AN INVESTIGATION OF THE OSEEN DIFFERENTIAL EQUATIONS FOR THE BOUNDARY LAYER

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By

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Abstract

The thesis is on an investigation of the Oseen partial differential equations for the problem of laminar boundary layer flow for the steady two-dimensional case of an incompressible, viscous fluid with the boundary conditions that the velocity at the surface is zero and outside the boundary layer is the free stream velocity.

It first shores-up some of the theory on using the Wiener-Hopf technique to determine the solution of the integral equation of Oseen flow past a semi-infinite flat plate. The procedure is introduced and it divides into two steps; first is to transform the Oseen equation (Oseen 1927) into an integral equation given by (Olmstead 1965), using the drag Oseenlet formula. Second is the solution of this integral equation by using the Wiener-Hopf technique (Noble 1958).

Next, the Imai approximation (Imai 1951) is applied to the drag Oseenlet in the Oseen boundary layer representation, to show it approximates to Burgers solution (Burgers 1930). Additionally, a thin body theory is applied for the potential flow. This solution is just the same as the first linearization in Kusukawa's solution (Kusukawa, Suwa et al. 2014) which, by applying successive Oseen linearization approximations, tends towards the Blasius/Howarth boundary layer (Blasius, 1908; Howarth, 1938).

Moreover, comparisons are made with all the methods by developing a finite-difference boundary layer scheme for different Reynolds number and grid size in a rectangular domain.

Finally, the behaviour of Stokes flow near field on the boundary layer is studied and it is found that by assuming a far-boundary layer Oseen flow matched to a near-boundary layer Stokes flow a solution is possible that is almost identical to the Blasius solution without the requirement for successive linearization.

Chapter 1 Introduction

1.1 Introduction

Oseen flow fluid dynamics has important applications in different aerodynamic applications, including modelling the flight of aeroplanes, birds and even balls, and in different hydrodynamic applications, including modelling the motion of ships and marine animals. Consequently, the investigation of the boundary layer for Oseen flow is an important branch of fluid dynamic research. The simplest model is that of laminar steady two-dimensional flow of an incompressible fluid over a semi-infinite flat plate.

In particular, recent research on modelling manoeuvring problems in fluids includes investigation of the far-field Oseen equations (Chadwick 1998), and continuing these equations into the near field (Chadwick 2002, Chadwick 2005, Chadwick 2006, Chadwick and Hatam 2007). It has shown the importance of including the viscous term in the Oseen model even for high Reynolds number. The motivation of the study is to investigate the viscous boundary layer in the content of this Oseen flow model (Olmstead 1965), in particular to consider the formulations given by Bhattacharya (1975) and (Gautesen 1971).

Comparisons will be made to the Blasius boundary layer form (Blasius 1908), supported by a Finite Difference numerical model.

1.2 The Aims and Objectives

The thesis is centred on the flow past a semi-infinite flat plate, in particular, Oseen's and Blasius' approximation, and is itemized in the following steps.

i. The critical study of the flow over a semi-infinite flat plate is performed by reviewing various literature both on Oseen's and Blasius' approximation.

ii. The Oseen integral equation, which represents flow over a flat plate, is considered.

iii. The Wiener-Hopf technique is described and applied to solve the integral equation of Oseen flow representation in the flat plate problem.

iv. The Oseen solution is derived in an integral form and the strength function is obtained from Noble (Noble & Peters, 1961) to achieve Gautesen solution (Gautesen 1971), also given in Bhattacharya (1975) study.

v. The Imai approximation (Imai, 1951) is applied to the drag Oseenlet in the Oseen integral representation and shown to be the same as Burgers' solution (Burgers, 1930), and also Kusukawa's solution (Kusukawa et al., 2014).

vi. The potential solution is derived by a thin-body theory applied to the Oseen integral representation.

vii. The behaviour of Stokes flow near field on the boundary layer is considered and on Oseen-Stokes matched boundary layer formulation is introduced.

viii. Numerical studies are carried out using boundary layer assumptions, Oseen's and Blasius' approximation of Navier-Stokes equation for various Reynolds number by the Finite Difference Method, then results are compared.

ix. The solutions are illustrated and plotted and these results discussed.

1.3 Thesis Outline

The remainder of this thesis is arranged as follows; Chapter 2 presents the literature review, which contain two sections Oseen and Blasius over the flat plate. Chapter 3 reviews the derivations of important equations related to our work such as the continuity equation, Navier–Stokes, Oseen, boundary layer then Blasius and finally the Oseen-Blasius equation.

Chapter 4 describes the general solution of the integral equation via the Wiener-Hopf technique containing some complex integral theorems, Fourier Transform and an application of this method. In Chapter 5 the integral representation of Oseen flow over the semi-Infinite flat plate is shown. The theory currently presented in literature is shored up by including more details of analysis and proofs.

In Chapter 6, the Finite Difference Method (FDM) is used to obtain numerical solutions for the problem of the two-dimensional steady flow over a flat plate for different approximations and various Reynolds numbers. We start with Boundary Layer equations, then Oseen

Boundary Layer equations, next the partial differential equation form of Blasius equation, and finally the Oseen-Blasius equation. Different grid sizes and Reynolds numbers are considered.

Chapter 7 focuses on the relationship between the Oseen and Blasius approximation in the boundary layer. First, the general form of the Oseen solution is presented. Then, for the solution of potential flow, a Thin Body Theory is presented and applied which is checked by the Laplacian Green function first. In addition, Imai's approximation for the drag Oseenlet velocity is given. Moreover, the Oseen-Blasius equation is derived from the Oseen integral representation, and then an analytic solution for this equation is obtained in two ways by both ordinary and partial differential equation form. Lastly, the solution of the Stokes Boundary Layer is obtained.

In Chapter 8 several comparisons have been performed. First, the comparison of the Numerical solutions for all the approximations of Navier-Stokes equations on flow over the flat plate; Boundary Layer equation, Oseen Boundary Layer, Blasius equation and Oseen-Blasius equation. Then, the analytical solution is compared with the Numerical solution of Oseen-Blasius. Next, the solution of Finite Difference Methods (FDM) of the partial differential equation form of Blasius equation is compared with the classical solution, and lastly, comparisons of the analytical solution of the Oseen–Blasius solution with Blasius solution is shown.

In Chapter 9, the matching is discussed of Oseen-Stokes Boundary Layer equations. First, it presents the Stokes Boundary Layer near field. Then, the matching with the far Boundary Layer solution to get the Oseen–Blasius solution is considered. Finally, Chapter 10 presents the Conclusion and Future Work.

Chapter 2 Literature Review

2.1 Oseen approximation for the flow past a semi-infinite flat plate

This section investigates and discusses several studies on the Oseen approximation of flow past a semi-infinite flat plate. Before proceeding to examine these studies, it will be necessary to start by formulating the problem. Thus the Oseen approximation of the Navier-Stokes equation of the steady two-dimensional flow of a viscous incompressible fluid of uniform velocity U in the x direction (Oseen, 1927, p.30-38) is analysed

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}\right\} \tag{2.1.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.1.2}$$

We consider the problem of flow past a semi-infinite flat plate in the half plane

$$y = 0, \quad 0 < x < \infty$$

where $u = u(x, y)$ and $v = v(x, y)$ denote the x-and y-components of the velocity respectively, $p = p(x, y)$ is the pressure, U a uniform stream velocity in the x direction, ν is kinematic viscosity. To satisfy the no-slip boundary condition on the flat plate we require that

$$u(x, 0) = v(x, 0) = 0, \qquad x > 0, \tag{2.1.3}$$

and the flow approaches uniform flow far from the flat plate,

$$\left.\begin{aligned} u(x, y \to \infty) \to U, \\ v(x, y \to \infty) \to 0. \end{aligned}\right\} \tag{2.1.4}$$

There are vector integral equations equivalent to the equations (2.1.1) and (2.1.2) which satisfy the condition (2.1.3), (Olmstead 1965, p. 242, eq. 3.6), given by

$$\left.\begin{aligned} u(x,y) &= U + \int_0^\infty u^D(x,y;\ s,0)\sigma(s)ds, \\ v(x,y) &= \int_0^\infty v^D(x,y;\ s,0)\sigma(s)ds \end{aligned}\right\} \tag{2.1.5}$$

where

$$-2\pi u^D(x,y;\ s,0) = \left\{\frac{x-s}{{r_s}^2} - ke^{k(x-s)}\left(\frac{x-s}{r_s}K_1(k\,r_s) + K_0(k\,r_s)\right)\right\},$$
$$-2\pi v^D(x,y;\ s,0) = \left\{\frac{y}{r_s} - ke^{k(x-s)}\frac{y}{r_s}K_1\,(k\,r_s)\right\}$$

(Olmstead 1965, p. 242, eq. 3.7).

where $r_s = \sqrt{(x-s)^2 + y^2}$, $k = U/2\nu$ and K_0, K_1 are modified Bessel functions of the second kind and u^D, v^D correspond to drag force, $\sigma(s)$ is the strength of the drag at $(s,0)$. The function $\sigma(s)$ must be determined from the boundary conditions. Once this quantity is known, then the solution to (2.1.1) is given in integral representation by (2.1.5). When imposing the boundary conditions (Olmstead 1968) (2.1.4) into (2.1.5) the vector integral equation can then be resolved into independent scalar equations as the following

$$u(x,0) = \int_0^\infty u^D(x,0)\,\sigma(s)ds, \quad x > 0. \tag{2.1.6}$$

Here

$$-2\pi u^D(x,0) = \left\{\frac{1}{x} - \left(\frac{x}{|x|}K_1(k\,x) + K_0(k\,x)\right)ke^{k(x)}\right\}.$$

There are several studies (Lewis and Carrier 1949, Olmstead and Byrne 1966, Gautesen 1971, Gautesen 1972, Bhattacharya 1975, Olmstead and Gautesen 1976) carried out to determine the solution to equations (2.1.1) and (2.1.2) with boundary conditions (2.1.3) and (2.1.4) by using the Olmstead vector integral equations (2.1.5) and from these studies the problem has been reduced to the solution of an integral representation for velocity and pressure.

The study Bhattacharya (1975) considered the equation

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\rho}\Gamma\delta(x, y; -b, 0) \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}\right\} \qquad (2.1.7)$$

with equation (2.1.2) and boundary conditions (2.1.3) and (2.1.4), where $\delta(x, y)$ is the Dirac delta function and the region D is the external to the semi-infinite plate: y=0, $0 < x < \infty$, and Γ is horizontal force strength in (x, y) plane along the positive x-axis, $b > 0$. Following from the analysis of (Olmstead 1965) the problem was considered by Bhattacharya (1975) when $\Gamma = 0$, so equation (2.1.7) became equation (2.1.1). The reason for using the Olmstead vector integral equations (2.1.5) is to obtain the following integral equation

$$2\pi U = k\int_0^\infty Q[k(x - s)]\ \sigma(s)ds, 0 < x < \infty \qquad (2.1.8)$$

(Bhattacharya 1975, p. 18, eq. 8), here

$$Q(kx) = \frac{1}{2\pi k}u^D(kx, 0).$$

Similarly, Gautesen studied symmetric Oseen flow past a semi-infinite plate and two force singularities at $(s, t)$and $(s, -t)$ in the (x, y) plane, which are defined Γ_1 is the source strength of the horizontal forces singularity, and Γ_2 is the source strengths of the vertical force singularities for the particular case such that

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\rho}\Gamma_1\{\delta(x - s, y - t) + \delta(x - s, y + t)\} \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{\rho}\Gamma_2\{\delta(x - s, y - t) + \delta(x - s, y + t)\} \end{aligned}\right\} (2.1.9)$$

(Gautesen 1971, p.144, eq.1)

In a similar way as Bhattacharya, Gautesen mentioned that when $\Gamma_1 = \Gamma_2 = 0$, the equations (2.1.9), (2.1.2) and (2.1.3) satisfy the Olmstead vector integral equations (2.1.5) (Gautesen 1971); However Gautesen used the following boundary condition

$$u \to U\,, v \to 0 \quad as \quad x^2 + y^2 \to \infty, \tag{2.1.10}$$

instead of the condition (2.1.4), which is inaccurate because $u \to 0$ for $x^2 \to \infty$, $y \to 0$ .To make this valid consider angle θ a constant and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ with $r = \sqrt{x^2 + y^2}$, then

$$u \to U\,, v \to 0 \quad as \quad r^2 \to \infty\,, \theta \neq 0\,.$$

Integral representations (2.1.5) for velocity are equivalent to the integral equation (2.1.6). The integral equation that (Gautesen 1971) obtains is similar to the integral equation (2.1.8) of Bhattacharya (1975) where the unknown function σ is the drag distribution along the plate, and both of them are represented by the integral equation

$$g(x) = \int_0^\infty h(x-s)\sigma(s)ds \;, x > 0, \tag{2.1.11}$$

here

$$h(x) = u^D(x,0), \qquad g(x) = 2\pi U.$$

According to Noble (1958) and Polyanin and Manzhirov (2012) the integral equation (2.1.11) is a Wiener–Hopf equation of the first kind, where g and h are known functions and σ is an unknown function.

(Gautesen 1971) derived the inverse of these integral equations and obtained the general solution (2.1.6) by using the technique developed in (Noble 1958) in chapter VI in the manner similar to (Olmstead 1966) and determined the function σ as

$$\sigma(x) = U\left(\frac{2}{\pi k x}\right)^{\frac{1}{2}}. \tag{2.1.12}$$

Also the solution in study of Bhattacharya (1975) is the same solution (2.1.12).

[The chapter four display the general solution of equation (2.1.11) at any kernel and next in the Chapter Five we derive the solution and compare between equations (2.1.6) and (2.1.8) with more details and derive in particular kernel which is drag Oseenlet; the drag distribution acts on the plate; to obtain the solution (2.1.12)].

Furthermore, (Olmstead 1968) considered several viscous flow problems and derived the solutions of classical result of problem (2.1.1) - (2.1.3) for flow past a finite flat plate in the half plane

$$y = 0, \qquad 0 \leq x \leq \beta, \qquad \beta > 0.$$

As above, Olmstead used the Olmstead vector integral equation (2.1.5) to express the velocity to the problem (2.1.1) in a finite interval, with the following boundary condition

$$u(x,0) = v(x,0) = 0, \qquad 0 \leq x \leq \beta.$$

He discussed this problem to achieve several viscous flows, one of them is uniform flow when β tends ∞ and the solution was obtained as

$$\sigma^*(x,a) = \frac{1}{\sqrt{2\pi^3 x}} + \frac{\sqrt{2}\,a}{\pi^2}\int_0^\infty \frac{e^{-xt^2}}{t^2 - a}dt, \qquad x > 0. \tag{2.1.13}$$

The special case of (2.1.13) when $a = 0$, and he mentioned that this special case is the problem of flow past a semi-infinite half-plane. The solution can be determined from this as

$$\sigma^*(x) = \frac{1}{\sqrt{2\pi^3 x}} = \frac{\sqrt{k}}{2\pi U}\sigma(x), \qquad x > 0. \tag{2.1.14}$$

In addition, the study by Olmstead and Gautesen (1976) investigated useful general properties and supplied some insights and method for integral representations which satisfy Oseen equations. It concluded the integral equation on a semi-infinite interval as the integral equation (2.1.6) and in symmetric Oseen flow past a semi-infinite plate the integral equation (2.1.6) becomes

$$\frac{1}{4\pi}\int_0^\infty h(x-s)\sigma(s)ds = 1, x > 0, \tag{2.1.15}$$

where the uniform flow velocity is $u(x, y \to \infty) = 2\pi$ and the force function $\sigma(x)$ was determined as

$$\sigma(x_1) = \left(\frac{8}{\pi x}\right)^{\frac{1}{2}}, x > 0. \tag{2.1.16}$$

The differences between the solutions (2.1.16) and (2.1.14) are due to the uniform flow velocity, the length and the pressure being normalized by U, L , $\rho U\nu/L$ respectively in equation (2.1.1) in (Olmstead and Gautesen 1976) whereas the length and the pressure have been normalized by $\rho U^2/2\,, 2\nu/U$ respectively in equation (2.1.1) in (Olmstead 1968) .

Finally, the study of Burgers (1930), developed an approximate velocity solution by applying Oseen theory to the boundary layer problems to obtain a vorticity solution. Our attention will be given on the part, which explains the first approximation of velocity solution. The study starts with the integral equation

$$\left.\begin{aligned} u(x,y)+U &= \frac{1}{2\pi\rho U}\int\left\{\frac{U\varepsilon}{\nu}-\frac{\partial}{\partial x}(\log r+\varepsilon)\right\}f(s)ds,\\ v(x,y) &= \frac{1}{2\pi\rho U}\int\left\{-\frac{\partial}{\partial x_2}(\log r+\varepsilon)\right\}f(s)ds, \end{aligned}\right\} \tag{2.1.17}$$

where $0<x<l$, $r=\sqrt{(x-s)^2+y^2}$, and $\varepsilon=ke^{k(x-s)}K_0(k\,r)$. For determinate the strength function $f(s)$, the boundary condition ($u=-U, v=0$ at the line) has been applied, in addition, the function between brackets has approximated by

$$\left\{\frac{U\varepsilon}{\nu}-\frac{\partial}{\partial x}(\log r+\varepsilon)\right\}=\sqrt{\frac{\pi U}{\nu(x-s)}}, \tag{2.1.18}$$

to get Abel's integral equation which was solved to obtain

$$f(x)=-\frac{2\rho\sqrt{\nu U^3}}{\sqrt{\pi x}}. \tag{2.1.19}$$

Furthermore, the function ε has approximated as $\sqrt{\frac{\pi U}{\nu(x-s)}}e^{-\frac{Uy^2}{(x-s)}}$, $x>s$. Then the solution is given by

$$u(x,y)+U=-U\left[1-\operatorname{erf}\left(\frac{U}{2}\frac{y}{\sqrt{\nu x}}\right)\right], \tag{2.1.20}$$

$$v(x,y)=\frac{\sqrt{\nu U}}{\sqrt{\pi y}}\frac{y}{|y|}\left(1-e^{-\frac{U\,y^2}{4\nu x}}\right), \tag{2.1.21}$$

(Burgers 1930, p.610, eq. 17-18), where $\operatorname{erf}\left(\frac{U}{2}\frac{x_2}{\sqrt{\nu x_1}}\right)$ is the Error function.

To summarize, the problem of flow over a flat plate via Oseen approximation has been presented by two lines of study:

First, the solution in an integral equation form by Wiener-Hopf technique where the equations (2.1.1) – (2.1.2) are equivalent to the vector integral equation the found in Olmstead (1965) and then resolved into scalar integral equations by satisfying the boundary condition (2.1.3) to obtain an integral equation of the Wiener-Hopf type in which the unknown function is the strength drag Oseenlet. The method developed in Noble (1958) is followed and proceeds in a manner similar to that of Olmstead (1966), where Gautesen (1971) derived the inverse of this integral equation.

Second, the study of Burgers (1930) obtained an approximate solution of problem, where the drag Oseenlet has been simplified to give the Abel's integral equation, which was solved to obtain the strength or force function then the solution was derived and given by the Error function.

2.2 Blasius approximation for the flow past a semi-infinite flat plate

The purpose of this section is to review the literature on the solution of the well-known Blasius equation for the steady two-dimensional viscous laminar flow over a semi-infinite flat plate, which can be expressed as $2f'''(\eta) + f(\eta)f''(\eta) = 0$ for a function f where a prime denotes differentiation with respect to the similarity variable η defined as $\eta = x_2(U/\nu\, x_1)^{1/2}$ with boundary conditions $f(0) = 0, f'(0) = 0, f'(\eta \to \infty) = 1$ (Blasius 1908). Over the past century a considerable amount of literature has been published on this problem, and has been carried out by using numerical and analytical methods.

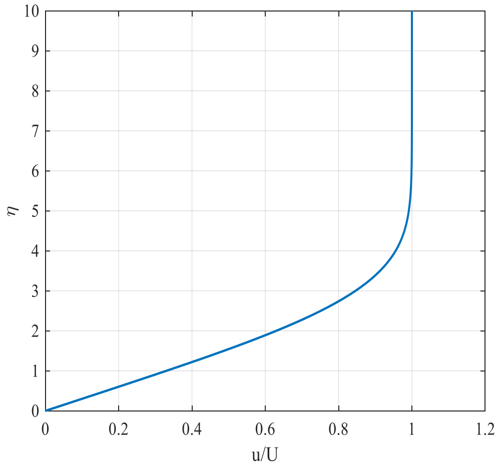
Various studies have been performed by many authors via numerical methods to obtain approximate solutions to the steady Blasius flow of viscous incompressible fluid, the study of Howarth is a pioneering work for the flat-plate flow, and solved the Blasius equation more accurately by numerical methods giving the value $f''(0) = 0.33206$ (Howarth 1938). In the same manner, Cortell presented a numerical study of the non-linear differential equation $af'''(\eta) + f(\eta)f''(\eta) = 0$ where $1 \leq a \leq 2$ with the same boundary conditions (Cortell 2005). It is clear that the case when $a = 2$ the problem returns to Blasius. Several numerical solutions were obtained using Runge-Kutta and the values of f , $f'$and f'' were computed when $a = 2$, agreeing with the numerical results of Howarth. However, the results are obtained in region$(0 \leq \eta \leq 10)$. Furthermore, the study by Xie deals with the same problem numerically by converting the Blasius equation to a pair of initial value problems, and then solving the initial value problem by Runge-Kutta method with Chebyshev interpolating point (Ming-Liang, XIONG et al. 2006), and it obtained good agreement with Howarth results.

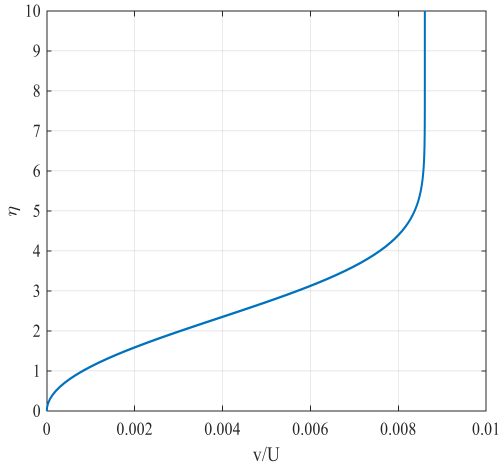
In addition, the study by Lien-Tsai and Cha'o-Kuang used the Differential Transformation method to solve the equation of Blasius equations numerically (Lien-Tsai and Cha'o-Kuang 1998). Also, the solution was compared with the Howarth solution; both results were in agreement with each other.

On the other hand, there are several studies to obtain an analytic solution to the Blasius equation with boundary conditions by using new kinds of analytic techniques for non-linear differential equation such as (Liao 1997, He 1998, Liao 1998, Liao 1999, He 2004, Kusukawa, Suwa et al. 2014), and we will illustrate the solutions from two different methods of these studies. The study of Liao presented the analytic solution for the above mentioned problem by using analytic technique in the whole region $0 \leq \eta < \infty$, this technique namely the Homotopy Analysis Method (HAM) (Liao 1997).

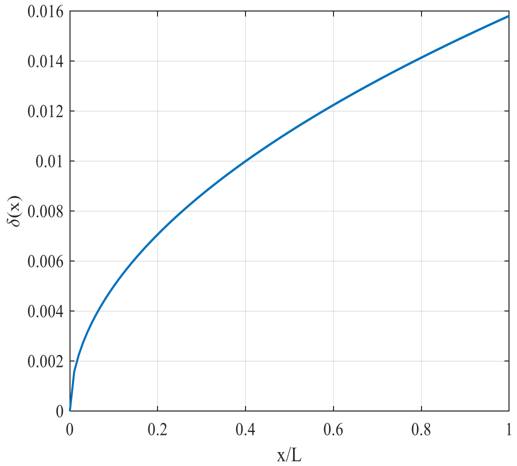
Table 2.1: η, f, f', f'' of Blasius solution (Shaughnessy, Katz et al. 2005)

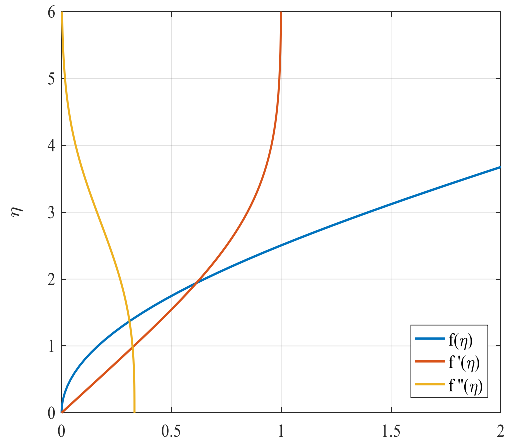
η	f	f'	f''
0	0	0	0.3320
0.5	0.0415	0.1659	0.3309
1	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8	6.2792	1.0000	0.0000





(a) **(b)**





(c) **(d)**

Figure 2.1 The Blasius Profile

This method is Runge-Kutta at Reynolds Number $Re = 10^5$, $x = 1$ and the Free stream Velocity U=1 with $\eta(x, y) = \frac{\sqrt{ReU}}{2}\frac{y}{\sqrt{x}}$, and $f''(0) = 0.3320$ obtained by Shooting-method (Puttkammer 2013, Sanders 2014)

(a) Velocity $u = f'(\eta)$ with η-axis.

(b) Velocity $v = \frac{\sqrt{Re}}{2}\left(\eta f'(\eta) - f(\eta)\right)$ with η -axis.

(c) Boundary Layer Thickness δ, evaluated by $u = 0.99$U, $\delta(x) = \frac{5\sqrt{x}}{\sqrt{Re}}$.

(d) The comparison of f, f', f'' with η-axis.

The study conducted by Kusukawa (Kusukawa, Suwa et al. 2014), proves that there is an analytical technique, namely an iteration method to solve the problem of the boundary layer flow over a flat plate which initially assumes the Oseen approximation as well as the Blasius approximation, and then iterates to find the full Blasius solution. This study deals with the Blasius equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

with boundary conditions $u(x,0)=v(x,0)=0,\ u(x,y\to\infty)=U.$ The iteration method used in this study assumes the velocity components u_i may be expanded into the ε-power series

$$u = U+\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\cdots$$

and

$$v = \varepsilon v_1+\varepsilon^2 v_2+\varepsilon^3 v_3+\cdots$$

where the parameter ε is the ratio of the boundary layer thickness. Next, substitute the ε-power series with boundary conditions into Blasius equations and rearrange the terms of the same order in ε, the following equations are obtained

$$U\frac{\partial u_1}{\partial x}-\nu\,\frac{\partial^2 u_1}{\partial y^2}=0, \tag{2.2.7}$$

and

$$U\frac{\partial u_n}{\partial x}-\nu\,\frac{\partial^2 u_n}{\partial y^2}=-\sum_{k=1}^{n-1}\left(u_k\frac{\partial u_{n-k}}{\partial x}-v_k\,\frac{\partial^2 u_{n-k}}{\partial y^2}\right),\ for\ n\geq 2 \tag{2.2.8}$$

$$\frac{\partial u_n}{\partial x}+\frac{\partial v_n}{\partial y}=0\,, \qquad for\ n\geq 1, \qquad (2.2.9$$

with boundary conditions

$$\left.\begin{array}{lr} u_1(x,0)=-\dfrac{U}{\varepsilon}, \qquad v_1(x,0)=0, & \\ u_n(x,0)=v_n(x,0)=0, & for\ n\geq 2 \\ u_n(x,y\to\infty)=0, & for\ n\geq 1. \end{array}\right\} \tag{2.2.10}$$

The first approximation (2.2.7) has been called a modified Oseen's equation. By using the Laplace transform, the first approximation analytical solution of equation (2.2.7) are obtained as

$$u_1(\eta) = \frac{U}{\varepsilon}\,[\operatorname{erf}(\eta/2\,) - \,1], \tag{2.2.11}$$

$$v_1(\eta) = \,\frac{1}{\varepsilon}\,\frac{\sqrt{\nu\,U}}{\sqrt{\pi x}}\left[1 - e^{\frac{-\eta^2}{4}}\right], \tag{2.2.12}$$

(Kusukawa, Suwa et al. 2014, p. 37, eq. 3.4 & eq. 3.6) where

$$\eta(x,y) = \frac{\sqrt{U}}{\sqrt{\nu}}\,\frac{y}{\sqrt{x}}.$$

Next, the second approximation is obtained by utilizing the continuity equation (2.2.9) for $n\ =\ 2$ and defining a stream function ψ as $u_2 = \partial\psi/\partial y$ and $v_2 = -\,\partial\psi/\partial x$, and by introducing the following stream function $f_2(\eta\,)$ which depends on η only such that

$$u_2(\eta) = Uf_2{}'(\eta\,),$$
$$v_2(\eta) = \frac{\sqrt{\nu\,U}}{2\sqrt{x}}[\,\eta\,f_2{}'(\eta\,) - f_2(\eta\,)\,].$$

In general, the approximation of each order can be defined as,

$$u_k(\eta) = Uf_k{}'(\eta\,). \tag{2.2.13}$$

The boundary conditions are

$$f_k(0\,) = f_k{}'(0\,) = 0, f_k{}'(\eta \to \infty\,) = 0, \qquad k\, = 1,2, \ldots \tag{2.2.14}$$

On the other hand, the velocity component u_1 of the original Blasius equations can be expressed as $u_1\, = Uf'(\eta\,)$, where $f\,(\eta\,)$ is a stream function of the Blasius flow. From the ε-power series we have

$$f\,(\eta) = \eta + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots.$$

Consequently, the derivation of u_k with respect to x and y where $k\, = 1,2, \ldots$, are substituted for each k individually into (2.2.8) applying the boundary conditions (2.2.14) to obtain approximation functions $f_1{}'f_2{}', f_3{}', \ldots$, and applying them into (2.2.13) to calculate u_k for each k. Finally, substitute $u_1, u_2, u_3, \ldots$, into the ε-power series to obtain the *x*-component of the velocity. By increasing the order of the approximation, the solution gradually tends to the result of Howarth.

However, by comparing the equation (2.2.7) with Oseen equation and Blasius equation, we observe that the equation (2.2.7) is the Navier-Stokes equation with both of Oseen's and Blasius approximation, so this equation may be called Oseen-Blasius equation, in Chapter 3 in section (3.4) we give the derivation. Therefore, the first approximation (2.2.11) and (2.2.12) are analytical solutions of Oseen-Blasius equation past a flat plate.

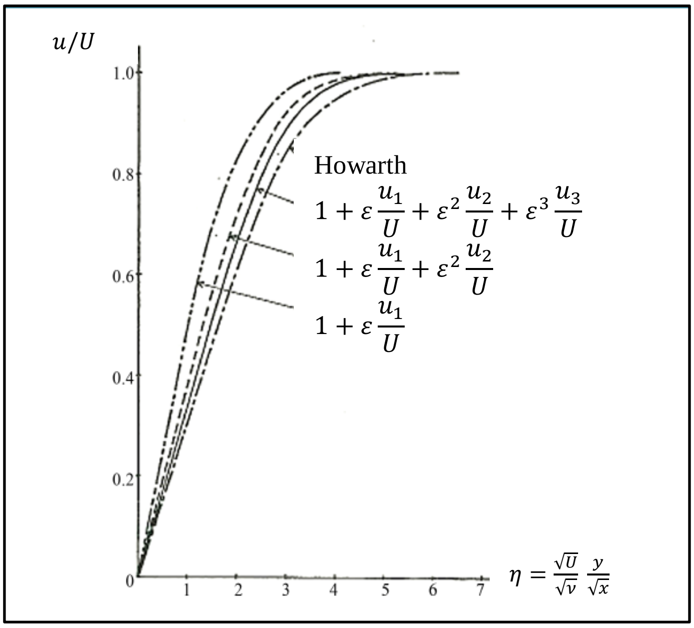


Figure 2.2 Kusukawa (2014) Result

Moreover, for potential solution, the boundary layer profile for outer flow has been considered by many authors via different methods to obtain potential solutions to the steady flow past a semi-infinite flat plate of viscous incompressible fluid, the study of Kou (1953) dealt with this problem and introduces the solution. Moreover, we will be focussing on the solution mentioned by Sobey (2000), where the potential velocity is defined by $u=\nabla\emptyset_i$ In 2-D the velocity potential ϕ and the stream–function ψ combines in a complex potential $\Phi=\phi+i\psi$ to acquire the potential solution given by

$$\Phi = z - i\beta \frac{\sqrt{2}}{\sqrt{Re}} z^{\frac{1}{2}},$$
$$\frac{d\Phi}{dz} = 1 - i \frac{\beta}{\sqrt{2Re}} z^{-\frac{1}{2}},$$

for a complex coordinate $z = x + iy$, $\frac{d\Phi}{dz} = u - iv$ can be used, where $\beta = \frac{\sqrt{2}}{\sqrt{\pi}}$, and the free stream velocity $U = 1$.

2.3 Summary

To summarize what is discussed in this chapter, the problem of the flow over a semi-infinite flat plate is considered under specific boundary conditions when the flow is steady, viscous and incompressible. We consider the problem in the half plane $y = 0, \quad 0 < x < \infty$ and flow has uniform stream velocity. This has been investigated via Oseen approximation and Blasius approximation of Navier-Stokes equations and there are several studies dealing with the Oseen approximation of this problem, the exact solution obtained by using a Wiener-Hopf technique.

Also, there are various studies that deal with Blasius approximation over a semi-infinite flat plate, some of them by analytical methods but most are numerical.

Chapter 3 Equations of Motion

In this chapter, we will derive some important equations of Motion, which are related to our work. We will start with Oseen equation followed by boundary layer equations, Blasius and Oseen-Blasius equations.

3.1 Oseen Equation

In this section we will derive Oseen equation from the Navier-Stokes equation by applying the Oseen approximation. First consider the steady two-dimensional Navier-Stokes equations for incompressible flow

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \end{aligned}\right\} \tag{3.1.1}$$

where ν is kinematic viscosity defined as $\nu \equiv \mu/\rho$. (Batchelor 1967, Shankar 2007).

According to Lamb (1932) Oseen's approximation to the fluid flow is that the velocity perturbation u^* is small compared to the uniform stream velocity U,

$$|u^*| \ll U, \qquad |v^*| \ll U. \tag{3.1.2}$$

Let the uniform stream velocity U be parallel to the x-axis. Hence the velocity u, v is given by

$$\begin{aligned} u &= U + u^*, \\ v &= v^*. \end{aligned}$$

The terms $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ can be rewritten in the Navier- Stokes equations (3.1.1) as

$$(U+u^*)\frac{\partial}{\partial x}+(v^*)\frac{\partial}{\partial y}=U\left(\frac{\partial}{\partial x}+\frac{u^*}{U}\frac{\partial}{\partial x}+\frac{v^*}{U}\frac{\partial}{\partial y}\right)$$

From Oseen approximation $\frac{|u^*|}{U}\ll 1$and $\frac{|v^*|}{U}\ll 1$, we obtain

$$(U+u^*)\frac{\partial}{\partial x}+(v^*)\frac{\partial}{\partial y}=U\frac{\partial}{\partial x}.$$

By applying into the Navier-Stokes equations (3.1.1) yields

$$\begin{aligned}U\frac{\partial u^*}{\partial x} &= -\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u^*}{\partial x^2}+\frac{\partial^2 u^*}{\partial y^2}\right),\\ U\frac{\partial v^*}{\partial y} &= -\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v^*}{\partial x^2}+\frac{\partial^2 v^*}{\partial y^2}\right).\end{aligned}$$

For brevity, u^* will be changed to u, the equation

$$\left.\begin{aligned}U\frac{\partial u}{\partial x} &= -\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\ U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),\end{aligned}\right\}\tag{3.1.3}$$

is the Oseen equation when the flow is steady viscous, incompressible fluid (Oseen 1927, Lamb 1932, Batchelor 1967, Shankar 2007).

3.2 Boundary Layer Equations

The boundary layer theory was introduced by Prandtl in 1904 to study the viscous fluid flow behaviour over a solid boundary (Prandtl 1904). It describes the steady two-dimensional boundary layer flow past a flat plate, which starts at $x=0$ and extends parallel to the x-axis having a length L. The x-momentum velocity u at flat plate is zero and at infinity far away from the boundary layer will be constant U. To derive the equations, we introduce some assumptions related to the boundary layer. The first one is that the boundary layer thickness δ is very small in comparison with the length in the x-direction L, let us introduce the following dimensionless variables

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta}.$$

So

$$\frac{\partial}{\partial x^*} = L\frac{\partial}{\partial x} \quad , \quad \frac{\partial}{\partial y^*} = \delta\frac{\partial}{\partial y}$$

which means

$$\frac{\partial}{\partial x} = O\left(\frac{\delta}{L}\right)\frac{\partial}{\partial y}$$

therefore

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}.$$

This has the consequence that the $\left[\frac{\partial^2}{\partial y^2}\right]$ will be much more dominant than the $\left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2}{\partial x^2}\right]$, so we can assume that

$$\frac{\partial^2 u}{\partial x^2} = 0. \qquad (3.2.1)$$

By introducing the non-dimensional variable $u^* = u/U, v^* = v/U$ and $p^* = p/\rho U^2$, the y-momentum equation of Navier-Stokes equations gives

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{L}{\delta}\right)^2\frac{\partial p^*}{\partial y^*} + \frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2\left(\left(\frac{\delta}{L}\right)\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right).$$

From boundary layer assumptions $\frac{\delta}{L}$ is very small, this gives $\frac{\partial p^*}{\partial y^*}$ is more dominant than the other terms in the equation above.

Therefore of Navier-Stokes equations reduce within the boundary layer to become:

The *x*-momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right). \qquad (3.2.2)$$

The y-momentum equation

$$-\frac{\partial p}{\partial y} = 0. \qquad (3.2.3)$$

The boundary conditions

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad u(x, y \to \infty) = U. \qquad (3.2.4)$$

The boundary bayer equation (3.2.2-4) are called Prandtl boundary layer equations.

