An Introduction to Complex and Algebraic Geometry- With focus on compact Riemann Surafces

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Contents

Part I

Differential Geometry of Real Surfaces

Chapter 1

Differential Geometry of Real Surafces

1.1 Fundamental of (2-dimensional) Riemannian Geometry

Let M be a smooth differential manifold of dimension m and let $p \in M$. The tangent space T_pM is a collection of tangent vectors v_p to M at the point p, here a tangent vector v_p is a map $v_p : C^\infty(M) \to \mathbf{R}$ such that (i) $v_p(af + bg) = av_p(f) + bv_p(g)$, (ii) $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$. Let (U, ϕ) a local coordinate for M at p with coordinate functions $x^k = \pi^k \circ \phi : U \to \mathbf{R}$ (so, for each $p \in U$, $\phi(p) = (x^1(p), \ldots, x^m(p)))$ (Note, sometime, we just write the local coordinate (U, ϕ) as $x : U \to \mathbf{R}^m$). Then we have special tangent vectors $\{\frac{\partial}{\partial x^k} |_p, 1 \le k \le m\}$ (called the **partial derivatives**)

$$\frac{\partial}{\partial x^k} |_p : C^\infty(X) \to \mathbf{R}$$

defined by

$$\frac{\partial}{\partial x^k} |_p (f) = D_k(f \circ \phi^{-1})(\phi(p)),$$

where $D_k(f \circ \phi^{-1})(\phi(p))$ means the ordinary x^k-partial derivative of the function $f \circ \phi^{-1}$ at the point $\phi(p)$. It is clear that $\{\frac{\partial}{\partial x^k} |_p, 1 \le k \le m\}$ forms a basis for T_pX, i.e. for every $v_p \in T_pX$,

$$v_p = \sum_{k=1}^{m} v_p(x^k) \frac{\partial}{\partial x^k} |_p .$$

Let M be a 2-dimensional real smooth manifold (surface). A vector field X assigns, at each point $p \in M$, a vector $X(p) \in T_pM$. Its dual is the differential

1-form ω. Locally, we can write $\omega = adu^1 + bdu^2$, with the following change of variables rule: let $u^1 = u^1(v^1, v^2), u^2 = u^2(v^1, v^2)$, then, for $1 \le j \le 2$,

$$du^i = \sum_{j=1}^{2} \frac{\partial u^i}{\partial v^j} dv^j.$$

1.2 Fundamental of Riemannian Geometry

Let M be a Riemannian manifold of dimension n. Let g be the Riemannian metric of M. The following theorem is called the fundamental theorem of Riemannian geometry:

Theorem. There exists a unique connection D (Levi-Civita connection) of M satisfies

1. (compatible with the metric) $Z < X, Y > = < D_Z X, Y > + < X, D_Z Y >$
2. (torsion free) $D_X Y - D_Y X = [X, Y]$

Let $\{X_i\}$ be a local orthonormal frame on M (local frame for TU). Let $\{\theta^i\}$ be the dual co-frame. Write

$$D_Z X_i = \sum_{j=1}^{m} \omega_i^j(Z) X_j$$

ω_i^j are called **connection forms** of D with respect to the local frame $\{X_i\}$. $\omega = (\omega_i^j)$ is the **connection matrix**.

Equivalently, if we use Christoffel sybmol, i.e. write

$$D_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

and write $[X_i, X_j] = \sum_k C_{ij}^k X_k$. Then

$$\Gamma_{ij}^k = \Gamma_{ik}^j, \quad \Gamma_{ij}^k - \Gamma_{ji}^k - C_{ij}^k = 0.$$

Let ω_j^i be 1-forms such that

$$\omega_j^k(X_i) = \Gamma_{ij}^k.$$

Then

$$DX_j = \sum_k X_k \omega_j^k$$

or

$$D\mathbf{X} = \mathbf{X}\omega.$$

The first structure equation

$$\omega_j^i + \omega_i^j = 0$$

$$d\theta^i = -\sum_{j=1}^{m} \omega_j^i \wedge \theta^j$$

or

$$d\theta = \omega \wedge \theta = 0.$$

The second structre equation is: define the curvature matrix

$$\Omega := d\omega + \omega \wedge \omega.$$

Write

$$\Omega_i^j = \frac{1}{2} \sum_{k,l=1}^{m} R_{ikl}^j \theta^k \wedge \theta^l,$$

where $R(X_k, X_l)X_i = R_{ikl}^j X_j$ which is called the curvature tensor.

In the change of coordinates

$$\tilde{\mathbf{X}} = \mathbf{X} \cdot \mathbf{A},$$

then

$$\tilde{\theta} = A^t \theta,$$
$$\tilde{\omega} = A^{-1}\omega A + A^{-1} dA$$
$$\tilde{\Omega} = A^{-1}\Omega A.$$

In the case when $\dim M = 2$: Since $\omega_j^i + \omega_i^j = 0$, $\omega_1^1 = \omega_2^2 = 0$, $\omega_1^2 = -\omega_2^1$. Hence the connection matrix is

$$\omega = \begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix}$$

and the curvature matrix is

$$\Omega = \begin{pmatrix} 0 & d\omega_2^1 \\ -d\omega_2^1 & 0 \end{pmatrix}.$$

Note that $\Omega_2^1 = d\omega_2^1$ is an exact form. According to the above "change of frame formula", $\tilde{\Omega} = A^{-1}\Omega A$, hence $\tilde{\Omega}_2^1 = (\det A)\Omega_2^1$. So Ω_2^1 is a globally defined 2-form. Define the Gauss curvature

$$K =< R(X_1, X_2)X_1, X_2 >= \Omega_2^1(X_1, X_2),$$

then

$$\Omega_2^1 = K d\theta^1 \wedge \theta^2 = K d\sigma.$$

1.3 Curves in the Surface, its Geodesic Curvature

Let C be a curve given by $\alpha : I \to M$ be a curve. Write $\alpha' = \sum_{i=1}^2 \xi^i \mathbf{e}_i$ with $\sum_{i=1}^2 (\xi^i)^2 = 1$. Let $\mathbf{T}(s) = \alpha'(s)$ be the tangent vector to the curve, $\mathbf{N} := -\xi^2\mathbf{e}_1 + \xi^2\mathbf{e}_2$. Recall that for a vector field $\mathbf{X} = \sum_{i=1}^2 \xi^i \mathbf{e}_i$ along the curve $(u(t), v(t))$, its covariant derivative along the curve is

$$\frac{D\mathbf{X}}{dt} = \sum_{i=1}^{2} \left(\frac{d\xi^i}{dt} + \sum_{j=1}^{2} \frac{\omega_j^i}{dt} \xi^j \right) \mathbf{e}_i.$$

The geodesic curvature of C is given by

$$\kappa_g := \left\langle \frac{D\mathbf{T}}{ds}, \mathbf{N} \right\rangle.$$

Note that

$$\left\langle \frac{D\mathbf{T}}{ds}, \mathbf{T} \right\rangle = \sum_{i=1}^{2} \left(\frac{d\xi^i}{ds}\xi^i + \frac{\omega_2^1}{ds} + \frac{\omega_2^1}{ds} \right) \xi^1 \xi^2 = 0.$$

We have that

$$\frac{D\mathbf{T}}{ds} = \kappa_g \mathbf{N}.$$

Write $\xi^1 = \cos\theta, \xi^2 = \sin\theta$. Then

$$\frac{D\mathbf{T}}{ds} = \left(-\xi^2 \frac{d\theta}{ds} + \frac{\omega_2^1}{ds}\xi^2 \right) \mathbf{e}_1 + \left(\xi^1 \frac{d\theta}{ds} + \frac{\omega_1^2}{ds}\xi^1 \right) \mathbf{e}_2 = \left(\frac{d\theta}{ds} - \frac{\omega_2^1}{ds} \right) \mathbf{N}.$$

Thus

$$\kappa_g = \frac{d\theta}{ds} - \frac{\omega_2^1}{ds}.$$

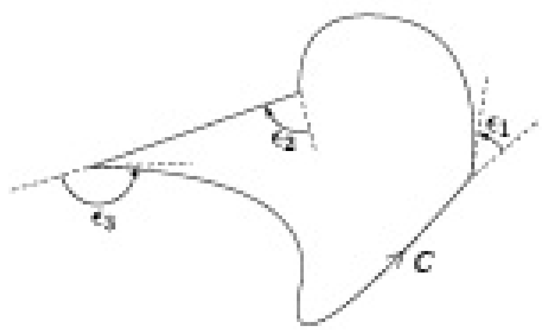
So on the curve C,

$$\omega_2^1 = d\theta - \kappa_g ds.$$

1.4 Gauss-Bonnet theorem

Theorem (Local Gauss-Bonnet). *Suppose that R is a simply connected region with piecewise smooth boundary in a parametrized surface. If $C = \partial R$ has exterior angles $\epsilon_j, j = 1, \ldots, q$, then*

$$\int_{\partial R} \kappa_g ds + \int\int_R K dA + \sum_{j=1}^{q} \epsilon_j = 2\pi.$$



Proof: Take C as a smooth piece of ∂M and the exterior angle ϵ_j at P_j gives the jump of theta as we cross P_j). Then, by Stokes' theorem, we have

$$\begin{aligned} \int\int_M K d\sigma &= -\int\int_M d\omega_{12} = -\int_{\partial M} \omega_{12} = -\int_{\partial M} (\bar{\omega}_{12} - d\theta) \\ &= -\int_{\partial M} \kappa_g ds + (2\pi - \sum \epsilon_j). \end{aligned}$$

When $R = T$ is a geodesic triangle on M (i.e. a region whose boundary consists of three geodesic segments), then it implies that (with $\epsilon_i = \pi - \alpha_i$)):

Theorem(Gauss Formula for embedded triangle) *Let M be a surface in $\mathbf{R}^3$ and let T be an embedded geodesic triangle on M (i.e. a region whose boundary consists of three geodesic segments) with interior angels $\alpha_1, \alpha_2, \alpha_3$. Then*

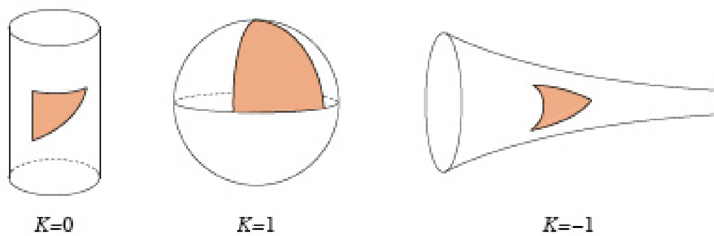
$$\int\int_T K dA = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Remark: The amount $\int\int_T K dA$ is call the *total Gaussian curvature* of T. and $\int_{\partial T} \kappa_g ds$ is called the *total geodesic curvature* of the boundary ∂T.

If the embedded triangle T is a geodesic triangle on M, i.e. it is formed by by the arcs of three geodesics on a surface M, and if A, B, C are interior angles, then the Gauss-Bonnet Formula reduces to what is known as the *Gauss formula*:

$$\int\int_T K dA = A + B + C - \pi.$$

If $K > 0$ on T, then the total sum of its interior angles **exceeds** π. $K < 0$ the total sum of its interior angles is less than π. If $K = 0$, then $A + B + C = \pi$,



GLOBAL VERSION OF THE GAUSS-BONNET THEOREM.

We now consider an oriented surface with piecewise-smooth boundary. T. Rado proved in 1925 that any such surface M can be triangulated. That is, we may write $M = \cup_{\lambda=1}^{m} \triangle_\lambda$ where

(i) $\triangle_\lambda$ is the image of a triangle under an (oriented-preserving) orthgonal parametrization;

(ii) $\triangle_\lambda \cap \triangle_\nu$ is either empty, or single vertex, or a single edge;

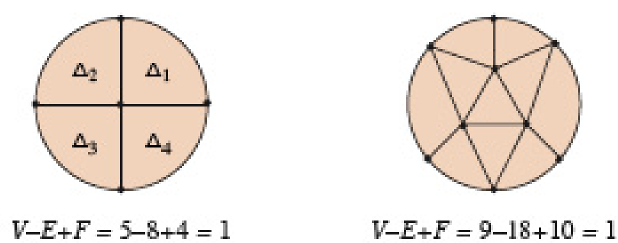
(iii) when $\triangle_\lambda \cap \triangle_\nu$ consists of a single edge, the orientations of the edge are opposite in $\triangle_\lambda$ and $\triangle_\nu$; and

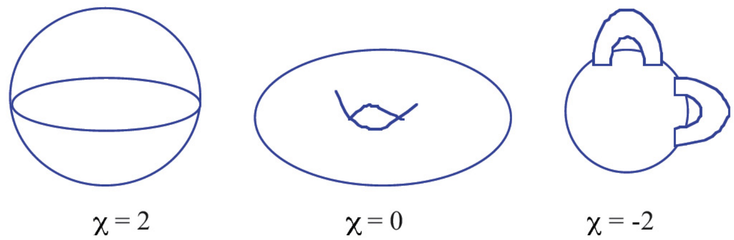
(iv) at most one edge of $\triangle_\lambda$ is contained in the boundary of M.

We now make a standard

Definition *Given a triangulation $\mathcal{T}$ of a surface M with V vertices, E edges, and F faces, we define the Euler characteristic $\chi(M, \mathcal{T}) = V - E + F$.*

We can triangulate a disk as





When M is compact (i.e. without the boundary), then we have the following neat formula:

Theorem (Gauss-Bonnet Formula for compact surface) *Let M be a surface compact surface in $\mathbf{R}^3$ without boundary. Then*

$$\int\int_M KdA = 2\pi\chi(M)$$

where K is the Gauss curvature, dA is the area measure, and $\chi(M)$ is the Euler characteristic of M.

The above theorem shows that the Euler characteristic $\chi(M, \mathcal{T})$ is indeed **independent** of the choice of the triangulation $\mathcal{T}$. It is the property of M itself. It is therefore legitimate to denote the Euler characteristic by $\chi(M)$.

Here is the proof of Gauss-Bonnet in the case that M is compact: Let $M = \cup\triangle_\lambda$ be a traingulation. Then

$$\int\int_M KdA = \sum_\lambda \int\int_{\triangle_\lambda} KdA.$$

Using the local Gauss-Bonnet for triangles $\triangle_\lambda$, we get

$$\int\int_{\triangle_\lambda} KdA + \int_{\partial\triangle_\lambda} \kappa_g ds = \sum_{j=1}^3 \ell_j - \pi,$$

where $\ell_j, 1 \leq j \leq 3$ are the three interior angles of the triangle $\triangle_\lambda$. By summing up, notice that the integrals $\int_{\partial\triangle_\lambda} \kappa_g ds$ cancel in Paris due to the opposite orientation, we have

$$\int\int_M KdA = \sum_\lambda \sum_{j=1}^3 \ell_j - \pi F,$$

where F is the number of triangles $\triangle_\lambda$ (i.e. the number of faces). Notice that at each vertice, the sum of all interior angles is 2π, so

$$\int\int_M KdA = 2\pi V - \pi F,$$

where $V = \#$ of vertices. Use the fact that M does not have boundary, every triangle has three edges, and each edge share with two triangles, hence $3F = 2E$, so

$$\begin{aligned}\int\int_M KdA &= 2\pi V - \pi F = 2\pi V - \pi(2F - 2E) \\ &= 2\pi(E + V - F) = 2\pi\chi(M).\end{aligned}$$

This proves our theorem.

Part II

The Theory of Compact Riemann Surfaces

Chapter 2

Basics about Riemann Surfaces

2.1 Riemann surfaces (and complex manifolds)

An n-dimensional complex manifold M is a Hausdorff paracompact topological space with a local coordinate covering $\{U_i, \Phi_i\}$ such such

(1) Each U_i is an open subset of M and $\cup U_i = M$,

(2) $\Phi_i : U_i \to U_i^0$ is a homeomorphism from U_i onto an open subset $U_i^0 \subset \mathbf{C}^n$,

(3) If $U_i \cap U_j \neq \emptyset$, then $\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$ are holomorphic.

A Riemann surface M is a (connected) complex manifold of dimension one. $\Phi : U \to \mathbf{C}$ is called a (coordinate) chart. $\Phi^{-1} : \Phi(U) \subset \mathbf{C} \to M$ is called a (local) parametrization.

Examples: The complex plane $\mathbf{C}$ is the first example of a Riemann surface. Its only chart is $U = \mathbf{C}$ with the identity map to $\mathbf{C}$. The Riemann sphere $\hat{\mathbf{C}}$ is the first example of a compact RS. Its atlas can be built from two charts (coordinate system): $U_0 = \hat{\mathbf{C}} - \infty = \mathbf{C}$ and Φ_0 is the identity map, $U_1 = \hat{\mathbf{C}} - \{0\}$ and $\Phi_1(z) = 1/z$ if $z \neq \infty$ and $\Phi_1(\infty) = 0$. Then $\Phi_0 \circ \Phi_1^{-1} : \mathbf{C}^* \to \mathbf{C}^* : \Phi_0 \circ \Phi_1^{-1}(z) = 1/z$. The sphere $\Sigma = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is also a compact RS where $U_0 = \Sigma - \{\text{north pole}\}$, $U_1 = \Sigma - \{\text{south pole}\}$, $\Phi_1(p_1, p_2, p_3) = \frac{p_1 + ip_2}{1 - p_3}$, $\Phi_0(p_1, p_2, p_3) = \frac{p_1 - ip_2}{1 - p_3}$, $\Phi_0 \circ \Phi_1^{-1} : \mathbf{C}^* \to \mathbf{C}^* : \Phi_0 \circ \Phi_1^{-1}(z) = 1/z$.

More examples:

(1) Complex projective space: $\mathbf{P}^1(\mathbf{C}) := \mathbf{C}^2 - \{0\}/\sim$ where (z_1, z_2) is equivalent to (w_1, w_2) if and only if $(w_1, w_2) = \lambda(z_1, z_2)$. Let $U_1 = [1, z_2]$, $\phi_1 : U_1 \to \mathbf{C}$ by $[1, z_2] \mapsto z_2$ and $U_2 = [z_1, 1]$, $\phi_2 : U_2 \to \mathbf{C}$ by $[z_1, 1] \mapsto z_1$.

(2) Complex Torus: $X = \mathbf{C}/\Lambda$. Let $\omega_1, \omega_2 \in \mathbf{C}$ be $\mathbf{R}$-linear independent. Consider the lattice $\Lambda := \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. We say that $z_1, z_2 \in \mathbf{C}$ are equivalent if

$z_1 - z_2 \in \Lambda$, so the quotient space $X := \mathbf{C}/\Lambda$ is well-defined (its elements are equivalent classes $[z], z \in \mathbf{C}$). Let $\pi : \mathbf{C} \to X$ be the natural projection (i.e. for $\pi : z \mapsto [z]$). We define $U \subset X$ to be open if $\pi^{-1}(U) \subset \mathbf{C}$ is open. This defines the topology on X. To find the chart of X, consider the parallelogramme $D = \{s\omega_1 + t\omega_2, 0 < s < 1, 0 < t < 1\}$. Then D has the following properties: (i) $\pi|_D$ is 1-1; (ii) $\pi|_{\bar{D}}$ is onto. In other words, every two points $z_1, z_2 \in D$ are not equivalent, and for every point $[z] \in X$ we can find its representation $z \in \bar{D}$. D is called the *fundamental region* of X. It is also easy to see that $\pi : \mathbf{C} \to X$ is locally one to one, i.e. there exists a $\delta > 0$ such that for every $w \in \mathbf{C}$, the map π, when being restricted to the δ-neighborhood of w, i.e. $V_w = \{z \in \mathbf{C} \mid |z - w| < \delta\}$, is one-to-one. Let $U_w = \pi(V_w), \phi_w = (\pi|_{V_w})^{-1}$. Then $\{U_w, \phi_w\}$ forms a coordinate system for X. Thus X is a Riemann surface.

2.2 Mappings between Riemann Surfaces

Let X and Y be two complex manifolds. A continuous map $f : X \to Y$ is called *a holomorphic map* if for each pair of charts $\phi : U \to \mathbf{C}, \psi : U \to \mathbf{C}$, the composition $\psi \circ f \circ \phi^{-1}$ is holomorphic. A holomorphic map $f : M \to \mathbf{C}$ is called *a holomorphic function*. Note that the notions of harmonic and sub-harmonic functions can also be extended to the RS.

Properties of holomorphic functions extend to manifolds:

(1) If M and N are Riemann surfaces (or complex manifolds) with M connected and $f, g : M \to N$ are holomorphic and coincide on a set with a limit point, then $f = g$ on M. Consider the set of points in which f, g coincide in a neighborhood. It is open (automatic). It is closed (given a sequence $\{z_k\}$ its tail lies in one chart). It is not empty, for it contains the limit point; so f, g must coincide everywhere on M.

(2) Suppose M is connected and f is holomorphic on M if $|f|$ has a relative maximum, it is constant. If $|f|$ has a relative maximum, in a neighborhood, it coincides with the constant function, use part (1).

From the maximal principle, every holomorphic map on a compact RS must be constant. As a result, meromorphic functions on a compact RS is more interesting.

Let $W \subset M$ be an open subset. We say a function f on W is meromorphic at $p \in W$ if f is is holomorphic on a punctured neighborhood of a point p and has either a pole or a removable singularity at p. The function $f : M \to \mathbf{C}$ is said to be a **meromorphic function** if there exists a discrete set $\{p_i\} \subset M$ such that $f : M \backslash \{p_j\}_{j=1}^{\infty} \to \mathbf{C}$ is holomorphic and f is meromorphic at each p_j. For example, consider the torus $X = \mathbf{C}/\Lambda$. We define a meromorphic function $\mathcal{P} : \mathbf{C} \to \mathbf{C}$ as follows:

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{0 \neq \omega \in L} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right).$$

Ignoring issues of convergence, observe that $\mathcal{P}(z+\omega) = \mathbf{P}(z)$ for all $\omega \in L$, thus $\mathcal{P}$ determined a unique meromorphic function on X, which (both) is called the Weierstrass $\mathcal{P}$-function. We also have the well-defined notion of order, which is denoted by $ord_p(f)$ (note: $ord_p(f) = k$ if p is a zero of f order k, and $ord_p(f) = -k$ if p is a pole of f order k).

2.3 Differential Forms

A 0-form on M is a function on M. A 1-form ω is an (ordered) assignment, for every local coordinate (U, z_U), $\omega = f_U dz_U + g_U d\bar{z}_U$, where f_U and g_U are two (local) functions, and is invariant under coordinate change, i.e. and for every (U, z_U) and (W, z_W), on $U \cap W$, we have $\omega = f_U dz_U + g_U d\bar{z}_U = f_W dz_W + g_W d\bar{z}_W$.

A 2-form Ω is an assignment, for every local coordinate (U, z_U), $\Omega = f_U dz_U \wedge d\bar{z}_U$, where f_U is a (local) function, and is invariant under coordinate change. Here we used the "exterior" multiplication of forms. This (wedge) multiplication satisfies the following: $dz \wedge dz = 0, dz \wedge d\bar{z} = -d\bar{z} \wedge dz, d\bar{z} \wedge d\bar{z} = 0$.

If f is a C^1 function on M, then $df := \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \partial f + \bar{\partial} f$ is a 1-form. d is called the exterior operator. The $d\omega$ for any 1-form ω is defined in a similar manner.

Lemma (Partition of Unit). *The existence of partitions of unity assumes two distinct forms: Given any open cover $\{U_i\}_{i \in I}$ of M.*

1. There exists a partition $\{\rho_i\}_{i \in I}$ indexed over the same set I such that supp $\rho_i \subset U_i$. Such a partition is said to be subordinate to the open cover $\{U_i\}_{i \in I}$.

2. There exists a partition $\{\rho_i\}_{i \in I}$ indexed over a possibly distinct index set J such that each supp ρ_j has compact support and for each $j \in J$, supp $\rho_j \subset U_i$ for some $i \in I$.

Thus one chooses either to have the supports indexed by the open cover, or the supports compact. If M is compact, then there exist partitions satisfying both requirements.

2.4 Integration of Differential Forms

Integration of 1-form: Let γ be piecewise smooth curve in M, and ω be a smooth 1-form on M. Let $\{(U_\alpha, \phi_\alpha\}_{\alpha \in A}$ be a collection of local coordinates (with $\cup_{\alpha \in A} U_\alpha = M$).

Case 1: Assume either γ lies in U_α or Supp $\omega \subset U_\alpha$ for some $\alpha \in A$ where Supp$\omega = \overline{\{p \in M \mid \omega(p) \neq 0\}}$. We define, write $\omega = f_\alpha dz_\alpha + g_\alpha d\bar{z}_\alpha$ on U_α,

$$\int_\gamma \omega := \int_a^b \left(f_\alpha(\phi_\alpha \circ \gamma) \frac{d\phi_\alpha \circ \gamma}{dt} + g_\alpha(\phi_\alpha \circ \gamma) \frac{d\overline{\phi_\alpha \circ \gamma}}{dt} \right) dt,$$

where $\gamma : [a, b] \to U_\alpha$ is a parameterization of the curve γ.

Case 2 (general case): In general, take a partition of unit $\{\rho_\alpha\}_{\alpha\in A}$, subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$, using $\sum_{\alpha\in A}\rho_\alpha \equiv 1$, we define

$$\int_\gamma \omega := \sum_{\alpha\in A}\int_\gamma (\rho_\alpha\omega).$$

Note that the key fact is that Supp $(\rho_\alpha\omega) \subset U_\alpha$, so $\int_\gamma(\rho_\alpha\omega)$ is defined in Case 1.

The integration of a two form Ω over a region $D \subset M$ is defined in a similar manner as above by using the partition of unit.

Stokes Theorem. *Let ω be a 1-form, $D \subset M$ is a closed domain with smooth boundary, then*

$$\int_{\partial D}\omega = \int_D d\omega.$$

2.5 Residues

Let $\omega = fdz$ be a meromorphic 1-form, and $p \in M$ be a pole of ω. Define $\mathrm{res}_p\omega := res_p(f)$, it is easy to check that the definition is independent of the choice of the coordinate. Alternatively, for a small disc D centered at p,

$$res_p(\omega) = \frac{1}{2\pi i}\int_{\partial D}\omega.$$

Theorem (Residue Theorem). *Let M be a RS and ω be a meromorphic 1-form on M. Let $D \subset M$ be an open subset whose closure is compact, ∂D is piecewise smooth, and ∂D does not contain the poles of ω. For any meromorphic 1-form ω,*

$$\int_{\partial D}\omega = \sum_{p\in D} res_p\omega$$

Proof. Note that since $\bar{D}$ is compact, the above sum is only a finite sum. Assume $p_1, \ldots, p_k$ are poles of ω in D. Let B_j be the small discs containing p_j only and mutually disjoint. Let $E = D - \cup_{j=1}^k B_j$, then ω is holomorphic on E, so $d\omega = 0$ on E. From the Stoke's theorem,

$$0 = \int_E d\omega = \int_{\partial E}\omega = \int_{\partial D}\omega - \sum_{j=1}^k \int_{\partial B_j}\omega$$

which proves the theorem.

Corollary. *If M is compact, then for any meromorphic 1-form ω,*

$$\sum_{p\in M} res_p\omega = 0.$$

Corollary. *Let M be RS and $D \subset M$ be an open subset whose closure is compact and whose boundary is piecewise smooth. If f is meromorphic on M with no zeros or poles on ∂D, then*

$$\frac{1}{2\pi}\int_{\partial D}\frac{df}{f} = \sum_{x\in D} ord_x(f).$$

Proof. By applying the above theorem with $\omega = df/f$.

Corollary. *Let M be a compact RS and f be meromorphic on M, then*

$$\sum_{x\in M} ord_x(f) = 0.$$

2.6 Holomorphic mappings between Riemann Surfaces

A meromorphic function f on M can be viewed as a holomorphic mapping $f : M \to \mathbf{P}^1$. Thus, it is important to study the properties for general holomorphic mappings between RS.

Let X any Y be two RS. A continuous map $f : X \to Y$ is called *a holomorphic map* (and we usually will not consider other maps between RS) if for each pair of charts $\phi : U \to \mathbf{C}, \psi : U \to \mathbf{C}$, the composition $\psi \circ f \circ \phi^{-1}$ is holomorphic.

Theorem (Normal Form Theorem). *Let $F : X \to Y$ be a holomorphic map between two RSs, and $x \in X$. Then there exist two coordinate charts $\phi_1 : U_1 \to V_1, \phi_2 : U_2 \to V_2$ at x and $F(x)$ respectively and a unique integer $m = m_x$ (which is called the multiplicity) such that $\phi_1(x) = \phi_2(F(x)) = 0$ and*

$$\phi_2 \circ F \circ \phi_1^{-1}(z) = z^m.$$

Proof. Choose any pair of coordinate charts. After translation, we assume that $\tilde{\phi}_1(x) = \phi_2(F(x)) = 0$. Then $\phi_2 \circ F \circ \tilde{\phi}_1^{-1}(\zeta) = \zeta^m e^{h(\zeta)}$. Let $\psi(\zeta) := \zeta e^{\frac{1}{m}h(\zeta)}$ which is locally 1-1. Let $\phi_1 := \psi \circ \tilde{\phi}_1$. This will serve our purpose.

Definition. *(1) We call* $m := Mult_x(F)$ *the multiplicity of* F *at* $x \in X$.

(2) If $Mult_x(F) \geq 2$, *we say that* F *is ramified at* x *and that* x *is a ramification point for* F.

(3) If $p \in X$ *is a ramification point for* F, *we call* $F(p)$ *a branch point of* F.

Degree of a holomorphic map.

Theorem. *Let* $F : X \to Y$ *be a holomorphic map between two connected compact RSs. Then*

$$\deg(F) := \sum_{x \in F^{-1}(y)} Mul_x(F)$$

is independent of y.

Riemann-Hurwitz Formula:

Definition. *Let* M *be a compact RS (regarded as a manifold of real-dimension 2) with smooth boundary (possibly empty),*

(1) A 0*-simplex, or vertex, is a point. A 1-simplex, or edge, is a set homeomorphic to a closed interval. A 2-simplex, or face, is a set homeomorphic to the triangle* $\{(x, y) \in [0, 1] \times [0, 1]; x + y \leq 1\}$.

(2) A triangulation of M *is a decomposition of* M *into faces, edges and vertices, such that the intersection of any two faces is a union of edges and the intersection of any two edges is a union of vertices.*

(3) Let M *have a triangulation with total number of faces equal to* F, *total number of edges equal to* E, *and total number of vertices equal to* V. *The number* $\chi(M) := F - E + V$ *is independent of the choices of the triangulation, which is called the Euler characteristic of* M. $\chi(M) := 2 - 2g$ *where* g *is called the genus of* M.

Theorem (Riemann-Hurwitz formula). $F : X \to Y$ *be a holomorphic map between two connected compact RSs. Then*

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{x \in X} (Mult_x(f) - 1).$$

Proof. Let $d = \deg(f)$. Take a triangulation of Y such that every branch point is a vertex. (There may, of course, be other vertices). Suppose this triangulation has F faces, E edges, V_u unbranched verticies, and V_b branched vertices.

Since the preimage of every unbranched point has d points, we obtain a triangulation of X with dF faces, dE edges and W vertices. To express W in terms of V and f, we observe that if $x \in X$ is a ramification point for f, then $Mult_x(f)$-many points are collapsed into one point, so that we have

$$W = dV - \sum_{y \in V_b} \sum_{x \in f^{-1}(y)} Mult_x(f) - 1 = dV - \sum_{x \in X} (Mult_x(f) - 1).$$

The last equality follows because $Mult_x(f) = 1$ for all unramified points x. This proves the theorem.

2.7 Automorphism groups of Complex Tori

Let $M = \mathbf{C}/\Lambda$, where $\Lambda := \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, and $\omega_1, \omega_2 \in \mathbf{C}$ are $\mathbf{R}$-linear independent.

Theorem. *$f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is a biholomorphic map if and only if there exists $F(z) = az + b$ with $a \neq 0$ such that F maps the equivalent classes w.r.t Λ_1 to equivalent classes w.r.t. Λ_2.*

The proof uses the lifting property (for universal coverings) from $f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ to get $F : \mathbf{C} \to \mathbf{C}$ and use the following result proved in last semester: If $F \in Aut(\mathbf{C})$ then $F = az + b$.

Corollary. *$\mathbf{C}/\Lambda_1$ is biholomorphic to $\mathbf{C}/\Lambda_2$ iff there exists $a \neq 0$ such that $F(z) = az$ sends an equivalent class with respect to Λ_1 to the equivalent class with respect to Λ_2.*

Hence,

$$a \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = F \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix},$$

and

$$F^{-1} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = B \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = F^{-1} \circ F \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = F^{-1}(A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = AF^{-1} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = AB \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Since ω_1 and ω_2 are real-linearly independent, $AB = I$. Hence $det(A)\det(B) = 1$. Since entries of A and B are integers, $det(A) = \pm 1$. Let $\tau = \omega_1/\omega_2$, $\tau' = \omega_1'/\omega_2'$. Then we have

Theorem. *Let $\Lambda = Span_{\mathbf{Z}}\{1, \tau\}, \Lambda' = Span_{\mathbf{Z}}\{1, \tau'\}$, with $Im\tau, Im\tau' > 0$. Then $\mathbf{C}/\Lambda$ is biholomorphic to $\mathbf{C}/\Lambda'$ if and only if*

$$\tau' = \frac{a_{11}\tau + a_{12}}{a_{21}\tau + a_{22}}, \qquad (*)$$

where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbf{Z}$ and $a_{11}a_{22} - a_{12}a_{21} = 1$.

We now introduce an equivalent relation as follows: $\mathbf{C}/\Lambda_1 \sim \mathbf{C}/\Lambda_2$ iff $\mathbf{C}/\Lambda_1$ is biholomorphic to $\mathbf{C}/\Lambda_2$, and denote by $\mathcal{A}_1$ the set of equivalent classes. So, from the theorem, $\Lambda_1 = \{1, \tau\}, \Lambda_2 = \{1, \tau'\}$, then they below to the same equivalent class if and only if (*) is satisfied. To describe clearly about $\mathcal{A}_1$. W consider $H = \{\tau \in \mathbf{C} \mid Im(z) > 0\}$ the upper-half plane on $\mathbf{C}$. Then (*) defines a map

$$\tau \mapsto \tau' = \frac{a_{11}\tau + a_{12}}{a_{21}\tau + a_{22}}, \ a_{11}a_{22} - a_{12}a_{21} = 1.$$

The set of such transformation becomes a group, and is denoted by $SL(2, \mathbf{Z})$ (called the modular group). We now define the fundamental domain $D \subset H$ of the modular group as the subset such that (i) every $\tau \in H$ is congruent to $\tau' \in D$ mod $SL(2, \mathbf{Z})$, (ii) Any two distinct points in D are not congruent mod $SL(2, \mathbf{Z})$.

A modular function is a holomorphic function or a meromorphic function defined on H which is invariant under the action of the group $SL(2, \mathbf{Z})$.

Chapter 3

The Theory of Differential Forms

3.1 The DeRham Cohomology $H^1_{DR}(M)$ and Its Pairing with $H_1(M, \mathbf{Z})$.

Differential Forms on a Riemann surface M: Recall that a 0-form on M is a function on M. A 1-form ω is an (ordered) assignment, for every local coordinate (U, z_U), $\omega = f_U dz_U + g_U d\bar{z}_U$, where f_U and g_U are two (local) functions, and is invariant under coordinate change, i.e. and for every (U, z_U) and (W, z_W), on $U \cap W$, we have $\omega = f_U dz_U + g_U d\bar{z}_U = f_W dz_W + g_W d\bar{z}_W$. A 2-form Ω is an assignment, for every local coordinate (U, z_U), $\Omega = f_U dz_U \wedge d\bar{z}_U$, where f_U is a (local) function, and is invariant under coordinate change. Here we used the "exterior" multiplication of forms. This (wedge) multiplication satisfies the following: $dz \wedge dz = 0, dz \wedge d\bar{z} = -d\bar{z} \wedge dz, d\bar{z} \wedge d\bar{z} = 0$. If f is a C^1 function on M, then $df := \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \partial f + \bar{\partial} f$ is a 1-form. d is called the exterior operator. The $d\omega$ for any 1-form ω is defined in a similar manner.

A 1-form ω is said to be d-closed (or just closed) if $d\omega = 0$. It is said to be d-exact if $\omega = df$ for some (global) smooth function f on M. Let $\Lambda_1(M)$ be the set of smooth closed 1-form on M. Two elements $\omega_1, \omega_2 \in \Lambda_1(M)$ are said to be equivalent if $\omega_1 - \omega_2$ is $d-$exact, i.e. $\omega_1 - \omega_2 = df$ for some smooth function f on M. Denote by $[\omega]$ the equivalent class of ω. The (free abelian) group (or a vector space) of the collection of all such equivalent classes is called the *de Rahm* cohomology, and is denoted by $H^1_{DR}(M)$, i.e.

$$H^1_{DR}(M) := \frac{\{\text{smooth closed 1-forms}\}}{\{\text{smooth exact 1-forms}\}}.$$

Pairing of $H_1(M,\mathbf{Z})$ and $H^1_{DR}(M)$: Define

$$([\gamma],[\omega]) \in \pi_1(M) \times H^1_{DR}(M) \mapsto \int_\gamma \omega \in \mathbf{C},$$

where $\pi_1(M)$ is the fundamental group of M. It is clear from the properties of integrals that the map is a homomorphism, and thus, since $\mathbf{C}$ is an abelian group, the kernel of this map must contain the commutation subgroup of $\pi_1(M)$. Define the quotient group

$$H_1(M,\mathbf{Z}) := \pi_1(M)/[\pi_1(M), \pi_1(M)].$$

$H_1(M,\mathbf{Z})$ is called **the first homology group of the surface** (it is a free-abelian group).

The pairing is non-degenerate, i.e. it satisfies that if $(\gamma,\omega) = 0$ for all d-closed ω, then $[\gamma] = 0$, and if $(\gamma,\omega) = 0 = 0$ for all $[\gamma] \in H_1(M,\mathbf{Z})$, then $\omega = 0$. Thus $\dim_{\mathbf{C}} H^1_{DR}(M)$ =rank of $H_1(M,\mathbf{Z}) = 2g$, where g is the genus of M.

More information about $H_1(M,\mathbf{Z})$ (topology of the RS): Here is an **alternative definition of $H_1(M,\mathbf{Z})$**: Recall

Definition. *Let M be a compact RS (regarded as a manifold of real-dimension 2).*

(1) A 0-simplex, or vertex, is a point. A 1-simplex, or edge, is a set homeomorphic to a closed interval. A 2-simplex, or face, is a set homeomorphic to the triangle $\{(x,y) \in [0,1] \times [0,1]; x+y \leq 1\}$.

(2) A triangulation of M is a decomposition of M into faces, edges and vertices, such that the intersection of any two faces is a union of edges and the intersection of any two edges is a union of vertices.

(3) Let M have a triangulation with total number of faces equal to F, total number of edges equal to E, and total number of vertices equal to V. The number $\chi(M) := F - E + V$ is independent of the choices of the triangulation, which is called the Euler characteristic of M. $\chi(M) := 2-2g$ where g is called the genus of M.

A n-chain is a finite combination of differential maps of a n-th dimensional simplex into M. A simplex carries an orientation: using this, we can define a boundary map ∂ on chains: if e.g. (p_1,p_2,p_3) is the oriented triangle bounded by the oriented edges $(p_1,p_2),(p_2,p_3)$ and (p_3,p_1), then

$$\partial < p_1,p_2,p_3 >=< p_2,p_3 > - < p_1,p_3 > + < p_1,p_2 > .$$

here the minus sign denotes the reversal of orientations, thus $-(p_1,p_3) = (p_3,p_1)$. Similarly,

$$\delta < p_1,p_2 >= p_2 - p_1.$$

Thus ∂ defined on simplices can be extended by linearity to a boundary operator on chains of M, and satisfies
$$\partial^2 = 0.$$
A chain C is called a cycle if $\partial C = 0$, and is called a boundary if $C' = \partial C$. The $j-$th homology group of M with coefficients in $\mathbf{Z}$ is defined as

$$H_j(M, \mathbf{Z}) := \frac{\{j\text{-dimensional cycles}\}}{\{j\text{-dimensional boundaries}\}}.$$

Observe that freely homotopic closed curves are homologous. Indeed, let $\gamma_0 : S^1 \to M$ and $\gamma_1 : S^1 \to M$ be two closed curves in M (S^1 being interval $[0, 1]$ with its end-points identified), and

$$H : S^1 \times [0, 1] \to M$$

is a homotopy between them (so that $H(t, 0) = \gamma_0, H(t, 1) = \gamma_1$). Then $\gamma_0 - \gamma_1 = \partial H(A)$, so that γ_0, γ_1 are homologous as asserted. The converse is however false in general: since homology groups are always abelian, any curves γ whose homotopy class of the form $aba^{-1}b^{-1}$ $(a, b, \in \pi_1(M, p_0))$ is always null-homologous, but not necessarily null-homotopic, since $\pi_1(M, p_0)$ is not abelian if $g \geq 2$. By the theorem of van Hampen,

$$H_1(M, \mathbf{Z}) := \pi_1(M)/[\pi_1(M), \pi_1(M)].$$

3.2 The Canonical Basis for $H_1(M, \mathbf{Z})$ and $H_{RD}(M)$

According to the uniformization theorem, every compact orientable 2-real dimensional manifold is hemeomorphic to g-torus (g is called the genus of M) with $g \geq 0$. We wish now to use the standard presentation of a compact R.S. of genus g. For $g = 0$, it is holomeomorphic to a sphere which is simply connected. For $g > 0$, there are $2g$ closed curves which have a common starting and end point, which is denoted by p_0, say $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, and M can be obtained from a $4g$-gon by identification of the edges defined by the word

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

With the common vertex of the sides as a base point p_0, one shows that $\pi_1(M)$ is generated by the simple loops $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ corresponding to the edges x_i and y_i, subject to one relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

Hence the holomogy group $H_1(M, \mathbf{Z})$ is free abelian group on the generators $[a_j], [b_j], j = 1, \ldots, g$. In particular, we get

$$H_1(M, \mathbf{Z}) = \mathbf{Z}^{2g}.$$

Let $a, b \in H_1(M, \mathbf{Z})$ represented by closed curves γ_1, γ_2 respectively. Then the **intersection number** of a, b is defined as

$$a \cdot b := \int_{\gamma_1} \eta_{\gamma_2} \left(= \int_M \eta_{\gamma_1} \wedge \eta_{\gamma_2} = - \int_{\gamma_2} \eta_{\gamma_1} \right),$$

where η_γ is the **the one-form defined by a closed curve** γ which can be constructed as follows: Since γ is compact, we can find an annular region Ω containing γ in its interior. Since γ is two sided (M is called orientable if all closed curves on M is two-sided), Ω will be separated by γ into a left side Ω^- (after an orientation of γ is given) and a right side Ω^+. We choose another smaller region Ω_0 containing γ which is contained in the interior of Ω. Let Ω_0^- denote the region to the left of γ in Ω_0. We now choose a real-valued C^∞ function on $M \backslash \gamma$ such that

$$f(z) = \begin{cases} 1 & z \in \Omega_0^- \text{ and } z \in \gamma \\ 0 & z \in M \backslash \Omega^- \end{cases}$$

and define

$$\eta_\gamma(z) := \begin{cases} df(z) & z \in \Omega \backslash \gamma \\ 0 & z \in \gamma \text{ or } z \in M \backslash \Omega. \end{cases}$$

The form η_γ is obviously closed, smooth and with compact support, although the function f itself has a jump of height 1 across γ. Although η_γ is closed, it is not in general exact (it turns out that η_γ is exact is γ is homologous to a point). The form η_γ has the following important property:
Claim: If $\omega \in L^2(M) \cap C^1$ is closed, then

$$\int_\gamma \omega = - \int_M \omega \wedge \eta_\gamma.$$

Proof of the claim. We compute, note that η_γ is real,

$$\begin{aligned} - \int_M \omega \wedge \eta_\gamma &= - \int_{\Omega^-} \omega \wedge df = \int_{\Omega^-} df \wedge \omega \\ &= \int_{\Omega^-} d(f\omega) - \int_{\Omega^-} f d\omega = \int_{\Omega^-} d(f\omega) = -int_{\partial\Omega^-} f\omega = \int_\gamma \omega. \end{aligned}$$

It is clear that $a \cdot b \in \mathbf{Z}$, $a \cdot b = -b \cdot a$ and $(a+b) \cdot c = a \cdot c + b \cdot c$.

Proposition. *The intersection pairing satisfies the following properties.*

1. The intersection $a \cdot b$ depends only on the homology classes of a and b.

2. One has $a \cdot b = -b \cdot a$.

3. $a \cdot b \in \mathbf{Z}$. In case the intersection points of the curves a and b are transversal, $a \cdot b$ is the (signed) number of intersection points.

Proof. The first property has already been explained: integrals of a closed form along homotopic paths are the same. The second property results from the anticommutativity of the multiplication of one-forms.

The third property can be checked for simple closed curves since any piecewise smooth closed curve is a finite union of simple closed curves. In this case $a \cdot b = \int_a \eta_b$ and we have to check that each intersection point of $a$with b contributes 1 or -1, depending on the orientation of the curves at the intersection point. Recall that η_b is defined as differential of a function f_b having a discontinuity along b. The function f_b is zero far away from b. Thus, the integral over a can be presented as a sum of the integrals over small segments of a_i of a containing the intersection points x_i of a with b. The integral $\int_{a_i} \eta_b$ has been already calculated once. The result was 1 or -1. This finishes the proof.

From above, we know that the pairing so defined counts the number of times a intersects b. A basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_1(M, \mathbf{Z})$ is siad to be a **canonical basis** if its intersection matrix looks like

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (*)$$

Let $\omega_j = \eta_{b_j},\ \omega_{g+j} = -\eta_{a_j}, j = 1, \ldots, g$. Then

$$\int_{\gamma_j} \omega_k = \delta_{jk}.$$

$\Psi([\omega_j]) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is at the j-th place. $\{\Psi([\omega_1]), \ldots, \Psi([\omega_{2g}])\}$ is called the *canonical basis* basis for $H^1_{DR}(M)$ (with respect to the $\{a_1, \ldots, a_g,$ $b_1, \ldots, b_g\}$).

3.3 The Hodge (theory) Decomposition

Though above pairing gives us a practical way of computing $H^1_{DR}(M)$ (i.e. $\dim_{\mathbf{C}} H^1_{DR}(M) = 2g$), it would be more convenient for computational purpose if a cohomology class is represented by a *unique* differential form rather than an equivalence class of differential forms. The Hodge theorem states that such is case: every equivalence class of differential forms is uniquely represented by the *harmonic* differential form (which is unique).

A 1-from $\omega \in C^1$ is called a *harmonic form* if locally we can write $\omega = df$ where f is harmonic. A 1-from $\omega \in C^1$ is called a *harmonic form* if locally we can write $\omega = df$ where f is harmonic. To further study harmonic forms, we introduce the star-operator: for ant 1-form $\omega = fdz + gd\bar{z}$, $\star\omega := -ifdz + igd\bar{z}$ (note that if $\omega = fdx + gdy$, then $\star\omega = -gdx + fdy$). Remark: We only defined the star-operator here for $1 - forms$ since the star operator is independent of the metric only for 1-forms on Riemann Surface. In general,,we need a metric $\lambda^2 dz d\bar{z}$ on M (it always exists) and define, for any 0-form f, $\star f := f\frac{i}{2}\lambda^2 dz \wedge d\bar{z}$

(here $\frac{i}{2}\lambda^2 dz \wedge d\bar{z}$ is called the Kahler (metric) form associated to the metric), and define, for any two form $\eta = h(z)\frac{i}{2}dz \wedge d\bar{z}$, $\star\eta(z) = \frac{1}{\lambda^2}h(z)$.. So in general, the star operator depends on the metric. The Laplace operator is $\triangle := 2i\partial\bar{\partial}$. It is easy to check that $\triangle = d \star d$.

A 1-form ω is harmonic if and only if ω is closed and is co-closed, i.e. $d\omega = 0$ and $d(\star\omega) = 0$. To see its proof. Obviously, $d\omega = 0$ is obvious since locally $\omega = df$. Moreover, since $\triangle = d \star d$ and f is harmonic, we see that ω is also co-closed.

