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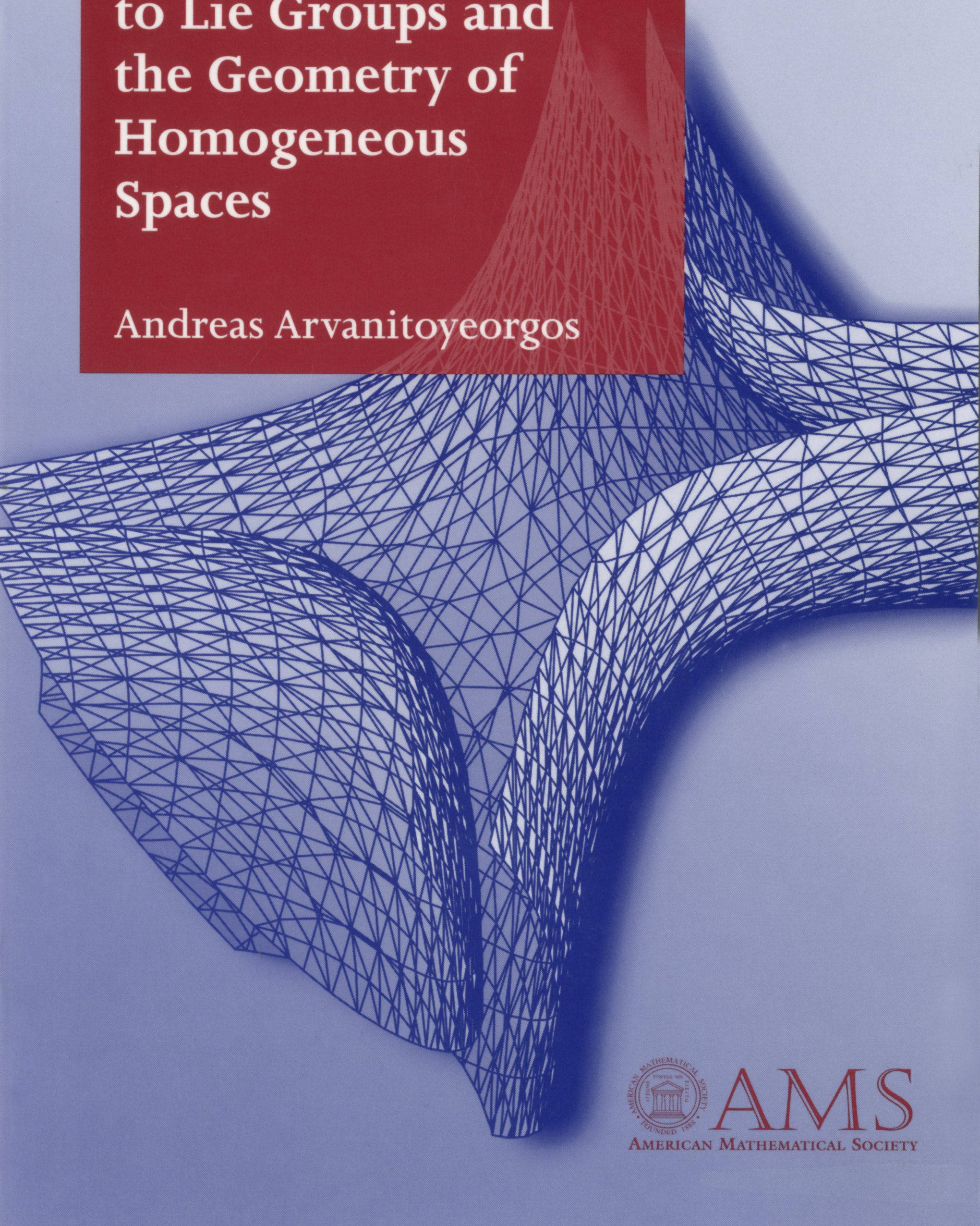
# An Introduction to Lie Groups and the Geometry of Homogeneous Spaces

Andreas Arvanitoyeorgos



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# Preface

The roots of this book lie in a series of lectures that I presented at the University of Ioannina, in the summer of 1997. The central theme is the geometry of Lie groups and homogeneous spaces. These are notions which are widely used in differential geometry, algebraic topology, harmonic analysis and mathematical physics. There is no doubt that there are several books on Lie groups and Lie algebras, which exhaust these topics thoroughly. Also, homogeneous spaces are occasionally tackled in more advanced textbooks of differential geometry.

The present book is designed to provide an introduction to several aspects of the geometry of Lie groups and homogeneous spaces, without becoming too detailed. The aim was to deliver an exposition at a relatively quick pace, where the fundamental ideas are emphasized. Several proofs are provided, when it is necessary to shed light on the various techniques involved. However, I did not hesitate to mention more difficult but relevant theorems without proof, in appropriate places. There are several references cited, that the reader can consult for more details.

The audience I have in mind is advanced undergraduate or graduate students. A first course in differential geometry would be desirable, but is not essential since several concepts are presented. Also, researchers from neighboring fields will have the chance to discover a pleasant introduction on a variety of topics about Lie groups, homogeneous spaces and related applications.