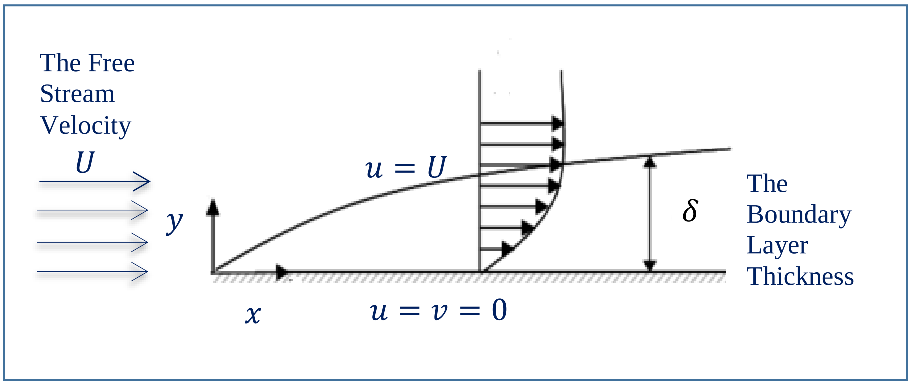


Figure 3.1 Velocity Boundary layer development on Flat Plate

3.3 Blasius Equation

The Blasius equation includes the boundary layer assumptions of the steady two-dimensional boundary layer flow, furthermore that the flow with zero pressure gradient is assumed

$$\frac{\partial p}{\partial x} = 0. \tag{3.3.1}$$

Under boundary layer theory and assumption (3.3.1), the Navier-Stokes equations (3.1.1) are simplified to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right), \tag{3.3.2}$$

with boundary conditions:

$$u(x,0) = 0, \quad v(x,0) = 0, \quad u(x, y \to \infty) = U. \tag{3.3.3}$$

The equation (3.3.2) is called the Blasius equation (Blasius 1908). Now we will derive the Blasius equation in the scale analyses, let

$$y \sim \delta \quad , \quad u_1 \sim U.$$

From the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

we get

$$v \approx \frac{\delta U}{x}.$$

Applying this equation with boundary $u(x, y \to \delta) \to U$ into equation (3.3.2) to define

$$\delta\, \delta \sim \sqrt{\frac{xv}{U}}. \quad (3.3.4)$$

Assume that the velocity u may be expressed as a function of the variable $\frac{y}{\delta}$. We have

$$\frac{u}{U} = g\left(\frac{y}{\delta}\right).$$

Let us introduce non-dimensional variable

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{U}{xv}}. \quad (3.3.5)$$

By (3.3.5), rewrite

$$u = Ug(\eta). \quad (3.3.6)$$

A stream function ψ is defined as

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}.$$

Now

$$\psi = \int u\, dy. \quad (3.3.7)$$

Substituting the differentiation of the equation (3.5.5) with respect to x_2 and equation (3.5.6) into (3.5.7) we obtain

$$\psi = U\sqrt{\frac{xv}{U}}\int g(\eta)d\eta\,.$$

Let us define the non-dimensional function $f(\eta)$ which is related to the stream function $\psi\ (x_1, x_2)$ as

$$f(\eta) = \int g(\eta)d\eta\,,$$

then

$$\psi = U\sqrt{\frac{xv}{U}}f(\eta).$$

Therefore, we obtain

$$u = U\, f'(\eta\,), \tag{3.3.8}$$

$$\frac{\partial u}{\partial x} \;=\; -\frac{U}{2x}\eta f''(\eta\,), \tag{3.3.9}$$

$$\frac{\partial u}{\partial y} \;=\; \frac{U\sqrt{U}}{\sqrt{x\nu}} f''(\eta\,), \tag{3.3.10}$$

$$\frac{\partial^2 u}{\partial y^2} \;=\; \frac{U^2}{x\nu} f'''(\eta\,). \tag{3.3.11}$$

and

$$v = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\;\left(\eta f'(\eta\,) - f(\eta\,)\right). \tag{3.3.12}$$

The governing the equations of (3.3.8), (3.3.9), (3.3.10), (3.3.11) and (3.3.12) can be substituting into (3.3.2) give

$$f'''(\eta\,) + \frac{1}{2} f(\eta\,) f''(\eta\,) = 0, \tag{3.5.13}$$

with boundary conditions:

$$f(0\,) = 0, \qquad f'(0\,) = 0, \qquad f'(\eta \to \infty\,) = 1\,. \tag{3.5.14}$$

In some texts a different η is defined as

$$\eta = \frac{x_2}{2}\sqrt{\frac{U}{x_1\nu}}. \tag{3.5.15}$$

In this case the equivalent to the equation (3.5.13) is

$$f'''(\eta\,) + 2f(\eta\,)f''(\eta\,) = 0. \tag{3.5.16}$$

The equation (3.5.13) or (3.5.16) is the well-known Blasius equation.

3.4 Oseen-Blasius Equation

In this section, we derive the Oseen-Blasius equation using both Blasius approximations (3.5.2), and Oseen approximation (3.3.1). The Navier-Stokes equations are simplified to

$$U\frac{\partial u}{\partial x} = \nu\ \frac{\partial^2 u}{\partial y^2}\ , \tag{3.4.1}$$

with the same boundary conditions mentioned in (3.5.3). Consider the ordinary differential form of this equation. Let

$$y\sim\delta \text{ and } u\sim U. \tag{3.4.2}$$

Thus, substituting (3.4.2) into (3.4.1), we have the equation

$$U\frac{U}{x} = \nu\frac{U}{\delta^2},$$

$$\delta = \sqrt{\frac{x\nu}{U}}. \tag{3.4.3}$$

Because (3.4.3) is the same as (3.3.4), the derivation follows the same as previous steps of the derivation of the Blasius equation. Therefore, substituting (3.3.9), and (3.3.11) into (3.4.1) we have

$$U\left(-\frac{U}{2x}\eta f''(\eta\,)\right) = \nu\frac{U^2}{\nu x}f'''(\eta\,).$$

Therefore

$$f'''(\eta\,) + \frac{1}{2}\eta f''(\eta\,) = 0, \tag{3.4.4}$$

with the boundary conditions that mentioned in (3.3.14). When defined η as in (3.3.15) it can be rewritten instead as

$$f'''(\eta\,) + 2\eta f''(\eta\,) = 0. \tag{3.4.5}$$

The equation (3.6. 5) will be considered further in chapter 7.

Chapter 4 The Wiener-Hopf Technique

The Wiener–Hopf technique is a mathematical method generally utilized in applied mathematics, it was initially formulated to deal with and solve certain classes of integral equations. Then, it has been widely used to solve partial differential equations with boundary conditions. In general, the method works by utilizing complex analytical properties (Noble 1958, Ho 2007).

In this chapter, we shall introduce the Wiener-Hopf method, which is used to solve the integral equation of Oseen flow over a semi-infinite flat plate. This method is based on mainly Fourier transform and some complex integrals, and comprehensive description is given for a complete understanding.

4.1 Gaussian integration

Gaussian integration is the integration of the exponential of a quadratic as

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,. \tag{4.1.1}$$

To prove this integral it needs to square it using variables x and y for the two integrals sequentially calculate the double integral via polar coordinates. Let

$$\begin{aligned} I \; &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \\ (I)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\, e^{-y^2}\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy. \end{aligned} \tag{4.1.2}$$

Let

$$r^2 = x^2 + y^2, \qquad dxdy = rdrd\theta,$$

$$-\infty < x < \infty \rightarrow 0 < r < \infty\,; \;\; -\infty < y < \infty \rightarrow 0 < \theta < 2\pi.$$

Substituting into (4.1.2)

$$(I)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{-1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta \; = \int_0^{2\pi}\frac{1}{2}d\theta = \pi.$$

From above we obtain

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

and

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \tag{4.1.3}$$

For a real constant $m > 0$,

$$\int_{-\infty}^{\infty} e^{-mx^2}\,dx = \sqrt{\frac{\pi}{m}}\,. \tag{4.1.4}$$

4.2 Jordan lemma

Let f a complex, continuous function defined on a the semi-circle contour H_R, shown in Figure (4.1), If the only singularities of $F(z)$ are poles, then

$$\lim_{R\to\infty}\int_{H_R} e^{imz} f(z)dz = 0 \tag{4.2.1}$$

provided that $m > 0$ and $|f(z)| \rightarrow 0$ as $R \rightarrow \infty$. If $m = 0$ then a faster convergence to zero is required for $f(z)$.

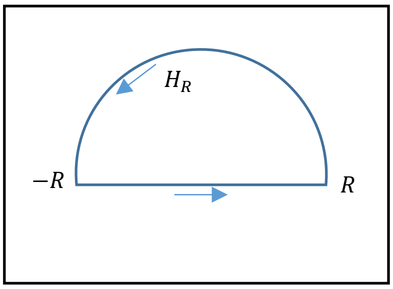


Figure 4.1 Integral Contour

Proof: Let $z = Re^{i\theta} = R(cos\theta + i\ sin\theta)$, and $dz = iRe^{i\theta}d\theta$.

Now

$$\lim_{R\to\infty}\left|\int_{H_R} e^{imz}f(z)\,dz\right| = \lim_{R\to\infty}\left|\int_{H_R} e^{imR(cos\theta+i\,sin\theta)}f(z)\,iRe^{i\theta}d\theta\right|$$

$$\leq \lim_{R\to\infty}\int_{H_R}\left|e^{imRcos\theta}\right|\left|e^{-Rm\,sin\theta}\right||f(z)||i||R|\left|e^{i\theta}\right|d\theta.$$

Since

$$\left|e^{i\theta}\right| = \sqrt{cos^2(\theta) +\ sin^2(\theta)} = 1, \qquad \left|e^{imRcos\theta}\right| = 1, \quad |i| = \left|e^{(\pi/2)i}\right| = 1.$$

So we obtain

$$\lim_{R\to\infty}\left|\int_{H_R} e^{imz}f(z)\,dz\right| \leq \lim_{R\to\infty}\int_{H_R} e^{-Rm\,sin\theta}|f(z)|Rd\theta\,. \tag{4.2.2}$$

On right hand side of (4.2.2) hence $\sin\theta > 0$ (in the upper half plane) and $m > 0$, then

$e^{-Rm\,sin\theta}$ is zero at $R \to \infty$, so we obtain

$$\lim_{R\to\infty}\int_{H_R} e^{imz}\,f(z)dz = 0.$$

4.3 An application of Jordon lemma and Gaussian integral

From Jordon lemma and Gaussian integral, the following results may be derived:

Theorem 4.3.1

$$\int_{\gamma} \frac{e^z}{\sqrt{z}} dz = 2i\sqrt{\pi}. \tag{4.3.1}$$

where $\gamma = (1 - i\infty, 1 + i\infty)$.

Proof: Let $z = re^{i\theta}$, then $f(z) = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{r}} e^{-i\theta/2}$, $-\pi \le \theta \le \pi$,

$$\oint_C \frac{1}{\sqrt{z}} dz = \left\{ \int_{\gamma} + \int_{C_r} + \int_{L_1} + \int_{C_\varepsilon} + \int_{L_2} \right\} \frac{1}{\sqrt{z}}. \tag{4.3.2}$$

The integral counter C is shown figure (4.4).

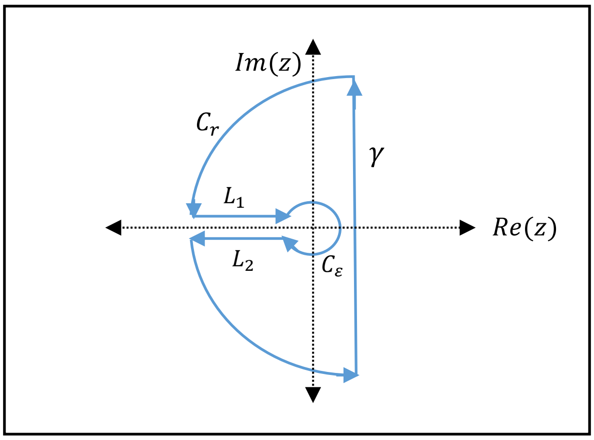


Figure 4.2 Integral Contour C

Since $f(z) = \frac{1}{\sqrt{z}}$ is analytic in the contour C, then apply Cauchy theorem on the left-hand side of (4.6.2) to obtain

$$\oint_C \frac{1}{\sqrt{z}} dz = 0.$$

(Brown, Churchill et al. 1996, Kodaira 2007)

Now let $r \to \infty$, $\varepsilon \to 0$, apply Jordan lemma $\int_{C_r} \to 0$, $\int_{C_\varepsilon} \to 0$.

We have got $\int_\gamma = -\int_{L_1} - \int_{L_2}$ Therefore

$$\int_\gamma \frac{1}{\sqrt{z}} e^z dz = -\int_\infty^0 \frac{1}{\sqrt{r\, e^{i\pi}}} e^{re^{i\pi}}\ e^{i\pi} dr - \int_0^\infty \frac{1}{\sqrt{r\, e^{-i\pi}}} e^{re^{-i\pi}}\ e^{-i\pi} dr$$

$$= \int_0^\infty \frac{1}{\sqrt{r}} e^{re^{i\pi}}\ \left(e^{\pi i}\right)^{1/2} dr - \int_0^\infty \frac{1}{\sqrt{r}} e^{re^{-i\pi}}\ \left(e^{\pi i}\right)^{-1/2} dr.$$

Since

$$e^{i\pi} = cos(\pi) + i\sin(\pi) = -1, \qquad e^{-i\pi} = cos(\pi) - i\sin(\pi) = -1,$$

and

$$e^{\pi i/2} = cos(\pi/2) + i\sin(\pi/2) = i, \quad e^{-\pi i/2} = cos(\pi/2) - i\sin(\pi/2) = -i.$$

Then

$$\int_\gamma \frac{1}{\sqrt{z}} e^z dz = \int_0^\infty \frac{1}{\sqrt{r}} e^{-r}\ idr + \int_0^\infty \frac{1}{\sqrt{r}} e^{-r}\ idr \ = 2i \int_0^\infty \frac{1}{\sqrt{r}} e^{-r}\ dr.$$

Let $r = x^2 \to \sqrt{r} = x$; $dr = 2xdx$

$$2i \int_0^\infty \frac{1}{\sqrt{r}} e^{-r}\ dr = 2i \int_0^\infty \frac{1}{x} e^{-x^2}\ 2xdx = 4i \int_0^\infty e^{-x^2}\ dx = 2i\sqrt{\pi}\,.$$

Then we obtain

$$\int_\gamma \frac{1}{\sqrt{z}} e^z dz = 2i\sqrt{\pi}. \tag{4.3.3}$$

In the same way we can obtain

$$\int_{1-i\infty}^{1+i\infty} \frac{1}{\sqrt{z}} e^{-z} dz = 2\sqrt{\pi}. \tag{4.3.4}$$

Theorem 4.3.2

$$\int_{-\infty}^{\infty} \frac{e^{-imz}}{(z+z_0)} dz = \begin{cases} 2\pi i Res_{z=-z_0}\left(\dfrac{e^{-imz}}{z+z_0}\right), & m > 0, \\ 0, & m < 0, \end{cases} \tag{4.3.5}$$

where z is a complex variable , m is a real number.

Proof:

(i) When $m > 0$, the function $f(z) = e^{-imz}/(z+z_0)$ has pole at $z = -z_0$, we could look at a closed contour C , Figure (4.3), in the lower half plane, the integral runs along the real x-axis from r to $-r$ and then along the semicircle $z = re^{i\theta}$ from $\theta = \pi$ to $\theta = 0$. There is a singularity at $z = -z_0$ inside the contour C.

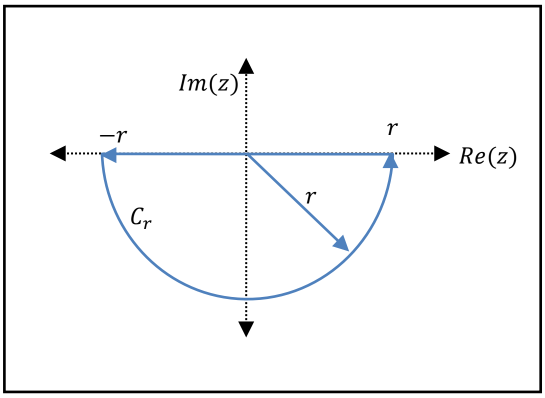


Figure 4.3 Integral Contour C

The integral (4.3.5) as can expressed as

$$\oint_c \frac{e^{-imz}}{(z+z_0)} dz = \int_{-\infty}^{\infty} \frac{e^{-imz}}{(z+z_0)} dz + \int_{C_r} \frac{e^{-imz}}{(z+z_0)} dz. \tag{4.3.6}$$

By Jordan lemma, the second integral of right hand side of (4.3.6) is zero so it reduces to

$$\oint_c \frac{e^{-imz}}{(z+z_0)} dz = \int_{-\infty}^{\infty} \frac{e^{-imz}}{(z+z_0)} dz.$$

By the residue theorem (Brown, Churchill et al. 1996, p.235) we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-imz}}{(z+z_0)} dz = 2\pi i Res_{z=-z_0}\left\{\frac{e^{-imz}}{(z+z_0)}\right\}.$$

(ii) If $m < 0$, in this case the integral runs from r to $-r$ and then around a semicircle in the upper half plane and the function $f(z)$ has no poles in this closed curve so the integral is automatically zero.

In the same manner

$$\int_{-\infty}^{\infty} \frac{e^{imz}}{(z-z_0)} dz = \begin{cases} 2\pi i Res_{z=-z_0}\left\{\dfrac{e^{imz}}{(z-z_0)}\right\}, & m > 0, \\ 0, & m < 0. \end{cases} \tag{4.3.7}$$

4.4 The Fourier Transform

The Wiener-Hopf technique is based mainly on Fourier transform in complex plane, and there are several equivalent forms of Fourier integral, but all previous literature which has dealt with this technique, it used the following formula of Fourier transform

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(x) e^{ix\alpha} \;\; dx \tag{4.4.1}$$

$$f\,(x) = \frac{1}{\sqrt{2\pi}} \int_{i\tau-\infty}^{i\tau+\infty} F(\alpha) e^{-ix\alpha} \;\; d\alpha \tag{4.4.2}$$

where α is a complex variable, the next two theorems prove it could obtain the equation (4.4.1) from equation (4.4.2) and vice versa with some conditions.

Theorem 4.4.1 (Noble 1958)

Let $f(x)$ be a function of real variable x such that $|f(x)| \leq Ae^{(\tau_{-}x)} as\ x \to \infty\ and\ |f(x)| \leq Be^{(\tau_{+}x)} as\ x \to -\infty$ where $\alpha = \sigma + i\tau$, with $\tau_{-} < \tau_{+}$:

Suppose that $\tau_{-} < \tau_0 < \tau_{+}$. Then if we define Fourier transform as given by (4.4.1) for $F(\alpha)$ is analytic function of α, regular in $\tau_{-} < \tau < \tau_{+}$, we can define the function $f(x)$ as given by (4.4.2) for any $\tau_{-} < \tau < \tau_{+}$.

Proof: Suppose that $g(x) = f(x)\, e^{-\tau x}$, substitute (4.4.1) in the right-hand side of (4.4.2), which becomes

$$I = \frac{1}{\sqrt{2\pi}} \int_{i\tau-\infty}^{i\tau+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{i\alpha x'} \ \ dx' \right) e^{-ix\alpha} \ \ d\alpha.$$

Substituting $\alpha = \sigma + i\tau$, the integral becomes

$$I = \frac{1}{2\pi} \int_{i\tau-\infty-i\tau}^{i\tau+\infty-i\tau} e^{-ix(\sigma+i\tau)} \int_{-\infty}^{\infty} f(x') e^{i(\sigma+i\tau)x'} \ \ dx'\, d\sigma.$$

From Fourier transform, we can say

$$G(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x')\, e^{i\sigma x'}\, dx'$$

where $g(x') = f(x')\, e^{-\tau x'}$ is given. We obtain

$$I = \frac{e^{\tau x}}{2\pi} \int_{-\infty}^{\infty} e^{-ix\sigma} \left\{ \sqrt{2\pi}\, G(\sigma) \right\} \ d\sigma.$$

Again, according to Fourier integral

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\sigma) e^{-ix\sigma} \ d\sigma.$$

Therefore

$$I = e^{\tau x} g(x) = e^{\tau x} f(x)\, e^{-\tau x} = f(x).$$

(Noble 1958 eq. 1.39) , the proof is completed.

Theorem 4.4.2 (Noble 1958)

Suppose that $F(\alpha), \alpha = \sigma + i\tau,$ is regular in $\tau_- < \tau < \tau_+$, be a function of real variable x such that $|F(\alpha)| \to 0$ uniformly as $|\sigma| \to \infty$ in $\tau_- + \varepsilon \le \tau_0 \le \tau_+ - \varepsilon,$ where ε is an arbitrary positive number. If we defined a function as given by (4.4.2) for a given τ,

$$\tau_- < \tau < \tau_+$$

and for any given x then we can define $F(\alpha)$ as given by (4.4.1).

Proof: Choose c, d such that $\tau_- < c < \tau < d < \tau_+$, and substitute (4.4.2) in the right-hand side of (4.7.1), which becomes

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{i\tau-\infty}^{i\tau+\infty} F(\beta) e^{-ix\beta} \ \ d\beta \right) e^{ix\alpha} \ \ dx.$$

By divided the interval the integral becomes

$$I \ = \frac{1}{2\pi} \left\{ \int_{-\infty}^{0} e^{ix\alpha} \int_{id-\infty}^{id+\infty} F(\beta) e^{-ix\beta} \ d\beta \, dx + \int_{0}^{\infty} e^{ix\alpha} \int_{ic-\infty}^{ic+\infty} F(\beta) e^{-ix\beta} d\beta \, e^{ix\alpha} \ dx \right\}.$$

Interchanging the orders to give

$$I \ = \frac{1}{2\pi} \left\{ \int_{id-\infty}^{id+\infty} F(\beta) \int_{-\infty}^{0} e^{i(\alpha-\beta)x} \ \ dx \ \ d\beta + \int_{ic-\infty}^{ic+\infty} F(\beta) \int_{0}^{\infty} e^{-i(\beta-\alpha)x} \ \ dx \ \ d\beta \right\}.$$

Since

$$\int_{-\infty}^{0} e^{i(\alpha-\beta)x} \ \ dx = \frac{1}{i(\alpha-\beta)},$$

and

$$\int_{0}^{\infty} e^{-i(\beta-\alpha)x} \ \ dx = \frac{1}{i(\beta-\alpha)},$$

then

$$I \ = -\frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \frac{F(\beta)}{\beta-\alpha} d\beta + \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{F(\beta)}{\beta-\alpha} \ \ d\beta.$$

Add and remove $\frac{1}{2\pi i} \int_{c}^{d} \frac{F(\beta)}{\beta-\alpha} d\beta$ and take the limit as $r \to \infty$ to obtain

$$I = \lim_{r\to\infty} \left\{ -\frac{1}{2\pi i} \int_{id-r}^{id+r} \frac{F(\beta)}{\beta-\alpha} d\beta + \frac{1}{2\pi i} \int_{ic-r}^{ic+r} \frac{F(\beta)}{\beta-\alpha} \ \ d\beta + \int_{c}^{d} \frac{F(\beta)}{\beta-\alpha} d\beta - \int_{c}^{d} \frac{F(\beta)}{\beta-\alpha} d\beta \right\}.$$

Then we have

$$I \ = \frac{1}{2\pi i} \lim_{r\to\infty} \left\{ \frac{1}{2\pi i} \int_{ic-r}^{ic+r} \frac{F(\beta)}{\beta-\alpha} \ \ d\beta + \int_{d}^{c} \frac{F(\beta)}{\beta-\alpha} d\beta + \int_{id+r}^{id-r} \frac{F(\beta)}{\beta-\alpha} d\beta + \int_{c}^{d} \frac{F(\beta)}{\beta-\alpha} d\beta \right\}.$$

The integral contour C is shown in Figure (4.6). Apply the residue theorem to contour C to give

$$I = \lim_{r\to\infty} \frac{1}{2\pi i}\int_C \frac{F(\beta)}{\beta-\alpha}d\beta = F(\alpha).$$

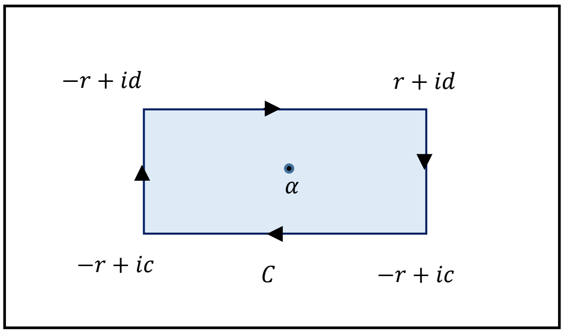


Figure 4.4 Integral Contour C

4.5 Fourier Transform of the modified Bessel function of the second kind

In this section, we determine the Fourier transform of modified Bessel function second kind, which is needed in this chapter and in the next one.

Theorem 4.5.1 The Fourier transform of $K_0\,(|x|)$ is $\frac{1}{\sqrt{2}}\left[\frac{\sqrt{\pi}}{\sqrt{\alpha^2+1}}\right]$.

Proof: From the Fourier integral formula

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} K_0\,(|x|)e^{ix\alpha}\;\; dx.$$

By splitting the Integral into two intervals, obtain

$$F(\alpha) \;\; = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} K_0\,(-x)e^{ix\alpha}\;\; dx + \int_{0}^{\infty} K_0\,(x)e^{ix\alpha}\;\; dx\right),$$

Swap the limits of the first integral with assuming $x=-x$ to get

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\left(\int_0^\infty K_0\,(x)e^{-ix\alpha}\;\;dx + \int_0^\infty K_0\,(x)e^{ix\alpha}\;\;dx\right)$$

For simplifying we suppose

$$I_1(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^\infty K_0\,(x)e^{ix\alpha}\;\;dx,$$

$$I_2(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^\infty K_0\,(x)e^{-ix\alpha}\;\;dx\,.$$

Since

$$I_2(\alpha) = I_1(-\alpha).$$

Therefore

$$F(\alpha) = I_1(\alpha) + I_2(\alpha) = I_1(\alpha) + I_1(-\alpha). \tag{4.5.1}$$

From Abramowitz and Stegun (1964, p.376, eq. 9.6.24)

$$K_0(z) = \int_0^\infty e^{-z\cosh t}dt = \frac{1}{2}\int_{-\infty}^\infty e^{-z\cosh t}dt. \tag{4.5.2}$$

We now evaluate the integral I_1, substitute (4.5.2) and interchange the order of integral to get

$$I_1(\alpha) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^\infty\int_0^\infty e^{-(\cosh t - i\alpha)x}\;\;dxdt.$$

Integrate the inner yield

$$I_1\,(\alpha) = \frac{-1}{2\sqrt{2\pi}}\int_{-\infty}^\infty \frac{1}{i\alpha - \cosh t}dt.$$

Multiply the integral by $\frac{t-(2\pi i+t)}{-2\pi i}$ and split it into two parts,

$$I_1(\alpha) = \frac{1}{\left(4\pi\sqrt{2\pi}\right)i}\left(\int_{-\infty}^\infty \frac{t}{i\alpha - \cosh t}dt - \int_{-\infty}^\infty \frac{t+2\pi i}{i\alpha - \cosh t}dt\right).$$

Let $s = t + 2\pi i$, substitute into the second one,

$$I_1(\alpha) = \frac{1}{\left(4\pi\sqrt{2\pi}\right)i}\left(\int_{-\infty}^\infty \frac{t}{i\alpha - \cosh t}dt - \int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{s}{i\alpha - \cosh(s-2\pi i)}ds\right).$$

Since $\cosh(s - 2\pi i) = \cosh s$ therefore

$$I_1(\alpha) = \frac{1}{(4\pi\sqrt{2\pi})i}\left(\int_{-\infty}^{\infty} \frac{t}{i\alpha - \cosh t} dt - \int_{-\infty+2\pi i}^{\infty+2\pi i} \frac{s}{i\alpha - \cosh s} ds\right).$$

Add and remove $\int_0^{2\pi} \frac{i\tau}{i\alpha - \cosh i\tau} i d\tau$, we can write

$$I_1(\alpha) = \frac{1}{(4\pi\sqrt{2\pi})i} lim_{\ r\to\infty}\left(\int_{-r}^{r} \frac{t}{i\alpha - \cosh t} dt + \int_0^{2\pi} \frac{i\tau}{i\alpha - \cosh i\tau} i d\tau\right.$$
$$\left. + \int_{r+2\pi i}^{-r+2\pi i} \frac{s}{i\alpha - \cosh s} ds + \int_{2\pi}^{0} \frac{i\tau}{i\alpha - \cosh i\tau} i d\tau\right).$$

By Cauchy Integral theorem yields

$$I_1(\alpha) = \frac{1}{(4\pi\sqrt{2\pi})i} lim_{\ r\to\infty}\left\{\oint_C \frac{z}{i\alpha - \cosh z} dz\right\}, \tag{4.5.3}$$

where the contour C is the closed rectangle shown from $-r$ to r along the real axis then going up to $r + 2\pi i$ reaching $-r + 2\pi i$ lastly returning down to $-r$. For integral path see Figure (4.7) .To look for the poles inside C, when $i\alpha + \cosh z = 0$, for $z = \sigma + i\tau$,

$$\cosh z = \cosh(\sigma + i\tau) = \cosh(\sigma)\cosh(i\tau) + \sinh(\sigma)\sinh(i\tau)$$

$$= \cosh(\sigma)\cos(\tau) + i\sinh(\sigma)\ \sin(\tau).$$

Substituting into $i\alpha - \cosh z = 0$ to obtain

$$\cos\tau\cosh\sigma + i\sin\tau\sinh\sigma = i\alpha.$$

That leads to

$$\cos\tau\cosh\sigma = 0$$

if $\cosh\sigma = 0$ then $\sigma = (\pi n + \frac{\pi}{2})i, n \in \mathbb{Z}$, but $\sigma = Re(z)$, so $\cosh(\sigma) \neq 0$ therefore

$\cos\tau = 0$ then $\tau = \frac{\pi}{2}, \frac{3\pi}{2}$, giving two poles.

When $\tau = \frac{\pi}{2}$, $i\sin\left(\frac{\pi}{2}\right)\sinh(\sigma) = i\alpha$, $\sigma = \sinh^{-1}(\alpha)$, so $z = \sinh^{-1}(\alpha) + i\frac{\pi}{2}$.

When $\tau = \frac{3\pi}{2}$, $i\sin\left(\frac{3\pi}{2}\right)\sinh(\sigma) = i\alpha$, $\sigma = -\sinh^{-1}(\alpha)$, so $z = -\sinh^{-1}(\alpha) + i\frac{3\pi}{2}$.

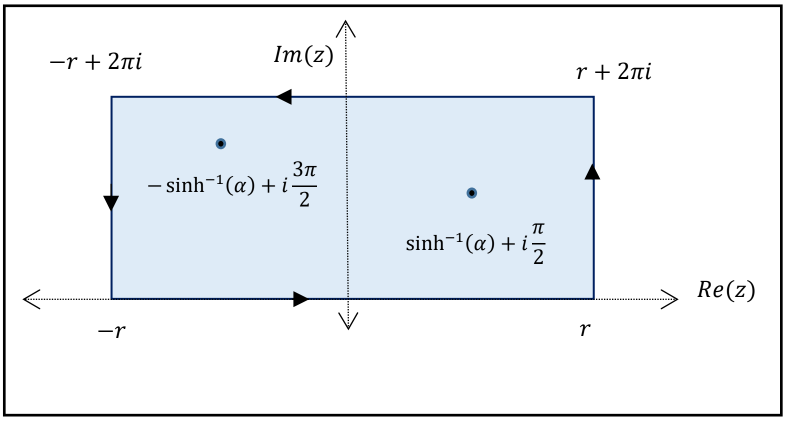


Figure 4.5 Integral Path

Therefore

$$I_1(\alpha) = \frac{2\pi i}{4\pi\sqrt{2\pi}\, i}\left[Res_{z=\sinh^{-1}\alpha+i\frac{\pi}{2}} \left\{\frac{z}{i\alpha - \cosh z}\right\} + Res_{z=-\sinh^{-1}\alpha+i\frac{3\pi}{2}} \left\{\frac{z}{i\alpha - \cosh z}\right\}\right].$$

Since $Res_{z=z_0}\{f(z)\} = \lim_{z\to z_0}(z-z_0)\, f(z)$, then we have

$$I_1(\alpha) = \frac{1}{2\sqrt{2\pi}}\left[lim_{z\to\sinh^{-1}(\alpha)+i\frac{\pi}{2}} \left\{\frac{z^2 - z\sinh^{-1}(\alpha) - i\frac{\pi}{2}z}{i\alpha - \cosh z}\right\} + lim_{z\to-\sinh^{-1}(\alpha)+i\frac{3\pi}{2}} \left\{\frac{z^2 + z\sinh^{-1}(\alpha) - i\frac{3\pi}{2}z}{i\alpha - \cosh z}\right\}\right].$$

The both of the limits is $0/0$, therefore the L'Hôpital Rule will be applied

$$I_1(\alpha) = \frac{1}{2\sqrt{2\pi}}\left[lim_{z\to\sinh^{-1}(\alpha)+i\frac{\pi}{2}} \left\{\frac{2z - \sinh^{-1}(\alpha) - i\frac{\pi}{2}}{-\sinh z}\right\} + lim_{z\to-\sinh^{-1}(\alpha)+i\frac{3\pi}{2}} \left\{\frac{2z + \sinh^{-1}(\alpha) - i\frac{3\pi}{2}}{-\sinh z}\right\}\right]. \qquad (4.5.4)$$

To calculate $\sinh\left(\sinh^{-1}(\alpha) + i\frac{\pi}{2}\right)$ we use:

$$\sinh(a+ib) = \sinh(a)\cos(b) + i\sin(b)\cosh(a)$$

so $\sinh\left(\sinh^{-1}(\alpha)+i\frac{\pi}{2}\right)=i\cosh(\sinh^{-1}\alpha)$, also we use

$$\cosh^2 z-\sinh^2 z=1\rightarrow \quad \cosh z=\sqrt{\sinh^2 z+1}.$$

Then

$$\cosh(\sinh^{-1}\alpha)=\sqrt{(\sinh[\sinh^{-1}(\alpha)])^2+1}=\sqrt{\alpha^2+1}.$$

We obtain

$$-\sinh\left(-\sinh^{-1}(\alpha)+i\frac{\pi}{2}\right)=-i\sqrt{\alpha^2+1}. \tag{4.5.5}$$

Similarly

$$-\sinh\left(-\sinh^{-1}(\alpha)+i\frac{3\pi}{2}\right)=i\sqrt{\alpha^2+1}, \tag{4.5.6}$$

and

$$2\left(\sinh^{-1}(\alpha)+i\frac{\pi}{2}\right)-\sinh^{-1}(\alpha)-i\frac{\pi}{2}=\sinh^{-1}(\alpha)+i\frac{\pi}{2}. \tag{4.5.7}$$

Also

$$2\left(-\sinh^{-1}(\alpha)+i\frac{3\pi}{2}\right)+\sinh^{-1}(\alpha)-\frac{3\pi}{2}i=-\sinh^{-1}(\alpha)+\frac{3\pi}{2}i. \tag{4.5.8}$$

Substitute (4.5.5), (4.5.6), (4.5.7) and (4.5.8) into (4.5.4) to obtain

$$I_1(\alpha)=\frac{1}{2\sqrt{2\pi}}\left[\frac{\sinh^{-1}(\alpha)+i\dfrac{\pi}{2}}{-i\sqrt{\alpha^2+1}}+\frac{-\sinh^{-1}(\alpha)+i\dfrac{3\pi}{2}}{i\sqrt{\alpha^2+1}}\right].$$

Then we obtain

$$I_1(\alpha)=\frac{1}{2\sqrt{2\pi}}\left[\frac{-2\sinh^{-1}(\alpha)+\pi i}{i\sqrt{\alpha^2+1}}\right]. \tag{4.5.9}$$

Since $I_2(\alpha)=I_1(-\alpha)$, therefore

$$I_2(\alpha)=\frac{1}{2\sqrt{2\pi}}\left[\frac{2\sinh^{-1}(\alpha)+\pi i}{i\sqrt{\alpha^2+1}}\right]. \tag{4.5.10}$$

Applying (4.5.9) and (4.5.10) into (4.5.1)

$$F(\alpha) = \frac{1}{2\sqrt{2\pi}}\left[\frac{2\pi i}{i\sqrt{\alpha^2+1}}\right] = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{\pi}}{\sqrt{\alpha^2+1}}\right]. \qquad (4.8.11)$$

By the same method we can find Fourier transform of $K_0\,(\lambda|x|)$, where $Re\,(\lambda) > 0,$ by assuming $\lambda x = t$ and $\alpha = \lambda\beta$. The equation (4.8.1) becomes

$$F(\alpha) = F(\lambda\beta) = \frac{1}{(\sqrt{2\pi})\lambda}\left(\int_0^\infty K_0\,(t)e^{it\beta}\ \ dt + \int_0^\infty K_0\,(x)e^{-it\beta}\ \ dt\right). \qquad (4.8.12)$$

By the same technique the equation (4.8.11) becomes

$$F(\lambda\beta) = \frac{1}{\sqrt{2}\lambda}\left[\frac{\sqrt{\pi}}{\sqrt{\beta^2+1}}\right]. \qquad (4.8.13)$$

Return to original symbols

$$F(\alpha) = \frac{1}{\sqrt{2}\lambda}\left[\frac{\sqrt{\pi}}{\sqrt{(\alpha/\lambda)^2+1}}\right] = \frac{1}{\sqrt{2}\lambda}\left[\frac{\sqrt{\pi}\lambda}{\sqrt{\alpha^2+\lambda^2}}\right].$$

We have

$$F(\alpha) = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{\pi}}{\sqrt{\alpha^2+\lambda^2}}\right]. \qquad (4.5.14)$$

This is the same result of Noble and Peters (1961), which it gives without proof. There is another technique to find the Fourier transform of $K_0\,(\lambda|x|)$ by the following. For

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty K_0\,(\lambda|x|)e^{i\alpha x}\ \ dx,$$

since $\lim_{\varepsilon\to 0}\left(e^{i\alpha x-\varepsilon|x|}\right) = e^{i\alpha x}$ then

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\int_{-\infty}^\infty K_0\,(\lambda|x|)e^{i\alpha x-\varepsilon|x|}\ \ dx.$$

Split the integral into two parts

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left(\int_0^\infty K_0\,(\lambda x)e^{-x(\varepsilon+i\alpha)}\ \ dx + \int_0^\infty K_0\,(\lambda x)e^{-x(\varepsilon-i\alpha)}\ \ dx\right).$$

From(Abramowitz and Stegun 1964, eq. 9.6.4 & eq. 9.6.5)

$$K_0\,(\lambda x) = \frac{\pi}{2} i[J_0(i\lambda x) + iY_0(i\lambda x)].$$

Substituting in the integral to get

$$F(\alpha) = \frac{\pi}{2}\frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\to 0}\Bigg(i\int_0^\infty J_0(i\lambda x)e^{-x(\varepsilon+i\alpha)}\ \ dx - \int_0^\infty Y_0(i\lambda x)e^{-x(\varepsilon+i\alpha)}\ \ dx$$
$$+\, i\int_0^\infty J_0(i\lambda x)e^{-x(\varepsilon-i\alpha)}\ \ dx - \int_0^\infty Y_0(i\lambda x)e^{-x(\varepsilon-i\alpha)}\ \ dx\Bigg).$$

According to Erdélyi (1954, p. 181, eq.1 & p.44, eq. 44)

$$\int_0^\infty J_0(\lambda x)e^{-xp}\,dx = \frac{1}{\sqrt{p^2+\lambda^2}}\,,\quad \int_0^\infty Y_0(\lambda x)\,e^{-xp}\,dx = -\frac{\pi}{2}\frac{\sinh^{-1}(p/\lambda)}{\sqrt{p^2+\lambda^2}},$$

which they are the Laplace transform of $J_0(\lambda x)$, $Y_0(\lambda x)$ respectively. We get

$$F(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{2}} \lim_{\varepsilon\to 0}\Bigg(\frac{i}{\sqrt{(\varepsilon+i\alpha)^2+(i\lambda)^2}} + \frac{\pi}{2}\frac{\sinh^{-1}((\varepsilon+i\alpha)/(i\lambda))}{\sqrt{(\varepsilon+i\alpha)^2+(i\lambda)^2}}$$
$$+\frac{i}{\sqrt{(\varepsilon-i\alpha)^2+(i\lambda)^2}} + \frac{\pi}{2}\frac{\sinh^{-1}((\varepsilon-i\alpha)/(i\lambda))}{\sqrt{(\varepsilon-i\alpha)^2+(i\lambda)^2}}\Bigg).$$

When $\varepsilon \to 0$, we obtain

$$F(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\Bigg(\frac{i}{\sqrt{(i\alpha)^2+(i\lambda)^2}} + \frac{\pi}{2}\frac{\sinh^{-1}(i\alpha/i\lambda)}{\sqrt{(i\alpha)^2+(i\lambda)^2}} + \frac{1}{\sqrt{(-i\alpha)^2+(i\lambda)^2}}$$
$$+\frac{\pi}{2}\frac{\sinh^{-1}(-\,i\alpha/i\lambda)}{\sqrt{(-i\alpha)^2+(i\lambda)^2}}\Bigg).$$

Since $\sinh^{-1}(-x) = -\sinh^{-1}(x)$ therefore

$$F(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\Bigg(\frac{2i}{\sqrt{-(\alpha^2+\lambda^2)}} + \frac{\pi}{2}\frac{\sinh^{-1}(\alpha/\lambda)}{\sqrt{-(\alpha^2+\lambda^2)}} - \frac{\pi}{2}\frac{\sinh^{-1}(\alpha/\lambda)}{\sqrt{-(\alpha^2+\lambda^2)}}\Bigg)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\Bigg(\frac{2i}{i\sqrt{\alpha^2+\lambda^2}}\Bigg).$$

Then

$$F(\alpha) = \frac{1}{\sqrt{2}}\Bigg(\frac{\sqrt{\pi}}{\sqrt{\alpha^2+\lambda^2}}\Bigg).$$

Theorem 4.5.2 The Fourier transforms of $\frac{x}{|x|}K_1\ (\lambda|x|)$ is $\left(\sqrt{\pi}/\sqrt{2}\lambda\right)\left(i\alpha/\sqrt{\alpha^2+\lambda^2}\right)$.