Hilbert Space Theory:

Weyl's Lemma. *Let* $D(0, R) = \{z \in \mathbf{C} \mid |z| < R\}$. *Then* $\phi \in L^2(D)$ *is a harmonic function if and only if*

$$\int_D \phi \triangle \eta = 0, \quad \forall \eta \in C_0^\infty(D).$$

Proof of the Weyl Lemma. For any given $\epsilon > 0$, choose a real-valued C^∞ function $\rho(r), r \in [0, +\infty)$ such that $\rho_\epsilon(r) \equiv 1$ for $r \in [0, \epsilon/2)$, $\rho_\epsilon(r) \equiv 0$ for $r \in (\epsilon, \infty)$, and $0 \le \rho_\epsilon(r) \le 1$ on $[\epsilon/2, \epsilon]$. Let

$$\Omega_\epsilon(r) = \frac{1}{\pi i}\rho_\epsilon(r)\log r.$$

For any function $\mu \in C_0^\infty(D)$, consider the function

$$\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z}.$$

When ϵ is small enough, η_ϵ has compact support. On the other hand, we can write it as

$$\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z|)\mu(z + \xi) dz \wedge d\bar{z}.$$

Hence η_ϵ is smooth, and

$$\frac{\partial^2}{\partial\bar{\xi}}\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \xi|)\frac{\partial}{\partial\bar{z}}\mu(z) dz \wedge d\bar{z}$$

$$\frac{\partial}{\partial\xi}\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \xi|)\frac{\partial}{\partial z}\mu(z) dz \wedge d\bar{z}.$$

We claim that

$$\frac{\partial^2}{\partial\xi\partial\bar{\xi}}\eta_\epsilon(\xi) = -\mu(\xi) + \int_{\mathbf{C}} \frac{\partial^2}{\partial\xi\partial\bar{\xi}}\Omega_\epsilon(|z - \xi|)\mu(z) dz \wedge d\bar{z}.$$

To prove the claim, fix $\xi_0 \in D$, and write

$$\eta_\epsilon(\xi) \equiv f(\xi) + g(\xi),$$

where ξ satisfies $|\xi - \xi_0| < \epsilon/4$ and

$$f(\xi) = \frac{1}{\pi i} \int_{|z-\xi_0|<\epsilon/4} \mu(z) \ln |z - \xi| dz \wedge d\bar{z}$$

$$g(\xi) = \frac{1}{\pi i} \int_{|z-\xi_0|>\epsilon/4} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z}.$$

It is easy to check that

$$\frac{\partial^2 f}{\partial \bar{\xi}} = -\mu(\xi).$$

When $|\xi - \xi_0| < \epsilon/4$ and $|z - \xi_0| < \epsilon/4$, $|\xi - z| < \epsilon/2$. Hence $\Omega_\epsilon(|z - \zeta|) = \ln |z - \xi|$ $(z \neq \xi)$, and is harmonic in ξ. Therefore,

$$\begin{aligned} \frac{\partial^2 g}{\partial \bar{\xi}} &= \int_{|z-\xi_0|>\epsilon/4} \frac{\partial^2}{\partial \bar{\xi}} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z} \\ &= \int_{\mathbf{C}} \frac{\partial^2}{\partial \bar{\xi}} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z}. \end{aligned}$$

This proves the claim. Assuming the claim holds, then, using $\eta = \eta_\epsilon$ the assumption gets

$$\begin{aligned} 0 &= \frac{1}{2i} \int_D \phi \triangle \eta_\epsilon \\ &= -\int_D \mu(\xi)\phi(\xi) d\xi \wedge d\bar{\xi} + \int_D \phi(\xi) d\xi \wedge d\bar{\xi} \int_{\mathbf{C}} \frac{\partial^2 \Omega_\epsilon(|z - \xi|)}{\partial \xi \partial \bar{\xi}} \mu(z) dz \wedge d\bar{z} \\ &= -\int_{\mathbf{C}} \mu(\xi) \left[\phi(\xi) - \int_D \phi(z) \frac{\partial^2 \Omega_\epsilon(|z - \xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z} \right] d\xi \wedge d\bar{\xi}. \end{aligned}$$

Since μ is arbitrary, we get

$$\phi(\xi) = \int_D \phi(z) \frac{\partial^2 \Omega_\epsilon(|z - \xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

When $|\xi - z| < \epsilon/2$,

$$\frac{\partial^2 \Omega_\epsilon(|z - \xi|)}{\partial z \partial \bar{z}} = 0,$$

hence

$$\phi(\xi) = \int_{D \setminus \triangle_{\epsilon/2}} \phi(z) \frac{\partial^2 \Omega_\epsilon(|z - \xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

Thus $\phi(\xi)$ is smooth. We have proved, in the remark, that if ϕ is C^2, then it is harmonic. This finishes the proof.

We use the Hilbert space theory to decompose the space of square integrable 1-forms (which is a Hilbert space) into closed subspaces. The basic tool is the above Weyl's lemma. A measurable 1-form is called square-integrable if

$$\|\omega\|^2 := \int_M \omega \wedge \star\bar{\omega} < +\infty.$$

Let $L^2(M)$ be the Hilbert space of all square-integrable 1-forms. On $L^2(M)$, we introduce an inner product

$$(\omega_1, \omega_2) := \int_M \omega_1 \wedge \star\overline{\omega_2}.$$

$L^2(M)$ becomes an Hilbert space under this inner product. Let E be the closure in $L^2(M)$ of the set $\{df \mid f \in C_0^\infty(M)\}$, and E^* be the closure in $L^2(M)$ of the set $\{\star df \mid f \in C_0^\infty(M)\}$. We have

$$L^2(M) = E \bigoplus E^\perp, \quad L^2(M) = E^* \bigoplus E^{*\perp}.$$

It is not hard to verify that

$$E^\perp = \{\omega \in L^2(M) \mid (\omega, df) = 0, \quad f \in C_0^\infty(M)\},$$

$$E^{*\perp} = \{\omega \in L^2(M) \mid (\omega, \star df) = 0, \quad f \in C_0^\infty(M)\},$$

Theorem.*Let $\omega \in L^2(M) \cap C^1(M)$. Then*
(i) $\omega \in E^{*\perp}$ if and only if ω is closed.
(ii) $\omega \in E^\perp$ if and only if ω is co-closed.

Proof. Assume that ω is closed. Let f be a smooth function on M with support inside D (D is compact). Then, using $d\omega = 0$,

$$(\omega, \star df) = -\int_D \omega \wedge d\overline{f} = -\int_D [d(\omega\bar{f}) - \bar{f}d\omega] = -\int_D d(\omega\bar{f}) = -\int_{\partial D} (\omega\bar{f}) = 0$$

where the last equality holds because f has compact support. Thus $\omega \in E^{*\perp}$. Conversely, we start from the third equality, and using $-\int_D d(\omega\bar{f}) = 0$, we get

$$\int_M \bar{f} d\omega = 0$$

for all smooth f on M with compact support it suffices to conclude that $d\omega = 0$. So ω is closed. This proves (i). The proof of (ii) is similar.

Corollary. *If ω is C^1, then ω is harmonic if and only if $\omega \in E^\perp \cap E^{*\perp}$.*

The Weyl lemma allows to remove the condition of "smoothness" in above, i.e. we have the following most important result about $L^2(M)$.

Theorem. *$E^\perp \cap E^{*\perp} = H$, where H is the set of harmonic forms (note, the definition of harmonic form requires C^1).*

Proof. If $\omega \in H$, then ω is smooth, closed, and co-closed, so from the theorem above, $\omega \in E^\perp \cap E^{*\perp}$.

For the converse, let $\omega \in E^\perp \cap E^{*\perp}$. Choose a coordinate chart (U, ϕ) on M and write locally $\omega = u(z)dz + v(z)d\bar{z}$. Consider a (any) smooth function on M with compact support in U with local expression $\eta = \eta(z)$. Let $f = \partial\bar{\eta}/\partial z$. Then from $0 = (\omega, df) = (\omega, \star df)$, we get $(\omega, \star df + idf) = 0$, i.e.

$$-\int_{\phi(U)} (u(z)dz + v(z)d\bar{z}) \wedge \eta_{z\bar{z}} dz = \int_{\phi(U)} v\eta_{z\bar{z}} dz \wedge d\bar{z} = 0.$$

By Weyl's theorem, v is harmonic on $\phi(U)$ hence is smooth. Applying this result to $\star\omega$ we see that u is also smooth. Hence ω is smooth. This finishes the proof.

From the definition, $E \subset E^{*\perp}$ and $E^* \subset E^\perp$, Thus elements in E and E^* are always orthogonal to each other. It then follows that the direct sum $(E^\perp \bigoplus E^*)^\perp$ is a closed, and therefore Thus

$$L^2(M) = E \bigoplus E^* \bigoplus (E \bigoplus E^*)^\perp.$$

It is easy to check that $(E \bigoplus E^*)^\perp = E^\perp \cap E^{*\perp}$. This proves

Theorem (Orthogonal Decomposition).

$$L^2(M) = E \bigoplus E^* \bigoplus H$$

where H is the set of all harmonic 1-forms.

The decomposition theorem for smooth differential forms: From above, for $\omega \in L^2(M) = E \bigoplus E^* \bigoplus H$, so every $\omega \in L^2(M)$, $\omega = \alpha + \beta + h, \quad \alpha \in E, \beta \in E^*, h \in H$. However, we need more information about α and β.

Lemma. *If $\omega \in E \cap C^1$, then ω is exact. If $\omega \in E^* \cap C^1$, then ω is co-exact.*

Proof. To prove ω is exact, it suffices to show that $\int_\gamma \omega = 0$. Let η_γ br the 1-form constructed earlier. We now prove
Claim (this is similar to the Riesz's representation theorem!): If $\omega \in L^2(M) \cap C^1$ is closed, then

$$\int_\gamma \omega = (\omega, \star\eta_\gamma).$$

Proof of the claim. We compute, note that η_γ is real,

$$\begin{aligned}(\omega, \star\eta_\gamma) &= -\int_M \omega \wedge \eta_\gamma = -\int_{\Omega^-} \omega \wedge df = \int_{\Omega^-} df \wedge \omega \\ &= \int_{\Omega^-} d(f\omega) - \int_{\Omega^-} f d\omega = \int_{\Omega^-} d(f\omega) = \int_{\partial\Omega^-} f\omega = \int_\gamma \omega.\end{aligned}$$

We now prove the lemma: From the assumption that $\omega \in E \cap C^1$, so $\omega \in E^{*\perp}$. Notice that η_γ has compact support, we can prove that $(\omega, \star\eta_\gamma) = 0$ From the claim above, we have that $\int_\gamma \omega = 0$. Hence ω is exact. This finishes the proof of the lemma.

Theorem (Hodge Decomposition theorem for smooth forms). *Let $\omega \in L^2(M) \cap C^1(M)$, then there exists C^2 functions f and g such that*

$$\omega = df + \star dg + h, \quad df \in E, \star dg \in E^*, h \in H.$$

Proof. Write

$$\omega = \alpha + \beta + h$$

with $\alpha \in E, \beta \in E^*, h \in H$. According to the result above, we only need to prove that α, β are C^1.

For any point $p_0 \in M$, take a coordinate chart (U, ϕ) with $p_0 \in U$. WLOG, assume that $\phi(U)$ is the unit disk $D(0,1)$ and $\phi(p_0) = 0$. Write locally $\omega = pdx + qdy$ (with $z = x + iy$). Let

$$G(z) = -\frac{1}{2\pi}\int_{D(0,1)} (p_\xi + q_\eta) \ln|\zeta - z| d\xi \wedge d\eta \quad (\zeta = \xi + i\eta).$$

Then it is easy to see that $G(z)$ is the solution of the equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = p_x + q_y$$

on the unit disk $D(0,1)$. Hence

$$d \star dG = 4\frac{\partial^2 G}{\partial z \partial \bar{z}} dx \wedge dy = 4(p_x + q_y) dx \wedge dy = d \star \omega.$$

Thus $d \star (\omega - dG) = 0$, i.e. $\omega - dG$ is co-closed. Hence, from the theorem above, $(\omega - dG) \perp E(U)$, where $E(U)$=closure of $\{df, f \in C_0^\infty\}$. From the decomposition theorem of $L^2(M)$ above,

$$\omega - dG = \beta' + h',$$

with $\beta' \in E^*(U)$, where $E^*(U)$=closure of $\{\star df, f \in C_0^\infty\}$, and h' is harmonic on U (hence smooth). From the smoothness of ω, dG and h', we conclude that

β' is smooth. Then, from $\omega = \alpha + \beta + h$, we get $\beta - \beta' = dG - \alpha + h' - h$. Notice that $\alpha \perp E^*(U)$ and $dG \perp E^*(U)$, we know that $\beta - \beta' \perp E^*(U)$. On the other hand, $\beta - \beta' \perp E(U)$. Hence $\beta - \beta' \in H$. Thus it is smooth. This implies that β is smooth. The similar argument also implies that α is smooth. This finishes the proof.

Hodge Theory: From the decomposition theorem, $H^1_{DR}(M) \cong H$, where H is the set of harmonic 1-forms on M. To see it, for every smooth closed 1-form ω, from the theorem we proved, $\omega \in {E^*}^{\perp}$. Hence, from the Hodge decomposition theorem, $\omega = df + h$. Thus ω are h belong to the same class. The map $\omega \mapsto h$ gives the isomorphism.

3.4 The Space of Holomorphic (meromorphic) 1-Forms

The principal question above the manifold is the existence of global objects. On the smooth category, one can always piece the local objects together by using the cut-off function to get a global one. However, it is hard to do it in the holomorphic category (since the cut-off functions are only smooth). From the maximal principle, every holomorphic map on a compact RS must be constant, As a result, meromorphic functions on a compact RS, or holomorphic (meromorphic) 1-forms are more interesting. The study of holomorphic form (resp. meromorphic) is THROUGH the study of harmonic 1-forms (with the Hodge Theory).

A 1-form ω is called a *holomorphic form* (resp. meromorphic) if locally $\omega = f dz$ where f is holomorphic (resp. meromorphic). A meromorphic 1-form is also called a *abel form*. Note that two meromorphic 1-forms ω_1, ω_2 produces a global meromorphic function ω_1/ω_2 on M. Denote by $H^0(M, \Omega^1)$ the space of holomorphic 1-forms on M.

The operator $\alpha \mapsto \frac{1}{2}(\alpha + i \star \alpha)$ transforms any harmonic form into a holomorphic form and acts identically on holomorphic forms. Its kernel consists of antiholomorphic forms since if $\alpha + i \star \alpha = 0$, one has $\bar{\alpha} - i \star \bar{\alpha} = 0$ which means that $\bar{\alpha}$ is holomorphic. This proves the following

Theorem. *One has a canonical decomposition*

$$H = H^0(M, \Omega^1) \bigoplus \overline{H^0(M, \Omega^1)}.$$

In particular $\dim H^0(M, \Omega^1) = g$.

Canonical basis for $H^0(M, \Omega^1)$: Let $a_1, b_1, \ldots, a_g, b_g$ be a canonical homology basis for M (i.e. for $H_1(M, \mathbf{Z})$). Let $\omega \in H^0(M, \Omega^1)$, the numbers $A_1 :=$

$\int_{a_1}\omega, \cdots, A_g := \int_{a_g}\omega$ (respectively $B_1 := \int_{b_1}\omega, \cdots, B_g := \int_{b_g}\omega$) are called the a-periods (resp. b-periods) of ω. Then

$$\omega - \sum_{j=1}^{g}(A_j\alpha_j + B_j\beta_j)$$

has zero $a-$periods and b-periods. Thus $\omega = \sum_{j=1}^{g}(A_j\alpha_j + B_j\beta_j) + df$ for some $f \in C^2$.

Proposition (Bilinear relation). *Let ω and $\tilde{\omega}$ be two smooth closed one-forms on M. Then*

$$\int_M \omega \wedge \tilde{\omega} = \sum_{j=1}^{g}\left(\int_{a_j}\omega\int_{b_j}\tilde{\omega} - \int_{a_j}\tilde{\omega}\int_{b_j}\omega\right).$$

Proof. From the above discuss, we have

$$\omega = \sum_{j=1}^{g}(A_j\alpha_j + B_j\beta_j) + df,$$

$$\tilde{\omega} = \sum_{j=1}^{g}(\tilde{A}_j\alpha_j + \tilde{B}_j\beta_j) + d\tilde{f}$$

where $A_1, \dots, A_g$ (resp. $\tilde{A}_1, \dots, \tilde{A}_g$) are the $a-$periods of ω (resp. $\tilde{\omega}$), and $B_1, \dots, B_g$ (resp. $\tilde{B}_1, \dots, \tilde{B}_g$) are the $b-$periods of ω (resp. $\tilde{\omega}$). Using the fact that M is compact, from Stoke's theorem,

$$\begin{aligned}\int_M \omega \wedge \tilde{\omega} &= \int_M (\omega - df) \wedge (\tilde{\omega} - d\tilde{f}) \\ &= \int_M (A_j\alpha_j + B_j\beta_j) \wedge (\tilde{A}_j\alpha_j + \tilde{B}_j\beta_j).\end{aligned}$$

Using the fact that

$$\int_M \alpha_j \wedge \beta_k = \int_{b_j}\beta_k = \int_{a_k}\alpha_j$$

and

$$\int_{a_j}\alpha_k = \int_{b_j}\beta_k = \delta_{jk}$$

it is easy to get the conclusion.

Corollary. *If ω is a holomorphic 1-form, and its $a-$periods are zero, then $\omega = 0$.*

Proof. From above, we have $\|\omega\|^2 = 0$. Hence $\omega = 0$.

Lemma. *Let $\phi_1, \dots, \phi_g$ be a basis of $H^0(M, \Omega^1)$. Then its a-period of matrix*

$$(a_{ij})_{g\times g} = \left(\int_{a_i} \phi_j\right)_{g\times g}$$

is of maximal rank.

Proof. Assume that $\sum_{j=1}^{g} \lambda_j a_{kj} = 0$ for $k = 1, \dots, g$ Let $\phi = \sum_{j=1}^{g} \lambda_j \phi_j$. Then the a-periods of ϕ is zero, thus, from the corollary above, $\phi = 0$. Hence, from the assumption that $\phi_1, \dots, \phi_g$ be a basis of $H^0(M, \Omega^1)$, we conclude that $\lambda_1 = \dots = \lambda_g = 0$. Thus the row vectors of the matrix are lienar independent. This proves the lemma.

From the above the lemma, the matrix $A := (a_{ij})_{g\times g}$ is invertible, so there exists a matrix C such that $AC = I$. Thus there is a (new) basis of $H^0(M, \Omega^1)$, say $\psi_1, \dots, \psi_g$ whose a-period matrix in I, the identical matrix, we call such basis a **canonincal basis** for $H^0(M, \Omega^1)$.

3.5 Bilinear Relation for Meromorphic 1-Forms

From the bilinear relation above, we have, for any two holomorphic 1-forms ω and $\tilde{\omega}$, we have

$$\sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \tilde{\omega} - \int_{a_j} \tilde{\omega} \int_{b_j} \omega\right) = \int_M \omega \wedge \tilde{\omega} = 0.$$

Now we want to extend this relation to meromorphic differential forms.

Theorem. *Let ω be a holomorphic 1-form and $\tilde{\omega}$ be a meremorphic 1-form which has only one pole at $p \in M$ with residue zero. Assume that locally*

$$\omega = (a_0 + a_1 z + \cdots)dz$$

$$\tilde{\omega} = \left(\frac{c_m}{z^m} + \cdots + \frac{c_{-2}}{z^2} + c_0 + c_1 z + \cdots\right) dz.$$

Then

$$\sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \tilde{\omega} - \int_{b_j} \omega \int_{a_j} \tilde{\omega}\right) = 2\pi i \sum_{n=2}^{m} \frac{c_{-n} a_{n-2}}{n-1}.$$

Note: The theorem is a key to the proof of Riemann-Roch theorem.

Proof. Note that $M_0 := M\backslash\{a_1, \dots, a_g, b_1 \dots, b_g\}$ is simply connected, so there exist smooth function f (defined as $f(p) = \int_{p_0}^{p} \omega$ for $p \in M_0$) such that $\omega = df$. Note that f can be extended to the boundary, but f may not have the same values on the boundary.

We first claim that

$$\int_{\partial M_0} f\tilde{\omega} = \sum_{j=1}^{g} \left(\int_{a_i} \omega \int_{b_i} \tilde{\omega} - \int_{b_i} \omega \int_{a_i} \tilde{\omega} \right).$$

To prove the claim, notice that for any $z \in a_i$, let $z' \in a_i^{-1}$ be the point which is equivalent to z, then

$$f(z') - f(z) = \int_z^{z'} \omega = \int_z^{p_0} \omega + \int_{b_i} \omega + \int_{p_0}^{z'} \omega = \int_{b_i} \omega,$$

since z' is equivalent to z and ω has the same value at the equivalent points. Hence

$$f(z') - f(z) = \int_{b_i} \omega.$$

Therefore, since $\tilde{\omega}$ has the same value at the equivalent points,

$$\int_{a_i} f\tilde{\omega} + \int_{a_i^{-1}} f\tilde{\omega} = \int_{a_i} (f(z) - f(z'))\tilde{\omega} = -\int_{b_i} \omega \int_{a_i} \tilde{\omega}$$

where $z' \in a_i^{-1}$ is the point which is equivalent to $z \in a_i$. Similarly,

$$\int_{b_i} f\tilde{\omega} + \int_{b_i^{-1}} f\tilde{\omega} = \int_{a_i} \omega \int_{b_i} \tilde{\omega}.$$

Thus,

$$\int_{\partial M_0} f\tilde{\omega} = \sum_{j=1}^{g} \left(\int_{a_i} \omega \int_{b_i} \tilde{\omega} - \int_{b_i} \omega \int_{a_i} \tilde{\omega} \right)$$

which proves the claim. On the other hand, we have the residue formula,

$$\int_{\partial M_0} f\tilde{\omega} = 2\pi i \sum Res(f \cdot \tilde{\omega}).$$

Now locally at p, $\omega = (a_0 + a_1 z + \cdots)dz$, so $f(z) = \int_0^z \omega = a_0 z + \frac{1}{2} a_1 z^2 + \cdots$, and

$$\tilde{\omega} = \left(\frac{c_m}{z^m} + \cdots + \frac{c_{-2}}{z^2} + c_0 + c_1 z + \cdots \right) dz,$$

Hence we have

$$\sum Res(f \cdot \tilde{\omega}) = \sum_{n=2}^{m} \frac{c_{-n} a_{n-2}}{n-1}.$$

This proves the theroem.

Chapter 4

Riemann-Roch Theorem and its Consequences

4.1 Divisors

A divisor D on a Riemann surface M is a locally finite subset $\{p_1, p_2, \ldots, ...\}$ of distinct points of M (it is useful to note that locally finite is not the same as isolated), together with a collection of integers $m_1, m_2, ...$ with m_i associated to p_j. The notation is

$$D = \sum_j m_j p_j.$$

The set of points $\{p_1, p_2, \ldots, ...\}$ is called the support of D. When the support of D is finite, the number

$$\deg(D) := \sum_j m_j$$

is called the degree of D. For example, Let f be a meromorphic function on M. Then we have a divisor

$$(f) := \sum_{p \in M} ord_p(f) p,$$

where $ord_p(f) = k$ is p is a zero of f with order k, and $ord_p(f) = -k$ is p is a pole of f of order k. From the theorem proved earlier, if M is compact, then $\deg(f) = 0$.

Example: Let $M = S^2 = \mathbf{P}^1$. Let $f([1 : z]) = z, f([0 : 1] = \infty$. Then $(f) = [1 : 0] - [0 : 1]$.

We say that D is effective if $m_j \geq 0$ for all j. Given two divisors D_1, D_2, we say that $D_1 \geq D_2$ if $D_1 - D_2$ is effective. The collection of divisors on M is denoted by $Div(M)$. It forms a group, so it is called the divisor group of M.

The purpose of introducing the concept of divisors is to study the meromorphic functions and meromorphic 1-forms. Given a divisor D, if $D = (f)$ for some meromorphic function f on M. We call such divisor a principal divisor. Two divisors D_1, D_2 are called linearly equivalent (denoted by $D_1 \cong D_2$) if $D_1 - D_2 = (f)$ for some meromorphic function f on M. The quotient group $\mathcal{D} := Div(M)/\sim$ is called the divisor class group.

Similarly, for a meromorphic 1-form ω, we can define

$$(\omega) := \sum_{p \in M} ord_p(\omega) p.$$

Such divisors are called canonical divisors. Denote by K a canonical divisor. For any two meromorphic 1-forms ω_1, ω_2, the ration ω_1/ω_2 is a meromorphic function on M. So (ω_1) and (ω_2) are always linearly equivalent (they belong to the same equivalent class).

Let D be a divisor, we define *the space of meromorphic functions with poles bounded by D* by

$$L(D) := \{f \mid f \text{ is a meromorphic function on } M, \text{ either } f \equiv 0 \text{ or } (f) + D \geq 0\}.$$

For example, if $D = 5p - q$, then $f \in L(D)$ means that f is meromorphic which has exactly one pole p with $|ord_p(f)| \leq 5$ and has exactly one zero at q with $ord_q(f) \geq 1$. The reason for the terminology is that following: For $D = \sum n_p p$, then $f \in L(D)$ meanas that $\text{ord}_p(f) \geq -n_p$. If $n_p > 0$, it means that f may have a pole of order n_p, but no worse. Similarly, if $n_p < 0$, then it means that f has a zero of order at least $(-n_p)$ at p. So either poles are being allowed (to specified order and no worse) or zeros being required (to at least some specified order), at the support of D. Another way to say the above definition is to use Laurent series. For any point p, choose a local coordinate z centered at p. Then $f \in L(D)$ is equivalent to saying that at all point $p \in Supp(D)$, the local Laurent series of f has no terms lower than z^{-n_p}.

Let

$$h^0(D) := \dim L(D).$$

We define

$$\Omega(D) := \{\omega \mid \omega \text{ is a meromorphic 1-form on } M, (\omega) - D \geq 0\}.$$

Write $i(D) = \dim_{\mathbf{C}} \Omega(D)$. Note that if $D_1 \cong D_2$, then $h(D_1) = h(D_2)$ and $i(D_1) = i(D_2)$. It is also easy to see that $i(D) = h^0(K - D)$ by, for fixed ω, the map $\eta \mapsto \frac{\eta}{\omega}$.

Lemma. *Let D be a divisor with* $\deg D < 0$. *Then* $h^0(D) = 0$.

Proof. For an $f \not\equiv 0$ in $L(D)$, we would have $(f) \geq -(D)$. Then $0 = \deg(f) \geq -\deg D > 0$ which is impossible. This proves the lemma.

4.2 The Riemann-Roch Theorem

Theorem (Riemann-Roch). *Let M be a compact Riemann surface of genus g. Let D be a divisor on M. Then*

$$h^0(D) = \deg D - g + 1 + h^0(K - D) = \deg D - g + 1 + i(D).$$

Corollary 1. $\deg(K) = 2g - 2$

Corollary 2. *Let M be a compact Riemann surface. Then $\mathcal{M}(M)$, the set of meromorphic functions on M, has infinite dimension as a complex vector space.*

Proof. Let $l > 0$ be any positive integer and fix $p \in M$. From RR,

$$h^0(l(p)) = l - g + 1 + i(D) \geq l - g.$$

Taking $l \to +\infty$, we get that the set of meromorphic functions on M, has infinite.

Corollary 3. *Let M be a compact Riemann surface with genus(M) > 0. Then, for every point $p \in M$, there exists a holomorphic 1-form ω with $\omega(p) \neq 0$.*

Proof. Assume the statement is false, then there is some $p \in M$ such that every $\omega \in H^0(M, \Omega^1)$ satisfies $\omega(p) = 0$. Thus $H^0(M, \Omega^1) \subset \Omega((p))$, i.e. $i((p)) = h^0(K - (p)) \geq \dim H^0(M, \Omega^1) = g$. Thus, from RR, $h^0(p) \geq 1 - g + 1 + g = 2$. This means that there is a meromorphic function on M which has only p as its pole. This function would give a biholomorphic map M into $\hat{\mathbf{C}}$, contradiction with the assumption that $g > 0$.

4.3 The Proof of Riemann-Roch Theorem:

The proof of the Riemann-Roch Theorem depends decisively on the following existence theorem.

Theorem (Existence). *Let M be a compact Riemann surface. Let $z_1, \ldots, z_n \in M$. Suppose a local chart has been choosen around zeach z_j. Then for any $t_1, \ldots, t_n \in \mathbf{C}$, there exists a unique meromorphic $1-$ forms τ_t on M ($t = (t_1, \ldots, t_n)$) with the following properties:*

(i) τ_t is holomorphic on $M \setminus \cup_{\nu=1}^{n} \{z_\nu\}$.

(ii) For each ν,

$$\tau_t(z) = (t_\nu z^{-2} + \textit{terms of order } \geq 0)dz$$

near z_ν, where z is a local coordinate at z_ν with $z(z_\nu) = 0$;

(iii)

$$\int_{a_i} \tau_t = 0, \quad i = 1, \ldots, g$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ being usual a canonical homology basis for M.

Proof. Consider $z_\nu \in U_0 \subset U_1 \subset M$. Take $\rho \in C^\infty(M)$ with $\rho = 1$ on U_0 and $\rho = 0$ on $M \backslash U_1$. Let z be a local coordinate in U_1 with $z(z_\nu) = 0$. Let

$$\theta := \left(-\frac{\rho t_\nu}{z} \right)$$

and $\psi := d\theta$. Notice that

$$\psi := d\left(-\frac{\rho t_\nu}{z} \right) = t_\nu \left(-\frac{\rho_z}{z} + \frac{\rho}{z^2} \right) dz - t_\nu \frac{\rho_{\bar{z}}}{z} d\bar{z}.$$

The $(0,1)-$part of ψ is smooth on M (so $\psi - i \star \psi$ is smooth on M), thus $\psi - i \star \psi = df + \star dg + h$ with h harmonic. Consider $\alpha_\nu := \psi - df = dw - df = \star dg + i \star dw + h$. This means that it is closed and c-closed on $M \backslash \{z_\tau\}$. Hence it is harmonic on $M \backslash \{z_\tau\}$. Thus

$$\sum_{nu=1}^{n} (\alpha_\nu + i \star \alpha_\nu)$$

satisfy the first two conditions of the lemma. Clearly two such forms differ only by a holomorphic form, and it follows that periods along $a_1, \ldots, a_g$ can be made to vanish by using the canonical basis for $H^0(M, \Omega^1)$. Conversely the form is uniquely determined (the uniqueness comes from the fact that any holomorphic 1-form whose a-periods vanish must be indentically zero). This finishes the proof of the existence theorem.

We now prove the Riemann-Roch theorem.

We first prove that case that D is effective (if D is trivial then there is nothing to prove), i.e. $D = \sum_{j=1}^n \alpha_j p_j$ with $\alpha_j > 0$. For simplicity of notation, we assume that $D = \sum_{\nu=1}^n z_\nu$. Consider V, the subspace of meromorphic 1-forms on M, which is given by

$$V = \{\omega \mid (\omega) + 2D \geq 0, \omega \text{ has zero periods and residues}\}$$

and the map

$$d : L(D) \to V, \text{by } f \mapsto df.$$

Note that if $\omega \in V$, then

$$f(z) := \int_{z_0}^{z} \omega \quad (z_0 \in M \text{ is fixed})$$

is well-defined, and $f \in L(D)$, so this map is onto. Clearly $df = df'$ if and only if f and f' differ by an additive constant, hence the kernel of the map is $\mathbf{C}$. Therefore we have

$$h^0(D) = \dim_{\mathbf{C}} V + 1.$$

To compute $\dim_{\mathbf{C}} V$, by identifying

$$\begin{array}{ll} \omega \in V & \longleftrightarrow t = (t_1, \ldots, t_n) \in \mathbf{C}^n \\ \| & \\ \omega = (t_j z^{-2} + \text{terms of order} \geq 0) dz, 1 \leq j \leq n & \end{array}$$

and for every such $t = (t_1, \ldots, t_n)$ by using the 1-form τ_t constructed above, we consider the linear map

$$l : \mathbf{C}^n \to \mathbf{C}^g$$

$$t \mapsto \left(\int_{b_1} \tau_t, \ldots, \int_{b_g} \tau_t \right).$$

Then clearly, by noticing that $\int_{a_i} \tau_t = 0, 1 \leq i \leq n$ by the construction,

$$V = \ker l.$$

If now $\alpha_1, \ldots, \alpha_g$ is the canonincal basis of $H^0(M, \Omega^1)$, so that $\int_{a_i} \alpha_j = \delta_{ij}$, we have, by the bilinear relation (note that $\int_{a_i} \tau_t = 0$)

$$\int_{b_j} \tau_t = 2\pi\sqrt{-1} \sum_{\nu} t_\nu \left(\frac{\alpha_j}{dz} \right) (z_\nu).$$

Thus l is defined by the matrix

$$A := (a_{ij}) = 2\pi\sqrt{-1} \begin{pmatrix} \left(\frac{\alpha_1}{dz}\right)(z_1) & \cdot & \cdot & \cdot & \left(\frac{\alpha_1}{dz}\right)(z_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left(\frac{\alpha_g}{dz}\right)(z_1) & \cdot & \cdot & \cdot & \left(\frac{\alpha_g}{dz}\right)(z_n) \end{pmatrix}.$$

Notice that if $\omega \in \Omega(D)$ (i.e. ω is holomorphic 1-form which vanish at al the $z_\nu, 1 \leq \nu \leq n$ with order one) and write $\omega = \lambda_1 \alpha_1 + \cdots + \lambda_g \alpha_g$, then $\lambda_1, \ldots, \omega_g$ is the solution of the system of linear equations

$$\sum_{k=1}^{g} \lambda_k a_{kj} = 0, \ 1 \leq j \leq n$$

and conversely, if $\lambda_1, \ldots, \omega_g$ is a solution of the system of above linear equations, then $\omega = \lambda_1 \alpha_1 + \cdots + \lambda_g \alpha_g \in \Omega(D)$. Hence, if we denote by $C(A)$ the column space and $N(A)$ the row space, then this means that $\dim N(A^t) = \dim \Omega(D) = i(D) = h^0(K - D)$. So $\dim C(A^t) = g - \dim N(A^t) = g - h^0(K - D)$. Thus,

$\dim V = \dim(kerl) = \dim N(A) = n - \dim C(A) = n - \dim C(A^t) = n - (g - h^0(K-D)) = n - g + h^0(K-D)$. Hence

$$h^0(D) = \dim(kerl) + 1 = n - g + h^0(K-D) + 1$$

which proves the theorem in the case that D is effective.

We now prove the general case for D.

Claim: *When $g > 1$, $\deg(\omega) = 2g - 2$ where ω is a holomorphic form on M.* Indeed, since (ω) is effective, use the the Riemann-Roch proved earlier with the assumption that $D = (\omega)$ is effective, we get

$$h^0((\omega)) = \deg(\omega) - g + 1 + h^0(0) = \deg(\omega) - g + 2.$$

Use the fact that $h^0(K-D) = i(D)$, we know immediately that $h^0(\omega)) = i(0) = g$, Thus $\deg(\omega) = 2g - 2$, which proves the claim.

We know that $r(D), i(D)$ and $\deg(D)$ depend only on the equivalent class of D, and we have proved the Riemann-Roch earlier in the case when D is equivalent to an effective divisor. We now prove that if D is a divisor with $K - D$ is effective, then Riemann-Roch still holds. Indeed, if $D' = K - D$ is equivalent to an effective divisor, then apply the Riemann-Roch to D' yields

$$h^0(K-D) = \deg(K-D) - g + 1 + h^0(D).$$

But $\deg(K) = 2g - 2$, so

$$h^0(D) = \deg(D) - g + 1 + h^0(K-D).$$

The above is in fact the Riemann-Roch to D.

It remains to the last case that both D and $K - D$ are not equivalent to effective divisors. In this, we'll have $h^(D) = 0$ and $h^(K-D) = 0$. In fact, if $h^0(D) \neq 0$, then there is a meromorphic function f with $(f) + D \geq 0$, contradicts with the assumption that D is not equivalent to an effective divisor. If $h^0(K-D) \neq 0$, then it contradicts with the assumption that $K - D$ is not equivalent to an effective divisor. Thus for such D, the Riemann-Roch result becomes

$$0 = \deg(D) - g + 1.$$

To prove above, write $D = D_1 - D_2$ with D_1, D_2 both effective, then $\deg(D) = \deg(D_1) - \deg(D_2)$. Applying the Riemann's inequality to D_1 yields

$$h^0(D_1) \geq \deg(D_1) - g + 1 = \deg(D_2) + \deg(D) - g + 1.$$

If $\deg(D) \geq g$, then $h^0(D) \geq (D_2) + 1$, thus there are at least $m = \deg(D_2) + 1$ meromorphic functions $f_1, \ldots, f_m \in L(D_1)$ which are linearly independent. We consider its linear combination $f = c_1 f_1 + \cdots + c_m f_m$. Since $m > \deg D_2$, we can choose siutable $c_1, \dots, c_m$ such that $f \not\equiv 0$, and every point in D_2 is a zero of f. Thus

$$f \in L(-D_2) = L(D)$$

which contradicts with the fact that $h^0(D) = 0$. Hence $\deg(D) < g$. Also, from $h^0(K - D) = 0$, similar to above and using $\deg(K) = 2g - 2$, we know $\deg(K - D) < g$, i.e. $\deg(D) > g - 2$. Hence we proved $g - 2 < \deg(D) < g$, i.e. $\deg(D) = g - 1$. This finishes the proof of Riemann-Roch.

4.4 Projective Embeddings

Let M be a compact RS. A map $\phi : M \to \mathbf{P}^N$ is said to be an embedding if it is injective and its differential $d\phi|_p i$ is injective at every point p of M)

Complete Linear System. A divisor D defineds a complete linear system

$$|D| := \{D' \geq 0| \; |D' \sim D\}.$$

Note that if $\deg D \leq 0$, then $|D|$ is empty. A point $p \in M$ is called a base point if $p \in \cap_{D' \in |D|} suppD'$. $|D|$ is said to be base point free if it does not have any base points. To each divisor D, we associate it with the map

$$\phi_D : M \to \mathbf{P}^{l-1},$$

$$P \mapsto [f_0(P) : \cdots : f_{l-1}(P)]$$

where $l = \dim L(D) = \dim |D|$ and $f_0, \dots, f_{l-1}$ is a basis of $L(D)$. If $|D|$ is base pont free, then ϕ_D is a well-defined holomorphic map. We are going to investigate for what kind of D the map ϕ_D is an embedding. D is called very ample if $|D|$ is base point free and the map $\phi_D : M \to \mathbf{P}^{l-1}$ is an embedding.

To do so, we need the following results above base points and embeddings:

Lemma(base point free criteria). *$p \in M$ is a base point of $|D|$ if and only $L(D - p) = L(D)$.*

Proof. $p \in M$ is a base point of $|D| \Leftrightarrow p \in D'$ for $\forall D' \in |D| \Leftrightarrow f(p) = 0$ for $\forall f \in L(D)$, since $D' = D + (f)$. Hence $L(D) \subset L(D - p)$. This proves the lemma.

Lemma(Injectivity). *ϕ_D is 1-1 if and only if for every pair of distinct points p, q, $h^0(D - p - q) < h^0(D - p) < h^0(D)$*

Proof. We only prove the only if part. From $h^(D - p - q) < h^0(D - p) < h^0(D)$, there is $f \in L(D)$ with $f(p) = 0, f(q) \neq 0$. Since $f = \sum_{j=0}^{l} a_j f_j$, we have $\sum_{j=0}^{l-1} a_j f_j(p) = 0, \sum_{j=0}^{l-1} a_j f_j(q) \neq 0$, which implies $\phi(p) = [f_0(p) : \cdots :$

$f_{l-1}(p)] \neq [f_0(q) : \cdots : f_{l-1}(q)]$ since otherwise we would have $f_i(p) = \lambda f_i(q)$. This proves the lemma.

Lemma(Local isomorphism). *ϕ_D is a local isomorphism at $p \in M$ if and only if if and only if $h^0(D-2p) < h^0(D-p) < h^0(D)$.*

Proof. We only prove the only if part. From $h^(D-2p) < h^0(D-p) < h^0(D)$, there is $f \in L(D)$ with $f(p) = 0, df(p) \neq 0$. Since $f = \sum_{j=0}^{l} a_j f_j$, we have $\sum_{j=0}^{l-1} a_j f_j(p) = 0, \sum_{j=0}^{l-1} a_j df_j(p) \neq 0$, which implies $d\phi(p) = [df_0(p) : \cdots : df_{l-1}(p)] \neq 0$ which means that $d\phi$ is a local isomorphism.

Therefore, to prove ϕ_D is an embedded, we only need to check that, for any points $z_1, z_2 \in M$ (need NOT to be distinct), the following (*) holds

$$0 < h^0(D - z_1 - z_2) < h^0(D - z_1) < h^0(D) \quad (*).$$

Theorem(Projective embedding theorem) *If D is a divisor on a compact Riemann surface of genus g. If $\deg(D) \geq 2g$, then $|D|$ is base point free. If $\deg(D) \geq 2g+1$, then $|D|$ is very ample.*

The proof is based on the followings:

1. The simple "vanishing thoerem": *If $\deg(D) < 0$, then $L(D) = \{0\}$.* It then implies the following

2. Vanishing Theorem: *If $\deg(D) \geq 2g-1$, then $h^0(K-D) = 0$.*

This implies that

Proposition. *(a) If $\deg(D) \geq 2g-1$, then $h^0(D) = \deg(D) + 1 - g$.*

(b) If $n > 0$, and $\deg(D) = g + n$, the $h^0(D) \geq n+1$.

Proof. $\deg(D) \geq 2g-1 \Longrightarrow \deg(K-D) = 2g-2-\deg(D) < 0$. Hence $h^0(K-D) = 0$. Thus (a) follows from the RR.

(b) By RR,

$$h^0(D) \geq \deg(D) + (1-g) \geq g + n + (1-g) = n + 1.$$

This proves the proposition.

We now prove the theorem. To check $|D|$ if base point free when $\deg(D) \geq 2g$, we notice that $h^0(D) \neq h^0(D-p)$ since from the above proposition, $h^0(D) = \deg(D) + 1 - g$ and $h^0(D-p) = \deg(D) - 1 + 1 - g$. So by the lemma above, $|D$ is base point free. To see ϕ_D is an embedding, we only need to check, as mentioned above,

$$0 < h^0(D - z_1 - z_2) < h^0(D - z_1) \quad (*).$$

By above proposition, since $\deg(D) = 2g+1$, $h^0(D-z_1-z_2) = \deg(D)-2+1-g$ and $h^0(D-z_1) = \deg(D)-1+1-g$, so (*) holds. This means that D is very ample.

By taking $D = (2g+1)p$, from above D is very ample, and $h^0(D) = 2g+1+1-g = g+2$, so we can can always imbed a compact RS M of genus g into $\mathbf{P}^{g+1}$.

If we concern about $D = (p)$, the we have

Lemma. *Let M be a compact Riemann surface. Suppose that for some point $p \in M$, $h^0(p) > 1$, Then M is isomorphic to the Riemann sphere (using ϕ_D with $D = (p)$.*

Proof. $h^0(p) > 1$ implies that there is a non-constant meromorphic function f which has a simple pole at p and no other poles. Thus $f : M \to \mathbf{C} \cup \{\infty\}$ has degree one, therefore is an isomorphism.

Theorem. *Let M be a compact Riemann surface of genus 0. Then M is isomorphic to the Riemann sphere, or equivalent, $D = (p)$ is very ample.*

Proof. Let $z_0 \in M$. From the above lemma, we only need to show that By RR using $D = z_0 \in M$

$$\begin{array}{ccccccc} h^0(D) - & i(D) & = & \deg(D) & + & (1-g) & . \\ & \| & & \| & & \| & \\ & -\dim\{\omega \mid (\omega) \geq (z_0)\} & & 1 & & 1 & \\ & \| & & & & & \\ & 0 & & & & & \end{array}$$

Thus, $h^0(z_0) = 2$. The theorem thus follows from the above lemma.

Canonical embedding: Take $D = K$, the canonical divisor, then ϕ_K is called the canonical map. We have the following result concerning about the canonical embedding: *If $g \geq 2$ and M is not hyperelliptic, then K is very ample.* More precisely,

Theorem. *Every compact Riemann surface admits a (holomorphic) embedding into a complex projective space. In fact, a compact Riemann surface of genus zero is biholomorphic to $\mathbf{P}^1$, a compact Riemann surface of genus one can be embedded into $\mathbf{P}^2$, and a compact Riemann surface of genus $g \geq 2$ can embedded by the tri-canonical map i_{3K} in $\mathbf{P}^{5g-6}$. If M is not hyperelliptic, then the canonical map i_K embeds M into $\mathbf{P}^{g-1}$.*

Proof. The case of $g = 0$ has been proved in above. Now suppose that $g > 0$ and let $\alpha_1, \ldots, \alpha_g$ be a basis for $H^0(M, \Omega^1)$. By Riemann-Roch, the α_i do not all vanish at any point of M. Hence, we get a well-defined map

$$i_K : M \to \mathbf{P}^{g-1}$$

by writing $\alpha_i = f_i dz$ and setting

$$i_K(z) := (f_1(z), \ldots, f_g(z)).$$

We now wish to investigate the conditions under which i_K will be an embedding (i.e. it is injective and its differential $di_K(P)$ is injective at every point P of M). It is not hard to see that i_K is injective precisely when, for any two distinct points $z_1, z_2 \in M$, there is $\alpha \in H^0(M, \Omega^1)$ with $\alpha(z_1) = 0, \alpha(z_2) \neq 0$. Similarly, i_K will have maximal rank at $z \in M$ precisely when there is $\alpha \in H^0(M, \Omega^1)$ for which z is a simple zero. Hence, i_K is an embedding precisely when, for any two not necessarily distinct points $z_1, z_2 \in M$,

$$0 < h^0(K - z_1 - z_2) < h^0(K - z_1). \qquad (*)$$

By Riemann-Roch, $h^0(z_1) = 1 - g + 1 + \deg h^0(K - z_1)$, and from the Lemma above, $h^0(z_1) = 1$ (note that otherwise we would have that M is isomorphic to the Riemann sphere, which contradicts with the assumption that $g > 0$). Thus $h^0(K - z_1) = g - 1$. On the other hand, by Riemann-Roch,

$$h^0(z_1 + z_2) = 2 - g + 1 + h^0(K - z_1 - z_2).$$

Hence the condition (*) is equivalent to

$$h^0(z_1 + z_2) = 1 \quad \text{(Recall that } h^0(D) \geq 1 \text{ for } D \text{ effective).}$$

And (*) fails, i.e. $h^0(K - z_1 - z_2) = h^0(K - z_1) = g - 1$, precisely when

$$h^0(z_1 + z_2) = 2$$

which means that there exists a non-constant meromorphic function g with $(g) + z_1 + z_2 \geq 0$, i.e. g has at most two simple poles or a double pole (according whether $z_1 \neq z_2$ or not). In any case, such g exhibits M as a branched holomorphic two-sheeted covering of S^2 via the map $g : M \to S^2$. Such map is called the hyperelliptic. Indeed, in above, we have proved the following statement: If $g \geq 2$, then i_K is an embedding or M is hyperelliptic.

It remains to deal with the hyperelliptic case. In the hyperelliptic case, we can show that it can embedded by the tri-canonical map i_{3K} in $\mathbf{P}^{5g-6}$ for $g \geq 2$. To do so, we consider the divisor mk with $m \geq 2$. We claim that $h^0(mK) = 0$ if $g = 0$, $h^0(mK) = 1$ if $g = 1$ and $h^0(mK) = (2m - 1)(g - 1)$ if $g \geq 2, m \geq 2$. Indeed, since $deg(mK) = -2m < 0$ if $g = 0$, we have that $h^0(mK) = 0$. If $g = 1$, the $\deg(K) = \deg(mK) = 0$, also since $1 = g = \dim H^0(M, \Omega^1)$, there is a holomorphic 1-form $fdz \not\equiv 0$ on M. Since $\deg(fdz) = \deg K = 0$, fdz can not have any zeros. Hence $f^m dz^m$ is nowhere zero. Hence if for any ϕ which is a m-canonical form, $\phi / f^m dz^m$ is a holomorphic function, hence is constant. This shows that $h^0(mK) = 1$ if $g = 1$. Finally, if $g \geq 2$, then $\deg(-K) = 2 - 2g < 0$. Hence $h^0(-K) = 0$. By Riemann-Roch,

$$h^0(mK) = 2mg - 2m - g + 1 = (2m - 1)(g - 1).$$

This proves the claim.

From the theorem we proved earlier, if $g \geq 1$, then there exists for each $z \in M$ an $\alpha \in H^0(M, \Omega^1)$ with $\alpha(z) \neq 0$. And then, α^m, defined locally by $f^m(z)dz^m$ if $\alpha = f(z)dz$, is so-called m-canonical form with

$$(\alpha^m) = mK.$$

Thus for each $z \in M$ there is an m-canonical form which does not vanish at z. Now let $\beta_1, \ldots, \beta_k$ $(k = (2m-1)(g-1))$ be a basis for $L(mK)$. Then by what has been said above,

$$i_{mK} : M \to \mathbf{P}^{k-1}$$
$$i_{mK}(z) := (\beta_1(z), \ldots, \beta_k(z))$$

gives a well-defined map. The condition that i_{mK} is an embedding is as before, for any two not necessarily distinct points $z_1, z_2 \in M$,

$$0 < h^0(mK - z_1 - z_2) < h^0(mK - z_1). \qquad (**)$$

We know already that

$$h^0(mK - z_1) = h^0(mK) - 1,$$

since not all m-canonical forms vanishes at z_1. Also

$$\deg(mK - z_1 - z_2) = m(2g-2) - 2.$$

Hence By Riemann-Roch,

$$h^0(mK - z_1 - z_2) = m(2g-2) - 2 - g + 1 + h^0(-(m-1)K + z_1 + z_2).$$

Thus if (**) fails, i.e.

$$h^0(mK - z_1 - z_2) = h^0(mK - z_1) = h^0(mK) - 1.$$

Then

$$h^0(-(m-1)K + z_1 + z_2) = 1.$$

Hence,

$$\deg(-(m-1)K + z_1 + z_2) \geq 0$$

i.e.

$$\deg((m-1) - z_1 - z_2) \leq 0$$

which is equivalent to

$$(m-1)(2g-2) - 2 \leq 0$$

or

$$(m-1)(g-1) \leq 1.$$

Since we are assuming that $m \geq 2, g \geq 2$, this happens if $m = 2, g = 2$. Thus we see that, if $g \geq 2$,

$$i_{3K} : M \to \mathbf{P}^{5g-6}$$

is always an embedding.

Chapter 5

Line bundles

Let M be a Riemann surface (or a general complex manifold). A *holomorphic line bundle* over M is a complex manifold L together with a surjective holomorphic map $\pi : L \to M$ having the following properties.

