bounded function, 92
- Bounded sequence, 40
- Bounded set, 12, 94
- Bounded variation, 159
- Bounds, 12
- Brouwer's Fixed Point Theorem, 127
- C , 22
- C_1 -summability, 474
- Cantor-Lebesgue Theorem, 313
- Cantor sets, 81, 312
- Carleman's Inequality, 63
- Carleman's Theorem, 523
- Carleson's Theorem, 530
- Cartesian products, 5, 6
- Category of a set, 109
- Cauchy Criterion, 51, 56, 412
- Cauchy product, 73, 473
- Cauchy sequence, 51, 108 uniform, 141

- Cauchy's Condensation Test, 398
- Cauchy's Estimate, 450
- Cauchy's Inequality, 24
- Center of a ball, 93
- Centroids, 395
- Cesàro summability, 474
- Chain Rule, 175, 324, 371
- Change of variables, 275, 325, 391
- Characters of \mathbb{R} , 235, 334
- Circle of convergence, 71
- Class C^1 , 367
- Closed ball, 112
- Closed interval, 28, 103
- Closed set, 96
- Closure of a set, 96
- Cluster point, 49
- Compact set (or space), 102
- Comparison tests, 60, 401
- Complement of a set, 2
- Complete metric space, 108
- Completeness of L_p , 291, 341
- Complex conjugate, 23
- Complex numbers, 22
- Component intervals, 99
- Composite relations, 3
- Concave function, 199
- Condensation test, 398
- Connected set, 107
 - arcwise, 126
- Continuity at a point, 115
- Continuity on a space, 116
- Continuous function, 116
- Convergence of sequences
 - of complex numbers, 30
 - of extended real numbers, 43
 - in measure, 315
 - pointwise, 140
 - in a topological space, 100
 - uniform, 140, 164
- Convergence of series of complex numbers, 54
- Convergence of series of functions, 140
- Convergence tests for series
 - Abel's Tests, 421
 - Alternating Series Test, 58
 - Bertrand's Test, 408
 - Cauchy's Condensation Test, 398
 - Cauchy's Criterion, 56
 - Comparison Tests, 60
 - Dini-Kummer Test, 406
 - Gauss' Test, 408
 - Geometric Series, 55
 - Integral Test, 399
 - Leibnitz's Test, 58
 - Limit Comparison Tests, 402
 - Log Tests, 405
 - n th Term Test, 54
 - Raabe's Test, 407
 - Ratio Test, 70
 - Root Test, 70
 - Weierstrass' Criterion, 434
 - Weierstrass' M-test, 141
- Convergent infinite products, 411
- Convex functions, 199, 334
 - logarithmically, 204
 - midpoint, 204, 307
- Convex sequence of cosine coefficients, 513
- Convolution of functions, 357-359
- Coordinate, 6
- Cosine function, 227
- Cotangent function (partial fraction expansion), 251, 519, 561
- Countable set, 29
- Covering theorem, Lebesgue's, 113
- Curve, space-filling, 145
- Darboux's Theorem, 186
- Decimal representation, 17, 66
- Decreasing function, 128
- Decreasing sequence, 43
- Dedekind completeness, 12
- Degree of an algebraic number, 187
- Denjoy-Young-Saks Theorem, 212
- Dense set, 96
- Density, points of, 211
- Denumerable set, 29
- Derivates, Dini, 171
- Derivative, 172
 - complex, 219
 - left, 171
 - partial, 366
 - right, 171
- Derivative of order n , 188
- Derivatives of integrals, 319, 380
- Diagonal sequence
 - argument, 166
- Diagonalization of matrices, 383
- Diameter of a set, 94
- Difference of two sets, 2
- Differentiable function, 171, 365
 - infinitely, 188
 - n times, 188
- Differential equations, 224
- Differentials on \mathbb{R}^n , 366
- Differentiation a. e., 206
- Differentiation term-by-term, 209, 214, 221, 519
- Differentiation theorem, Lebesgue's, 207
 - Fubini's, 209
- Dini derivates, 171
- Dini's Test, 545
- Dini's Theorem, 143
- Dini-Kummer Test, 406
- Dini-Lipschitz Test, 550
- Dirichlet-Jordan Theorem, 547
- Dirichlet kernel, 231, 534
- Dirichlet problem (solution for circle), 540
- Discontinuity, simple, 128