Proof: First, we will start with $\frac{x}{|x|}K_1\ (|x|)$, from Fourier transform integral formula

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{|x|}K_1\ (|x|)\ e^{i\alpha x}\ \ dx$$

$$=\frac{1}{\sqrt{2\pi}}\left(-\int_0^{\infty}K_1\ (x)e^{-i\alpha x}\ \ dx+\int_0^{\infty}K_1\ (kx)e^{i\alpha x}\ \ dx\right)=I_1+I_2, \qquad (4.5.15)$$

with

$$I_1=\frac{-1}{\sqrt{2\pi}}\int_0^{\infty}K_1\ (x)e^{-i\alpha x}\ \ dx\ , \quad I_2=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}K_1\ (x)e^{i\alpha x}\ \ dx.$$

From Abramowitz and Stegun (1964, p. 376, eq. 9.6.24)

$$K_1(x)=\int_0^{\infty}\cosh t\ e^{-x\cosh t}dt=\frac{1}{2}\int_{-\infty}^{\infty}\cosh t\ e^{-x\cosh t}dt. \qquad (4.5.16)$$

Inserting equation (4.5.16) into first part of equation (4.5.15),

$$I_1=\frac{-1}{\sqrt{2\pi}}\int_0^{\infty}\left(\frac{1}{2}\int_{-\infty}^{\infty}\cosh t\ e^{-x\cosh t}dt\right)e^{-i\alpha x}dx.$$

Interchange the order of integral and integrate the inner to get

$$I_1=\frac{-1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\cosh t}{i\alpha+\cosh t}\ dt.$$

Multiply by $\frac{t-(t+2\pi i)}{-2\pi i}$, and split it into two integrals,

$$I_1=\frac{1}{4\pi i\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\frac{t\cosh t}{i\alpha+\cosh t}\ dt-\int_{-\infty}^{\infty}\frac{(t+2\pi i)\cosh t}{i\alpha+\cosh t}\ dt\right).$$

Let $s=(t+2\pi i)$, substitute into the second one, since $\cosh(s-2\pi i)=\cosh(s)$, then

$$I_1=\frac{1}{4\pi i\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\frac{t\cosh t}{i\alpha+\cosh t}\ dt+\int_{\infty+2\pi i}^{-\infty+2\pi i}\frac{s\cosh s}{i\alpha+\cosh s}\ ds\right).$$

By Cauchy Integral theorem (Brown, Churchill et al. 1996) we can say

$$I_1 = \frac{1}{4\pi i\sqrt{2\pi}} \oint_C \frac{z\cosh z}{i\alpha + \cosh z}\, dz.$$

There are two poles at $z = -\sinh^{-1}(\alpha) + \frac{\pi}{2}i$ and $z = \sinh^{-1}(\alpha) + \frac{3\pi}{2}i$ inside C.

$$I_1 = \frac{1}{4\pi i\sqrt{2\pi}} \oint_C \frac{z\cosh z}{i\alpha - \cosh z}\, dz$$

$$= \frac{2\pi i}{4\pi i\sqrt{2\pi}} \left[Res_{z=-\sinh^{-1}(\alpha)+\frac{\pi}{2}i} \left\{\frac{z\cosh z}{i\alpha + \cosh z}\right\} + \; Res_{\sinh^{-1}(\alpha)+\frac{3\pi}{2}i} \left\{\frac{z\cosh z}{i\alpha + \cosh z}\right\}\right].$$

This gives

$$I_1 = \frac{1}{2\sqrt{2\pi}}\, lim_{z\to -\sinh^{-1}(\alpha)+\frac{\pi}{2}i} \left\{\frac{[z + \sinh^{-1}(\alpha) - (\pi/2)i]z\cosh z}{i\alpha + \cosh z}\right\}$$
$$+ \frac{1}{2\sqrt{2\pi}}\, lim_{z\to \sinh^{-1}(\alpha)+\frac{3\pi}{2}i} \left\{\frac{[z - \sinh^{-1}(\alpha) - (3\pi/2)i]z\cosh z}{i\alpha + \cosh \mathrm{z}}\right\}.$$

Both of the limits are $0/0$, therefore the L'Hôpital Rule will be applied, give

$$lim_{z\to z_0} \left\{\frac{(z-z_0)z\cosh z}{i\alpha + \cosh z}\right\} = lim_{z\to z_0} \left\{\frac{(z-z_0)\cosh z + (z-z_0)z\sinh z + z\cosh z}{\sinh \mathrm{z}}\right\}.$$

Therefore when $z \to -\sinh^{-1}(\alpha) + \frac{\pi}{2}i$ the first limits becomes

$$\frac{\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right)\cosh\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right)}{\sinh\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right)}.$$

Since

$$\cosh\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right) = -i\alpha \quad , \quad \sinh\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right) = i\sqrt{\alpha^2+1}$$

yield

$$\frac{\left(-\sinh^{-1}(\alpha) + \frac{\pi}{2}i\right)(-i\alpha)}{i\sqrt{\alpha^2+1}} = \frac{\left(\alpha\sinh^{-1}(\alpha) - \frac{\pi}{2}i\alpha\right)}{\sqrt{\alpha^2+1}}.$$

Whereas the second limits when $z \to \sinh^{-1}(\alpha) + \frac{3\pi}{2}i$ gives

$$\frac{\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)\cosh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)}{\sinh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)}.$$

Also

$$\cosh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)=-i\alpha\,,\qquad \sinh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)=-i\sqrt{\alpha^2+1},$$

we have

$$\frac{\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)\cosh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)}{\sinh\left(\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\right)}=\frac{\left(\alpha\sinh^{-1}(\alpha)+\frac{3\pi}{2}i\alpha\right)}{\sqrt{\alpha^2+1}}.$$

Then

$$I_1=\frac{1}{4\pi i\sqrt{2\pi}}\oint_C\frac{zdz}{i\alpha-\cosh z}=\frac{1}{2\sqrt{2\pi}}\left\{\frac{\alpha\sinh^{-1}(\alpha)-\frac{\pi i\alpha}{2}+\alpha\sinh^{-1}(\alpha)+\frac{3\pi i\alpha}{2}}{\sqrt{\alpha^2+1}}\right\}$$

$$=\frac{1}{2\sqrt{2\pi}}\left\{\frac{2\alpha\sinh^{-1}(\alpha)+\pi i\alpha}{\sqrt{\alpha^2+1}}\right\},$$

which gives the first part. I_2 follows similarly as

$$I_2=\frac{1}{4\pi i\sqrt{2\pi}}\oint_C\frac{z\cosh z\,dz}{i\alpha-\cosh z}=\frac{1}{2\sqrt{2\pi}}\left\{\frac{-\alpha\sinh^{-1}(\alpha)-\frac{\pi i\alpha}{2}-\alpha\sinh^{-1}(\alpha)+\frac{3\pi i\alpha}{2}}{\sqrt{\alpha^2+1}}\right\}$$

$$=\frac{1}{2\sqrt{2\pi}}\left\{\frac{-2\alpha\, sinh^{-1}(\alpha)+i\pi\alpha}{\sqrt{\alpha^2+1}}\right\},$$

which gives the second part. Further,

$$I_1+I_2=\left(\frac{1}{2\sqrt{2\pi}}\right)\left(\frac{2\alpha\sinh^{-1}(\alpha)+i\pi\alpha-2\alpha\sinh^{-1}(\alpha)+i\pi\alpha}{\sqrt{\alpha^2+1}}\right)=\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{i\alpha}{\sqrt{\alpha^2+1}}.$$

Substitute $I_1+\ I_2$ into equation (4.5.15),

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{|x|}K_1\,(|x|)\,e^{i\alpha x}\ \ dx=\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{i\alpha}{\sqrt{\alpha^2+1}}.\qquad(4.5.17)$$

In the same mothed we can evaluate Fourier transform of $\frac{x}{|x|} K_1 (\lambda|x|)$ by assuming $\lambda x = s$ and $\alpha = \lambda\beta$ which the right hand side of the equation (4.8.15) becomes

$$\frac{1}{(\sqrt{2\pi})\lambda}\left(-\int_0^\infty K_1 (s)e^{-i\beta s} \; ds + \int_0^\infty K_1 (s)e^{i\beta s} \; ds\right).$$

Following the same technique, the right hand side of the equation (4.5.17) becomes

$$\frac{\sqrt{\pi}}{(\sqrt{2\pi})\lambda}\frac{i\beta}{\sqrt{\beta^2+1}}.$$

Lastly, returning to the origin variables,

$$\frac{\sqrt{\pi}}{(\sqrt{2})\lambda}\frac{i(\alpha/\lambda)}{\sqrt{(\alpha/\lambda)^2+1}} = \frac{\sqrt{\pi}}{(\sqrt{2})\lambda}\frac{i(\alpha/\lambda)\sqrt{\lambda^2}}{\sqrt{\alpha^2+\lambda^2}} = \frac{\sqrt{\pi}}{(\sqrt{2})\lambda}\frac{i\alpha}{\sqrt{\alpha^2+k^2}}.$$

So we obtain

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{x}{|x|}K_1 (\lambda|x|)\, e^{i\alpha x} \; dx = \frac{\sqrt{\pi}}{(\sqrt{2})\lambda}\frac{i\alpha}{\sqrt{\alpha^2+\lambda^2}}. \tag{4.5.18}$$

There is a different way to get the same result of the equation (4.5.17) as following

$$\frac{\partial}{\partial x}\{K_0(|x|)\} = \int_0^\infty -\frac{x}{|x|}\; e^{-|x|\cosh(t)} \; \cosh(t)\, dx.$$

But from the equation (4.5.17)

$$K_1 (|x|) = \int_0^\infty e^{-|x|\cosh(t)} \; \cosh(t)\, dx.$$

Therefore

$$\frac{\partial}{\partial x}\{K_0(|x|)\} = -\frac{x}{|x|}K_1 (|x|). \tag{4.5.19}$$

Applying Fourier transform to both sides of (4.5.19)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}K_0(|x|)\right\}\,e^{i\alpha x}\quad dx=\frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{|x|}K_1\,(|x|)\,e^{i\alpha x}\quad dx. \tag{4.5.20}$$

Integration by parts method is performed to the left side hand to get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}K_0(|x|)\right\}\,e^{i\alpha x}\,dx$$
$$=\frac{1}{\sqrt{2\pi}}\left\{\left[-\frac{x}{|x|}e^{i\alpha x}K_1\,(|x|)\right]_{x=-\infty}^{x=\infty}-\int_{-\infty}^{\infty}i\alpha K_0\,(|x|)\,e^{i\alpha x}\quad dx\right\}.$$

From definition of modified Bessel function $K_1\,(|x|)\rightarrow 0$ as $x\rightarrow\infty$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}K_0(|x|)\right\}\,e^{i\alpha x}\,dx=\frac{-i\alpha}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_0\,(|x|)\,e^{i\alpha x}\quad dx. \tag{4.5.21}$$

From equation (4.8.20) and (4.8.21) we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{|x|}K_1\,(|x|)\,e^{i\alpha x}\quad dx=\frac{i\alpha}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_0\,(|x|)\,e^{i\alpha x}\quad dx.$$

Finally, from the equation (4.8.11)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{|x|}K_1\,(|x|)\,e^{i\alpha x}\quad dx=\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{i\alpha}{\sqrt{\alpha^2+1}}.$$

The proof is complete.

4.6 Shift theorem

The Fourier transform with respect to x of $f(x-\ y)$ is $F(\alpha)e^{i\alpha y}$.

Proof: Let $\mathcal{F}\{f(x)\}=F(\alpha),\quad \mathcal{F}^{-1}\{F(\alpha)\}=f(x)$. Now

$$\mathcal{F}\{f(x-\ y)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-\ y)e^{i\alpha s}dx. \tag{4.6.1}$$

Multiplying the right-hand side of (4.6.1) by $\left(e^{i\alpha y}.\,e^{-i\alpha y}=1\right)$

$$\mathcal{F}\{f(x-\ y)\}\ =\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-y)e^{i\alpha x}\,e^{i\alpha y}\,e^{-i\alpha y}dx\ =\frac{e^{i\alpha y}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-y)e^{i\alpha(x-y)}\,dx.$$

Substituting $u = x - y$ and $du = dx$ yields

$$\mathcal{F}\{f(x-y)\} = e^{i\alpha y}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\alpha u}\,du.$$

By using of The Fourier transform definition we obtain

$$\mathcal{F}\{f(x-y)\} = e^{i\alpha y}F(\alpha). \tag{4.6.2}$$

4.7 Convolution theorem

$$\int_0^{\infty} l(x-y)f(y)\,dy = \int_{-\infty}^{\infty} L(\alpha)F_+(\alpha)e^{-i\alpha x}\,d\alpha. \tag{4.7.1}$$

Where $F_+(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(y)e^{i\alpha y}\,dy.$

Proof: Applying Fourier transformation of left hand side of (4.7.1) yield

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty} l(x-y)f(y)\,dy\,e^{i\alpha x}\right)dx.$$

Interchange the order of integration and by shift theorem (4.6.2) obtian

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(y)\left(\int_{-\infty}^{\infty} l(x-y)e^{i\alpha x}\,dx\right)dy = L(\alpha)\left(\frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(y)e^{i\alpha y}\,dy\right).$$

By Fourier transform definition, we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty} l(x-y)f(y)\,dy\right)e^{i\alpha x}\,dx = L(\alpha)\,F_+(\alpha). \tag{4.7.2}$$

Applying inverse Fourier transform to both sides of (4.7.2), the result follows. (Bracewell, 1978, p. 108; Brigham, 1988, p. 60).

4.8 The Wiener-Hopf Technique.

In this section we will show the general solution of integral equation by using Wiener-Hopf technique. This method was developed by Noble (1958) to obtain the general solution to

$$\int_0^{\infty} l(x-\ y) f(y)\, dy\ = g(x);\ 0\ <\ x\ <\ \infty \tag{4.8.1}$$

for l, g as known functions and f unknown function.

Define

$$\left.\begin{aligned} L(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l(x) e^{i\alpha x}\ dx, \\ F_+(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x) e^{i\alpha x}\ dx, \end{aligned}\right\} \tag{4.8.2}$$

where $\alpha = \sigma + i\tau$, $F_+(\alpha)$ is regular for $\tau > q$; $L(\alpha)$ is regular and non-zero in $-p < \tau < p$ and $-p < q < p$,

First we will apply convolution theorem (4.7.1) on the left-hand side of (4.8.1)

$$\int_0^{\infty} l(x-\ y) f(y)\, dy = \int_{-\infty}^{\infty} L(\alpha) F_+(\alpha) e^{-i\alpha x}\ d\alpha, \tag{4.8.3}$$

We suppose that L can be decomposed in the form

$$L(\alpha) =\ L_+(\alpha) L_-(\alpha), \tag{4.8.4}$$

where $L_+(\alpha)L_-(\alpha)$ are regular and non-zero in $\tau > -p,\ \tau < p$ respectively in upper (lower) half regions of the complex α-plane.

By using (4.8.3) and (4.8.4), the equation (4.8.1) can be expressed as

$$\int_{-\infty}^{\infty} L_+(\alpha) L_-(\alpha) F_+(\alpha) e^{-i\alpha x}\ d\alpha\ = g(x);\quad 0\ <\ x\ <\ \infty\,. \tag{4.8.5}$$

Now we replace x by $x + s$ with $x > 0, s > 0$

$$\int_{-\infty}^{\infty} L_+(\alpha)L_-(\alpha)F_+(\alpha)e^{-i\alpha(x+s)}\ d\alpha\ = g(x+s), x\ >\ 0\,.$$

Multiply both sides by $n(s)$

$$n(s)\int_{-\infty}^{\infty} L_+(\alpha)L_-(\alpha)F_+(\alpha)e^{-i\alpha(x+s)}d\alpha\ = n(s)g(x+s)\,, x\ >\ 0.$$

where

$$n(s) = \frac{1}{2\pi}\int_{0}^{\infty} N_-(\alpha)e^{i\alpha s}d\alpha\,,\qquad N_-(\alpha) = \int_{0}^{\infty} n(s)e^{-i\alpha s}ds. \tag{4.8.6}$$

Integrate with respect to s from 0 to ∞ to achieve

$$\int_{0}^{\infty} n(s)\int_{-\infty}^{\infty} L_+(\alpha)L_-(\alpha)F_+(\alpha)e^{-i\alpha s}e^{-i\alpha x}d\alpha\, ds\ = \int_{0}^{\infty} n(s)g(x+s)ds\ , x\ >\ 0.$$

Interchange the orders of integration of left-hand side

$$\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} n(s)e^{-i\alpha s}ds\right)L_+(\alpha)L_-(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha\ = \int_{0}^{\infty} n(s)g(x+s)ds\ , x\ >\ 0$$

Using (4.8.6) gives

$$\int_{-\infty}^{\infty} N_-(\alpha)L_+(\alpha)L_-(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha\ = \int_{0}^{\infty} n(s)g(x+s)ds\ , x\ >\ 0. \tag{4.8.7}$$

Choose

$$N_-(\alpha) = \frac{1}{(\alpha-u)L_-(\alpha)}, \tag{4.8.8}$$

where u is arbitrary constant such that Im $u\ \geq p,\ N_-(\alpha)$ is regular and non-zeroin $\tau < p$.

Insertion of (4.8.8) into (4.8.7) gives

$$\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)\frac{e^{-i\alpha x}}{(\alpha-u)}d\alpha = \int_{0}^{\infty} n(s)g(x+s)ds\,, x>0.$$

Multiply this equation by e^{iux}

$$e^{iux}\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)\frac{e^{-i\alpha x}}{(\alpha-u)}d\alpha = e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds.$$

Differentiate with respect to x

$$\frac{d}{dx}\left(\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)\frac{e^{(u-\alpha)ix}}{(\alpha-u)}d\alpha\right) = \frac{d}{dx}\left(e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds\right).$$

We have

$$\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)\frac{-(\alpha-u)ie^{(u-\alpha)ix}}{(\alpha-u)}d\alpha = \frac{d}{dx}\left(e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds\right),$$

then we obtain

$$\left(-ie^{iux}\right)\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha = \frac{d}{dx}\left(e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds\right).$$

Multiply this equation by $\frac{1}{\sqrt{2\pi}(-ie^{iux})}$ gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha = \frac{ie^{-iux}}{\sqrt{2\pi}}\frac{d}{dx}\left(e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds\right). \qquad (4.8.9)$$

Let

$$G(x) = \frac{ie^{-iux}}{\sqrt{2\pi}}\frac{d}{dx}\left(e^{iux}\int_{0}^{\infty} n(s)g(x+s)ds\right), (x>0, s>0). \qquad (4.8.10)$$

Now, the equation (4.8.9) becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha = G(x), \qquad (x>0). \tag{4.8.11}$$

Applying inverse Fourier transform to equation (4.8.11)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} L_+(\alpha)F_+(\alpha)e^{-i\alpha x}d\alpha\right)e^{i\alpha y}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(y)e^{i\alpha y}dy,$$

gives

$$L_+(\alpha)F_+(\alpha) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0}\big(G(y)\big)\,e^{i\alpha y}dy + \int_{0}^{\infty}\big(G(y)\big)\,e^{i\alpha y}dy\right). \tag{4.8.12}$$

The first term of the right hand side of (4.8.12) is automatically zero when $y<0$, so we obtain

$$F_+(\alpha) = \frac{1}{\sqrt{2\pi}L_+(\alpha)}\int_{0}^{\infty}\big(G(y)\big)\,e^{i\alpha y}dy.$$

Again applying the inverse transform of this integral equation, we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\frac{1}{L_+(\alpha)}\int_{0}^{\infty} G(y)e^{i\alpha y}dy\right)e^{-ix\alpha}d\alpha\ ,$$

$$2\pi f(x) = \int_{-\infty}^{\infty}\frac{e^{-ix\alpha}}{L_+(\alpha)}\int_{0}^{\infty} G(y)e^{i\alpha y}dy\,d\alpha.$$

Interchanging the order of integration and multiplying by $\left(e^{-ivx}\,e^{ivx}\right)$, where v is arbitrary constant in a lower half –plane,

$$2\pi f(x) = e^{-ivx}\int_{-\infty}^{\infty}\frac{e^{-i(\alpha-v)x}}{L_+(\alpha)}\int_{0}^{\infty} G(y)e^{i\alpha y}dy\,d\alpha,$$

since

$$\frac{d}{dx}\left(\frac{ie^{-i(\alpha-v)x}}{(\alpha-v)}\right) = e^{-i(\alpha-v)x}$$

then

$$2\pi f(x) = ie^{-ivx}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{e^{-i(\alpha-v)x}}{(\alpha-v)L_+(\alpha)}\int_{0}^{\infty} G(y)e^{i\alpha y}dy\, d\alpha.$$

Now, we can express f as

$$2\pi f(x) = ie^{-ivx}\frac{d}{dx}e^{ivx}\int_{-\infty}^{\infty}\frac{e^{-i\alpha x}}{(\alpha-v)L_+(\alpha)}\int_{0}^{\infty} G(y)e^{i\alpha y}dy\, d\alpha.$$

Interchange the order of integration gives

$$2\pi f(x) = ie^{-ivx}\frac{d}{dx}e^{ivx}\int_{0}^{\infty} G(y)\int_{-\infty}^{\infty}\frac{e^{-i(x-y)\alpha}}{(\alpha-v)L_+(\alpha)}d\alpha dy. \tag{4.8.13}$$

From theorem 4.3.2 the inner integral of (4.11.13) is zero when $x-y<0$, so $x-y>0$. The limits of integration of the outer one have to change. The equation (4.8.13) can be written as

$$f(x) = \frac{ie^{-ivx}}{2\pi}\frac{d}{dx}e^{ivx}\int_{0}^{x} G(y)\int_{-\infty}^{\infty}\frac{e^{-i(x-y)\alpha}}{(\alpha-v)L_+(\alpha)}d\alpha dy,$$

(Noble and Peters 1961, p. 119-121). Simplifying,

$$f(x) = \frac{ie^{-ivx}}{2\pi}\frac{d}{dx}e^{ivx}\int_{0}^{x} m(x-y)G(y)dy,$$

here

$$m(s) = \int_{-\infty}^{\infty}\frac{e^{-i\alpha s}}{(\alpha-v)L_+(\alpha)}d\alpha, \qquad (s>0). \tag{4.8.14}$$

The general solution of the Integral equation (4.8.1) is

$$f(x) = \frac{-e^{-ivx}}{\sqrt{(2\pi)^3}}\frac{d}{dx}e^{ivx}\int_{0}^{x} m(x-y)e^{-iuy}\left(\frac{d}{dy}e^{iuy}\int_{y}^{\infty} n(t-y)g(t)dt\right)dy. \tag{4.8.15}$$

(Noble and Peters 1961, p.121, eq. 9)

4.9 An application of the Wiener-Hopf Technique

Consider the following integral equation

$$\int_0^\infty K_0\,(\lambda|x-y|)f(y)\,dy \;=g(x);\; 0\;<\;x\;<\;\infty \qquad (4.9.1)$$

where K_0 is the modified Bessel function of the second kind. To obtain the general solution we need find the following $L_+(\alpha)$, $L_-(\alpha)$, $n(x)$, $m(x)$.

By the equation (4.5.14) the Fourier transform of $K_0\,(\lambda|x|)$ is

$$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{\pi}}{\sqrt{\alpha^2+\lambda^2}}\right).$$

To obtain (4.8.4) a suitable decomposition is given by

$$L_+(\alpha)=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{\pi}}{\sqrt{\alpha+i\lambda}}\right),\qquad L_-(\alpha)=\frac{1}{\sqrt{\alpha-i\lambda}}.$$

by choosing $u=-v=i\lambda$, and applying (4.8.6) gives

$$n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{\sqrt{\alpha-i\lambda}}d\alpha.$$

We know that when $x<0$ the integral is zero, to evaluate this integral when $x>0$, let

$$z=i(\alpha-i\lambda)x,\qquad (\alpha-i\lambda)=\frac{z}{ix},\quad d\alpha=\frac{dz}{ix},\qquad i\alpha x=z-\lambda x$$

$$n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{\sqrt{\alpha-i\lambda}}d\alpha=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{z}{ix}\right)^{-1/2}e^{z-\lambda x}\frac{dz}{ix}$$

$$=\frac{1}{2\pi}\left(\frac{1}{\sqrt{i}}\right)\left(\frac{1}{\sqrt{x}}\right)e^{-x\lambda}\int_{-\infty}^{\infty}\frac{1}{\sqrt{z}}e^{z}\,dz.$$

The equation (4.3.1) has been used to evaluate the integral, it gives

$$n(x) = \frac{\sqrt{i}}{\sqrt{\pi}} \frac{e^{-x\lambda}}{\sqrt{x}}.$$

Since $i = e^{\pi i/2}$ therefore

$$n(x) = \frac{e^{\pi i/4}}{\sqrt{\pi}} \frac{e^{-x\lambda}}{\sqrt{x}}. \tag{4.9.2}$$

Also applying $-v = i\lambda$ in (4.8.14) gives

$$m(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{\sqrt{\alpha + i\lambda}} d\alpha.$$

In same method we can calculate this integral by letting $z = i(\alpha + i\lambda)x$ to obtain

$(\alpha + i\lambda) = \frac{z}{ix},\ d\alpha = \frac{dz}{ix},\ i\alpha x = \lambda x + z,$ Substituting gives

$$m(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{\sqrt{\alpha + i\lambda}} d\alpha = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-i\infty}^{i\infty} \frac{\sqrt{ix}}{\sqrt{z}} e^{-\lambda x - z} \frac{dz}{ix}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{e^{-x\lambda}}{\sqrt{xi}} \int_{-i\infty}^{i\infty} \frac{1}{\sqrt{z}} e^{-z} dz.$$

From (4.3.4) and $\frac{1}{\sqrt{i}} = e^{-\pi i/4}$, we obtain

$$m(x) = 2\sqrt{2}\, e^{-\pi i/4} \frac{e^{-x\lambda}}{\sqrt{x}}. \tag{4.9.3}$$

In general if $H(\alpha) = \frac{1}{\alpha^q}$ then

$$h(x) = \int_{-\infty}^{\infty} \frac{1}{\alpha^q} e^{\mp ix\alpha}\, d\alpha = \begin{cases} \dfrac{2\pi}{\Gamma(q)} (x)^{q-1} e^{\mp iq\pi/2} & , x > 0, \\ 0 & , x < 0. \end{cases}$$

(Noble 1958, p. 88, ex. 2.4).

Substitute (4.9.2) and (4.9.3) into (4.8.15) giving

$$f(x) = \frac{-e^{-\lambda x}}{\sqrt{(2\pi)^3}} \frac{d}{dx} e^{\lambda x} \int_0^x e^{\lambda y} \; 2\sqrt{2} e^{-\pi i/4} \frac{e^{-(x-y)\lambda}}{\sqrt{x-y}}$$

$$\times \left(\frac{d}{dy} e^{-\lambda y} \int_y^\infty \frac{e^{\pi i/4}}{\sqrt{\pi}} \frac{e^{-(t-y)\lambda}}{\sqrt{t-y}} g(t) dt \right) dy.$$

Simplifying gives

$$f(x) = \frac{-e^{-\lambda x}}{\pi^2} \frac{d}{dx} e^{\lambda x} \int_0^x e^{\lambda y} \; \frac{e^{-\lambda x} \, e^{\lambda y}}{\sqrt{x-y}} \left(\frac{d}{dy} e^{-\lambda y} \int_y^\infty \frac{e^{-\lambda t} e^{\lambda y}}{\sqrt{t-y}} g(t) dt \right) dy,$$

$$= \frac{-e^{-\lambda x}}{\pi^2} \frac{d}{dx} \int_0^x \; \frac{e^{2\lambda y}}{\sqrt{x-y}} \left(\frac{d}{dy} \int_y^\infty \frac{e^{-\lambda t}}{\sqrt{t-y}} g(t) dt \right) dy.$$

This agrees with Noble and Peters (1961).

Chapter 5 The Oseen Solution in Integral Form

The solution in integral form of Oseen flow over the semi-Infinite flat plate is derived. The results in this chapter are well-known and we give them in more details than given in the literature. The main difference between what will discuss here and with Gautesen (Gautesen 1971) is we will derive the Oseen solution in integral form by Wiener-Hopf Integral by utilizing the general solution which is given in the previous chapter for any kernel. Then, in this chapter we will insert the Fourier transform of the particular function which relates to the Oseen problem. This is different from the study of Gautesen (Gautesen 1971) who derived the solution when the kernel is drag Oseenlet directly.

Moreover, the Gautesen study did not mention how to calculate the Fourier transform of the kernel which is an alternative form of the drag Oseenlet function. So in this chapter we will prove this alternative function is equivalent to drag Oseenlet in the limit when epsilon tends to zero. Then we will compute the Fourier transform.

5.1 Integral representation of Oseen flow past a flat plate

In this section we will derive the integral representation in Gautesen (1971) and in Bhattacharya (1975).

First we will start with Bhattacharya:

The Oseen approximation of the steady two dimensional flow of a viscous incompressible fluid of uniform velocity U in the x-direction is given in the equation (2.1.1) with the continuity equation (2.1.2) with boundary conditions (2.1.3) and (2.1.4). According to Olmstead (1965) the solution of (2.1.1) and (2.1.2) satisfying (2.1.3) is given by

$$u(x,y) = U + \int_0^\infty u^D(x,y;\ x_0,0)\,\sigma(x_0)dx_0, \tag{5.1.1}$$

$$v(x,y) = \int_0^\infty v^D(x,y;\ x_0,0)\,\sigma(x_0)dx_0, \tag{5.1.2}$$

(Olmstead 1965, p. 242, eq. 3.6), where

$$\begin{aligned} -2\pi u^D(x,y;\ x_0,0) &= \left\{\frac{x-x_0}{{r_s}^2} - ke^{k(x-x_0)}\left(\frac{x-x_0}{r_s}K_1(k\,r_s) + K_0(k\,r_s)\right)\right\}, \\ -2\pi v^D(x,y;x_0,0) &= \left\{\frac{y}{r_s} - ke^{k(x-x_0)}\frac{y}{r_s}K_1\,(k\,r_s)\right\}, \end{aligned} \tag{5.1.3}$$

(Olmstead 1965, p. 242, eq. 3.7), where ${r_{x_0}}^2 = (x-x_0)^2 + y^2$ and K_0 , $K_1$are the modified Bessel functions of the second kind in order zero, one respectively, and $k = \rho U/2\mu = U/2\nu$.

By using the condition (2.1.3), the first equation of (5.1.3) becomes:

$u^D(x-x_0,0) =$

$$-\frac{1}{2\pi}\left\{\frac{1}{|x-x_0|} - \left(\frac{x-x_0}{|x-x_0|}\ K_1\,(k|x-x_0|) + K_0\,(k\,|x-x_0|)\right)\,ke^{k(x-x_0)}\right\}, \tag{5.1.4}$$

on $y = 0$ then the equation (5.1.1) reduces to

$$U = \int_0^\infty u^D(x,0;\ x_0,0)\sigma(x_0)dx_0, \tag{5.1.5}$$

Now substituting the equation (5.1.4) into (5.1.5) we obtain

$$2\pi U = \int_0^\infty \frac{1}{x-x_0} - \left(\frac{x-x_0}{|x-x_0|}\ K_1\,(k\,|x-x_0|) + K_0\,(k\,|x-x_0|)\right)ke^{k(x-x_0)}\sigma(x_0)dx_0.$$

Let us say

$$h(x-x_0)=\frac{1}{(x-x_0)}-\left(\frac{x-x_0}{|x-x_0|}\,K_1\,(k\,|x-x_0|)+K_0\,(k\,|x-x_0|)\right)ke^{k(x-x_0)}. \quad (5.1.6)$$

The integral equation then becomes

$$2\pi U=\int_0^\infty h(x-x_0)\,\sigma(x_0)dx_0. \quad (5.1.7)$$

The integral equation (5.1.7) is called the Wiener-Hopf Integral (Noble 1958, Noble and Peters 1961, Polyanin and Manzhirov 2012). Now we will derive the integral equation that established in Bhattacharya. We will start with the equation (5.1.5), inserting equation (5.1.4) in this equation yields

$$2\pi U=\;k\int_0^\infty Q[k(x-x_0)]\sigma(x_0)dx_0, \quad (5.1.8)$$

where

$$Q[k(x-x_0)]=\frac{1}{k}h(x-x_0).$$

For convenience the following variables are introduce

$$s=kx, \qquad t=\;kx_0\,, \qquad \widetilde{\sigma(t)}=\frac{\sigma(x_0)}{2\pi U},$$

gives

$$Q(s-t)=\frac{1}{s-t}-\left(\frac{s-t}{|s-t|}\;K_1\,(|s-t|)+K_0\;\,(|s-t|)\right)\,e^{s-t}\,.$$

Substitute $dx_0=dt/k$, $\widetilde{\sigma(t)}=\sigma(x_0)/2\pi U$ and $Q(s-t)$ into (5.1.7) to obtain

$$2\pi U=k\int_0^\infty \big(Q(s-t)\big)\;2\pi U\widetilde{\sigma(t)}\frac{dt}{k}.$$

Finally, the integral equation becomes

$$1=\int_0^\infty Q(s-t)\quad \widetilde{\sigma(t)}dt. \quad (5.1.9)$$

The integral equation (5.1.8) is also called the Wiener-Hopf integral as well as the integral equation (5.1.6).

5.2 Oseen Solution of Integral Equation of the Basic Wiener-Hopf Problem

In this section we will drive the solution of Oseen flow past semi-infinite flat plate of integral equation (5.1.7) by using Wiener-Hopf technique.

Now, we resolve the integral equation (5.1.6) that was solved by Gautesen (Gautesen 1971) but with more details. He treated (5.1.7) by using ideas of Olmstead (Olmstead and Byrne 1966) that use auxiliary integral equation which is introduced in the following way

$$2\pi U = \int_0^\infty l(x-s)\,\sigma_\varepsilon(s)ds \quad , \quad x>0 \tag{5.2.1}$$

where

$$l(x) = e^{(k-\varepsilon)x}k\left[K_0\,(k|x|) + \frac{x}{|x|}\;K_1\,(k|x|)\right] - \varepsilon\left[K_0\,(\varepsilon|x|) + \frac{x}{|x|}K_1\,(\varepsilon|x|)\right],$$

and $0 < \varepsilon < k$.

Gautesen mentioned $\sigma_\varepsilon(x) \to \sigma(x)$ as $\varepsilon \to 0$. This means $l(x) \to h(x)$ as $\varepsilon \to 0$. To verify we just need to prove

$$\lim_{\varepsilon\to 0} \varepsilon\left[K_0\,(\varepsilon|x|) + \frac{x}{|x|}K_1(\varepsilon|x|)\right] = \frac{1}{x}.$$

According to Abramowitz and Stegun (1964, p. 375, eq. 9.6.8)

$$K_0(z) \to -lnz \text{ as } z \to 0.$$

Let $z = \varepsilon|x|$, since $x > 0$, $\frac{x}{|x|} = 1$ and $z = \varepsilon x$, so

$$\lim_{\varepsilon\to 0} K_0(\varepsilon x) = -\,ln(\varepsilon x)$$

then

$$\lim_{\varepsilon\to 0} \varepsilon K_0(\varepsilon x) = 0.$$

Differentiate both sides of $\lim_{\varepsilon\to 0} K_0(\varepsilon x) = -\,ln(\varepsilon x)$ with respect to x to get

$$\lim_{\varepsilon \to 0} \left\{ \frac{\partial}{\partial x} K_0(\varepsilon x) \right\} = \frac{-1}{x},$$

$$\lim_{\varepsilon \to 0} \left\{ -\frac{\partial}{\partial x} K_0(\varepsilon x) \right\} = \frac{1}{x}.$$

From the equations (4.5.2)

$$K_0(\varepsilon x) = \int_0^\infty e^{-\varepsilon x \cosh(t)} \; dt$$

$$\frac{\partial}{\partial x} \{K_0(\varepsilon x)\} = -\varepsilon \int_0^\infty e^{-\varepsilon x \cosh(t)} \; \cosh(t)\, dt$$

gives

$$-\frac{\partial}{\partial x} [K_0(\varepsilon x)] = \varepsilon K_1(\varepsilon x).$$

Therefore

$$\lim_{\varepsilon \to 0} \{\varepsilon K_1(\varepsilon x)\} = \lim_{\varepsilon \to 0} \left\{ -\frac{\partial}{\partial x} [K_0(\varepsilon x)] \right\} = -\frac{\partial}{\partial x} \lim_{\varepsilon \to 0} [K_0(\varepsilon x)] = -\frac{\partial}{\partial x} \{-\, ln(\varepsilon x)\} = \frac{1}{x}.$$

So we have

$$\lim_{\varepsilon \to 0} \varepsilon [K_0\,(\varepsilon x) + K_1(\varepsilon x)] = \lim_{\varepsilon \to 0} \varepsilon K_0\,(\varepsilon x) + \lim_{\varepsilon \to 0} \varepsilon K_1\,(\varepsilon x) = \frac{1}{x}.$$

We obtain

$$\lim_{\varepsilon \to 0} \varepsilon \left[K_0(\varepsilon |x|) + \frac{x}{|x|} K_1(\varepsilon |x|) \right] = \frac{1}{x}, \qquad x > 0. \tag{5.2.2}$$

Then

$$\lim_{\varepsilon \to 0} l(x) \to h(x). \tag{5.2.3}$$

We apply Wiener-Hopf technique on equation (5.2.1) instead of equation (5.1.7) and after that $\sigma_\varepsilon(x) \to \sigma(x)$ as $\varepsilon \to 0$. To obtain the solution we just need find the following $L_+(\alpha)$,

$L_-(\alpha)$, $m(x)$ and $n(x)$ (where $n(x)$ and $m(x)$ have defined in the equations (4.8.6), (4.8.14) respectively) as in the sections (4.8) and (4.9) from chapter four.