(i) (Locally triviality) For $\forall p \in M$ there is a neighborhood U of p and a map $\phi_U : \pi^{-1}(U) \to \mathbf{C}$ such that the map

$$\phi_U : \pi^{-1}(U) \ni v \mapsto (\pi(v), f_U(v)) \to U \times \mathbf{C}$$

is a diffeomorphism.

(ii) (Global linear structure) For each pair of such neighborhoods U_α and U_β there is a map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}^*$$

such that $\phi_{U_\alpha} \circ \phi_{U_\beta}^{-1}(x, \lambda) = (x, g_{\alpha\beta}\lambda)$.

The map ϕ_U is also called the locally trivialization of the line bundle. The maps $g_{\alpha\beta}$ are called transition functions. The set $L_x := \pi^{-1}(x), x \in M$ is called the fiber of the line bundle at x.

If one can choose $\phi_U : \pi^{-1}(U) \to \mathbf{C}$ to be holomorphic, then L is called a holomorphic line bundle.

The transition functions $\{g_{\alpha\beta}\}$ satisfy $g_{\alpha\alpha} = Id$, $g_{\alpha\beta}g_{\beta\alpha} = Id$ on $U_\alpha \cap U_\beta$, $g_{\alpha\alpha}g_{\beta\gamma}g_{\gamma\alpha} = Id$ $U_\alpha \cap U_\beta \cap U_\gamma$. Conversely, if holomorphic functions $\{g_{\alpha\beta}\}$ satisfy the above properties. Then let

$$L := \cup(U_\alpha \times \mathbf{C})/\sim$$

where $\sim$ is an equivalent relation defined by

$$(x, \lambda_\alpha) \sim (x, \lambda_\beta) \leftrightarrow \lambda_\beta = g_{\alpha\beta}\lambda_\alpha, \quad \forall x \in U_\alpha \cap U_\beta.$$

We denote by $[x, \lambda_\alpha]$ the equivalent calss of (x, λ_α). Then L is a manifold whose coordinate charts are $\{W_\alpha, \Psi_\alpha\}$ where

$$W_\alpha := \{[x, \lambda_\alpha] \mid (x, \lambda_\alpha) \in U_\alpha \times \mathbf{C}\}$$

and

$$\Phi_\alpha : W_\alpha \to U_\alpha \times \mathbf{C}$$
$$[x, \lambda_\alpha] \mapsto (x, \lambda_\alpha).$$

Then L is a holomorphic line bundle over M with the trivializations

$$\Psi_\alpha : \pi^{-1}(U_\alpha \to U_\alpha \times \mathbf{C}$$
$$[x, \lambda_\alpha] \mapsto (x, \lambda_\alpha).$$

Hence

$$L \longleftrightarrow \{U_\alpha, g_{\alpha\beta}\}.$$

A (holomorphic) section of L is a holomorphic map $s : M \to L$ such that $\phi \circ s = id$. Write $s = s_\alpha e_\alpha$ on U_α, where $e_\alpha(p) = \phi^{-1}(p, 1)$ Then $s_\alpha = g_{\alpha\beta} s_\beta$. So a (holomorphic) section s assigns, on every U_α, a holomorphic function s_α with the property that $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$. Let $L \Leftrightarrow \{U_i, g_{ij}\}$ be a line bundle. A meromorphic section of L is a collection $s = \{s_i \in |calM(U_i)\}$ satisfying $s_i = g_{ij} s_j$. So the divisor (s) is well-defined by letting $\mathrm{ord}_p(s) := ord_p(s_i)$.

Consider $L = \mathcal{O}_{\mathbf{P}^n}(-1)$, *tautological line bundle* on $P^n(\mathbf{C})$ (which some books called it *the universal line bundle*). The fiber of $\mathcal{O}_{\mathbf{P}^n}(-1)$ over a point $p = [z_0 : \cdots z_n]$ consists of the complex line spanned by $(z_0, \ldots, z_n)$ (passign through the origin). To find its trivilization and transition functions, take the standard covering $\mathbf{P}^n = \cup_{i=0}^n U_i$ with $U_i = \{[z_0 : \cdots : z_n] \mid z_i \neq 0\}$. The points in the fiber of L over $[z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n]$ has the form $([z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n], \lambda(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)) \subset \mathbf{P}^n \times \mathbf{C}^{n+1}$. We define the trivialization of $\mathcal{O}_{\mathbf{P}^n}(-1)$ over U_i is given by

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbf{C},$$
$$([z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n], \lambda(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n))$$
$$\mapsto ([z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n], \lambda).$$

Since on $U_i \cap U_j \neq \emptyset$, for any $p = [z_0 : \cdots : z_n]$,

$$\begin{aligned} \psi_j^{-1}(p, 1) &= ([z_0 : \cdots : z_n], (z_0/z_j, \ldots, z_{j-1}/z_j, 1, z_{j+1}/z_j, \ldots, z_n/z_j)) \\ &= ([z_0 : \cdots : z_n], (z_i/z_j)(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) \end{aligned}$$

Hence $\psi_i \circ \psi_j^{-1}(p, 1) = (p, z_i/z_j)$. So the transition functions are $g_{ij} = \frac{z_i}{z_j}$.

The line bundle of hyperplane of $\mathbf{P}^n$: The dual of $\mathcal{O}_{\mathbf{P}^n}(-1)$, denoted by $\mathcal{O}_{\mathbf{P}^n}(1)$ is called the hyperplane line bundle. Its transition functions are $g_{\alpha\beta} = \frac{z^\beta}{z^\alpha}$. On

U_α, consider $s_\alpha = a_1\frac{z^1}{z^\alpha} + \cdots + a_{\alpha-1}\frac{z^{\alpha-1}}{z^\alpha} + a_\alpha + a_{\alpha+1}\frac{z^{\alpha+1}}{z^\alpha} + \cdots + a_n\frac{z^n}{z^\alpha}$. Then $s_\alpha = \frac{z^\beta}{z^\alpha}s_\beta$. So s_α defined a holomorphic section $s = a_0z_0 + \cdots a_nz_n$. It zero is the hyperplane $H = \{[z^0, \cdots, z^n] \in \mathbf{P}^n \mid \sum_{\alpha=0}^{n} a_\alpha z^\alpha = 0\}$ in $\mathbf{P}^n$. This is where the name of hyperplane line bundle of $\mathbf{P}^n$ comes from. We sometimes also denoted it by $[H]$.

Holomorphic tangent bundle $\pi : T^{(1,0)}M \to M$. Let $\{W_\alpha\}$ be a local coordinate covering of M with coordinate functions $\{z_\alpha : W_\alpha \to W_\alpha^0 \subset \mathbf{C}\}$. Then, for any $p \in W_\alpha$, $\pi^{-1}(p) = \{a\frac{\partial}{\partial z_\alpha}|_p \mid a \in \mathbf{C}\}$. We define

$$\psi_\alpha : \pi^{-1}(W_\alpha \to W_\alpha \times \mathbf{C} \to W_\alpha^0 \times \mathbf{C}$$

$$a\frac{\partial}{\partial z_\alpha}|_p \mapsto (p, a) \mapsto (z_\alpha(p), a).$$

$T^{(1,0)}M$ becomns a complex manifold of dimension 2 with coordinate covering $\pi^{-1}(W_\alpha)\}$ and coordinate map$\{\psi_\alpha\}$. On $W_\alpha \cap W_\beta \neq \emptyset$,

$$\psi_\alpha^{-1}(x, y_\alpha) = \psi^{-1}(x, y_\beta) \Longleftrightarrow y_\beta = y_\alpha\frac{\partial z_\beta}{\partial z_\alpha}$$

Hence the transition functions are $g_{\alpha\beta} = \frac{\partial z_\alpha}{\partial z_\beta}$.

Holomorphic tangent bundle $\pi : T^{(1,0)^*}M \to M$ *Canonical line bundle on* M: Let M be a Riemann surface. Let $\{U_\alpha\}_{\alpha\in I}$ be a holomorphic coordinate covering of M, $(z_{(\alpha)})$ be a local coordinate system of U_α. Then, for any $p \in W_\alpha$, $\pi^{-1}(p) = \{adz_\alpha|_p \mid a \in \mathbf{C}\}$. We define

$$\psi_\alpha : \pi^{-1}(W_\alpha) \to W_\alpha \times \mathbf{C} \to W_\alpha^0 \times \mathbf{C}$$

$$adz_\alpha|_p \mapsto (p, a) \mapsto (z_\alpha(p), a).$$

$T^{(1,0)}M$ becomns a complex manifold of dimension 2 with coordinate covering $\pi^{-1}(W_\alpha)\}$ and coordinate map$\{\psi_\alpha\}$. On $W_\alpha \cap W_\beta \neq \emptyset$,

$$\psi_\alpha^{-1}(x, y_{|alpha}) = \psi^{-1}(x, y_\beta) \Longleftrightarrow y_\alpha dz_\alpha = y_\alpha dz_\beta \quad \text{or} \quad y_\beta = y_\alpha\frac{dz_\alpha}{dz_\beta}.$$

Hence the transition functions are $g_{\alpha\beta} = \frac{\partial z_\beta}{\partial z_\alpha}$. Sections of K_M are $(1,0)$-forms $\omega = adz_\alpha$.

Operators on line bundles: Let $L \leftrightarrow \{U_\alpha, g_{\alpha\beta}\}$, $L' \leftrightarrow \{U_\alpha, g'_{\alpha\beta}\}$. We define $L + L'$ or $L \otimes L'$ to be the line bundle given by $\{U_\alpha, g_{\alpha\beta}g'_{\alpha\beta}\}$ and its dual bunlde L^{-1} (or $-L$) by $\{U_\alpha, \frac{1}{g_{\alpha\beta}}\}$.

We call a bioholomorphic map $h : L_1 \to L_2$ a bundle isomorphism if the following diagram commutes:

$$\begin{array}{ccc} L_1 & \xrightarrow{h} & L_2 \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ M & = & M \end{array}$$

and (1) h preserves the fibers, (2) $h_{\pi^{-1}(z)}$ is a vector space isomorphism.

Lemma. *Holomorphic line bundles L and L' are isomorphic $\Longleftrightarrow$ there is a common open refinement $\{W_\alpha\}$ such that L and L' are given by $\{W_\alpha, g_{\alpha\beta}\}$ and $\{W_\alpha, g'_{\alpha\beta}\}$ respectively, and holomorphic functions $\phi_i \in \mathcal{O}^*(U_\alpha)$ such that, on $U_\alpha \cap U_\beta \neq \emptyset$,*

$$g'_{\alpha\beta} = \frac{\phi_\alpha}{\phi_\beta} g_{\alpha\beta}.$$

Divisors and Line bundles: Let $s : M \to L$ be a meromorphic section, then (s) is a divisor. On the other hand, let $D = \sum_{p\in M} D(p)p$ be a divisor on M and fix an atlas $\{U_\alpha, z_\alpha\}$ for M such that $U_\alpha \subset\subset M$ for all α. For each α, fix a function $f_\alpha \in \mathcal{M}(U_\alpha)$ such that

$$ord(f_\alpha) = D|_{U_\alpha} := \sum_{p\in U_\alpha} D(p)p.$$

(For example, one could take $f_\alpha = \prod_{p\in U_\alpha}(z-p)^{D(p)}$.). Then we obtain a collection of functions

$$g_{\alpha\beta} : \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

It gives a holomorphic line bundle $[D]$ by $\{W_\alpha, g_{\alpha\beta} := f_\alpha/f_\beta\}$. Note that $\{f_\alpha\}_{\alpha\in\Lambda}$ is a meromorphic section over M. Moreover, if D is effective, then there is a holomorphic section a $s \in H^0(M, [D])$ such that $D = D_s$. Note that $s = \{f_i\}_{i\in I}$ if $D \cap U_i = (f_i)$. This section is called the *canonical section* and is denoted by s_D. If $D = H$ is a hyperplane, then $[H] = \mathcal{O}_{\mathbf{P}^n}(1)$. The mapping $D \to [D]$ is a homomorphism from the group of divisors on M to the group of line bundles. Denote by $\mathcal{L}$ the abelian group of line bundles, up to an isomorphism and $\mathcal{D}$ be the abelian group of divisors on M, uo to a linear equivalence.

Theorem. $\mathcal{D} \cong \mathcal{L}$.

Proof. We send $D \in \mathcal{D}$) to $[D] \in \mathcal{L}$, by let D be a divisor of M given by $\{W_\alpha, f_\alpha \in \mathcal{M}(W_\alpha)\}$ then it gives a holomorphic line bundle $[D]$ by $\{W_\alpha, g_{\alpha\beta} := f_\alpha/f_\beta\}$. It is well-defined, since if D is given by another $\{W_\alpha, f'_\alpha \in \mathcal{M}(W_\alpha)\}$ then it gives a holomorphic line bundle $[D]$ by $\{W_\alpha, g'_{\alpha\beta} := f'_\alpha/f'_\beta\}$. Then

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = g_{\alpha\beta}\frac{\phi_\alpha}{\phi_\beta}$$

with $\phi_\alpha = \frac{f'_\alpha}{f_\alpha} \in \mathcal{O}^*(U_\alpha)$. Therefore we get a line bundle isomorphism. The map is obviously a group homomorphism. We now prove this map is onto. Let $L \in \mathcal{L}$ be a line bundle with transition functions $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Then there exists some (not identically vanishing) $f_1 \in \mathcal{M}(U_1)$ with $f_1|_{U_1 \cap U_2} = g_{12}$. Having defined f_i we find $f_{i+1} \in \mathcal{M}(U_{i+1})$ with $f_{i+1}|_{U_i \cap U_{i+1}} = \frac{g_{i+1,i}}{f_i}$. Since g_{ij} satisfies the co-cycle rules, the collection $\{U_i, f_i)\}$ defined some divisor D with $[D] = L$, and D is determined up to linear equivalence. This ptoves the theorem.

To summarize, here is the correspondence between divisors and line bundles:

Theorem. *If $D \in \mathcal{D}$, then there is a meromorphis section s of $[D]$ such that $(s) = D$ (such section is called the canonical section. Conversely, if $L \in \mathcal{L}$, and s is any meromorphic section s of L (always exists from above), then $L = [(s)]$.*

Proof. Indeed, for any divisir $D = \{U_\alpha, f_\alpha\}$, we can associate a line bundle $[D]$, with $s_D = \{f_\alpha\}$ being a meromorphis section (called the canonical section) of $[D]$. Conversely, for any line bunlde L, let s be any meromorphic section s of L (always exists from above). Write $s = \{s_\alpha\}$, then $s_\alpha = g_{\alpha\beta} s_\beta$, where $g_{\alpha\beta}$ are transition functions of L. On the other hand, from the discussion abiove, the transition functions of $[(s)]$ are also s_α/s_β. Hence $L = [(s)]$.

Lemma. *For $\forall D \in \mathcal{D}$, $H^0(M, [D]) \cong L(D)$.*

Proof. Let $[D] = \{U_\alpha, f_\alpha\}$. We define

$$i : H^0(M, [D]) \to L(D)$$

$$s = \{s_\alpha\} \mapsto s_\alpha/f_\alpha,$$

and

$$j : L(D) \to H^0(M, [D])$$

$$f \mapsto \{f f_\alpha\}.$$

This proves the lemma.

Similarly, we can prove

Lemma. *For any $L \in \mathcal{L}$, and $D \in \mathcal{D}$,*

$$H^0(M, L - [D]) \cong \{s = \textit{meromorphic section of } L \mid (s) - D \geq 0\}.$$

Corollary. *Assume that $L \in \mathcal{L}$, and there is some $D \in \mathcal{D}$ such that $\dim H^0(M, L - [D]) > 0$. Then there is $D_0 \in \mathcal{D}$ such that $L = [D_0]$.*

Proof. Since $\dim H^0(M, L - [D]) > 0$, from above there is a not identically vanishing meromorphic section s on L, and this implies that $L = [(s)]$.

The above corollary will be used later to give another proof of $\mathcal{D} \cong \mathcal{L}$.

The preceding concepts allow the reformulation of Riemann-Roch theorem as

Corollary *Let L be a line bundle over a compact Riemann surface M of genus g. Then*

$$\dim H^0(M, L) = \deg L - g + 1 + \dim H^0(M, K \otimes L^{-1}),$$

where $\deg L := \deg(s)$*, where s is any meromorphic section of L (independent of the choice if s), $H^0(M, L)$ is the space of all holomorphic sections of L and K is the canonical bundle over M.*

Chapter 6

Sheaves and cohomology

6.1 Sheaves

A *Sheaf* $\mathcal{F}$ over a complex manifold X consists of, for each open set $U \subset X$, an abelian group (or vector spaces, rings, or any desired object) $\mathcal{F}(U)$ (also denoted $\Gamma(\mathcal{F}, U)$ and called the set of sections over U), and a collection of restriction maps such that for each $U \subset V \subset X$, $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, and satisfy:

(1) *Identity*: $\rho_{U,U} = id|_{\mathcal{F}(U)}$,

(2) *Compatibility*: If $U \subset V \subset W \subset X$, then $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$;

(3) *Sheaf axiom (gluing)*: Let $U = \mathcal{U}p_\alpha U_\alpha$ and $\sigma_\alpha|_{U_\alpha \cap U_\beta} = \sigma_\beta|_{U_\alpha \cap U_\beta}$ for all α, β, then there exists a (unique) $\sigma \in \mathcal{F}(U)$ such that $\sigma_\alpha = \sigma|_\alpha$ for all α.

If only (1) and (2) are satisfied, then $\mathcal{F}$ is call a **presheaf**. Elements in $\mathcal{F}(U)$ is called a local section on U, and Elements in $\mathcal{F}(X)$ is called a global section.

Examples:

1. $\mathcal{O}_X$ (the sheaf of holomorphic functions on X): $\mathcal{O}(U) = \{$holomorphic functions on $U\}$.

2. $\mathcal{O}(L)$: $\mathcal{O}(L)(U) = \{$holomorphic sections of L on $U\}$, where L is a holomorphic line bundle over X.

3. $\mathcal{O}_X^*$: $\mathcal{O}_X^*(U) = \{$holomorphic nowhere zero functions on $U\}$.

4. $\mathcal{M}_X$: $\mathcal{M}_X(U) = \{$meromorphic functions on $U\}$.

5. $\mathcal{O}_X(D)$: $\mathcal{O}_X(D)(U) = \{f \mid f$ is a meromorphic function on U, $\mathrm{ord}_p(f) \geq -D(p)$ for $p \in U\}$. Note: as a vector space, $\mathcal{O}_X(D) = L(D)$.

6. $\mathcal{E}_X^1$: $\mathcal{E}_X^1(U) = \{$ smooth 1-forms on $U\}$.

7. Ω_X^1: $\Omega^1(U) = \{$ holomorphic 1-forms on $U\}$.

8. $\Omega_X^1[-D]$(the sheaf of holomorphic 1-forms vanishing along D): $\Omega_X^1[-D](U) = \{$ holomorphic 1-forms ω with $\mathrm{ord}_p(\omega) \geq D(p)$ for $p \in U\}$.

9. The *skyscraper* sheaf $\mathbf{C}_p$: $\mathbf{C}_p(U) = \mathbf{C}$ if $p \in U$, and $\mathbf{C}_p(U) = 0$ if $p \notin U$ along with the natural restriction maps.

10. Locally constant Sheaves. Note that the peroperty of being constant is not a local property for a function. Specially, if an open set is disjoint of

the subsets, then a functiion may be constant on each of the subsets, but with different values, it is not constant on the whole set. So **C** (or in general, an abelian group G) is not a sheaf, only a presheaf. We now modify it by considering functions which are *locally constant*: $f : U \subset M \to G$ is locally constant, if for every point $p \in U$, there is $p \in V \subset U$ such that f is constant on V. The locally constant functiuons into a group G forms a sheaf, and is denoted by $\underline{G}$. For example, we have sheaves $\underline{\mathbf{Z}}, \underline{\mathbf{R}}, \underline{\mathbf{C}}$, etc.. (without confusion, we just denote it by G).

6.2 Cech Cohomology

Origins: The Mittage-Leffler Problem: Let M be a Riemann surface, not necessarily compact, $p \in M$ with local coordinate z centered at p. A *principal part* at p is the polar part $\sum_{k=1}^{n} a_k z^{-k}$ of Laurent series. If $\mathcal{O}_p$ is the local ring of holomorphic functions around p, $\mathcal{M}_p$ the field of meromorphic functions arounf p, a principal is just an element of the quotient group $\mathcal{M}_p/\mathcal{O}_p$. The *Mittage-Leffler question* is, given a discrete set $\{p_n\}$ of points in M and a principal part at p_n for each n, does there exist a meromorphic function f on S, holomorphic outside $\{p_n\}$, whose principal part at each p_n is the one specified? The question is clearly trivial locally, and so the problem is one of passage from local to global data. Here are two approaches, both lead to cohomology theories.

Cech: Take a covering $\mathcal{U} = \{U_\alpha\}$ of M by open sets such that each U_α contains at most one point p_n, and let f_α be a meromorphic function on U_α solving the problem in U_α. Set

$$f_{\alpha\beta} = f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta).$$

In $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0.$$

Solving the problem globally is equivalent to finding $\{g_\alpha \in \mathcal{O}(U_\alpha)\}$ such that $f_{\alpha\beta} = g_\beta - g_\alpha$ in $U_\alpha \cap U_\beta$: given that g_α, $f = f_\alpha + g_\alpha$ is globally defined function satisfying the conditions, and conversely. In the Cech theory,

$$Z^1(\{U_\alpha\}, \mathcal{O}) = \{\{f_{\alpha\beta}\} : f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\}$$

$$\delta C^0(\{U_\alpha\}, \mathcal{O}) = \{\{f_{\alpha\beta}\} : f_{\alpha\beta} = g_\beta - g_\alpha, \text{ some } \{g_\alpha\}\}$$

and the *first Cech cohomology group*

$$H^1(\{U_\alpha\}, \mathcal{O}) = Z^1(\{U_\alpha\}, \mathcal{O})/\delta C^0(\{U_\alpha\}, \mathcal{O})$$

is the obstruction to solving the problem. The direct limit of $H^1(\{U_\alpha\}, \mathcal{O})$ is denoted by $H^1_{Cech}(M, \mathcal{O})$ defines a cohomology group, which only depends on M, which is called the *first Cech cohomology group* of M with coefficient $\mathcal{O}$.

Dolbault. As before, take f_α be a meromorphic function on U_α solving the problem in U_α, and let ρ_α be a bump function (partition of unit), 1 in a neighborhood of $p_n \in U_\alpha$ and having compact support in U_α. Then

$$\phi = \sum_\alpha \overline{\partial}(\rho_\alpha f_\alpha)$$

is a $\overline{\partial}$-closed c^∞-(0,1)-form on M ($\phi \equiv 0$ in a neighborhood of p_n). If $\phi = \overline{\partial}\eta$ for $\eta \in C^\infty(M)$, then the function

$$f = \sum_\alpha \rho_\alpha f_\alpha - \eta$$

satisfies the conditions of the problem: thus the obstruction to solving the problem is in $H^{0,1}_{Dol}(M)$, the *Dolbault*-cohomology.

Note that these two different approaches exactly give what the Dolbault theorem is.

Cech cohomology: Let $\mathcal{F}$ be an abelian group sheaf over a complex manifold X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of topological space X. We denote by

$$U_{i_0,i_1,\dots,i_n} := U_{i_0} \cap \cdots \cap U_{i_n}.$$

The deletion of one of the indices is indicated with the use of a "$\widehat{i_k}$".

An p-cochain for the sheaf $\mathcal{F}$ over $\mathcal{U}$ is a collection of sections of $\mathcal{F}$, one over each $U_{i_0,i_1,\dots,i_p}$ (If $U_{i_0} \cap \cdots \cap U_{i_p} = \emptyset$, we take $f_{i_o\cdots i_p} = 0$). We use $C^p(\mathcal{U}, \mathcal{F})$ to denote the set of all p-cochains of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. Thus

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,i_1,\dots,i_p)} \mathcal{F}(U_{i_0,i_1,\dots,i_p}).$$

For $\forall\{f_{i_0\cdots i_p}\},\ \{g_{i_0\cdots i_p}\} \in C^p(\mathcal{U}, \mathcal{F})$, defining the addition operation

$$\{f_{i_0\cdots i_p}\} + \{g_{i_0\cdots i_p}\} = \{f_{i_0\cdots i_p} + g_{i_0\cdots i_p}\}$$

then $C^p(\mathcal{U}, \mathcal{F})$ becomes an abelian group, we called $C^p(\mathcal{U}, \mathcal{F})$ p-dimensional **cochains group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$.

Now we define the operator

$$\delta_p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F}) : f \mapsto \delta_p f$$

where

$$(\delta_p f)_{i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k f_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}. \tag{1}$$

In the right hand side of (1), each $f_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}$ restricts to $U_{i_0} \cap \cdots \cap U_{i_{p+1}}$ and proceeds the addition operation in $\Gamma(U_{i_0} \cap \cdots \cap U_{i_{p+1}}, \mathcal{F})$. It is easy to verify

δ_p is a homeomorphism of group, and $\delta_{p+1} \circ \delta_p = 0$; $p \geq 1$. $Z^p(\mathcal{U}, \mathcal{F}) := Ker\ \delta_p \subset C^p(\mathcal{U}, \mathcal{F})$, $p \geq 0$, is called the p-dimensional **cocycles group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$, and $B^p(\mathcal{U}, \mathcal{F}) = Im\ \delta_{p-1}$, $p \geq 1$, is called the p-dimensional **coboundaries group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$, and $B^0(\mathcal{U}, \mathcal{F}) \equiv 0$. From $\delta_{p+1} \circ \delta_p \equiv 0$, $B^p(\mathcal{U}, \mathcal{F}) \subset Z^p(\mathcal{U}, \mathcal{F})$. Define

$$H^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F}), \quad \text{for } p \geq 1$$

and

$$H^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) \quad p = 0$$

$H^p(\mathcal{U}, \mathcal{F})$ is called the p-dimensional cohomology gy group of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. Define

$$H^p(X, \mathcal{F}) = \lim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F}).$$

6.3 Sheaf Maps

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over M. Suppose there is $\{\phi|_U\}, \phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that, for any open set $U \subset V$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \stackrel{\phi_U}{\to} & \mathcal{G}(U) \\ \rho_V^U \downarrow & & \rho_V^U \downarrow \\ \mathcal{F}(V) & \stackrel{\phi_V}{\to} & \mathcal{G}(V) \end{array}.$$

We call such map a **sheaf map**.

Examples:
1. Inclusion maps: $\mathcal{C} \subset \mathcal{O}_X \subset \mathbf{M}_X$.
2. Differentiation maps:

$$d : \mathcal{C}_X^\infty \to \mathcal{E}_X^1.$$

$$d(= \partial) : \mathcal{O}_X \to \Omega_X^1.$$

3. Restriction or Evaluation Maps:

$$div : \mathcal{M}_X^* \to Div_X.$$

$$eval_p : \mathcal{O}_X[D] \to \mathbf{C}_p$$

$$f = \sum_{n \geq -D(p)} c_n z^n \mapsto c_{-D(p)}.$$

4. The exponenial maps. $exp(2\pi i -) : \mathcal{O}_X \to \mathcal{O}_X^*$.

The Kernel of the sheaf map: Suppose that $\phi : \mathcal{F} \to \mathbf{G}$ is a sheaf map Define Let

$$\mathcal{K}(U) := ker_\phi(U) = ker\{\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)\} \subset \mathcal{F}(U),$$

then it is a well-defined sheave.

One-to-one and onto: We say that ϕ is one-to-one, or injective, if every point p and open set U with $p \in M$, there is an open set $V \subset U$ containing p such that β_V such that ϕ_V is 1-1. We say that ϕ is onto, or surjective, if for every p and open set U with $p \in M$, and every $f \in \mathcal{G}(U)$, there is an open set $V \subset U$ containing p such that ϕ_V hits the restriction of f to V. Note that we don't require that all ϕ_U to be 1-1 or onto, but only "eventually" 1-1 or onto, in the sense above, although we have the following lemma regarding the 1-1:

Lemma. *The following are equivalent for sheaf map $\phi : \mathcal{F} \to \mathbf{G}$ (i) ϕ is 1-1, (ii) ϕ_U is 1-1 for every open subset $U \subset M$, (iii) the kernel sheaf $\mathcal{K}$ is identically zero sheaf.*

The analogous lemma **is not true** for onto maps of sheaves. For example, take $M = \mathbf{C}^*$, and consider $exp(2\pi i -) : \mathcal{O}_X \to \mathcal{O}_X^*$. $g(z) = 1/z \in \mathcal{O}_X^*$, there is no $f \in \mathcal{O}_X$ with $exp(2\pi i f) = g$. But, from the definition above, this map is onto.

Short Exact seqeuence: We say that a sequence of sheaf maps

$$0 \to \mathcal{K} \to \mathcal{F} \stackrel{\phi}{\to} \mathcal{G} \to 0$$

is a short exact sequence if ϕ is onto, and the sheaf $\mathcal{K}$ is the kernel sheaf of phi. Or equivalently, we can use the the quotient sheaf $\mathcal{G}/Im\phi$ to define it: the quotient sheaf $\mathcal{G}/Im\phi$ defined as follows: a section $s \in (\mathcal{G}/Im\phi)(U)$ if and only if there is an open covering of U: $U = \cup_\alpha U_\alpha$ and $s_\alpha \in \mathcal{G}(U_\alpha)$ such that for all $U_\alpha \cap U_\beta \neq$,

$$s_\alpha|_{U_\alpha \cap U_\beta} - s_\beta|_{U_\alpha \cap U_\beta} \in \phi_{U_\alpha \cap U_\beta}(\mathcal{F}(U_\alpha \cap U_\beta)).$$

A sequence of sheaf maps

$$0 \to \mathcal{F}_1 \stackrel{\phi}{\to} \mathcal{F}_2 \stackrel{\beta}{\to} \mathcal{F}_3 \to 0$$

is a short exact sequence if $Im(\alpha) = ker(\beta)$ and $\mathcal{F}_3 = \mathcal{F}_2/Im(\alpha)$.

Remark: For a short exact sequence

$$0 \to \mathcal{F}_1 \stackrel{\phi}{\to} \mathcal{F}_2 \stackrel{\beta}{\to} \mathcal{F}_3 \to 0,$$

by the definition of the quotient sheaf, it does not imply that

$$0 \to \mathcal{F}_1(U) \stackrel{\phi_U}{\to} \mathcal{F}_2(U) \stackrel{\beta_U}{\to} \mathcal{F}_3(U) \to 0.$$

It only implies the following: if for every section $\sigma \in \mathcal{F}_3(U)$, and every $p \in U$, there is an open set $V_p \subset U$ containing p such that σ_V is the image of β_V.

Examples of short exact sequences:

1.
$$0 \to \mathbf{C} \to \mathcal{O} \overset{d=\partial}{\to} \Omega^1_X \to 0$$

2.
$$0 \to \mathcal{Z} \to \mathcal{O} \overset{\exp(2\pi i -)}{\to} \mathcal{O}^* \to 0$$

3.
$$0 \to \mathcal{O} \to \mathcal{C}^\infty \overset{\bar\partial}{\to} \mathcal{E}^{0,1} \to 0$$

4. For any divisor D,
$$0 \to \mathcal{O}[D-p] \to \mathcal{O}[D] \overset{Eval_p}{\to} \mathcal{C}_p \to 0$$

5. For any divisor D,
$$0 \to \Omega^1[p-D] \to \Omega^1[-D] \overset{Res_p}{\to} \mathcal{C}_p \to 0$$

Definition. Let
$$\mathcal{F}_1 \overset{\alpha}{\to} \mathcal{F}_2 \overset{\beta}{\to} \mathcal{F}_3$$
be a sequence. This sequence is exact at $\mathcal{F}_2$ if, firstly, the composition of the map is zero, and secondly, for every open set U and every point $p \in U$ and every section $g \in \mathcal{F}_2(U)$ which is in the kernel of β_U, there is an open set $V \subset U$ containing p such that α_V such that $\rho^U_V(g)$ is in the image of α_V.

Proposition. *Let*
$$0 \longrightarrow \mathcal{F} \overset{\lambda}{\longrightarrow} \mathcal{G} \overset{\mu}{\longrightarrow} \mathcal{H} \longrightarrow 0 \quad (*)$$
be an exact sequence of sheaves. Then for $\forall U \subset X$*,*
$$0 \longrightarrow \Gamma(U,\mathcal{F}) \overset{\lambda_U}{\longrightarrow} \Gamma(U,\mathcal{G}) \overset{\mu_U}{\longrightarrow} \Gamma(U,\mathcal{H}) \quad (**)$$
is an exact sequence of section groups.

Proof $Ker(\lambda_U) = 0$, since $\forall f \in \Gamma(U,\mathcal{F}),\ \lambda_U(f) = 0$, i.e., for $\forall x \in U$, $\lambda\big(f(x)\big) = 0$, since λ is injective, $f(x) = 0$. $\forall x \in U,\ f \equiv 0$, therefore the sequence (**) is exact at $\Gamma(U,\mathcal{F})$. Since $\mu \circ \lambda = 0$, $\mu_U \circ \lambda_U = 0$ by the definition of μ_U and λ_U, therefore $Im(\lambda_U) \subset Ker(\mu_U)$. For $\forall g \in \Gamma(U,\mathcal{G})$, if $\mu_U(g) = 0$, that is $\mu\big(g(x)\big) = 0$, for $\forall x \in U$. By the exactness of (*) , $g(x) \in Im(\lambda)$, $\forall x \in U$, i.e., $Im\ g \subset Im(\lambda)$, hence there exists $f \in \Gamma(U,\mathcal{F})$ such that $\lambda_U(f) = g$. This finishes the proof.

In general, the μ_U is not necessarily surjective. We provide an example to elucidate the fact.

Example: $X = \Delta^* = \{z \in \mathbf{C}^1 | 0 < |z| < 1\}$ is the punctured unit disc in $\mathbf{C}^1, \mathcal{O}$ is the sheaf of germs of holom orphic functions, $|calO^*$ is the sheaf of germs of holomorphic functions without the zero, Z is the sheaf of germs of integral numbers, then we have following exact sequence of sheaves

$$0 \longrightarrow Z \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$$

where i is inclusion homomorphism, $e(\mathbf{f}_x) = (exp\ 2\pi i f)_x$, where $\mathbf{f}_x$ is the germ of f at x and f is a holomorphic on a neighborhood of x, $(exp\ 2\pi i f)_x$ in the germ of $exp\ 2\pi i f$ at x. It is easy to verify (4) is an exact sequence of sheaves. Now we consider the following sequence of group homomorphisms,

$$0 \longrightarrow \Gamma(\Delta^*, Z) \xrightarrow{i_{\Delta^*}} \Gamma(\Delta^*, \mathcal{O}) \xrightarrow{e_{\Delta^*}} \Gamma(\Delta^*, \mathcal{O}^*) \longrightarrow 0.$$

For the holomorphic function $z \in \Gamma(\Delta^*, \mathcal{O}^*)$, there is no $g \in \Gamma(\Delta^*, \mathcal{O})$ such that $exp(2\pi i g) = z$. In fact, the only solution is $g = \frac{1}{2\pi i} log\ z$, but $\frac{1}{2\pi i} log\ z$ is not the unique valued holomorphic functions on Δ^*.

The Connecting Homomorphism. Suppose $\phi : \mathcal{F} \to \mathbf{G}$ is an onto map of sheaves. Let $\mathcal{K}$ be the kernel sheaf for ϕ. We define a map, called the *Connecting Homomorphism*

$$\delta : H^0(X, \mathcal{G})(\cong \mathcal{G}(X)) \to H^1(X, \mathcal{K})$$

as follows: Take $g \in \mathcal{G}(X)$. Since ϕ is onto, for every point $p \in X$, there is an open neighborhood U_p of p such that $g = phi(f_p)$ on U_p. Note that the collection $\mathcal{U} = \{U_p\}$ is an open cover of X: let $h_{pq} := f_q - f_p \in \mathcal{F}(U_p \cap U_q)$. It is clear that (h_{pq}) is a 1-cocycle for the sheaf ; moreover, $\phi(h_{pq}) = 0$ since the difference is essentially $g - g$. Therefore (h_{pq}) is a 1-cocycle for the kernel sheaf $\mathcal{K}$, and represent a cohomology class in $H^1(\mathcal{U}, \mathcal{K})$. Its image in $H^1(X, \mathcal{K})$ will be denoted by $\delta(g)$. It can be proved that the construction of $\delta(g)$ is independent of the choice of covering $\mathcal{U}$ and the choice of preimage f_p.

The purpose of the Connecting Homomorphism δ is to give a criterion for when a given global section $g \in \mathcal{G}(X)$ is hit by a global section of $\mathcal{F}$.

Lemma. *Suppose that $g \in \mathcal{G}(X)$ is a global section. Then there is a global section of $f \in \mathcal{F}$ such that $\phi(f) = g$ if and only if $\delta(g) = 0$.*

Proof. "$\Longrightarrow$". Suppose that $\phi(s) = g$ for some $s \in \mathcal{F}$. Then in the definition of the connecting homomorphism, we may choose $U_p = X$ for every $p \in X$ and $f_p = s$. Using the notation above, $h_{pq} = 0$ for every p, q so this the identically zero 1-cocycle, which if course induces the zero element in cohomology.

"$\Longleftarrow$". Suppose that $\delta(g) = 0$ in $H^1(X, \mathcal{K})$. Using the definition above, this means that $h_{pq} = 0$ is a boundary, and we may write $h_{pq} = k_q - k_p$ for some 0-cohain k_p) for $\mathcal{K}$. Set $s_p := f_p - f_q$, where f_p is the preimage of g under ϕ locally on the set U_p. On $U_p \cap U_q$, we have

$$s_p - s_q = (f_p - k_p) - (f_q - k_q) = (k_q - k_p) - (f_q - f_p) = k_q - k_p - h_{pq} = 0$$

and son, by the sheaf axiom the section $\{s_p\}$ patch together to give a global section $s \in \mathcal{F}(X)$. This finishes the proof.

Corollary. *Let $\phi : \mathcal{F} \to \mathbf{G}$ be an onto map of sheaves with kernel sheaf $\mathcal{K}$. Then the map $\phi(X) : \mathcal{F}(X) \to \mathbf{G}(X)$ is onto if $H^1(X, \mathcal{K}) = 0$.*

The Long Exact Sequence of Cohomology.

Theorem*Let $\phi : \mathcal{F} \to \mathbf{G}$ be an onto map of sheaves with kernel sheaf $\mathcal{K}$. Then the sequence*

$$0 \to \mathcal{K}(X) \stackrel{inc}{\to} \mathcal{F}(X) \stackrel{\phi_X}{\to} \mathcal{G}(X) \stackrel{\delta}{\to} H^1(M, \mathcal{K}) \stackrel{inc_*}{\to} H^1(X, \mathcal{F} \stackrel{\phi_*}{\to} H^1(X, \mathcal{G})$$

is exact at every step.

Proof. The exactness at $\mathcal{K}(X)$ and $\mathcal{F}(X)$ is just the defiiton of the kernel sheaf. The exactness at $\mathcal{G}(X)$ is, as mentioend above, exactly the content of the above Lemma.

To see the image $(\delta) \subset Ker(inc_*)$, suppose that $g \in \mathcal{G}(X)$. The first step in defining $\delta(g)$ is to choose an open covering $\{U_i\}$ and find elements $f_i \in \mathcal{F}(U_i)$ with $\phi_{U-i}(f_i) = g|_{U_i}$, then $\delta(g)$ is defined by the 1-cocycle $f_i - f_j$ for the sheaf $\mathcal{K}$. But this cocycle is obviously a coboundary in the sheaf $\mathcal{F}$.

To finish the exactness at $H^1(M, \mathcal{K})$, we must check that $Ker(inc_*) \subset image(\delta)$. Suppose that (k_{ij}) is a 1-cocycle for the sheaf $\mathcal{K}$ ehich represents a class in the kernel of inc_*. Then (k_{ij}) is a coboundary, considered as a 1-cocycle for the sheaf $\mathcal{F}$, and so there is a 0-cochain (f_i) such that $k_{ij} = f_i - f_j$ on $U_i \cap U_j$ for every i, j. Consider the 0-cochain (g_i) for $\mathcal{G}$, where $g_i = \phi(f_i)$. Note that

$$g_i - g_j = \phi(f_i - f_j) = \phi(k_{ij})$$

on $U_i \cap U_j$, so by the sheaf axiom for $\mathcal{G}$ there is a global section $g \in \mathcal{G}(X)$ such that $g|_{U_i} = g_i$ for every i. It is clear from the definition of δ that $\delta(g)$ is the class of (k_{ij}).

Finally we must check the exactness at $H^1(M, \mathcal{F})$. It is clear that $inc_* \circ \phi_* = 0$, so we only need to check that $ker(\phi_*) \subset image(inc_*)$. Let c be a class in $ker(\phi_*)$, and represent c by a 1-cocycle (f_{ij}) with respect to some open covering $\mathcal{U}$ of X. Since $\phi_*(c) = 0$, we have that the 1-cocycle $(\phi(f_{ij}))$ represents $A0$ in $H^1(M, \mathcal{G})$. Therefore it is a coboundary; there is a 0-cocycle (g_i) with respect to the open covering $\mathcal{U}$ such that $\phi(f_{ij}) = g_i - g_j$ for every i, j. After refining $\mathcal{U}$ further we may assume, since ϕ is an onto map of sheaves, that each g_i is equal to $\phi(f_i)$ for some element $f_i \in \mathcal{F}(U_i)$. Let $h_{ij} = f_{ij} - f_i - f_j \in \mathcal{F}(U_i \cap U_j)$, this is clearly a 1-cocycle since (f_{ij}) is. Appying ϕ, we see that

$$\phi(h_{ij}) = \phi(f_{ij}) - g_i - g_j = 0,$$

so that $\phi(h_{ij})$ is actually a 1-cocycle for the kernel sheaf $\mathcal{K}$. Since it differs from the cocycle (f_{ij}) by the coboundary of the 0-cocycle (f_i), it also gives the original class c in cohomology. Thus c is in the image of inc_*. This finishes the proof.

The above theorem is usually expressed as saying "a short exact sequences of heaves gives a long exact sequences in cohomology". Paracompactness is the property which ensures it is true. In general we can prove, in a similar way:

Theorem(from short to long exact sequence) *Assume that*

$$0 \to \mathcal{E} \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{G} \to 0$$

is exact. Then there are connecting homomorphisms $\delta : H^n(X, \mathcal{G}) \to H^{n+1}(X, \mathcal{F})$ for every $n \geq 0$ such that the sequence of cohomology groups

$$0 \to H^0(M, \mathcal{K}) \xrightarrow{i^*} H^0(M, \mathcal{F}) \xrightarrow{j_*} H^0(M, \mathcal{G}) \xrightarrow{\delta}$$

$$\to H^1(M, \mathcal{K}) \xrightarrow{i^*} H^1(M, \mathcal{F}) \xrightarrow{j_*} H^1(M, \mathcal{G}) \xrightarrow{\delta^*}$$

$$\to H^2(M, \mathcal{K}) \xrightarrow{i^*} H^2(M, \mathcal{F}) \xrightarrow{j_*} H^2(M, \mathcal{G}) \xrightarrow{\delta^*} \cdots$$

is exact.

6.4 Sheaves and Line bundles

An **invertible sheaf** is a coherent sheaf $\mathcal{L}$ on M such that each point $x \in M$ has an open neighborhood $U \subset M$ such that $\mathcal{L}(U) \cong \mathcal{O}_U$ as $\mathcal{O}_M$-modules.

Recall that a holomorphic line bundle L defines a coherent analytic sheaf (of sections) $\mathcal{L}$ over X by $\mathcal{L}(U) = \{$ (local) holomorphic sections of L on $U\}$. It is **an invertible sheaf** since

$$\mathcal{L}(U_\alpha) \cong \mathcal{O}_{U_\alpha}.$$

Conversely, let $\mathcal{L}$ be an invertible sheaf, and let $\phi_\alpha : \mathcal{L}(U_\alpha) \cong \mathcal{O}_{U_\alpha}$ be the local trivializations. Then $g_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ gives the line bundle L. Hence, we also call **invertible sheaf** as **line bundle** (or an invertible sheaf on M (any irreducible algebraic variety) is simply the sheaf of holomorphic sections of some holomorphic line bundle, the structure sheaf of holomorphic functions $\mathcal{O}$ corresponds to the trivial line bundle).

Given a line bundle L over M, and given an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M with U_α being the trivialization neighborhood of L. Then its transition function $\phi_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$, where $\mathcal{O}_M^*(U_\alpha)$ is the sheaf of nonwhere vanishing holomorphic functions on M. So $\{\phi_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{O}_M^*)$. Further, the compatible conditions imply that $\{\phi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}_M^*)$. We we get a map $L \mapsto [\{\phi_{\alpha\beta}\}] \in H^1(\mathcal{U}, \mathcal{O}_M^*)$. In this way, we can prove the following important statement: *There is one-to-one correspondence between the equivalent classes of holomorphic line bundles on M and the elements of the cohomology group $H^1(M, \mathcal{O}_M^*)$.*

The concept of line bundle is intimitely related to the concept of **divisors**, which originated from the Riemann surfaces. On a Riemann surface, poles and zeros of meromorphic functions are isolated points. We use $p_1, \cdots p_n$ to denote these isolate points. Then the formal sum, $\sum n(p_i)p_i$, is called a divisor, where $n(p_i) \in Z$. Those $n(p_i) \in Z^+$ denote the multiplicities of the zeros p_i, and those $n(p_i) \in Z^-$ denote the multiplicities of the poles p_i. So, in fact, $\sum_i n(p_i)p_i$ reflects a meromorphic function with the given poles and zeros, counting multiplicities.

For a complex manifold M, the divisor is a complex submanifold with codimention 1, which is locally defined by the set of zeros of a holomorphic function. Alternatively (Weil's divisor)

Definition A divisor D on M is a formal linear combination

$$D = \sum a_i[Y_i]$$

where $Y_i \subset M$ irreducible hypersurfaces and a_i are integers. The divisor group $Div(X)$ is the set of all divisors endowed with the natural group structure. A divisor D is called effective if $a_i \geq 0$ for all i.

Let D be a divisor on M, and $\{U_i\}_{i\in I}$ be an open covering of M such that on each U_i; $i \in I$, $D \cap U_i = \{f_i = 0\}$, where f_i is a holomorphic function on U_i. When $U_i \cap U_j \neq \phi$

$$\phi_{ij} := \frac{f_i}{f_j} \quad U_i \cap U_j,$$

then $\phi_{ij} \neq 0$ on $U_i \cap U_j$ and $\phi_{ij} \cdot \phi_{ji} = 1$; on $U_i \cap U_j$, $\phi_{ij}\phi_{jk}\phi_{ki} = 1$ on $U_i \cap U_j \cap U_k$, so $\{\phi_{ij}\}_{i\in I}$ is a transitive function, which defines a line bundle L. We call L the line bundle associated to the divisor D, and denote it by $L = [D]$. If D is defined by $D \cap U_i = \{f_i = 0\}$, where $\{U_i\}_{i\in I}$ is an open covering of M and f_i is holomorphic function, then $\{f_i\}_{i\in I}$ is a holomorphic section over M, i.e. $f \in \Gamma(M, [D])$

$$f \mid U_i = f_i.$$

Obviously the zeros of f is just the divisor D. This section is called the *canonical section* and is denoted by s_D.

We need to point that the $[D]$ is unique in the isomorphic sense of line bundles. If there is another system of holomorphic functions defining D, then $\frac{f_i}{f_i'} \neq 0$ on U_i; $\forall i \in I$, then

$$u_i = \frac{f_i}{f_i'} : u_i \longrightarrow \mathbf{C}^* = \mathbf{C}\backslash\{\mathbf{0}\}$$

so that

$$\phi_{ij} = \frac{f_i}{f_j} = \frac{u_i}{u_j} \cdot \frac{f_i'}{f_j'} = \frac{u_i}{u_j} = \phi_{ij}'.$$

Hence the line bundles defined by $\{\phi_{ij}\}$ and $\{\phi'_{ij}\}$ are equivalent.

Let's take $H : a_0z_0+\cdots+a_nz_n = 0$ be a hyperplane in $\mathbf{P}^n$. let $\mathbf{P}^n = \cup_{i=0}^n U_i$ be the standard open covering. Then on U_i, we have $f_i = a_0\frac{z_0}{z_i} + \cdots + a_n\frac{z_n}{z_i}$, hence $\phi_{ij} := \frac{f_i}{f_j} = \frac{z_j}{z_i} \quad U_i \cap U_j$.

A Cartier divisor on X is a family $(U_i, g_i), i \in I$, where $\{U_i\}_{i\in I}$ is an open covering of X, and g_i are meromorphic functions such that g_i/g_j is holomorphic on each intersections $U_i \cap U_j$. The functions g_i are called local equations of the divisor. More precisely, a Cartier divisor is an equivalence class of such data. Two collections (U_i, g_i) and (U'_i, g'_i) are equivalent if their union is still a divisor. Cartier divisors can be added by multiplying their local equations. Thus they form a group, denoted by $Div(X)$. The divisor $(U_i, g_i), i \in I$, is called effective if every g_i is holomorphic. Let $\mathcal{M}_X$ be the sheaf of meromorphic functions on M. $\mathcal{M}(U) = \mathbf{C}(U)$. To every Cartier divisor $(U_i, g_i), i \in I$, we can attach a subsheaf $\mathcal{O}_X(D) \subset \mathcal{M}_X$. Namely, on U_i, it is defined as $g_i^{-1}\mathcal{O}_{U_i}$. On the intersections $U_i \cap U_j$, the sheaves $g_i^{-1}\mathcal{O}_{U_i}$ and $g_j^{-1}\mathcal{O}_{U_j}$ coincide since g_i/g_j is invertible. Therefore, the sheaves can be pasted together into a sheaf $\mathcal{O}_X(D) \subset \mathcal{M}_X$. It is an invertible sheaf since multiplication by g_i gives an isomorphism $\mathcal{O}_{U_i}(D)$ and $\mathcal{O}_{U_i}$. A nonzero section of $\mathcal{O}_X(D)$ is a meromorphic function on X such that fg_i are holomorphic on U_i, in other words, $(f) + D$ is effective. If D itself is effective, then the sheaf $\mathcal{O}_X(D)$ has a canonical section s_D, which corresponds to the constant function 1. By contrast, the sheaf $\mathcal{O}_X(-D)$, for an effective D, is an ideal sheaf in $\mathcal{O}_X$. The sections of invertible sheaf define some divisors. Let $s \in H^0(X, \mathcal{L})$ be a non-trivial section, then after choosing some trivilizations $\phi_i : \mathcal{L}_{U_i} \sim \mathcal{O}_{U_i}$, we obtain an effective divisor $(U_i, \phi_i(s_i))$, which we denoted by $div(s, \mathcal{L})$. For instance, the canonical section of s_D defines D.