I would like to express my sincere thanks to the editors for their thorough suggestions on the manuscript, as well as my gratitude to Professors Jurgen Berndt, Martin A. Guest, Lieven Vanhecke, and McKenzie Y. Wang for their kindness in making comments on it.

Andreas Arvanitoyeorgos

Athens, August 2003

# Introduction

There are several terms which are included in the title of this book, such as "Lie groups", "geometry", and "homogeneous spaces", so it maybe worthwhile to provide an explanation about their relationships. We will start with the term "geometry", which most readers are familiar with.

Geometry comes from the Greek word "γεωμετρείν", which means to measure land. Various techniques for this purpose, including other practical calculations, were developed by the Babylonians, Egyptians, and Indians. Beginning around 500 BC, an amazing development was accomplished, whereby Greek thinkers abstracted a set of definitions, postulates, and axioms from the existing geometric knowledge, and showed that the rest of the entire body of geometry could be deduced from these. This process led to the creation of the book by Euclid entitled *The Elements.* This is what we refer to as *Euclidean geometry.*

However, the fifth postulate of Euclid (the parallel postulate) attracted the attention of several mathematicians, basically because there was a feeling that it would be possible to prove it by using the first four postulates. As a result of this, new geometries appeared (elliptic, hyperbolic), in the sense that they are consistent without using Euclid's fifth postulate. These geometries are known as *Non-Euclidean Geometries*, and some of the mathematicians that contributed to their development were N. I. Lobachevsky, J. Bolyai, C. F. Gauss, and E. Beltrami.

A detailed theory of surfaces in three-dimensional space was developed by C. F. Gauss. His main result was the *Theorema Egregium*, which states that the curvature of a surface is an "intrinsic" property of the surface. This means it can be measured and "felt" by someone who is on the surface, rather than only by observing the surface from outside.

However, the fundamental question "What is geometry?" still remained. There are two directions of development after Gauss. The first, is related to the work of B. Riemann, who conceived a framework of generalizing the theory of surfaces of Gauss, from two to several dimensions. The new objects are called *Riemannian manifolds*, where a notion of curvature is defined, and is allowed to vary from point to point, as in the case of a surface. Riemann brought the power of calculus into geometry in an emphatic way as he introduced metrics on the spaces of tangent vectors. The result is today called *differential geometry.*

The other direction is the one developed by F. Klein, who used the notion of a transformation group to define geometry. According to Klein, the objects of study in geometry are the invariant properties of geometrical figures under the actions of specific transformation groups. Hence, the consideration of different transformation groups leads to different kinds of geometry, such as Euclidean geometry, affine geometry, or projective geometry. For example, Euclidean geometry is the study of those properties of the plane that remain invariant under the group of rigid motions of the plane (the Euclidean group). The groups that were available at that time, and which Klein used to determine various geometries, were developed by the Norwegian mathematician Sophus Lie, and are now called *Lie groups.*

This brings us to the other terms of the title of this book, namely "Lie groups" and "homogeneous spaces". The theory of Lie has its roots in the study of symmetries of systems of differential equations, and the integration techniques for them. At that time, Lie had called these symmetries "continuous groups". In fact, his main goal was to develop an analogue of Galois theory for differential equations.

The equations that Lie studied are now known as equations of Lie type, and an example of these is the well-known Riccati equation. Lie developed a method of solving these equations that is related to the process of "solution by quadrature" (cf. [**Fr-Uh, pp. 14, 55**], [**Ku**]). In Galois' terms, for a solution of a polynomial equation with radicals, there is a corresponding finite group. Correspondingly, to a solution of a differential equation of Lie type by quadrature, there is a corresponding continuous group.

The term "Lie group" is generally attributed to E. Cartan (1930). It is defined as a manifold $G$ endowed with a group structure, such that the maps $G \times G \to G$ $(x, y) \mapsto xy$ and $G \to G$ $x \mapsto x^{-1}$ are smooth (i.e. differentiable). The simplest examples of Lie groups are the groups of isometries of $\mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{H}^n$ ($\mathbb{H}$ is the set of quaternions). Hence, we obtain the orthogonal group $O(n)$, the unitary group $U(n)$, and the symplectic group $Sp(n)$.

An algebra $\mathfrak{g}$ can be associated with each Lie group $G$ in a natural way; this is called the *Lie algebra* of $G$. In the early development of the theory, $\mathfrak{g}$ was referred to as an "infinitesimal group". The modern term is attributed by most people to H. Weyl (1934). A fundamental theorem of Lie states that every Lie group $G$ (in general, a complicated non-linear object) is "almost" determined by its Lie algbera $\mathfrak{g}$ (a simpler, linear object). Thus, various calculations concering $G$ are reduced to algebraic (but often non-trivial) computations on $\mathfrak{g}$.