- Discrete metric, 91
- Discrete topology, 95
- Disjoint sets, 2
- Distance
 - between two sets, 111
 - from a point to a set, 94
- Distance function, 91
- Distributional derivative, 339
- Divergence, 40
 - to zero, 412
- Divergent Fourier series, 529, 532, 557
- Divergent infinite products, 411
- Divergent series, 54, 473
- Division algorithm, 17
- Divisor, 37
- Domain, 3
- Dominated Convergence Theorem, 268, 291
- Double series, 61, 446, 450
- Dyadic expansion, 66
- Dyadic n -cube, 347

- e , 46, 67
 - irrational, 68
 - transcendental, 284
- Egorov's Theorem, 302
- Element of a set, 1
- Empty set, 1
- Endpoints, 28
- Enumeration of a set, 29
- Equicontinuity, 165
 - uniform, 168
- Equivalence class, 4
- Equivalence relation, 3
- Equivalent sets, 29
- Essential supremum, 306
- Euclidean metric, 92
- Euclidean norm, 92
- Euclidean topology, 95
- Euler's constant, 401, 410, 433
- Euler's cotan expansion, 251
- Euler's formulas, 227
- Euler's numbers, 453
- Euler's Summation Formula, 432

- Exp, 76, 134, 226
- Exponent, 136
- Exponential function, 76, 134, 226
- Exponents, laws of, 16, 136
- Extended real numbers, 27

- F_σ , 110
- Factorial, 33
- Family of sets, 2
 - pairwise disjoint, 2
- Fatou's Lemma, 289
- Fejér-Lebesgue Theorem, 539
- Fejér's divergent Fourier series, 532
- Fejér's kernel, 231, 534
- Fejér's Lemma, 312
- Fejér's polynomials, 530
- Fejér's sums, 536
- Fejér's Tauberian Theorem, 498
- Fejér's Theorem (on C_1 -summability), 536
- Fibonacci numbers, 89
- Field, 9
 - ordered, 11
- Finite sequence, 17
- Finite set, 29
- Finite variation, 159
- First category, 109
- Fixed point theorem,
 - Banach's, 113
 - Brouwer's, 127
- Fourier coefficients, 506
- Fourier series, 506
 - absolutely convergent, 519-526
 - convergence tests for, 545-550
 - divergent, 529, 532, 557
 - summability of, 536-540
 - term-by-term differentiation of, 519-520
 - term-by-term integration of, 511, 561
- Fourier transforms on \mathbb{R}^n , 359

- Fubini's Differentiation Theorem, 209
- Fubini's Theorem (on multiple integrals), 352
- Function, 4
- Function algebra, 146
- Fundamental Theorem of Algebra, 127, 240
- Fundamental Theorem of Calculus, 274, 311, 320

- G_δ , 110
- Gamma Function, 205, 394, 461
 - logarithm of, 467
- Gauss' kernel, 359
- Gauss' Multiplication Formula, 470
- Gauss' Test, 408
- Gelfand-Schneider Theorem, 242
- Geometric Mean-
 - Arithmetic Mean Inequality, 27, 183, 184, 344
- Geometric progression, 18
- Geometric series, 55
- Gibbs phenomenon, 554
- Goffman, 279

- Hahn, 133
- Hardy-Littlewood Tauberian Theorem, 495
- Hardy's Tauberian Theorem, 498, 500
- Harmonic mean, 36, 184
- Harmonic series, 55
- Hausdorff space, 99
- Heine-Borel Theorem, 32, 104
- Hilbert space, sequential, 93
- Hölder's Inequality, 340
 - generalized, 344
- Hölder summability, 489
- Holomorphic function, 219
- Hyperbolic functions, 139
- Hypergeometric series, 409, 437