Suppose now X is a projective variety in $\mathbf{P}^n$, then any sheaf $\mathcal{O}(d)$ can be restricted on X, thus we get a sheaf $\mathcal{O}_X(d)$ for any d. In particular, we have a restriction homomorphism of global sections $H^0(\mathbf{P}^n, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$. This map is not injective if and only if X is degenerate (i.e. X is contained in some hyperplane). Its image is a vector subspace $W \subset H^0(X, \mathcal{O}_X)$ with the following obvious property: for any $x \in X$, there is $s \in W$ with $s(x) \neq 0$. Clearly, the divisors of the form $div(s, \mathcal{O}_X(1)), s \in W$ are just the hyperplanes sections of $\mathcal{H}$.

In general, if X is a variety with an invertible sheaf $\mathcal{L}$, then any family of divisors $|W|$ of the form $div(s, \mathcal{L}), s \in W$ is called a linear systems of divisors.

The Divisor Group and the Picard Group:

Recall that when $n = 1$, a divsor is $D = \sum_p n_p p$. When $n > 1$, a divisor is $D = \sum_V n_V V$, where V are irreducible analytic hypersurfaces. Denote $Div(M) = H^0(M, \mathcal{M}^*/\mathcal{O}^*)$, also called the *group of divisors*. In fact, locally D

is given by $f_\alpha \in \mathcal{M}(U_\alpha)$, then $f := \frac{f_\alpha}{f_\beta} \in H^0(M, \mathcal{M}^*/\mathcal{O}^*)$ is a global meromorphic section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$.

For $\forall f \in \mathcal{M}(M)$, $(f) = \sum_p ord_p(f)p \in \mathcal{D}$. Denote by $\mathcal{P} = \{(f), f \in \mathcal{M}(M)\}$. When $n > 1$, $\forall f \in \mathcal{M}(M)$, $(f) = \sum_V ord_V(f)V$, where V are irreducible analytic hypersurfaces. $\mathcal{P} = \{(f), f \in \mathcal{M}(M)\}$. Then $\mathcal{P} \cong H^0(M, \mathcal{M}^*)$.

A line bundle $L \Leftrightarrow \{U_\alpha, g_{\alpha\beta}\}$ can be regarded as an element in $H^1(M, \mathcal{O}^*)$ (since $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and satisfies $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$)). The groups of the line bundles up to isomorphisms is called the Picard group, and is denoted by $Pic(M)$.

when $n > 1$, from the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0,$$

one has the exact sequence

$$\begin{array}{cccc} H^0(M, \mathcal{M}^*) \to & H^0(M, \mathcal{M}^*/\mathcal{O}^*) \to & H^1(M, \mathcal{O}^*) \to & H^1(M, \mathcal{M}^*) \\ \| & \| & \| & \| \\ \mathcal{P} & Div(M) & Pic(M) & \text{it may not be empty} \end{array}.$$

In the case that $H^1(M, \mathcal{M}^*) \neq 0$, $Div(M)/\mathcal{P}$ may not be isomorphic to $Pic(M)$ (not in the case $n = 1$, we have $Div(M)/\mathcal{P} \cong Pic(M)$).

When $n = 1$ and for any divisor $D = \sum n_p p$, we have the formula $\deg(D) = c_1([D])(M) = \frac{1}{2\pi}\int_M \Theta$ where $c_1([D])$ is the first Chern class of the line bundle $[D]$ (see below) and Θ is the curvature form. When $n > 1$, for any divisor D, the first Chern class $c_1([D]) \in H^2(M, \mathbf{Z})$. This come from the short exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

and hence $H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \to H^2(M, \mathbf{Z})$. When $n = 1$, we have $\deg(f) = 0$ for any $f \in \mathcal{M}(M)$ by the residue theorem. When $n > 1$, we also have $c_1((f)) = \int_M \Theta = 0$ because $(f) \in \mathcal{P}$ means that $[(f)] = 0$ in $Div(M)/\mathcal{P}$ and $\delta : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbf{Z})$ is a group homomorphism.

6.5 Cohomology Computations

There are at least three basic ways to use vanishing of cohomology groups to make the conclusion about the other cohomology groups, using the long exact sequence. The most trivial one is that if

$$0 = A \to B \to C = 0$$

then $B = 0$.

A second is if

$$0 = A \to B \to C \to D = 0$$

then one concludes that $B \cong C$.

A third is that if one knows that in a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \stackrel{\phi}{\to} \mathcal{G} \to 0$$

the $H^1(X, \mathcal{F})$ in the middle sheaf is zero. One then conclude that

$$H^1(X, \mathcal{K}) \cong \frac{\mathcal{G}(X)}{\phi(\mathcal{F}(X))}.$$

The vanishing of H^1:

1. The vanishing of H^1 for C^∞ sheaves: We have, for any $n \geq 1$,

$$H^n(X, \mathcal{C}^\infty) = 0,$$

$$H^n(X, \mathcal{E}^1) = 0.$$

2. The vanishing of H^1 for C^∞ skycraper sheaves: Let $\mathbf{C}_p$ be the skyscraper sheaf. Then (i) $H^0(M, \mathbf{C}_p) = \mathbf{C}$, (ii) $H^1(M, \mathbf{C}_p) = 0$. The assertion of (i) is trivial. As for (ii), consider a cohomology class $\xi \in H^1(M, \mathbf{C}_p)$, which is represented by a cocycle in $Z(\mathcal{U}, \mathbf{C}_p)$. The covering $\mathcal{U}$ has a refinement $\mathcal{B} = \{V_\alpha\}$ such that the point p is contained in only one V_α. But then $Z(\mathcal{U}, \mathbf{C}_p) = 0$ and hence $\xi = 0$. This finishes the proof.

3. **Cohomology of locally constant sheaves**. Let X be a compact Riemann surface of genus g. Let G be an abelian group. Then

(a) $H^0(X, G) \cong G$,
(b) $H^1(X, G) \cong G^{2g}$,
(c) $H^2(X, G) \cong G$ and
(d) $H^n(X, G) = 0$ for $n \geq 3$.

4. **The vanishing of** $H^2(X, \mathcal{O}_X[D])$. Let M be a compact Riemann surfae and D be a divisor. Then $H^n(X, \mathcal{O}_X[D]) = 0$ for any $n \geq 2$.

6.6 The DeRham and Dobeault Theorem

De Rham cohomology. Recall that the De Rham Cohomology groups are defined using the smooth forming and noticing that $d \circ d = 0$.

$$H^k_{DR}(M) := \frac{\{\text{smooth closed } k\text{-forms}\}}{\{\text{smooth exact } k\text{-forms}\}}.$$

Note that $H^0_{DR}(M) \cong \mathbf{C}$ the space of constant functions on M.

Theorem(DeRham Theorem). *Let X be a compact complex manifold. Then, for any $n \geq 0$,*

$$H^n_{DR}(M) \cong H^n(M, \mathbf{C}).$$

Proof. The result is clear for $n = 0$, as well as for $n \geq 3$ (both are zero). To udnerstand $H^1_{DR}(M)$, recall the exact sequence

$$0 \to \mathbf{C} \to \mathcal{C}^\infty \xrightarrow{d} \mathcal{K} = ker(d : \mathcal{E}^1 \to \mathcal{E}^2) \to 0$$

see that, from the long-exact sequence of Cohomology and by noticing that $H^1(X, \mathcal{C}^\infty) = 0$ (using partition of unit) that

$$H^1(M) \cong \mathcal{K}(M)/d(\mathcal{C}^\infty(M)).$$

Note also that

$$H^n(X, \mathcal{K}) \cong H^{n+1}(M, \mathbf{C})$$

for every $n \geq 1$, again, , from the long-exact sequence of Cohomology and by noticing that $H^n(X, \mathcal{C}^\infty) = 0$ (using partition of unit), for all $n \geq 1$.

The analysis of the $H^1_{DR}(M)$ is similar. By Poincare's lemma, the sheaf map $d : \mathcal{E}^1 \to \mathcal{E}^2$ is onto with the kernal $\mathcal{K}$. We then have the long-exact sequence of Cohomology; this gives that

$$H^n(X, \mathcal{K}) = 0 \quad \text{for } n \geq 2$$

and

$$0 \to \mathcal{K}(M) \to \mathcal{E}^1(M) \xrightarrow{d} \mathcal{E}^2(M) \to H^1(M, \mathcal{K}) \to 0$$

sinc e$H^n(X, \mathcal{C}^\infty) = 0$ (using partition of unit), for all $n \geq 1$. Thus we have that

$$H^2_{DR}(M) \cong H^1(M, \mathcal{K}) \cong H^2(M, \mathbf{C}).$$

This proves the theorem.

The Dolbeault Theorem. Recall the definition of the Dolbeault cohomology

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\ker\ \bar{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)}{\text{image}\ \bar{\partial} : \mathcal{E}^{p,q-1}(X) \to \mathcal{E}^{p,q}(X)}.$$

Define the sheaf of holomorphic p-forms Ω^p_M by

$$\Omega^p_M(U) := \Gamma(U, \Omega^p_M) := \{\omega \in \mathcal{A}^{p,0}(U), \bar{\partial}\omega = 0\},$$

the set of holomorphic p-forms on U.

The ordinary Poincare lemma that every closed form on $\mathbf{R}^n$ is exact ensures the de Rham groups are locally trivial. Analogously, a fundamental fact about the Dolbeault cohomology groups is the

Theorem ($\bar{\partial}$-Poincare lemma). *For $\triangle$ a polycyliner in $\mathbf{C}^n$,*

$$H^{(p,q)}_{Dol}(\triangle) = 0, \quad q \geq 1.$$

Similar to the deRahm theorem above, we have

Theorem(Dolbeault Theorem). *Let X be a compact complex manifold. Then*

$$H^{p,q}_{\bar{\partial}}(M) = H^q(M, \Omega^p_M),$$

where Ω^p_M is the sheaf of holomorphic p-forms.

Remark: Note that $\mathcal{E}^{p,q} = 0$ if $p+q > 2$, so have have only 4 possible cases:

$$H^{0,0}_{\bar{\partial}}(M) = \mathcal{O}(M),$$

$$H^{1,0}_{\bar{\partial}}(M) = \Omega^1(M),$$

$$H^{0,1}_{\bar{\partial}}(M) = \frac{\mathcal{E}^{0,1}(X)}{\text{image } \bar{\partial} : \mathcal{C}^\infty(X) \to \mathcal{E}^{0,1}(X)},$$

$$H^{1,1}_{\bar{\partial}}(M) = \frac{\mathcal{E}^{2}(X)}{\text{image } \bar{\partial} : \mathcal{E}^{1,0}(X) \to \mathcal{E}^{2}(X)},$$

Its proof is similar to above, using $d = \bar{\partial} + \partial$, and spliting the usual deRham sequence above in $\bar{\partial}$), i.e we consider

$$0 \longrightarrow \mathcal{O} \to \mathcal{C}^\infty \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0$$

which gives the long exact sequence

$$0 \longrightarrow \mathcal{O}(M) \to \mathcal{C}^\infty(M) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(M) \to H^1(M, \mathcal{O}) \to 0.$$

We see immediately that

$$H^{0,1}_{\bar{\partial}}(M) \cong H^1(M, \mathcal{O}).$$

Similarly, consider the short exact sequence

$$0 \to \Omega^1 \to \mathcal{E}^{1,0} \xrightarrow{\bar{\partial}} \mathcal{E}^2$$

which gives the long exact sequence

$$0 \longrightarrow \Omega^1(M) \to \mathcal{E}^{1,0}(M) \xrightarrow{\bar{\partial}} \mathcal{E}^2(M) \to H^1(M, \Omega^1) \to 0.$$

Therefore we have

$$H^{1,1}_{\bar{\partial}}(M) \cong H^1(M, \Omega^1).$$

6.7 Serre's Duality

Theorem(Serr's Duality). Consider the Dolbeault exact seqeunce

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0$$

or more general the L-valued forms (where L is a holomporphic line bundle over M) (it is called the L-twisting)

$$0 \to \mathcal{O}(L) \to \mathcal{E}^{0,0}(L) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(L) \to 0$$

we get (using the long exact sequence, similar to above)

$$H^1(M, \mathcal{O}(L)) \cong \mathcal{E}^{0,1}(L)(M)/\bar{\partial}(\mathcal{E}^{0,0}(L)(M)).$$

Let L be a holomorphic line bundle on a compact Riemann surface M. Then

$$H^q(M, \Omega^p(L)) \cong (H^{1-q}(M, \Omega^{1-p}(-L)))^*.$$

Proof. We only prove the case when $p = 0$ and $q = 1$, i.e.

$$H^1(M, \mathcal{O}(L)) \cong (H^0(M, \Omega^1(-L)))^*.$$

Let $\phi \in H^0(M, \mathcal{E}^{0,1}(L)), \psi \in H^0(M, \Omega^1(-L))$, then $\phi \wedge \psi \in H^0(M, \mathcal{E}^{1,1})$ (indeed, on $U_\alpha \cap U_\beta$, $\phi_\alpha = g_{\alpha\beta}\phi_\beta$ and $\psi_\alpha = g_{\alpha\beta}^{-1}\psi_\beta$, this implies that $\phi_\alpha \wedge \psi_\alpha = \phi_\beta \wedge \psi_\beta$.) Now since M is compact, so $\int_M \phi \wedge \psi \in \mathbf{C}$. We get a bilinear map

$$H^0(M, \mathcal{E}^{0,1}(L)) \times H^0(M, \Omega^1(-L)) \to \mathbf{C}.$$

If $\phi \in \bar{\partial}(H^0(M, \mathcal{E}^{0,0}(L))) \subset H^0(M, \mathcal{E}^{0,1}(L))$, so that $\phi = \bar{\partial} f$, and $\psi \in H^0(M, \Omega^1(-L)) \subset H^0(M, \mathcal{E}^{1,0}(-L))$, then, by Stoke's theorem,

$$(\phi, \psi) = \int_M (\bar{\partial} f) \wedge \psi = \int_M d(f\psi) = \int_{\partial M} f\psi = 0.$$

So we get the pairing

$$\begin{array}{ccccc} H^0(M, \mathcal{E}^{0,1}(L))/\bar{\partial}(H^0(M, \mathcal{E}^{0,0}(L))) & \times & H^0(M, \Omega^1(-L)) & \to \mathbf{C} \\ \| & & \| & \\ H^1(M, \mathcal{O}(L)) & \times & H^0(M, \Omega^1(-L)) & \to \mathbf{C}. \end{array}$$

This pairing yields the duality $H^1(M, \mathcal{O}(L)) \cong (H^0(M, \Omega^1(-L)))^*$. This prove the theorem.

We can re-formualate the RR as follows **RR** *Let L be a line bundle over a compact Riemann surface M of genus g. Then*

$$\chi(L) := \dim H^0(M, L) - \dim H^1(M, L) = \deg L - g + 1,$$

where $\deg L := \int_M c_1(L)$.

6.8 A new (Sheaf Method) Proof of Riemann-Roch Theorem

Some fact about exact sequence of vector spaces. A sequence of finite dimensional spaces

$$A_1 \stackrel{a_1}{\to} A_2 \stackrel{a_2}{\to} A_3 \stackrel{a_3}{\to} \cdots$$

is exact if Image(a_j) =Kernel(a_{j+1}) for all j. We have the following result: Let

$$0 \to A_0 \stackrel{a_0}{\to} A_1 \stackrel{a_1}{\to} \cdots A_{N+1} \stackrel{a_{N+1}}{\to} 0$$

be exact. Then

$$\sum_{j=0}^{N} (-1)^j \dim(A_j) = 0.$$

Here is the proof: Let $I_k := Image(a_k)$ and $K_k = Ker(a_k)$. Then $A_k = I_k + K_k$ by dimension theorem, and $K_{k+1} = I_k$. Hence

$$\begin{aligned} \sum_{j=0}^{N} (-1)^j \dim(A_j) &= \dim I_0 + \sum_{j=1}^{N-1} (\dim I_j + \dim K_j) + (-1)^N \dim K_N \\ &= \dim I_0 + \sum_{j=1}^{N-1} (\dim I_j + \dim I_{j-1}) + (-1)^N \dim I_{N-1} = 0. \end{aligned}$$

The new (sheaf-method) proof of the Riemann-Roch: Let $D = \sum D(p)p$ be a divisor on the compact RS M and $p \in M$ be a point. Then there is a natrual inclusion map $\mathcal{O}(D) \to \mathcal{O}(D+p)$. Define the sheaf homomorphisn $\beta : (D+p) \to \mathbf{C}_p$ as follows: for $f \in (D+p)(U)$ locally write $f = \sum_{n=-(D(p)+1)}^{\infty} c_n z^n$, and define $\beta_U(f) := c_{-(D(p)+1)} \in \mathbf{C}$. We get the short exact sequence

$$0 \longrightarrow \mathcal{O}(D) \to \mathcal{O}(D+p) \stackrel{\beta}{\to} \mathbf{C}_p \to 0.$$

We now prove the Riemann-Roch Theorem. The case when $D = 0$ is obtained by the fact that $\dim H^0(M, \Omega^1) = g$. Now let D be a divisor on the compact RS M and $p \in M$ be a point. Let $D' = D + p$. Then the above short exact sequence leads to a long exact seqeuence

$$0 \to H^0(M, \mathcal{O}(D)) \to H^0(M, \mathcal{O}(D')) \to H^0(M, \mathbf{C}_p) = \mathbf{C}$$

$$\to H^1(M, \mathcal{O}(D)) \to H^1(M, \mathcal{O}(D') \to H^1(M, \mathbf{C})_p = 0.$$

Hence

$$\dim H^0(M, \mathcal{O}(D)) - H^0(M, \mathcal{O}(D')) + \dim \mathbf{C}$$

$$- \dim H^1(M, \mathcal{O}(D)) + \dim H^1(M, \mathcal{O}(D')) = 0.$$

Thus

$$\dim H^0(M, \mathcal{O}(D')) - H^1(M, \mathcal{O}(D')) - \deg D'$$
$$= \dim H^0(M, \mathcal{O}(D)) - \dim H^1(M, \mathcal{O}(D)) - \deg D.$$

This proves the case $D \geq 0$. In general, we can write $D = P_1 + \cdots + P_m - P_{m+1} - \cdots - P_n$, and this case also can be proved by repeating the above argument.

6.9 A New Proof of the Embedding Theorem

We now use the exact sequence (with directly using the RR) plus the vanishing theorem to reprove the embedding theorem (this gives an insight of the proof of its geberalization to higher-dimensional case by Kodaria).

Theorem(Vanishing theorem). *Let L be a holomorphic line bundle. Then*

(a) If $d(L) > 0$, then $H^1(M, \Omega^1(L)) = 0$,

(b) If $d(L) > 2g - 2$, then $H^1(M, \mathcal{O}(L)) = 0$..

Proof. (a) From Serre's duality,

$$\dim H^1(M, \Omega^1(L)) = \dim H^0(M, \mathcal{O}(-L)).$$

Since $\deg(-L) = -\deg(L) < 0$, we have $\dim H^0(M, \mathcal{O}(-L)) = 0$. This proves (a). The proof of (b) is similar.

We now re-prove the embedding theorem: *If D is a divisor on a compact Riemann surface of genus g. Let $D = (2g + 1)p$. Then $\phi_D : M \to \mathbf{P}^N$ is an embedding.*

Proof: Consdier $L(D)$. As we discussed above, we only need to check (i) For any $q \in M$, there is $f \in L(D)$ such that $f(q) \neq 0$ (base point free), (ii) For any distinct $p, q \in M$, there is $f \in L(D)$ with $f(p) = 0, f(q) \neq 0$, (iii) For any $q \in M$, there is $f \in L(D)$ with $df(p) \neq 0$.

(i): Conisder the short exact sequence

$$0 \to \mathcal{O}(L - q) \to \mathcal{O}(L) \to \mathbf{C}_q \to 0.$$

It then induces a long exact sequence

$$0 \to H^0(M, \mathcal{O}(L - q)) \to H^0(M, \mathcal{O}(L)) \to H^0(M, \mathbf{C}_q) \to H^1(M, \mathcal{O}(L - q)).$$

Since $\deg(L - K - q) = (2g + 1) - (2g - 2) - 1 = 2 > 0$, we have, from the vanishing theroem above, $\dim H^1(M, K + L - K - q) = 0$. Hence we have

$$0 \to H^0(M, \mathcal{O}(L - q)) \to H^0(M, \mathcal{O}(L)) \to H^0(M, \mathbf{C}_q) \to 0.$$

In other words, there is $f \in H^0(M, \mathcal{O}(L))$ with $f(q) \neq 0$. This proves (i).

(ii) and (iii) Take $q, q' \in M$ (may be the same points) and conisder $L_1 = L - q, L_2 = L - q - q'$. Then similar as above, $H^1(M, \mathcal{O}(L_1)) = 0, H^1(M, \mathcal{O}(L_2)) = 0$. Conisder the short exact sequence

$$0 \to \mathcal{O}(L_1) \to \mathcal{O}(L) \to \mathbf{C}_q \to 0$$

and

$$0 \to \mathcal{O}(L_2) \to \mathcal{O}(L_1) \to \mathbf{C}_{q'} \to 0.$$

We obtain that

$$0 \to H^0(M, L_1) \to H^0(M, L) \to^{\pi} H^0(M, \mathbf{C}_q) \to 0,$$

$$0 \to H^0(M, L_2) \to H^0(M, L_1) \to H^0(M, \mathbf{C}_q) \to 0.$$

By indentifying $H^0(M, L_1)$ with $ker(\pi)$, we see that $H^0(M, L_1)$ is a proper subspace of $H^0(M, L)$ and from the second exact sequence, we have that $H^0(M, L_2)$ is a proper subspace of $H^0(M, L_1)$. This shows that ϕ_D is one-to-one and local diffeomorphism, which finishes the proof.

Chapter 7

Complex Geometry of Riemann Surfaces

7.1 Hermitian metric on complex manifolds

Let M be a complex manifold. For $p \in M$, let $(z_1, \ldots, z_n)$ be a local coordinates. Define

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right) \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right),$$

$$\partial = \sum \frac{\partial}{\partial z^i} \otimes dz^i, \quad \bar{\partial} = \sum \frac{\partial}{\partial \bar{z}^i} \otimes d\bar{z}^i, \qquad \textit{and} \quad d = \partial + \bar{\partial}.$$

The *complexified tangent space* is

$$\begin{aligned} T_{\mathbf{C},p}(M) =: \mathbf{C} \otimes \mathrm{T_p(M)} &= \left\{ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{a^i} \frac{\partial}{\partial \mathrm{x^i}}|_{\mathrm{p}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{b^i} \frac{\partial}{\partial \mathrm{y^i}}|_{\mathrm{p}} \mid \mathrm{a^i}, \mathrm{b^i} \in \mathbf{C} \right\} \\ &= \mathbf{C}\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\}. \end{aligned}$$

The *holomorphic tangent space* $T_p^{1,0}(M)$ and the *antiholomorphic tangent space* $T_p^{0,1}(M)$, for $p \in M$, are given by

$$T_p^{1,0}(M) = \mathbf{C}\left\{ \frac{\partial}{\partial z^i}|_p \right\}_{i=1}^n, \qquad T_p^{0,1}(M) = \mathbf{C}\left\{ \frac{\partial}{\partial \bar{z}^i}|_p \right\}_{i=1}^n,$$

so that

$$T_{\mathbf{C},p}(M) = T_p^{1,0}(M) \oplus T_p^{0,1}(M).$$

$T^{(1,0)}(M) = \cup_{p \in M} T_p^{1,0}(M)$ is called the *holomorphic tangent bundle*.

$\Gamma(M, T^{(1,0)}(M)$ is the set of smooth sections of $T^{(1,0)}(M)$, which is also called the *smooth vector fields*. When M is a Riemann surface, $T^{(1,0)}(M)$ is a holomorphic line bundle.

A *Hermitian metric* on M, denoted by ds^2, is a set of Hermitian inner-product $\{\langle\cdot,\cdot\rangle_p\}_{p\in M}$ on $T_p^{(1,0)}(M)$ such that If ξ, η are C^∞ section of $T^{1,0}(M)$ over an open set U, then $\langle \xi, \zeta\rangle$ is the C^∞ function on U. If $z^1, \cdots, z^n$ is a local coordinate system of M, we write

$$ds^2 = \sum g_{i,\bar{j}} dz^i \otimes d\bar{z}^j.$$

In the case of RS, a conformal Riemannian metric (Hermitian) on a Riemann surface M is given by in local coordinates by

$$\lambda^2(z) dz d\bar{z}, \quad \lambda(z) > 0$$

(we assume that λ is C^∞). If $w \mapsto z(w)$ is a transformation of local coordinates, then the metric should transform to

$$\lambda^2(z) \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} dw d\bar{w}.$$

The length of a curve $\gamma : [0,1] \to M$ is given by

$$l(\gamma) := \int_\gamma \lambda(z) d|z|,$$

and the area of a measuarable subset B of M by

$$Area(B); = \int_B \lambda^2(z) \frac{i}{2} dz \wedge \bar{z}.$$

Note that length and area no not depend on the local coordinate.

7.2 Hermitian Line bundles

Instead of $T^{(1,0)}(M)$, we can put a Hermitian metric on (any) line bundle L. An Hermitian metric for a line bnudle $L \to M$ is a smooth section h of the line bundle $l^* \otimes \bar{L}^* \to \mathbf{C}$ such that the function $h : L \otimes \bar{L} \to \mathbf{C}$ defined by

$$h(v, \bar{w}) : h(v \otimes \bar{w})$$

satisfies $h(v, \bar{w}) = \overline{h(w, \bar{v})}$, $h(v, \bar{v}) \geq 0$ and $h(v, \bar{v}) = 0$ iff $v = 0$.

Let L be a line bundle over M with transition functions g_{ij}. Write $h_i = h(e_i, \bar{e}_i)$. Then $h_j = |g_{ij}|^2 h_i$. Hence, a Hermtian metric h on L is a collection of positive smooth real vlaued functions h_i such that $h_j = |g_{ij}|^2 h_i$. Let $s \in$

$H^0(M,L)$ and write $s = s_i e_i$, then $\|s\|^2 = |s_i|^2 h_i = |s_j|^2 h_j$ is well-defined on M. It is called the norm of the holomorphic section s.

For example, on the hyperplane line bundle of hyperplane line bundle of $\mathbf{P}^n$. We endow with a Hermitian metric h on line bundle $[H]$, $h = (h_\alpha)_{0\leq\alpha\leq n}$, where h_α is the local expression of h on U_α.

$$h_\alpha = \frac{|z^\alpha|^2}{|z|^2} = \frac{1}{\sum\limits_{\alpha\neq\beta}^{m} |\frac{z^\beta}{z^\alpha}|^2 + 1}.$$

Connection: A connection is a map $D : \Gamma(M,L) \longrightarrow \Gamma(M, \mathcal{E}^1 \otimes L)$ (note that $\mathcal{E}^k$ is the sheaf of smooth k-forms on M), so $\sigma \in \Gamma(M, \mathcal{E}^1 \otimes L)$ is called the smooth E-valued k-form) such that $D(s+s') = D(s) + D(s')$ and $D(fs) = df \otimes s + fDs$. Let ξ be a local frame of L over an open subset U, i.e. a section of L over U which such that $\xi(x) \neq 0$ for all $x \in U$. Since $D\xi$ is an L-valued form, we can write $D\xi = \omega \otimes \xi$ for some differential form ω that depends on ξ. We call ω is the connection form of D with respect to the local frame ξ. Any section s of L is $s = f\xi$, we we have

$$D(s) = D(f\xi) = df \otimes \xi + \omega \otimes (f\xi).$$

Remark: In the literature one often finds the expression $D = d + \omega$ or

$$Ds = ds + \omega s.$$

These expressions depend on the choice of frame, but often the frame is not explictly mentioned.

If we change the frame ξ to another frame ξ', i.e. $\xi' = f\xi$. Then

$$\omega' \otimes \xi' = D(\xi') = D(f\xi) = df \otimes \xi + f\omega \otimes xi = \left(\frac{df}{f} + \omega\right) \otimes \xi',$$

therefore,

$$\omega' = \omega + \frac{df}{f}.$$

Hence ω is not globally defined. Notice that, however, $d\omega$ is a globally defined 2-form on M (independent of the choice of the local frame). The form $d\omega$ is called the *curvature form* of the connection D.

Example. Let M be a Riemann surface.

(1) The exterior derivative d is a connection for the trivial bundle $\mathcal{O} \to M$.

(2) (Non-example). It is mistakenly asserted in a number of sources that the operator $\bar{\partial} : f \mapsto \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$ is a connection of trivial bundle $\mathcal{O} \to M$. In fact, it is not the case, since

$$\bar{\partial}(fg) \neq df \otimes g + f\bar{\partial}g,$$

so that the Lebniz Rule is not satisfied.

Let L_1, L_2 be two complex line bundles with connections D_1, D_2. Then

$$D_1 \otimes D_2)(\xi_1 \otimes xi_2) = (D_1 \otimes D_2)\xi_1) \otimes xi_2) + \xi_1 \otimes D_2(\xi_2)$$

defines a connection on $L_1 \otimes L_2$. In particular, given L with the connection D, let ξ be a local fram and ξ^* be its dual, notice that $\xi \otimes \xi^*$ is the identity map of the section of the line bundle $L \otimes L^*$, it induced the connection D^* with

$$D^*(\xi^*) = -\xi^* \otimes D(\xi) \otimes \xi.$$

Let L be a complex line bundles with connections D. Its complex conjugate $\bar{D}$ gives a connection on $\bar{L}$ given by

$$\bar{D}(\bar{\xi}) = \overline{D(\xi)}.$$

The Hermitian Connection (or Chern connection) for holomorphic Hermitian line bundles: Since $\mathcal{E}^1 = \mathcal{E}^{(1.0)} \oplus \mathcal{E}^{(0,1)}$, we can decompose D into $D = D' + D''$ where $D' : \Gamma(M, L) \longrightarrow \Gamma(M, \mathcal{E}^{(1,0)} \otimes L)$ and $D'' : \Gamma(M, L) \longrightarrow \Gamma(M, \mathcal{E}^{(0,1)} \otimes L)$. For a general complex line bundle, this splitting is not particularly helpful. However, when the underlying line bundle is holomorphic, this splitting plays a crucial role. The main difference in the setting of holomorphic vector bundles is the ability to define the $\bar{\partial}$-operator for sections of holomorphic line bnudles.

Definition. *Let $L \to M$ be a holomorphic line bundle.* We define $\bar{\partial} : \Gamma(M, L) \longrightarrow \Gamma(M, \mathcal{E}^{(0,1)} \otimes L)$ as follows: choose a holomorphic local frame (section)

$$\bar{\partial}(f\xi) := \bar{\partial} f \otimes \xi.$$

It is easy to see that it is well-defined (independent of the choice of ξ.

Given an Hermitian metric on L, there is a canonical connection (called *Hermitian connection*) $D : \Gamma(M, L) \longrightarrow \Gamma(M, \mathcal{E}^1 \otimes L) = \mathcal{E}^1(L)$ which is

(i) compatible with the complex structure, i.e. in some **holomorphic** local frame e_α, D is type (1, 0), namely $De_\alpha = \theta_\alpha e_\alpha$ with θ_α being a $(1, 0)$ form), or equivalently $D'' = \bar{\partial}$.

(ii) compatible with the Hermitian metric on L (i.e. $d < e_\alpha, e_\alpha >=< De_\alpha, e_\alpha > + < De_\alpha, e_\alpha >$). Such connection is called the Chern connection (or canonical connection).

From $d < e_\alpha, e_\alpha >=< De_\alpha, e_\alpha > + < e_\alpha, De_\alpha >$ we get

$$dh_\alpha = \theta_\alpha h_\alpha + \overline{\theta_\alpha} h_\alpha.$$

Hence

$$\theta = \partial h_\alpha \cdot {h_\alpha}^{-1} = \partial log\ h_\alpha,$$

which is called the connection form. The curvature form is

$$\Theta = d\theta_\alpha = \bar{\partial}\partial \log h_\alpha = \bar{\partial}\partial \log h_\beta, \quad on\ U_\alpha \cap U_\beta.$$

So Θ is a global (1.1)-form on M.

Remark: We have chosen an ad hoc definition for the curvature of the Chern connection, but to give thiis definition some additional meaning, we present the following discussion. The Chen connection for a holomorphic Hermitian line bunlde (L, h), being a $(1, 0)$-form, can be written as

$$D = D' + \bar{\partial},$$

where $D's = \partial s - (\partial \log h)s$. If we think of "$D$" as a "twisted" version of the exterior derivative, designed to map the sections of the line bundle L to L-valued 1-forms, we can consider extending this twisted exterior derivative to differential forms with values in L. Since we are on a Riemann surface, we only need to to L-valuied 1-forms. We define

$$D(\alpha \otimes s) := d\alpha \otimes s - \alpha \wedge Ds,$$

note that the minus sigen in the second term is the usual one obtained by extending the Lebniz Rule to forms of higher degree. The similarity with exterior derivative **ends** when we compute two consective derivatives; we find $DDs \neq 0$. In fact, use the local formula $D = d + \theta$,

$$DDs = D(ds + \theta \otimes s) = d(\theta \otimes s) + \theta \wedge (ds + \theta \otimes s) = (d\theta) \otimes s$$

here we have used $\theta \wedge \theta = 0$. The failure of the second covariant derivative to vanish means that the order of the covrariant partial derivative matters, and therefore suggets that the sections see the space on which they are defined as somewhat "curved". The curvature operator, which measures this failure of the commutativity of mixed partials, is a 0^{th}-order differential opearator (also called the "multipliier") with valued in $\mathcal{E}^{(1,1)}$.

Define the first Chern form of the Hermitian line bundle (L, h) as $c_1(L, h) = \frac{\sqrt{-1}}{2\pi}\Theta = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h_\alpha$. If $\{h'_\alpha\}$ is another metric, then $\Theta' = \bar{\partial}\partial \log h'_\beta$. Hence

$$\Theta - \Theta' = \bar{\partial}\partial(\log h_\beta - \log h'_\alpha = \bar{\partial}\partial \log(h_\beta/h'_\alpha).$$

It is easy to check (since h_α, h'_α satisfy the same transition rule), $(h_\alpha/h'_\alpha) = (h_\beta/h'_\beta)$, so $\gamma := (h_\alpha/h'_\alpha)$ is a globally defined smooth function. Hence

$$\Theta - \Theta' = \bar{\partial}(\partial \log \gamma) = d(\partial \log \gamma).$$

Thus, from the definition of De-Rham cohomology and the DeRham theorem, $c_1(L) \in H^2(M, \mathbf{C})$ and called the *first Chern class of L*.

A (1,1)-form ω is real $\Longleftrightarrow$ locally, $\omega = f\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$ with f being a real valued function. ω is said be be positive(denoted by $\omega > 0$ if $f > 0$. Since for an Hermitian line bunlde L with metric $\{h_\alpha\}$,

$$c_(L,h) = \frac{\sqrt{-1}}{2p}\Theta = -\frac{1}{\pi}\frac{\partial^2 \log h_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha}\left(\frac{\sqrt{-1}}{2}dz_\alpha \wedge d\bar{z}_\alpha\right)$$

which is a real $(1,1)$-form. If M is compact, then $c_1(L,h)(M) := \int_M c_1(L,h) \in \mathbf{R}$ which is called the Cehrn number.

Theorem. *lLet M be a compact Riemann surface, and let h be a Hermitian metric for a holomorphic line bundle L. Then the number*

$$c(L) := \int_M c_1(L,h)$$

is independent of the choice of the metric h.

Proof. If h and h' are two metrics on L, then h/h' is a metric for the trivial bundle and is thus a smooth function on M with no zeros. Denote it by e^{-f}, then

$$\Theta_h - \Theta_{h'} = \sqrt{-1}\partial\bar{\partial}f = d(\sqrt{-1}\bar{\partial}f).$$

Thus by Stokes' theorem, we see that $c(L)$ is independent of the choice of the metric h. This finishes the proof.

Below we shall prove that $c_1(L,h) \in \mathbf{Z}$. L is said to be positive (or ample), denoted by $L > 0$ if there is an hermitian metric h on M such that $c_1(L,h) > 0$.

For example, on the hyperplane line bundle of hyperplane line bundle of $\mathbf{P}^n$. We endow with a Hermitian metric h on line bundle $[H]$, $h = (h_\alpha)_{0\le\alpha\le n}$, where h_α is the local expression of h on U_α.

$$h_\alpha = \frac{|z^\alpha|^2}{|z|^2} = \frac{1}{\sum\limits_{\alpha\neq\beta}^{m} |\frac{z^\beta}{z^\alpha}|^2 + 1}.$$

$$c_1([H]) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \|z\|^2 > 0.$$

so $[H]$ is positive line bundle. It is easy to see that $[H]$ is, in fact, independent of the choice of H, so we denote it by $\mathcal{O}_{\mathbf{P}^n}(1)$.

Theorem. *Let M be a compact Riemann surface and let (L,h) be a holomorphic Hermitian line bundle over M. Let s be a meromorphic section of L. Then*

$$\int_M c_1(L,h) = \#([s=0]) - \#([s=\infty])$$

where $\#([s=0])$ *is the number of zeros, counting multiplicities, and* $\#([s=\infty)]$ *is the number of poles, counting multiplicities.*

Proof. Write $M_s := \{x \in M, ord_x(s) = 0\}$ (so on which s has no zeros or poles). Let $M_{s,\epsilon}$ be the subset of M obtained by removing the coordinate discs $|z_j| < \epsilon$ about the points of $M - M_s$ from M. By stokes theorem,

$$\int_{M_{s,\epsilon}} dd^c \log \|s\|^2 = -\sum_{j=1}^{k} \int_{|z_j|=\epsilon} d^c(\log |z_j|^{2m_j} - h_j).$$

A simple calculation shows that (recall that $d^c = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)$)

$$\int_{|z|=\epsilon} d^c \log |z|^2 = 2\pi.$$

On the other hand, on M_s, we have

$$dd^c \log \|s\|^2 = c_1(L, h).$$

Hence, by letting $\epsilon \to 0$, we get

$$\int_M c_1(L, h) = \#([\sigma = 0]) - \#([\sigma = \infty])$$

which proves the theorem.

The above theorem shows that the Chern number $\int_M c_1(L, h)$ is independent of the choice of the metric on L. Also it means that

Corollary *Let* D *be a divisor on* M*. Then the first Chern class* $c_1([D])$ *is Poincare dual to* D *in the sense that*

$$\int_M c_1([D]) = \deg D.$$

As we see from above, the reason for introducing the line bundles is that it affords us a good technique for localizing and utilizing metric methods in the study of divisors.

We laso have $\deg L = \int_M c_1(L)$.

Corollary *If* L *is a line bundle with* $\deg(L) < 0$*. Then* L *has no non-trivial holomorphic sections.*

Example: The holomorphic line bundle $T_M^{(1,0)}$.

Let M be a real oriented surface with a Riemannian metric g. Since an isothermal coordinates on M always exist, we can choose a complex atlas to make M a Riemann surface, such that in local coordinate $z = x + \sqrt{-1}y$,

$$g == \frac{r}{2}(dx \otimes dx + dy \otimes dy) = r\frac{1}{2}dz \otimes d\bar{z}.$$

Noice that this Riemannian metric for M is noe a Hermitian metric for the holomorphic line bundle $T_M^{(1,0)}$. Recall that the function r depends on z, if z' is another coordinates, then

$$g = r\frac{1}{2}dz \otimes d\bar{z} = r\left|\frac{\partial z}{\partial z'}\right| dz' \otimes d\bar{z}'.$$

Hence the differential $(1,1)$-form

$$\omega_g := \frac{\sqrt{-1}}{2} r dz \wedge d\bar{z}$$

is globally defined. This form is called the metric form, or the area formm associated to g.

It turns out the Chern connection of $T_M^{(1,0)}$ with the Hermitian metric g agrees with the Levi-Civita connection of the Riemannian metric g on M, after we indentify $T_M^{(1,0)}$ with TM by sending a $(1,0)$-vector to its tewice of its real part.

A Hermitian manifold X of arbitrary dimension whose Chern connection of $T_M^{(1,0)}$ with the Hermitian metric g agrees with the Levi-Civita connection of the Riemannian metric g on M, after we indentify $T_M^{(1,0)}$ with TM by sending a $(1,0)$-vector to its tewice of its real part, is called a Kahler manifold. It turns out that being Kahler is equivalent to the property that $d\omega_g = 0$, which holds trivially on Riemann surafce.

The fact that a Hermtian metric on a Riemann surafce is automatically Kahler is one of relative feww low-dimensional accidents that account for the extraodinary rich structure of Riemann surfaces.

7.3 The Gauss-Bonnet Theorem

Let M be a Riemann surafce, and $L = T^{(1,0)}(M)$. Write the metric as $\sigma := r_\alpha dz_\alpha \otimes d\bar{z}_\alpha$ where

$$r_\alpha = \left\langle \frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\alpha} \right\rangle.$$

Then $\Omega = r_\alpha \frac{\sqrt{-1}}{2} dz_\alpha \wedge d\bar{z}_\alpha$ on U_α is the well-defined volume form on M. Let Θ be the curvature form of the metric σ, then we can write

$$K := -\frac{\sqrt{-1}\Theta}{\Omega}$$

is called the Gauss curvature of M with metric σ. Note that K is a globally defined function on M. By direct computation,

$$K = -\triangle \log r_\alpha,$$

where he Laplace-Beltrami operator with respect to the metric σ is defined by

$$\triangle := \frac{4}{r_\alpha^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

For example, on the unit disc $\{|z| < 1\}$, the Poicare metric is given by

$$\frac{4}{(1-|z|^2)^2}dzd\bar{z}.$$

Then $K = -1$.

Theorem (Gauss-Bonnet). *Let M be a compact Riemann surface of genus g, with a metric $\lambda^2(z)dzd\bar{z}$. Then*

$$\int_M K\lambda^2(z)\frac{i}{2}dz \wedge d\bar{z} = 2\pi(2-2g).$$

The Gauss-Bonnet theorem is the special case of RR when taking $L = K$, the canonical bundle of M.

7.4 The Negative Curvature Method

Theorem (Ahlfors-Schwarz Lemma). *Let M be a Riemann surfac with a metric $\lambda^2(z)dzd\bar{z}$ whose curvature K satisfies $K \leq -\kappa < 0$. Then for any holomorphic map $f : D(0,1) \to M$ we have*

$$\lambda^2(f(z)f_z\overline{f_{\bar{z}}} \leq \frac{1}{\kappa}\rho(z),$$

where

$$\rho^2(z))dzd\bar{z} := \frac{4}{(1-|z|^2)^2})dzd\bar{z}$$

is the Poincare metric on the unit-disc.

Proof. Let $\mathbf{D}_r$ be the disc of radius $r < 1$ with the Poincaré metric ds^2 of curvature -1 given by

$$ds^2 = 2a_r(z)dzd\bar{z} \text{ where } a_r(z) = \frac{2r^2}{(r^2-|z|^2)^2}.$$

We compare this metric with $d\sigma^2 = 2b(z)dzd\bar{z}$. Put

$$\mu(z) = \log\frac{b(z)}{a_r(z)}.$$

Since $\mu(z) \to -\infty$ as $z \to \partial \mathbf{D}_r$, there is a point $z_0 \in \mathbf{D}_r$ such that

$$\mu(z_0) = \sup\{\mu(z); z \in \mathbf{D}_r\} > -\infty.$$

Then $b(z_0) > 0$. Since z_0 is a maximal point of $\mu(z)$,

$$0 \geq \frac{\partial^2 \mu}{\partial z \partial \bar{z}}(z_0).$$

On the other hand, since the Gausssian curvature of the Poincaré metric is -1 and the curvature of $d\sigma^2$ is bounded above by -1,

$$\frac{\partial^2 \log a_r}{\partial z \partial \bar{z}} = a_r(z) \text{ and } \frac{\partial^2 \log b}{\partial z \partial \bar{z}}(z) \geq b(z).$$

So

$$0 \geq \frac{\partial^2 \mu}{\partial z \partial \bar{z}}(z_0) = \frac{\partial^2 \log b}{\partial z \partial \bar{z}}(z_0) - \frac{\partial^2 \log a_r}{\partial z \partial \bar{z}}(z_0) \geq b(z_0) - a_r(z_0).$$

Hence $a_r(z_0) \geq b(z_0)$ and so $\mu(z_0) \leq 0$. By the choice of z_0, we have $\mu(z) \leq 0$ on $\mathbf{D}_r$, that is

$$a_r(z) \geq b(z).$$

The Theorem is proven by letting $r \to 1$.

let $M = \mathbf{P}^1(\mathbf{C}) - \{a_i\}_{i=1}^q$ and let $\|z, a\|$ denote the spherical distance of $\mathbf{P}^1(\mathbf{C})$. Define a hermitian metric $d\sigma^2$ on M by

$$d\sigma^2 = \frac{1}{\prod_{i=1}^q \|z, a_i\|^2 (\log c\|z, a_i\|^2)^2} \cdot \frac{4}{(1+|z|^2)^2} dz d\bar{z}$$

where $c > 0$ is a constant. Taking small $c > 0$, one finds that the Gaussian curvature $K_{d\sigma^2} \leq -k < 0$ with a constant $k > 0$ So the Schwarz lemma implies that *The Riemann sphere* $\mathbf{P}^1(\mathbf{C})$ *minus at least three points is Kobayashi hyperbolic.*

Note that in the proof of Theorem above, we see that the theorem holds if $d\sigma^2$ is only continuous at zero points of $d\sigma^2$ and is twice differentiable at the points where it is positive(and hence the curvature is defined). This allows Ahlfors to extend Theorem 5.1.2 to non-smooth metrics. Let $d\sigma^2$ be an upper semi-continuous Hermitian pseudo-metric on the unit disc $\mathbf{D}$. A pseudo-Hermitian metric $d\sigma_0^2$ is called a **supporting pseudo metric** for $d\sigma^2$ at $z_0 \in \mathbf{D}$ if it is defined and of class C^2 in a neighborhood U of z_0 and satisfies the following condition:

$$d\sigma^2 \geq d\sigma_0^2 \text{ on } U \quad \text{and} \quad d\sigma^2 = d\sigma_0^2 \text{ at } z_0.$$

We define

$$K_{d\sigma^2}(z_0) = \inf K_{d\sigma_0^2}(z_0),$$

where the infimum is taken over all supporting pseudo metric $d\sigma_0^2$ for $d\sigma^2$ at z_0. Theorem 5.1.2 is generalized to the following theorem.

Theorem *Let ds^2 denote the Poincaré metric on the unit disc* $\mathbf{D}$*. Let $d\sigma^2$ be an upper semi-continuous Hermitian pseudo-metric on* $\mathbf{D}$ *whose curvature is bounded above by -1. Then*

$$d\sigma^2 \leq ds^2.$$

Corollary *Let X be a Riemann surface with a Hermitian pseudo-metric ds^2_X whose curvature (wherever defined) is bounded above by -1. Then every holomorphic map $f : \mathbf{D} \to X$ is distance-decreasing, i.e.,*

$$f^* ds^2_X \leq ds^2,$$

where ds^2 is the Poincaré metric on the unit disc $\mathbf{D}$*.*

Proof. Set $d\sigma^2 = f^* ds^2_X$. Then $d\sigma^2$ is a Hermitian pseudo-metric on $\mathbf{D}$. If we denote the curvature of ds^2_X by K_X, then the curvature of $d\sigma^2$ is given by $f^* K_X$. Now the Corollary follows from Theorem above. □

The classical Schwarz-Pick Lemma immediately follows from Corollary.

Schwarz-Pick Lemma Let $\mathbf{D}$ be the unit disc with the Poicaré metric ds^2. Then every holomorphic map $f : \mathbf{D} \to \mathbf{D}$ is distance-decreasing, i.e.,

$$f^* ds^2 \leq ds^2, \quad \text{or equivalently}$$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \quad \text{for} \quad z \in \mathbf{D}.$$

7.5 Holomorphic 1-forms and Metrics on compact Riemann surfaces

Theorem. *M be a compact Riemann surface of genus g, and let $\alpha_1, \ldots, \alpha_g$ ne a basis for $H^0(M, \Omega^1)$. Then*

$$\sum_{i=1}^{g} \alpha_i(z) \bar{\alpha}_i(z)$$

defined a metric on M with nonpositive curavture, the so-called the Bergman metric. If $g \geq 2$, then the curvature vanishes at most in a finite number of points.