First, the Fourier Transform of $l(x)$ is $L(\alpha)$ as the following

$$\begin{aligned} L(\alpha) = \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_0\,(k|x|)\, e^{(k-\varepsilon)x} e^{i\alpha x}\ \ dx + \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{|x|}\ K_1\,(k|x|)\, e^{(k-\varepsilon)x} e^{i\alpha x}\ \ dx \\ - \frac{\varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_0\,(\varepsilon|x|)\, e^{i\alpha x}\ \ dx - \frac{\varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_1\,(\varepsilon|x|)\, e^{i\alpha x}\ \ dx \\ = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

With

$$I_1 = \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_0\,(k|x|)\, e^{(k-\varepsilon)x} e^{i\alpha x}\ \ dx = \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_0\,(k|x|)\, e^{i(\alpha - ik + i\varepsilon)x}\ \ dx.$$

$$I_2 = \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{|x|}\ K_1\,(k|x|)\, e^{(k-\varepsilon)x} e^{i\alpha x}\, dx = \frac{k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{|x|}\ K_1\,(k|x|)\, e^{i(\alpha - ik + i\varepsilon)x}\ \ dx.$$

$$I_3 = -\frac{\varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_0\,(\varepsilon|x|)\, e^{i\alpha x}\ \ dx, \quad I_4 = -\frac{\varepsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{|x|} K_1\,(\varepsilon|x|)\, e^{i\alpha x}\ \ dx.$$

From (4.5.14) and (4.5.18) we find

$$I_1 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) \frac{k}{\sqrt{(\alpha - ik + i\varepsilon)^2 + k^2}}\ .$$

$$I_2 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) \frac{i(\alpha - ik + i\varepsilon)}{\sqrt{(\alpha - ik + i\varepsilon)^2 + k^2}}\ .$$

$$I_3 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) \frac{-\varepsilon}{\sqrt{\alpha^2 + \varepsilon^2}}.$$

$$I_4 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) \frac{-i\alpha}{\sqrt{\alpha^2 + \varepsilon^2}}.$$

Then

$$I_1 + I_2 + I_3 + I_4 = \frac{\sqrt{\pi}}{\sqrt{2}} \left\{ \frac{2k + i\alpha - \varepsilon}{\sqrt{(\alpha - ik + i\varepsilon)^2 + k^2}} + \frac{-\varepsilon - i\alpha}{\sqrt{\alpha^2 + \varepsilon^2}} \right\}$$

It could be simplified the denominator of the first term between brackets as the following

$$\sqrt{(\alpha - ik + i\varepsilon)^2 + k^2} = \sqrt{\alpha^2 + 2\alpha(-ik + i\varepsilon) + (-ik + i\varepsilon)^2 + k^2}$$

$$= \sqrt{\alpha^2 + 2\alpha i\varepsilon - \varepsilon^2 - 2\alpha ik + 2\varepsilon k}$$

$$= \sqrt{(\alpha + i\varepsilon)^2 - 2ik(\alpha + i\varepsilon)}.$$

Inserting above to obtain

$$I_1 + I_2 + I_3 + I_4 = \frac{\sqrt{\pi}}{\sqrt{2}}\left\{\frac{2k + i\alpha - \varepsilon}{\sqrt{(\alpha + i\varepsilon)^2 - 2ik(\alpha + i\varepsilon)}} + \frac{-\varepsilon - i\alpha}{\sqrt{\alpha^2 + \varepsilon^2}}\right\}.$$

Consider

$$\frac{2k + i\alpha - \varepsilon}{\sqrt{(\alpha + i\varepsilon)^2 - 2ik(\alpha + i\varepsilon)}} = \frac{i(-2ik + \alpha + i\varepsilon)}{\sqrt{(\alpha + i\varepsilon)(\alpha + i\varepsilon - 2ik)}} = \frac{i\sqrt{(\alpha + i\varepsilon - 2ik)}}{\sqrt{(\alpha + i\varepsilon)}},$$

also

$$\frac{-\varepsilon - i\alpha}{\sqrt{\alpha^2 + \varepsilon^2}} = \frac{-i(-i\varepsilon + \alpha)}{\sqrt{(\alpha + i\varepsilon)(\alpha - i\varepsilon)}} = \frac{-i\sqrt{(\alpha - i\varepsilon)}}{\sqrt{(\alpha + i\varepsilon)}}.$$

That leads to

$$I_1 + I_2 + I_3 + I_4 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{i}{\sqrt{(\alpha + i\varepsilon)}}\left\{\sqrt{(\alpha + i\varepsilon - 2ik)} - \sqrt{(\alpha - i\varepsilon)}\right\}.$$

Multiplying by $\left[\sqrt{\alpha - 2ik + i\varepsilon} + \sqrt{\alpha - i\varepsilon}\,\right]/\left[\sqrt{\alpha - 2ik + i\varepsilon} + \sqrt{\alpha - i\varepsilon}\,\right]$ to give

$$I_1 + I_2 + I_3 + I_4 = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\frac{i}{\sqrt{(\alpha + i\varepsilon)}}\left\{\frac{-2i(k - \varepsilon)}{\sqrt{(\alpha - 2ik + i\varepsilon)} + \sqrt{(\alpha - i\varepsilon)}}\right\}$$

$$= \left(\sqrt{2\pi}\right)\frac{k - \varepsilon}{\sqrt{(\alpha + i\varepsilon)}}\left\{\frac{1}{\sqrt{(\alpha - 2ik + i\varepsilon)} + \sqrt{(\alpha - i\varepsilon)}}\right\}.$$

Then,

$$L(\alpha) = \sqrt{2\pi}(k - \varepsilon)(\alpha + i\varepsilon)^{-\frac{1}{2}}\left[(\alpha - 2ik + i\varepsilon)^{\frac{1}{2}} + (\alpha - i\varepsilon)^{\frac{1}{2}}\right]^{-1}.$$

This agrees with Gautesen (1971, p.147).

The Wiener-Hopf procedure depend on finding a product factorisation for the Fourier-transformed kernel, in the form

$$L(\alpha) = L_+(\alpha)L_-(\alpha).$$

Suitable choices for $L_+(\alpha)$ and $L_-(\alpha)$ are

$$L_+(\alpha) = (\alpha + i\varepsilon)^{-1/2},$$

$$L_-(\alpha) = (k-\varepsilon)\sqrt{2\pi}\left[(\alpha - 2ik + i\varepsilon)^{1/2} + (\alpha - i\varepsilon)^{1/2}\right]^{-1}.$$

Insert $L_+(\alpha)$ into (4.8.14) by choosing $-v = i\varepsilon$ to compute $m(s)$ as

$$m(s) = \int_{-\infty}^{\infty} \frac{e^{-i\alpha s}}{(\alpha + i\varepsilon)(\alpha + i\varepsilon)^{-\frac{1}{2}}} d\alpha = \int_{-\infty}^{\infty} \frac{e^{-i\alpha s}}{(\alpha + i\varepsilon)^{\frac{1}{2}}}\, d\alpha.$$

By equation (4.12.3) we have

$$m(s) = 2\sqrt{\pi}\, e^{-\pi i/4} \quad \left(\frac{1}{x}\right)^{1/2} e^{-x\varepsilon}.$$

Also, by choosing $u = 2ik - i\varepsilon$, and applying u and $L_-(\alpha)$ in (4.11.7) to give

$$n(x) = \frac{1}{2\pi\sqrt{2\pi}(k-\varepsilon)} \int_{-\infty}^{\infty} \frac{1}{(\alpha - 2ik + i\varepsilon)\left[\sqrt{\alpha - 2ik + i\varepsilon} + \sqrt{\alpha - i\varepsilon}\right]^{-1}} e^{i\alpha x} d\alpha.$$

Since

$$\frac{1}{(\alpha - 2ik + i\varepsilon)\left[\sqrt{\alpha - 2ik + i\varepsilon} + \sqrt{\alpha - i\varepsilon}\right]^{-1}} = \frac{1}{\sqrt{\alpha - 2ik + i\varepsilon}} + \frac{\sqrt{\alpha - i\varepsilon}}{\alpha - 2ik + i\varepsilon}.$$

Also

$$\frac{\sqrt{\alpha - i\varepsilon}}{\alpha - 2ik + i\varepsilon} = \frac{(\alpha - 2ik + i\varepsilon)}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} + \frac{2i(k-\varepsilon)}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}}.$$

Therefore

$$n(x) = \frac{1}{2\pi\sqrt{2\pi}(k-\varepsilon)}\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{\alpha-2ik+i\varepsilon}}e^{i\alpha x}d\alpha + \int_{-\infty}^{\infty}\frac{1}{\sqrt{\alpha-i\varepsilon}}e^{i\alpha x}d\alpha\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{2i(k-\varepsilon)}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}e^{i\alpha x}d\alpha\right\}.$$

Applying equation (4.12.2) into the first and second integrals gives

$$n(x) = \frac{1}{2\pi\sqrt{2\pi}(k-\varepsilon)}\left\{2\sqrt{\pi}e^{i\pi/4}\frac{e^{-2(k-\varepsilon)x}}{\sqrt{x}} + 2\sqrt{\pi}e^{i\pi/4}\frac{e^{-\varepsilon x}}{\sqrt{x}}\right.$$

$$\left. + \int_{-\infty}^{\infty}\frac{2i(k-\varepsilon)}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}e^{i\alpha x}d\alpha\right\}. \qquad (5.2.4)$$

It remains to calculate the third integral, consider the following integral

$$\oint_C \frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}d\alpha,$$

where C is the contour as shown, Figure (5.1), as the following

$$\oint_C \frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}d\alpha = \int_{-R}^{R} + \int_{C_R} + \int_{L_1} + \int_{C_\delta} + \int_{L_2}\left\{\frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}d\alpha\right\}$$

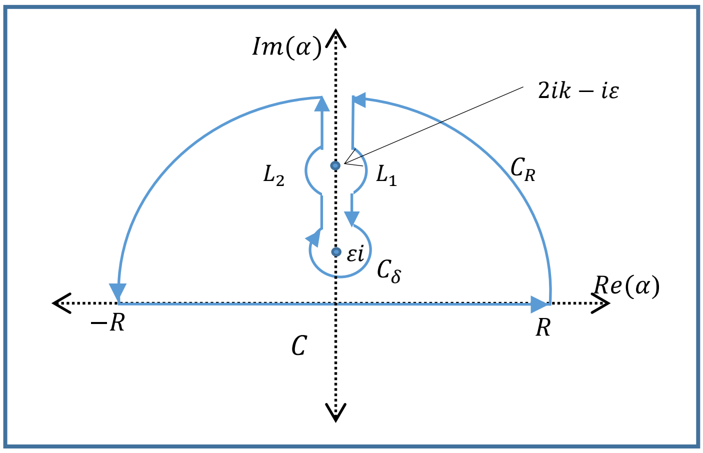


Figure 5.1 Integral Contour C

where

$$\int_{L_1} f(\alpha)d\alpha = -\int_{iR}^{2ik-i(\varepsilon+\delta)} - \int_{2ik-i(\varepsilon+\delta)}^{2ik-i(\varepsilon-\delta)} - \int_{2ik-i(\varepsilon+\delta)}^{i(\varepsilon+\delta)} \{f(\alpha)d\alpha\},$$

$$\int_{L_2} f(\alpha)d\alpha = \int_{i(\varepsilon+\delta)}^{2ik-i(\varepsilon-\delta)} + \int_{2ik-i(\varepsilon-\delta)}^{2ik-i(\varepsilon+\delta)} + \int_{2ik-i(\varepsilon+\delta)}^{iR} \{f(\alpha)d\alpha\}.$$

Let R tends to ∞ and δ tends to 0

$$\int_{L_1} f(\alpha)d\alpha = -\int_{i\infty}^{i\varepsilon} f(\alpha)d\alpha = \int_{i\varepsilon}^{i\infty} f(\alpha)d\alpha, \qquad \int_{L_2} f(\alpha)d\alpha = \int_{i\varepsilon}^{i\infty} f(\alpha)d\alpha.$$

Applying Jordan lemma , $\int_{C_R} \to 0$, $\int_{C_\delta} \to 0$, the counter C has a pole at $\alpha = 2ik - i\varepsilon$.

Thus

$$\int_{-\infty}^{\infty} \frac{e^{ix\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} d\alpha$$

$$= \oint_C \frac{e^{ix\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} d\alpha - \int_{i\varepsilon}^{i\infty} \frac{2e^{ix\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} d\alpha. \qquad (5.2.5)$$

Because of $i\varepsilon \notin$C, therefore $\frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}$ has one a pole in C which is $(2ik - i\varepsilon)$, then

$$\oint_C \frac{e^{ix\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} d\alpha = 2\pi i Res_{\alpha=2ik-i\varepsilon} \left\{\frac{e^{xi\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}}\right\}$$

$$= 2\pi i \lim_{\alpha \to 2ik-i\varepsilon} \left\{\frac{(\alpha - 2ik + i\varepsilon)e^{xi\alpha}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}}\right\}$$

$$= \frac{2\pi\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}. \qquad (5.2.6)$$

The second integral of (5.2.6) would be calculated, multiplying by$(e^{-x\varepsilon}.e^{x\varepsilon})$ gives

$$\int_{i\varepsilon}^{i\infty} \frac{2e^{x\alpha i}}{(\alpha - 2ik + i\varepsilon)\sqrt{\alpha - i\varepsilon}} d\alpha = 2e^{-x\varepsilon} \int_{i\varepsilon}^{i\infty} \frac{e^{xi(\alpha-\varepsilon i)}}{[(\alpha - \varepsilon i) - 2i(k - \varepsilon)]\sqrt{\alpha - i\varepsilon}} d\alpha.$$

Setting $iz = -i(\alpha - \varepsilon i)$, and multiplying the integral by $\sqrt{i}/\sqrt{i}$ gives

$$2\sqrt{i}e^{-x\varepsilon}\int_0^\infty \frac{e^{-x\,z}}{[z+2(\varepsilon-k)]\sqrt{z}}dz.$$

According to Abramowitz and Stegun (1964)

$$2\sqrt{i}e^{-x\varepsilon}\int_0^\infty \frac{e^{-x\,z}}{[\,z+2(\varepsilon-k)]\sqrt{\,z}}d\,z = 2\sqrt{i}e^{-x\varepsilon}\left\{\frac{\pi e^{-2x(k-\varepsilon)}}{\sqrt{-2(k-\varepsilon)}}\operatorname{erfc}\left(\sqrt{-2x(k-\varepsilon)}\right)\right\},$$

$$= \left\{\frac{-2\pi\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\left(1-\operatorname{erf}\left(i\sqrt{2x(k-\varepsilon)}\right)\right)\right\}. \tag{5.2.7}$$

The Error function and the complementary Error function are defined as (Abramowitz and Stegun 1964, eqs. 7.1.1 &7.1.2).

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt\,, \qquad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt = 1-\operatorname{erf}(z),$$

Applying (5.2.7) and (5.2.6) into (5.2.5) gives

$$\int_{-\infty}^\infty \frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}\,d\alpha = \frac{2\pi\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\left(\operatorname{erf}\left(i\sqrt{2x(k-\varepsilon)}\right)\right).$$

Also

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}dt,$$

(Ng and Geller 1969,p. 3, eq. 3), we obtain

$$\int_{-\infty}^\infty \frac{e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}\,d\alpha = \frac{2\pi\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\left\{\frac{2i}{\sqrt{\pi}}\int_0^{\sqrt{2(k-\varepsilon)x}} e^{t^2}dt\right\}$$

$$= \frac{4i\sqrt{\pi}\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\int_0^{\sqrt{2(k-\varepsilon)x}} e^{t^2}dt.$$

Let $t^2 = 2(k-\varepsilon)s, t = \sqrt{2(k-\varepsilon)s},\ dt = 2(k-\varepsilon)/2\sqrt{2(k-\varepsilon)s}\,ds,$

when $t=0 \rightarrow s=0$ and

$$t=\sqrt{2(k-\varepsilon)x} \rightarrow s=\frac{\left(\sqrt{2(k-\varepsilon)x}\right)^2}{2(k-\varepsilon)}=\frac{2(k-\varepsilon)x}{2(k-\varepsilon)}=x,$$

also

$$\int_0^{\sqrt{2(k-\varepsilon)x}} e^{t^2}dt=\int_0^x e^{2(\varepsilon-k)s}\frac{2(k-\varepsilon)}{2\sqrt{2(k-\varepsilon)s}}ds=\frac{\sqrt{2(k-\varepsilon)}}{2}\int_0^x \frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds.$$

Then we find

$$\int_{-\infty}^{\infty}\frac{2i(k-\varepsilon)e^{ix\alpha}}{(\alpha-2ik+i\varepsilon)\sqrt{\alpha-i\varepsilon}}d\alpha$$
$$=-\frac{8\sqrt{\pi}\sqrt{i}(k-\varepsilon)e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\left(\frac{\sqrt{2(k-\varepsilon)}}{2}\int_0^x\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right)$$

$$=-4\sqrt{\pi}(k-\varepsilon)e^{i\pi/4}e^{-(2k-\varepsilon)x}\int_0^x\frac{e^{2(\varepsilon-k)s}}{\sqrt{s}}ds.$$

$$=\frac{4i\sqrt{\pi}\sqrt{i}e^{-(2k-\varepsilon)x}}{\sqrt{2(k-\varepsilon)}}\int_0^{\sqrt{2(k-\varepsilon)x}} e^{t^2}dt.$$

Thus,

$$n(x)=\frac{1}{2\pi\sqrt{2\pi}(k-\varepsilon)}\left\{2\sqrt{\pi}e^{i\pi/4}\frac{e^{-(2k-\varepsilon)x}}{\sqrt{x}}+2\sqrt{\pi}e^{i\pi/4}\frac{e^{-\varepsilon x}}{\sqrt{x}}\right.$$

$$\left.-4\sqrt{\pi}(k-\varepsilon)e^{i\pi/4}e^{-(2k-\varepsilon)x}\int_0^x\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\},$$

$$=\frac{\sqrt{2}e^{i\pi/4}}{2\pi(k-\varepsilon)}\left\{\frac{e^{-(2k-\varepsilon)x}}{\sqrt{x}}+\frac{e^{-\varepsilon x}}{\sqrt{x}}-2(k-\varepsilon)e^{-(2k-\varepsilon)x}\int_0^x\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}. \tag{5.2.8}$$

Finally, to obtain the solution, insert the $u, v, g(t), m(s)$ and $n(s)$ into (4.11.15) gives

$$\sigma_\varepsilon(x)=\frac{-e^{-i(-i\varepsilon)x}}{\sqrt{(2\pi)^3}}\frac{d}{dx}e^{i(-i\varepsilon)x}\int_0^x e^{-i(2ik-i\varepsilon)y}\ 2\sqrt{\pi}e^{-\pi i/4}$$

$$\times \frac{e^{-(x-y)\varepsilon}}{\sqrt{x-y}}\left(\frac{d}{dy}e^{i(2ik-i\varepsilon)y}\int_{y}^{\infty}\frac{\sqrt{2}}{2\pi}\frac{e^{i\pi/4}}{(k-\varepsilon)}\left\{\frac{e^{-(2k-\varepsilon)(t-y)}}{\sqrt{(t-y)}}\right.\right.$$

$$\left.\left.+\frac{e^{-\varepsilon(t-y)}}{\sqrt{(t-y)}}-2(k-\varepsilon)e^{-(2k-\varepsilon)(t-y)}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}2\pi U dt\right)dy$$

$$=\frac{-4\sqrt{2}\pi\sqrt{\pi}Ue^{-i(-i\varepsilon)x}}{2\pi\sqrt{(2\pi)^3}(k-\varepsilon)}\frac{d}{dx}e^{i(-i\varepsilon)x}\int_0^x e^{-i(2ik-i\varepsilon)y}\,\frac{e^{-(x-y)\varepsilon}}{\sqrt{x-y}}\frac{d}{dy}e^{i(2ik-i\varepsilon)y}$$

$$\times\int_y^{\infty}\left\{\frac{e^{-(2k-\varepsilon)(t-y)}}{\sqrt{(t-y)}}+\frac{e^{-\varepsilon(t-y)}}{\sqrt{(t-y)}}-2(k-\varepsilon)e^{-(2k-\varepsilon)(t-y)}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}dt\,dy$$

$$=\frac{-Ue^{-\varepsilon x}}{\pi(k-\varepsilon)}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^{\infty}e^{-(2k-\varepsilon)y}\,e^{(2k-\varepsilon)y}$$

$$\times\left\{\frac{e^{-(2k-\varepsilon)t}}{\sqrt{(t-y)}}+\frac{e^{-\varepsilon t}e^{-2(k-\varepsilon)y}}{\sqrt{(t-y)}}-2(k-\varepsilon)e^{-(2k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}dtdy$$

$$=\frac{-Ue^{-\varepsilon x}}{\pi(k-\varepsilon)}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^{\infty}e^{-\varepsilon t}$$

$$\times\left\{\frac{e^{-2(k-\varepsilon)t}}{\sqrt{(t-y)}}+\frac{e^{-2(k-\varepsilon)y}}{\sqrt{(t-y)}}-2(k-\varepsilon)e^{-2(k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}dtdy$$

$$=\frac{-Ue^{-\varepsilon x}}{\pi(k-\varepsilon)}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^{\infty}e^{-\varepsilon t}$$

$$\times\left\{\frac{e^{-2(k-\varepsilon)t}}{\sqrt{(t-y)}}+\frac{e^{-2(k-\varepsilon)y}}{\sqrt{(t-y)}}-2(k-\varepsilon)e^{-2(k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right\}dtdy.$$

Now, since

$$\frac{d}{dt}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds=\frac{e^{2(k-\varepsilon)(t-y)}}{\sqrt{t-y}},$$

then, the product rule is applied to give

$$\frac{d}{dt}\left(e^{-2(k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right)=\frac{e^{-2(k-\varepsilon)y}}{\sqrt{t-y}}-2(k-\varepsilon)e^{-2(k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds.$$

Substituting to obtain

$$\sigma_\varepsilon(x)=\frac{-Ue^{-\varepsilon x}}{\pi(k-\varepsilon)}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^\infty e^{-\varepsilon t}$$

$$\times\left\{\frac{e^{-2(k-\varepsilon)t}}{\sqrt{(t-y)}}+\frac{d}{dt}\left(e^{-2(k-\varepsilon)t}\int_0^{(t-y)}\frac{e^{2(k-\varepsilon)s}}{\sqrt{s}}ds\right)\right\}dtdy.$$

When ε tends to zero, we get $\sigma_\varepsilon \to \sigma$

$$\sigma(x)=\frac{-U}{\pi k}\frac{d}{dx}\int_0^x\left[\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^\infty\left\{\frac{e^{-2kt}}{\sqrt{(t-y)}}+\frac{d}{dt}\left(e^{-2kt}\int_0^{(t-y)}\frac{e^{2ks}}{\sqrt{s}}ds\right)\right\}dt\right]dy.$$

Since

$$\frac{d}{dy}\int_y^\infty f(t)dt=-f(y),\quad \frac{d}{dy}\int_y^\infty\frac{d}{dt}\left(\int_0^{t-y}f(s)ds\right)dt=0,$$

therefore

$$\sigma(x)=\frac{-U}{\pi k}\frac{d}{dx}\int_0^x\left[\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\int_y^\infty\frac{e^{-2kt}}{\sqrt{(t-y)}}dt\right]dy,$$

also, it can be rewritten

$$\frac{e^{-2kt}}{\sqrt{(t-y)}}=\frac{d}{dt}\left(e^{-2ky}\int_0^{(t-y)}\frac{e^{-2ks}}{\sqrt{s}}ds\right).$$

This gives

$$\sigma(x)=\frac{-U}{\pi k}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\frac{d}{dy}\left(e^{-2ky}\int_y^\infty\frac{d}{dt}\int_0^{(t-y)}\frac{e^{-2ks}}{\sqrt{s}}ds\,dt\right)dy.$$

The product rule can be applied to the term between brackets gives

$$\sigma(x)=\frac{-U}{\pi k}\frac{d}{dx}\int_0^x\frac{e^{2ky}}{\sqrt{x-y}}\left(-2ke^{-2ky}\int_y^\infty\frac{d}{dt}\int_0^{(t-y)}\frac{e^{-2ks}}{\sqrt{s}}ds\,dt\right.$$

$$\left.+e^{-2ky}\frac{d}{dy}\int_y^\infty\frac{d}{dt}\int_0^{(t-y)}\frac{e^{-2ks}}{\sqrt{s}}ds\,dt\right)dy.$$

Also,

$$\frac{d}{dy}\left(\int_y^\infty \frac{d}{dt}\int_0^{(t-y)} \frac{e^{-2ks}}{\sqrt{s}}\,ds\,dt\right) = 0,$$

then

$$\sigma(x) = \frac{-U}{\pi k}\frac{d}{dx}\int_0^x \left[\frac{e^{2ky}}{\sqrt{x-y}}\left(-2ke^{-2ky}\int_y^\infty \frac{d}{dt}\int_0^{(t-y)} \frac{e^{-2ks}}{\sqrt{s}}\,ds\,dt\right)\right]dy$$

$$= \frac{2U}{\pi}\frac{d}{dx}\int_0^x \left[\frac{e^{2ky}}{\sqrt{x-y}}\left(e^{-2ky}\int_y^\infty \frac{e^{-2k(t-y)}}{\sqrt{(t-y)}}\,dt\right)\right]dy.$$

Then we have

$$\sigma(x) = \frac{2U}{\pi}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-y}}\int_y^\infty \frac{e^{2k(y-t)}}{\sqrt{(t-y)}}\,dtdy. \tag{5.2.9}$$

Let $x^2 = 2k(t-y)$ and $xdx = kdt$, the equation (5.2.9) can be expressed as

$$\sigma(x) = \frac{2U}{\pi}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-y}}\int_0^\infty \frac{\sqrt{2k}e^{-x^2}}{x}\frac{xdx}{k}\,dy$$

$$= \frac{2\sqrt{2}U}{\pi\sqrt{k}}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-y}}\int_0^\infty e^{-x^2}\,dxdy.$$

By using Gaussian integral (4.1.4)

$$\sigma(x) = \frac{2\sqrt{2}U}{\pi\sqrt{k}}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-y}}\left(\frac{\sqrt{\pi}}{2}\right)dy.$$

Finally, the integration gives$(2\sqrt{x})$, then differentiate it with respect to *x* gives the strength function of the drag Oseenlet

$$\sigma(x) = U\left(\frac{2}{\pi x k}\right)^{\frac{1}{2}}. \tag{5.2.10}$$

This solution agrees with Lewis and Carrier (1949), Gautesen (1971), Bhattacharya (1975), Olmstead and Gautesen (1976).

Chapter 6 Numerical Study

The main objective of this chapter is to present numerical solutions for the problem of the 2D steady flow past a flat plate with various Reynolds numbers. To achieve this aim, Boundary Layer Theory is applied to different approximations of the dimensionless Navier-Stokes equations, Finite Difference Method (FDM) has been used with uniform grid to solve these conservation equations with boundary conditions that the velocity is zero on the flat plate and the free stream velocity outside boundary layer. We start with Boundary Layer equations then Oseen Boundary Layer equations, Blasius equation and Oseen-Blasius equation for different grid size and Reynolds numbers is determined.

6.1 Navier-Stokes equations in dimensionless form

The two dimensional steady Navier-Stokes equations with the continuity equation are

$$\rho\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*}+\mu\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right),$$

$$\rho\left(u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*}+\mu\left(\frac{\partial^2 v^*}{\partial {x^*}^2}+\frac{\partial^2 v^*}{\partial {y^*}^2}\right),$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*} = 0.$$

(Batchelor 1967, Shankar 2007)

Let us make the variables in the equation above dimensionless. Introduce the non-dimensional variables as follows

$$x^* = Lx, \qquad y^* = Ly, \qquad u^* = Uu\,, \qquad v^* = Uv, \qquad p^* = \rho U^2 p \tag{6.1.1}$$

where L is the length of the flat plate and ρ and U are the density and the free stream velocity of the fluid respectively. Substituting these parameters in the x-momentum and y-momentum Navier-Stokes equations to obtain

$$\frac{\rho U^2}{L}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\rho U^2}{L}\frac{\partial p}{\partial x}+\frac{\mu U}{L^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{\rho U^2}{L}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\rho U^2}{L}\frac{\partial p}{\partial y}+\frac{\mu U}{L^2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).$$

Divide both sides by $(\rho U^2/L)$ to obtain

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}+\frac{\mu}{\rho UL}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}+\frac{\mu}{\rho UL}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \end{aligned}\right\} \tag{6.1.2}$$

The non-dimensional parameter Reynolds number is defined by

$$Re = \frac{\rho UL}{\mu}. \tag{6.1.3}$$

The Navier-Stokes equations can be rewritten as

x-momentum,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{6.1.4}$$

y-momentum,

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \tag{6.1.5}$$

Similarly, the dimensionless continuity equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{6.1.6}$$

6.2 The Finite Difference Method (FDM)

The explicit finite difference method (FDM) will be applied for the numerical solution. The derivatives are approached by finite differences approximated using the Taylor series expansion of functions as

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}(h)^2 + \frac{f''(x)}{3!}(h)^3 + \cdots$$

Dividing across by h gives

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

This is an approximation for the first derivative of the function f.

The discretization approximates the derivative, Δx and Δy denote the mesh spacing in the directions of increasing x and y respectively, the grid that will be used is illustrates in figure 6.1.

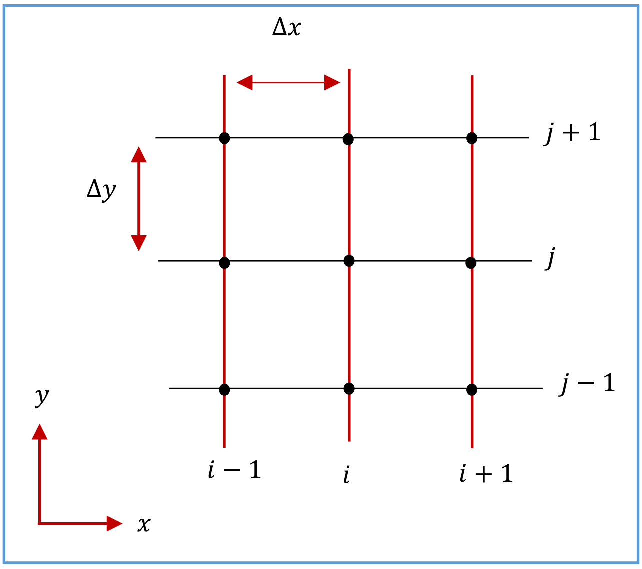


Figure 6.1 Graphic view of grid where i runs along x-axis and j runs along y-axis

The centred space difference expressions will be used to approximate the first and second derivatives for u with respect to y in the equations

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}. \tag{6.2.1}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}. \tag{6.2.2}$$

However, the forward difference method will be applied to approximate the derivative u with respect to x in the equations

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x}. \tag{6.2.3}$$

Similarly, the forward difference will be adopted for the derivative p with respect to x and y in the equations

$$\left(\frac{\partial p}{\partial x}\right)_{i,j} \approx \frac{p_{i+1,j} - p_{i,j}}{\Delta x}. \tag{6.2.4}$$

$$\left(\frac{\partial p}{\partial y}\right)_{i,j} \approx \frac{p_{i,j+1} - p_{i,j}}{\Delta y}. \tag{6.2.5}$$

It remains the derivative v with respect to y where the backward difference is applied

$$\left(\frac{\partial v}{\partial y}\right)_{i+1,j} \approx \frac{v_{i+1,j} - v_{i+1,j-1}}{\Delta y}. \tag{6.2.6}$$

6.3 Boundary Layer equation in Cartesian coordinates

The Navier–Stokes equations have been simplified significantly by the Boundary Layer approximations. In the present chapter the numerical study will be implemented for Boundary Layer equation by Finite Difference Method (FDM).

The dimensionless Boundary Layer equation in Cartesian coordinate is written as

– The x-momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} +\frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right). \tag{6.3.1}$$

– The y-momentum

$$\frac{\partial p}{\partial y} = 0, \;\; p = p(x). \tag{6.3.2}$$

– Boundary Conditions

$$\left.\begin{array}{l} u(x,0\,) = 0 \\ u(x,\ y \to \infty\,) = U \\ v(x,0\,) = 0. \end{array}\right\} \tag{6.3.3}$$

The numerical study will be performed to solve the two-dimensional Boundary Layer equation (6-3.1) - (6.3.3) with the continuity equation (6.1.6) in the rectangle domain $D = [a,b] \times [c,d]$.

In some references the boundary layer equations (6-3.1) - (6.3.3) are called Prandtl's Boundary Layer equation.

6.4 Boundary conditions of Boundary Layer

Set up the boundary conditions as shown in equation (6.3.3). The figure 6.1 shows the specification of different sides of boundaries. ABCD is the domain of the problem where the flow is from left to right side from AD to BC. Furthermore, AB represents the flat plate with velocity zero, and reaches the free stream velocity U at the surface DC which it located as far-field boundary. Thus if δ is the thickness of boundary layer and L is the length of flat plate then $\delta/L \ll 1$ so the distance between the surface DC and flat plate AB is sufficiently larger than δ. For the boundary of the pressure p, on the flat plate AB the equations (6.3.1) and the boundary condition equation (6.3.3) could be used, whereas the equation (6.3.2) could be applied in the sides DA and CB.

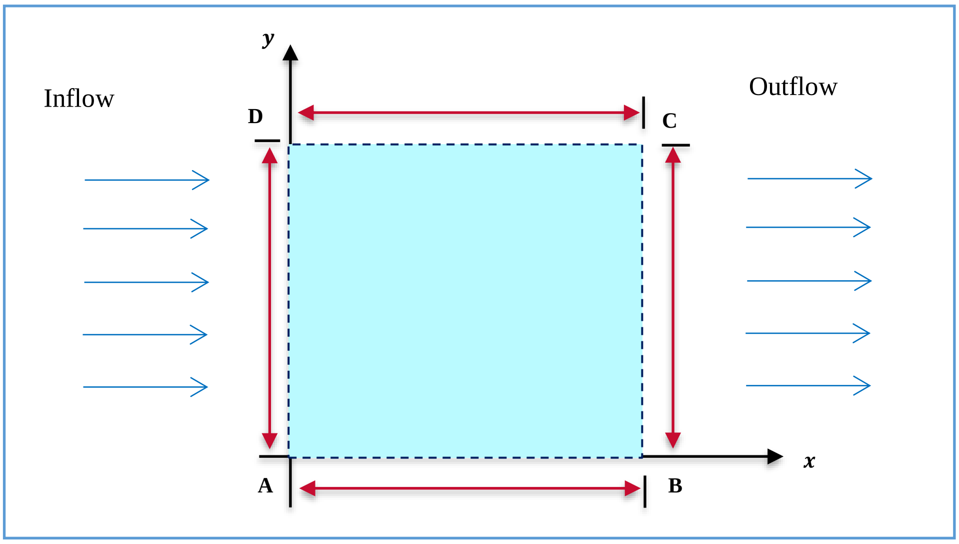


Figure 6.2 Illustration of different sides of boundaries in the rectangle domain ABCD

The following give boundary conditions

1) Boundary on The flat plate $(A < x < B)$

$$\left.\begin{aligned} u &= 0, \\ v &= 0, \\ \frac{\partial p}{\partial x} &= \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right). \end{aligned}\right\} \tag{6.4.1}$$

2) Boundary in the up side $(D < x < C\ )$

$$u = U \tag{6.4.2}$$

3) Boundary in the left side $(\ A < y < D)$

$$\left.\begin{aligned} u &= U, \\ v &= 0, \\ \frac{\partial p}{\partial y} &= 0. \end{aligned}\right\} \tag{6.4.3}$$

6.5 The implementation process steps

The implementation of the explicit finite difference method of the Boundary Layer equations (6.3.1)-(6.3.3) with the continuity equation (6.1.6) and boundary conditions (6.4.1)-(6.4.4) on a rectangular grid point $M \times N$ (where the domain divided into M points in x-axis and N in y-axis) could be accomplished by the following:

Step 1. Pressure calculation

- Assume $p_{1,1} = 0$, then calculate the pressure at point $(i+1,1)$ in the grid on flat plate AB from the pressure boundary condition (6.4.1) using equations (6.2.2) and (6.2.4):

$$p_{i+1,1} = p_{i,1} + \frac{1}{Re}\left[\frac{u_{i,1} - 2u_{i,2} + u_{i,3}}{(\Delta y)^2}\right]. \tag{6.5.1}$$

- The y-momentum equation (6.3.2) via the equation (6.2.5) will be applied to compute pressure at point $(i+1, j)$

$$p_{i+1,j} = p_{i+1,j-1} \quad , j = 2, \dots, N . \tag{6.5.2}$$

Step 2. Velocity calculation

- The velocity $u_{i+1,j}$ has calculated by substituting (6.2.1), (6.2.2), (6.2.3) and (6.2.4) into x-momentum equation (6.3.1)

$$u_{i+1,j} = u_{i,j} - \frac{\Delta x}{2\Delta y}\left[\frac{v_{i,j}\left(u_{i,j+1} - u_{i,j-1}\right)}{u_{i,j}}\right] + \frac{\Delta x}{(\Delta y)^2 Re}\left[\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{u_{i,j}}\right]$$
$$- \frac{\left(p_{i+1,j} - p_{i,j}\right)}{u_{i,j}} , \qquad j = 2, \dots, N . \tag{6.5.3}$$

- The continuity equation (6.1.6) and the equations (6.2.6) and (6.2.3) will be used for y-momentum velocity $v_{i+1,j}$

$$v_{i+1,j} = v_{i+1,j-1} - \frac{\Delta y}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right). \quad j = 2, \dots, N \tag{6.5.4}$$

This procedure must be repeated for all grid point, $M \times N$ times.

6.6 Oseen Boundary Layer Equations

According to equation (3.1.3) the two dimensional of Oseen equation is

$$U\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

The dimensionless Oseen equation in Cartesian coordinates is given as follows:

$$U\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

Applying the boundary layer assumptions, the Oseen Boundary Layer equations satisfy

$$U\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right), \tag{6.6.1}$$

$$\frac{\partial p}{\partial y} = 0, \tag{6.6.2}$$

with boundary conditions defined in the equation (6.3.3).

For implementation process steps, use the previous procedure in section 6.5 to perform the computation of Oseen Boundary Layer equation (6.6.1), (6.6.2) with the continuity equation (6.1.6) on a rectangular domain (Figure 7.2) via the Finite Difference Method (FDM). The only change required is to replace the equation (6.5.3) by the following equation

$$u_{i+1,j} = u_{i,j} - \frac{1}{U}\left(p_{i+1,j} - p_{i,j}\right) + \frac{\Delta x}{URe}\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2}\right). \tag{6.6.3}$$

6.7 Blasius equation

The partial differential equation form of two-dimensional Blasius equation has been discussed in section 3.3 chapter 3, and it stated as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right).$$

The dimensionless Blasius equation in Cartesian coordinates has been defined in the x-direction as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right). \tag{6.7.1}$$

The numerical solution of Blasius equation (6.7.1) with finite difference methods (FDM) uses the procedures mentioned in section (6.5). The implementation is summarized in two steps; first the Blasius equation (6.7.1) for computing the x-momentum velocity u; then the continuity equation (6.1.6) for calculating the y-momentum velocity v where the Boundary conditions (6.4.1)-(6.4.4) has been applied on the domain D.

- The x-momentum velocity u calculation:

  The velocity $u_{i+1,j}$ has to calculate by substituting (6.2.1), (6.2.2) and (6.2.3) into equation (6.7.1)

$$u_{i+1,j} = u_{i,j} - \frac{\Delta x}{2\Delta y}\left[\frac{v_{i,j}\left(u_{i,j+1} - u_{i,j-1}\right)}{u_{i,j}}\right] + \frac{\Delta x}{(\Delta y)^2 Re}\left[\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{u_{i,j}}\right] \tag{6.7.2}$$

  where $j = 2, \ldots, N$.

- The y-momentum velocity v calculation:

  The continuity equation (6.1.6) and the equations (6.2.6) and (6.2.3) will be used for y-momentum velocity $v_{i+1,j}$

$$v_{i+1,j} = v_{i+1,j-1} - \frac{\Delta y}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right). \quad j = 2, \ldots, N. \tag{6.7.3}$$

6.8 Oseen-Blasius equation

The two-dimensional of Oseen-Blasius equation is given as

$$U\frac{\partial u}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right).$$

This equation includes both Oseen and Blasius approximations. The numerical solution by Finite Difference Method (FDM) is illustrated as follows. According to section 6.1, the dimensionless Oseen-Blasius equation in Cartesian coordinates is

$$U\frac{\partial u}{\partial x} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right), \tag{6.8.1}$$

with boundary conditions defined in the equation (6.2.3).