- Image of a set, 5
- Imaginary number, 23
- Imaginary part, 23
- Implicit Function Theorem, 381
- Increasing function, 128
- Increasing sequence, 43
- Indeterminate forms, 179
- Induction, 14
- Inequality
 - Bernoulli's, 26
 - Bessel's, 517
 - Carleman's, 63
 - Cauchy's, 24
 - Čebyšev's, 35
 - Geometric mean–
 - Arithmetic Mean, 27, 183, 184, 344
 - Hausdorff-Young, 518
 - Hölder, 340
 - Jensen, 201
 - Minkowski, 25, 340
 - Schwarz, 340
 - Triangle, 24, 91
- Infimum (inf), 12, 28
- Infinite products, 411
- Infinite sequence. *See* Sequence
- Infinite series. *See* Series
- Infinite set, 29
- Infinitely differentiable, 188
- Infinitely differentiable approximate unit, 357
- Infinitely differentiable on \mathbb{R}^n , 357
- Infinity (∞), 28
- Inner regularity, 296
- Integers, 16
- Integrable function, Lebesgue, 264, 290, 304
 - Riemann, 270
- Integral, abstract, 284
 - iterated, 350
 - Lebesgue, 258, 263, 264, 288, 290, 304
 - Lebesgue-Stieltjes, 283
 - Riemann, 270
 - Riemann-Stieltjes, 281
- Integral over a set, 304
- Integral Test, 399
- Integration by parts, 275
 - 283, 323
- Integration by substitution, 275, 325, 391
- Integration of Fourier series, 511
- Interior of a set, 96
- Interior point, 96
- Intermediate Value Theorem, 123
 - for derivatives, 186
 - for Lebesgue measure, 307
- Intersection of sets, 2
- Intervals, 28, 103, 346
- Inverse Function Theorem, 374
- Inverse image of a set, 5
- Inverse relation (or function), 3
- Inverse sine, 441
- Inverse tangent, 243
- Inversion of Fourier transforms, 360
- Irrational numbers, 21
- Isolated point, 96
- Iterated integrals, 350
- Jacobian, 366
- Jensen's Inequality, 201
- Jordan's Decomposition Theorem, 160
- Jordan's Test, 335, 546
- Katznelson, 217
- Kolmogorov's divergent Fourier series, 530
- Kronecker's Approximation Theorem, 363
- Kummer's Test, 406
- L_p , 344
- L_p -norm, 341
- L_p -space, 341
- λ -null set, 79
- Lagrange multipliers, 382
- Lambert series, 456
- Laplacian in polar coordinates, 380
- Laws of exponents, 16, 137
- Lebesgue constants, 551, 562
- Lebesgue covering theorem, 113
- Lebesgue integrable function, 264, 290
- Lebesgue integral, 258, 263, 264, 288, 290, 315–316
- Lebesgue measurable function, 285, 290
- Lebesgue measurable set, 293
- Lebesgue measure, 77, 294, 346
- Lebesgue point, 539
- Lebesgue singular function, 130
- Lebesgue summable. *See* Lebesgue integrable
- Lebesgue's Decomposition Theorem, 331
- Lebesgue's Differentiation Theorem, 207
- Lebesgue's divergent Fourier series, 557
- Lebesgue's Dominated Convergence Theorem, 268, 291
- Lebesgue-Gerges Test, 547
- Lebesgue-Stieltjes Integral, 283
- Legendre's Duplication Formula, 470
- Leibnitz's Formula, 189, 378
- Leibnitz's Rule, 380
- Leibnitz's Test, 58
- Length of an arc, 329
- Length of an interval, 77
- L'Hospital's Rule, 180, 188, 225
- Limaçon, 428
- Limit comparison tests for integrals, 278
 - for series, 402
- Limit inferior, 47
 - one-sided, 170
- Limit of a function at a point, 114
- Limit of a sequence. *See*

- Convergence of sequences
- Limit point, 96
- Limit superior, 47
 - one-sided, 170
- Lindemann, 241
- Linear mapping, 364**
- Liouville numbers, 187
- Lipschitz condition, 160, 337, 531
- Littlewood's Tauberian Theorem, 498
- Local extrema, 177, 193, 379
- Locally compact space, 153
- log, 135
- Log, 236
- Log series, 242
- log tests, 405
- Logarithm, complex, 236
- Logarithm of the Gamma Function, 467
- Logarithm, real, 135
- Lower bound, 12
- Lower envelope, 132
- Lower function, 271
- Lower semicontinuous function, 132
- Luzin's Theorem, 303

- Maclaurin series, 190. *See* Power series
- Marcinkiewicz's theorem on derivatives, 316
- Maximum, local, 177, 379
- Mean of order p , 184, 344
- Mean value theorems, 178, 197, 373
 - for integrals, 281, 328, 334
- Measurable function, 285, 290
- Measurable set, 293
- Measure
 - dense set, 307
 - Lebesgue, 77, 294, 346
- Member of a set, 1
- Menshov's set of multiplicity, 565
- Mertens' Theorem, 75