Corollary. *Every compact Riemann surface of genus $g \geq 2$ admits a metric with negative curvature, hence it is hyperbolic.*

Chapter 8

Hodge Theorem revisited

8.1 The Laplacian Operator

Let M be a Riemann surface and let $G = r_\alpha dz_\alpha d\bar{z}_\alpha = r_\alpha(dx_\alpha \otimes dx_\alpha + dy_\alpha \otimes dy_\alpha)$ be the Riemannian metric on M, i.e.

$$G(\partial/\partial z_\alpha, \partial/\partial z_\alpha) = r_\alpha.$$

The metric G on $T^{(1,0)}$ induces a metric on $T^{(1,0)^*}$ (hence on the space of smooth (p,q)-forms) as follows: $\{\frac{1}{\sqrt{r_\alpha}} \frac{\partial}{\partial z_\alpha}\}$ is an orthonormal basis of $T^{(1,0)}M$. By declaring $\{\sqrt{r_\alpha} dz_\alpha\}$ being an orthonormal basis of $T^{(1,0)^*}M$, it induces a metric on $T^{(1,0)^*}M$, and we have

$$G(dz_\alpha, dz_\alpha) = \frac{1}{r_\alpha} G(d\bar{z}_\alpha, d\bar{z}_\alpha) = \frac{1}{r_\alpha},$$

$$G(dz_\alpha \wedge d\bar{z}_\alpha, dz_\alpha \wedge d\bar{z}_\alpha) = \frac{1}{r_\alpha^2}.$$

Let $\Omega_\alpha = \frac{\sqrt{-1}}{2} r_\alpha dz_\alpha \wedge d\bar{z}_\alpha$ be the volume form. It is easy to check that, on $U_\alpha \cap U_\beta \neq \emptyset$, $\Omega_\alpha = \Omega_\beta$, so it is a blobally defined 2-form on M, which is called the volume form, denoted by Ω. Then $G(\Omega, \Omega) = 1$. Denote by $A^p = \mathcal{E}^p(M) = \{C^\infty$ p-forms on $M\}$ and $A^{p,q}(M) = \mathcal{E}^{p,q}(M) = \{C^\infty (p,q)\text{-forms on } M\}$. The metric $G = r_\alpha dz_\alpha d\bar{z}_\alpha$ induces a metric in $A^{p,q}$ as mentioned above.

The Star Operator: Define the operator $\star : A^{p,q} \to A^{1-q,1-p}$ (and hence $\star : A^k \to A^{2-k}$) by $\phi \wedge \overline{\star\psi} = G(\phi, \psi)\Omega$ for any $\phi \in A^{1-q,1-p}, \psi \in A^{p,q}$, or equivalently, in the local coordinate, $\star 1 = \Omega, \star\Omega = 1$ and on the Riemann surface, $\star dz_\alpha = -i dz_\alpha, \star d\bar{z}_\alpha = i d\bar{z}_\alpha$. It can be easily checked that

(1)

$$\star\star : A^{p,q} \to A^{p,q}, \star\star = (-1)^{p+q},$$

$$\star\star : A^p \to A^p, \star\star = (-1)^p,$$

(2)

$$G(\star\phi, \star\psi) = G(\phi, \psi),$$

(3) $\star$ is real , $\star\bar{\phi} = \overline{\star\phi}$.

The (global) Inner Product: For a given Hermitian line bundle L and for $\sigma \in A^{p,q}(L)$, write locally $\sigma = \omega^{(\alpha)} \otimes s^{(\alpha)}$. We define $\star\sigma = (\star\omega^{\alpha}) \otimes s^{\alpha}$. For $\sigma_1, \sigma_2 \in A^{p,q}(L)$, we define an inner product as follows: Write locally $\sigma_j = \omega_j s_j$ on W_α, $j = 1, 2$, we define

$$(\sigma_1, \sigma_2) = \int_M < s_1, s_2 > \omega_1 \wedge \star\bar{\omega}_2.$$

Then (,) induces an inner product on $A(L) := \oplus A^{p,q}(L)$.

The adjoint of $\bar{\partial}$:

Definition. *Let $T_1, T_2 : A(L) \to A(L)$ be two linear operators such that $(T_1\sigma, \eta) = (\sigma, T_2\eta)$, $\forall \sigma, \eta$ with compact support. We call T_1, T_2 are adjoint to each other. We write $T_2 = T_1^*$ or $T_1 = T_2^*$.*

For example, $\star$ and $\star^{-1}$ are adjoint to each other.

We need to find the adjoint of $\bar{\partial}$. First define

$$D'_L : A^{p,q}(L) \to A^{p+1,q}(L)$$

$$\sigma = \omega_\alpha e_\alpha \mapsto (\partial\omega_\alpha + (-1)^{p+q}\omega_\alpha \wedge \theta_\alpha)e_\alpha,$$

where θ_α is the connection form (with respect to the given metric on L), i.e. $De_\alpha = \theta_\alpha e_\alpha$. Note that $\theta_\alpha = \partial \log h_\alpha$. In the case when L is trivial, then $D' = \partial$.

Remark: Let $L = \{U_\alpha, \phi_{\alpha\beta}\}$ be a Hermitian line bundle over a compact *Kähler* manifold, and h be its Hermitian metric. As a well-known fact, if $\omega \in \Gamma(M, \varepsilon^{p,q}(L))$, then $\bar{\partial}\omega \in \Gamma(M, \varepsilon^{p,q+1}(L))$. Indeed, if $\omega \in \Gamma(M, \varepsilon^{p,q}(L))$ i.e., $\omega_\alpha \in \Gamma(M, \varepsilon^{p,q}(L))$, $\alpha \in I$, $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M consists of the trivialization neighborhoods of L, then

$$\omega_\alpha = \phi_{\alpha\beta}\omega_\beta; \quad \textit{on } U_\alpha \cap U_\beta.$$

Since $\phi_{\alpha\beta}$ is holomorphic,

$$\bar{\partial}\omega_\alpha = \phi_{\alpha\beta}\bar{\partial}\omega_\beta; \quad \textit{on } U_\alpha \cap U_\beta.$$

Thus $\bar{\partial}\omega \in \Gamma(M, \varepsilon^{p,q+1}(L))$. **However for the operator ∂, $\partial\omega$ is no longer a L-valued differential form**, since if $\omega_\alpha = \phi_{\alpha\beta}\omega_\beta$ on $U_\alpha \cap U_\beta$, then

$$\partial\omega_\alpha = \partial\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta; \quad \textit{on } U_\alpha \cap U_\beta,$$

and, in general $\partial\phi_{\alpha\beta} \neq 0$, so $\partial\omega$ is no longer a L-valued differential form. For this reason, we introduce $D'_L : \Gamma(M, \varepsilon^{p,q}(L)) \longrightarrow \Gamma(M, \varepsilon^{p+1,q}(L))$, which is a differential operator of degree $(1,0)$ on L-valued forms, by letting

$$D'_L\omega_\alpha = \partial\omega_\alpha + (\partial \log\ h_\alpha)\omega_\alpha = h_\alpha^{-1}\partial(h_\alpha\omega_\alpha).$$

Then

$$\begin{aligned} D'_L\omega_\alpha &= \partial\omega_\alpha + \partial log\ h_\alpha\omega_\alpha \\ &= \partial(\phi_{\alpha\beta}\omega_\beta) + \partial log\ (h_\beta|\phi_{\beta\alpha}|^2)\phi_{\alpha\beta}\omega_\beta \\ &= \partial\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log\ h_\beta + (\partial log\ \phi_{\beta\alpha}))\phi_{\alpha\beta}\omega_\beta \\ &= \partial\phi_{\alpha\beta}\phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log\ h_\beta\omega_\beta)\phi_{\alpha\beta} + \partial log\ \phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta \\ &= \partial log\phi_{\alpha\beta}\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log\ h_\beta\omega_\beta)\phi_{\alpha\beta} + \partial log\ \phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta \\ &= \phi_{\alpha\beta}(\partial\omega_\beta + \partial log\ h_\beta\omega_\beta) = \phi_{\alpha\beta}D'_L\omega_\beta \quad . \end{aligned}$$

Theorem

$$\bar{\partial}^* = - \star D'_L \star .$$

Proof. $\forall\sigma_1 = \omega_1 e_\alpha \in A^{p,q-1}(L), \forall\sigma_2 = \omega_2 e_\alpha \in A^{p,q}(L)$,

$$(\bar{\partial}\sigma_1, \sigma_2) = \int_M (e_\alpha, e_\alpha)\omega_1 \wedge \star\bar{\omega}_2 = \int_M h_\alpha\omega_1 \wedge \star\bar{\omega}_2.$$

Notice that, since $\omega_1 \wedge \star\bar{\omega}_2 h_\alpha$ is a $(1,0)$-form,

$$\begin{aligned} d(\omega_1 \wedge \star\bar{\omega}_2 h_\alpha) &= \bar{\partial}(\omega_1 \wedge \star\bar{\omega}_2 h_\alpha) \\ &= \bar{\partial}(\omega_1) \wedge \star\bar{\omega}_2 h_\alpha + (-1)^{p+q-1}\omega_1 \wedge \bar{\partial} \star \bar{\omega}_2 h_\alpha - \omega_1 \wedge \star\bar{\omega}_2 \wedge \bar{\partial}h_\alpha. \end{aligned}$$

By Stoke's theroem, since M is compact,

$$\int_M d(\omega_1 \wedge \star\bar{\omega}_2 h_\alpha) = 0$$

hence,

$$\int_M h_\alpha\bar{\partial}(\omega_1) \wedge \star\bar{\omega}_2 = -\int_M \left[(-1)^{p+q-1}\omega_1 \wedge \bar{\partial} \star \bar{\omega}_2 h_\alpha - \omega_1 \wedge \star\bar{\omega}_2 \wedge \bar{\partial}h_\alpha\right].$$

Thus

$$\begin{aligned}
(\bar\partial\sigma_1,\sigma_2) &= \int_M h_\alpha \bar\partial\omega_1\wedge\star\bar\omega_2 \\
&= -\int_M \left((-1)^{p+q-1}\omega_1\wedge\bar\partial\star\bar\omega_2 h_\alpha - \omega_1\wedge\star\bar\omega_2\bar\partial h_\alpha\right) \\
&= -\int_M (-1)^{p+q-1}h_\alpha\omega_1\wedge\left(\bar\partial\star\bar\omega_2 h_\alpha + (-1)^{p+q}\star\bar\omega_2\wedge\frac{\bar\partial h_\alpha}{h_\alpha}\right) \\
&= -\int_M (-1)^{p+q-1}h_\alpha\omega_1\wedge\left(\overline{\bar\partial\star\omega_2 h_\alpha + (-1)^{p+q}\star\omega_2\wedge\theta_\alpha}\right) \\
&= -\int_M h_\alpha\omega_1\wedge\star\star\left(\overline{\bar\partial\star\omega_2 h_\alpha + (-1)^{p+q}\star\omega_2\wedge\theta_\alpha}\right) \\
&= -\int_M h_\alpha\omega_1\wedge\overline{\star\star\left(\bar\partial\star\omega_2 h_\alpha + (-1)^{p+q}\star\omega_2\wedge\theta_\alpha\right)} \\
&= (\sigma_1, -\star D_L'\star\sigma_2)
\end{aligned}$$

here in above, we used the following fact: $\star\star = (-1)^{p+q-1}$. This shows that $\bar\partial^* = -\star D_L'\star$. which proves the theorem.

The Laplace operator $\Box$.

Now we have

$$\bar\partial : A^{p,q}(L)\to A^{p,q+1}(L),\quad \bar\partial^* : A^{p,q+1}(L)\to A^{p,q}(L)$$

with $\bar\partial^2=0, \bar\partial^{*2}=0$. Let

$$\Box := -\bar\partial^*\bar\partial + \bar\partial\bar\partial^* = (\bar\partial+\bar\partial^*)^2$$

which is called the Laplacian operator with respect to (L,h) and (M,G).

We remark that if $L=\mathcal{O}$ is the trivial line bundle with the trivial metric, then

$$\Box = -2\frac{\partial^2}{\partial z\partial\bar z}.$$

This is why we call $\Box$ Laplacian operator. We have

$$(\Box\sigma_1,\sigma_2) = (\sigma_1,\Box\sigma_2).$$

Lemma.

$$\Box\phi = 0 \Longleftrightarrow \bar\partial\phi = 0 \textit{ and } \bar\partial^*\phi = 0.$$

Proof. Notice

$$(\Box\phi,\phi) = (\bar\partial\phi,\bar\partial\phi) + (\bar\partial^*\phi,\bar\partial^*\phi).$$

The lemma can thus be easily verified.

The expression of the Laplace operator $\Box$.

Next, we compute the local expression of $\Box$, i.e. $\Box f$ for any $f \in A^{p,q}(L)$. On W_α, e_α is local frame for L, $h_\alpha =< e_\alpha, e_\alpha >$ and on $T^{(1.0)}M$, the metric $G = r_\alpha dz_\alpha d\bar{z}_\alpha$. Write the linear differential operator

$$\Box_0 = -\frac{2}{r_\alpha}\left(\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha} + \frac{\partial \log h_\alpha}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha}\right).$$

Let $f = f_\alpha \phi_\alpha e_\alpha$ with $f_\alpha \in C^\infty(W_\alpha)$, and $\phi_\alpha := 1$ if $(p,q) = (0,0)$; $:= dz_\alpha$ if $(p,q) = (1,0)$; $:= d\bar{z}_\alpha$ if $(p,q) = (0,1)$; $:= \Omega$ if $(p,q) = (1,1)$. Here $\phi_\alpha e_\alpha$ is a basis of $A(L)$ over W_α. Denote by K the Gauss curvature of the metric $\{h_\alpha\}$ on L, i.e. $\Theta = K\Omega$. By direct computation, we have the following formulas: For $f \in A^{0,0}(L)$,

$$\Box f = (\Box_0 f_\alpha)\phi_\alpha e_\alpha.$$

For $f \in A^{1,0}(L)$,

$$\Box f = \left((\Box_0 + \frac{2}{r_\alpha}\frac{\partial \log r_\alpha}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha})f_\alpha)\right)\phi_\alpha e_\alpha.$$

For $f \in A^{0,1}(L)$,

$$\Box f = \left((\Box_0 + \frac{2}{r_\alpha}\frac{\partial \log r_\alpha}{\partial z_\alpha}\frac{\partial}{\partial \bar{z}_\alpha} + [K + \frac{2}{r_\alpha}\frac{\partial \log r_\alpha}{\partial \bar{z}_\alpha}\frac{\partial \log h_\alpha}{\partial z_\alpha}])f_\alpha\right)\phi_\alpha e_\alpha.$$

For $f \in A^{1,1}(L)$,

$$\Box f = \{(\Box_0 + K)f_\alpha\}\phi_\alpha e_\alpha.$$

The above computations are starightfoward, but the above(last) formula is very important in the proof of the vanishing theorems, so we derive this formula here: Let $f = f_\alpha \Omega e_\alpha$. From the definition

$$\begin{aligned}
\Box f &= (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})f = \bar{\partial}\bar{\partial}^* f = -\bar{\partial} \star D'_L(f_\alpha e_\alpha) \\
&= -\bar{\partial} \star (\partial f_\alpha + f_\alpha \theta_\alpha)e_\alpha \\
&= \sqrt{-1}(\bar{\partial}\partial f_\alpha + \bar{\partial} f_\alpha \wedge \theta_\alpha + f_\alpha \bar{\partial}\theta_\alpha)e_\alpha \\
&= \sqrt{-1}\left(\frac{\partial^2 f_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_\alpha + \frac{\partial \log h_\alpha}{\partial z_\alpha}\frac{\partial f_\alpha}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_\alpha + f_\alpha \Theta\right)e_\alpha \\
&= (\Box_0 + K)f_\alpha \Omega e_\alpha.
\end{aligned}$$

In summary, for any $f = f_\alpha \phi_\alpha e_\alpha$,

$$\Box f = \tilde{f}_\alpha \phi_\alpha e_\alpha$$

where

$$\tilde{f}_\alpha = -\frac{2}{r_\alpha}\left(\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha} + k_1 \frac{\partial}{\partial z_\alpha} + k_2 \frac{\partial}{\partial \bar{z}_\alpha} + k_3\right) f_\alpha$$

In above, the principal part is

$$-\frac{2}{r_\alpha}\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha} = - - \frac{2}{r_\alpha}\left(\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2}\right).$$

Since $-\frac{2}{r_\alpha} < 0$, $\Box$ is an elliptic operator. This is why the Hodge theory works.

8.2 The Hodge Theorem

Harmonic forms: Write

$$\mathcal{H}^{p,q}(L) = \{f \in A^{p,q}(L) \mid \Box f = 0\}.$$

$\mathcal{H}^{p,q}(L)$ is called the space of harmonic (p,q)-forms. Denote

$$\mathcal{H}(L) = \oplus \mathcal{H}^{p,q}(L).$$

Theorem (Hodge theorem). *Let (L, H) be a Hermitian line bundle over a Hermitian compact Riemann surface (M, G). Then*

(1) $\mathcal{H}(L)$ is a finite dimensional space.

(2) There is an operator G, called the Green operator of $\Box$, $G : A(L) \to A(L)$ such that $ker(G) = \mathcal{H}(L)$, $G(A^{p,q}) \subset A^{p,q}$ (i.e. G keeps the type, G commutes with $\bar{\partial}, \bar{\partial}^*$, $\Box G(\omega) = G\Box(\omega)$ for $\forall \omega \in \mathcal{H}^\perp$.

(3) $A(L) = \mathcal{H}(L) \oplus \Box G A(L) = \mathcal{H}(L) \oplus G\Box A(L)$.

Remark: The above decomposition means that for any $\sigma \in A(L)$, $(\sigma - G\Box\sigma) \in \mathcal{H}(L)$. If we define $H\sigma := \sigma - G\Box\sigma$, then it is the orthogonal projection $A(L) \to \mathcal{H}(L)$. Hence we can write

$$\sigma = H\sigma + G\Box\sigma.$$

Such expression is unique. Since $\bar{\partial}G = G\bar{\partial}$ and $\bar{\partial}^* G = G\bar{\partial}^*$, we have

$$\sigma = H\sigma + G\Box\sigma = H\sigma + \bar{\partial}(\bar{\partial}^* G\sigma) + \bar{\partial}^*(\bar{\partial}G\sigma).$$

Hence we have the following decomposition

$$A^{p,q}(L) = \mathcal{H}^{p,q}(L) \oplus \bar{\partial}A^{p,q-1}(L) \oplus \bar{\partial}^* A^{p,q+1}(L).$$

Corollary

$$H^q(M, \Omega^p(L)) = \mathcal{H}^{p,q}(L).$$

Proof. By Dolbeauly theorem,

$$H^q(M, \Omega^p(L)) \cong \frac{\bar{\partial} \text{ closed } L \text{ valued smooth } (p,q) - \text{forms}}{\bar{\partial} \text{ exact } L \text{ valued smooth } (p,q) - \text{forms}}.$$

When $q = 0$, by above, $H^0(M, \Omega^p(L)) \cong \{f \in A^{p,q}(L) \mid \bar{\partial} f = 0\}$. In this case, $f \in A^{p,-1}(L) = \{0\}$ so that $\bar{\partial}^* f = 0$. Hence $f \in \mathcal{H}^{p,0}(L)$. Thus $H^0(M, \Omega^p(L)) \cong \mathcal{H}^{p,0}(L)$. When $q = 1$, By Dolbeauly theorem,

$$H^1(M, \Omega^p(L)) \cong A^{p,1}(L)/\bar{\partial} A^{p,0}(L).$$

Notice any $f \in A^{p,1}(L)$ must be $\bar{\partial}$-closed by consideration of degree. By Hodge theorem,

$$\begin{aligned} A^{p,1}(L) &= \mathcal{H}^{p,1}(L) \oplus G(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*)A^{p,1}(L) \\ &= \mathcal{H}^{p,1}(L) \oplus G\bar{\partial}\bar{\partial}^* A^{p,1}(L) \\ &= \mathcal{H}^{p,1}(L) \oplus \bar{\partial}\bar{\partial}^* G A^{p,1}(L) \subset \mathcal{H}^{p,1}(L) \oplus \bar{\partial} A^{p,0}(L) \end{aligned}$$

because $\bar{\partial}^* G A^{p,1}(L) \subset \bar{\partial}^* A^{p,1}(L) \subset A^{p,0}(L)$. Since $\mathcal{H}^{p,1}(L) \oplus \bar{A}^{p,0}(L) \subset A^{p,1}(L)$, we have

$$A^{p,1}(L) = \mathcal{H}^{p,1} \oplus \bar{\partial} A^{p,0}(L).$$

Therefore

$$H^1(M, \Omega^p(L)) \cong \mathcal{H}^{p,1} \oplus \bar{\partial} A^{p,0}(L),$$

which finishes the proof.

Recall for any divisor D on M,

$$h^0(D) = \dim H^0(M, \mathcal{O}([D])),$$

$$i(D) = \dim H^0(M, \Omega^1(-[D]).$$

We have, from the Hodge theorem, that

$$h^0(D), i(D) < \infty.$$

8.3 The Proof of the Hodge Theorem

To prove the theorem, basically we need to show two things: (1): $\mathcal{H}(L)$ **is a finite dimensional vector space**, (2): Write $(L) = \mathcal{H}(L) \oplus \mathcal{H}^\perp(L)$, where $\mathcal{H}^\perp(L)$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ ,\ )$, we need to show that $\Box : \mathcal{H}^\perp \to \mathcal{H}^\perp$ **and** $\Box$ **is one-to-one and onto.** (note that: for every $\phi \in A(L), \psi \in \mathcal{H}$, $(\Box\phi, \psi) = (\phi, \Box\psi) = 0$, so $\Box\phi \in \mathcal{H}^\perp$. Hence $\Box : \mathcal{H}^\perp \to \mathcal{H}^\perp$). Once (1) and (2) are proved, then we take $G|_{\mathcal{H}} = 0$, and $G|_{\mathcal{H}^\perp} = \Box^{-1}$. This will prove the Hodge theorem. To do so, we first note that the operator $\Box$, as shown above, is positive (i.e. its eigenvalues are all positive). So $\Box$ is an elliptic self-adjoint operator. We therefore use the "theory of elliptic (self-adjoint) differential operator" (the Hodge theorem holds for general elliptic (self-adjoint) differential operators, not only to $\Box$, this is part of the PDE theory). To do so, we need first introduce the concept of "Sobolov space".

Let $\Omega \subset \mathbf{R}^n$ be an open subset. Let $L^2(G)$ be the space of complex valued functions with

$$\int_G \|f\|^2 dx < \infty.$$

It is a Hilbert space. For $f \in L^2(G)$, if there is $g \in L^2(G)$ such that for any $h \in C_0^\infty(G)$ (test function) such that

$$(f, D^\alpha h) = (-1)^{|\alpha|}(g, h)$$

where $(f,g) = \int_G f\bar{g}dx$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha h = \frac{\partial^\alpha h}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}$, $|\alpha| = \sum_{i=1}^n \alpha_i$, then we say g is the α-th order weak (or general) derivative, and is still denoted by $D^\alpha f$. Let s be a nonnegative integer. Because $C_0^\infty(G)$ is dense in $L^2(G)$, we can define a norm on $C_0^\infty(G)$, $\| \ \|_s$ by

$$\|f\|_s^2 := \sum_{|\alpha| \le s} \|D^\alpha f\|^2.$$

The complete extension of $C_0^\infty(G)$ with respect to the norm $\| \ \|_s$ in $L^2(G)$ is denoted by $H_s(\Omega)$ is called the Sobolev space. The definition extends trivially on on $A(L)$.

We use the following three facts(proofs are omitted):

- **Garding's inequality**: *There exist constant* $c_1, c_2 > 0$*, such that for every* $f \in A(L)$*, we have*

  $$(\Box f, f) \ge c_1\|f\|_1^2 - c_2\|f\|_0^2.$$

  Remark: This is a variant of so-called *Bocher technique.*

  To state the second fact, we introduce the concept of *weak derivative*: Write $P = \partial + \bar{\partial}^*$ and $\Box = P^2$. For $\phi \in H_s(M)$ and $\psi \in H_t(M)$, we say $P\phi = \psi$(weak), if for every test form $f \in A(L)$ (i.e. smooth with compact support), we have $(\phi, Pf) = (\psi, f)$. If $\phi \in H_s(M)$, $\psi \in H_t(M)$, and $P\phi = \psi$(weak), we denote it by $P\phi \in H_t(M)$.

- **Regularity of the operator** $\bar{\partial} + \bar{\partial}^*$: *If* $f \in H_0(L)$*,* $g \in A(L)$*, and* $(\bar{\partial} + \bar{\partial}^*)f = g$*, then* $f \in A(L)$*.*

- **Rellich Lemma**: *If* $\{\phi_i\} \subset A(L)$ *is bounded in the* $\| \ \|_1$*, then it has a Cauchy subsequence with respect to the norm* $\| \ \|_0$*.*

  The above theorem about the regularity of the operator $\bar{\partial} + \bar{\partial}^*$ implies the following lemma

- **The weak form of the Wyle lemma**: *If* $\phi \in H_1(M)$*, and* $g \in A(L)$ *with* $\Box f = g$*(weak) with* $f \in A(L)$*.*

Proof of the Hodge Theorem: We first prove that $\mathcal{H}(L)$ is a finite dimensional vector space. If not, there exists an infinite orthonormal set $\{\omega_1, \ldots, \omega_n, \cdots\}$. By Garding's inequality, there exist constants c_1, c_2 such that for all i, we have

$$\|\omega_i\|_1^2 \leq \frac{1}{c_1}\{(\Box\omega_i, \omega_i) + c_2\|\omega_i\|_0^2\} = \frac{c_2}{c_1}.$$

Thus $\{\omega_i\}$ is bounded set with respect to $|\ |_1$. By Rellich Lemma, $\{\omega_i\}$ must have a Cauchy subsequence with respect to the norm $\|\ \|_0$, which is impossible, since $\|\omega_i - \omega_j\|_0^2 = 2$ for $i \neq j$. This proves that $\mathcal{H}$ **is a finite dimensional vector space**.

Next, write

$$A(L) = \mathcal{H} \oplus \mathcal{H}^\perp,$$

where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ ,\ )$. We now prove a simpler version of Garding's inequality.

Garding's Lemma *Let $\mathcal{H}^\perp(L)$ is the orthogonal complement of $\mathcal{H}(L)$ in $A(L)$ with respect to the inner product. Then there exists a constant C_0 such that*

$$|f|_1^2 \leq C_0(\Box f, f), \quad \forall f \in \mathcal{H}^\perp(L).$$

Proof. If not, there exists a sequence $f_i \in \mathcal{H}^\perp$ with $\|f_i\|_1 = 1$ and $(\Box f_i, f_i) \to 0$. From Rellich lemma, we assume, WLOG, that f_i is convergent with respect to $\|\ \|_0$, i.e. there exists $F \in H_0(M)$ such that $\lim_{i\to+\infty} \|F - f_i\|_0 = 0$. We claim that $F = 0$. In fact, from above, $(\Box f_i, f_i) = \|Pf_i\|_0^2 \to 0$, hence for every $\phi \in A(L)$,

$$(F, P\phi) = \lim_{i\to+\infty}(f_i, P\phi) = \lim_{i\to+\infty}(Pf_i - \phi) = 0.$$

Hence $PF = 0$ (weak). From the regularity of P, we have $F \in A(L)$. Hence

$$\Box F = P(PF) = 0,$$

so $F \in \mathcal{H}$. Also, since $f_i \in \mathcal{H}^\perp$, we have, for every $\phi \in \mathcal{H}$,

$$(F, \phi) = \lim_{i\to+\infty}(f_i, \phi) = 0,$$

so $F \in \mathcal{H}^\perp$. Thus $F \in \mathcal{H} \cap \mathcal{H}^\perp$. This implies that $F = 0$. This means that $\lim_{i\to+\infty} \|f_i\|_0 = 0$. Now, by the Garding inequality, There exist constant $c_1, c_2 > 0$, such that

$$(\Box f_i, f_i) \geq c_1\|f_i\|_1^2 - c_2\|f_i\|_0^2.$$

Because, from above, both $(\Box f_i, f_i)$ and $\|f_i\|_0^2$ converge to zero, so $\lim_{i\to+\infty} \|f_i\|_1 = 0$, which contradicts the assumption that $\|f_i\|_1 = 1$. This proves Garding's lemma.

We now prove that $\Box : \mathcal{H}^\perp \to \mathcal{H}^\perp$ *and* $\Box$ *is one-to-one and onto.*

First we show that $\Box : \mathcal{H}^\perp \subset \mathcal{H}^\perp$. In fact, for every $\phi \in A(L), \psi \in \mathcal{H}$,

$$(\Box\phi, \psi) = (\phi, \Box\psi) = 0,$$

so $\Box\phi \in \mathcal{H}^\perp$. To show $\Box$ is one-to-one, let $\phi_1, \phi_2 \in \mathcal{H}^\perp$, and assume that $\Box\phi_1 = \Box\phi_2$. Then, from one hand, $\phi_1 - \phi_2 \in \mathcal{H}^\perp$. On the other hand, since $\Box(\phi_1 - \phi_2) = 0$, $\phi_1 - \phi_2 \in \mathcal{H}$. Hence $\phi_1 = \phi_2$. It remains to show that $\Box$ is onto. i.e. for every $f \in \mathcal{H}^\perp$, there exists $\phi \in \mathcal{H}^\perp$ such that $\Box\phi = f$. This gets down to solve the differential equation $\Box\phi = f$ (with unknown ϕ). Let B be the closure of $\mathcal{H}^\perp$ in $H_1(M)$. From Wyle's theorem, we only need to solve $\Box\phi = f$ in the weak sense, i.e. there exists $\phi \in B$ such that, for every $g \in A(L)$ with compact support,

$$(\phi, \Box g) = (f, g).$$

Since $A(L) = \mathcal{H} \oplus \mathcal{H}^\perp$, we can write $g = g_1 + g_2$ where $g_1 \in \mathcal{H}, g_2 \in \mathcal{H}^\perp$. So the above identity is equivalent to every $g_2 \in \mathcal{H}^\perp$,

$$(\phi, \Box g_2) = (f, g_2).$$

So the proof is reduced to the following statement: *for every $f \in \mathcal{H}^\perp$, there exists $\phi \in B$ such that, for every $g \in \mathcal{H}^\perp$,*

$$(\phi, \Box g) = (f, g).$$

We now use the **Riesz representation** theorem to prove this statement. In fact, for every $\phi, \psi \in \mathcal{H}^\perp$, define $[\phi, \psi] = (\phi, \Box\psi)$, and consider the linear transformation $L : B \to \mathbf{R}$ defined by $l(g) = (f, g)$ for every $g \in B$. Our goal is to show that we can extend $[\ ,\ ]$ to B such that l is continuous with respect to $[\ ,\ ]$ (or bounded). Then by **Riesz representation** theorem, there exists $\phi \in B$ such that, for every $g \in B$ (in particular for $g \in \mathcal{H}^\perp$),

$$l(g) = [\phi, g].$$

This will prove our statement. To extend $[\ ,\ ]$, we compare $[\ ,\ ]$ with $(\ ,\ )_1$. From definition, $[\ ,\ ]$ is bilinar. From Garding's inequality, for every $\phi \in \mathcal{H}^\perp$,

$$[\phi, \phi] = (\phi, \Box\phi) \geq \frac{1}{c_0}\|\phi\|_1^2.$$

On the other hand,

$$[\phi, \phi] = (\phi, \Box\phi) = \|P\phi\|_0.$$

By direct verification, we have, for every $\phi \in A(L)$,

$$\|P\phi\|_0^2 \leq c\|\phi\|_1^2.$$

Hence

$$[\phi, \phi] \leq c\|\phi\|_1^2.$$

So $[\ ,\ ]$ and $(\ ,\ )_1$ are equivalent on $\mathcal{H}^\perp$. So there exists an unique continuation on B, and for every $g \in B$, we have

$$[g, g] \geq \frac{1}{c_0} \|g\|_1^2.$$

To show that l is continuous with respect to $[\ ,\ ]$(or bounded), we notice that

$$|l(g)| = |(f, g)| \leq \|f\|_0 \|g\|_0 \leq \|f\|_0 \|g\|_1 \leq \sqrt{c_0} \|f\|_0 \sqrt{[g, g]}.$$

So the claim is proved. This finishes the proof that $\Box$ is onto.

To prove Hodge's theorem, since, from above, $\Box : \mathcal{H}^\perp \to \mathcal{H}^\perp$ is one-to-one and onto, we let $G : A(L) \to A(L)$ be defined as follows: $G|_{\mathcal{H}} = 0$, and $G|_{\mathcal{H}^\perp} = \Box^{-1}$. Then we see that $\ker G = \mathcal{H}$ and $I = \mathcal{H} + \Box \circ G$. The rest of properties are also easy to verify. This finishes the proof.

Chapter 9

Some Further Results

9.1 The computation of $H^2(M, \mathbf{C})$

Recall that, for an Hermitian line bundle (L, h), we have $c_1(L, h) = \frac{\sqrt{-1}}{2p}\Theta$ and for any metric h, h' on L, $\Theta - \Theta'$ is d-exact, hence $c(L) := [c_1(L, h)] \in H^2_{DR}(M) \cong H^2(M, \mathbf{C})$. We now define a "evaluation" homomorphism

$$[M] : H^2(M, \mathbf{C}) \to \mathbf{C}$$

$$\eta \mapsto \eta[M] = \int_M \eta.$$

It is well-defined since if $[\eta] = [\eta']$, then $\eta - \eta' = d\omega$, hence

$$\int (\eta - \eta') = 0$$

by stokes theorem.

Theorem *Let M be a compact Riemann surface, then $H^2(M, \mathbf{C}) \cong \mathbf{C}$.*

To prove the above theroem, it is easy to see the map $[M]$ is onto, since if Ω is the volume form of an Hermitian metric on M, then $\Omega > 0$ so $\int_M \Omega = v > 0$. So for any $t \in \mathbf{C}$,

$$\left(\frac{t}{v}\Omega\right)[M] = t.$$

To prove it is one-to-one, we need the following $\bar{\partial}\partial$-lemma.

To state and prove the $\bar{\partial}\partial$-lemma, we first recall the Hodge theory: Fix a Hermitian metric G on M and $H = \{h_\alpha\}$ on L, and let Ω be the volume form of G. we claim that, for any L-valued $(1,1)$-form ω,

$$\Box\omega = \Omega(\Box \star \omega) + \sqrt{-1}\Theta(\star\omega), \qquad (1)$$

where Θ is the curvature form of $\{h_\alpha\}$. Here is the proof: Write $\omega = \omega_\alpha \Omega e_\alpha$, then from what we have proved earlier, $\Box\omega = (\Box_0\omega_\alpha)\Omega e_\alpha + K\omega_\alpha\Omega e_\alpha$. But $\star\omega = \omega_\alpha e_\alpha, K\Omega = \sqrt{-1}\Theta$, and in the third formula proved earlier, $\Box\star\omega = (\Box_0\omega_\alpha)e_\alpha$. Hence the claim holds.

We now consider the special case of the Hodge theorem when $L = \mathcal{O}$, the trivial line bundle. In this case, we have

$$A = \mathcal{H} \oplus \Box GA$$

where A is the set of all smooth forms. Since L is trivial, $\Theta = 0$, and $D' = \partial$, so $\bar{\partial}^* = -\star\partial\star$, so if ω is a smooth (1,1)-form and write $\omega = f\Omega$ where f is a smooth function on M, then (1) implies that

$$\Box\omega = (\Box f)\Omega.$$

Hence $\Box\omega = 0 \Leftrightarrow \Box f = 0 \Leftrightarrow f$ is constant since M is compact. Hence

$$\mathcal{H} = \{S\Omega : S \in \mathbf{C}\},$$

where $\mathcal{H}$ is the set of all harmonic 1-forms.

Lemma($\bar{\partial}\partial$-lemma). *Let M be a compact Riemann surface, ϕ is a real (1,1)-form and $\int_M \phi = 0$. Then there is a real valued function h on M such that $\phi = \bar{\partial}\partial(ih)$.*

Proof. We first prove that $\phi \perp \mathcal{H}$. To do so, we only need to check, by above discussion, $(\phi, \Omega) = 0$. From the definition,

$$(\phi, \Omega) = \int_M \phi \wedge \star\bar{\Omega} = \int_M \phi \wedge \star\Omega = \int_M \phi = 0.$$

So $\phi \in \Box GA$, i.e. there is a smooth (1,1)-form ϕ_0 such that $\phi = \Box G\phi_0$. Since G preserves the type, $\phi_1 := G\phi_0$ is still a (1,1)-form, and $\phi = \Box\phi_1$. Because $\bar{\partial}\phi = 0$ (there is no (1,2)-forms on M),

$$\phi = \Box\phi_1 = \bar{\partial}\bar{\partial}^*\phi_1 = -\bar{\partial}\star\partial\star\phi_1.$$

Let $k := \star\phi_1$, then k is a function, and since ∂k is (1,0)-form, $\star k = -i\partial k$ by definition. Thus $\phi = \bar{\partial}\partial(ik)$. Now we use the fact that ϕ is real, so write $k = h + ih'$, then, since $\bar{\partial}\partial = -\partial\bar{\partial}$, we have

$$\phi = \bar{\partial}\partial(ih) - \bar{\partial}\partial h'$$

$$\bar{\phi} = \bar{\partial}\partial(ih) + \bar{\partial}\partial h'.$$

By adding the above two together and using $\bar{\phi} = \phi$, we get $\phi = \bar{\partial}\partial(ih)$. This proves the theorem.

Corollary. *If ϕ is a (1,1)-form and $\int_M \phi = 0$. Then ϕ is exact.*

We now ready to finish the proof that

$$[M] : H^2(M, \mathbf{C}) \to \mathbf{C}$$

$$\eta \mapsto \eta[M] = \int_M \eta$$

is an isomorphism. It remains to prove that $[M]$ is 1-1. Let ϕ with $[M](\phi) = 0$, then from above Corollary, ϕ is exact, so $[\phi] = 0$, This proves that $[M]$ is 1-1. So the proof is finished.

9.2 Existence of Positivity of Hermitian line bundles

A (1,1)-form ω is real $\Longleftrightarrow$ locally, $\omega = f\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$ with f being a real valued function. ω is said be be positive(denoted by $\omega > 0$ if $f > 0$.

Recall that for an Hermitian line bunlde L with metric $\{h_\alpha\}$, its curvature from is $\Theta = \bar{\partial}\partial \log h_\alpha$, hence the first Chern form is

$$c_1(L, h) := \frac{\sqrt{-1}}{2p}\Theta = -\frac{1}{\pi}\frac{\partial^2 \log h_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha}\left(\frac{\sqrt{-1}}{2}dz_\alpha \wedge d\bar{z}_\alpha\right)$$

which is a real $(1,1)$-form. L is said to be positive, denoted by $L > 0$ if there is an hermitian metric h on M such that $c_1(L, h) > 0$. The following discuss various equivalent notions of positivity.

Lemma. *Let M be a compact RS, and L be a line bundle. Let $\psi \in C(L)$ be a real (1,1) form, then there is a (smooth) Hermitian metric h on L such that its curvature form Ψ satisfies $\frac{\sqrt{-1}}{2\pi}\Psi = \psi$.*

Proof. Let $h = \{h_\alpha\}$ be an Hermitian metric on L and Θ is its curvature form. Then ψ and $\frac{\sqrt{-1}}{2\pi}\Theta$ belongs to the same class in $C(L)$. Hence $\psi - \frac{\sqrt{-1}}{2\pi}\Theta$ is an exact real (1.1)-form. By the $\bar{\partial}\partial$-lemma, there is real-valued function $\tilde{f}$ on M such that

$$\psi = \frac{\sqrt{-1}}{2p}\Theta + \bar{\partial}\partial(i\tilde{f}).$$

Let $f := exp(2\pi\tilde{f})$, then

$$\psi = \frac{\sqrt{-1}}{2\pi}\Theta + \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log f = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log(f h_\alpha).$$

Since $f > 0$, fh_α is also a metric on L. This proves the lemma.

Using the above lemma, let Ω be the volume form of the an Hermitian metric G on M, then, for a compact Riemm surface M, we have the following alternative definition about the positivity of L:

Theorem; *Let L be a line bundle on M. Then the following are equivalent:*

(a) $L > 0$,

(b) $C(L)$ (the Chern class) has a positive (1,1)-form.

(c)There is $S > 0$ such that $S\Omega \in C(L)$ where Ω is the volume form of an Hermitian metric G on M.

(d) $C(L)(M) > 0$.

Proof. We shall prove that $(a) \Leftrightarrow (d) \Leftrightarrow (d) \Leftrightarrow (c) \Leftrightarrow (b) \Leftrightarrow (a)$. $(a) \Leftrightarrow (d)$ is true directly from the definition. To show $(d) \Leftrightarrow (c)$, Let $C(L)[M] = t > 0$, and let $v = \int_M$, $S = t/v$. Then $[S\Omega][M] = t$, from the fact that $[M]$ is an isomorphism, $[S\Omega] = C(L)$. $(c) \Leftrightarrow (b)$ is trivial. $(b) \Leftrightarrow (a)$ can be derive from above lemma. This finished the proof.

9.3 The vanishing theorem

Theorem(Vanishing theorem). *Let L be a holomorphic line bundle. Then*

(a) If $L > 0$, then $H^1(M, \Omega^1(L)) = 0$,

(b) If $L - K > 0$, then $H^1(M, \mathcal{O}(L)) = 0$..

Proof. Assume G is an Hermitian metric on M, and Ω is its volume form. Since $L > 0$, from Lemma above, there is $S \in \mathbf{R}$, $S > 0$ such that $S\Omega \in C(L)$. So, from lemma, there is a metric $\{g_\alpha\}$ on L such that its curvature form Θ satisfies

$$\frac{\sqrt{-1}}{2\pi}\Theta = S\Omega.$$

From the Hodge theorem, we only need to show that any L-values harmonic $(1,1)$-form ω vanishes. In deed, from

$$0 = \Box\omega = \Omega(\Box \star \omega) + 2\pi S\Omega(\star\omega) = \Omega\{(\Box \star \omega) + 2\pi S(\star\omega)\}.$$

Hence $\Box \star \omega + 2\pi S \star \omega = 0$. Thus

$$0 = ((\Box \star \omega + 2\pi S \star \omega, \star\omega) = (\Box \star \omega) + 2\pi S(\star\omega, \star\omega)$$

$$= (\bar{\partial} \star \omega, \bar{\partial} \star \omega) + (\bar{\partial}^* \star \omega, \bar{\partial}^* \star \omega) + 2\pi S(\star\omega, \star\omega).$$

Thus $2\pi S(\star\omega, \star\omega) = 0$. Since $S > 0$, this implies that $\star\omega = 0$. So $\omega = 0$. This proves (a).

(b) Notice that $\mathcal{O}(L) = \mathcal{O}(K - K + L) = \Omega^1(L - K)$, hence $H^1(M, \mathcal{O}(L)) = H^1(M, \Omega^1(L - K)) = 0$.. This finishes the proof.

We using this vanishing theorem, as we discussed before, we can prove the imbedding theorem (by using the exact-seqeunce): *If $L > 0$, then there is an integer $m > 0$ such that ϕ_{mL} gives M an embedding.* The higher dimensional result is due to Kodaira and his method is similar to what we have discussed here.

Part III

The Theory of Complex Geometry

Chapter 10

Differential Geometry of Complex Manifolds

10.1 Hermitian Metrics; Kahler Structure

Definition 10.1. **A Hermitian metric**, *denoted by* ds^2, *is a set of inner-product* $\{\langle\cdot,\cdot\rangle_p\}_{p\in M}$ *such that*

(1). For $\forall p \in M$, $\langle\cdot,\cdot\rangle_p$ *is a Hermitian inner product on* $T_p^{(1,0)}(M)$, *i.e.* $\forall \eta,\zeta \in T_p^{1,0}(M), \forall c_1, c_2 \in C$, $\langle\xi,\xi\rangle > 0$, *as* $\xi \neq 0$; $\langle c_1\xi + c_2\xi,\zeta\rangle = c_1\langle\xi,\zeta\rangle + c_2\langle\eta,\zeta\rangle$ *and* $\langle\xi,\eta\rangle = \overline{\langle\eta,\xi\rangle}$.

(2). If ξ,η *are* C^∞ *section of* $T^{1,0}(M)$ *over an open set* U, *then* $\langle\xi,\zeta\rangle$ *is the* C^∞ *function on* U.

If $z^1,\cdots,z^n$ is a local coordinate system of M, then $\frac{\partial}{\partial z^i}$, $1 \leq i \leq n$, are holomorphic sections on this local coordinates neighborhood U, and

$$g_{i,\bar{j}} = \langle\frac{\partial}{\partial z^i},\frac{\partial}{\partial z^j}\rangle; \quad 1 \leq i,j \leq n$$

is the C^∞ function on U with $g_{i,\bar{j}} = \overline{g_{j,\bar{i}}}$. We can write this Hermitian metric as $ds^2 = \sum\limits_{i,j=1}^{n} g_{i,\bar{j}} dz^i \otimes d\bar{z}^j$. Since $\langle\xi,\xi\rangle > 0$ for $\xi \neq 0$, the matrix $g = (g_{i,\bar{j}})_{1\leq i,j\leq n} > 0$, i.e., $g = (g_{i,\bar{j}})_{1\leq i,j\leq n}$ is a positive definite Hermitian matrix.

A complex manifold with a given Hermitian metric is said to be a *Hermitian manifold*.

We can prove that given any complex manifold M, we can introduce an Hertimitian metric on M.

Definition 10.2. *The linear operator* $D : \Gamma(M, T^{(1,0)}) \longrightarrow \Gamma(M, \mathcal{A}^1(M) \otimes T^{(1,0)})$ *is a connection if* D *satisfies*

$$D(fs) = df \otimes S + fDs,$$

for $\forall s \in \Gamma(M, T^{(1,0)})$ *and* f *is a smooth function on* M*, where* $\mathcal{A}^1(M)$ *is the set of smooth* 1*-forms on* M*.*

In terms of local coordinate $(z^1, \ldots, z^n)$, we write

$$D\frac{\partial}{\partial z^i} = \sum_{j=1}^{n} \omega_i^j \frac{\partial}{\partial z^j},$$

where $\omega = (\omega_i^j)$ is a $n \times n$ matrix whose entries are all 1-forms. ω is called the *connection matrix*. For $\xi \in \Gamma(M, T^{(1,0)})$, in terms of local coordinate $(z^1, \ldots, z^n)$, write $\xi = \sum\limits_{i=1}^{n} \xi^i \frac{\partial}{\partial z^i}$. Then

$$D\xi = \sum_{i=1}^{n} d\xi^i \frac{\partial}{\partial z^i} + \sum_{i,j=1}^{n} \xi^i \omega_i^j \frac{\partial}{\partial z^j}.$$

We can make the requirements that dictate a canonical choice of the connection: (1). If we split $\mathcal{A}^1(M) = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ and write $D = D' + D''$, where $D' : \Gamma(M, T^{(1,0)}) \longrightarrow A^{1,0} \otimes \Gamma(M, T^{(1,0)})$. We say a connection D is **compatible with the complex structure** if $D'' = \bar{\partial}$.

(2). D is said to be compatible with the Hermitian metric if

$$d < \xi, \eta > = < D\xi, \eta > + < \xi, D\eta >$$

where $\xi, \eta \in \Gamma(M, T^{(1,0)})$. Write $D' = dz^i \otimes \bigtriangledown_i$, then $\bigtriangledown_i$ is called the *covariant derivative*.

This connection D is called a *Hermitian connection* on M (with respect to the metric g). We can show that such connection exists and is unique. Furthermore, we claim that curvature matrix ω under the natural frame $(\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n})$ is

$$\omega = \partial g \cdot g^{-1}.$$

Proof. Since

$$\begin{aligned} dg_{i\bar{j}} &= \left\langle D\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right\rangle + \left\langle \frac{\partial}{\partial z^i}, D\frac{\partial}{\partial z^j} \right\rangle \\ &= \left\langle \omega_k^i \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^j} \right\rangle + \left\langle \frac{\partial}{\partial z^i}, \omega_k^j \frac{\partial}{\partial z^k} \right\rangle \\ &= \omega_k^i g_{k\bar{j}} + \bar{\omega}_k^j g_{i\bar{k}}. \end{aligned}$$

D is the Hermitian connection implies that ω is the matrix of forms of $(1,0)$-type, so the above yields $\partial g = \omega g$, or $\omega = \partial g \cdot g^{-1}$. This finishes the proof.

In terms of local coordinate $(z^1, \ldots, z^n)$, we write

$$\omega_j^i = \Gamma_{jk}^i dz^k$$

where the functions Γ_{ji}^k are called the *Christoffel symbols*. From above, $\Gamma_{ik}^j = \frac{\partial g_{i\bar{l}}}{\partial z^k} g^{\bar{l}j}$. For $\xi \in \Gamma(M, T^{(1,0)})$, in terms of local coordinate $(z^1, \ldots, z^n)$, write $\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial z^i}$. Then

$$\begin{aligned} D'\xi &= \sum_{i=1}^n \partial \xi^i \frac{\partial}{\partial z^i} + \sum_{i,j=1}^n \xi^i \omega_i^j \frac{\partial}{\partial z^j} \\ &= \sum_{i=1}^n \frac{\partial \xi^j}{\partial z^k} dz^k \otimes \frac{\partial}{\partial z^j} + \sum_{i,j=1}^n \xi^i \Gamma_{ik}^j dz^k \otimes \frac{\partial}{\partial z^j} \\ &= \big(\sum_{i,k=1}^n \big(\frac{\partial \xi^j}{\partial z^k} + \xi^i \Gamma_{ik}^j \big) dz^k \big) \otimes \frac{\partial}{\partial z^j}. \end{aligned}$$

or

$$\bigtriangledown_k \xi = \big(\sum_{i=1}^n \big(\frac{\partial \xi^j}{\partial z^k} + \xi^i \Gamma_{ik}^j \big) \big) \big) \otimes \frac{\partial}{\partial z^j}.$$

If we write

$$\bigtriangledown_k \xi^j = \frac{\partial \xi^j}{\partial z^k} + \sum_{i=1}^n \Gamma_{ik}^j \xi^i$$

then

$$\bigtriangledown_k \xi = \sum_{j=1}^n \big(\bigtriangledown_k \xi^j \big) \frac{\partial}{\partial z^j}.$$

Note that, for covariant tensor field $\{\xi^j\}$, the resulting $\{\bigtriangledown_k \xi^j\}$ (when i is fiexed) is still a covariant tensor field.