So the scheme 6.5 is used with the Oseen-Blasius equation (6.8.1). This gives

- The x-momentum velocity u calculated from Oseen-Blasius equation (6.8.1) at the pint $(i+1, j)$ as

$$u_{i+1,j} = u_{i,j} + \frac{\Delta x}{URe}\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2}\right). \quad j = 2, \dots, N. \tag{6.8.2}$$

- The y-momentum velocity v obtained from the continuity equation at the pint $(i+1, j)$ as

$$v_{i+1,j} = v_{i+1,j-1} - \frac{\Delta y}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right). \quad j = 2, \dots, N. \tag{6.8.3}$$

6.9 Numerical Result

The Numerical study has been implemented for the case the free stream velocity U is 1 ms^{-1} with the different Re. The step sizes $\Delta x, \Delta y$ are chosen according to following equation

$$\frac{2\Delta x}{(\Delta y)^2} < URe \tag{6.9.1}$$

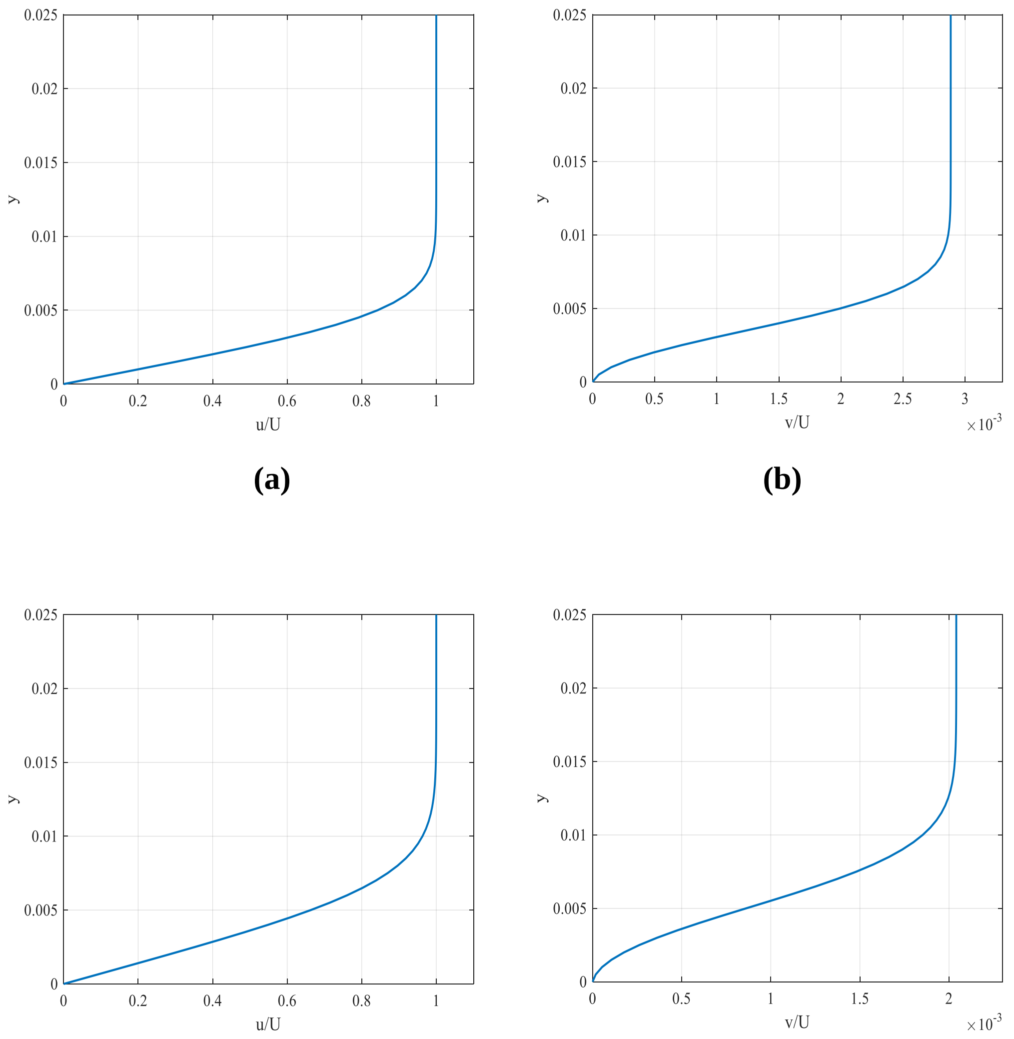
to obtain stability criterion for this procedure (Courant, Friedrichs et al. 1967). The numerical study has been carried out for various values of Reynolds numbers with different sizes of grids. For Re=10^5 the grid size is 2000 × 2000 to give $\Delta x = \Delta y = 0.001$ and Re=10^4, grid size 500 × 500 ($\Delta x = \Delta y = 0.002$) has been chosen. Moreover the grid size 100 × 100 is selected for Re=10^3 where $\Delta x = \Delta y = 0.01$, where the length ($L$) of the flat plate is 1m, which the computational domain D is $(0,1) \times (0,1)$.

The computational results by MATLAB programme are obtained for the flow over flat plate problem using Boundary Layer equations (6.3.1), (6.3.2), Oseen Boundary Layer equations (6.6.1), (6.6.2), the partial differential form of Blasius equation (6.7.1) and Oseen-Blasius equation (6.8.1).

Figures (6.3)-(6.5), (6.9)-(6.11), (6.15)-(6.17) and (6.21)-(6.23) describe the dimensionless velocities (x-momentum u/U and y-momentum $/U$) profiles with y-axis in centre and end in x interval. Furthermore, the velocity u surface in xy plane are presented in Figures (6.6), (6.12), (6.18) and (6.24).

The Boundary Layer with x-axis is given in figures (6.7), (6.13), (6.19) and (6.25), with thickness, rising from zero to merge or reach the outer stream. The figures (6.8), (6.14), (6.20) and (6.26) compare the thickness for various Reynolds Numbers Re =10^5,$10^4$and 10^3, the main characteristic that can be observed is that the Boundary Layer thickness δ is inversely proportional to Reynolds Number.

All velocities $u, v,$ surface of velocity u and Boundary thickness δ are plotted for each equation.



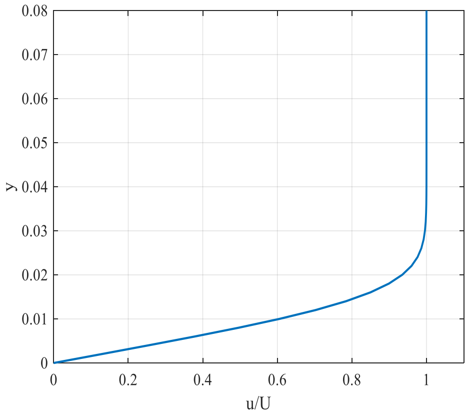
(a) (b)

(c) (d)

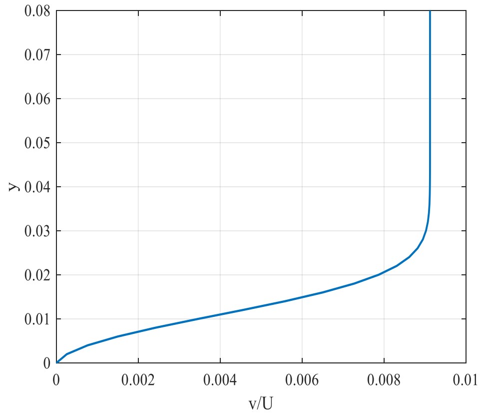
Figure 6.3 Numerical solution of the Boundary Layer equation at $Re = 10^5$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $D = (0,1) \times (0,1)$.

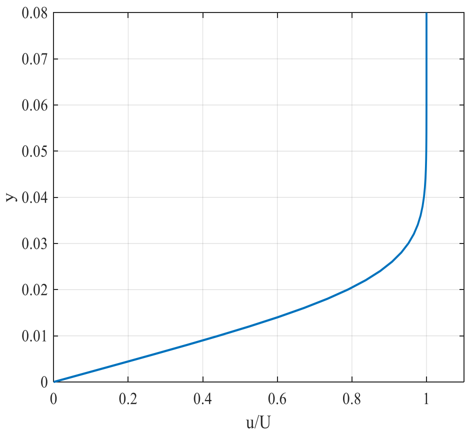
(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.



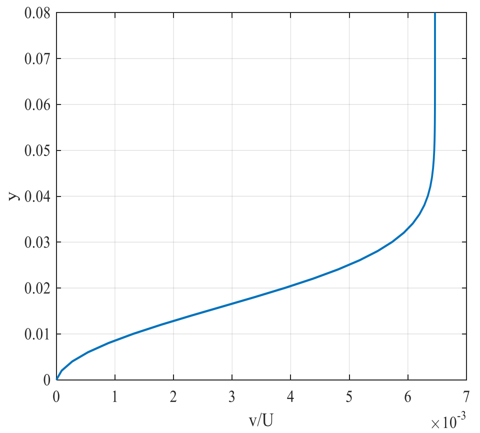
(a)



(b)



(c)



(d)

Figure 6.4 Numerical solution of the Boundary Layer equation at $\mathrm{Re} = 10^4$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.

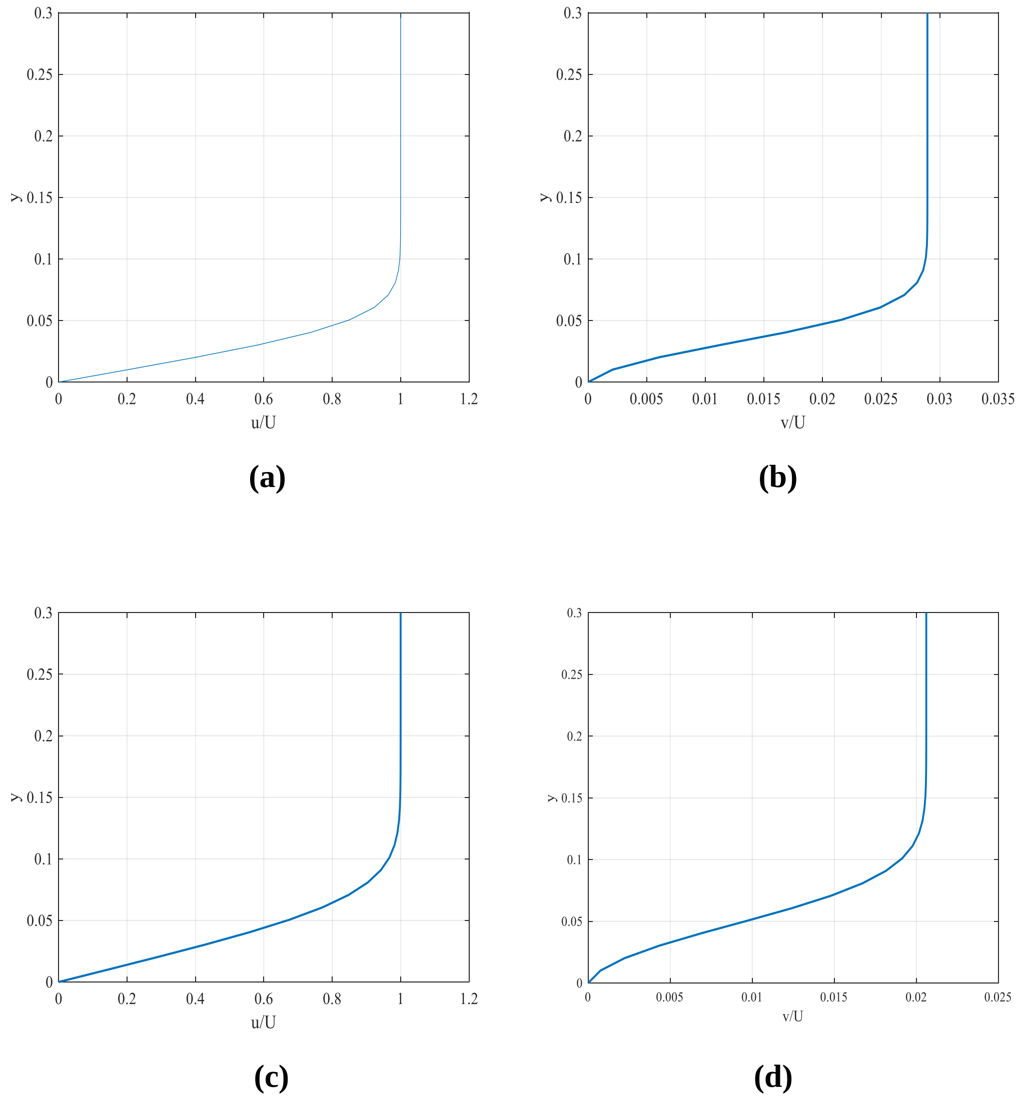


Figure 6.5 Numerical solution of the Boundary Layer equation at $\mathrm{Re} = 10^3$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.

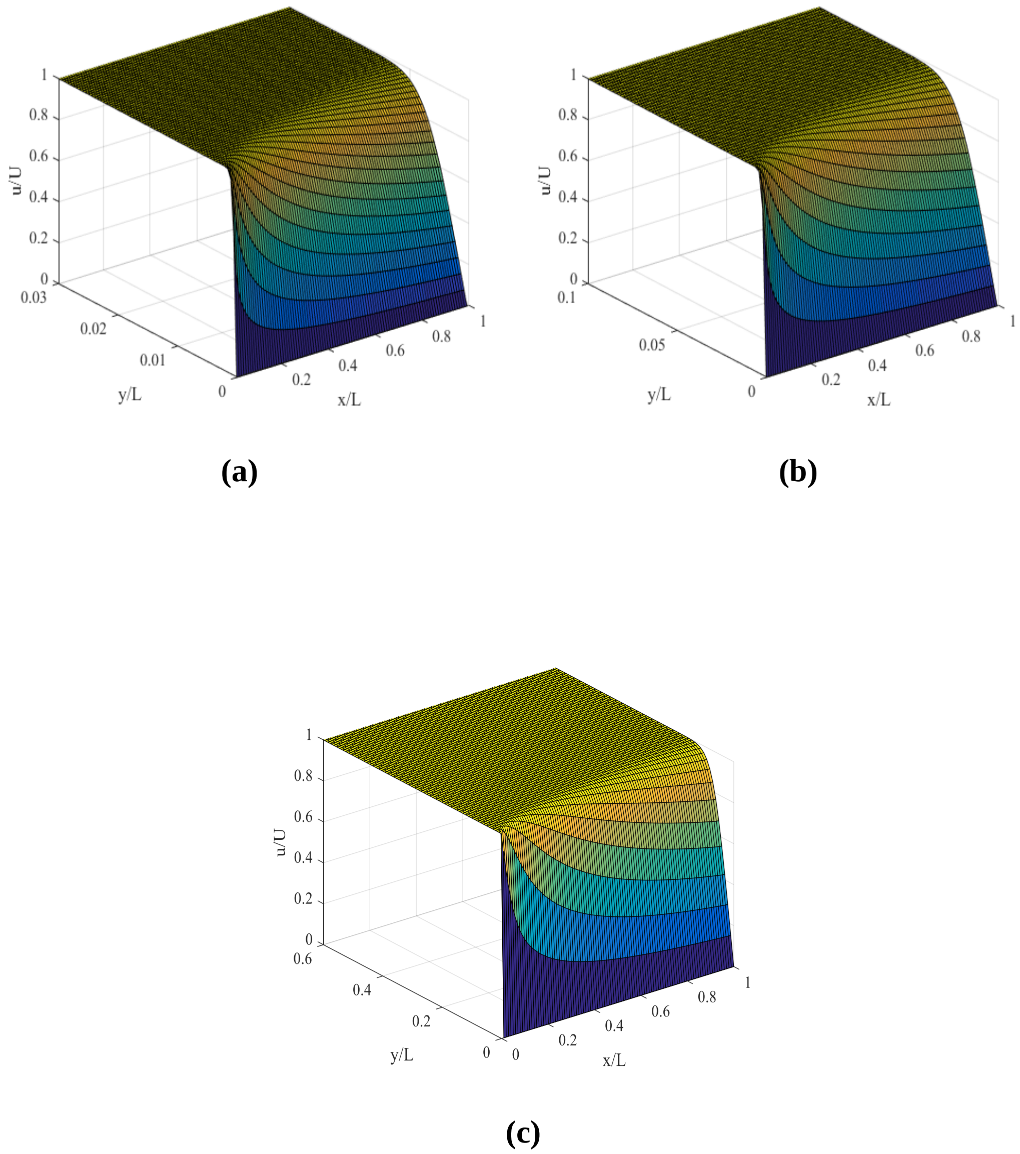
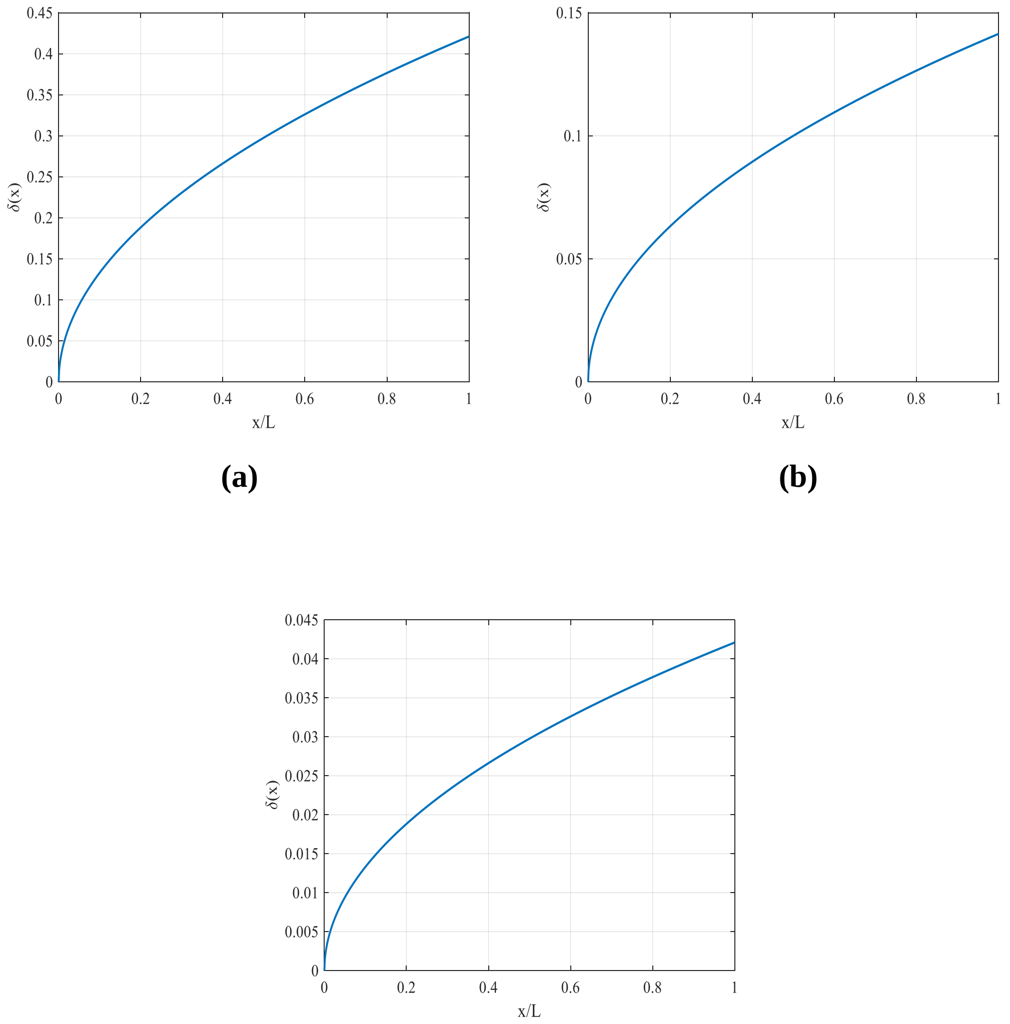


Figure 6.6 Velocity u Surface with yx-plane of the Boundary Layer equation.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) in the rectangle domain $D = (0,1) \times (0,1)$.

(a) $Re = 10^5$, grid size $2000 \times\ 2000$, $\Delta x = \Delta y =\ 0.0005$.
(b) $Re = 10^4$, grid size 500×500, $\Delta x = \Delta y =\ 0.002$.
(c) $Re = 10^3$, grid size $100 \times\ 100$, $\Delta x = \Delta y =\ 0.01$.

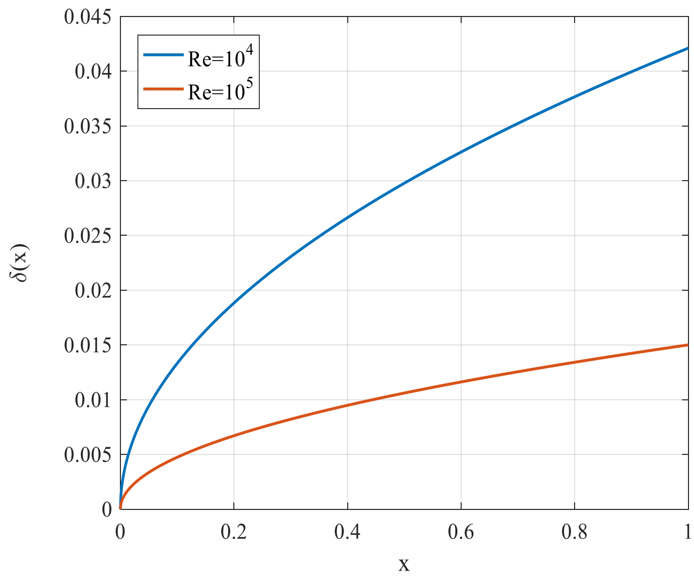


(c)

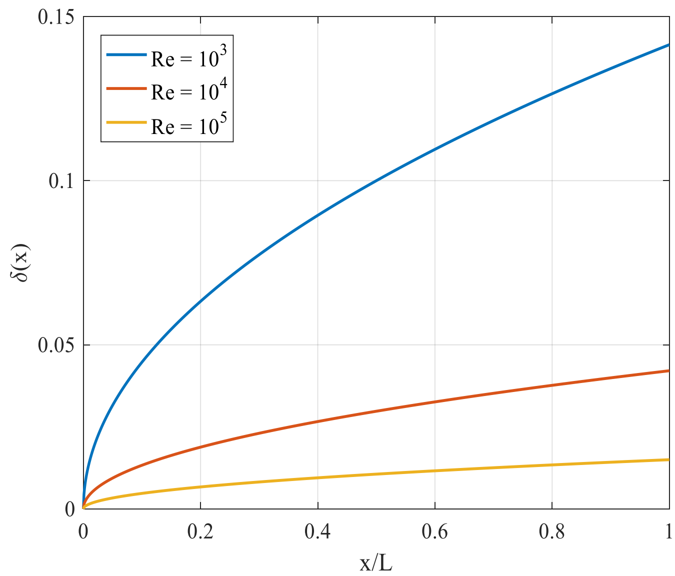
Figure 6.7 Boundary Layer Thickness δ with x-axis of the Boundary Layer equation.

It is evaluated by $u = 0.99U$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$ by Finite Difference Method (FDM) with the free stream velocity $U = 1$.

(a) $Re = 10^5$, grid size 2000×2000, $\Delta x = \Delta y = 0.0005$.
(b) $Re = 10^4$, grid size 500×500, $\Delta x = \Delta y = 0.002$.
(c) $Re = 10^3$, grid size 100×100, $\Delta x = \Delta y = 0.01$.



(a)



(b)

Figure 6.8 Thickness δ of Boundary Layer equation in different Re.

Thickness δ with x-axis, the free stream velocity $U = 1$ evaluated by $u = 0.99U$ by Finite Difference Method (FDM)

(a) $Re = 10^4$ and $Re = 10^5$.
(b) $Re = 10^3$, $Re = 10^4$ and $Re = 10^5$.

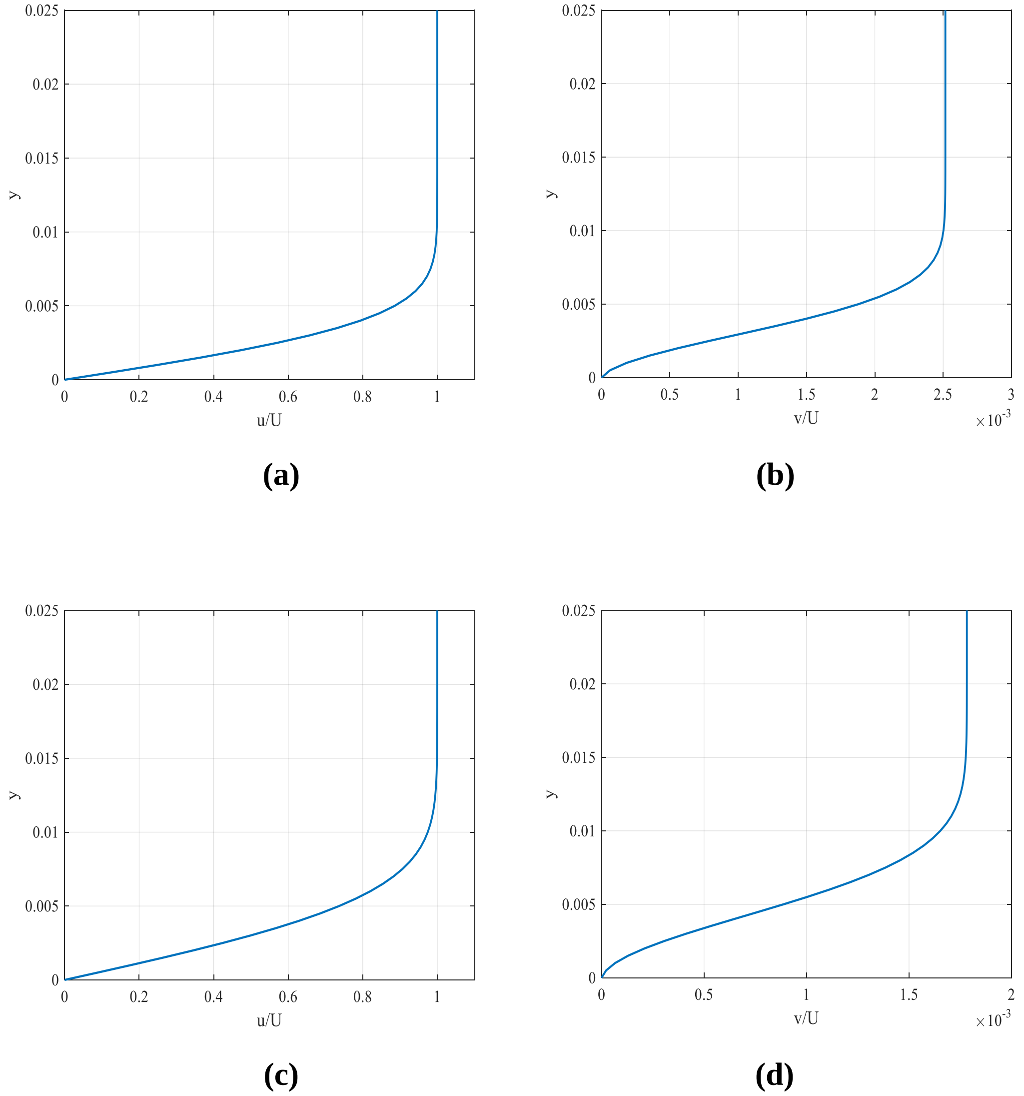
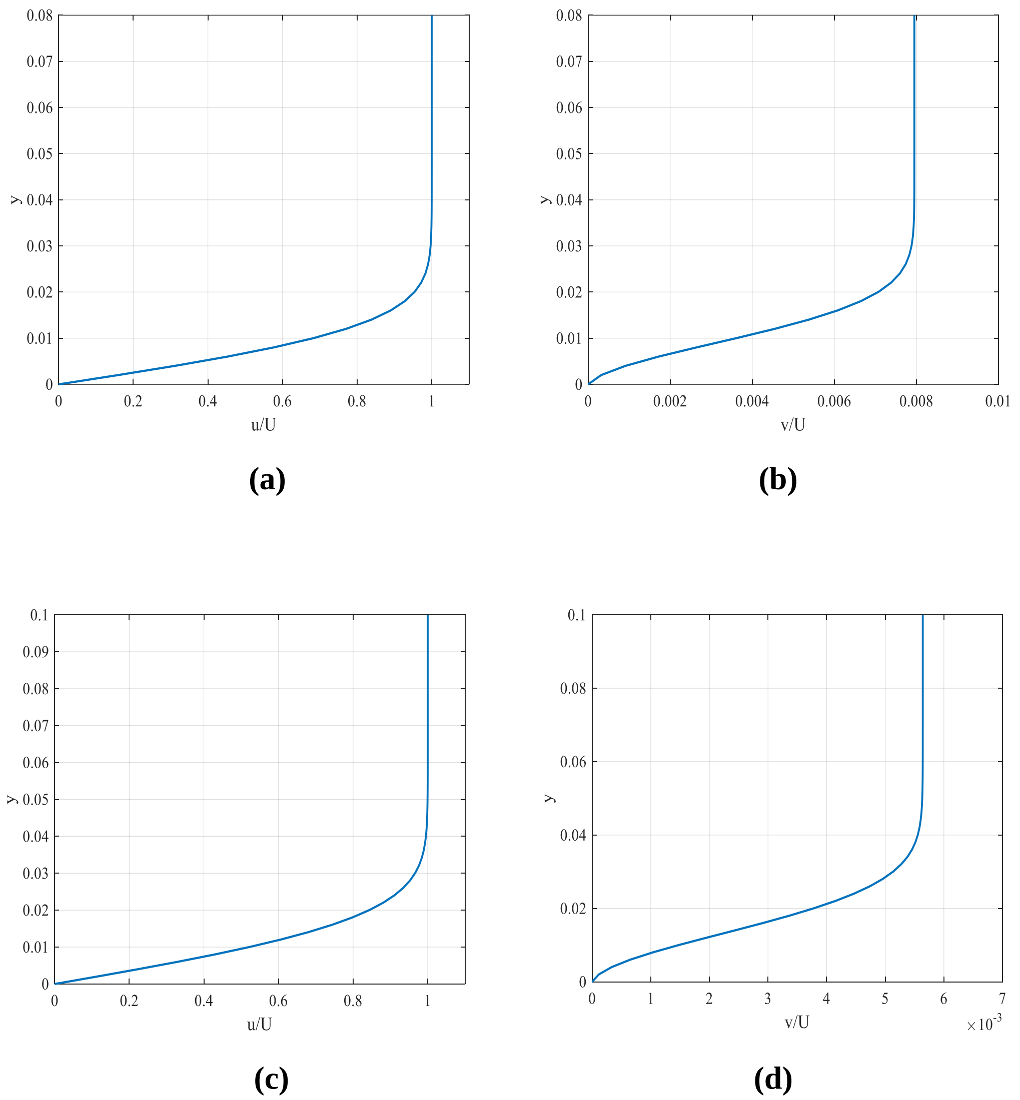


Figure 6.9 Numerical solution of the Oseen Boundary Layer equation at $Re = 10^5$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.



(a) (b)

(c) (d)

Figure 6.10 Numerical solution of the Oseen Boundary Layer equation at $Re = 10^4$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.

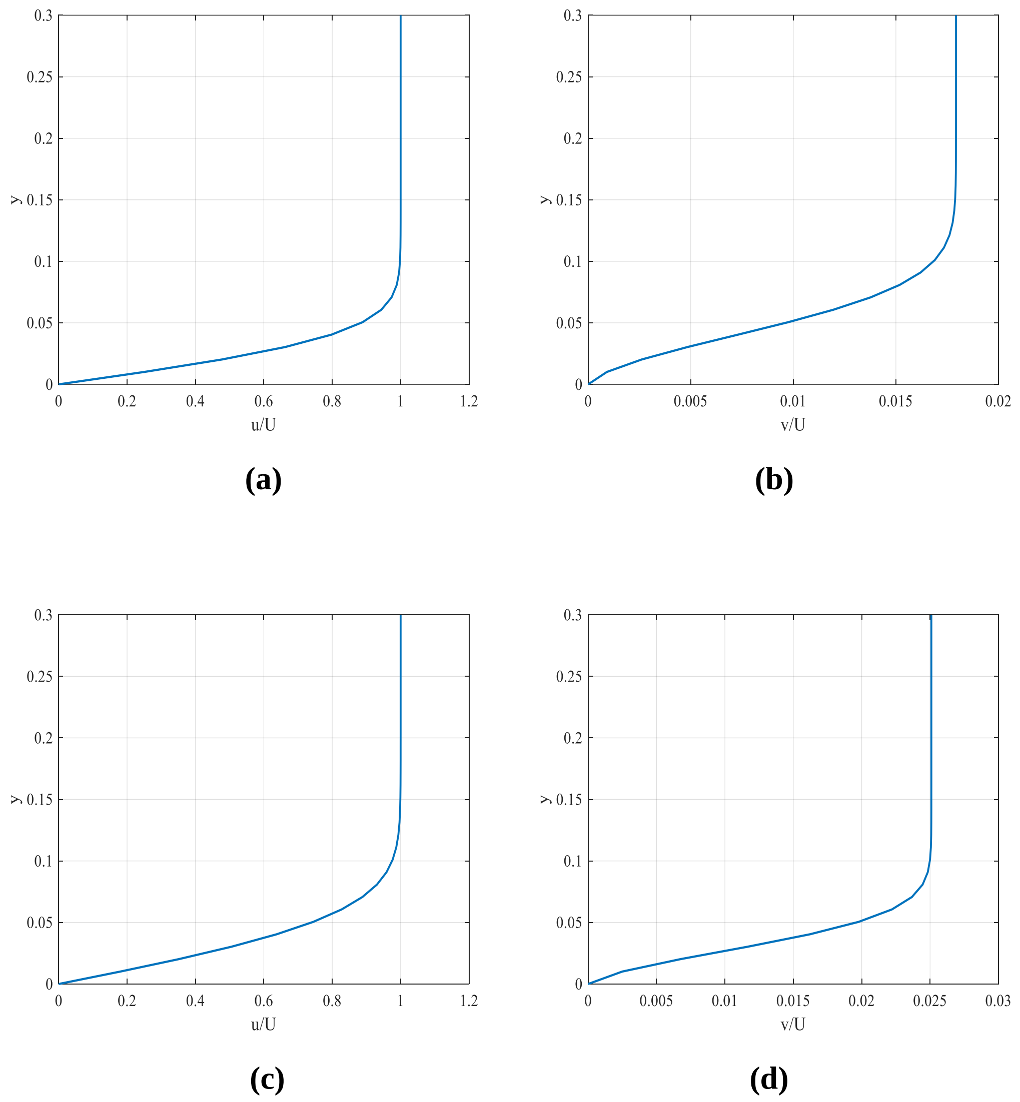
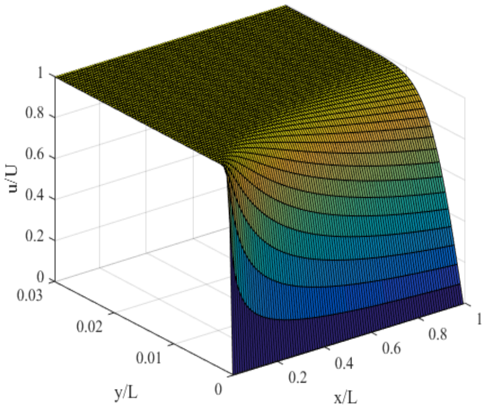
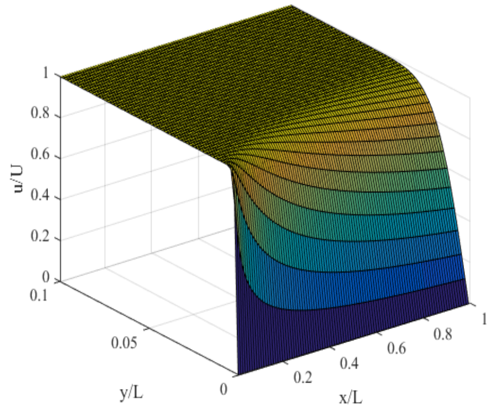


Figure 6.11 Numerical solution of the Oseen Boundary Layer equation at $Re = 10^3$.

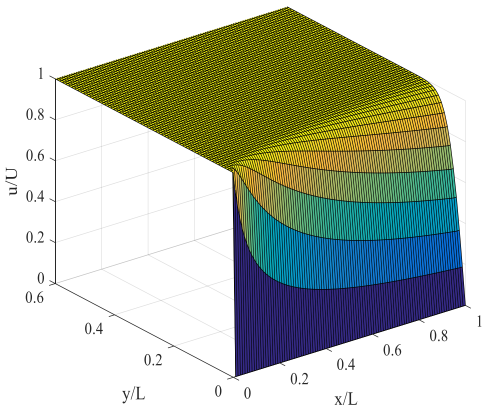
The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.





(a) **(b)**

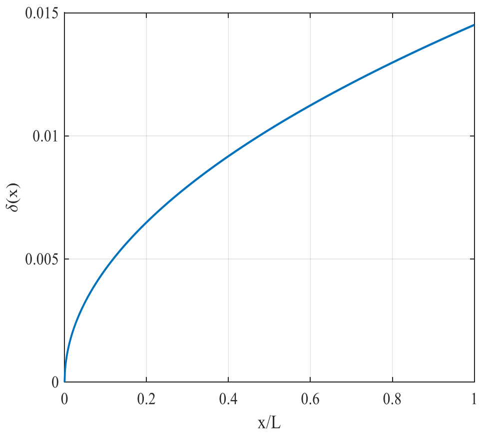


(c)

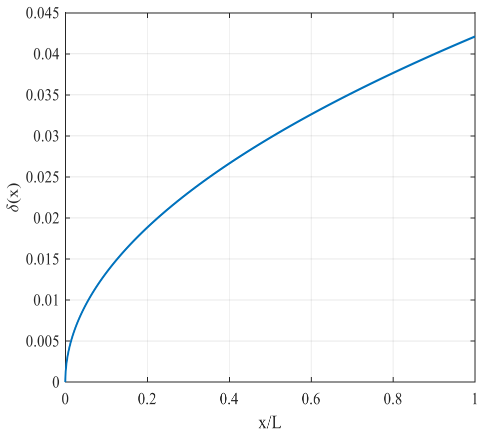
Figure 6.12 Velocity u Surface with yx-plane of the Oseen Boundary Layer equation.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) in the rectangle domain $D = (0,1) \times (0,1)$.

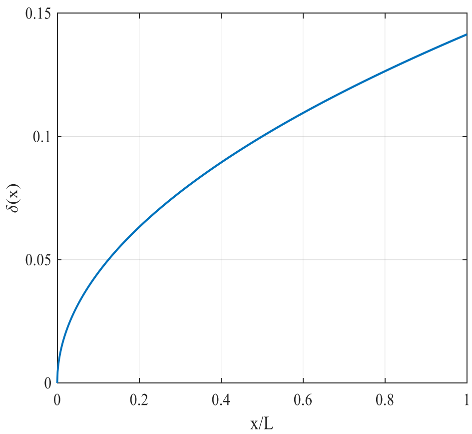
(a) $\mathrm{Re} = 10^5$, grid size 2000×2000, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.0005$.
(b) $\mathrm{Re} = 10^4$, grid size 500×500, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.002$.
(c) $\mathrm{Re} = 10^3$, grid size 100×100, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.01$.