- Mesh of a subdivision, 270
- Metric, 91
- Metric space, 91
- Midpoint convex function, 204, 307
- Minimum, local, 177, 379
- Minkowski's Inequality, 25, 340
- Modulus
 - of continuity, 520
 - of a number, 23
- Monotone Convergence Theorem, 266, 288
- Monotone function, 128
- Monotone sequence, 43
- Multiindex, 378
- Multiplication of series, 72, 449
- Multiplicity, set of, 565

- \mathbb{N} , 14
- N-Functions, 333
- Natural numbers, 13
- Neighborhood, 49, 96
- Nested Interval Principle, 30
- Nonmeasurable sets, 298
- Nontangential limit, 428
- Nörlund summability, 491
- Norm
 - L_p , 341
 - uniform, 92
- Nowhere dense set, 109
- Nowhere differentiable continuous functions, 174, 562
- Nowhere monotone functions, differentiable but, 217, 337
- Null set, 79

- One-to-one, 5
- Onto, 4
- Open cover, 102
- Open interval, 28
- Open set, 93, 95
- Ordered field, 11
- Ordered pair, 3

- Oscillation function, 120
- Osgood's Theorem, 120
- Outer regularity, 296

- π , 228
 - is irrational, 240
 - rational approximations to, 245
- Pairwise disjoint family, 2
- Parseval's Identities, 517
- Partial derivatives, 366
- Partial sums, 54
 - of Fourier series, 506, 543
 - of trigonometric series, 503
- Perfect set, 96
- Pi, 228
- Pigeon-hole Principle, 239
- Plancherel transform, 361
- Plancherel's Theorem, 362
- Point of a set, 1
- Pointwise convergence, 140
- Pointwise limit, 140
- Poisson kernel, 535
 - on \mathbb{R}^n , 361
- Poisson Summation Formula, 559
- Polar Coordinates, 369, 392
- Power series, 71
 - $(1 + h)^a$, 197
 - $(1 + z)^a$, 438
 - Arcsin z , 442
 - Arctan z , 244
 - $\exp(z)$, 76
 - $\log(1 + h)$, 197
 - $\log(1 + z)$, 242
 - $\sec z$, 453
 - $\sin z$, $\cos z$, 227
 - $z \cot z$, $\tan z$, $z \csc z$, 432
- Powers, 16, 136, 238
- Pringsheim's Theorem, 223
- Product of two series, 72, 449
- Product symbol, 18
- Proper subset, 1

- \mathbb{Q} , 21
 Quadratic mean, 184

 \mathbb{R} , 8–12
 \mathbb{R}^n , 27
 Raabe's Test, 407
 Radial functions, 393
 Radius of a ball, 93
 Radius of convergence, 71
 Range, 3
 Ratio test, 70
 Rational numbers, 21
 Real analytic function, 223
 Real number system, 8–12
 extended, 27
 Real part, 23
 Rearrangement of a series, 64,
 444, 445, 448
 Regular summation matrix,
 481
 Regularity of Lebesgue
 measure, 296
 Relation, 3
 equivalence, 3
 Relative topology, 101
 Relatively open, 101
 Remainder forms, 195
 Residual set, 109
 Restriction of a function, 5
 Riemann integrable, 270
 Riemann integral, 270
 Riemann-Lebesgue Lemma,
 313, 510
 Riemann Localization Princi-
 ple, 544
 Riemann-Stieltjes Integral,
 281
 Riemann's Theorem on rear-
 ranging series, 444
 Riesz-Fischer Theorem, 513
 Rolle's Theorem, 178
 Roots, n th, 20, 238
 Root test, 70