The connection also extends naturally to all kind of tensors (using the musical isomorphisms). In particular, if, for $\omega = \sum_{j=1}^n f_j dz^j$ (contra-variant tensor field), then

$$\bigtriangledown_i \omega = \left(\frac{\partial f_j}{\partial z^i} - \sum_{k=1}^n \Gamma_{ij}^k f_k \right) dz^j,$$

or simply

$$\bigtriangledown_i f_j = \frac{\partial f_j}{\partial z^i} - \sum_{k=1}^n \Gamma_{ij}^k f i_k.$$

We extend the connection operator $D : \Gamma(M, \mathcal{A}^k(M) \otimes T^{(1,0)}) \longrightarrow \Gamma(M, \mathcal{A}^{k+1} \otimes T^{(1,0)})$, $1 \le k \le 2n$, using the Lebnitz's rule

$$D(\psi \otimes \xi) = d\psi \otimes \xi + (-1)^k \psi \wedge D\xi$$

where $\psi \in \mathcal{A}^k(M)$ is a smooth k-form and $\xi \in \Gamma(M, T^{(1,0)})$.

In particular, we discuss $D^2 : \Gamma(M, T^{(1,0)}) \longrightarrow \Gamma(M, T^{(1,0)} \otimes T^{2*}_{\mathbf{C}})$. Let $f \in C^\infty(M)$ and $\sigma \in \Gamma(M, T^{(1,0)})$, then

$$\begin{aligned} D^2(f\sigma) &= D(df \otimes \sigma + f D\sigma) \\ &= -df \otimes D\sigma + df D\sigma + f D^2\sigma = f D^2 \sigma, \end{aligned}$$

which indicated an important property that D^2 is linear over $C^\infty(M)$.

In terms of local coordinate $(z^1, \ldots, z^n)$, we write

$$D^2 \frac{\partial}{\partial z^i} = \sum_{j=1}^n \Omega^i_j \frac{\partial}{\partial z^j},$$

where Ω is called the *connection matrix*.

Write $\xi = (\partial/\partial z^1, \ldots, \partial/\partial z^n)^t$, then $D\xi = \omega \otimes \xi$ and

$$\begin{aligned} D^2\xi = D(\omega \otimes \xi) &= d\omega \otimes \xi - \omega \wedge D\xi \\ &= d\omega \otimes \xi - (\omega \wedge \omega) \otimes \xi. \end{aligned}$$

Hence

$$\Omega = d\omega - \omega \wedge \omega = \bar{\partial}(\partial g \cdot g^{-1}) = \bar{\partial}(\omega),$$

where $g = (g_{ij})$ is the Hermitian metric matrix on M.

Under the local coordinate $(z^1, \ldots, z^n)$, $\Omega = (\Omega^i_j)$ where Ω^i_j is $(1,1)$-form. So

$$\Omega^i_j = R^i_{j\bar{h}l} d\bar{z}^h \wedge dz^l = R^i_{jl\bar{h}} d\bar{z}^l \wedge d\bar{z}^h,$$

where $R^i_{jl\bar{h}} = -R^i_{j\bar{h}l}$ and

$$\Omega_{\bar{i}j} := g_{s\bar{i}} \Omega^s_j = R_{\bar{i}j\bar{k}l} d\bar{z}^k \wedge dz^l.$$

$R_{\bar{i}j\bar{k}l}$ is call the *curvature tensors*, and $R_{\bar{k}l} : R_{\bar{i}j\bar{k}l} g^{\bar{i}j}$ is called the *Ricci tensor*, where $g^{\bar{i}j}$ is the entries of the inverse matix of g.

From $\Omega = \bar{\partial}(\omega)$,

$$\Omega^i_j = \bar{\partial}(\sum \Gamma^i_{jl} dz^l) = \sum \bar{\partial}_k \Gamma^i_{jl} d\bar{z}^k \wedge dz^l.$$

Hence
$$R^i_{j\bar{k}l} = \bar{\partial}_k \Gamma^i_{jl},$$
where
$$\bar{\partial}_k \Gamma^i_{jl} := \frac{\partial \Gamma^i_{jl}}{\partial \bar{z}^k}.$$
From above, $\Gamma^i_{lj} = g^{\bar{t}i}\partial_l g_{j\bar{t}}$. Hence
$$R^i_{j\bar{k}l} = \bar{\partial}_k \Gamma^i_{lk} = \sum_t g^{\bar{t}i}\bar{\partial}_k \partial_l g_{j\bar{t}} + \sum_t \bar{\partial}_k g^{\bar{t}i}\partial_l g_{j\bar{t}}.$$
Then,
$$\sum_i g_{i\bar{s}} R^i_{j\bar{k}l} = \sum_{t,i} g_{i\bar{s}}\bar{\partial}_k g^{\bar{t}i}\partial_l g_{j\bar{t}} + \bar{\partial}_k \partial_l g_{j\bar{s}}.$$
Since
$$\sum_i g_{i\bar{s}} g^{\bar{t}i} = \delta^{\bar{t}}_{\bar{s}},$$
$$\sum_i g_{i\bar{s}}\bar{\partial}_k g^{\bar{t}i} = -\sum_i g_{\bar{t}i}\bar{\partial}_k g^{i\bar{s}}.$$
Thus
$$R_{\bar{s}j\bar{k}l} = \bar{\partial}_k \partial_l g_{\bar{s}j} - \sum_{i,t} g^{\bar{t}i}\bar{\partial}_k g_{i\bar{s}}\partial_l g_{j\bar{t}}.$$

We also have the so-called *Bianchi Equality*:
$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

Proposition 10.3. *Let E be a Hermitian vector bundle on a complex manifold M. For $\forall p \in M$ there exists a holomorphic local frame e such that*
(1) $h(z) = I + O(|z|^2)$,
(2) $\Omega(0) = \bar{\partial}\partial h(0)$.

Proof. We first choose a local coordinates $z^1, \cdots, z^n$ such that $z(p) = (z^1(p), \cdots, z^n(p)) = 0$. There is a non-singular matrix B, such that $h(0) = B\bar{B}^t$. Take the new frame $f = B^{-1}e$, then $\tilde{h}(0) = I$ with the respect to frame f, and
$$\tilde{h}(z) = I + S(z) + O(|z|^2),$$
where $S(z)$ is a $r \times r$ matrix, whose entries are linear functions of $z^1, \cdots, z^n$, and $\bar{z^1}, \cdots, \bar{z^n}$. Since $\tilde{h} = \bar{\tilde{h}}^t$, $S(z) = \overline{S(z)}^t$. Decomposing $S(z) = S_1(z) + S_2(\bar{z})$, the entries of $S_1(z)$ and $S_2(\bar{z})$ are linear functions of $z^1, \cdots, z^n$ and $\bar{z^1}, \cdots, \bar{z^n}$ respectively. Since
$$\overline{S(z)}^t = \overline{S_1(z)}^t + \overline{S_2(\bar{z})}^t = S_1(z) + S_2(\bar{z}),$$
$S_1(z) = \overline{S_2(\bar{z})}^t$ and $S_2(\bar{z}) = \overline{S_1(z)}^t$.

We now take the new frame $e' = (I - S_1(z))f$. We use h' to denote the metric matrix with respect to the frame e', then

$$\begin{aligned} h' &= (I - S_1(z))(I + S_1(z) + \overline{S_1(z)}^t + O(|z|^2))(I - \overline{S_1(z)}^t) \\ &= I + O(|z|^2), \end{aligned}$$

and it is easy to verify $(h')^{-1} = I + O(|z|^2)$ in an open neighborhood of p. So

$$\Omega(z) = \bar\partial(\partial h' \cdot {h'}^{-1}) = \bar\partial\partial h + O(|z|)$$

especially

$$\Omega(0) = \bar\partial\partial h(0).$$

□

Definition 10.4. *Let M be a Hermitian manifold with the metric $ds^2 = g_{i\bar j} dz^i \otimes d\bar z^j$. If the Kähler* form

$$\Phi = \frac{\sqrt{-1}}{2} g_{i\bar j} dz^i \wedge d\bar z^j$$

is closed, i.e., $d\Phi = 0$, then we call M is a **Kähler manifold.**

Proposition 10.5. *For a Hermitian manifold M, the following condition is equivalent*

(1) M is Kähler;

(2) If $w^i_j = \Gamma^i_{jk} dz^k$ is local expression of connection forms, then $\Gamma^i_{jk} = \Gamma^i_{kj}$;

(3) For $\forall p \in M$, there is a C^∞ function ϕ on an open neighborhood of p, such that $\Phi = \sqrt{-1}\partial\bar\partial\phi$;

(4) For $\forall p \in M$, there exists a local holomorphic coordinate system $z^1, \cdots, z^n$, such that $g_{ij}(p) = \delta_j$, $dg_{ij}(p) = 0$. Such a coordinate is said to be **normal at** *p*

.

Proof. (1) ⇔ (2). Since $d\Phi = \frac{\sqrt{-1}}{2}\frac{\partial g_{i\bar j}}{\partial z^k} dz^k {}_\wedge dz^i {}_\wedge dz^{\bar j} + \frac{\partial g_{i\bar j}}{\partial z^k} d\bar z^k_\wedge dz^i {}_\wedge d\bar z^j$, $d\Phi = 0$ is equivalent to

$$\frac{\partial g_{i\bar j}}{\partial z^k} = \frac{\partial g_{k\bar j}}{\partial z^i}, \quad \textit{and} \quad \frac{\partial g_{i\bar j}}{\partial \bar z^k} = \frac{\partial g_{i\bar k}}{\partial \bar z^j} \quad \forall 1 \le i, j \le n. \tag{2.1.1}$$

Since $\omega^j_i = \frac{\partial g_{i\bar t}}{\partial z^k} g^{\bar t j} dz^k$, $\Gamma^j_{ik} = \frac{\partial g_{i\bar t}}{\partial z^k} g^{\bar t j} = \frac{\partial g_{k\bar t}}{\partial z^i} g^{\bar t j} = \Gamma^j_{ki}; \forall 1 \le i, j \le r$.

(2) ⟶ (1).

$$g_{j\bar s}\Gamma^j_{ik} = \frac{\partial g_{i\bar t}}{\partial z^h} g^{\bar t j} g_{j\bar s} = \frac{\partial g_{i\bar s}}{\partial z^k} = g_{j\bar s}\Gamma^j_{ki} = g_{j\bar s}\frac{\partial g_{k\bar t}}{\partial z^i} g^{\bar t j} = \frac{\partial g_{k\bar s}}{\partial z^i},$$

so we have

$$\frac{\partial g_{i\bar s}}{\partial z^k} = \frac{\partial g_{h\bar s}}{\partial z^i}; \quad 1 \le i, j, s \le n$$

the conjugate of above equality is

$$\frac{\partial g_{s\bar{l}}}{\partial \bar{z}^h} = \frac{\partial g_{s\bar{k}}}{\partial \bar{z}^i}; \quad 1 \le i, j, s \le n$$

so (2.1.1) is valid , i.e. $d\Phi = 0$.

(1) $\Leftrightarrow$ (3) since Φ is a real closed (1,1) form, by *Porincaré* theorem, there is a 1-form H defined in a neighborhood of p such that $\Phi = dH$, $H = H^{0.1} + H^{1.0}$ is its decomposition of (0,1) form and (1, 0) form. Since Φ is real,

$$H^{0.1} = \bar{H}^{1.0}$$

$$\begin{aligned}\Phi = dH &= (\partial + \bar{\partial})(H^{0.1} + H^{1.0}) \\ &= \partial H^{0.1} + \bar{\partial} H^{0.1} + \partial H^{1.0} + \bar{\partial} H^{0.1}.\end{aligned}$$

However, Φ is (1.1) form, so $\partial H^{1.0} = \bar{\partial} H^{0.1} = 0$. Hence, according to the Dolbeault-Grodendick Lemma, there exists a C^∞ function F defined in a neighborhood of p, such that

$$H^{0.1} = \overline{\partial} F \quad and \quad H^{1.0} = \partial \overline{F}.$$

Then

$$\Phi = \overline{\partial} H^{1.0} + \partial H^{0.1} = \overline{\partial}\partial\overline{F} + \partial\overline{\partial} F = \partial\overline{\partial}(F - \overline{F}) = \sqrt{-1}\partial\overline{\partial}\phi,$$

where $\phi = 2ImF$ is a real C^∞ function.

(3) $\Rightarrow$ (1) is trivial.

(1) $\Leftrightarrow$ (4) By a constant linear change of coordinate if necessary, we may assume that the $z^i(p) = 0$; $1 \le i \le n$ and $g_{ij}(p) = \delta_{ij}$, $1 \le i, j \le n$. Now we define a new holomorphic coordinate $(\tilde{z_1}, \cdots, \tilde{z_n})$ by

$$\tilde{z_j} = z_j + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial g_{i\bar{j}}}{\partial z^k}(p) z^k z^i.$$

We use $\tilde{g}$ to denote the metric matrix under $(\tilde{z_1}, \cdots, \tilde{z_n})$. Setting

$$b_{ij} = \frac{\partial \tilde{z}^j}{\partial z^i} = \delta_{ji} + \frac{1}{2} \sum_{h,s=1}^{n} \frac{\partial g_{s\bar{j}}}{\partial z^k}(p)(\delta_{si} z^k + \delta_{ik} z^s) \tag{2.1.2}$$

$$= \delta_{ji} + \frac{1}{2}\left(\sum_k \frac{\partial g_{ij}}{\partial z^k} z^k + \frac{1}{2} \sum_s \frac{\partial g_{sj}}{\partial z^i} z^s \right)$$

$$= \delta_{ji} + \sum_k \frac{\partial g_{ij}}{\partial z^k}(p) z^k$$

and $B = (b_{ij})$ is the $n \times n$ matrix, then $\tilde{g} = B^{-1} g \overline{B^{-1}}^t$. Since $B(p) = B^{-1}(p) = g(p) = I$,

$$d\tilde{g}(p) = (dB^{-1})(p) + dg(p) + (d\overline{B^{-1}}^t)(p)$$

$$= -dB(p) + dg(p) - d\overline{B}^t(p)$$
$$= -\partial B(p) + \partial g(p) + \overline{\partial g}^t - \overline{\partial B}^t(p)$$
$$= 0,$$

the last equality holds because, by (2.1.2), $\partial g(p) = \partial B(p)$.

On the other hand, for $\forall p \in M$, there exists a local holomorphic coordinate coordinates $\tilde{z_1}, \cdots, \tilde{z_n}$, such that $dg(p) = 0$. Then $d\Phi(p) = \frac{\sqrt{-1}}{2} d\tilde{g}_{i\bar{j}}(p) d\tilde{z}^i \wedge d\bar{\tilde{z}}^j = 0$.

□

From proposition 2.5, we know that, at any point of a *Kähler* manifold, the local difference between the *Kähler* metric and Euclidean metric of C^n is the 2 orders infinitsimal, so under the suitable local holomorphic coordinates $z^1, \cdots, z^n$, $\forall p \in M$, $z^i(p) = 0$, $1 \le i \le n$ and

$$g(p) = I, \quad dg(p) = 0$$

i.e. $\partial g(p) = \bar{\partial} g(p) = \bar{\partial} g^{-1}(p) = g^{-1}(p) = 0$, and

$$\Omega(p) = (\bar{\partial}\partial g)(0).$$

By (3) in proposition 2.5, there is a real C^∞ function ϕ on the local neighborhood of p, such that

$$\Phi = i\partial\bar{\partial}\phi$$

so that

$$g_{l\bar{k}} = 2\frac{\partial^2 \phi}{\partial \bar{z}^k \partial z^l}, \quad 1 \le l, k \le n.$$

Therefore

$$R_{\bar{i}j\bar{k}l} = 2\frac{\partial^4 \phi}{\partial \bar{z}^i \partial z^j \partial \bar{z}^h \partial z^l}(0). \tag{2.1.3}$$

So for a *Kähler* manifold, we always have

Proposition 10.6.

$$R_{\bar{i}j\bar{k}l} = R_{\bar{k}j\bar{i}l} = R_{\bar{k}l\bar{i}j} = R_{\bar{i}lk\bar{j}}$$
$$\overline{R_{\bar{i}j\bar{k}l}} = R_{\bar{j}i\bar{l}k}.$$

Proposition 2.6 can be proved by using (2.1.3) and the equality of tensors is independent on the choice of the frame

$$R_{\bar{i}j} = R_{\bar{i}j\bar{k}l} g^{\bar{k}l} = R_{\bar{k}l\bar{i}j} g^{\bar{k}l} = \bar{\partial}_{\bar{i}}((\partial_j g_{l\bar{k}}) g^{\bar{k}l}). \tag{2.1.4}$$

Proposition 10.7. $R_{\bar{i}j} = \partial_{\bar{i}} \partial_j (\log \det g)$

Proof. Let $g = (g_{i\bar{j}})$. We use $A_{i\bar{j}}$ to denote the cofactor of $g_{i\bar{j}}$, then $\det g = \sum_{i,j=1}^{n} g_{i\bar{j}} A_{i\bar{j}}$. Hence

$$\frac{\partial \det g}{\partial g_{i\bar{j}}} = A_{i\bar{j}} = \det g \cdot g^{\bar{j}i}.$$

So

$$\partial_j \det g = \frac{\partial \det g}{\partial g_{i\bar{k}}} \frac{\partial g_{i\bar{k}}}{\partial z^j} = \det g \cdot g^{\bar{k}i} \frac{\partial g_{i\bar{k}}}{\partial z^j}.$$

Therefore,

$$\partial_j \log \det g = \sum_{i,k=1}^{n} \frac{\partial g_{i\bar{k}}}{\partial z^j} g^{\bar{k}i}$$

$$R_{\bar{i}j} = [\partial_{\bar{i}}(\sum_{l,k=1}^{n} \partial_j \Gamma_{lk}^{l})] = (\partial_{\bar{i}} \partial_j \det \log g).$$

□

Definition 10.8. *For* $\forall \xi, \eta \in T_p^{1.0}(M)$*, the* **holomorphic bisectional sectional curvature** is

$$R(\xi_\wedge \eta) = -R_{\bar{i}j\bar{k}l} \bar{\xi}^i \xi^j \bar{\eta}^k \eta^l / \langle \xi, \xi \rangle_p \langle \eta, \eta \rangle_p \tag{2.1.5}$$

where $\xi = \xi^i \frac{\partial}{\partial z^i}$, $\eta = \eta^i \frac{\partial}{\partial z^i}$, and $R_{\bar{i}j\bar{k}l}$ is the curvature tensors under the natural frame $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}$. The holomorphic **sectional curvature** is

$$R(\xi) = -R_{\bar{i}j\bar{k}l}(p) \bar{\xi}^i \xi^j \bar{\eta}^k \eta^l / \langle \xi, \xi \rangle_p^2. \tag{2.1.6}$$

The Ricci curvature is

$$Ric(\xi) = -R_{\bar{i}j}(p) \bar{\xi}^i \xi^j / \langle \xi, \xi \rangle_p, \tag{2.1.7}$$

and the *scalar curvature* at $p \in M$ is

$$R = -R_{\bar{i}j} g^{\bar{i}j}.$$

For Kahler manifold, we also have, for any (p,q)-form ω on M, $\partial \omega = D' \omega$.

10.2 Hermitian Line and vector bundles

The above concepts can be extended from the tangent bundle $T^{(1,0)}(M)$ to a general vector bundle.

Recall that a *holomorphic vector bundle* E over M is a topological space together with a continuous mapping $\pi : E \longrightarrow M$ such that (i) $E_p = \pi^{-1}(p)$; $\quad \forall p \in$

M, is a linear space with rank r; (ii) There exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of M and biholomorphic maps ϕ_α with

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times C^r, \quad \forall \alpha \in I$$

and such that

$$\phi_\alpha : E_P \xrightarrow{\sim} \{p\} \times C^r \xrightarrow{\sim} C^r, \quad \forall p \in U_\alpha$$

is a **C** linear isomorphism between complex vector space. On $U_\alpha \cap U_\beta \neq \emptyset$, let $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$. Then, for $p \in U_\alpha \cap U_\beta$, $\phi_{\alpha\beta}(p) : \{p\} \times C^r \longrightarrow \{p\} \times C^r$ is a linear map, with its matrix representation $g_{\alpha\beta}$ such that $\phi_{\alpha\beta}(p, w) = (p, g_{\alpha\beta}(p)w)$. The map $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r, C)$ is holomorphic, which is called the **transitive function** of E; (iii) The $g_{\alpha\beta}$ satisfies the compatible conditions: $g_{\alpha\beta}(p)g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$ and $g_{\alpha\beta}(p) = g_{\beta\alpha}(p)^{-1}; \quad p \in U_\alpha \cap U_\beta$.

The holomorphic tangent bundle $T^{(1,0)}(M)$ is a vector bundle of rank $n = \dim M$ with the trivialization

$$\phi_\alpha \left(p, \sum_{j=1}^{n} a_j(p) \frac{\partial}{\partial z_j}|_p \right) = (p, (a_1(p), \ldots, a_n(p)) \in U_\alpha \times C^n.$$

A (holomorphic) section s of E is a (holomorphic map) $s : M \to E$ such that $\pi \circ s = id$. When $r = 1$ (line bundle), let $\{U_\alpha\}_{\alpha\in I}$ be trivialization neighborhoods of L, and take a local frame of $L|_{U_\alpha}$ (for example, take $e_\alpha(p) = \phi_\alpha^{-1}(p, 1)$), we can write $s = s_\alpha e_\alpha$, where s_α is holomorphic function on U_α. We have

$$s_\alpha = g_{\alpha\beta} s_\beta,$$

where $g_{\alpha\beta}$ are transition functions. We sometimes just write $s = \{s_\alpha\}$.

A vector bundle E is called a *Hermitian vector bundle* if there is an Hermitian inner product on each fiber E_p for $p \in M$. Similar to above, with the given Hermitian metric, there is a canonical connection (called *Hermitian connection*) $D : \Gamma(M, E) \longrightarrow \Gamma(M, \mathcal{A}^1(M) \otimes E)$ which is compatible with the complex structure and with the Hermitian metric on E. Let $\{e_1, \ldots, e_r\}$ be a local holomorphic frame, and $h_{ij} =< e_i, e_j >, h = (h_{ij}) = h_e$. Write $De_i = \sum_j \omega_i^j e_j$, or write $De = \omega e$. As the calculation above, we have the following expression of the connection matrix $\omega = \partial h \cdot h^{-1}$, so it is of type $(1, 0)$. Write $D^2 = \Omega e$. Then, as above,

$$\Omega = \bar{\partial}\omega = -\partial\bar{\partial}h \cdot h^{-1} + \partial h \cdot h^{-1} \wedge \bar{\partial}h \cdot h^{-1}$$

so Ω is of type (1,1).

For simplicity, we only focus on the line bundle $E = L$, i.e. $r = 1$. Let $\{U_\alpha\}_{\alpha\in I}$ be trivialization neighborhoods of L. Let h be a Hermitian metric on

L. Let $e_\alpha(p) = \phi_\alpha^{-1}(p,1)$ be a local frame of $L|_{U_\alpha}$. Write $h_\alpha = h(e_\alpha, e_\alpha)$. Then the Hermitian metric $\{h_\alpha\}_{\alpha\in I}$ is a set of positive functions with $h_\alpha = |g_{\beta\alpha}|^2 h_\beta$ on $U_\alpha \cap U_\beta$. where $g_{\beta\alpha}$ are transition functions. Its connection form is

$$\theta = \partial h_\alpha \cdot {h_\alpha}^{-1} = \partial log\ h_\alpha,$$

and the curvature form is

$$\Theta = \bar\partial\partial \log h_\alpha = \bar\partial\partial \log h_\beta, \quad on\ U_\alpha \cap U_\beta.$$

So Θ is a global (1.1)-form on M. Define the *first Chern* form of the Hermitian line bundle (L,h) as $c_1(L,h) = \frac{\sqrt{-1}}{2\pi}\Theta = \frac{\sqrt{-1}}{2\pi}\bar\partial\partial \log h_\alpha$.

(L,h) is said to be *positive* (or *ample*) if $c_1(L,h)$ is positive.

Example *The line bundle of hyperplane of* $\mathbf{P}^n$: Let $H = \{[z^0, \cdots, z^n] \in \mathbf{P}^n \mid \sum_{\alpha=0}^{n} a_\alpha z^\alpha = 0\}$ be a hyperplane in $\mathbf{P}^n$. On the coordinate neighborhood $U_\alpha = \{z \in \mathbf{P}^n \mid z^\alpha \neq 0\}$, $s_\alpha = a_1\frac{z^1}{z^\alpha} + \cdots + a_{\alpha-1}\frac{z^{\alpha-1}}{z^\alpha} + a_\alpha + a_{\alpha+1}\frac{z^{\alpha+1}}{z^\alpha} + \cdots + a_n\frac{z^n}{z^\alpha}$ is a defining function of H, where $\frac{z^1}{z^\alpha} \cdots \frac{z^{\alpha-1}}{z^\alpha}\ \frac{z^{\alpha+1}}{z^\alpha} \cdots \frac{z^n}{z^\alpha}$ is a local coordinate system of $\mathbf{P}^n$ in U_α. Then $g_{\alpha\beta} = \frac{s_\alpha}{s_\beta} = \frac{z^\beta}{z^\alpha} : U_\alpha \cap U_\beta \longrightarrow C^*$ are the transitive functions of $[H]$, the **hyperplane line bundle** of $\mathbf{P}^n$. We now endow with a Hermitian metric h on line bundle $[H]$, $h = (h_\alpha)_{0\leq\alpha\leq n}$, where h_α is the local expression of h on U_α.

$$h_\alpha = \frac{|z^\alpha|^2}{|z|^2} = \frac{1}{\sum_{\alpha\neq\beta}^{m} |\frac{z^\beta}{z^\alpha}|^2 + 1}.$$

$$c_1([H]) = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log h_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \|z\|^2 > 0.$$

so $[H]$ is positive line bundle. It is easy to see that $[H]$ is, in fact, independent of the choice of H, so we denote it by $\mathcal{O}_{\mathbf{P}^n}(1)$.

The above construction can be extended to any divisors. A divisor D on M is a formal linear combination

$$D = \sum n_i[Y_i]$$

where $Y_i \subset M$ irreducible hypersurfaces and n_i are integers. A divisor D is called *effective* if $n_i \geq 0$ for all i. Any divisor D induces $\mathcal{O}(D)$, the line bundle associated to D, in a canonical way: If D is a hypersurface locally defined by $f_\alpha = 0$ on U_α, then $\phi_{\alpha\beta} = f_\alpha/f_\beta$ are the transition functions for $\mathcal{O}(D)$. The section $\{s_\alpha := f_\alpha\}$ is called the *canonical section*, and is denoted by s_D.

Let $L \to M$ be a holomorphic line bundle with the transition functions $\{g_{\alpha\beta}\}$. Let m be a positive integer and $s^1, \ldots, s^N$ be sections of mL. Write $s = s_\alpha e_\alpha$, and define

$$h_\alpha = \frac{1}{(|s_\alpha^1|^2 + \cdots + |s_\alpha^N|^2)^{1/m}}.$$

Then it satisfies $h_\alpha = |g_{\beta\alpha}|^2 h_\beta$. This defined a (possible) singular metric on L which blows up exactly on the common zeros of the sections $s^1, \ldots, s^N$. If L is ample, then this defines a metric on L.

Example. *Canonical line bundle on* M: Let $\{U_\alpha\}_{\alpha \in I}$ be a holomorphic coordinate covering of M, $(z^1_{(\alpha)}, \cdots, z^n_{(\alpha)})$ be a local coordinate system of U_α. The canonical line bundle K_M is the line bundle with the transitive functions $\phi_{\alpha\beta} = det\, \frac{\partial(z^1_{(\beta)}, \cdots, z^n_{(\beta)})}{\partial(z^1_{(\alpha)}, \cdots, z^n_{(\alpha)})}$. Sections of K_M are $(n,0)$-forms $\omega = a^\alpha dz^1_{(\alpha)} \wedge \cdots \wedge z^n_{(\alpha)}$.

With Hermitian metric $ds^2 = g^{(\alpha)}_{ij} dz^i_{(\alpha)} \otimes dz^j_{(\alpha)}$ on M, $\det\, g^{(\alpha)} = \det\, (g^{(\alpha)}_{ij})$ is an Hermitian metric of $\det(T^{(1,0)}(M))$, thus $\det\ \ {g^{(\alpha)}}^{-1}$ is the Hermitian metric of K_M. The connection form of K_M is thus $\theta_{(\alpha)} = \partial \det\ g^{-1}_{(\alpha)} \cdot \det\ g_{(\alpha)} = -\partial \log\ \det g_{(\alpha)}$, and the curvature form is $\Omega_{(\alpha)} = -\bar{\partial}\partial \log\ \det g_{(\alpha)} = R_{\bar{j}i} dz^i \wedge d\bar{z}^j$.

Chapter 11

Bochner-Kodaira Formula

The proof of Bochner-Kodaira Formula relies on the calculation of $\Box\omega$, where ω is a E-valued differential form. We first deal with the case when E is trivial.

11.1 The Hilbert Spaces

Let M be a n-dimensional complex manifold with the Hermitian metric $ds^2 = g_{i\bar{j}}dz^i \otimes d\bar{z}^j$. The associated *Kähler* form is $\Phi = \frac{\sqrt{-1}}{2}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$, a real (1.1)-form. The volume form is

$$\frac{1}{n!}\Phi^n = (-1)^{\frac{n(n-1)}{2}} g dx^1{}_\wedge dx^2{}_\wedge \cdots {}_\wedge dy^{n-1}{}_\wedge dy^n.$$

Let ϕ be a smooth (p, q)-form, If we choose local coordinate z, then we can write $\phi = \sum \phi_{I_p\bar{J}_q} dz_{I_p} \wedge d\bar{z}_{J_q}$. It follows that the quantity

$$\langle\phi, \psi\rangle = \frac{1}{p!q!}\phi_{I\bar{J}}\overline{\psi_{K\bar{L}}}g^{i_1\bar{k}_1}\cdots g^{i_p\bar{k}_p}g^{l_1\bar{j}_1}\cdots g^{l_q,\bar{j}_q}$$

is independent of the choice of local coordinates, where $g^{\bar{j}s}$ are the entries of the g^{-1}, the inverse matrix of metric g.

Remark: It is sometimes convenient to employ the notation

$$\psi^{J\bar{I}} := \psi_{K\bar{L}}g^{k_1\bar{i}_1}\cdots g^{k_p\bar{i}_p}g^{j_1\bar{l}_1}\cdots g^{j_q,\bar{l}_q}.$$

We also use, for simplicity, $(\phi)_{I_p\bar{J}_q}$ (as C^∞ covariant tensor field) to denote the coefficient $\phi_{I_p\bar{J}_q}$ and $(\phi)^{\bar{I}_pJ_q}$ (as C^∞ contra-variant tensor field) to denote $g^{\bar{I}_pS_p}g^{\bar{T}_qJ_q}\phi_{S_p\bar{J}_q}$. Then we can write $\langle\phi, \psi\rangle = \phi_{I\bar{J}}\overline{\psi^{J\bar{I}}}$. We define the (global) inner product as

$$(\phi, \psi) = \int_M \langle\phi, \psi\rangle dV,$$

where dV is the volume form of Hermitian manifold.

Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$. Let $\Box = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$.

11.2 Covaraint Derivatives

1. For the vector field $V = a^i \frac{\partial}{\partial z^i}$,

$$DV = da^i \frac{\partial}{\partial z^i} + a^i \theta_i^j \frac{\partial}{\partial z^j}.$$

Hence, for any $\frac{\partial}{\partial z^k}$,

$$\begin{aligned} D_{\frac{\partial}{\partial z^k}} V &= da^i \left(\frac{\partial}{\partial z^k}\right) \frac{\partial}{\partial z^i} + a^i \theta_i^j \left(\frac{\partial}{\partial z^k}\right) \frac{\partial}{\partial z^j} \\ &= \frac{\partial a^i}{\partial z^k} \frac{\partial}{\partial z^i} + a^i \Gamma_{ik}^j \frac{\partial}{\partial z^j} \\ &= \left(\frac{\partial a^i}{\partial z^k} + a^l \Gamma_{lk}^i\right) \frac{\partial}{\partial z^i} \end{aligned}$$

where $\theta_i^j = \Gamma_{ik}^j dz^k$, $\bigtriangledown_k a^i := \frac{\partial a^i}{\partial z^k} + a^l \Gamma_{lk}^i$ is called the covariant derivative of a^i with respect to $\frac{\partial}{\partial z^k}$, and is also denoted by ${a^i}_{;k}$, i.e.

$$DV = {a^i}_{;k} \frac{\partial}{\partial z^i} \otimes dx^k,$$

in ${a^i}_{;k}$ we use a semicolon to separate indices resulting from diffferentiation from the preceding indices. We note that $\{{a^i}_{;k}\}_{1 \le i,k \le n}$ is a tensor. Note that

$$\Gamma_{ik}^j = \frac{\partial g_{i\bar{l}}}{\partial z^k} g^{\bar{l}j}$$

When M is Kahler, $\Gamma_{ik}^j = \Gamma_{ki}^j$.

Definition. *For a smooth vector field $V = a^i \frac{\partial}{\partial z^i}$, we define*

$$\bigtriangledown_k a^i := \frac{\partial a^i}{\partial z^k} + \Gamma_{lk}^i a^l,$$

where

$$\Gamma_{ik}^j = \frac{\partial g_{i\bar{l}}}{\partial z^k} g^{\bar{l}j}.$$

2. For $\omega = \sum_{j=1}^n f_j dz^j$ (contra-variant tensor field), then

$$D_{\frac{\partial}{\partial z^k}} \omega = \left(\frac{\partial f_j}{\partial z^k} - \sum_{p=1}^n \Gamma_{kj}^p f_p\right) dz^j,$$

or simply

$$\bigtriangledown_k f_j = \frac{\partial f_j}{\partial z^k} - \sum_{t=1}^n \Gamma_{kj}^t f_t.$$

Definition. *For a smooth* $(1,0)$*-form* $\omega = \sum_{j=1}^n f_j dz^j$*, we define*

$$\bigtriangledown_k f_j = \frac{\partial f_j}{\partial z^k} - \Gamma_{kj}^t f_t.$$

3. For $p = 2$, $q = 0$, i.e. $\phi = \phi_{i_1 i_2} dz^{i_1} \wedge dz^{i_2}$,

$$D_{\frac{\partial}{\partial z^k}} \phi = \left(\frac{\partial \phi_{i_1 i_2}}{\partial z^k} - (\phi_{t i_2} \Gamma_{k i_1}^t + \phi_{i_1 t} \Gamma_{k i_2}^t) \right) dz^{i_1} \wedge dz^{i_2}.$$

3. For general p, q, and $\phi = \sum \phi_{I_p \bar{J}_q} dz^{I_p} \wedge d\bar{z}^{J_q}$,

$$D'_{\frac{\partial}{\partial z^k}} \phi = \left(\frac{\phi_{i_1 \cdots i_p \bar{J}_q}}{\partial z^k} - \sum_{s=1}^p \phi_{i_1 \cdots (t)_s \cdots i_p} \Gamma_{k i_s}^t \right) dz^{i_1} {}_\wedge \cdots {}_\wedge dz^{i_p} {}_\wedge dz^{\bar{J}_q},$$

where $(t)_s$ means that the index t is at the s-th place.

Definition. *For a smooth (anti) vector field* $\eta = \eta^i \frac{\partial}{\partial \bar{z}^i}$*, we define*

$$\bigtriangledown_k \eta^i := \frac{\partial \eta^i}{\partial z^k}.$$

Definition. *For a smooth* $(0,1)$ *form* $\phi = \phi_i d\bar{z}^i$*, we define*

$$\bigtriangledown_k \phi_i := \frac{\partial \phi_i}{\partial z^k}.$$

Similarly, for the definition of $\bigtriangledown_{\bar{k}}$, we summerize as follows: for $V = a^i \frac{\partial}{\partial z^i}$, $\bigtriangledown_{\bar{k}} a^i = \frac{\partial a^i}{\partial \bar{z}^k}$, for a smooth $(1,0)$-form $\omega = \sum_{j=1}^n f_j dz^j$, $\bigtriangledown_{\bar{k}} f_j = \frac{\partial f_j}{\partial \bar{z}^k}$, a smooth (anti) vector field $\eta = \eta^i \frac{\partial}{\partial \bar{z}^i}$, $\bigtriangledown_{\bar{k}} \eta^i = \frac{\partial \eta^i}{\partial \bar{z}^k} + \overline{\Gamma_{kt}^i} \eta^t$, For a smooth $(0,1)$ form $\phi = \phi_i d\bar{z}^i$, $\bigtriangledown_{\bar{k}} \phi_i = \frac{\partial \phi_i}{\partial \bar{z}^k} - \overline{\Gamma_{ki}^t} \phi_t$.

We sometimes also write $\bigtriangledown_{\bar{k}}$ as $\overline{\bigtriangledown_k}$.

3. The reason we convert ∂ to $\bigtriangledown_i$ is that we need to deal with the metric (the connection is compatibel with the metric). In particular, we have

Theorem. *For the metric tensor* $g_{i\bar{j}}$ *and its inverse* $g^{\bar{j}i}$*, we have* $\bigtriangledown_k g_{i\bar{j}} = 0$ *and* $\bigtriangledown_k g^{\bar{j}i} = 0$.

Proof.

$$\bigtriangledown_k g_{i\bar{j}} = \frac{\partial g_{i\bar{j}}}{z_k} - g_{l\bar{j}}\Gamma^l_{ik} = \frac{\partial g_{i\bar{j}}}{z_k} - g_{l\bar{j}}\frac{\partial g_{i\bar{s}}}{\partial z^k} g^{\bar{s}l} = 0.$$

Also

$$\bigtriangledown_k g^{\bar{j}i} = \frac{\partial g_{\bar{j}i}}{z_k} + g_{l\bar{j}}\Gamma^l_{ik} = \frac{\partial g_{i\bar{j}}}{z_k} - g^{\bar{j}s}\frac{\partial g_{s\bar{l}}}{\partial z^k} g^{\bar{l}i},$$

so $\bigtriangledown_k g^{\bar{j}i} = 0$. This finishes the proof.

Note that the above theorem is actually due to the fact that the connection is compatibel with the metric. The theorem can be proved directly by using the fact that the connection is compatibel with the metric. It shows that you don't have to worry about $\bigtriangledown_k g_{i\bar{j}}$ (it is zero) when you use the connection (covariant derivative) to differentiate the forms, rather than using the exterior derivative d. The following propostion shows that in the case that M is Kahler, there is indeed no difference.

Proposition 1 *Assume that M is Kahler. for any*

$$\phi = \sum \phi_{I_p \bar{J}_q} dz^{I_p} \wedge d\bar{z}^{J_q},$$

we have

$$\partial\phi = \sum \nabla_i \phi_{I_p \bar{J}_q} dz^i \wedge dz^{I_p} \wedge d\bar{z}^{J_q}.$$

Proof. To get the idea why the Propsition works, let us first consider the case when $q = 0, p = 1$ and dimension $M = 2$, i.e. $\omega = \sum_{j=1}^2 f_j dz^j$. Then from the above,

$$\begin{aligned}
\sum_{j,k=1}^{2} \bigtriangledown_k f_i dz^k \wedge dz^j &= \sum_{k,j=1}^{2}\left(\frac{\partial f_i}{\partial z^k} - \sum_{t=1}^{2}\Gamma^t_{kj} f_t\right) dz^k \wedge dz^j \\
&= \sum_{k,j=1}^{2} \frac{\partial f_j}{\partial z^k} dz^k \wedge dz^j - \sum_{t=1}^{2} f^t(\Gamma^t_{12} - \Gamma^t_{21}) dz^1 \wedge dz^2 \\
&= \sum_{k,j=1}^{2} \frac{\partial f_j}{\partial z^k} dz^k \wedge dz^j \\
&= \partial\omega,
\end{aligned}$$

where we used the fact $\Gamma^t_{12} = \Gamma^t_{21}$ since M is Kahler.

In the general case, by definition,

$$\bigtriangledown_i \phi_{i_1\cdots i_p \bar{J}_q} dz^i \wedge dz^{i_1} {}_\wedge \cdots {}_\wedge dz^{i_p} {}_\wedge dz^{\bar{J}_q} = \left(\frac{\phi_{i_1\cdots i_p\bar{J}_q}}{\partial z^i} - \sum_{s=1}^{p} \Gamma^t_{i_s i}\phi_{i_1\cdots(t)_s\cdots i_p\bar{J}_q}\right) dz^i \wedge dz^{i_1} {}_\wedge \cdots {}_\wedge dz^{i_p} {}_\wedge dz^{\bar{J}_q}$$

$$= \frac{\partial \phi_{i_1 \cdots i_p \bar{J}_q}}{\partial z^i} dz^i{}_\wedge dz^{i_1}{}_{\wedge \cdots \wedge} dz^{i_p}{}_\wedge d\bar{z}^{J_q} = \partial \phi$$

where, in the last equality, we used the fact that $\Gamma^t_{i_s i} = \Gamma^t_{i i_s}$ on the *Kähler* manifold and $dz^i{}_\wedge dz^{i_1}{}_{\wedge \cdots \wedge} dz^{i_p}$ are anti-symmetric when interchage the orders.

Similarly, by taking the conjugate,

Proposition 2. *Let*

$$\phi = \sum \phi_{I_p \bar{J}_q} \otimes dz^{I_p} \wedge d\bar{z}^{J_q}$$

be a smooth (p, q)*-form. Then*

$$\bar{\partial}\phi = (-1)^p \sum_{k=0}^{q} (-1)^k \frac{\bar{\partial}\phi_{I_p \bar{j}_0 \bar{j}_1 \cdots \hat{\bar{j}}_k \cdots \bar{j}_q}}{\partial \bar{z}^{j_k}} dz^{I_p} \wedge d\bar{z}^{j_0} \wedge \cdots \wedge d\bar{z}^{j_q}$$

$$= (-1)^p \sum_{k=0}^{q} (-1)^k \nabla_{\bar{j}_k} \phi_{I_p \bar{j}_0 \bar{j}_1 \cdots \hat{\bar{j}}_k \cdots \bar{j}_q} dz^{I_p} \wedge d\bar{z}^{j_0} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

11.3 The formula for $\bar{\partial}^*$

We now derive the formula for $\bar{\partial}^*$.

Proposition 3. *In the compact Kähler case, Let*

$$\phi = \sum \phi_{I_p \bar{J}_q} \otimes dz^{I_p} \wedge d\bar{z}^{J_q}$$

be a smooth (p, q)*-form. Then*

$$(\bar{\partial}^* \phi)_{I_p \bar{j}_1 \cdots \bar{j}_{q-1}} = (-1)^{p+1} g^{\bar{j} i} \nabla_i \phi_{I_p \bar{j} \bar{j}_1 \cdots \bar{j}_{q-1}}.$$

Proof. One has

$$\begin{aligned}
(\bar{\partial}^* \phi, \psi) &= (\phi, \bar{\partial}\psi) \\
&= \frac{1}{p!q!} \int_M \phi_{I_p \bar{j}_1 \cdots \bar{j}_q} (-1)^p \sum_{k=1}^{q} (-1)^{k+1} \overline{g^{j_k \bar{j}'_k} \nabla_{\bar{j}'_k} \psi^{j_1 \cdots \hat{j}_k \cdots j_q \bar{I}_p}} dA \\
&= \frac{1}{p!q!} \int_M (-1)^{p+1} \sum_{k=1}^{q} g^{j'_k \bar{j}_k} \left(\nabla_{j'_k} (-1)^{k+1} \phi_{I_p \bar{j}_1 \cdots \bar{j}_q} \right) \overline{\psi^{j_1 \cdots \hat{j}_k \cdots j_q \bar{I}_p}} dA \\
&= \frac{1}{p!(q-1)!} \int_M \left((-1)^{p+1} g^{i\bar{j}} \nabla_i \phi_{I_p \bar{j} \bar{j}_1 \cdots \bar{j}_{q-1}} \right) \overline{\psi^{j_1 \cdots j_{q-1} \bar{I}_p}} dA.
\end{aligned}$$

where the second-to-last inequality follows form the metric compatibility of the connection. This finishes the proof.

Note that we can also use the $\star$-operator to express $\bar{\partial}^*$, similar to the Riemann surface case, we can prove that $\bar{\partial}^* = - \star \partial \star$.

11.4 The Bochner-Kadaira formula

We first deal with the case that the line bundle is trivial.

Theorem (Weitzenbook identity) *Let M be a compact Kähler manifold,*

$$\omega = \frac{1}{p!q!} a_{I_p \bar{j}_1 \cdots \bar{j}_q} dz^{I_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \in \varepsilon^{p,q}(M).$$

Then

$$\begin{aligned}(\Box\omega)_{I_p \bar{J}_q} &= -\sum g^{\bar{j}i} \nabla_i \bar{\nabla}_j a_{I_p \bar{J}_q} \\ &+ \sum_{t=1}^{p} \sum_{s=1}^{q} R^{k}_{i_t \bar{j}_s}{}^{\bar{l}} a_{i_1 \cdots i_{s-1} k i_{s+1} \cdots i_p \bar{j}_1 \cdots \bar{j}_{t-1} \bar{l} \bar{j}_{t+1} \cdots \bar{j}_q} \\ - &\sum_{s=1}^{q} \sum_{l=1}^{q} R^{\bar{l}}_{\bar{j}_s} a_{I_p \bar{j}_1 \cdots \bar{j}_{s-1} \bar{l} \bar{j}_{s+1} \cdots \bar{j}_q},\end{aligned}$$

where $R^{\bar{l}}_{\bar{j}} = R_{\bar{j}k} g^{\bar{l}k}$, $R^{k}_{i\bar{j}}{}^{\bar{l}} = g^{\bar{t}k} g^{\bar{l}s} R_{ti\bar{j}s}$

Remarks:

1. We sometime write it into the crude form

$$(\Box\omega)_{I_p \bar{J}_q} = -\sum g^{\bar{j}i} \nabla_i \bar{\nabla}_j a_{I_p \bar{J}_q} + A^1(\omega),$$

where $A^1(\omega)$ only involves first order differentiation. In other words, modulo lower-order terms, the global Laplacian on forms looks like the Euclidean Laplacian $-\sum_k \partial^2/\partial z_k \partial \bar{z}_k$.