(a)



(b)

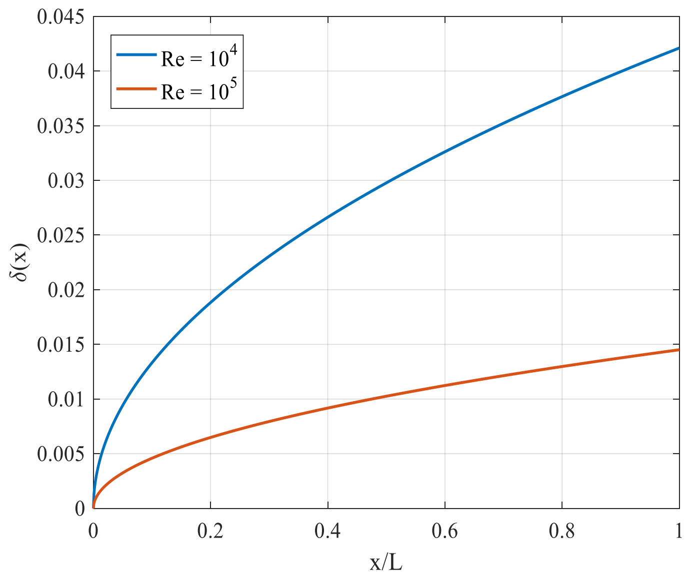


(c)

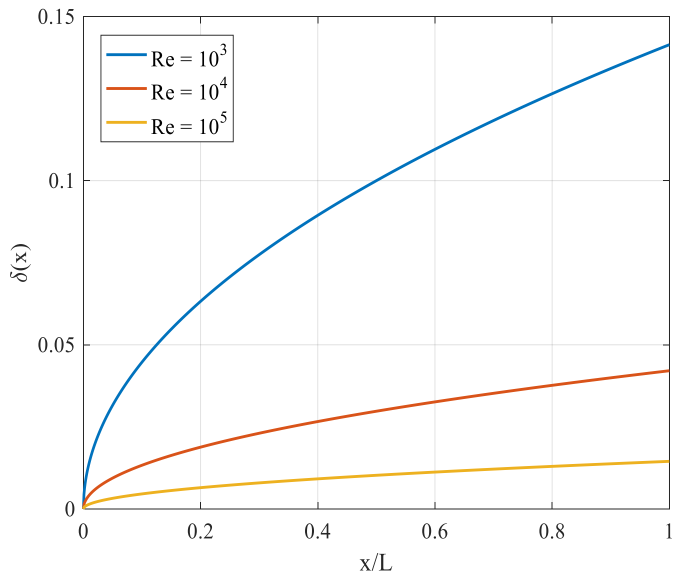
Figure 6.13 Boundary Layer Thickness δ of the Oseen Boundary Layer equation.

Thickness δ with x-axis is evaluated by $u = 0.99U$ in the rectangle domain $D = (0,1) \times (0,1)$ by Finite Difference Method (FDM) with free stream velocity $U = 1$.

(a) $Re = 10^5$, grid size 2000×2000, $\Delta x = \Delta y = 0.0005$.
(b) $Re = 10^4$, grid size 500×500, $\Delta x = \Delta y = 0.002$.
(c) $Re = 10^3$, grid size 100×100, $\Delta x = \Delta y = 0.01$.



(a)

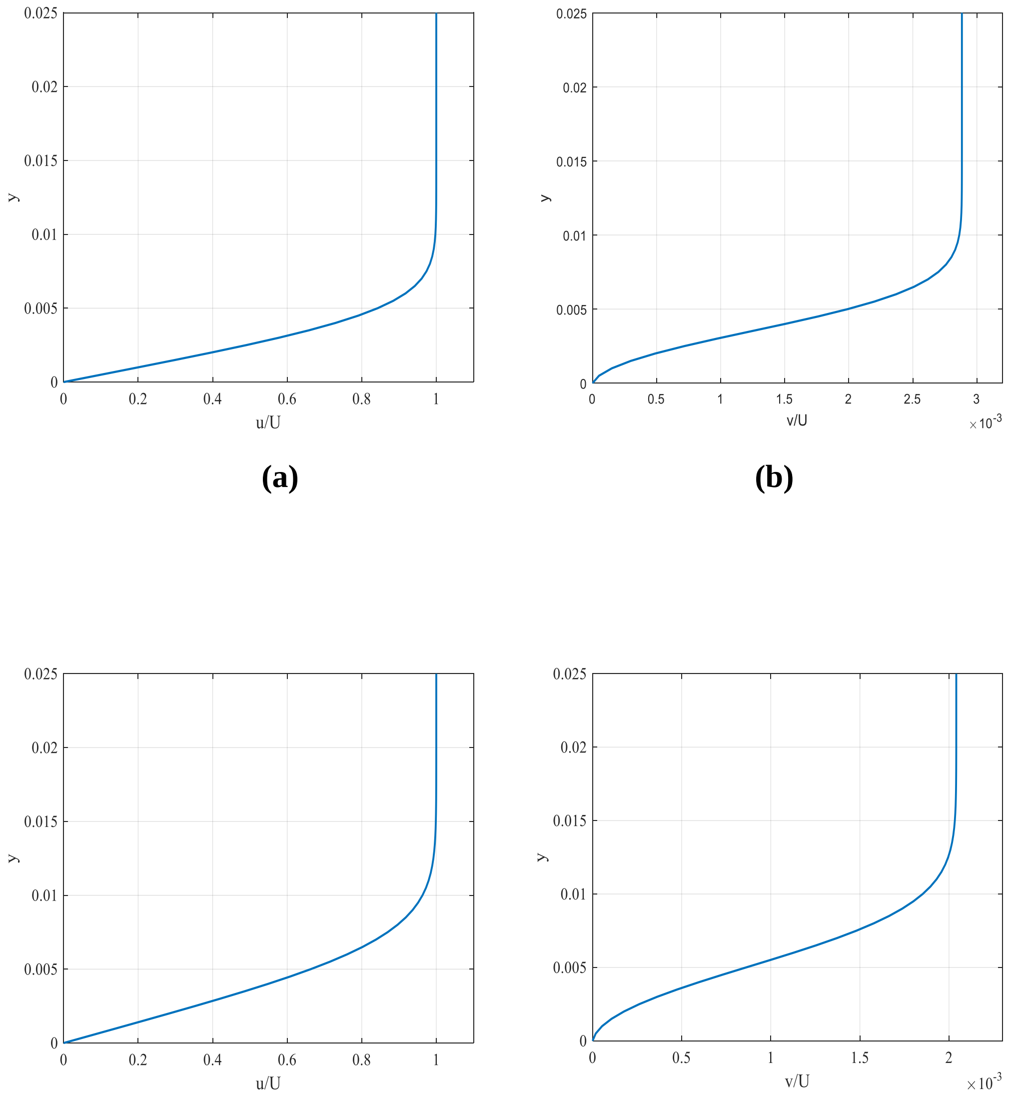


(b)

Figure 6.14 Thickness δ of The Oseen Boundary Layer equation in different Re.

Thickness δ with x-axis, the free stream velocity $U = 1$ is evaluated by $u = 0.99U$ by Finite Difference Method (FDM).

(a) $Re = 10^4$ and $Re = 10^5$.
(b) $Re = 10^3$, $Re = 10^4$ and $Re = 10^5$.



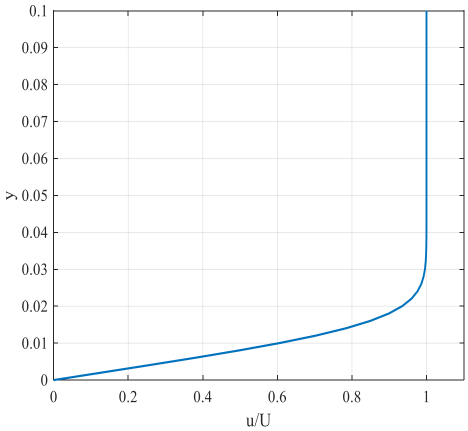
(a) **(b)**

(c) **(d)**

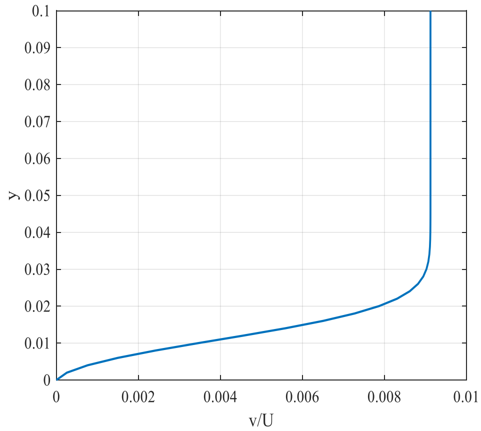
Figure 6.15 Numerical solution of Blasius equation at $Re = 10^5$.

The free stream velocity $U = 1$ Finite Difference Method (FDM) with grid size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

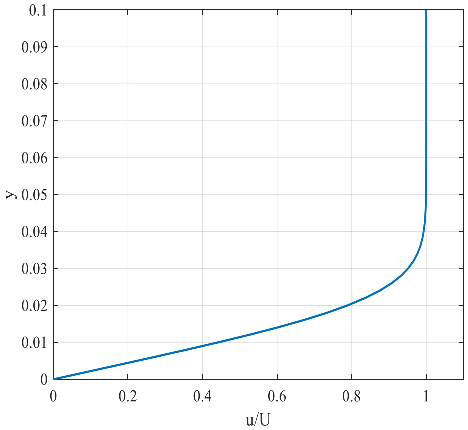
(a) Velocity u with y-axis, $x = 0.5$.
(b) Velocity v with y-axis, $x = 0.5$.
(c) Velocity u with y-axis, $x = 1$.
(d) Velocity v with y-axis, $x = 1$.



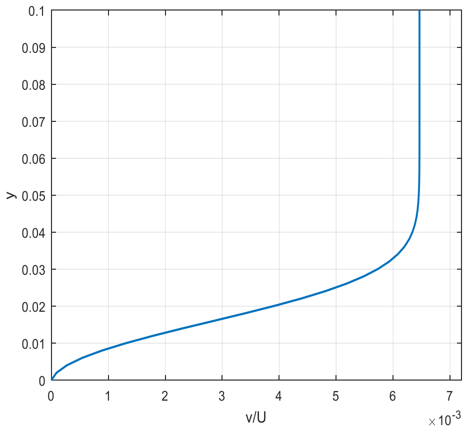
(a)



(b)



(c)



(d)

Figure 6.16 Numerical solution of Blasius equation at $Re = 10^4$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

(a) Velocity u/U with y-axis, $x = 0.5$.
(b) Velocity v/U with y-axis, $x = 0.5$.
(c) Velocity u/U with y-axis, $x = 1$.
(d) Velocity v/U with y-axis, $x = 1$.

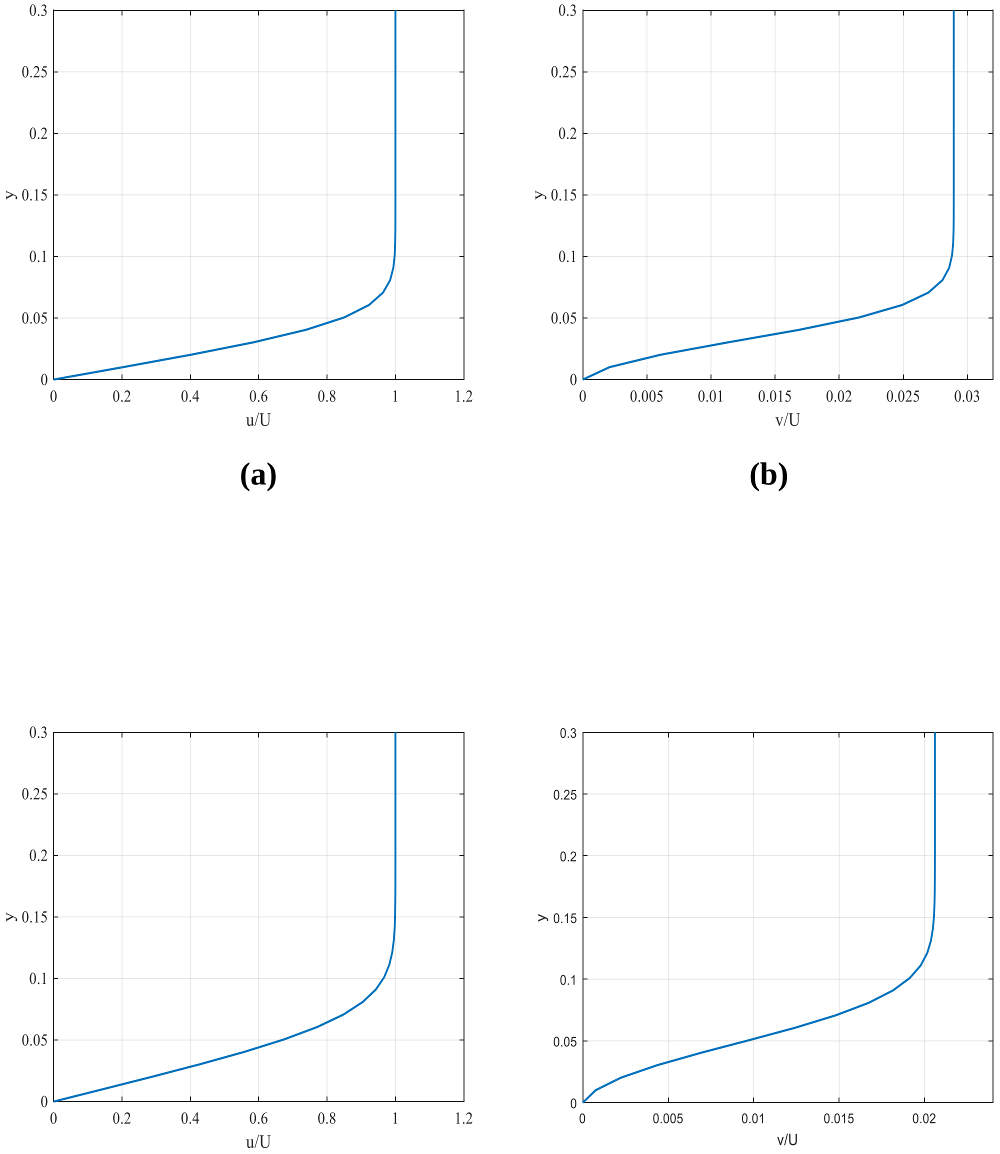
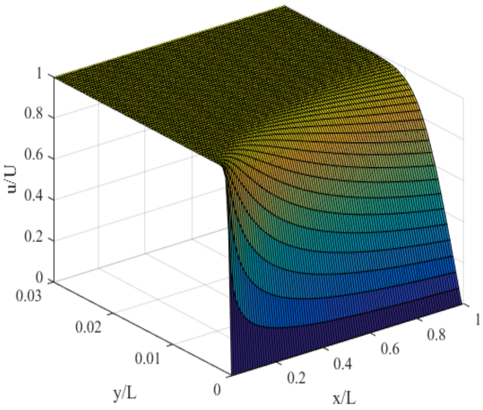


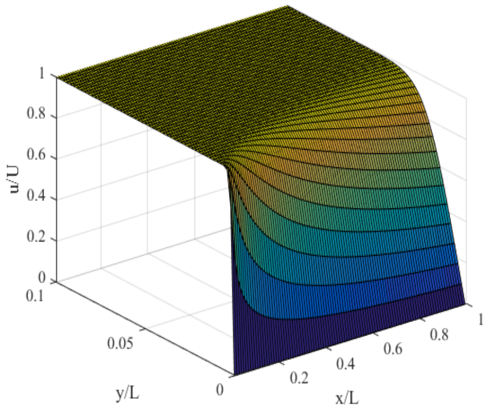
Figure 6.17 Numerical solution of Blasius equation at $Re = 10^3$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $D = (0,1) \times (0,1)$.

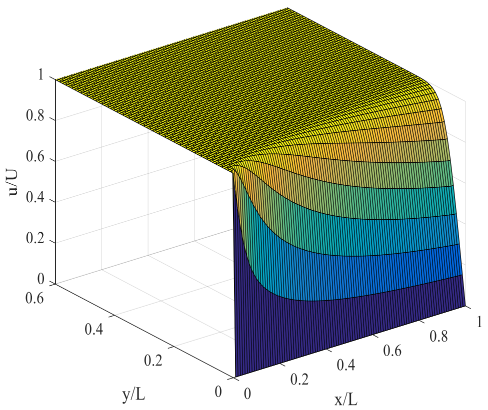
(a) Velocity u/U with y-axis, $x = 0.5$.
(b) Velocity v/U with y-axis, $x = 0.5$.
(c) Velocity u/U with y-axis, $x = 1$.
(d) Velocity v/U with y-axis, $x = 1$.



(a)



(b)

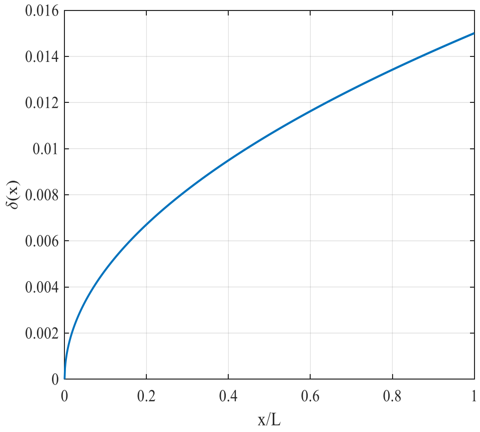


(c)

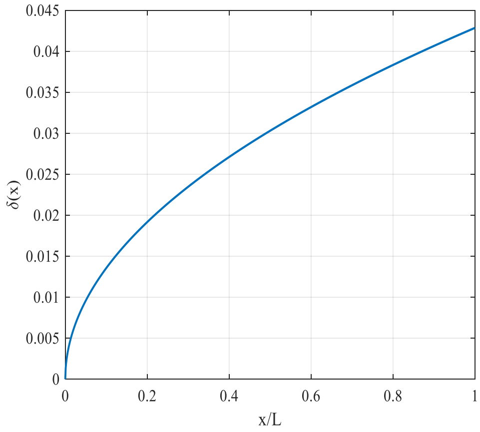
Figure 6.18 Velocity u/U Surface with yx-plane Blasius equation.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) in the rectangle domain $D = (0,1) \times (0,1)$.

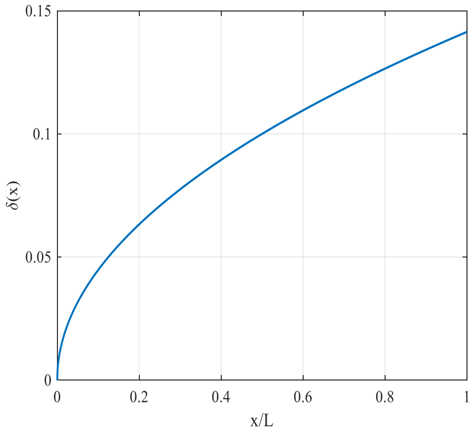
(a) $\mathrm{Re} = 10^5$, grid size 2000×2000, $\Delta x = \Delta y = 0.0005$.
(b) $\mathrm{Re} = 10^4$, grid size 500×500, $\Delta x = \Delta y = 0.002$.
(c) $\mathrm{Re} = 10^3$, grid size 100×100, $\Delta x = \Delta y = 0.01$.



(a)



(b)

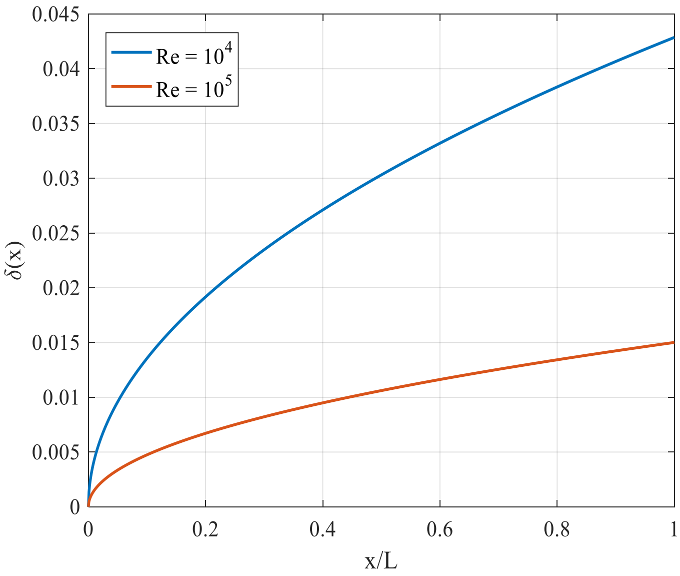


(c)

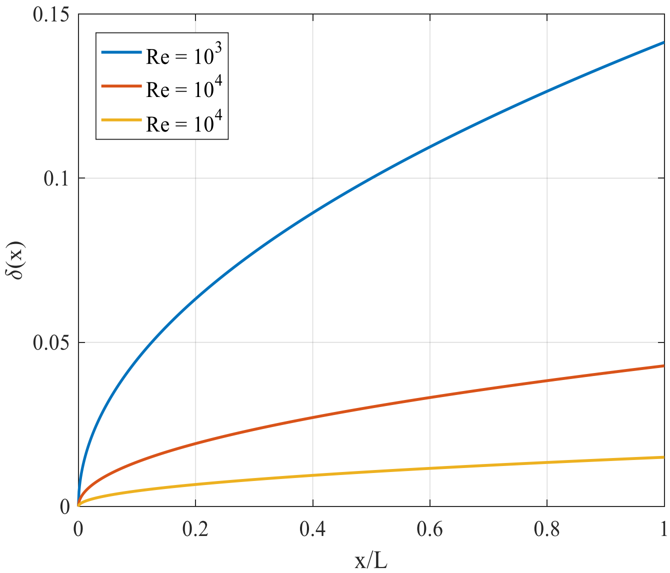
Figure 6.19 Boundary Layer Thickness δ with x-axis of Blasius equation.

The Thickness δ is evaluated by $u = 0.99U$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$ by Finite Difference Method (FDM) with the free stream velocity $U = 1$.

(a) Re $= 10^5$, grid size 2000×2000, $\Delta x = \Delta y = 0.0005$.
(b) Re $= 10^4$, grid size 500×500, $\Delta x = \Delta y = 0.002$.
(c) Re $= 10^3$, grid size 100×100, $\Delta x = \Delta y = 0.01$.



(a)

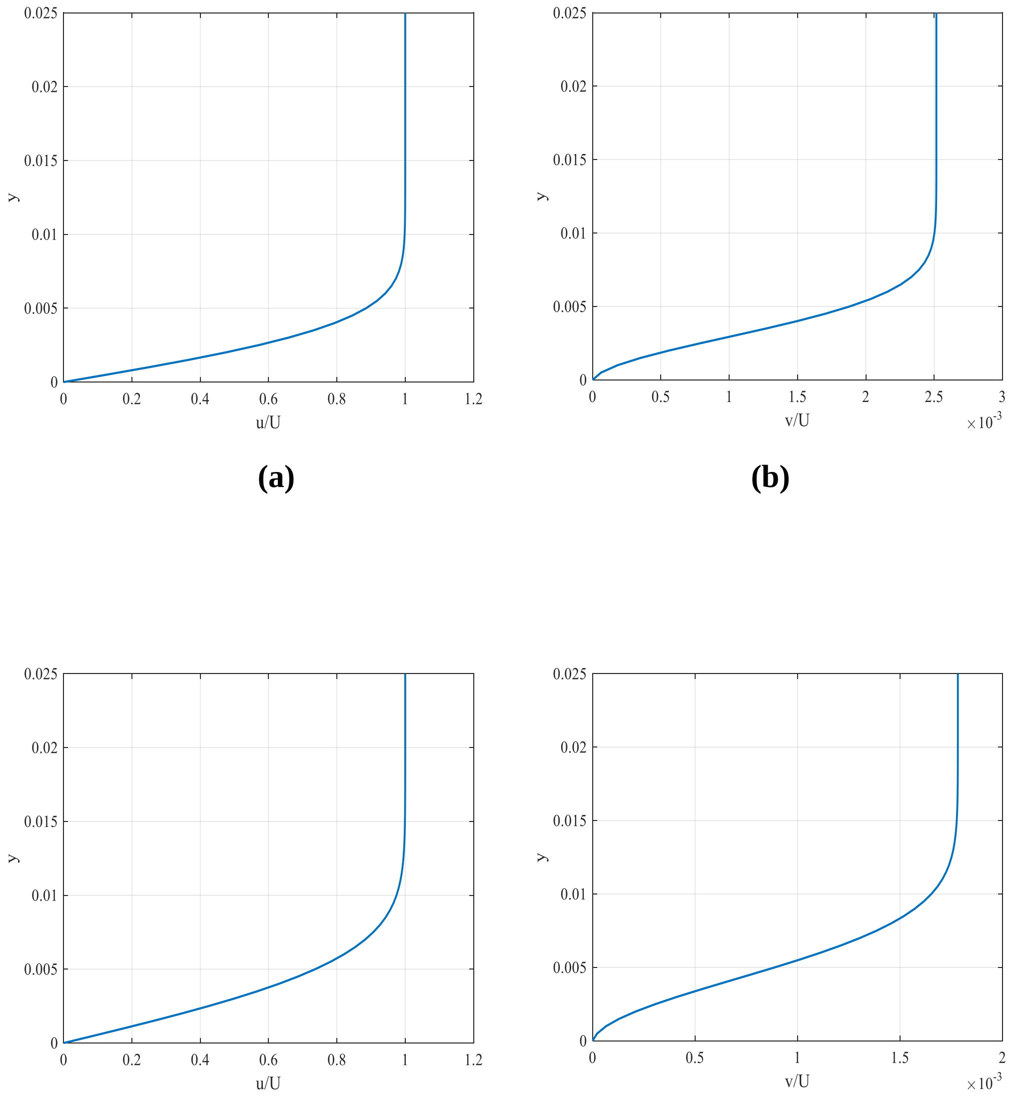


(b)

Figure 6.20 Boundary Layer thickness of Blasius equation in different Re.

Thickness δ with x-axis with the free stream velocity $U = 1$ is evaluated by $u = 0.99U$ by Finite Difference Method (FDM).

(a) $\mathrm{Re} = 10^4$ and $\mathrm{Re} = 10^5$.
(b) $\mathrm{Re} = 10^3$, $\mathrm{Re} = 10^4$ and $\mathrm{Re} = 10^5$.



(a) **(b)**

(c) **(d)**

Figure 6.21 Numerical solution of Oseen-Blasius equation at $Re = 10^5$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with the grid size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u/U with y-axis, $x = 0.5$.
(b) Velocity v/U with y-axis, $x = 0.5$.
(c) Velocity u/U with y-axis, $x = 1$.
(d) Velocity v/U with y-axis, $x = 1$.

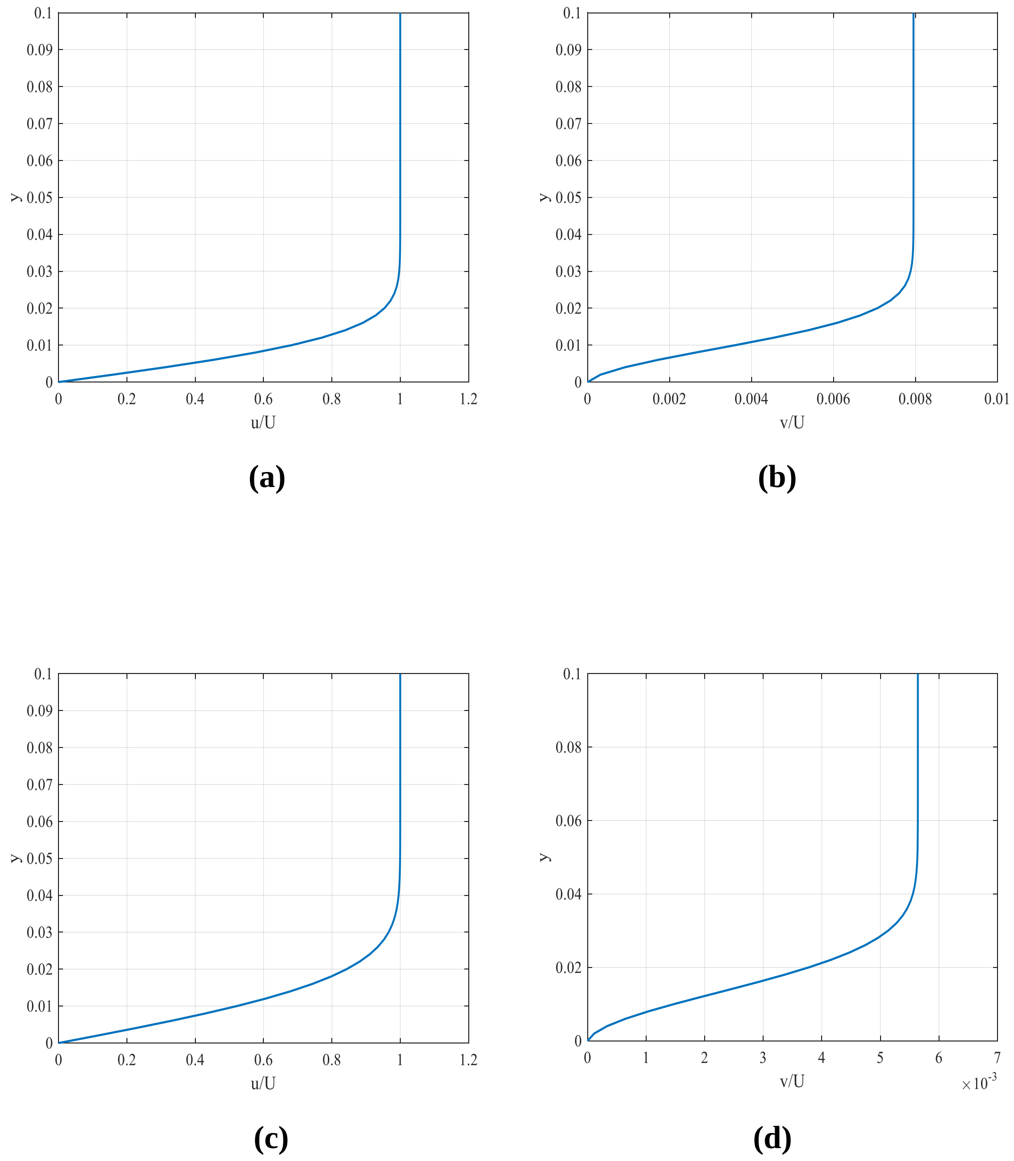
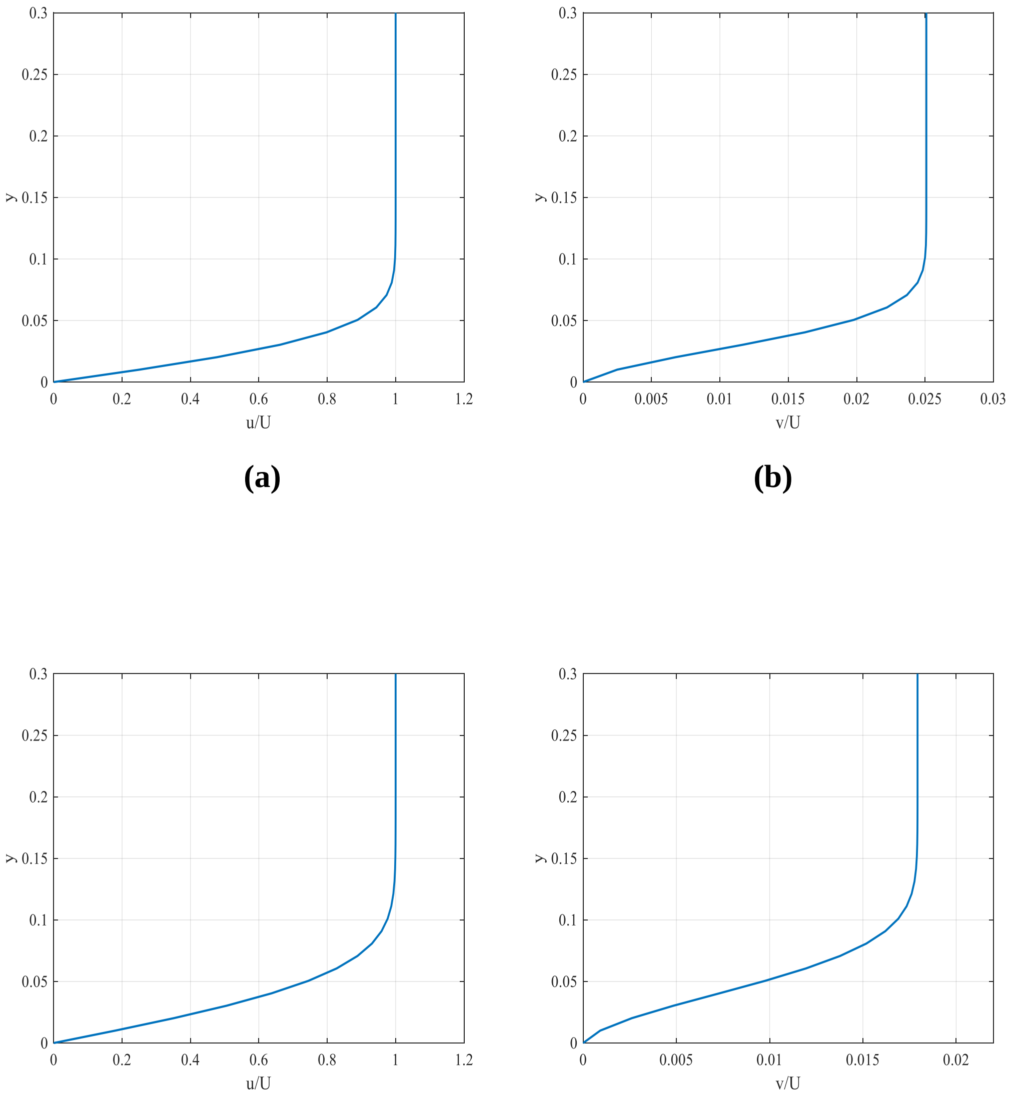


Figure 6.22 Numerical solution of Oseen-Blasius equation at $Re = 10^4$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with the grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u/U with y-axis, $x = 0.5$.
(b) Velocity $v/U$with y-axis, $x = 0.5$.
(c) Velocity u/U with y-axis, $x = 1$.
(d) Velocity v/U with y-axis, $x = 1$.

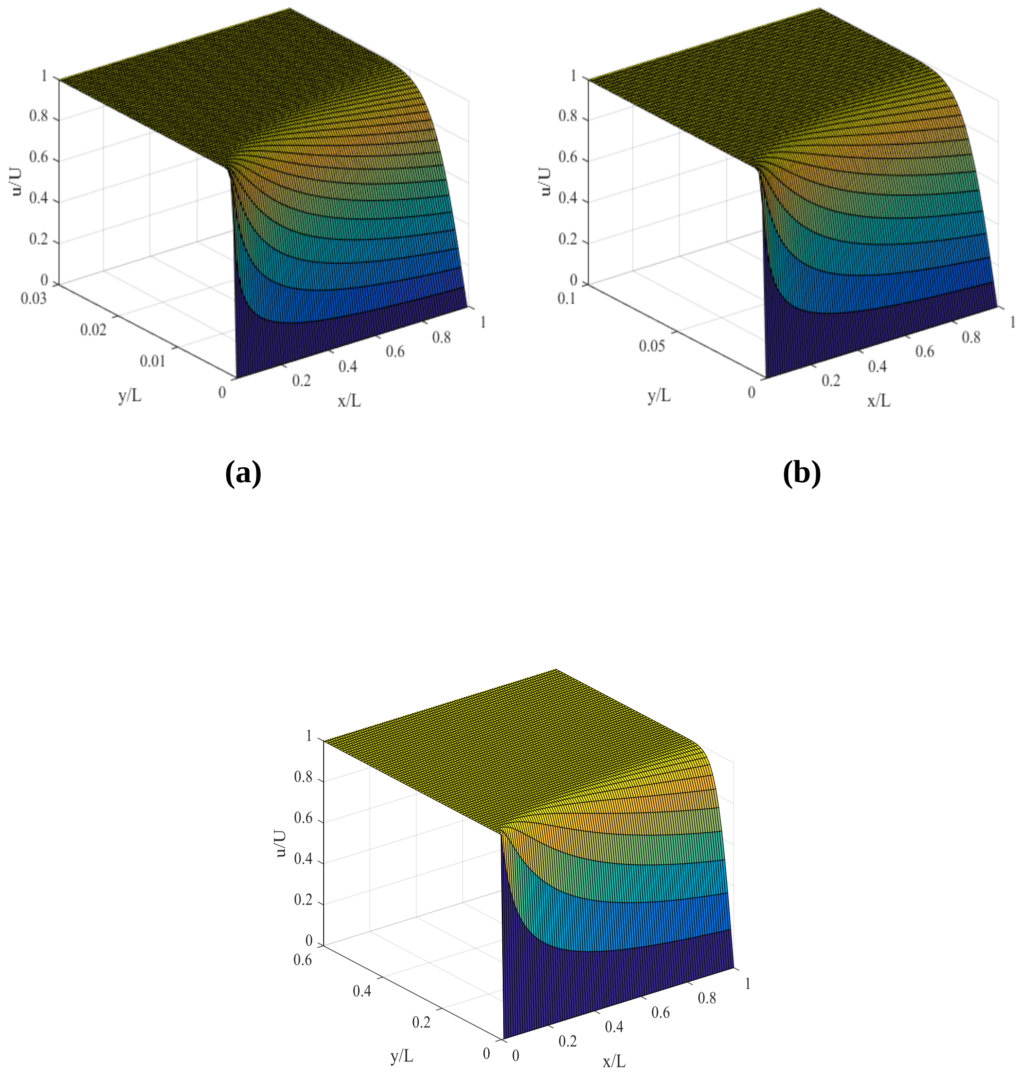


(a) (b) (c) (d)

Figure 6.23 Numerical solution of Oseen-Blasius equation at $Re = 10^3$.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) with the grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $D = (0,1) \times (0,1)$.

(a) Velocity u/U with y-axis, $x = 0.5$.
(b) Velocity v/U with y-axis, $x = 0.5$.
(c) Velocity u/U with y-axis, $x = 1$.
(d) Velocity v/U with y-axis, $x = 1$.



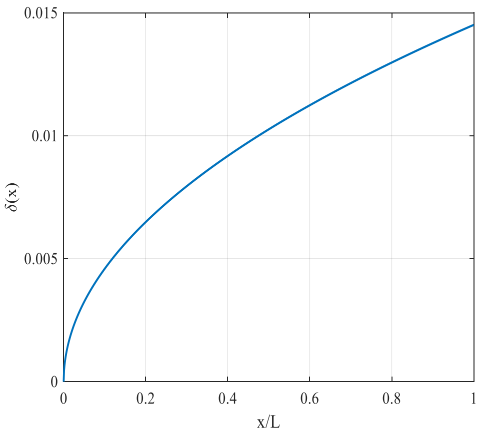
(a) (b)

(c)

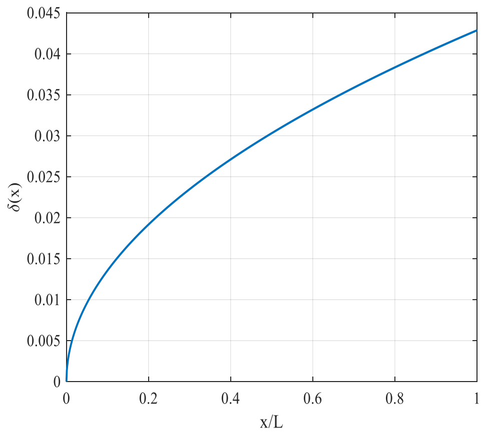
Figure 6.24 Velocity u Surface with yx-plane of Oseen-Blasius equation.

The free stream velocity $U = 1$ by Finite Difference Method (FDM) in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

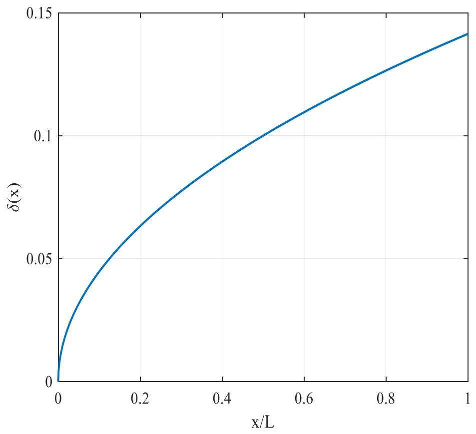
(a) $Re = 10^5$, grid size 2000×2000, $\Delta x = \Delta y = 0.0005$.
(b) $Re = 10^4$, grid size 500×500, $\Delta x = \Delta y = 0.002$.
(c) $Re = 10^3$, grid size 100×100, $\Delta x = \Delta y = 0.01$.