A *homogeneous space* is a manifold $M$ on which a Lie group acts transitively. As a consequense of this, $M$ is diffeomorphic to the coset space $G/K$, where $K$ is a (closed Lie) subgroup of $G$. In fact, if we fix a base point $m \in M$, then $K$ is the subgroup of $G$ that consists of the points in $G$ that fix $m$ (it is called the *isotropy subgroup of* $m$). As mentioned above, these are the geometries according to Klein, in the sense that they are obtained from a manifold $M$ and a transitive action of a Lie group $G$ on $M$. The advantage is that instead of studying a geometry with base point $m$ as the pair $(M, m)$ with the group $G$ acting on $M$, we could equally study the pair $(G, K)$.

One of the fundamental properties of a homogeneous space is that, if we know the value of a geometrical quantity (e.g. curvature) at a given point, then we can calculate the value of this quantity at any other point of $G/K$ by using certain maps (translations). Hence, all calculations reduce to a single point which, for simplicity, can be chosen to be the identity coset $o = eK \in G/K$. Furthermore, in an important special case where the homogeneous space is *reductive*, then the tangent space of $G/K$ at $o$ can be identified in a natural way with a subspace of $\mathfrak{g}$.

As a consequence of this, many hard problems in homogeneous geometry can be formulated in terms of the group $G$ and the subgroup $K$, and then in terms of their corresponding infinitesimal objects $\mathfrak{g}$ and $\mathfrak{k}$. Such an infinitesimal approach enables us to use linear algebra to tackle non-linear problems (from geometry, analysis, or theory of differential equations). For example, the equations satisfied by an Einstein metric (these, according to general relativity, describe the evolution of the universe) are a complicated non-linear system of partial differential equations. However, for $G$-invariant metrics on a homogeneous space, this system reduces to a system of algebraic equations, which can be solved in many cases.

There is a large variety of applications of Lie groups in mathematics. They appear in various ways beyond differential geometry, such as algebraic topology, harmonic analysis, and differential equations, to name a few. They also possess important applications in physics, since they become involved in field theories in many ways. In fact, certain classical Lie groups appear as the building blocks in various physical theories of matter. Homogeneous spaces, in turn, have been employed in the physics of elementary particles as models called *supersymmetric sigma models*. Also, what physicists call *coherent states*, are in one-to-one correspondence with elements in a homogeneous space.

Before we proceed to the description of the chapters of this book, we would like to mention that the two generalizations of Euclidean geometry that we mentioned, namely that of Riemann and that of Klein, were unified by E. Cartan in his theory of *espaces généralizés*. In Cartan's geometry, at each point $m$ of $M$, there is a Klein-style geometry in the tangent space. That is to say, Cartan took Klein's geometry and made it local to each tangent space.

Chapter 1 starts with a simple example of a Lie group that exhibits the manifold and group structure. Then we give a brief review of manifolds, and then we proceed with the definition of a Lie group. We define the Lie algebra of a Lie group as the tangent space at the identity element of the group, and alternatively as the set of its one-parameter subgroups. We also list a simplified version of Lie's theorems.

In Chapter 2, after discussing a few elementary concepts about representations, we develop the appropriate tools that are needed for the classification of the compact and connected Lie groups. These are the adjoint representation, and the maximal torus of a Lie group. We also introduce a very useful tool, the Killing form, and we provide a brief insight through the complex semisimple Lie algebras.

Chapter 3 starts with a brief review of Riemannian manifolds, and then discusses a way to make a Lie group into a Riemannian manifold. The metrics which are important here are the bi-invariant metrics, and with respect to such metrics we give formulas for the connection and the various types of curvatures.

In Chapter 4 we define the notion of a homogeneous space and provide several examples. We discuss the reductive homogeneous spaces, and the isotropy representation of such a space.

The geometry of a homogeneous space is discussed in Chapter 5, where we show how a homogeneous space $G/K$ can become a Riemannian manifold (so we obtain a *Riemannian homogeneous space*). The important metrics here are the $G$-invariant metrics. Formulas are presented for the connection and the various types of curvatures.

In Chapters 6 and 7 we discuss two important, and generally non-overlapping, classes of homogeneous spaces, which are the symmetric spaces and the generalized flag manifolds. One of the most significant advances of the twentieth century mathematics is Cartan's classification of semisimple Lie groups. This leads to the classification of these two classes of homogeneous spaces. These spaces have many applications in real and complex analysis, topology, geometry, dynamical systems, and physics.

In Chapter 8 we give three applications of homogeneous spaces. The first is about homogeneous Einstein metrics. These are Riemannian metrics whose Ricci tensor is proportional to the metric. The second refers to symplectic geometry, which is rooted in Hamilton's laws of optics. Here we present a Hamiltonian system on generalized flag manifolds. A Hamiltonian system is a special case of an integrable system, which is a subject that has attracted much attention recently. The third application deals with homogeneous geodesics in homogeneous spaces. Geodesics are important not only in geometry, being length minimizing curves, but also have important applications in mechanics since, for example, the equation of motion of many systems reduces to the geodesic equation in an appropriate Riemannian manifold. Here, we present some results about homogeneous spaces, all of whose geodesics are homogeneous, that is, they are orbits of one-parameter subgroups. These are usually known in the literature as g.o. spaces.

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