2. For the application of proving the vanishing theorem, we only use the formula when $p = 0$. In this case, the term

$$\sum_{t=1}^{p} \sum_{s=1}^{q} R^{k}_{i_t \bar{j}_s}{}^{\bar{l}} a_{i_1 \cdots i_{s-1} k i_{s+1} \cdots i_p \bar{j}_1 \cdots \bar{j}_{t-1} \bar{l} \bar{j}_{t+1} \cdots \bar{j}_q}$$

will disappear, so for

$$\omega = \frac{1}{q!} a_{\bar{j}_1 \cdots \bar{j}_q} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

Then

$$(\Box\omega)_{\bar{J}_q} = -\sum g^{\bar{j}i} \nabla_i \bar{\nabla}_j a_{I_p \bar{J}_q} - \sum_{s=1}^{q} \sum_{l=1}^{q} R^{\bar{l}}_{\bar{j}_s} a_{I_p \bar{j}_1 \cdots \bar{j}_{s-1} \bar{l} \bar{j}_{s+1} \cdots \bar{j}_q}.$$

When write

$$Ric(\omega) = -\sum_{k,t=1}^{q} R^{\bar{t}}_{\bar{j}_k} (\omega)_{\bar{j}_1 \cdots (\bar{t})_k \cdots \bar{j}_q} \qquad t \text{ in } k\text{-th spot},$$

Then

$$(\Box\omega)_{\bar{J}_q} = -\sum g^{\bar{j}i}\nabla_i\bar{\nabla}_j a_{I_p\bar{J}_q} + Ric(\omega).$$

Proof. By proposition 2

$$(\bar{\partial}\omega)_{I_p\bar{j}_0\cdots\bar{j}_q} = (-1)^p\sum_{t=0}^{q}(-1)^t\bar{\nabla}_{j_t}a_{I_p\bar{j}_0\cdots\hat{\bar{j}}_t\cdots\bar{j}_q}$$

and by proposition 3

$$(\bar{\partial}^*\bar{\partial}\omega)_{I_p\bar{j}_1\cdots\bar{j}_q} = (-1)^{p+1}g^{\bar{j}i}\nabla_i(\bar{\partial}\omega)_{I_p\bar{j}\bar{j}_1\cdots\cdots\bar{j}_q}$$

$$= -g^{\bar{j}i}\nabla_i\bar{\nabla}_j a_{I_p\bar{j}_1\cdots\bar{j}_q} - \sum_{s=1}^{q}(-1)^s g^{\bar{j}i}\nabla_i\bar{\nabla}_{j_s}a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}.$$

Similarly

$$(\bar{\partial}^*\omega)_{I_p\bar{j}_1\cdots\bar{j}_q} = (-1)^{p+1}g^{\bar{j}i}\nabla_i a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_q}$$

$$(\bar{\partial}\bar{\partial}^*\omega)_{I_p\bar{j}_1\cdots\bar{j}_q} = (-1)^p\sum_{s=1}^{q}(-1)^{s+1}\bar{\nabla}_{j_s}(\bar{\partial}^*\omega)_{I_p,\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}$$

$$= -\sum_{s=1}^{q}(-1)^{s+1}\bar{\nabla}_{j_s}(g^{\bar{j}i}\nabla_i a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q})$$

$$= -\sum_{s=1}^{q}g^{\bar{j}i}(-1)^{s+1}\bar{\nabla}_{j_s}\nabla_i a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}$$

so that

$$(\Box\omega)_{I_p\bar{j}_1\cdots\bar{j}_q} = (\bar{\partial}\bar{\partial}^*\omega)_{I_p\bar{j}_1\cdots\bar{j}_q} + (\bar{\partial}^*\bar{\partial}\omega)_{I_p\bar{j}_1\cdots\bar{j}_q}$$

$$= -g^{\bar{j}i}\nabla_i\nabla_j a_{I_p\bar{j}_1\cdots\bar{j}_q} - \sum_{s=1}^{q}g^{\bar{j}i}(-1)^s(\nabla_i\bar{\nabla}_{js} - \bar{\nabla}_{js}\nabla_i)a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}.$$

Note that up to here, the I_p part is unchanged since we are performing $\bar{\partial}$ and $\bar{\partial}^*$ only, so you may letting $p = 0$ in the above computations for simplicity. It remains to compute $[\nabla_i, \bar{\nabla}_{js}]\ a_{I_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}$. For simplicity we only need to compute $[\nabla_i, \bar{\nabla}_j]$ for (1, 0)-forms $a_k dz^k$ and $b_{\bar{k}}\overline{dz^k}$. Since the result $[\nabla_i, \bar{\nabla}_j]$ acting on $a_{i_1\cdots i_p\bar{j}\bar{j}_1\cdots\bar{j}_{s-1}\hat{\bar{j}}_s\bar{j}_{s+1}\cdots\bar{j}_q}$, for each index among $i_1\cdots i_p$, is similar to those for each index among $\bar{j}, \bar{j}_1, \cdots\bar{j}_{s-1}\bar{j}_{s+1}\cdots\bar{j}_q$.

$$\text{(13.3.1)}\qquad [\nabla_i, \bar{\nabla}_j]a_k = \nabla_i\bar{\nabla}_j a_k - \bar{\nabla}_j\nabla_i a_k = \nabla_i\partial_j a_k - \bar{\nabla}_j(\partial_i a_k - \Gamma^t_{ki}a_t)$$

$$= \partial\bar{\partial}_j a_k - \bar{\partial}_j a_t\Gamma^t_{ki} - \bar{\partial}\partial a_k + \bar{\partial}_j(\Gamma^t_{ki}a_t)$$

$$= -\bar{\partial}_j a_t\Gamma^t_{ki} + \bar{\partial}_j a_t\Gamma^t_{ki} + \bar{\partial}_j\Gamma^t_{ki}a_t$$

$$= \bar{\partial}_j\Gamma^t_{ki}a_t = R^t_{k\bar{j}i}a_t$$

and

$$(13.3.2)\qquad [\nabla_i, \bar\nabla_j]b_{\bar k} = \nabla_i\bar\nabla_j b_{\bar k} - \bar\nabla_j\nabla_i b_{\bar k} = \nabla_i(\bar\partial_j b_{\bar k} - \overline{\Gamma^t_{kj}}b_{\bar t}) - \bar\nabla_j(\partial_i b_{\bar k})$$
$$= \partial_i\bar\partial_j b_{\bar k} - \partial_i\overline{\Gamma^t_{kj}}b_{\bar t} - \overline{\Gamma^t_{ij}}\partial_i b_{\bar t} - \bar\partial_j\partial_i b_{\bar k} + \partial_i b_{\bar t}\overline{\Gamma^t_{kj}}$$
$$= -\partial_i\overline{\Gamma^t_{kj}}b_{\bar t} = -\overline{\bar\partial_i\Gamma^t_{kj}}b_{\bar t} = -\overline{R^t_{kj\bar i}}b_{\bar t} = -R_{\bar k}{}^{\bar t}{}_{\bar j i}b_{\bar t}.$$

Applying (13.3.1) and (13.3.2) to $[\nabla_i, \bar\nabla_{js}]a_{i_1\cdots i_p\bar j\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\cdots\bar j_q}$, we have

$$[\nabla_i, \bar\nabla_{js}]a_{i_1\cdots i_p\bar j\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\cdots\bar j_q}$$
$$= \sum_{k=1}^{p} R^l_{i_k\bar j_s i}a_{i_1\cdots l(i_k)\cdots i_p\bar j\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\cdots\bar i_q} - R_{\bar j}{}^{\bar l}{}_{\bar j_s i}a_{I_p\bar l\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\cdots\bar i_q}$$
$$-\sum_{k<s} R_{\bar j_k}{}^{\bar l}{}_{\bar j_s i}a_{I_p\bar j\bar j_1\cdots\bar l(\bar j_k)\cdots\hat{\bar j}_s\cdots\bar i_q} - \sum_{k>s} R_{\bar j_k}{}^{\bar l}{}_{\bar j_s i}a_{I_p\bar j\bar j_1\cdots\hat{\bar j}_s\cdots l(\bar j_k)\cdots\bar i_q}.$$

Since $g^{\bar j i}R_{\bar j}{}^{\bar l}{}_{\bar j_s i} = R_{\bar j_s}{}^{\bar l}$ and $g^{\bar j i}R_{\bar j_k}{}^{\bar l}{}_{\bar j_s i} = R_{\bar j_k}{}^{\bar l}{}_{\bar j_s}{}^{\bar j}$,

$$(3)\qquad g^{j\bar i}\sum_{s=1}^{q}(-1)^s[\nabla_i, \bar\nabla_j]a_{I_p\bar j\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\bar j_q}$$
$$= \sum_{s=1}^{q}(-1)^s\sum_{t=1}^{p}R^l{}_{i_t\bar j_s}{}^{\bar j}a_{i_1\cdots i_{t-1}li_{t+1}i_p\bar j\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\cdots\bar j_q}$$
$$-\sum_{s=1}^{q}(-1)^s R_{\bar j_s}{}^{\bar l}a_{I_p\bar l\bar j_1\cdots\bar j_{s-1}\bar j_{s+1}\bar j_q}$$
$$-\sum_{s=1}^{q}(-1)^s\sum_{k<s}R_{\bar j_k}{}^{\bar l}{}_{\bar j_s}{}^{\bar j}a_{I_p\bar j\bar j_1\cdots\bar j_{k-1}\bar l(\bar j_k)\bar j_{k+1}\cdots\hat{\bar j}_s\cdots\bar j_q}$$
$$-\sum_{s=1}^{q}(-1)^s\sum_{k>s}R_{\bar j_k}{}^{\bar l}{}_{\bar j_s}{}^{\bar j}a_{I_p\bar j\bar j_1\cdots\hat{\bar j}_s\cdots\bar j_{k-1}\bar l(\bar j_k)\bar j_{k+1}\cdots\bar j_q},$$

where symbol $\bar l(\bar j_k)$ ($l(j_k)$) denote the $\bar l$ instead of $\bar j_k$, and because of $R_{\bar j_k}{}^{\bar l}{}_{\bar j_s}{}^{\bar j} = R_{\bar j k}{}^{\bar j}{}_{\bar j_s}{}^{\bar l}$, so the last two terms in (3) are vanishing. Therefore we obtained the expression formula of complex Laplacian.

11.5 The general case

Let L be a Hermitian line bundle over a compact *Kähler* manifold, and h be its Hermitian metric. We want to derive a similar formula for $\Box_L$ acting on

$\Gamma(M, \varepsilon^{p,q}(L))$. A form $\omega \in \Gamma(M, \varepsilon^{p,q}(L))$ corresponds to a family of (p,q)-forms ω_α on $\{U_\alpha\}$, where $\{U_\alpha\}$ is an open covering consists of the trivialization neighborhoods of L. Let $\{\phi_{\alpha\beta}\}$ be the transitive functions of L, then

$$\omega_\alpha = \phi_{\alpha\beta}\omega_\beta; \quad on \;\; U_\alpha \cap U_\beta.$$

Let $\omega, \eta \in \Gamma(M, \varepsilon^{p,q}(L))$, then

$$(\omega, \eta) = \int_M h_\alpha < \omega_\alpha, \eta_\alpha > .$$

As a well-known fact, if $\omega \in \Gamma(M, \varepsilon^{p,q}(L))$, then $\bar{\partial}\omega \in \Gamma(M, \varepsilon^{p,q+1}(L))$. If $\omega \in \Gamma(M, \varepsilon^{p,q}(L))$ i.e., $\omega_\alpha \in \Gamma(M, \varepsilon^{p,q}(L))$, $\alpha \in I$, $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M consists of the trivialization neighborhoods of L, then

$$\omega_\alpha = \phi_{\alpha\beta}\omega_\beta; \quad on \;\; U_\alpha \cap U_\beta.$$

Since $\phi_{\alpha\beta}$ is holomorphic,

$$\bar{\partial}\omega_\alpha = \phi_{\alpha\beta}\bar{\partial}\omega_\beta; \quad on \;\; U_\alpha \cap U_\beta.$$

But for the operator ∂, $\partial\omega$ is no longer a L-valued differential form, since if $\omega_\alpha = \phi_{\alpha\beta}\omega_\beta$ on $U_\alpha \cap U_\beta$, then

$$\partial\omega_\alpha = \partial\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta; \quad on \;\; U_\alpha \cap U_\beta,$$

and, in general $\partial\phi_{\alpha\beta} \neq 0$, so $\partial\omega$ is no longer a L-valued differential form. Let $h = (h_\alpha)$ be the Hermitian metric of L. We introduce $D_L : \Gamma(M, \varepsilon^{p,q}(L)) \longrightarrow \Gamma(M, \varepsilon^{p+1,q}(L))$, which is a differential operator of degree $(1,0)$ on L-valued forms, by letting

$$D_L\omega_\alpha = \partial\omega_\alpha + \partial \log \; h_\alpha\omega_\alpha = h_\alpha^{-1}\partial(h_\alpha\omega_\alpha).$$

Then

$$\begin{aligned}
D_L\omega_\alpha &= \partial\omega_\alpha + \partial log \; h_\alpha\omega_\alpha \\
&= \partial(\phi_{\alpha\beta}\omega_\beta) + \partial log \; (h_\beta|\phi_{\beta\alpha}|^2)\phi_{\alpha\beta}\omega_\beta \\
&= \partial\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log \; h_\beta + (\partial log \; \phi_{\beta\alpha}))\phi_{\alpha\beta}\omega_\beta \\
&= \partial\phi_{\alpha\beta}\phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log \; h_\beta\omega_\beta)\phi_{\alpha\beta} + \partial log \; \phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta \\
&= \partial log\phi_{\alpha\beta}\phi_{\alpha\beta}\omega_\beta + \phi_{\alpha\beta}\partial\omega_\beta + (\partial log \; h_\beta\omega_\beta)\phi_{\alpha\beta} + \partial log \; \phi_{\beta\alpha}\phi_{\alpha\beta}\omega_\beta \\
&= \phi_{\alpha\beta}(\partial\omega_\beta + \partial log \; h_\beta\omega_\beta) = \phi_{\alpha\beta}D_L\omega_\beta .
\end{aligned}$$

new: global calculation. It is easy to check that the operator D_L satisfies

$$\partial(\eta \wedge \bar{\xi}h) = \partial\eta \wedge \bar{\xi}h + (-1)^{\deg\eta}\eta \wedge \overline{D_L\xi}h,$$

so it also proves that the D_L is well defined.

The Bochner-Kodaira Formula:

Similar to what we have proved, we can prove that (see the book by Morrow and Kodaria: Complex Manifolds)

Theorem (The Bochner-Kadaira formula). *Let L be an Hermitian line bundle over M. Then for any L-valued $(0,q)$-form*

$$\phi = \frac{1}{q!}\phi_{\bar{j}_1\cdots\bar{j}_q} d\bar{z}^{j_1} \wedge d\bar{z}^{j_q}$$

$$(\Box_L \phi)_{\bar{j}_1\cdots\bar{j}_q} = -g^{\bar{j}i}\nabla_i^{(L)}\nabla_{\bar{j}}\phi_{\bar{j}_1\cdots\bar{j}_q} + \sum_{k=1}^{q}\sum_t (\Omega^{\bar{t}}_{\bar{j}_k} - R^{\bar{t}}{}_{\bar{j}_k})\phi_{\bar{j}_1\cdots(\bar{t})_k\cdots\bar{j}_q}.$$

where $\nabla_i^{(L)} = \partial_i + \partial_i log\ h_\alpha$ *and* $\Omega_{\bar{j}}^{\bar{i}} = -\nabla_{\bar{j}}\ g^{\bar{i}k}\partial_k log\ h_\alpha = g^{\bar{i}k}\bar{\nabla}_j\partial_k log\ h_\alpha$

We can also formulate the The Bochner-Kadaira formula as follows

$$\Box = -\mathrm{Trace}(\nabla^{(L)}\overline{\nabla}) + T_g(\Omega - Ric(R))$$

where

$$\mathrm{Trace}(\nabla^{(L)}\overline{\nabla}) := g^{i\bar{j}}\nabla_i^{(L)}\nabla_{\bar{j}}$$

and

$$T_g(\Omega - Ric(R)) = \sum_{k=1}^{q} g^{i\bar{t}}\Omega_{i\bar{j}_k}\phi_{\bar{j}_1\cdots(\bar{t})_k\cdots\bar{j}_q} - \sum_{k=1}^{q} g^{i\bar{t}}R_{i\bar{j}_k}\phi_{\bar{j}_1\cdots(\bar{t})_k\cdots\bar{j}_q},$$

$\Omega_{i\bar{j}} = \sum_t g_{i\bar{t}}\Omega^{\bar{t}}_{\bar{j}} = -\bar{\nabla}_j\partial_i \log h_\alpha = -\partial_i\partial_{\bar{j}}\log h_\alpha$ is the curvature of the metric $\{h_\alpha\}$, $R_{\bar{i}j} = \partial_{\bar{i}}\partial_j(\log\det(g))$ and $c_1(K_M) = \frac{\sqrt{-1}}{2\pi}R_{\bar{i}j}d\bar{z}^i \wedge dz^j$.

Theorem *Let M be a compact kahler and let L be an Hermitian line bundle over M. Then for any L-valued $(0,q)$-form ω,*

$$(\Box\omega,\omega) = \|\overline{\nabla}\omega\|^2 + ((T_g(\Omega - Ric(R)))\omega,\omega).$$

Proof. Write locally $\omega = \omega_\alpha e_\alpha$ where e_α is a local frame for L. We introduce the following $(0,1)$-form on M

$$\Psi_{U_\alpha} = h_\alpha\bar{\nabla}_j\omega_{\alpha\bar{J}_q}\phi_\alpha^{\bar{J}_q}d\bar{z}^j.$$

It is indeed global define, since

$$\Psi_{U_\alpha} = h_\alpha(\bar{\nabla}_j\omega_{\alpha\bar{J}_q})\overline{\omega_\alpha^{J_q}}d\bar{z}^j = |\phi_{\beta\alpha}|^2 h_\beta\bar{\nabla}_j(\omega_{\beta\bar{J}_q})|\phi_{\alpha\beta}|^2\overline{\omega_\beta^{J_q}}d\bar{z}^j$$

$$= \Psi_{U_\beta}, \quad on \;\; U_\alpha \cap U_\beta \qquad .$$

We use the fact that if Ψ is a 1-form on M, then

$$\int_M \bar\partial^* \Psi dV_M = 0,$$

this is because $\bar\partial^*\Psi$ is a global function, and

$$\int_M \bar\partial^* \Psi dV_M = (\bar\partial^*\Psi, 1) = (\Psi, \bar\partial 1) = 0.$$

On the other hand, $\bar\partial^*\Psi$ can be calculated as follows

$$\begin{aligned}
\bar\partial^*\Psi = & -g^{\bar{j}i}\nabla_i(h_\alpha \bar\nabla_j \omega_{\alpha \bar{J}_q} \overline{\omega_\alpha^{J_q}}) \\
& = -g^{\bar{j}i}(\nabla_i(h_\alpha h_\alpha^{-1}) h_\alpha \bar\nabla_j \omega_{\alpha \bar{J}_q} \overline{\omega_\alpha^{J_q}} + g^{\bar{j}i} h_\alpha \nabla_i \bar\nabla_j \omega_{\alpha \bar{J}_q} \overline{\omega_\alpha^{J_q}} \\
& \quad -g^{\bar{j}i} h_\alpha \bar\nabla_j \omega_{\alpha \bar{J}_q} \nabla_i \overline{\omega_\alpha^{J_q}} \\
& = -g^{\bar{j}i} h_\alpha \nabla_i^L \bar\nabla_j \omega_{\alpha \bar{J}_q} \overline{\omega_\alpha^{J_q}} - g^{\bar{j}i} h_\alpha \overline{\nabla}_j \omega_{\alpha \bar{J}_q} \nabla_i \overline{\omega_\alpha^{J_q}} \\
& = -g^{\bar{j}i} h_\alpha \nabla_i^L \bar\nabla_j \omega_{\alpha \bar{J}_q} \overline{\omega_\alpha^{J_q}} - g^{\bar{j}i} h_\alpha \overline{\nabla}_j \omega_{\alpha \bar{J}_q} \overline{\bar\nabla_i \omega_\alpha^{J_q}}.
\end{aligned}$$

Thus, from the above Bochner-Kodaira Formula, we get

$$(\Box\omega, \omega) = \|\overline{\nabla}\|^2 + ((T_g(\Omega - Ric(R)))\omega, \omega).$$

This finishes the proof.

Recall that, from Proposition 2.7,

$$R_{\bar{i}j} = \partial_{\bar{i}} \partial_j (\log \det(g)).$$

So

$$c_1(K_M) = \frac{\sqrt{-1}}{2\pi} R_{\bar{i}j} d\bar{z}^i \wedge dz^j.$$

Also Ω_L is the curvature form of L. Therefore, if $L \otimes K_M^*$ is positive, then $-Ric\omega + \Omega_L$ is positive, so $H^q(M, \mathcal{O}(L))$ must vanish. Here is the proof:

Theorem (Kodaira's vanishing theorem). *Let M be a n-dimensional compact Kähler manifold, and L be a line bundle with the Hermitian metric h. If $L \otimes K_M^*$ is positive, then*

$$H^q(M, \mathcal{O}(L)) = 0 \quad for \quad q \geq 1.$$

Proof. The condistion that $L \otimes K_M^*$ is positive means the matrix $(X_{j\bar{i}} - R_{\bar{i}j})$ is positive definition, so $((T_g(\Omega - Ric(R)))\omega, \omega) > 0$ for all $\omega \neq 0$. Hence $(\Box\omega, \omega) > 0$ for all $\omega \neq 0$. This implies that $\mathcal{H}^{(0,q)}(M, L) = 0$ bcause for any $\omega \in \mathcal{H}^{(0,q)}(M, L)$, $0 = (\Box\omega, \omega) > 0$ if $\omega \neq 0$. Thus $H^q(M, L) = 0$ by Hodge's theorem. This finishes the proof.

Note that we can actually prove Kodaira's vanishing theorem by bypass the Hodge theory. By using Dolbeault theorem, $H^q(M, L \otimes K_M) = 0$ for $q \geq 1$ if we can solve

$$\bar{\partial}\omega = \psi$$

for any $\bar{\partial}$-closed (n, q)-form ψ. It can be achieved by using the fact that

$$\|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2 = (\Box\phi, \phi)$$

and the **Lax-Milgram Lemma** that *If* $||g||^2 \leq c(||T^*g||^2 + ||Sg||^2)$, *then* $Tu = f$ *has a solution to* $f \in Ker\ S$. *This solution* u *satisfies the estimate*

$$||u|| \leq c^{\frac{1}{2}}||f||, \quad u \in (Ker\ T)^{\perp}$$

where we consider Hilbert spaces:

$$H_1 \to^T H_2 \to^S H_3$$

where $H_1,\ H_2,\ H_3$ are all Hilbert spaces, T, S are linear, closed, densely defined operators with $ST = 0$.

This leads the materials on solving $\bar{\partial}$-equations for domains $\Omega \subset \mathbf{C}^n$ with flat metric, but with boundaries (the theory is discussed in the next chapter).

Chapter 12

L^2 ESTIMATES

We will present the method of L^2 estimates in this section. The method is to use the Hilbert space to prove the existence of the solution to the $\overline{\partial}$ problem on a pseudoconvex domain, based on a priori estimate. The tool is is to use so-called *Lax-Milgram* lemma. The trick to deal with the boundary is called *Morrey trick*. Using the L^2 estimates, we can solve the Levi's problem: The pseudoconvex domain is the domain of holomorphy.

12.1 Problem and the Formulation

Let $\Omega \subset \mathbf{C}^n$ be a bounded domain, $f = \sum f_v d\overline{z}^v$ be a form of type $(0,1)$ defined on Ω and satisfy $\overline{\partial} f = 0$. The question is whether

$$\overline{\partial} u = f \tag{3.1}$$

has a solution. If we use the theory of Hilbert space, considering

$$L^2_{(0,0)}(\Omega) \to L^2_{(0,1)}(\Omega) \to L^2_{(0,2)}(\Omega), \tag{3.2}$$

then the above problem is equivalent to: Whether the kernel of the second $\overline{\partial}$ is equal to the image of the first $\overline{\partial}$.

We summerize the above discussion in terms of the model of Hilbert spaces:

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3 \tag{3.3}$$

where $H_1,\ H_2,\ H_3$ are all Hilbert spaces, and T, S are linear, closed, densely defined operators. Assume $ST = 0$, the problem is whether, for $\forall f \in Ker\ S$, the solution to

$$Tu = f \tag{3.4}$$

exists.

12.2 Basic Facts from the Theory of Hilbert Spaces

As we mentioned above, we now consider

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3 \tag{3.3}$$

where $H_1,\ H_2,\ H_3$ are all Hilbert spaces, and T, S are linear, closed, densely defined operators. Assume $ST = 0$, the problem is whether, for $\forall f \in Ker\ S$, the solution to

$$Tu = f \tag{3.4}$$

exists.

First, note a simple fact: $Tu = f$ *is equivalent to*

$$(3.5). \qquad (Tu,\ g) = (f,\ g), \quad \forall g \in \ \textit{some dense subset}$$

This is because because (3.5) $\iff$ $(Tu - f,\ g) = 0$, $\forall g \in$ some dense subset $\iff$ $(Tu - f,\ H_2) = 0 \iff Tu = f$.

Let T^* be an adjoint operator of T. By the theory of functional analysis that T^* is a closed operator, and $(T^*)^* = T$ if and only if T is closed. Here we recall the definition of T^*: Let $y \in H_2$. If there exists a $y^* \in H_1$ such that for $\forall x \in Dom\ T$, we have

$$(Tx,\ y) = (x,\ y^*), \tag{3.6}$$

then $y \in Dom\ T^*$, and we define $T^* y = y^*$. By (3.6),

$$(Tx,\ y) = (x,\ T^* y). \tag{3.7}$$

Next we will write out the expression of T^* on $C^\infty(\Omega)$, where $C^\infty(\Omega)$ is the set of infinitely differentiable functions on some neighborhood of $\overline{\Omega}$, so $Dom\ T^*$ is dense in H_2. In other words, T^* is also a linear closed densely defined operator.

From (3.5), $(Tu,\ g) = (f,\ g)$, $\forall g \in$ some dense subset. If this dense subset $\subset Dom\ T^*$, then, noticing $(Tu,\ g) = (u,\ T^* g)$,

$$\begin{aligned} &Tu = f \iff (Tu,\ g) = (f,\ g) \iff \\ &(u,\ T^* g) = (f,\ g), \forall g \in \text{ some dense subset in } Dom\ T^*. \end{aligned} \tag{3.8}$$

The existence of u thus could be possibly found by applying the Rietz Representation theorem as follows: let $T^* g \to (f,\ g)$ be a linear functional defined on a subset of H_1(i.e. $\{T^* g |\ g \in$ some dense subset in $Dom\ T^*\}$). If we can extend the above functional to a bounded linear functional on entire H_1, then an application of Rietz Representation theorem to (3.8) will thus show that the problem $Tu = f$ is solved. Recall that the Rietz Representation theorem states that *if* $\lambda : H \longrightarrow \mathbf{C}$ *is a bounded linear functional on a Hilbert space* H*, then there exists* $u \in H$ *such that* $\lambda(x) = (x,\ u)$ *for* $\forall x \in H$. Hence the main step is whether we can extend $T^* g \to (f,\ g)$ to a bounded linear functional on entire H_1.

Lemma 12.1. *If there exists a constant c_f depending only on f such that*

$$|(g,\ f)| \le c_f ||T^*g||, \tag{3.9}$$

then $T^*g \to (g,\ f)$ can be extended to a bounded linear functional on H_1.

Proof. First note that, under (3.9), the definition $T^*g \to (g,\ f)$ is well-defined, since if $T^*g_1 = T^*g_2$, then $|(g_1 - g_2,\ f)| \le c_f||T^*(g_1 - g_2)|| = 0$, i.e., $(g_1,\ f) = (g_2,\ f)$.

Next we extend $T^*g \longrightarrow (g,\ f)$ to $\overline{\{T^*g\}}$ the closed envelope of $\{T^*g | g \in Dom(T^*)\}$. If $x \in \overline{\{T^*g | g \in Dom(T^*)\}}$, then there exists g_v such that $x = \lim T^*g_v$, by (3.9),

$$|(g_v - g_u,\ f)| \le c_f||T^*g_v - T^*g_u|| \longrightarrow 0 (v, u \to \infty).$$

Hence $lim(g_v,\ f)$ exists and it is the value of this functional at x.

Finally, for a general $x \in H_1$, if we denote P by the projective operator $H_1 \longrightarrow \overline{\{T^*g | g \in Dom(T^*)\}}$ (this is a closed subspace), then we can define the value of this functional at x by that at Px, and the latter is signficative above. □

In the above discussion, we however **only used the front half** of

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3.$$

However, since we only need to solve the equation $Tu = f$ or $(T^*g,\ u) = (g,\ f)$ for $f \in Ker\ S$, it is unnecessary to prove (3.9) for all $f \in H_2$, rather we just need to prove (3.9) for $f \in Ker\ S$. In this case, we hope that g in (3.9) belongs to some dense subset in $Dom\ T^*$ due to the proceeding proof.

The method of proving $|(g, f)| \le c_f||T^*g||$ is through proving a more general equality:

$$||g||^2 \le c(||T^*g||^2 + ||Sg||^2) \quad \forall g \in Dom\ T^* \cap Dom\ S.$$

First we note, in our problem, $Dom\ T^*$ and $Dom\ S$ contain $C_0^\infty(\Omega)$—- the set of infinitely differentiable functions whose supports in Ω, hence $Dom\ T^* \cap Dom\ S$ is dense on both $Dom\ T^*$ and H_2. Now we need

Lemma 12.2. *If*

$$||g||^2 \le c(||T^*g||^2 + ||Sg||^2) \quad \forall g \in Dom\ T^* \cap Dom\ S \tag{3.10}$$

then

$$|(g,\ f)| \le c^{\frac{1}{2}}||f||\ ||T^*g||, \quad \forall f \in Ker\ S,\ g \in Dom\ T^* \cap Dom\ S. \tag{3.11}$$

Proof. For every $g \in Dom\ T^* \cap Dom\ S$, g can be decomposed orthogonally along the closed subspace $Ker\ S$ and its orthogonal complement $(Ker\ S)^\perp$, that is,

$$g = g_1 + g_2, \quad g_1 \in Ker\ S, \quad g_2 \in (Ker\ S)^\perp.$$

Since $ST = 0$, $(Ker\ S)^\perp \subset (ImT)^\perp$, and if $x \in (ImT)^\perp$,then $(x,\ Ty) = 0,\ \forall y \in Dom\ T$. By the definition of T^*, $0 = (x,\ Ty) = (T^*x,\ y),\ \forall y \in Dom\ T$, then $T^*x = 0$, so we have $(Ker\ S)^\perp \subset (ImT)^\perp \subset Ker\ T^*$. Thus $g_1 = g - g_2 \in Dom\ T^*,\ g_2 = g - g_1 \in Dom\ S \cap Dom\ T^*$, hence $g_1,\ g_2$ are both in $Dom\ T^* \cap Dom\ S$. Hence

$$\begin{aligned}
|(g,\ f)| &= |(g_1,\ f)| && (f \in Ker\ S, g_2 \in (Ker\ S)^\perp)\\
&\le ||f|| \cdot ||g_1|| && \text{(Schwartz inequality)}\\
&\le c^{\frac{1}{2}}||f||(||T^*g_1||^2 + ||Sg_1||^2)^{\frac{1}{2}} && ((3.10), g_1 \in Dom\ T^* \cap Dom\ S)\\
&\le c^{\frac{1}{2}}||f|| \cdot ||T^*g_1|| && (g_1 \in Ker\ S)\\
&\le c^{\frac{1}{2}}||f|| \cdot ||T^*g|| && (g_2 \in Ker\ T^*, T^*g_2 = 0)
\end{aligned}$$

□

Applying Lemma 3.2, we have that if $||g||^2 \le c(||T^*g||^2 + ||Sg||^2)$ for all $g \in Dom\ T^* \cap Dom\ S$, then $|(g,\ f)| \le c^{\frac{1}{2}}||f|| \cdot ||T^*g||$ for $\forall f \in Ker\ S,\ g \in Dom\ T^* \cap Dom\ S$. Hence, by Lemma 3.1, $T^*g \longrightarrow (g,\ f)$ can be extended to be a bounded linear functional on H_1, whose bound is $c^{\frac{1}{2}}||f||$. By Rietz's representation theorem, there exists $u \in H_1$ such that $(T^*g,\ u) = (g,\ f)$ for $\forall g \in Dom\ T^* \cap Dom\ S$. Since $Dom\ T^* \cap Dom\ S$ is dense in H_2, we have $(g,\ Tu) = (g,\ f)$, for $\forall g \in H_2$. By (3.8), the equation $Tu = f$ has a solution In addition, from the Rietz Representation theorem, we have

$$||u|| \le c^{\frac{1}{2}}||f||, \quad \text{and} \qquad u \in (Ker\ T)^\perp.$$

In fact, $||u|| \le c^{\frac{1}{2}}||f||$ is the direct conseqeunce of Rietz's representation theorem; to see $u \in (Ker\ T)^\perp$, note that, according to the way t hat $T^*g \longrightarrow (g,\ f)$ is extended to a bounded linear functional on entire H_1, this functional vanishes on the orthogonal complement of $\overline{\{T^*g|g \in Dom(T^*)\}}$, thus $u \in \overline{\{T^*g|g \in Dom(T^*)\}}$. If $u = \lim\limits_{v\to\infty} T^*g_v$, then for every $x \in Ker\ T$, we have

$$(x,\ u) = \lim_{v\to\infty}(x,\ T^*g_v) = \lim_{v\to\infty}(Tx,\ g_v) = 0,$$

hence, $u \in (Ker\ T)^\perp$.

In general, the solution to $Tu = f$ is not unique, since $\forall u_1 \in Ker\ T$, then

$$\begin{aligned}
&(T^*g,\ u + u_1) = (T^*g,\ u) + (T^*g,\ u_1)\\
&= (T^*g,\ u) + (g,\ Tu_1) = (T^*g,\ u)
\end{aligned}$$

and $u,\ u + u_1$ are both the solution to $Tu = f$. However, $u \in (Ker\ T)^\perp$ is the condition to assure that the above solution to $Tu = f$ is unique.

From the above discussion, we have proved **the follolwing important result**:

Lemma 12.3. *(Lax-Milgram Lemma)* *If* $||g||^2 \le c(||T^*g||^2 + ||Sg||^2)$*, then* $Tu = f$ *has a solution to* $f \in Ker\ S$*. This solution* u *satisies the estimate*

$$||u|| \le c^{\frac{1}{2}}||f||, \quad u \in (Ker\ T)^{\perp} \tag{3.12}$$

Note: If $T = \overline{\partial}$*, then* (3.12) *implies* u *is orthogonal to all analytic functions.*

12.3 Solving $\bar{\partial}$-equations.

Now we return to practise problem that we discussed above. Assume $H_1 = L^2_{(0,0)}(\Omega,\ \varphi)$, $H_2 = L^2_{(0,1)}(\Omega,\ \varphi)$, $H_3 = L^2_{(0,2)}(\Omega,\ \varphi)$, where $\varphi \in C^\infty(\overline{\Omega})$ and the norm of L^2 space is denoted by $||\cdot||$. We define

$$||f||^2 = \int_\Omega |f|^2 e^{-\varphi} dx.$$

To the forms of types (0,1), (0,2), there are integrations of square sums of their components (relative to the factor $e^{-\varphi}$). For example $f = \sum f_i d\overline{z}^i$, then

$$||f||^2 = \int_\Omega \sum |f_i|^2 e^{-\varphi} dx$$

It will manifest gradually the importance of weight function $e^{-\varphi}$ in the following deduction. In fact, it is relative to the metric of ordinararily line bundle $\Omega \times \mathbf{C}$ on Ω. We will explain it in detail on the section of Kodaira vanishing theorem in the latter part of this book. On the other hand, T and S are closed extensions of $\overline{\partial}$ (on $C^\infty(\overline{\Omega})$ and $C^\infty_{(0,1)}(\overline{\Omega})$) on $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$. By lemma 4.3, the solution to $\overline{\partial}-$problem depends on the proof of the inequality (3.12).

To prove this basic inequlity, we require the following steps:

1. The formally adjoint operator of $T = \overline{\partial}$.

First, for all $f \in C^\infty_{(0,0)}(\overline{\Omega}) \subset Dom\ T$, we have

$$(Tf,\ g) = (f,\ T^*g).$$

If $g = \sum g_i d\overline{z}^i \in C^\infty_{(0,1)}(\overline{\Omega})$, the above equality becomes

$$\sum_i \int_\Omega (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} = (Tf,\ g) = (f,\ T^*g) = \int f(\overline{T^*g})e^{-\varphi}.$$

is valid to all $f \in C^\infty_{(0,0)}(\overline{\Omega})$, especially to f with compact support. If $Supp\ f \subset \Omega$, then due to integration by parts

$$\begin{aligned}
\sum_i \int_\Omega (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} &= -\sum_i \int_\Omega f\overline{\partial}_i(\overline{g}_i e^{-\varphi}) \\
&= -\sum_i \int_\Omega f\overline{e^{\varphi}\partial_i(g_i e^{-\varphi})}e^{-\varphi} \\
&= -\sum_i \int_\Omega f\overline{\delta_i g_i}e^{-\varphi},\ \delta_i g_i = e^{\varphi}\partial_i(e^{-\varphi}g_i)
\end{aligned}$$

$$T^*g = -\sum_i \delta_i g_i. \tag{3.13}$$

This equality is the form of T^*g (when $g \in C^\infty_{(0,1)}(\overline{\Omega})$), we call it the **formally adjoint operator of** T.

2. Determing *Dom* T^*.

Does $C^\infty_{(0,1)}(\overline{\Omega})$ belong to *Dom* T^*? From above, when $g \in C^\infty_{(0,1)}(\overline{\Omega})$ and in *Dom* T^*, then $T^*g = -\sum \delta_i g_i$. Does this T^*g satisfy $(Tf,\ g) = (f,\ T^*g)$ to all $f \in C^\infty_{(0,0)}(\overline{\Omega})$? Not at all, we shall add some conditions to g.

Before continuing discuss, we prove a fomula which is badically the divergence theorem.

Proposition 12.4. *If the boundary* $\partial\Omega = \{r = 0\}$ *of a bounded domain* $\Omega = \{r < 0\}$ *is differentiable,* $|dr| = 1$*, and* $L = \sum a_i \frac{\partial}{\partial x_i}$ *is a differentiable operator of 1-order with constant coefficients, then*

$$\int_\Omega Lf = \int_{\partial\Omega} (Lr)f.$$

Proof. By usual Stokes fomula,

$$\int_\Omega \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n = \int_{\partial\Omega} f dx_2 \wedge \cdots \wedge dx_n$$

where $\frac{\partial}{\partial x_1}$ can be replaced by every $\frac{\partial}{\partial x_i}$. Let $p \in \partial\Omega$, r be one of local ordinates near p because $|dr| = 1$. We assume local ordinates of $\partial\Omega$ be $\theta_1, \cdots, \theta_{n-1}$, and $d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$ the volumn element of $\partial\Omega$, and $dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$ the unit volumn element near p, that is,

$$dx_1 \wedge \cdots \wedge dx_n = dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{n-1},$$

here we can do it because $|dr| = 1$. Hence,

$$dx_2 \wedge \cdots \wedge dx_n = dr \wedge \omega + \alpha d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$$

where ω is some $(n-2)-$degree form. Then

$$dr \wedge dx_2 \wedge \cdots \wedge dx_n = \alpha dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{n-1},$$

$$\frac{\partial r}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n = \alpha dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$$

$$\alpha = \frac{\partial r}{\partial x_1}.$$

So

$$\begin{aligned}
&\int_\Omega \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n = \int_{\partial\Omega} f dx_2 \wedge \cdots \wedge dx_n \\
&= \int_{\partial\Omega} f(dr \wedge \omega + \alpha d\theta_1 \wedge \cdots \wedge d\theta_{n-1}) \\
&= \int_{\partial\Omega} f\alpha d\theta_1 \wedge \cdots \wedge d\theta_{n-1} = \int_{\partial\Omega} f \frac{\partial r}{\partial x_1}.
\end{aligned}$$

Likewise we have

$$\int_\Omega Lf = \int_{\partial\Omega} (Lr)f$$

where $L = \sum a_i \frac{\partial}{\partial x_i}$. It is still true when $a_i \in \mathbf{C}$, $\frac{\partial}{\partial x_i}$ is replaced by $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial \bar{z}_i}$. This completes the proof. $\square$

Now we compute $(Tf,\ g)$ for $f \in C^\infty_{(0,0)}(\overline{\Omega})$, $g \in Dom\ T^* \cap C^\infty_{(0,1)}(\overline{\Omega})$. First,

$$\overline{\partial}_i(f\overline{g}_i e^{-\varphi}) = (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} + f\overline{\partial_i(g_i e^{-\varphi})}.$$

Integrating on Ω,

$$\int_\Omega \overline{\partial}_i(f\overline{g}_i e^{-\varphi}) = \int_\Omega (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} + \int_\Omega f\overline{\partial_i(g_i e^{-\varphi})}.$$

By proposition 3.4,

$$\int_\Omega \overline{\partial}_i(f\overline{g}_i e^{-\varphi}) = \int_{\partial\Omega} (\overline{\partial}_i r)f\overline{g}_i e^{-\varphi},$$

$$\int_\Omega f\overline{\partial_i(g_i e^{-\varphi})} = -\int_\Omega (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} + \int_{\partial\Omega} (\overline{\partial}_i r)f\overline{g}_i e^{-\varphi}.$$

Summing up the above for i, the first term of the right–hand side becomes $(-1)(Tf,\ g)$, while the left–hand side is

$$\sum \int_\Omega f\overline{\partial_i(g_i e^{-\varphi})} = \sum \int f\overline{e^{\varphi}\partial_i(g_i e^{-\varphi})}e^{-\varphi} = (-1)(f,\ T^*g).$$

But $(Tf,\ g) = (f,\ T^*g)$, so, for $g \in Dom\ T^* \cap C^\infty_{(0,1)}(\overline{\Omega})$, we have

$$\sum \int_{\partial\Omega} f(\overline{\partial}_i r)\overline{g}_i e^{-\varphi} = 0. \tag{3.14}$$

Since $f \in C^\infty(\overline{\Omega})$ is arbitrary, the above equation is equivalent to

$$\sum_i (\partial_i r)g_i|_{\partial\Omega} = 0. \tag{3.15}$$

Thus we get the sufficient and necessarry condition (3.15) of $g \in Dom\ T^* \cap C^\infty_{(0,1)}(\overline{\Omega})$. So if g is infinitely differentiable with compact support $\subset \Omega$, then $g \in Dom\ T^*$.

3. Computing $||T^*g||^2 + ||Sg||^2$, as $g \in Dom\ T^* \cap Dom\ S \cap C^\infty_{(0,1)}(\overline{\Omega})$.

We can reduce the deduced fomula above,

$$\sum \int_\Omega f\overline{\partial_i(g_i e^{-\varphi})} = -\sum \int_\Omega (\overline{\partial}_i f)\overline{g}_i e^{-\varphi} + \sum \int_{\partial\Omega} (\overline{\partial}_i r)f\overline{g}_i e^{-\varphi}$$

to a fomula: If $f, g \in C^\infty(\overline{\Omega})$, then

$$(f,\ \delta_i g) = -(\overline{\partial}_i f,\ g) + ((\overline{\partial}_i r) f,\ g)_{\partial\Omega} \tag{3.16}$$

where the signification of δ_i is as same as (3.13), and $(\cdot,\cdot)_{\partial\Omega}$ indicates the integral on $\partial\Omega$ relative to weight factor $e^{-\varphi}$.

Now computing

$$||T^*g||^2 = \int_\Omega |\sum_i \delta_i g_i|^2 e^{-\varphi} = \sum_{i,j}\int_\Omega (\delta_i g_i)\overline{(\delta_j g_j)} e^{-\varphi},$$

$$\begin{aligned} ||Sg||^2 &= \int_\Omega \sum_{i<j} |\overline{\partial}_i g_j - \overline{\partial}_j g_i|^2 e^{-\varphi} \\ &= \sum_{i<j}\int_\Omega (|\overline{\partial}_i g_j|^2 - \overline{\partial}_i g_j \cdot \overline{\overline{\partial}_j g_i} - \overline{\partial}_j g_i \cdot \overline{\overline{\partial}_i g_j} + |\overline{\partial}_j g_i|^2) e^{-\varphi} \\ &= \sum_{i,j}\int_\Omega (|\overline{\partial}_i g_j|^2 - (\overline{\partial}_i g_j)(\partial_j \overline{g}_i)) e^{-\varphi}. \end{aligned}$$

So

$$||T^*g||^2 + ||Sg||^2 = \sum_{i,j}\int_\Omega |\overline{\partial}_i g_j|^2 e^{-\varphi} + \sum_{i,j}\int_\Omega ((\delta_j g_j)\cdot\overline{(\delta_i g_i)} - (\overline{\partial}_i g_j)\cdot(\partial_j \overline{g}_i)) e^{-\varphi}.$$

By (3.16),

$$\int_\Omega (\delta_j g_j)\overline{(\delta_i g_i)} e^{-\varphi} = -(\overline{\partial}_i \delta_j g_j,\ g_i) + ((\overline{\partial}_i r)\delta_j g_j,\ g_i)_{\partial\Omega}$$

$$\int_\Omega (\overline{\partial}_i g_j)\overline{(\overline{\partial}_j g_i)} e^{-\varphi} = -(g_j,\ \delta_i \overline{\partial}_j g_i) + ((\overline{\partial}_i r) g_j,\ \overline{\partial}_j g_i)_{\partial\Omega}.$$

Noting $\sum_{i,j}\int_i g_j \partial_j \overline{g}_i e^{-\varphi} = \sum_{i,j}\int \partial_i \overline{g}_{jj} g_i e^{-\varphi} = \sum_{i,j}\overline{\int_i g_j \partial_j \overline{g}_i e^{-\varphi}} = -\sum_{i,j}(\delta_{ij} g_i, g_j) +$ $\sum_{i,j}(\partial_i r \overline{g}_j, \partial_j \overline{g}_i)_{\partial\Omega}$, and substituting it to the formulas of $||T^*g||^2 + ||Sg||^2$, then

$$\begin{aligned} ||T^*g||^2 + ||Sg||^2 &= \sum_{i,j}\int_\Omega |\overline{\partial}_i g_j|^2 e^{-\varphi} + \sum_{i,j}((\delta_i \overline{\partial}_j - \overline{\partial}_j \delta_i) g_i,\ g_j) \\ &- \sum_{i,j}\int_{\partial\Omega} (\overline{\partial}_i r)(\delta_j g_j)\overline{g}_i e^{-\varphi} - \sum_{i,j}(\partial_i r)\overline{g}_j \overline{(\partial_j \overline{g}_i)} e^{-\varphi}. \end{aligned}$$

The following equality obtained by direct computation,

$$\begin{aligned} (\delta_i \overline{\partial}_j - \overline{\partial}_j \delta_i)\omega &= e^\varphi \partial_i((\overline{\partial}_j \omega) e^{-\varphi}) - \overline{\partial}_j(e^\varphi \partial_i(\omega e^{-\varphi})) \\ &= (\overline{\partial}_j \partial_i \varphi)\omega. \end{aligned}$$

At the same time, $g \in Dom\ T^* \cap C^\infty_{(0,1)}(\overline{\Omega})$, thus $\sum_i (\partial_i r) g_i|_{\partial\Omega} = 0$, hence

$$\sum_{i,j}\int_{\partial\Omega} (\overline{\partial}_i r)\delta_j g_j \cdot \overline{g}_i \cdot e^{-\varphi} = \sum_j \int_{\partial\Omega} \delta_j g_j \cdot \sum_i (\overline{\partial}_i r)\overline{g}_i e^{-\varphi} = 0.$$

Therefore, we have

$$\begin{aligned}||T^*g||^2+||Sg||^2&=\sum_{i,j}\int_\Omega|\overline{\partial}_i g_j|^2e^{-\varphi}\\ &+\sum_{i,j}\int_\Omega(\overline{\partial}_j\partial_i\varphi)g_i\overline{g}_je^{-\varphi}-\sum_{i,j}\int_{\partial\Omega}(\partial_i r)\overline{g}_j\cdot\overline{\partial}_jg_ie^{-\varphi}.\end{aligned}\tag{3.17}$$

4. The domination of the boundary term – Morrey's trick.

In the history development of $\overline{\partial}-$operator in L^2 method, it was difficult to dominate the last term in (3.17), i.e., the boundary term

$$-\sum_{i,j}\int_{\partial\Omega}(\partial_i r)\overline{g}_j(\overline{\partial}_jg_i)e^{-\varphi}$$

for a long time, untill 1958, when Morrey successfully overcame this difficulty(See C. B. Morrey, Ann. of Math. 68(1958)). The method he presented is called **Morrey's trick** now. The method is: Let $g\in Dom\ T^*\cap C^\infty_{(0,1)}(\overline{\Omega})$, $r=0$ define the boundary of Ω, and the defining function r be differentiable. Thus

$$\sum(\partial_i r)g_i$$

are local functions, differentiable at every point. By (3.15), these functions vanish at $r=0$, i.e., on $\partial\Omega$. By Taylor expansion, it can be written as

$$\sum(\partial_i r)g_i=\lambda\cdot r$$

where λ is some differentiable function. Taking $\overline{\partial}_j$ to both sides to yield

$$\sum_i(\overline{\partial}_j\partial_i r)g_i+\sum_i(\partial_i r)(\overline{\partial}_jg_i)=(\overline{\partial}_j\lambda)r+\lambda\overline{\partial}_jr.$$

Multiplying $\overline{g}_j$ and summing up for j,

$$\sum_{i,j}(\overline{\partial}_j\partial_i r)g_i\overline{g}_j+\sum_{i,j}(\partial_i r)(\overline{\partial}_jg_i)\overline{g}_j=\sum_j r(\overline{\partial}_j\lambda)\overline{g}_j+\lambda\sum_j(\overline{\partial}_jr)\overline{g}_j.$$

Integrating on $\partial\Omega$, noting $r=0$ on $\partial\Omega$, $\sum(\partial_j r)g_j=0$, to get

$$-\sum_{i,j}\int_{\partial\Omega}(\partial_i r)(\overline{\partial}_jg_i)\overline{g}_je^{-\varphi}=\sum_{i,j}\int_{\partial\Omega}(\overline{\partial}_j\partial_i r)g_i\overline{g}_je^{-\varphi}.$$

By (3.17), we get

$$\begin{aligned}||T^*g||^2+||Sg||^2&=\sum_{i,j}\int_\Omega|\overline{\partial}_ig_j|^2e^{-\varphi}+\sum_{i,j}\int_\Omega(\overline{\partial}_j\partial_i\varphi)g_i\overline{g}_je^{-\varphi}\\ &+\sum_{i,j}\int_{\partial\Omega}(\overline{\partial}_j\partial_i r)g_i\overline{g}_je^{-\varphi}.\end{aligned}\tag{3.18}$$

Note that we have not made any special restrictions to Ω and to the choice of φ so far. Now we assume

(i) Ω is a pseudoconvex domain, i.e.

$$\sum_{i,j}(\overline{\partial}_j\partial_i r)\xi_i\overline{\xi}_j \geq 0, \qquad \forall \sum(\partial_i r)\xi_i = 0; \tag{3.19}$$

(ii) φ satisfies that complex Hessian is strictly positive definite, that is, there exists $c > 0$ so that

$$\sum_{i,j}(\partial_i\overline{\partial}_j\varphi)\xi_i\overline{\xi}_j \geq c\sum|\xi_i|^2. \tag{3.20}$$

Under the above two assumptions, the first term in the right - hand side of (3.18) is nonnegative, the third term is also nonnegative because the boundary condition $\sum(\partial_i r)g_i|_{\partial\Omega} = 0$ and (3.19), and the second term satisfies

$$\sum_{i,j}\int_\Omega(\overline{\partial}_j\partial_i\varphi)g_i\overline{g}_j e^{-\varphi} \geq c\sum_i\int_\Omega|g_i|^2e^{-\varphi} = c||g||^2.$$

Hence we proved the following theorem:

Theorem 12.5. *Let Ω be a pseudoconvex domain. Given a real valued function $\varphi \in C^\infty(\overline{\Omega})$ satisfies $\sum(\partial_i\overline{\partial}_j\varphi)\xi_i\overline{\xi}_j \geq c\sum|\xi_i|^2$, $c > 0$, then for $g \in Dom\ T^* \cap Dom\ S \cap C^\infty_{(0,1)}(\Omega)$, we have*

$$c||g||^2 \leq ||T^*g||^2 + ||Sg||^2. \tag{3.21}$$

Recall that in the previous discussion, if for all $g \in Dom\ T^* \cap Dom\ S$, we have $c||g||^2 \leq ||T^*g||^2 + ||Sg||^2$, then the $\overline{\partial}$ - problem of a pseudoconvex domain has a solution. However, (3.21) implies that $c||g||^2 \leq ||T^*g||^2 + ||Sg||^2$ holds for all **infinitely differentiable functions** in $Dom\ T^* \cap Dom\ S$. To prove this estimate holds for all g in $Dom\ T^* \cap Dom\ S$, it sufficies to show that, for $\forall g \in Dom\ T^* \cap Dom\ S$ there exists a sequence $g_v \in C^\infty_{(0,1)}(\overline{\Omega})$ such that

$$g_v \longrightarrow g,\ T^*g_v \longrightarrow T^*g,\ Sg_v \longrightarrow Sg.$$

Note that it is important to prove that these convergence holds at the same time. It is easy to prove the first and the third holds (because S is a closed operator, by the definition of a closed operator, if $g \in Dom\ S$, then it implies there exists $g_v \in C^\infty_{(0,1)}(\overline{\Omega})$ such that $g_v \to g$, $Sg_v \to Sg$). The question becomes to show that the second holds at the same time. The method is called the regularization method of K. Friedrichs, first due to K. Friedrichs in 1944 (Trans, Amer. Math. Soc. 55(1944)), P. 132 - 151), later Hörmander further developped it (basically, by convolution with mollifiers, i.e. smooth functions with compact support and total integral 1, one can approximate L^2-functions by smooth, compactly supported functions).