(a)



(b)

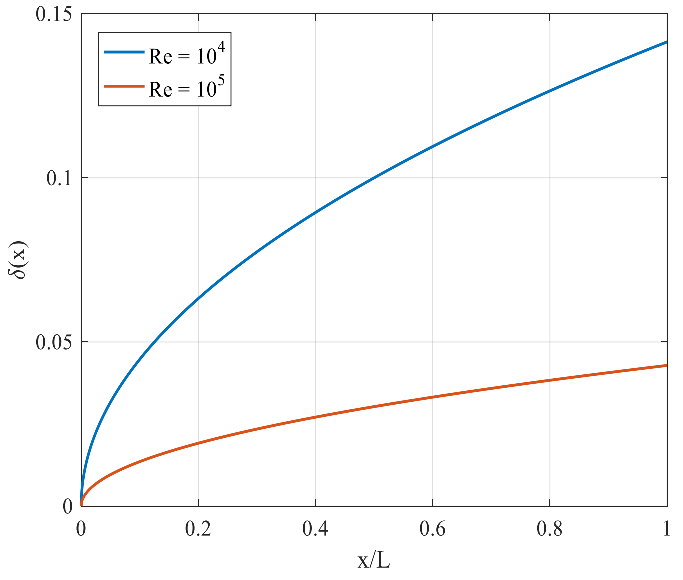


(c)

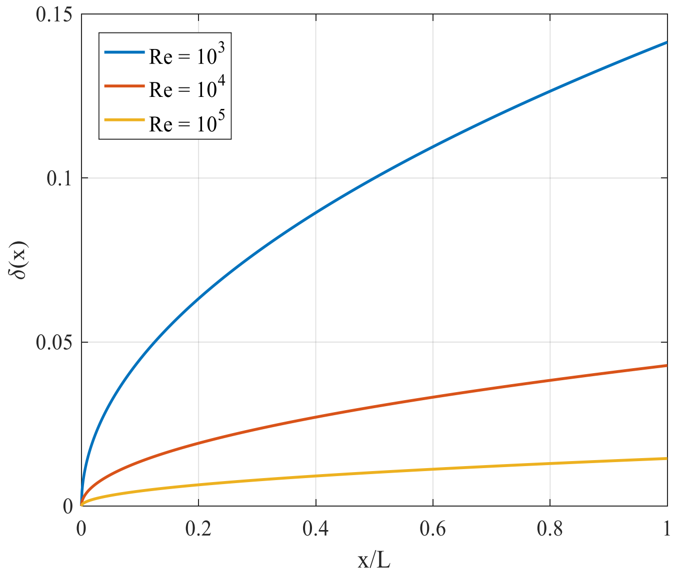
Figure 6.25 Boundary Layer Thickness of the Oseen-Blasius equation.

The Thickness δ with x-axis is evaluated by $u = 0.99U$ in the rectangle domain $D = (0,1) \times (0,1)$ by Finite Difference Method (FDM) with the free stream velocity $U = 1$.

(a) $\mathrm{Re} = 10^5$, grid size 2000×2000, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.0005$.
(b) $\mathrm{Re} = 10^4$, grid size 500×500, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.002$.
(c) $\mathrm{Re} = 10^3$, grid size 100×100, $\Delta\mathrm{x} = \Delta\mathrm{y} = 0.01$.



(a)



(b)

Figure 6.26 Boundary Layer thickness of the Oseen-Blasius equation in different Re.

The thickness δ with x-axis is evaluated by $u = 0.99U$ by Finite Difference Method (FDM) with the free stream velocity $U = 1$.

(a) $\mathrm{Re} = 10^4$ and $\mathrm{Re} = 10^5$.
(b) $\mathrm{Re} = 10^3$, $\mathrm{Re} = 10^4$ and $\mathrm{Re} = 10^5$.

Chapter 7 Analytical Study

7.1 Introduction

The analytic study focused on Oseen and Blasius approximation in the Boundary layer. the Oseen strength function obtained in chapter 5 is used in the Oseen integral representation, with the Imia's approximation of the drag Oseenlet. Moreover, for the solution of potential flow, a Thin Body Theory is applied which is checked by Laplacian Green function first. In addition, the analytic solution of Oseen-Blasius equation is discussed in two ways, ordinary differential equation and partial differential equation form. Finally, the behaviour of Stokes flow near field on the boundary layer is considered.

7.2 General form of Oseen solution

The general solutions of equations of the steady Oseen flow of an incompressible fluid are derived by decomposing the velocity of fluid into a potential velocity and a viscous velocity (Oseen 1927, Goldstein 1929, Goldstein 1931, Lamb 1932) such that

$$\left.\begin{aligned} u(x,y) &= \frac{\partial}{\partial x}\phi(x,y) + \omega_1(x,y), \\ v(x,y) &= \frac{\partial}{\partial y}\phi(x,y) + \omega_2(x,y). \end{aligned}\right\} \tag{7.2.1}$$

The wake velocity ω is obtained from the boundary velocity potential χ. The functions ϕ, χ satisfied the following

$$\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\chi = 0, \tag{7.2.2}$$

$$\nabla^2\phi = 0, \tag{7.2.3}$$

respectively where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The velocity is

$$u(x,y) = 1 + \int_0^\infty u^D(x-s,y)\sigma(s)ds, \tag{7.2.4}$$

$$v(x,y) = \int_0^\infty v^D(x-s,y)\sigma(s)ds, \tag{7.2.5}$$

where u^D, $v^D$is the drag Oseenlet velocity is given as

$$u^D(x,y) = \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}[\ln r + e^{kx}K_0(kr)] - 2ke^{kx}K_0(kr)\right\}, \tag{7.2.6 }$$

$$v^D(x,y) = \frac{1}{2\pi}\left\{\frac{\partial}{\partial y}[\ln r + e^{kx}K_0(kr)]\right\}, \tag{7.2.7}$$

where $r = \sqrt{x^2 + y^2}$, (Oseen 1927, Lagerstrom 1964).

The potential velocity ϕ of the drag Oseenlet is

$$\phi^D(x,y) = \frac{1}{2\pi} lnr. \tag{7.2.8}$$

The wake (boundary) velocities ω_1, ω_2 satisfies

$$u = \omega_1 = \frac{\partial\psi}{\partial y}, \qquad v = \omega_2 = -\frac{\partial\psi}{\partial x} \tag{7.2.9}$$

where $\psi(x,y)$ is the stream function (Oseen 1927, Goldstein 1929, Goldstein 1931, Lamb 1932).

Also, from equation (5.2.10), Chapter 5, the strength Oseen function has been derived via Wiener-Hopf technique, it can be rewritten as

$$\sigma(x) = \frac{2}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}. \tag{7.2.10}$$

7.3 The Green's function of the Laplacian on 2-D domain

In the section, we derive the fundamental solution for the Green's function of the Laplacian operator on 2-D domain, which satisfies

$$\nabla^2 \varphi = \delta\,(r). \tag{7.3.1}$$

Here $r = \sqrt{(x-\xi)^2 + y^2}$ and $\delta\,(r)$ is Dirac delta function defined by:

$$\delta\,(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

and satisfy the identity $\int_{-\infty}^{\infty} \delta\,(x) dx = 1$ (Nikodým 1966, p. 724, eq. 1 & p. 725, eq. 2). The equation (7.3.1) is also called the Poisson equation (Polyanin and Nazaikinskii 2015). First, the general solution is considered for the 2-D Laplace equation, and then divergence theorem will applied on equation (7.3.1) to determine the particular solution. The Laplace equation in the Cartesian coordinate system is given by:

$$\nabla^2\, \varphi(x, y) = 0.$$

Suppose that $\varphi\,(x, y)$ is symmetric, such that is $\varphi = \varphi(r)$, the Laplace equation can be represented by the polar coordinate system as follows

$$\frac{\partial \varphi}{\partial x} = \varphi_r \frac{x-\xi}{r} \text{ and } \frac{\partial^2 \varphi}{\partial x^2} = \varphi_{rr} \frac{(\xi - x)^2}{r^2} + \varphi_r \frac{r^2 - (x-\xi)^2}{r^3},$$

where $\varphi_r = \frac{\partial \varphi}{\partial r}$ and $\varphi_{rr} = \frac{\partial^2 \varphi}{\partial r^2}$, also

$$\frac{\partial \varphi}{\partial y} = \varphi_r \frac{y}{r} \text{ and } \frac{\partial^2 \varphi}{\partial y^2} = \varphi_{rr} \frac{y^2}{r^2} + \varphi_r \frac{r^2 - y^2}{r^3}.$$

Therefore

$$\nabla^2 \varphi = \varphi_{rr} \frac{(\xi - x)^2}{r^2} + \varphi_r \frac{r^2 - (x-\xi)^2}{r^3} + \varphi_{rr} \frac{y^2}{r^2} + \varphi_r \frac{r^2 - y^2}{r^3}.$$

Rearranging and substituting into original equation gives

$$\varphi_{rr} + \frac{1}{r} \varphi_r = 0.$$

This is second order linear homogeneous differential equations of non-constant coefficients.

Lets $\omega = \varphi_r$ leads to $\omega_r = \varphi_{rr}$ substitute them into the equation to get

$$\omega_r + \frac{1}{r}\omega = 0.$$

It can be solved using the integrating factor method, the general solution define as

$$\omega(r) = \frac{A}{r}.$$

Integrating again gives

$$\varphi(r) = A\, ln\, r + B,$$

where A, B are constants. To obtain potential velocity of the drag Oseenlet let assume $B = 0$. We will find A, let origin be the center of a disc Σ with radius ε, then integrating over disc Σ gives

$$\iint_{\Sigma} \nabla^2\varphi\, d\Sigma = \iint_{\Sigma} \delta\,(r) d\Sigma = 1.$$

The Divergence Theorem will be used, let $\partial\Sigma$ be the boundary of Σ , we have

$$1 = \iint_{\Sigma} \nabla^2\varphi\, d\Sigma = \int_{\partial\Sigma} \varphi.\mathbf{n} d\partial\Sigma.$$

Now

$$\nabla\varphi.\mathbf{n} = \nabla\varphi.\frac{\nabla\varphi}{|\nabla\varphi|} = \left(\frac{A}{r}\boldsymbol{i}\right).\left(\frac{A/r}{\sqrt{(A/r)^2}}\boldsymbol{i}\right) = \frac{A}{r}.$$

With circumference of a circle $\partial\Sigma$ is $2\pi\varepsilon$ gives $\nabla\varphi.\mathbf{n}|_{C_\varepsilon} = \frac{A}{\varepsilon}$, thus

$$1 = \int_{C_\varepsilon} \frac{A}{\varepsilon} dC_\varepsilon = A2\pi$$

Hence

$$\varphi(r) = \frac{1}{2\pi}\ \ln r. \tag{7.3.1}$$

This is called the fundamental solution for the Green's function of the Laplacian on 2-D domains(Guenther and Lee 1996, Myint-U and Debnath 2007), and this also the potential velocity of the drag Oseenlet (7.2.8)

7.4 A Thin-Body Theory for the Green function of 2-D Laplacian operator

In this section a Thin Body Theory is applied for the Green's function of 2-D Laplace operator (7.3.1) which is equivalent to the drag Oseenlet function (7.2.8) for potential flow ϕ to obtain the solution of the Oseen integral representation.

Consider the integral equation

$$\emptyset(x,y) = \int_{-\infty}^{\infty} h(s) \frac{\ln r_s}{2\pi}\, ds, \tag{7.4.1}$$

where $r_s = \sqrt{(x_s)^2 + y^2}$ and $z_s = x_s + iy$ where $x_s = x - s$ and $z = x + iy$.

Now, observe that

$$\begin{aligned} \frac{\partial}{\partial x}\{z_s \ln z_s - z_s\} &= z_s \left\{\frac{\partial}{\partial x} \ln z_s\right\} + \ln z_\xi \left\{\frac{\partial z_s}{\partial x}\right\} - \frac{\partial z_s}{\partial x} \\ &= z_s \left(\frac{1}{z_s}\right) + \ln z_s - 1 \ = \ln z_s . \end{aligned} \tag{7.4.2}$$

From exponential form of complex numbers

$$z_s = re^{i\theta}, \ \ \theta = \tan^{-1}\frac{y}{x-s}, \quad z_s = \ r_s e^{i\left(\tan^{-1}\frac{y}{x-s}\right)}.$$

Therefore

$$\ln z_s = \ln[x_s + iy] = \ln r_s e^{i\left(\tan^{-1}\frac{y}{x_s}\right)} = \ln r_s + i\tan^{-1}\frac{y}{x_s} . \tag{7.4.3}$$

From (7.4.2) and (7.4.3), notice that

$$\begin{aligned} Real\left\{\frac{\partial}{\partial x}\{z_s \ln z_s - z_s\}\right\} &= Real\,\{\ln z_s\} \\ &= Real\left\{\ln r_s + i\tan^{-1}\frac{y}{x_s}\right\} = \ln r_s . \end{aligned} \tag{7.4.4}$$

Applying (7.4.3) into the integral equation (7.4.1) gives

$$\emptyset(x,y) = Real\left\{\int_{-\infty}^{\infty} \frac{h(s)}{2\pi}\frac{\partial}{\partial x}\{z_s \ln z_s - z_s\} ds\right\}.$$

Since $\frac{\partial}{\partial x}\{h(x-s)\} = -\frac{\partial}{\partial s}h(x-s)$ yields

$$\emptyset(x,y) = Real\left\{-\int_{-\infty}^{\infty}\frac{h(s)}{2\pi}\frac{\partial}{\partial s}\{z_s\ln z_s - z_s\}ds\right\}.$$

The integration by parts will be required to evaluate $\emptyset$

$$\emptyset(x,y) = Real\left\{\left[\frac{-h(s)}{2\pi}(z_s\ln z_s - z_s)\right]_{s=-\infty}^{s=\infty} + \int_{-\infty}^{\infty}\frac{h'(s)}{2\pi}\{z_s\ln z_s - z_s\}ds\right\}. \qquad (7.4.5)$$

Now, the real part of the integral equation (7.4.5) is determined, we have two cases.

First when $x_s > 0$ yields

$$\begin{aligned} z_s\ln z_s - z_s &= (x_s+iy)\ln(x_s+iy) - (x_s+iy)\\ &= (x_s+iy)\ln\left\{x_s\left(1+i\frac{y}{x_s}\right)\right\} - (x_s+iy)\\ &= (x_s+iy)\left\{\ln x_s + \ln\left(1+i\frac{y}{x_s}\right)\right\} - (x_s+iy). \end{aligned}$$

To simplify, the power series representation for $\ln(1+x)$ is

$$\begin{aligned} \ln\left(1+\frac{iy}{x_s}\right) &= i\frac{y}{x_s} - \frac{(iy/x_s)^2}{2} + O\left(\left(\frac{iy}{x_s}\right)^3\right)\\ &\approx i\frac{y}{x_s} + O\left(\frac{y^2}{x_s}\right), \qquad y \ll x\,. \end{aligned}$$

This gives

$$\begin{aligned} z_s\ln z_s - z_s &\approx (x_s+iy)\left\{\ln x_s + i\frac{y}{x_s}\right\} - (x_s+iy)\\ &= (x_s)\left\{\ln x_s + i\frac{y}{x_s}\right\} + (iy)\left\{\ln x_s + i\frac{y}{x_s}\right\} - (x_s+iy)\\ &= x_s\ln x_s - \frac{y^2}{x_s} + iy + iy\ln x_s - (x_s+iy). \end{aligned}$$

Thus,

$$Real\{z_s\ln z_s - z_s\} = x_s\ln x_s - x_s + O\left(\frac{y^2}{x_s}\right). \qquad (7.4.6)$$

Second, for $x_s < 0$, since

$$(-x_s - iy)e^{\pm i\pi} = (-x_s - iy)(\cos\pi \pm i\sin\pi) = (-x_s - iy)(-1) = (x_s + iy),$$

then

$$\begin{aligned} z_s \ln z_s - z_s &= (x_s + iy)\ln\{(-x_s - iy)e^{\pm i\pi}\} - (x_s + iy) \\ &= (x_s + iy)\ln\left\{-x_s\left(1 + i\frac{y}{x_s}\right)e^{\pm i\pi}\right\} - (x_s + iy) \\ &= (x_s + iy)\left\{\ln(-x_s) \pm i\pi + \ln\left(1 + i\frac{y}{x_s}\right)\right\} - (x_s + iy). \end{aligned}$$

The series expansion of $\ln\left(1 + i\frac{y}{x_s}\right)$ is $\left(i\frac{y}{x_s} + \frac{\left(i\frac{y}{x_s}\right)^2}{2} + \cdots\right)$ will be used as

$$\begin{aligned} z_s \ln z_s - z_s &\approx (x_s + iy)\left\{\ln(-x_s) \pm i\pi + i\frac{y}{x_s}\right\} - (x_s + iy) \\ &= \left\{x_s\ln(-x_s) \mp \pi y - \frac{y^2}{x_s}\right\} + i\{y\ln(-x_s) + \pi x + y\} - (x_s + iy). \end{aligned}$$

Then, it gives

$$Real\ \{z_s \ln z_s - z_s\} \approx x_s\ln(-x_s) - x_s \mp \pi y + 0\left(\frac{y^2}{x_s}\right). \tag{7.4.7}$$

Applying (7.4.7) and (7.4.6) into integral equation (7.4.5) we obtain

$$\begin{aligned} \emptyset(x,y) = &\left[\frac{-h(s)}{2\pi}(x_s\ln(x_s) - x_s - \theta y)\right]_{s=-\infty}^{s=\infty} + \int_x^\infty \frac{h'(s)}{2\pi}\{x_s\ln x_s - x_s\}ds \\ &+ \int_{-\infty}^x \frac{h'(s)}{2\pi}\{x_s\ln(-x_s) - x_s \mp \pi y\}ds\ . \end{aligned}$$

Setting $\left[\frac{-h(s)}{2\pi}(x_s\ln(x_s) - x_s - \theta y)\right]_{s=-\infty}^{s=\infty} = 0$, and

$$f(x) = \int_x^\infty \frac{h'(s)}{2\pi}\{x_s\ln x_s - x_s\}ds + \int_{-\infty}^x \frac{h'(s)}{2\pi}\{x_s\ln(-x_s) - x_s\}ds.$$

This gives

$$\emptyset(x,y) = \int_{-\infty}^{x} \frac{h'(\xi)}{2\pi} |\pi y| ds + f(x) = \begin{cases} \frac{y}{2} h(x) + f(x), y > 0 \\ -\frac{y}{2} h(x) + f(x), y < 0 \end{cases}$$

So

$$\emptyset(x,y) = \frac{|y|}{2} h(x) + f(x). \qquad (7.4.8)$$

Setting $h(x) = 1$ then $f(x) = 0$ therefore 1-D source is obtained

$$\emptyset(x,y) = \frac{|y|}{2}. \qquad (7.4.9)$$

Letting Σ be a region in the plane with boundary $\partial\Sigma$, then the divergence theorem is written in the form

$$\iint_{\Sigma} (\nabla^2 . \emptyset)\, d\Sigma = \int_{\partial\Sigma} (\emptyset . \mathbf{n}) d\partial\Sigma.$$

So, by choosing suitable Σ, it could be written

$$\int_{\partial\Sigma} \emptyset . \mathbf{n}\, d\partial\Sigma = \left(-\frac{1}{2}\right) . (-1)\Big|_{y<0} + \left(\frac{1}{2}\right) . (1)\Big|_{y>0} = 1.$$

So, it could be written

$$\int_{\partial\Sigma} \emptyset . \mathbf{n}\, d\partial\Sigma = \iint_{\Sigma} \delta\,(r) d\Sigma,$$

where δ is the Dirac delta function.

It can be verified that $\emptyset(x,y) = \frac{|y|}{2}$ by applying the fundamental solution for the Green's function of the Laplacian on 1-D from the equation (7.3.1) and assuming $s(x) = 1$, so the equation (7.4.1) becomes

$$I = FPI \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} ln\, r_s\, ds \right\},$$

where *FPI* is known as the Finite-Part Integral. Since $r_s = \sqrt{(x_s)^2 + y^2}$ then

$$I = FPI \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} ln \sqrt{(x_s)^2 + y^2}\, ds \right\}. \qquad (7.4.10)$$

Differentiating the equation (7.4.10) with respect to y gives

$$\frac{\partial I}{\partial y}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{y}{(x_s)^2+y^2}ds, \qquad (7.4.11)$$

where $\frac{\partial}{\partial y}\left(\ln\sqrt{(x_s)^2+y^2}\right)=\frac{y}{(x_s)^2+y^2}$.

Simplifying the problem, the equation (7.3.11) is solved instead. Assuming $t=\frac{x_s}{y}$, consequently

$$\frac{\partial I}{\partial y}=\pm\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^2}dt.$$

Let $t=\tan\theta,\ dt=\sec^2\theta\,,\ \theta\to-\frac{\pi}{2}\ as\ t\ \to-\infty,\ \theta\to\frac{\pi}{2}as\ t\ \to\infty$. We have

$$\frac{\partial I}{\partial y}=\pm\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta=\pm\frac{1}{2\pi}(\pi)=\pm\frac{1}{2}.$$

The integral (7.3.11) also could be evaluated in complex plane by applying residues theorem, since $\frac{1}{z^2+1}$ has a pole at $z=i$, therefore

$$\frac{\partial I}{\partial y}=\pm\frac{2\pi i}{2\pi}\left\{\underset{z=i}{Res}\frac{1}{z^2+1}\right\}=\pm i\left\{\lim_{z\to i}\frac{(z-i)}{(z+i)(z-i)}\right\}=\pm\frac{1}{2}.$$

Since $\frac{\partial I}{\partial y}=\pm\frac{1}{2}$ then

$$I=FPI\left\{\int_{-\infty}^{\infty}\frac{1}{2\pi}\ln r_\xi\,d\xi\right\}=\frac{|y|}{2}.$$

Here $\frac{|y|}{2}$ is 1-D Source.

7.5 A Thin-Body potential velocity for outer flow

Consider the steady flow generated by a uniform free velocity field past a thin body flat plate, whose major axis is aligned nearly to the uniform stream direction. Let us suppose Oseen flow in the boundary layer. A Thin Body Theory in Oseen flow is presented and applied for the drag Oseenlet function for potential flow Φ, and the strength function σ of the drag Oseenlet to obtain the functions $\emptyset$ and ψ for the solution of outer flow of the Boundary Layer. In particular the drag Oseenlet in far-fields (7.2.8) and the strength Oseen function (7.2.10) are applied to the finite-part integral (FPI) of Oseen integral representation (7.2.4) to give the velocity profile outer flow on a semi-infinite flat plate by applying complex variable theory $\Phi = \phi + i\psi$. Consider

$$\phi(x,y) = \frac{1}{\pi\sqrt{\pi Re}} \int_0^\infty \frac{ln\, r_s}{\sqrt{s}} ds \tag{7.5.1}$$

where $r_s = \sqrt{(x-s)^2 + y^2}$.

For complex potential let $z = x + iy$ and $z_s = (x-s) + iy$ then

$$\Phi(x,y) = \phi(x,y) + i\psi(x,y) = \frac{1}{\pi\sqrt{\pi Re}} \int_0^\infty \frac{ln\, z_s}{\sqrt{s}} ds = \frac{1}{\pi\sqrt{\pi Re}} \int_0^\infty \frac{ln(z-s)}{\sqrt{s}} ds.$$

Consider $p(s,q)$ a complex function defined as $p^2 = s + iq$. On the x-axis in the complex plane $p(s, q = 0)$ yields $p^2 = s$ then $ds = 2pdp$ substituting gives

$$\Phi = \frac{1}{\pi\sqrt{\pi Re}} \int_0^\infty \frac{ln(z-p^2)}{p} 2pdp = \frac{2}{\pi\sqrt{\pi Re}} \int_0^\infty ln(z-p^2)\, dp\, .$$

Since $(z-p^2) = (\sqrt{z}-p)(\sqrt{z}+p)$ then

$$\Phi = \frac{2}{\pi\sqrt{\pi Re}} \left\{ \int_0^\infty ln(\sqrt{z}-p)\, dp + \int_0^\infty ln(\sqrt{z}+p)\, dp \right\}. \tag{7.5.2}$$

Assume $p = -p^*$applying in the second term of right hand side to get

$$\int_0^\infty ln(\sqrt{z}+p)\, dp = \int_0^{-\infty} ln(\sqrt{z}-p^*)\,(-dp^*) = \int_{-\infty}^0 ln(\sqrt{z}-p^*)\, dp^*\, .$$

Substituting into (7.5.2) obtain

$$\Phi = \frac{2}{\pi\sqrt{\pi Re}} \int_{-\infty}^{\infty} ln(\sqrt{z} - p)\, dp. \qquad (7.5.3)$$

Consider $Im(\sqrt{z}) > 0$ the line of integration will be raised from 0 to $Im(\sqrt{z})$ (Figure 7.1) so the integration contour consists of three segments; from $(-\infty + Im(\sqrt{z}))$ to $\sqrt{z}$ and on the arc c_z around the branch cut $\sqrt{z}$ then from $\sqrt{z}$ to $\infty + Im(\sqrt{z})$; the equation (7.5.3) becomes

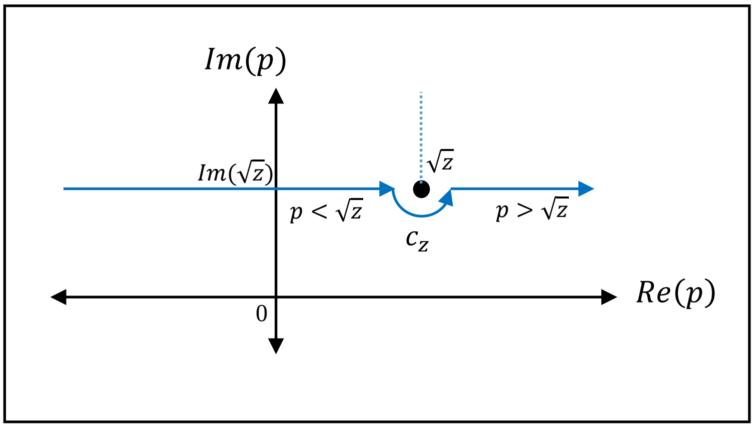


Figure 7.1 Integration Path

$$\int_{-\infty}^{\infty} ln(\sqrt{z} - p)\, dp = \int_{-\infty+\sqrt{iy}}^{\sqrt{z}} + \oint_{C_z} + \int_{\sqrt{z}}^{\infty+\sqrt{iy}} \{ln(\sqrt{z} - p)\, dp\}.$$

The integral $\oint_{C_z} ln(\sqrt{z} - p)\, dp$ will be vanish when the radius of C_z reaches zero by representing p as $\sqrt{z} - re^{i\theta}$ then $\int ln(\sqrt{z} - p) = \int ln\, re^{i\theta} \to 0$ as $r \to 0$.

Since $p - \sqrt{z} \leq 0$ when $R(p) \in [R(\sqrt{z}), \infty]$ then we have

$$\Phi = \frac{2}{\pi\sqrt{\pi Re}} \left\{ \int_{-\infty+\sqrt{iy}}^{\sqrt{z}} ln(\sqrt{z} - p)\, dp + \int_{\sqrt{z}}^{\infty+\sqrt{iy}} ln[e^{i\pi}(p - \sqrt{z})]\, dp \right\}. \qquad (7.5.4)$$

Since $\ln[e^{i\pi}(p - \sqrt{z})] = i\pi + \ln(p - \sqrt{z})$ then

$$\Phi = \frac{2}{\pi\sqrt{\pi Re}} \left\{ -i\pi\sqrt{z} + \int_{-\infty+\sqrt{iy}}^{\sqrt{z}} ln(\sqrt{z} - p)\, dp + \int_{\sqrt{z}}^{\infty+\sqrt{iy}} ln(p - \sqrt{z})\, dp \right\}.$$

Putting $p^* = \sqrt{z} - p$, gives

$$\Phi = \frac{1}{\pi\sqrt{\pi Re}}\left\{-i\pi\sqrt{z} + \int_0^\infty ln\, p^*\, dp^* + \int_{-\infty}^0 ln(-p^*)\, dp^*\right\}.$$

Integration by parts and Finite-Part Integrals (*FPI*) will be applied gives

$$\Phi = \phi + i\psi = \frac{2}{\pi\sqrt{\pi Re}}\{-i\pi\sqrt{z}\}. \tag{7.5.5}$$

using exponential form of complex number with Euler formula

$$-i\sqrt{z} = \sqrt{r}\left(-i\, cos\frac{\theta}{2} + sin\frac{\theta}{2}\right).$$

For small θ, it can deduce (Figure 7.2)

$$\sin(\theta/2) \approx \frac{\theta}{2} \approx \frac{1}{2} sin\theta = \frac{1}{2}\frac{y}{r}, \qquad \cos(\theta/2) \approx 1\,, \qquad r \approx x\ .$$

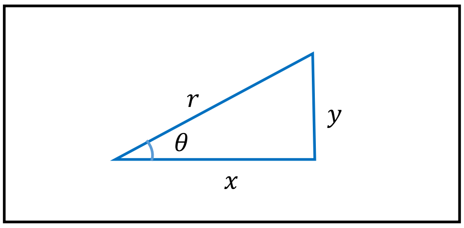


Figure 7.2 Relation between *x*, *y*, *r* and θ

Hence $-i\sqrt{z} \approx \sqrt{x}\left(-i + \frac{1}{2}\frac{y}{x}\right) = \frac{y}{2\sqrt{x}} - i\sqrt{x}$. Substituting in equation (7.5.5) to obtain

$$\phi(x,y) \approx \frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}}. \tag{7.5.6}$$

$$\psi(x,y) \approx -\frac{2}{\sqrt{\pi Re}}\sqrt{x}\,. \tag{7.5.7}$$

This solution agrees with the Sobey outer flow solution (Sobey 2000, eq. 2.54). The potential solutions (7.5.6) - (7.5.7) could also be verified via the result of the thin theory for Laplacian operator (7.4) by applying the strength Oseen function $\sigma(s)$ into (7.4.8) to give

$$\emptyset(x,y) = \frac{y}{2}\sigma(x) = \frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}},$$

since $\frac{\partial\emptyset}{\partial y} = -\frac{\partial\psi}{\partial x}$, therefore

$$\psi(x,y) = -\frac{2}{\sqrt{\pi Re}}\sqrt{x}.$$

7.6 Imai approximation for Drag Oseenlet Velocity

The aim of this section is to derive new representations more convenient to our work by defining of new stream function ψ_i and potential $\emptyset$ with variable η by using error function are defined by

$$\emptyset(x,y) = \frac{\theta}{2\pi}, \tag{7.6.1}$$

$$\psi(x,y) = -\frac{1}{2}\operatorname{erf}(\eta), \tag{7.6.2}$$

respectively, where

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2}\, d\eta',$$

$$\eta(x,y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\,. \tag{7.6.3}$$

The equations (7.6.1) - (7.6.2) are called Imai stream function (Imai 1951). The stream function related to potential flow is $\emptyset$, and viscous flow is ψ, so the total stream function is

$$\Psi = \phi + \psi,$$

by applying the equations (7.2.9); differentiate the stream function ψ with respect to y and to x to get velocity $\omega_1$and ω_2 respectively. Furthermore, for the function of potential ϕ; into these equations (7.6.1) - (7.6.2), the following equations are required to proof to reach to our aim.

For potential flow

$$\frac{\partial}{\partial y}\left[\frac{\theta}{2\pi}\right] = \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}(\ln r)\right\}, \tag{7.6.4}$$

$$\frac{\partial}{\partial x}\left[\frac{\theta}{2\pi}\right] = -\frac{1}{2\pi}\left\{\frac{\partial}{\partial y}(\ln r)\right\}. \tag{7.6.5}$$

For boundary layer (wake) flow

$$\frac{\partial}{\partial y}\left[-\frac{1}{2}(\operatorname{erf}(\eta))\right] \approx \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}\left[e^{\frac{Re}{2}x}K_0\left(\frac{Re}{2}r\right)\right] - Ree^{\frac{Re}{2}x}K_0\left(\frac{Re}{2}r\right)\right\}, \tag{7.6.6}$$

$$\frac{\partial}{\partial x}\left[\frac{1}{2}(\operatorname{erf}(\eta))\right] \approx \frac{1}{2\pi}\left\{\frac{\partial}{\partial y}\left[e^{\frac{Re}{2}x}K_0\left(\frac{Re}{2}r\right)\right]\right\}. \tag{7.6.7}$$

From exponential form of complex number and Euler formula yields

$$y = r\sin\theta \rightarrow \frac{\partial\theta}{\partial x} = -\frac{y}{r^2}, \qquad x = r\cos\theta \rightarrow \frac{\partial\theta}{\partial y} = \frac{x}{r^2},$$

where

$$r = \sqrt{x^2+y^2}, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

So we have

$$\frac{\partial}{\partial y}\left(\frac{\theta}{2\pi}\right) = \frac{1}{2\pi}\frac{x}{r^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\theta}{2\pi}\right) = -\frac{1}{2\pi}\frac{y}{r^2}.$$

Also, it is easy to see that

$$\frac{\partial}{\partial x}\left(\frac{\ln r}{2\pi}\right) = \frac{1}{2\pi}\frac{x}{r^2},$$

$$\frac{\partial}{\partial y}\left(\frac{\ln r}{2\pi}\right) = \frac{1}{2\pi}\frac{y}{r^2}.$$

This completes the proof of the equations (7.6.4) - (7.6.5). Now the left-hand side of equations (7.6.6) and (7.6.7) will be treated first, from the Error function and η function (7.6.3) it may to be concluded that

$$\frac{\partial}{\partial y}\left(-\frac{1}{2}\operatorname{erf}(\eta)\right) = -\frac{\sqrt{Re}}{2\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\eta^2}, \tag{7.6.8}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}\operatorname{erf}(\eta)\right) = -\frac{\sqrt{Re}}{4\sqrt{\pi}}\frac{y}{x\sqrt{x}}\; e^{-\eta^2}. \tag{7.6.9}$$

In order to complete the proof the simplifying of the term $\left(e^{kx}K_0(kr)\right)$ is needed, the behaviour of the Bessel functions for large arguments is required. So the following formula is applied (Abramowitz and Stegun 1964).

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \,,$$

where x is large, setting $\nu = 0$ yields $K_0\left(\frac{Re}{2}r\right) \approx \sqrt{\frac{\pi}{Re\,r}}e^{-\frac{Re}{2}r}$, now for small y and large x it can be written as following

$$e^{\frac{Re}{2}x}K_0\left(\frac{Re}{2}r\right) \approx \sqrt{\frac{\pi}{Re\,r}}e^{-\frac{Re}{2}(r-x)}.$$

Since

$$\begin{aligned} r - x = \sqrt{x^2+y^2} - x &= x\left\{1+\frac{y^2}{x^2}\right\}^{\frac{1}{2}} - x = x\left\{1+\frac{y^2}{2x^2}+\cdots\right\} - x \\ &= \left(x+\frac{y^2}{2x}+\cdots\right) - x \\ &\approx \frac{y^2}{2x}. \quad \text{for } x > 0. \end{aligned}$$

Therefore it can be written in the form

$$e^{kx}K_0(kr) \approx \sqrt{\frac{\pi}{Rex}}e^{-\frac{Re}{4}\frac{y^2}{x}} = \sqrt{\frac{\pi}{Rex}}e^{-\eta^2}. \qquad (7.6.10)$$

Now employing (7.6.10) into the left hand side of (7.6.6) leads us to

$$\begin{aligned} \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}\left(\sqrt{\frac{\pi}{Rex}}e^{-\eta^2}\right) - Re\sqrt{\frac{\pi}{Rex}}e^{-\eta^2}\right\} &= \frac{1}{2\pi}\left\{\left(\frac{1}{x}\eta^2 - \frac{1}{2x} - Re\right)\sqrt{\frac{\pi}{Rex}}e^{-\eta^2}\right\} \\ &\approx -\frac{1}{2}\frac{\sqrt{Re}}{\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\eta^2}. \qquad (7.6.11) \end{aligned}$$

Since the right hand side of (7.6.11) is same as the right hand side of (7.6.8) therefore (7.6.6) is proved. In a similar fashion we can prove the right hand side of equation (7.6.7) as

$$\frac{1}{2\pi}\left\{\frac{\partial}{\partial y}\left[e^{\frac{Re}{2}x}K_0\left(\frac{Re}{2}r\right)\right]\right\} \approx \frac{1}{2\pi}\left\{\frac{\partial}{\partial y}\sqrt{\frac{\pi}{Rex}}e^{-\eta^2}\right\} = -\frac{\sqrt{Re}}{4\sqrt{\pi}}\frac{y}{x\sqrt{x}}e^{-\eta^2}. \qquad (7.6.12)$$

By comparing the right hand side of (7.6.12) with the right hand side of (7.6.9) then (7.6.7) is proved.

7.7 The Boundary Layer (wake) velocity from Oseen integral equation and Imai's approximation

In this section, boundary layer (wake) velocity would be derived from Oseen integral representation (7.2.4) - (7.2.4) and the strength Oseen function (7.2.10) utilizing Imai's approximation of Drag Oseen let

$$\left.\begin{aligned} \omega_1(x,y) &= \int_0^\infty u^\omega(x-s,y)\sigma(s)ds\ , \\ \omega_2(x,y) &= \int_0^\infty v^\omega(x-s,y)\sigma(s)ds\ , \end{aligned}\right\} \tag{7.7.1}$$

where u^ω, v^ω are the velocity representations of the viscous part of a drag Oseenlet. According to section (7.6), the drag Oseenlet for boundary layer (wake) viscous velocity can be approximated as

$$\left.\begin{aligned} u^\omega(x,y) &\approx -\frac{1}{2\pi}\frac{\sqrt{Re}}{\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-\frac{Re\,y^2}{4x}}, \\ v^\omega(x,y) &\approx -\frac{1}{4\pi}\frac{\sqrt{Re}}{\sqrt{\pi}}\frac{y}{x\sqrt{x}}e^{-\frac{Re\,y^2}{4x}}. \end{aligned}\right\} \tag{7.7.2}$$

First, the limitation of the integral representation (7.7.1) may be reduced, let $x \in (0,\infty)$, then

$$\left.\begin{aligned} \omega_1(x,y) &= \int_0^x u^\omega(x-s,y)\sigma(s)ds + \int_x^\infty u^\omega(x-s,y)\sigma(s)ds\ , \\ \omega_2(x,y) &= \int_0^x v^\omega(x-s,y)\sigma(s)ds + \int_x^\infty v^\omega(x-s,y)\sigma(s)ds\,. \end{aligned}\right\} \tag{7.7.3}$$

The contribution of the second term of ω_1 and ω_2 in (7.7.3) are negligible because it lies outside the boundary layer (wake) of drag Oseenlet, and so exponentially small (see Figure 7.3).