 σ -algebra, 308
 Salem's Theorem, 523
 Schwarz's Inequality, 340
 Schwarz's theorem on con-
 vexity, 206
 Second category, 109
 Second countable space, 112
 Self-adjoint, 152
 Semicontinuity, 132, 310
 Separable space, 167
 Separating point, 148
 Sequence, 39
 decreasing, 43
 finite, 17
 increasing, 43
 monotone, 43
 Sequential Hilbert space, 93
 Series
 of complex terms, 53
 double, 61, 446, 450
 geometric, 55
 of nonnegative extended
 real terms, 59
 power, 71
 rearrangement of, 64,
 444, 445, 448
 Set, 1
 Sigma algebra, 308
 Signum function (sgn), 128
 Simple discontinuity, 128
 Simple function, 300, 310
 Sine function, 227
 infinite product for the,
 249, 519
 Smooth functions that wig-
 gle everywhere, 217,
 337
 Space-filling curve, 145
 Spherical coordinates in
 \mathbb{R}^n , 369
 Star-shaped sets, 220
 Steinhaus' Theorem, 297
 Step functions, 257, 349
 Stieltjes Integral, 281
 Stirling's Formula, 253, 468
 Stolz Limit Theorem, 428
 Stone-Weierstrass Theo-
 rems, 146–154
 compact-complex, 152
 compact-real, 148
 lattice version, 157
 locally-compact case, 154
 Subalgebra, 146
 Subbase, 96
 Subcover, 102
 Subdivision, 159
 Subsequence, 52
 Subset, 1
 Subspace (topological), 10
 Sum of a series, 54, 60
 Summability methods
 Abel, 474
 Borel, 493
 C_1 (Cesàro), 474
 C_∞ , 486
 Euler-Knopp, 483
 H_p (Hölder), 489
 logarithmic means, 482
 N_p (Nörlund), 491
 Summable function. *See*
 Integrable function
 Summation by parts, 421
 Summation symbol, 17
 Superset, 1
 Support of a function, 156
 Supremum (sup), 12, 28
 Supremum Principle, 12
 Supremum, essential, 306

 Tauberian hypothesis, 495
 Tauberian theorems, 495
 Fejér's, 498
 Hardy's, 498, 500
 Hardy-Littlewood, 495
 Littlewood's, 498
 Tauber's, 500
 Taylor polynomials, 193
 Taylor's Theorem, 194,
 281, 378
 Taylor series, 189. *See*
 Power series
 Telescoping sum, 56
 Term
 of an infinite product,
 413
 of a sequence, 17, 39
 of a series, 54
 Term-by-term differentia-
 tion, 209, 214, 221,
 519

- Term-by-term integration, 267, 269, 276, 278, 282, 288, 289, 291, 511
- Ternary expansion, 66
- Ternary set, Cantor's, 81
- Tietze's Extension Theorem, 134, 154, 311
- Toeplitz matrix, 481
- Tonelli's Theorem, 353
- Topological space, 95
- Topology, 95
- Total variation, 159
- Totally bounded metric space, 112
- Totally disconnected set, 113
- Tower of powers, 185
- Transcendental number, 37, 187, 241
- Triangle inequalities, 24, 91
- Trigonometric functions, 227
- Trigonometric polynomial, 503
- Trigonometric series, 503

- Unconditional convergence of series, 448
- Uncountable set, 29
- Uniform Cauchy sequence, 141
- Uniform closure, 146
- Uniform continuity, 123
- Uniform convergence, 140, 164
- Uniform limit, 140
- Uniform metric, 92
- Uniform norm, 92
- Uniformly distributed, 363
- Union of sets, 2
- Uniqueness theorem for differential equations, 224
- Uniqueness theorem for Fourier coefficients, 508
- Upper bound, 12
- Upper envelope, 132
- Upper function, 271
- Upper semicontinuous function, 132
- Usual metric on \mathbb{R} , 92
- Usual topology, 95, 96

- Value of an infinite product, 411
- Van der Waerden, 174
- Vanish at infinity, 153
- Vanishes nowhere, 148
- Variation
 - bounded, 159
 - finite, 159
 - total, 159
- Variation norm, 163
- Vieta's product, 419
- Vitali-Carathéodory Theorem, 310
- Vitali's Covering Theorem, 335
- Void set, 1
- Volterra's example, 312
- Volume
 - of an n -ball, 394
 - of an n -cone, 397
 - n -dimensional, 346
 - of an n -simplex, 393
 - of revolution, 396

- Wallis Formulas, 250, 280
- Weierstrass Approximation Theorem, 149, 156, 157
- Weierstrass Criterion, 434
- Weierstrass Double Series Theorem, 450
- Weierstrass M-test, 141
- Weierstrass' nowhere differentiable continuous functions, 562
- Well-ordering, 15
- Weyl's Theorem on uniform distribution, 362
- Wiener-Levy Theorem, 525
- Wiener's Theorem on reciprocals, 526

- Zygmund's Theorem on absolute convergence, 521

ISBN: 978-1-4704-2544-9



CHEL/376.H