So we have proved that, for a pseudoconvex domain Ω, if $\varphi \in C^\infty(\overline{\Omega})$ satisfies $\sum(\partial_i\overline{\partial}_j\varphi)\xi_i\overline{\xi}_j \geq c\sum|\xi_i|^2$, then we have

$$c||g||^2 \leq (||T^*g||^2 + ||Sg||^2)$$

for all $g \in Dom\ T^* \cap Dom\ S$. Combining the former part of this section, we solved the $\overline{\partial}$ – problem of pseudoconvex domains in the sense of distributions: for all $f \in L^2_{(0,1)}(\Omega,\varphi)$, $\overline{\partial}f = 0$, there exists $u \in L^2(\Omega,\varphi)$ such that

$$\overline{\partial}u = f \text{ (extended)}, \quad ||u|| \leq \frac{1}{\sqrt{c}}||f|| \tag{3.41}$$

and u is orthogonal to all holomorphic functions in $L^2(\Omega,\varphi)$.

The next problem is the regularity properties of the solution u, i.e., when f have enough differentiability, the solution u to $\overline{\partial}u = f$ must also have appropriate differentiability. In this respect the weaker result is:

Theorem 12.6 (Inner regularity property theorem)**.** *For a pseudoconvex domain Ω with differentiable boundary, $\overline{\partial}u = f$. If $f \in C^\infty_{(0,1)}(\Omega)$, then $u \in C^\infty(\Omega)$.*

And stronger result is:

Theorem 12.7 (Kohn theorem)**.** *For a strictly pseudoconvex domain Ω, $\overline{\partial}u = f$. If $f \in C^\infty_{(0,1)}(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.*

We only discuss inner regularity property theorem in this study material. Setting

$$L^2(\Omega, loc) = \{g|\text{ for all } K \subset\subset \Omega, \text{ then } g \in L^2(K)\}, \tag{3.42}$$

we call it **Local L^2 space**.

Lemma 12.8. *If $\overline{\partial}u = f \in L^2(\Omega, loc)$, then 1-order differential of $u \in L^2(\Omega, loc)$.*

Proof. Obviously we only need to prove $\partial_i u\ (i = 1,\cdots,n) \in L^2(\Omega, loc)$. First we may assume u has a compact support in Ω. We know, from Friedrichs regularization, that there exist $u_\epsilon \in C^\infty$, which still have the compact support in Ω such that

$$u_\epsilon \longrightarrow u, \qquad \overline{\partial}_i u_\epsilon \longrightarrow \overline{\partial}_i u.$$

So

$$\begin{aligned}\int |\partial_i u_\epsilon|^2 &= \int (\partial_i u_\epsilon)\overline{(\partial_i u_\epsilon)} = -\int (\overline{\partial}_i\partial_i u_\epsilon)\overline{u_\epsilon}\\ &= -\int (\partial_i\overline{\partial}_i u_\epsilon)\overline{u_\epsilon} = \int |\overline{\partial}_i u_\epsilon|^2.\end{aligned}$$

Since $\overline{\partial}_i u_\epsilon \longrightarrow \overline{\partial}_i u$, there exists a constant c such that $\int|\overline{\partial}_i u_\epsilon|^2 < c$ independent on ϵ. But bounded sets in a Hilbert space are sequence compact, that is, the subsequence weakly converges. So we can assume that $\partial_i u_\epsilon \longrightarrow g$ (weak). Then, for every function $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{array}{l}(\partial_i u_\epsilon,\ \varphi) \longrightarrow (g,\ \varphi)\\ \quad\ \ \| \\ -(u_\epsilon,\ \overline{\partial}_i\varphi) \longrightarrow -(u, \overline{\partial}_i\varphi)\ (u_\epsilon \to u).\end{array}$$

So

$$(g,\ \varphi) = -(u,\ \partial_i\varphi).$$

Hence $g = \partial_i u$ exists, and $\partial_i u$ is local L^2. Later we can choose a cut-off function $\rho \geq 0$ with compact support in Ω such that $\rho \equiv 1$ in a more smaller compact set, thus

$$\overline{\partial}(\rho u) = (\overline{\partial}\rho)u + \rho\overline{\partial}u = (\overline{\partial}\rho)u + \rho f.$$

Obviously it is still in L^2, and ρu has compact support, then $\partial_i(\rho u) \in L^2$. Hence we have proved, to every compact support $K \subset \Omega$, we shall choose ρ so that $K \subset \{x|\rho \equiv 1\}$, then $\partial_i u \in L^2(K)$, i.e., $\partial_i u$ is local L^2. □

Now we'll prove inner regularity property theorem.

Proof. Let $\overline{\partial}u = f$. If f is differentiable up to order s (in the distribution sense) and local L^2, then

$$D^s f = D^s\overline{\partial}u = \overline{\partial}(D^s u).$$

From above lemma, we have $\partial(D^s u) \in L^2(\Omega, loc)$, which indicates u is differentiable up to order $\leq s+1$ in the distribution sense and local L^2. Then derivatives of all orders of f are local L^2, so u have derivatives of all orders which is local L^2. From famous Sobolev lemma, any function with derivative of order $\geq s + \frac{n}{2}$ in the distribution sense, and local L^2, is contained in $\mathbf{C}^s(\Omega)$ so that $u \in C^\infty(\Omega)$. □

Note: in this section, we only proved $\overline{\partial}u = f$, and f is the form of type (0,1), by using L^2 method of solving the $\overline{\partial}$ problem in a pseudoconvex domain. In fact, when f is the form of type $(0,\ p)$ $(p \leq n)$, $\overline{\partial}u = f$; u is the form of type $(0,\ p-1)$, one can still sovle it, using a similar proof.

12.4 Levi Problem

In this seciton, we will discuss Levi problem by applying $\overline{\partial}$ probl em. In history, the solution of Levi problem was first obtained by the method of coherent sheaf, then the method of L^2 estimate appeared. The advantage of L^2 estimate is that its solution posses naturally L^2 estimate, but it can not be applied to the spaces with singularity. The third method is using integral representation, its solution also has L^∞ estimate. It will be discussed in §5.

Problem 12.9 (Levi problem)**.** *If $\Omega \subset \mathbf{C}^n$ is a bounded domain, $\partial\Omega$ is differentiable, pseudoconvex, then Ω is a domain of holomorphy.*

Before prove Theorem 3.11, we recall the assumption of $\overline{\partial}$ problem on a pseudoconvex domain: If $\Omega \subset \mathbf{C}^n$, bounded, pseudoconvex and $\varphi \in C^\infty(\overline{\Omega})$,

$$\sum (\partial_i \overline{\partial}_j \varphi)\xi_i \overline{\xi}_j \geq c \sum |\xi_i|^2;\ c > 0, \tag{3.43}$$

the n the $\overline{\partial}$ problem has solutions and if $f \in C^\infty(\Omega)$, so $u \in C^\infty(\Omega)$. Now we first explain that the condition (3.43) can be reduced to that φ is plurisubharmonic(p.s.h.).

Lemma 12.10. *Let Ω be a pseudoconvex domain, and φ be p.s.h. in some neighborhood of $\overline{\Omega}$. If f is a $\overline{\partial}$ closed form of type* (0,1) *satisfying*

$$\int_\Omega |f|^2 e^{-\varphi - |z|^2} < +\infty \quad (|z|^2 = \sum z_i \overline{z_i}) \tag{3.44}$$

then there exists u such that $\overline{\partial} u = f$ and

$$\int_\Omega |u|^2 e^{-\varphi-|z|^2} \leq \int_\Omega |f|^2 e^{-\varphi - |z|^2}. \tag{3.45}$$

Proof. If $\varphi \in C^\infty$ and φ is p.s.h., then $(\partial_i \overline{\partial}_j \varphi) \geq 0$, so

$$\sum \partial_i \overline{\partial}_j (\varphi + |z|^2)\xi_i \overline{\xi}_j \geq \sum |\xi_i|^2. \tag{3.46}$$

In the solution to $\overline{\partial}$-pro blem, we replace $\varphi + |z|^2$ by φ. Next we note that (3.46) is equivalent to $c = 1$ in (3.43), it completes (3.45).

If we only assume φ is p.s.h. in some neighborhood of $\overline{\Omega}$, we can use the convolution $\varphi_\epsilon = \varphi * \chi_\epsilon$ so that $\varphi_\epsilon \searrow \varphi$. Let χ be a C^∞ function of $|z|$ and its support in $|z| \leq 1$, $\chi \geq 0$, $\int \chi = 1$. Set $\chi_\epsilon = \frac{1}{\epsilon^{2n}}\chi(\frac{z}{\epsilon})$. We only prove the theorem in the case $n = 1$, since it is similiar in the case $n > 1$. Let $t = re^{i\theta}$ and $d\sigma_t$ is the volume element of $\mathbf{C}'$,

$$\begin{aligned}
(\varphi * \chi_\epsilon)(z) &= \int \varphi(z-t)\chi_\epsilon(t) d\sigma_t = \int \varphi(z - re^{i\theta})\chi_\epsilon(r) r dr d\theta \\
&= \int_0^\epsilon \int_0^{2\pi} \varphi(z - r^{i\theta}) d\theta r \chi_\epsilon(r) dr \\
&\geq \left(2\pi \int_0^\epsilon \chi_\epsilon(r) r dr\right) \varphi(z) \\
&= \varphi(z) \int_0^\epsilon \int_0^{2\pi} \chi_\epsilon(r) r dr d\theta = \varphi(z).
\end{aligned}$$

Since φ is upper semicontinuous, it is locally bounded. Let $\underset{a\ neighborhood\ of\ \overline{\Omega}}{Sup} \varphi = M$, then

$$\varphi_\epsilon(z) = (\varphi * \chi_\epsilon)(z) = \int \varphi(z-t)\chi_\epsilon(t) \leq M \int \chi_\epsilon(t) = M.$$

We have $\varphi_\epsilon \longrightarrow \varphi$ ($\epsilon \to 0$), so $\varphi(z) \le \varphi_\epsilon(z) \le M$, and φ_ϵ are p.s.h., since

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \varphi_\epsilon(z+re^{i\theta}w)d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int \varphi(z+re^{i\theta}w-t)\chi_\epsilon(t)d\sigma_t d\theta \\ &= \int\left(\frac{1}{2\pi}\int_0^{2\pi}\varphi(z-t+te^{i\theta}w)\chi_\epsilon(t)d\theta\right)d\sigma_t \\ &\ge \int \varphi(z-t)\chi_\epsilon(t)d\sigma_t = \varphi_\epsilon(z).\end{aligned}$$

We choose a sequence φ_v in φ_ϵ so that $\varphi_v \longrightarrow \varphi$ ($\varphi \to \infty$). Applying $\overline{\partial}$-problem to φ_v (since they are C^∞), there exist u_v such that $\overline{\partial}u_v = f$ and

$$\int_\Omega |u_v|^2 e^{-\varphi_v-|z|^2} \le \int_\Omega |f|^2 e^{-\varphi_v-|z|^2} \le \int_\Omega |f|^2 e^{-\varphi-|z|^2} < +\infty.$$

Since $\varphi_v \le M$,

$$\int_\Omega |u_v|^2 e^{-M-|z|^2} \le \int_\Omega |u_v|^2 e^{-\varphi_v-|z|^2} < +\infty.$$

Hence $\{u_v\}$ is uniformly bounded in L^2 . But

$$\overline{\partial}(u_v - u_1) = 0.$$

This means that $\{u_v - u_1\}$ is a family of analytic functions on Ω. Also $\int |u_v - u_1|^2 \le \int |u_v|^2 + |u_1|^2 < +\infty$. To each compact set K in Ω, $\{u_v - u_1\}$ is uniformly bounded on K, so $\{u_v - u_1\}$ is a normal family.Hence there exists a subsequence (without loss of generality, we still assume $\{u_v - u_1\}$) which converges uniformly to an analytic function $u - u_1$ on any compact set of Ω, i.e.,

$$u_v - u_1 \longrightarrow u - u_1,$$

so

$$\overline{\partial}u = \overline{\partial}(u-u_1) + \overline{\partial}u_1 = f.$$

By Fatou lemma (the lemma is: $\int lim|f_n| \le \underline{lim}\int |f_n|$),

$$\begin{aligned}\int_\Omega |u|^2 e^{-\varphi-|z|^2} &= \int \lim_v |u_v|^2 e^{-\varphi_v-|z|^2} \\ &\le \underline{lim}\int |u_v|^2 e^{-\varphi_v-|z|^2} \\ &\le \int |f|^2 e^{-\varphi-|z|^2}.\end{aligned}$$

□

The lemma below indicates that the condition of φ can be reduced to that φ is only a p.s.h function on Ω.

Lemma 12.11. *The assumptions and results are as the same as lemma* 3.12*, except that φ is p.s.h. on a neighborhood of $\overline{\Omega}$ replaced by* Ω.

Proof. The difference with the proof of lemma 3.12 is that we can not use the method of $\varphi * \chi_\epsilon$. because φ is only defined on Ω and the definition domain of $\varphi * \chi_\epsilon$ is outside Ω.

By a result in §2, if Ω is a pseudoconvex domain, then $-log\ d \in C^\infty(\Omega)$ and p.s.h.. Let

$$\Omega_c = \{-log\ d < c\}.$$

By Sard theorem ($f:\ \Omega \longrightarrow \mathbf{R}^n$ is differentiable, the measure of the point set $\{f(x)|x \in \Omega,\ df(x) = 0\}$ is zero), for almost all c, the differential of $(-log\ d - c) \neq 0$ on $\partial\Omega_c$, since $-log\ d$ is p.s.h., and $\big(\partial_i\overline{\partial}_j(-1)log\ d\big) \geq 0$. So for these c, Ω_c are pseudoconvex domains.

Now if f is a one-form of type (0,1), satisfying $\overline{\partial}f = 0,\ \int |f|^2 e^{-\varphi-|z|^2} < +\infty$, then by lemma 3.12, there exist u_c on Ω_c such that

$$\overline{\partial}u_c = f\ |_{\Omega_c},$$

and

$$\int_{\Omega_c} |u_c|^2 e^{-\varphi-|z|^2} \leq \int_{\Omega_c} |f|^2 e^{-\varphi-|z|^2} \leq \int_\Omega |f|^2 e^{-\varphi-|z|^2}.$$

Obviously, if $d > c$, then $\Omega_c \subset \Omega_d,\ \overline{\partial}(u_c - u_d)\ |_{\Omega_c} = 0$, that is, $u_c - u_d$ is analytic on Ω_c.

Because φ is upper semicontinuous, it has superior limit on every compact subsets in Ω, so $e^{-\varphi-|z|^2}$ has inferior limit on every compact subset. Same as in the proof of lemma 3.12, for every compact subset $K \subset \Omega_c \cap \Omega_d,\ \{u_c - u_d\}$ is uniformly bounded on K. So, for every fixed c, $\{u_d - u_c\}_{d>c}$ is a normal family of holomorphic functions on Ω_c.

Now choose c_1 arbitrarily. In $\{u_d - u_{c_1}\}_{d>c_1}$, we choose a subsequence $\{u_{d_1} - u_{c_1}, u_{d_2} - u_{c_1}, \cdots\}$ converges on Ω_{c_1}. Let c_2 be sufficiently large in $\{d_1, d_2, \cdots\}$. In $\{u_{d_m} - u_{c_2}\}_{d_m > c_2}$, we choose a subsequence $\{u_{e_1} - u_{c_2}, u_{e_2} - u_{c_2}, \cdots\}$ converges on Ω_{c_2}. Note $\{e_i\}$ is a subsequence of $\{d_i\}$. Still let c_3 sufficiently large in $\{e_1, e_2, \cdots\}$, with the similar methods , we can get $c_j \to \infty$ and for every fixed i, $\{u_{c_j} - u_{c_i}\}$ $(j \to \infty)$ converges to an analytic function on Ω_{c_i}. Then, $u = \lim_j u_{c_j}$ exists and $(u - u_{c_i})$ is holomorphic on Ω_{c_i} for each i. So on Ω_{c_i}, $\overline{\partial}u = \overline{\partial}(u - u_{c_i} + u_{c_i}) = f$, hence $\overline{\partial}u = f$ for entire Ω.

By Fatou lemma,

$$\int_{\Omega_c} |u|^2 e^{-\varphi-|z|^2} \leq \underline{lim}_j \int_{\Omega_c} |u_{c_j}|^2 e^{-\varphi-|z|^2} \leq \int_\Omega |f|^2 e^{-\varphi-|z|^2}.$$

Let $c \to \infty$, then

$$\int_\Omega |u|^2 e^{-\varphi-|z|^2} \leq \int_\Omega |f|^2 e^{-\varphi-|z|^2}.$$

□

Now we can prove the Levi problem.

Theorem 12.12 (Levi conjecture)**.** *If $\Omega \subset \mathbf{C}^n$ is a bounded domain, $\partial\Omega$ is differentiable and pseudoconvex, then Ω is a domain of holomorphy.*

Proof. We need prove that, for each point a^* in $\partial\Omega$, there exists an analytic function on Ω which can not be analytically extension over a^*. Fix $a^* \in \partial\Omega$ arbitrarily. Let points a_i in Ω satisfy $a_i \longrightarrow a^*$. We shall construct an analytic function F satisfying $F(a_i) = i$ $(i = 1, 2, \cdots)$. Then the proof is completed.

For each a_i, we choose a neighborhood U_i, $a_i \in U_i$, and the intersection of every two of U_i is void. Let functions $\rho_i \in C^\infty$, $\rho_i \geq 0$, $Supp\ \rho_i \subset U_i$, and be equal to 1 in a more smaller neighborhood of a_i. Let $f = \overline{\partial}(\sum i\rho_i) \in C^\infty_{(0,1)}(\overline{\Omega})$ to solve $\overline{\partial}u = f$. By the solvability of $\overline{\partial}$-problem on a pseudoconvex domain and inner regularity theorem, the solution u exists and $u \in C^\infty(\Omega)$. Let $F = \sum i\rho_i - u$, then $\overline{\partial}F = \overline{\partial}(\sum i\rho_i) - \overline{\partial}u = f - \overline{\partial}u = 0$. Hence F is analytic on Ω. If we can prove $u(a_i) = 0$, then

$$F(a_i) = (\sum i\rho_i - u)(a_i) = i.$$

To prove $u(a_i) = 0$, we must use the estimate of $\overline{\partial}$-solution. By lemma 3.13, if a p.s.h. function φ on Ω satisfies

$$\int_\Omega |f|^2 e^{-\varphi-|z|^2} < +\infty,$$

then the solution u to $\overline{\partial}u = f$ has the estimate

$$\int_\Omega |u|^2 e^{-\varphi-|z|^2} \leq \int_\Omega |f|^2 e^{-\varphi-|z|^2} < +\infty.$$

If we can choose this p.s.h. function φ such that φ descends fast enough in a neighborhood of a_i, $\varphi(a_i) = -\infty$ (note p.s.h. function can have value $-\infty$), then by $e^{-\varphi-|z|^2}(a_i) = +\infty$ (fast enough), if $u(a_i) \neq 0$ (note u is C^∞), it contradicts to $\int |u|^2 e^{-\varphi-|z|^2} < +\infty$. Hence the choice of φ must satisfy $\int |f|^2 e^{-\varphi-|z|^2} < +\infty$ at the same time. By $f = \overline{\partial}(\sum i\rho_i)$, $\sum i\rho_i$ is equal to i near a_i, so f vanishes near a_i, it is possible to choose this φ. How do we choose φ? First we let $\psi = \sum_m \rho_m log|z - a_m|$, $\psi \in C^\infty$ except $z = a_m$ $(m = 1, 2, \cdots)$, its support $\subset \bigcup_m Supp\ \rho_m$. Usually ψ is not p.s.h.. Consider $\chi = -log\ d + |z|^2$. Since Ω is pseudoconvex, $-log\ d$ is C^∞ and p.s.h., then $-log\ d + |z|^2$ is strictly p.s.h., obviously, as $z \to \partial\Omega$, $\chi(z) \longrightarrow \infty$. Choose a function $\sigma : R \longrightarrow R$ with $\sigma' \geq 0$, $\sigma'' \geq 0$. We will show that $\varphi = \sigma \circ \chi + \psi$ satisfying the conditions mentioned above:

1^0 φ is p.s.h.

In fact,

$$\partial_i\overline{\partial}_j(\sigma \circ \chi) = (\sigma' \circ \chi)\partial_i\overline{\partial}_j\chi + (\sigma'' \circ \chi)\partial_i\chi \circ rline\partial_j\chi;$$

$$\begin{aligned}\sum \partial_i\overline{\partial}_j(\sigma\circ\chi)\xi_i\overline{\xi}_j &= \sum(\sigma'\circ\chi)(\partial_i\overline{\partial}_j\chi)\xi_i\overline{\xi}_j + (\sigma''\circ\chi)\left|\sum(\partial_i\chi)\xi_i\right|^2\\ &\geq (\sigma'\circ\chi)\sum|\xi_i|^2,\end{aligned}$$

where, in above, we used the property that $\sigma' \geq 0$, and $(\partial_i\overline{\partial}_j\chi) = \left(\partial_i\overline{\partial}_j(-log\ d + |z|^2)\right) \geq I$. Therefore, when $z \notin Supp\ \psi \subset \bigcup_m Supp\ \rho_m$,

$$\begin{aligned}\sum(\partial_i\overline{\partial}_j\varphi)\xi_i\overline{\xi}_j &= \sum \partial_i\overline{\partial}_j(\sigma\circ\chi+\psi)\xi_i\overline{\xi}_j\\ &= \sum \partial_i\overline{\partial}_j(\sigma\circ\chi)\xi_i\overline{\xi}_j \geq 0.\end{aligned}$$

But z is in a sufficiently small neighborhood near a_m, and $\rho_m \equiv 1$,

$$\left(\partial_i\overline{\partial}_j 2log|z-a_m|\right) = \frac{\delta_{ij}}{|z-a_m|^2} - \frac{1}{|z-a_m|^4}\left((z_i-a_m^i)(\overline{z}_j-\overline{a}_m^j)\right);$$

$$\begin{aligned}&\sum_{i,j}\partial_i\overline{\partial}_j 2log|z-a_m|\xi_i\overline{\xi}_j = \frac{1}{|z-a_m|^2}\sum|\xi_i|^2 - \frac{1}{|z-a_m|^4}\left|\sum(z_i-a_m^i)\xi_i\right|^2\\ &\geq \frac{1}{|z-a_m|^2}\sum|\xi_i|^2 - \frac{1}{|z-a_m|^4}\left(\sum|\xi_i|^2\right)|z-a_m|^2\\ &\geq 0.\end{aligned}$$

At other points in $Supp\ \rho_m$, $(\partial_i\overline{\partial}_j\psi)$ may be negative definite, but they are bounded. If σ' increase fast enough (σ'' larger), then

$$(\sigma'\circ\chi)(\sum|\xi_i|^2) + \sum(\partial_i\overline{\partial}_j\psi)\xi_i\overline{\xi}_j \geq 0.$$

Thus $\varphi = \sigma\circ\chi+\psi$ is p.s.h..

2^0

$$\int |f|^2 e^{-\varphi-|z|^2} < +\infty.$$

In fact, because $f \equiv 0$ in the sufficientlly small neighborhoods of every a_m, $|f|^2e^{-\psi-|z|^2}$ are locally bounded except for these small neig hborhoods. If we choose σ satisfying $\sigma(x) \longrightarrow +\infty$ fast enough as $x \to +\infty$, then, as $z \to \partial\Omega$, $\chi(z) \longrightarrow \infty$. So $(\sigma\circ\chi)(z) \longrightarrow +\infty$, $e^{-\sigma\circ\chi(z)} \longrightarrow 0$, and

$$\int |f|^2 e^{-\psi-|z|^2}\cdot e^{-\sigma\circ\chi} = \int |f|^2 e^{-\varphi-|z|^2} < +\infty.$$

The concrete constrution is choosing σ' first, then defining σ by $\sigma = \int\sigma'$. The proof of the theorem is completed. □

12.5 Homander's Theorem

The above method of using Lax-Milgram Lemma and the Morrey Tirck can be extended to any "psudoconvex" domain in a Kahler manifold to solve $\bar{\partial}$-equation for L-value forms where L is a Hermitian line bundle, using the notions diesucced in the previous chapter.

We first discuss the notion of *pesudoconvexity*. Let X be a complex manifold and $Y \subset\subset X$ an open subset whose boundary ∂Y is smooth and of real codimension 1. For each $x \in \partial Y$ thre is a neighborhood U on $x \in$ and a smooth function $\rho : \bar{U} \to \mathbf{R}$ such that $U \cap \partial Y = \{\rho = 0\}$ and $d\rho|_{U\cap\partial Y}$ is nowhere zero. The *complex tangent space* to ∂Y at $x \in \partial Y$ is the collection of all vectors $v \in T_{X,x}$ such that $v \in T_{\partial Y,x}$ and $Jv \in T_{\partial Y,x}$, where J is the almost complex structure on X associated to the complex structure. We write

$$v \in T^{1,0}_{\partial Y,x}.$$

Note that if $v \in T^{1,0}_{\partial Y,x}$ then $d\rho|_{U\cap\partial Y}(x)v = 0$ and $Jd\rho|_{U\cap\partial Y}(x)Jv = 0$, and thus

$$\partial\rho(x)v = 0.$$

Conversely, if $\partial\rho(x)v = 0$, then $\partial\rho(x)Jv = J\partial\rho(x)v = 0$, and thus we see that

$$T^{1,0}_{\partial Y,x} = \text{Kernel } \partial\rho(x).$$

Next we pursue a notion of curvature of the boundarythat is a ppropriate in complex geoemtry. With this pursit in mind, consider the $(1,1)$-form $\partial\bar{\partial}\rho(x)$ on the boundary ∂Y. We say that the point $x \in \partial Y$ is *pseudoconvex boundary point* if for all $v \in T^{1,0}_{\partial Y,x}$,

$$\partial\bar{\partial}\rho(x)(v,\bar{v}) \geq 0.$$

Observe that if ρ is replaced by $h\rho$ for some smooth positive function h, then

$$\partial\bar{\partial}(h\rho) = h\partial\bar{\partial}\rho + \bar{\partial}\rho \wedge \overline{\partial h} + \bar{\partial}h \wedge \overline{\partial\rho}.$$

It follows that for $v, w \in T^{1,0}_{\partial Y,x}$,

$$\partial\bar{\partial}(h\rho(x))(v,\bar{v}) = h(x)\partial\bar{\partial}\rho(x)(v,\bar{v}).$$

Thus the notion of pseudoconvexity does not depend on the choice of the function ρ. The form

$$\mathcal{L}_x := \partial\bar{\partial}\rho(x)(v,\bar{w})$$

is called the *Levi form.*

Recall that from Bochner-Kodaira's formula (see previous section) for any smooth E-valued $(0,q)$-form ϕ,

$$(\Box\phi,\phi) = \|\bar{\nabla}\phi\|_M + (Ric\phi,\phi)_M + (\Omega\phi,\phi)_M, \tag{7.5}$$

where

$$\|\bar{\nabla}\phi\|_M = \int_M g^{i\bar{j}}\overline{\nabla}_j\phi^{\alpha}_{\bar{J}_q}\overline{\overline{\nabla}_i\phi^{j_1\cdots j_q}_{\bar{\alpha}}},$$

$$(Ric\phi,\phi)_M = -\sum_{k=1}^{q}\int_M R^{\bar{s}}_{\bar{j}_k}\phi^{\alpha}_{\bar{j}_1\cdots(\bar{s})_k\cdots\bar{j}_q}\overline{\phi^{j_1\cdots j_q}_{\bar{\alpha}}},$$

$$(\Omega\phi,\phi)_M = \sum_{k=1}^{q}\int_M \Omega^{\bar{l}}_{\bar{j}_k}\phi^{\alpha}_{\bar{j}_1\cdots(\bar{l})_k\cdots\bar{j}_q}\overline{\phi^{j_1\cdots j_q}_{\bar{\alpha}}},$$

where Ω is the curvature form of E. Notice that

$$(\Box\phi,\phi) = \|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2,$$

so the Lax-Milgram Lemma can be applied. Using the same Morrey's trick, we can get

Theorem (Hormander) *Let (X,g) be a Kahler manifold and let $L \to X$ be a holomorphic line bundle with Hemitian metric h having the curvature Ω such that*

$$(Ric\phi,\phi)_M + (\Omega\phi,\phi)_M \geq c\|\phi\|^2$$

for some positive constant c. Let Y be a pesudoconvex domain in X. Then, for each L-valued (p,q) form ω such that

$$\int_Y |\omega|^2_{h,g}dV < +\infty \quad \text{and} \quad \bar{\partial}\omega = 0$$

in the sese of distribution, there exists a L-valued $(p,q-1)$ form u such that

$$\bar{\partial}u = \omega \quad \text{and} \quad \int_Y |u|^2_{h,g}dV \leq \frac{1}{c}\int_Y |\omega|^2_{h,g}dV.$$

As a consequence of Hormander's theormre, we re-proves the Kodaria's vanishing theorem.

Chapter 13

Positive Closed Currents Theory

13.1 Plurisubharmonic functions

When $n = 1$, for a C^2-function u defined on an open subset $\Omega \subset \mathbf{C}$, we recall that u is harmonic on $\Omega \iff \triangle u = 0 \iff$ locally $u = Re(f)$ for $f \in \mathcal{O} \iff dd^c u = 0 \iff \forall a \in \Omega, \triangle(u, |\zeta|) \subset \Omega$ such that $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + \zeta e^{i\theta})d\theta$. And u is subharmonic on Ω if and only if $u; \Omega \to [-\infty, +\infty)$ is semicontinuous such that

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + \zeta e^{i\theta})d\theta.$$

As an example, for any local holomorphic function f, $\log |f|$ is subharmonic.

When $n \geq 1$, for any C^2-function u defined on an open subset $\Omega \subset \mathbf{C}^n$, we define

$$u \text{ is harmonic on } \Omega \iff u \in H(\Omega) \iff \triangle u = 0.$$

$$u \text{ is subharmonic on } \Omega \iff u \in SH(\Omega) \iff \triangle u \geq 0.$$

$$u \text{ pluriharmonic on } \Omega \iff u \in PH(\Omega) \iff dd^c u = 0.$$

$$u \text{ is plurisubharmonic on } \Omega \iff u \in PSH(\Omega) \iff dd^c u \geq 0.$$

We have

$$PH(\Omega) \underset{\neq}{\subset} H(\Omega)$$

$$PSH(\Omega) \underset{\neq}{\subset} SH(\Omega) \subset L^1_{loc}(\Omega)$$

$$PH(\Omega) \underset{\neq}{\subset} PH(\Omega), H(\Omega) \underset{\neq}{\subset} SH(\Omega).$$

The condition of C^2-smooth is in general not required to define harmonic functions.

Definition *Let $u : \Omega \subset \mathbf{R}^m \to \mathbf{R}$ be continuous. u is said to be harmonic if $u \not\equiv -\infty$ on each connected component of Ω, and $\forall B(a,r) \subset \Omega$,*

$$u(a) = A(u, ar) := \frac{1}{\lambda(B(0,1))r^m} \int_{B(a,r)} u(x) d\lambda(x)$$

where λ is the Lebesgue mesaure on $\mathbf{R}^m$.

Definition *Let $u : \Omega \subset \mathbf{R}^m \to [-\infty, \infty)$ be upper semicontinuous. u is said to be subharmonic if $u \not\equiv -\infty$ on each connected component of Ω, and $\forall B(a,r) \subset \Omega$,*

$$u(a) \leq A(u, ar) := \frac{1}{\lambda(B(0,1))r^m} \int_{B(a,r)} u(x) d\lambda(x)$$

where λ is the Lebesgue mesaure on $\mathbf{R}^m$.

Definition *Let $u : \Omega \subset \mathbf{R}^m \to \mathbf{R}$ be upper semicontinuous. u is said to be plurisubharmonic on Ω if $u \not\equiv -\infty$ on each connected component of Ω, and for every complex line l, $u|_{\Omega \cap l}$ is subharmonic or $u|_{\Omega \cap l} \equiv -\infty$.*

Remarks:

- $\log |f|^2 \in PSH(\Omega)$, for any $f \in \mathcal{O}(\Omega)$. Notice that $\log(|f_1|^2 + \cdots + |f_m|^2)$ may not be in $PH(\Omega)$ for $m > 1$. But we always have

$$\log(|f_1|^2 + \cdots + |f_m|^2) \in PSH(\Omega) \subset SH(\Omega) \subset L^1_{loc}(\Omega).$$

- If $u \in C^2(\Omega)$, then

$$u \in PSH(\Omega) \Longleftrightarrow \left(\frac{\partial^2}{\partial z_j \bar{\partial} z_k}\right) \text{is semipositive definite matrix.}$$

- If $u_k \in PSH(\Omega), u_k \searrow u$, then $u = \lim_k u_k \in PSH(\Omega)$.
- Let $u \in PSH(\Omega)$. Then $u \star \rho_\epsilon \in C^\infty \cap PSH(\Omega_\epsilon)$ and $\Omega_\epsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \epsilon\}$.

13.2 Currents

Recall that if $f, g \in C^0[0,1]$, then $f \equiv g$ if and only if $\int_0^1 f(x)\phi(x) = \int_0^1 g(x)\phi(x)$ for every $\phi \in C_0^\infty[0,1]$. Also for closed intervals $A, B \subset \mathbf{R}$, $A = B$ if and only if $\int_A \phi(x) = \int_B \phi(x)$. Here "functions" and "subsets" can be regarded as **linear functional forms** on $C_0^\infty[0,1]$. These concepts are unified by a general concept of *currents*: Let M be a real differentiable manifold with $\dim M = m$. A *current* of degree $q = m - p$ (or dimension p) is a real linear map $T : \mathcal{D}^p(M) \to \mathbf{R}$, such that for any compact subset K of M, there exists constant C_K with

$$|T(\phi)| \leq C_K \sup_K |\phi|_N, \quad \forall \phi \in \mathcal{D}^p(M) \ \mathrm{supp}(\phi) \subset K$$

where $|\phi|_N = \sum_{|I|\leq N} |D^I\phi|$, and where $\mathcal{D}^p(M)$ is the set of smooth p-forms on M with compact support. The set of currents of degree $q = m - p$ is denoted by $\mathcal{D}'^q(M) = \mathcal{D}'_p(M)$. Typical examples are smooth or L^1_{loc} q-forms β with $T = [\beta]$ is defined by $T(\phi) = \int_M \beta \wedge \phi$ for $\phi \in \mathcal{D}^p(M)$ with $q = m - p$, as well as p-dimensional oriented submanifold $S \subset M$ with $T = [S]$ defined as $T(\phi) = \int_S \phi$.

For any $T \in \mathcal{D}'^q(M)$, define $dT \in \mathcal{D}'^{q+1}(M)$ by

$$dT(\phi) = (-1)^{q+1}T(d\phi), \forall \phi \in \mathcal{D}_{m-q-q}(M).$$

We say that T is *closed* if $dT = 0$.

Notice that Stoke's theorem implies that

$$\int_S d\phi = \int_{\partial S} \phi$$

meaning that

$$d[S] = (-1)^{m-p+1}[\partial S].$$

Now consider a complex manifold X with $n = \dim X$. We define

$$\mathcal{D}^{p,q}(X) = \text{the set of smooth } (p,q) - \text{forms with compact support},$$

and

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X) = \text{the set of all } (p,q) - \text{currents}.$$

$T \in \mathcal{D}'_{n-p,n-q}(X)$ is called *(weekly) positive* if $\forall(1,0)$-form $\alpha_1, \dots, \alpha_p$ on X,

$$T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge \alpha_p \wedge \bar{\alpha}_p$$

is a positive measure, i.e. $(T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge \alpha_p \wedge \bar{\alpha}_p)(\phi) \geq 0, \forall \phi \in C_0^\infty(X)$ with $\phi \geq 0$. We denote $T \geq 0$.

Example If $u \in C^2(\Omega) \cap PSH(\Omega)$ where $\Omega \subset \mathbf{C}^n$, then the matrix

$$\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}\right) \geq 0, \quad \text{i.e. semipisitive definite,} \Longleftrightarrow \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}, \forall \zeta(\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$$

so that $T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u \geq 0$ is semipositive definite, and hence is a positive current.
Example If $u \in L^1_{loc}(\Omega)$, then

$$u \in PSH(\Omega) \Longleftrightarrow T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u \geq 0$$

as positive current.

Theorem *(1) If $u \in PSH(\Omega)$, then $T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ is a positive current.*

(2) ($\partial\bar{\partial}$-Poincare lemma) *Let T be a closed positive $(1,1)$-curent. Then locally*

$$T = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$$

for some $u \in PSH(\Omega)$.

Theorem (Poincare-Lelong) formula: *Let X be a complex manifold and $f \in \mathcal{O}(X)$ be a holomorphic function. Then*

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f|^2 = [f=0] \in \mathcal{D}'^{1,1}(X)$$

holds as currents.

Proof. We only prove $n = 1$. It is then a local problem so that we can consider $f(z) = z^m g(z)$, where g is defined on a neighborhood U of 0 and $g(z) \neq 0$ on U. Then

$$\partial\bar{\partial}\log|f|^2 = \partial\bar{\partial}\log|z|^{2m}$$

So we only need to show that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^{2m} = [zero(z^m)] \in \mathcal{D}'^{1,1}(\mathbf{C}).$$

In fact, $\forall\phi$,

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^{2m}(\phi) = \frac{\sqrt{-1}}{2\pi}\int_{\mathbf{C}}(\partial\bar{\partial}\log|z|^{2m})\phi$$

$$= -\frac{\sqrt{-1}}{2\pi}\int_{\mathbf{C}}\bar{\partial}\partial\log|z|^{2m})\phi \quad \text{using } \partial\bar{\partial}+\bar{\partial}\partial = 0$$

$$= -\frac{\sqrt{-1}}{2\pi}\int_{\mathbf{C}} d(\partial\log|z|^{2m})\phi \quad \text{using } \partial^2 = 0$$

$$= -\frac{\sqrt{-1}m}{2\pi}\int_{\mathbf{C}}(\partial\log|z|^2)\wedge\bar{\partial}\phi \quad \text{using the fact that } \mathrm{supp}(\phi) \subset \triangle \subset \mathbf{C}$$

$$= -\frac{\sqrt{-1}m}{2\pi}\int_{\mathbf{C}}\frac{\bar{z}}{z}\wedge\frac{\partial\phi}{\partial\bar{z}}d\bar{z} = -\frac{\sqrt{-1}m}{2\pi}\int_{\mathbf{C}}\frac{\partial\phi}{\partial\bar{z}}\frac{dz\wedge d\bar{z}}{z-0}$$

$$= m\phi(0). \quad \text{by Cauchy's integral formula}$$

On the other hand,

$$[zero(z^m)](\phi) = m[\{0\}](\phi) = m\phi(0).$$

This proves the theorem for the case $n = 1$. The case when $n > 1$ is similar.

13.3 Simgular metric

Kodaria's vanishing theorem has been extended by Nadel to line bundles with singular metric (i.e. $h = \{h_\alpha\}$, where h_α may be singular). We write $h_\alpha = e^{-\kappa_\alpha}$, here we usually write $\kappa := \kappa_\alpha$ if no risk of confusion, then the curvature $\Theta_h := \partial\overline{\partial}\kappa$ is not a smooth differential form anymore if the metrics singular(it is in fact is called *current*). We say that e^κ has non-negative (reps. positive) curvature current if Θ_h is a non-negative (reps. (1,1)- current, or equivalently, the local representatives κ are plurisubharmonic.

- Currents: Recall that if $f, g \in C^0[0,1]$, then $f \equiv g$ if and only if $\int_0^1 f(x)\phi(x) = \int_0^1 g(x)\phi(x)$ for every $\phi \in C_0^\infty[0,1]$. Also for closed intervals $A, B \subset \mathbf{R}$, $A = B$ if and only if $\int_A \phi(x) = \int_B \phi(x)$. Here "functions" and "subsets" can be regarded as **linear functional forms** on $C_0^\infty[0,1]$. These concepts are unified by a general concept of *currents*: Let M be a real differentiable manifold with $\dim M = m$. A *current* of degree $q = m - p$ (or dimension p) is a real linear map $T : \mathcal{D}^p(M) \to \mathbf{R}$, where $\mathcal{D}^p(M)$ is the set of smooth p-forms on M with compact support. Typical examples are smooth or L^1_{loc} q-forms β with $T = [\beta]$ is defined by $T(\phi) = \int_M \beta \wedge \phi$ for $\phi \in \mathcal{D}^p(M)$ with $q = m - p$, as well as p-dimensional oriented submanifold $S \subset M$ with $T = [S]$ defined as $T(\phi) = \int_S \phi$.

- Poincare-Lelong formula: Let $f \in \mathcal{O}(M)$ be a holomorphic function. Then

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f|^2 = [f = 0]$$

  holds as currents.

- Example of singular metric: Let $L \to M$ be a holomorphic line bundle. Let m be a positive integer and $s^1, \ldots, s^N$ be sections of mL. Write $s = s_\alpha e_\alpha$, and define

$$\kappa_\alpha = \frac{1}{m}\log(|s^1_\alpha|^2 + \cdots + |s^N_\alpha|^2).$$

  This singular metric blows up exactly on the common zeros of the sections $s^1, \ldots, s_N$.

- Let $U \subset M$ be an open subset, and let ϕ be a locally integrable function on U. We define

$$\mathcal{I}(U) := \{f \in \mathcal{O}_M(U) : |f|^2 e^{-\phi} \in L^1_{loc}(U)\}.$$

  The corresponding sheaf of germs $\mathcal{I}_\phi$ is called the multiplier ideal sheave associated to ϕ.

- Nadel proved that if ϕ is a plurisubharmonic, then the multiplier ideal sheave $\mathcal{I}_\phi$ is a coherent sheaf of ideals.

- Nadel's Vanishing Theorem: *Let X be a compact Kahler manifold with Kahler form ω, and let F be a line bundle with singular Hermitian metric $h = e^{-\phi}$ such that $\sqrt{-1}\partial\bar{\partial}\phi \geq \epsilon\omega$ for some continuous function $\epsilon > 0$ (in the sense of distribution). Then, for $q \geq 1$, $H^q(X, \mathcal{O}_{\mathcal{X}}(\mathcal{K}_{\mathcal{X}} + \mathcal{F}) \otimes \mathcal{I}_\phi)$ =0 where K_X is the canonical line bundle of X.*

  The point of the proof is that *any plurisubharmonic function is the limit of a decreasing sequence of smooth plurisubharmonic functions*, so eventually it can be reduced to the smooth case.

- Lelong numbers of plurisubharmonic functions: The zero of the ideal sheaf $\mathcal{I}_\phi$ is then the set of points where $e^{-\phi}$ is not locally integrable. Such points only occur where ϕ has poles, but the poles need to have a sufficiently high order. If $\phi = \frac{1}{m}\log(|s_\alpha^1|^2 + \cdots + |s_\alpha^N|^2)$ as in the example earlier, then one has a notion of (log)-pole order. In general, the pole orders are defined using the so-called Lelong numbers: Let X be a complex manifold and ϕ a plurisubharmonic function in a neighborhood U of $x \in X$. Fix a coordinate chart U near x, and let $z$be a local coordinates vanishing on x. The Lelong number of ϕ is defined to be the number

$$\upsilon(\phi, x) := \liminf_{z \to x} \frac{\phi(z)}{\log|x - z|^2}.$$

  We also set

$$E_c(\phi) := \{x \in X; \upsilon(\phi, x) \geq c\}.$$

- A famous paper of Siu showed that $E_c(\phi)$ is a complex analytic set.

- The Lelong number information $\upsilon(\phi, x)$ gives the information about the vanishing order of f at x for $f \in \mathcal{I}_{\phi,x}$ which is stated as the lemma of Skoda: *Let ϕ a plurisubharmonic function on an open set U of X containing x. Then (1). If $\upsilon(\phi, x) < 1$, then $e^{-\phi}$ is integrable in a neighborhood of x. In particular, $\mathcal{I}_{\phi,x} = \mathcal{O}_{U,x}$; (2). If $\upsilon(\phi, x) \geq n + s$ for some positive integer, then the estimate $e^{-\phi} \geq C|z - x|^{-2(n+s)}$ holds in a neighborhood of x, In particular, one obtains that $\mathcal{I}_{\phi,x} \subset m_{U,x}^{s+1}$, where $m_{U,x}$ is the maximal ideal of $\mathcal{O}_{U,x}$; 3. The zero variety $V(\mathcal{I}_\phi)$ of $\mathcal{I}_\phi$ satisfies $E_{2n}(\phi) \subset V(\mathcal{I}_\phi) \subset E_2(\phi)$.*

- Nadel's vanishing theorem plus Skoda's lemma gives a new proof (without using blow-ups) of Kodaira's embedding theorem: *Let X be a compact Kähler manifold. Assume there exists a positive line bundle L over X, then X can be embedded in projective space $\mathbf{P}^N$.*

- To prove the embedding theorem, it gets down to construct holomorphic sections. Consider the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathcal{I}_\phi \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_\phi \to 0$$

twisted by $\mathcal{O}(K_x \otimes L)$, and apply Nadel's vanishing theorem of the first H^1 group, we'll have: *Let X be a weakly pseudo-convex Kahler manifold with Kahler form ω, and let F be a line bundle with singular Hermitian metric $h = e^{-\phi}$ such that $\sqrt{-1}\partial\bar{\partial}\phi \geq \epsilon\omega$ for some continuous function $\epsilon > 0$. Let $x_1, \ldots, x_N$ be isolated points in the zero variety $V(\mathcal{I}_\phi)$. Then there is a surjective map*

$$H^0(X, K_X \otimes L) \to \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X \otimes L)_{x_j} \otimes (\mathcal{O}_X/\mathcal{I}_\phi)_{x_j}.$$

- Exercise: Assume that X is compact and L is a positive line bundle. Let $\{x_1, \ldots, x_N\}$ be a finite set. Show that there are constants $a, b \geq 0$ depending only on L and N such that $H^0(X, L^{\otimes m})$ generates jets of any order s at all points x_j for $m \geq as + b$,

  Hint. Apply the above Corollary to $L' = K_X^{-1} \otimes L^{\otimes m}$, with a singular metric on L of the form $h = h_0 e^{-\epsilon\psi}$, where h_0 is smooth of positive curvature, $\epsilon > 0$ small and

  $$\psi(z) = \sum \chi_j(z)(n + s - 1) \log \sum |w^{(j)}(z)|^2$$

  with respect to coordinate systems $(w_k^{(j)}(z))_{1 \leq k \leq n}$ centered at x_j. The cut-off functions χ_j can be taken of a fixed radius (bounded away from 0) with respect to a finite collection of coordinate patches covering X. It is easy to see such h serves our purposes.

- Taking $s = 2$ and m with $m \geq 2a + b$ as in the Exercise, then the sections of $H^0(X, L^{\otimes m})$ generates any pair of $L_x \oplus L_y$ for distinct points $x \neq y$ in X, as well as 1-jets of L at any point $x \in X$. The existence of the section of $H^0(X, L^{\otimes m})$ which generates any pair of $L_x \oplus L_y$ for distinct points $x \neq y$ in X implies that F is injective. Now, we use the fact that there is a section s of $H^0(X, L^{\otimes m})$ which generates 1-jets of L at any point $x \in X$, ie. the section s vanishes to the second order. Choosing sections $s^1, \cdots, s^n$ such that the function $s^1/s, \cdots, s^n/s$ have independent differential at x, then the holomorphic map

  $$\left(\frac{s^1}{s}, \cdots, \frac{s^n}{s}\right)$$

  defined in a neighborhood of x is an immersion near x. This complete the proof of Kodaira's imbedding theorem.