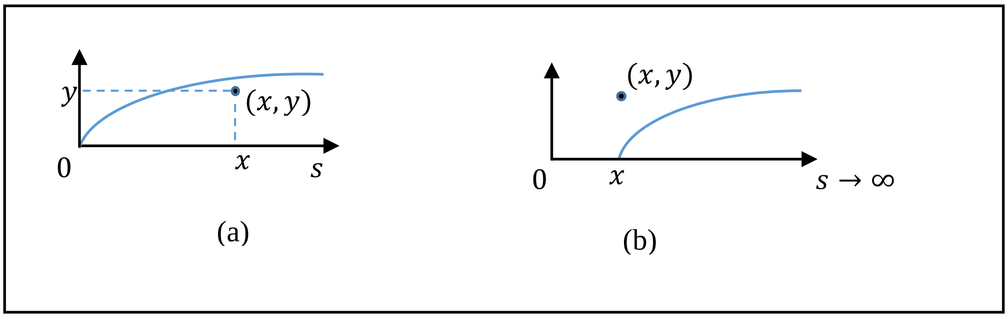


Figure 7.3 Limitation of Integral

So equation (7.7.3) becomes

$$\left.\begin{aligned} \omega_1(x,y) &\approx \int_0^x u^{\omega}(x-s,y)\sigma(s)ds\ , \\ \omega_2(x,y) &\approx \int_0^x v^{\omega}(x-s,y)\sigma(s)ds\ . \end{aligned}\right\} \tag{7.7.4}$$

Inserting $\sigma(x)$ from equation (7.2.10) and substituting (7.7.2) into (7.7.4) gives

$$\omega_1(x,y) = -\frac{1}{\pi}\int_0^x \frac{1}{\sqrt{s(x-s)}} e^{-\frac{Re\,y^2}{4(x-s)}} ds, \tag{7.7.5}$$

$$\omega_2(x,y) = -\frac{1}{2\pi}\int_0^x \frac{y}{\sqrt{s(x-s)^3}} e^{-\frac{Re\,y^2}{4(x-s)}} ds. \tag{7.7.6}$$

It has been observed that

$$\frac{\partial}{\partial y}\left\{\frac{1}{\pi}\int_0^x \frac{1}{\sqrt{s(x-s)}} e^{-\frac{Re\,y^2}{4(x-s)}} ds\right\} = -\frac{Re}{2\pi}\int_0^x \frac{y}{\sqrt{s(x-s)^3}} e^{-\frac{Re\,y^2}{4(x-s)}} ds,$$

so it can be rewritten

$$\frac{\partial \omega_1(x,y)}{\partial y} = -Re\,\omega_2(x,y). \tag{7.7.7}$$

ω_2 can be evaluated first, then the relationship (7.7.7) may be used to find ω_1. Using the transformation

$$s = \frac{xr^2}{r^2+1}. \tag{7.7.8}$$

Therefore,

$$\left.\begin{aligned}
&(x-s)=\left(\frac{x}{r^2+1}\right),\\
&\sqrt{s(x-s)^3}=\frac{x^2 r}{(r^2+1)^2}\,,\\
&\frac{ds}{dr}=\frac{2xr}{(r^2+1)^2}.
\end{aligned}\right\} \tag{7.7.9}$$

When $s=0$ gives $r\to 0$ and when $s=x$ gives $r\to\infty$. Substituting into ω_2 to obtain

$$\omega_2(x,y)=-\frac{y}{\pi x}e^{-\frac{Re\,y^2}{4x}}\int_0^\infty e^{-\left(\frac{Re\,y^2}{4x}\right)r^2}dr.$$

Since $\left(\frac{Re\,y^2}{4x}\right)>0$ then the Gaussian integral equation(4.1.4) Chapter 4 may be applied, so we have

$$\omega_2(x,y)=-\frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}e^{-\frac{Re\,y^2}{4x}}\,. \tag{7.7.10}$$

It remains to evaluate the velocity ω_1, so the relationship (7.7.7) can be used by integrating with respect to y' from y to ∞, applying boundary condition $\omega_1(x,\infty)=0$, yields

$$\omega_1(x,y)=\frac{\sqrt{Re}}{\sqrt{\pi}}\frac{1}{\sqrt{x}}\int_\infty^y e^{-\frac{Re\,{y'}^2}{4x}}dy'.$$

Letting $t=\frac{\sqrt{Re}}{2\sqrt{x}}y'$ then $dy'=\frac{2\sqrt{x}}{\sqrt{Re}}\,dt$, and the integration is split into two intervals with exchanging the limits of the first one

$$\omega_1(x,y)=-\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt\;+\frac{2}{\sqrt{\pi}}\int_0^{\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right)} e^{-t^2}dt\,.$$

From the Gaussian integral equation(4.1.4) Chapter 4 and Error function (7.6.3), it could be obtained

$$\omega_1(x,y)=-1+\operatorname{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right). \tag{7.7.11}$$

7.8 The total solution (Boundary and Potential) Velocity

In this section the total solution will be found by combine the boundary layer (wake) velocity solution to the potential velocity solution for outer flow which is obtained by the Thin Body Theory.

First, the *x*-momentum velocity $u(x,y)$, for potential velocity ϕ, from (7.2.1) and (7.5.6) gives

$$\frac{\partial}{\partial x}\phi(x,y)=\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}}\right\}=\frac{-1}{2\sqrt{\pi Re}}\frac{y}{x\sqrt{x}},$$

the potential velocity ϕ very small when *x* very large with compare with ω_1, then there is no significant contribution of the potential velocity inside boundary layer region, therefore it is may be neglected, in this way, the boundary layer velocity solution ω_1 equation (7.7.11) would be substituted into the integral representation of Oseen solution (7.2.4), the total *x* -momentum velocity $u(x,y)$ becomes

$$u(x,y)=\operatorname{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right). \tag{7.8.1}$$

Next, to evaluate $v(x,y)$, the total *y*-momentum velocity, $\omega_2(x,y)$ and $\phi(x,y)$ the boundary layer (wake) velocity solution (7.7.10) and the potential velocity solution for outer flow (7.5.6) respectively could be inserted into the *y*-momentum integral representation of Oseen solution (7.2.5), so that it can be written the velocities $v(x,y)$ as

$$v(x,y)=\frac{\partial}{\partial y}\phi(x,y)+\omega_2(x,y)=\frac{\partial}{\partial y}\left\{\frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}}\right\}-\frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}e^{-\frac{Re\,y^2}{4x}}$$

$$=\frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}\left(1-e^{-\frac{Re\,y^2}{4x}}\right). \tag{7.8.2}$$

The solutions $u(x,y)$ and $v(x,y)$ agree with the Burgers' solution (Burgers 1930, p. 610, eq. 17 & eq. 18).

7.9 Derivation of the Oseen-Blasius equation from Oseen integral representation

This section shows the Oseen flow over flat plate is matching Blasius problem in boundary layer, we will obtain Oseen-Blasius equation from Oseen integral representation (7.2.4) by applying the Imai representation of the drag Oseenlet (7.6.2), and reducing the limitation of the integral representation. We have the following

$$\Psi(x,y) = \int_0^x \left(-\frac{1}{2}\operatorname{erf}(\eta_s)\right)\frac{2}{\sqrt{\pi Re\, s}}\,ds, \tag{7.9.1}$$

where

$$\eta_s(x-s,y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x-s}}. \tag{7.9.2}$$

Let $\alpha = \frac{s}{x}$, where $s < x$, applying into (7.9.2) gives

$$\eta_\alpha = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x-\alpha x}} = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\frac{1}{\sqrt{1-\alpha}} = \frac{\eta}{\sqrt{1-\alpha}}. \tag{7.9.3}$$

Now $ds = x d\alpha$. Substituting into (7.9.1) gives

$$\Psi(x,y) = \int_0^\alpha \left(-\frac{1}{2}\operatorname{erf}(\eta_\alpha)\right)\frac{2}{\sqrt{\pi Re\, \alpha' x}}\,x d\alpha'.$$

Then we have

$$\Psi(x,y) = -\frac{\sqrt{x}}{\sqrt{\pi Re}}\int_0^\alpha \frac{1}{\sqrt{\alpha'}}\left[\operatorname{erf}\left(\frac{\eta}{\sqrt{1-\alpha'}}\right)\right]d\alpha'. \tag{7.9.4}$$

Let

$$f(\eta) = -\frac{1}{\sqrt{2\pi}}\int_0^\alpha \frac{1}{\sqrt{\alpha}}\left[\operatorname{erf}\left(\frac{\eta}{\sqrt{1-\alpha'}}\right)\right]d\alpha', \tag{7.9.5}$$

where

$$\eta(x,y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}.$$

Then the equation (7.9.4) can be expressed as

$$\Psi(x,y) = \frac{\sqrt{2x}}{\sqrt{Re}}\, f(\eta)\,. \tag{7.9.6}$$

The main contribution in (7.9.4) is when α small, so $1-\alpha \approx 1$, this gives

$$f(\eta) \approx -\frac{1}{\sqrt{2\pi}}\int_0^{\alpha} \frac{1}{\sqrt{\alpha'}}\operatorname{erf}(\eta)\, d\alpha' = -\frac{\sqrt{2\alpha}}{\sqrt{\pi}}\operatorname{erf}(\eta). \tag{7.9.7}$$

The derivatives of a function f with respect to η provide polynomial, it could be approximated to give Oseen-Blasius equation as the following

$$f'(\eta) = -\frac{2\sqrt{2\alpha}}{\pi}\, e^{-\eta^2},$$

where $\frac{\partial}{\partial\eta}\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}e^{-\eta^2}$, and

$$f''(\eta) = \frac{4\sqrt{2\alpha}}{\pi}\, \eta e^{-\eta^2},$$

the third derivative is

$$\begin{aligned} f'''(\eta) &= \frac{4\sqrt{2\alpha}}{\pi}\, e^{-\eta^2} - \frac{8\sqrt{2\alpha}}{\pi}\, \eta^2 e^{-\eta^2} \\ &= \frac{f''(\eta)}{\eta} - 2\eta f''(\eta). \end{aligned} \tag{7.9.8}$$

For large η, the equation (7.9.8) can be rewritten as

$$f'''(\eta) \approx -2\eta f''(\eta)$$

this gives the equation

$$f'''(\eta) + 2\eta f''(\eta) = 0.$$

Therefore, the Oseen-Blasius equation (3.4.5) is obtained.

7.10 The analytic solution of the Oseen-Blasius equation

The solution of Oseen-Blasius equation in the flow over flat plate will be obtained analytically from two different forms first the ordinary differential equation and second the partial differential equation. It has been solved by Laplace transform method.

7.10.1 The solution of ordinary differential equation form of the Oseen-Blasius equation

First, the equation with boundary conditions is

$$f'''(\eta) + 2\eta f''(\eta) = 0,$$

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\eta \to \infty) = 1,$$

where the free stream velocity $U = 1$, $\eta(x, y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}$.

To solve this ordinary differential equation, reduction of order method may be applied, by assumption $g(\eta) = f''(\eta)$ the equation becomes

$$g'(\eta) + 2\eta\, g(\eta) = 0.$$

The general solution of this equation can be written as

$$f''(\eta) = f''(0)e^{-\eta^2}. \tag{7.10.1.1}$$

To evaluate $f''(0)$, the equation (7.10.1.1) is integrated with respect to η from 0 to ∞, by utilizing the boundary conditions $f'(0) = 0$, $f'(\eta \to \infty) = 1$ and Gaussian integral (4.1.4) to give

$$f''(0) = \frac{2}{\sqrt{\pi}}. \tag{7.10.1.2}$$

Inserting (7.10.1.2) into equation (7.10.1.1) and integrating from 0 to η, from the Error function integral representation and boundary conditions $f'(0) = 0$ we have

$$f'(\eta) = \operatorname{erf}(\eta). \tag{7.10.1.3}$$

In order to find f, we integrate f' from 0 to η and integration by parts method is applied for Error function gives

$$f(\eta) = \eta \operatorname{erf}(\eta) + \frac{2}{\sqrt{\pi}} e^{-\eta^2} - \frac{2}{\sqrt{\pi}}. \tag{7.10.1.4}$$

Finally, inserting $f'(\eta)$ and $f(\eta)$ into equations (3.5.8), (3.5.12) (Section 3.5 Chapter 3) respectively, therefore the velocities profiles $u(x,y)$ and $v(x,y)$ are

$$u(x,y) = \operatorname{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right), \tag{7.10.1.5}$$

$$v(x,u) = \frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}\left(1 - e^{-\frac{Re}{4}\frac{y^2}{x}}\right). \tag{7.10.1.6}$$

It can easily be noticed that (7.10.1.5) and (7.10.1.5) are exactly same (7.8.1) and (7.8.2) respectively.

7.10.2 The solution of partial differential equation form of the Oseen-Blasius equation

The procedure of this section is taken from the study of Kusukawa (Kusukawa et al., 2014) where the aim of this study is Blasius equation, Kusukawa study proves that there is an analytical technique, namely an iteration method to solve the problem of the boundary layer flow over a flat plate. He obtained several approximations of the solution of this equation and the ε power series has been applied to expand the velocity components (see chapter 2), whereas we focused on the first approximation which is the solution to the Oseen-Blasius equation. In the present section the equation is expressed when the free stream velocity $U = 1$ as

$$\frac{\partial u^*}{\partial x} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial y^2}.$$

It has both Oseen and Blasius approximation, (in some references called modified Oseen equation).

From Oseen approximation (3.1.2) the velocity components $(u\,,v\,)$ are expanded such as

$$u = 1 + \varepsilon\, u^*\,, \qquad v = \epsilon\, v^*\,, \tag{7.10.2.1}$$

where and the parameter ε is the ratio of the boundary layer thickness to the plate length. From (7.10.2.1) the boundary conditions become

$$u^*(x,0) = -\frac{1}{\varepsilon}, \qquad u^*(x, y \to \infty) = 0, \qquad v^*(x,0) = 0. \tag{7.10.2.2}$$

The Laplace transform with respect to x considering y as a parameter can be applied to solve the equation with boundary conditions (7.10. 2.2) which it is defined by

$$\overline{u^*}(s,y) = \int_0^\infty u^*(x,y) e^{-sx} dx. \tag{7.10.2.3}$$

To obtain the Laplace transform of the derivative of $u(x,y)$ with respect to x, integration by parts will be applied into the Laplace Transform equation (7.10. 2.3) as follows

$$\int_0^\infty u^*(x,y)\, e^{-sx} dx = \left[\frac{-u^*(x,y)e^{-sx}}{s}\right]_{x=0}^{x=\infty} - \frac{1}{s}\int_0^\infty \frac{\partial u^*(x,y)}{\partial x} e^{-sx} dx.$$

So we have

$$\int_0^\infty \frac{\partial u^*(x,y)}{\partial x} e^{-sx} dx = s\int_0^\infty u^*(x,y)\, e^{-sx} dx = s\,\overline{u^*}(s,y) \tag{7.10.2.4}$$

where $u^*(0,y) = 0$, and

$$\frac{\partial^2 \overline{u^*}(s,y)}{\partial y^2} = \int_0^\infty \frac{\partial^2 u^*(s,y)}{\partial y^2} e^{-sx} dx. \tag{7.10.2.5}$$

Then applying the Laplace transform to the partial differential equation form of Oseen-Blasius equation, we have

$$\frac{\partial^2 \overline{u^*}(s,y)}{\partial y^2} - (Re)(s)\overline{u^*}(s,y) = 0.$$

The general solution of this equation as

$$\overline{u^*}(s,y) = C_1\, e^{(\sqrt{Res})y} + C_2 e^{-(\sqrt{Res})y}. \tag{7.10.2.6}$$

Evaluating the constant C_1 and C_2 using the boundary conditions gives

$$\overline{u^*}(s,o) = -\frac{1}{\varepsilon}\int_0^\infty e^{-sx}dx = -\frac{1}{\varepsilon}\left(\frac{1}{s}\right),$$

and $\overline{u^*}(s, y \to \infty) = 0$, thus we have $C_1 = 0,\ C_2 = -\frac{1}{\varepsilon}\left(\frac{1}{s}\right)$, inserting in the general solution (7.10.2.6) gives

$$\overline{u^*}(s,y) = -\frac{e^{-\left(\sqrt{(Re)(s)}\right)y}}{\varepsilon s}.$$

The inverse Laplace transform of $\left(\frac{1}{s}e^{-a\sqrt{s}}\right)$ is $1-\mathrm{erf}\left(\frac{a}{2\sqrt{x}}\right)$; (Abramowitz & Stegun, 1964, p.1026, eq. 29.3.82 & p.297, eq. 7.1.2); yields

$$u^*(x,y) = \frac{1}{\varepsilon}\left[\mathrm{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right) - 1\right]. \tag{7.10.2.7}$$

To complete the process, the y-momentum velocity v^* will be obtained by substituting u^* in the continuity equation and integrate with respect to y from 0 to y and from boundary condition $v^*(x,0) = 0$, yielding

$$v^*(x,y) = -\int_0^y \frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\left[\mathrm{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right) - 1\right]\right)dy.$$

Since

$$\frac{\partial}{\partial x}\ \mathrm{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right) = -\frac{\sqrt{Re}}{\sqrt{\pi}}\frac{y}{x\sqrt{x}}\ e^{-\frac{Re}{4x}y^2},$$

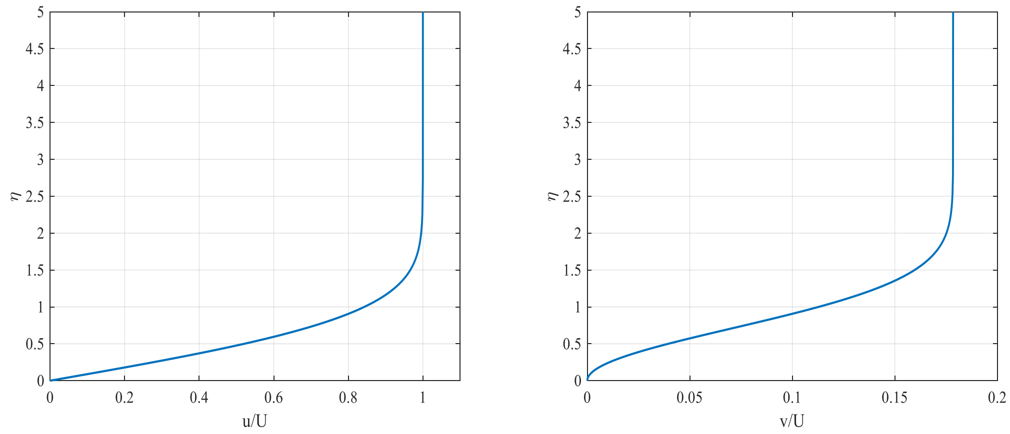
therefore

$$v^*(x,y) = \frac{1}{\varepsilon}\ \frac{1}{\sqrt{\pi x Re}}\left(1 - e^{-\frac{Re\,y^2}{4x}}\right). \tag{7.10.2.8}$$

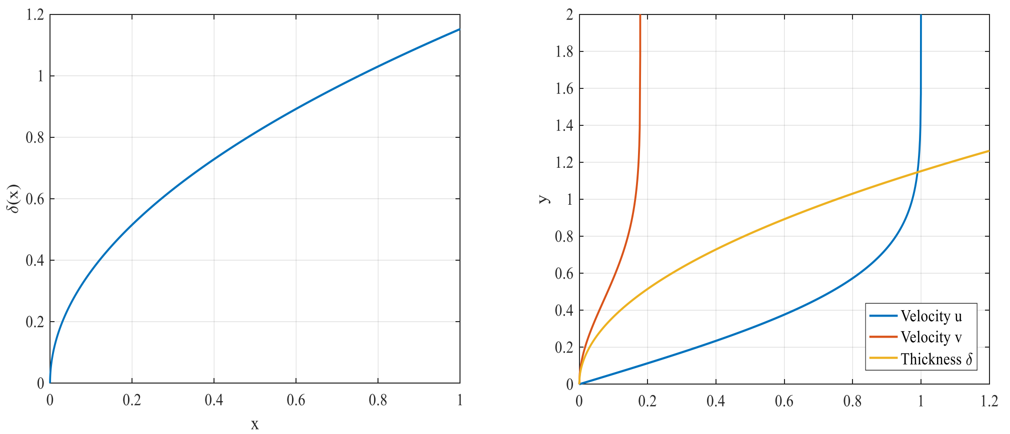
The solution of Oseen-Blasius equations in variables x, y are

$$u(x,y) = \mathrm{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right),$$
$$v(x,y) = \frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}\left(1 - e^{-\frac{Re\,y^2}{4x}}\right).$$

So that the solution for ordinary form (7.10.1.5) and (7.10.1.6) are verified.



(a) **(b)**



(b) **(d)**

Figure 7.4 The analytic solution of Oseen-Blasius equation

where: $Re = 10,\ x = 1$ and $U = 1$ with $\eta(x, y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}$

(a) Velocity u with η-axis .

(b) Velocity v with η -axis.

(c) Boundary Layer Thickness δ, evaluated by $u = 0.99$U,

$$\delta(x) = \frac{2(1.8213)\sqrt{x}}{\sqrt{Re}}.$$

(d) Comparison between u, v, δ with y-axis.

7.11 Stokes Boundary Layer

The Stokes boundary layer is considered, it refers to the boundary layer near to the flat plate for our case of laminar steady flow over a flat plate in the direction parallel to the x-axis. In the near field, the Oseen flow tends to Stokes flow (Batchelor 1967), and we will use this fact to obtain the solution of boundary layer problem by Stokes equation in near-field. In the similar procedure in the section (7.7), we can express the velocity by the integral equation as

$$u(x,y) = \int_0^\infty u^D(x-s,y)\sigma(s)ds.$$

Here the Drag Stokeslet is defined by

$$u^D(x,y) = \frac{Re}{4\pi}\left[\ln(\sqrt{x^2+y^2}) - \frac{x^2}{x^2+y^2}\right], \tag{7.11.1}$$

(Batchelor 1967). Since

$$y\frac{\partial}{\partial y}\left[\ln\left(\sqrt{x^2+y^2}\right)\right] = \frac{y^2}{x^2+y^2},$$

we can rewrite (7.11.1) as

$$u^D(x,y) = \frac{Re}{4\pi}\left\{\ln(\sqrt{x^2+y^2}) - 1 + y\frac{\partial}{\partial y}\left[\ln\left(\sqrt{x^2+y^2}\right)\right]\right\}. \tag{7.11.2}$$

Therefore we obtain

$$u(x,y) = \frac{Re}{4\pi}\left\{\int_0^\infty \left[\ln\left(\sqrt{(x-s)^2+y^2}\right) - 1\right]\sigma(s)ds \right.$$
$$\left. + y\frac{\partial}{\partial y}\int_0^\infty \ln(\sqrt{(x-s)^2+y^2})\,\sigma(s)ds\right\}.$$

It can be expressed as

$$u(x,y) \approx \frac{Re}{4\pi}\left\{\int_0^\infty \ln\left(\sqrt{(x-s)^2+y^2}\right)\sigma(s)ds + y\frac{\partial}{\partial y}\int_0^\infty \ln\left(\sqrt{(x-s)^2+y^2}\right)\sigma(s)ds\right\}.$$

Since $r_s = \sqrt{(x-s)^2+y^2}$, and the strength Oseen function $\sigma(s) = \frac{2}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}$, it can be rewritten as

$$u(x,y) = \frac{Re}{2}\left\{\frac{1}{\pi\sqrt{\pi Re}}\int_0^\infty \frac{\ln r_s}{\sqrt{s}}ds + y\frac{\partial}{\partial y}\left(\frac{1}{\pi\sqrt{\pi Re}}\int_0^\infty \frac{\ln r_s}{\sqrt{s}}ds\right)\right\}.$$

Apply the thin body theory result (7.5.6) gives

$$u(x,y) = \frac{Re}{2}\left\{\frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}} + y\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\pi Re}}\frac{y}{\sqrt{x}}\right)\right\}.$$

By differentiating the second term with respect to y gives

$$u(x,y) = \frac{\sqrt{Re}}{\sqrt{\pi}}\frac{y}{\sqrt{x}}. \tag{7.11.3}$$

Since $\eta(x,y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}$, substituting into (7.11.3), the Stokes solution (Figure 7.7); becomes

$$u(x,y) = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right) = \frac{2}{\sqrt{\pi}}\eta. \tag{7.11.4}$$

There is alternative technique to obtain the solution (7.11.4) by apply the boundary layer theory on steady Stokes equation to give Stokes Boundary Layer equation as

$$\frac{1}{Re}\frac{\partial^2 u}{\partial y^2} = 0\,. \tag{7.11.5}$$

(Batchelor, 1967, p. 190).

The solution of this equation could be calculated by integrated twice with respect to y with applied the boundary condition as

$$u(x, y) = a(x)y,$$

set $a(x) = \frac{\sqrt{Re}}{2}\frac{A}{\sqrt{x}}$ to obtain

$$u(x, y) = \mathrm{A}\,\eta(x, y), \tag{7.11.6}$$

where A is a constant.

The stream velocity function $\psi(x, y)$ could also be applied into Stokes Boundary Layer (7.11.5) to obtain same solution as

$$\frac{\partial^3\psi}{\partial y^3} = 0.$$

By following same procedure of Oseen-Blasius equation, so we can say

$$f'''(\eta) = 0. \tag{7.11.6}$$

Integrating gives

$$f''(\eta) = A, \qquad u(x, y) = f'(\eta) = A\eta + B.$$

Since $u = 0$ as $\eta = 0$ then $B = 0$ therefore

$$u(x, y) = f'(\eta) = A\eta.$$

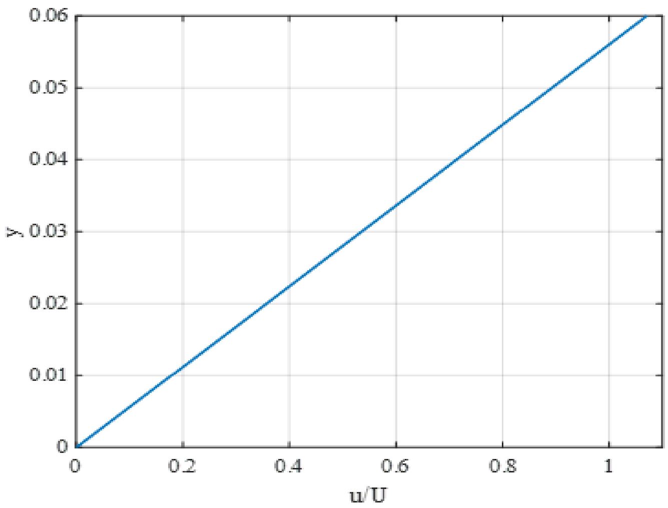


Figure 7.5 The solution of velocity u of Stokes boundary layer equation where $x = 1$and $Re = 10^3$.

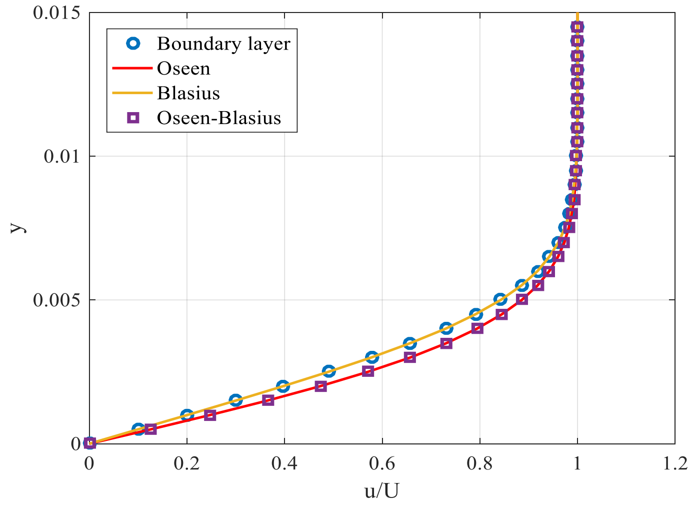
Chapter 8 Numerical and analytical Comparison

8.1 Comparisons of Numerical solution

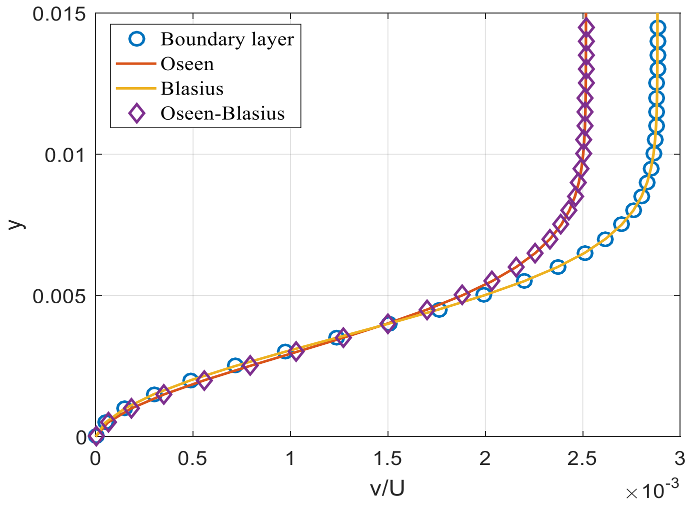
In this section, the profile solutions for four approximate equations of Navier-Stokes equations: Boundary Layer, Oseen Boundary Layer, Blasius and Oseen-Blasius are compared.

The Figures (8.1)-(8.8) illustrate the graphs of x-momentum and y-momentum velocities of these approximate equations at x=0.5 and at x=1 by Finite Difference Method (FDM) for various Reynolds number $Re = 10^4\ , 10^3, 10^3$ and 10^5 and the free stream velocity $U = 1$ with different grid size in rectangle domain $D = \ (0,1) \times (0,1)$.

The numerical results show that Blasius is more close to Boundary Layer equation than the others. In addition, the Oseen-Blasius matches Oseen Boundary Layer equation. Therefore, it demonstrates that the pressure does not play a main role in the flow over a flat plate and can be ignored. Furthermore, we can conclude that the simplest way to formulate the Oseen problem is Oseen-Blasius equation, where the study of Kusukawa (Kusukawa, Suwa et al. 2014) proves that the Oseen-Blasius solution is the first approximation to Blasius solution (Figure 2.1).

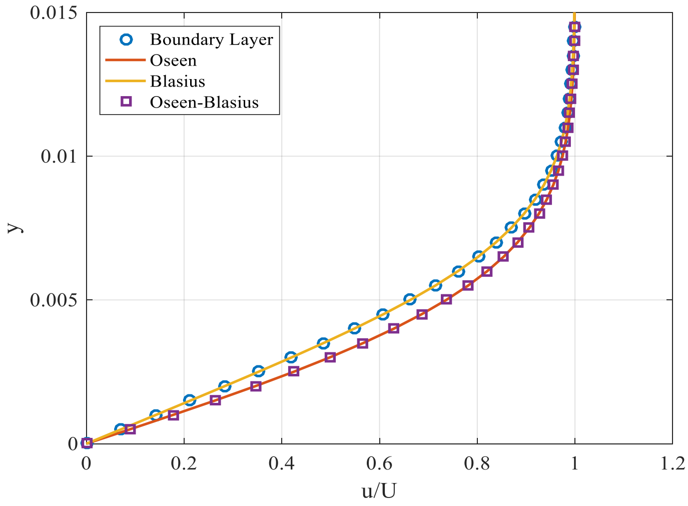


(a) The x-momentum velocity u/U with y-axis.

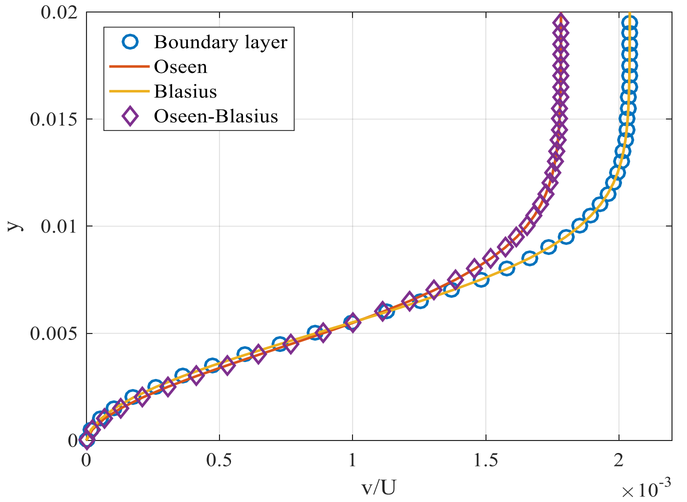


(b) The y-momentum velocity v/U with y-axis

Figure 8.1 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $Re = 10^5$, $x = 0.5$. The comparison is evaluated at $Re = 10^5$ and the free stream velocity $U = 1$ by Finite Difference Method (FDM) with grid size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $D = (0,1) \times (0,1)$.

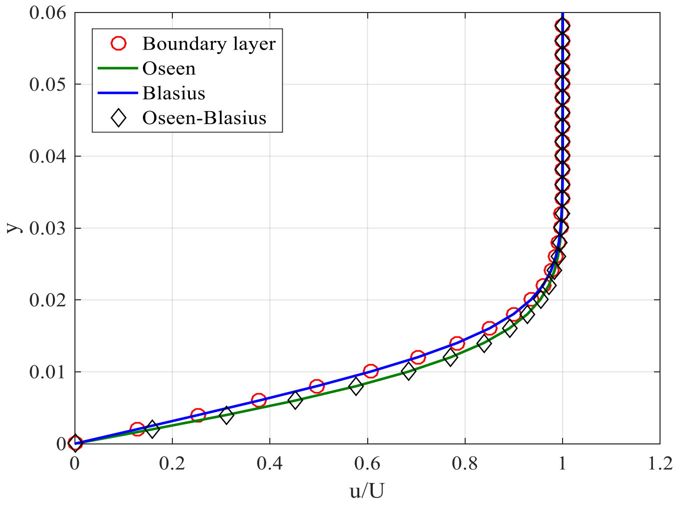


(a) The x-momentum velocity u/U with y-axis.

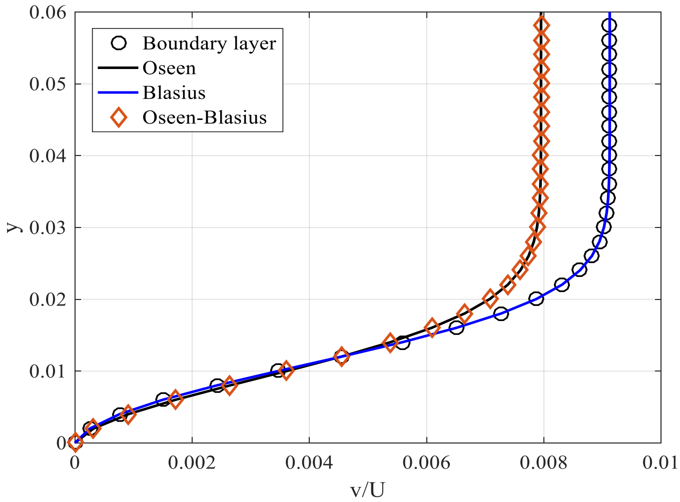


(b) The y-momentum velocity v/U with y-axis.

Figure 8.2 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $Re = 10^5$, x=1.
The Comparison is evaluated by Finite Difference Method of 2D flow over flat plate, the free stream velocity $U = 1$ with grid size size 2000×2000, $\Delta x = \Delta y = 0.0005$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.



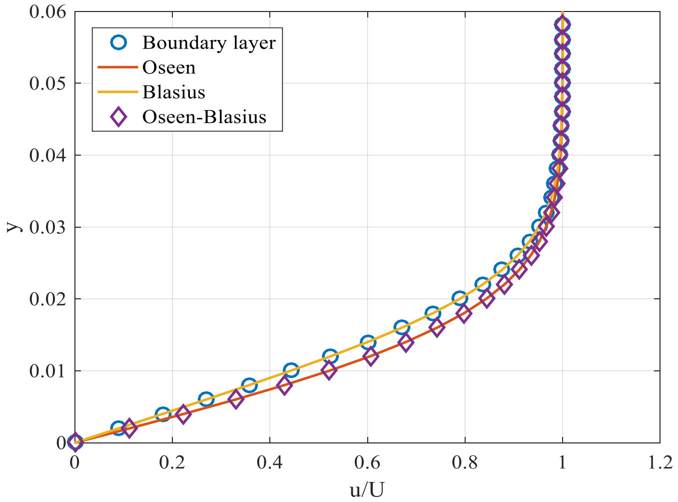
(a) The x-momentum velocity u/U with y-axis.



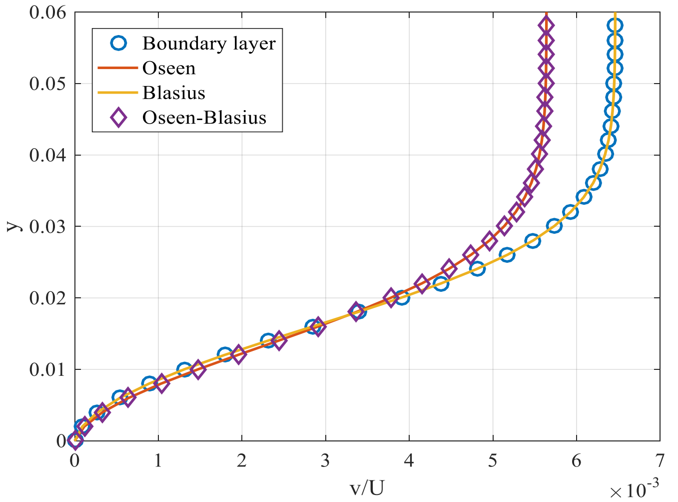
(b) The y-momentum velocity v/U with y-axis.

Figure 8.3 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $\mathrm{Re} = 10^4$, x=0.5.

The Comparison is evaluated by Finite Difference Method of 2D flow over flat plate, the free stream velocity $U = 1$ with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.



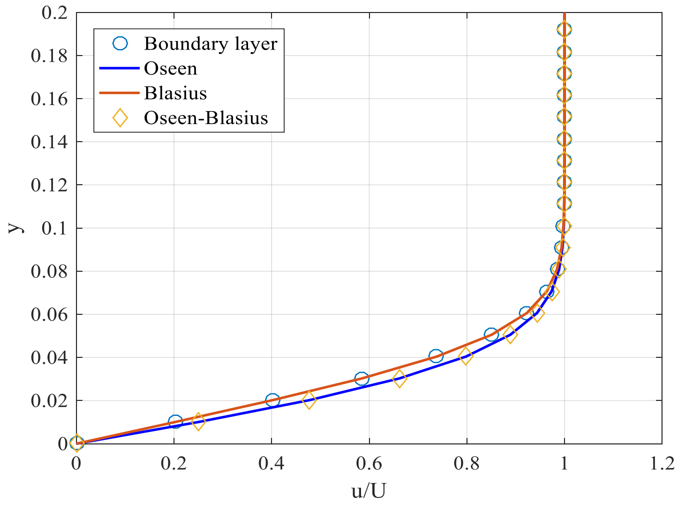
(a) The x-momentum velocity u/U with y-axis.



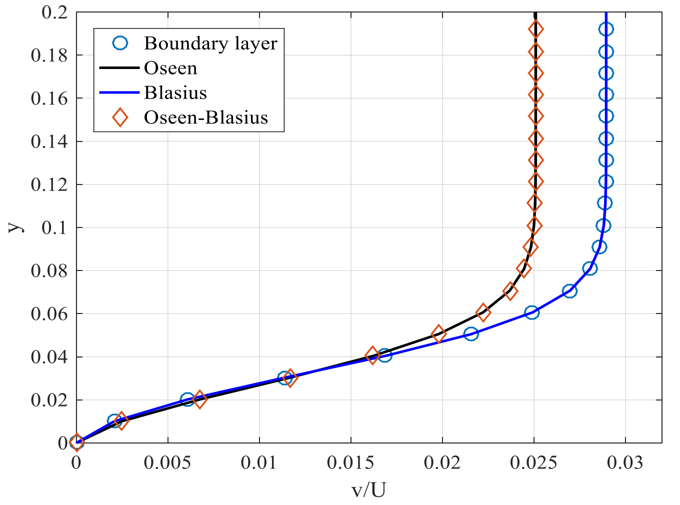
(b) The y-momentum velocity v/U with y-axis.

Figure 8.4 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $\mathrm{Re} = 10^4$, x=1.

The Comparison is evaluated by Finite Difference Method of 2D flow over flat plate, the free stream velocity $U = 1$ with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $D = (0,1) \times (0,1)$.

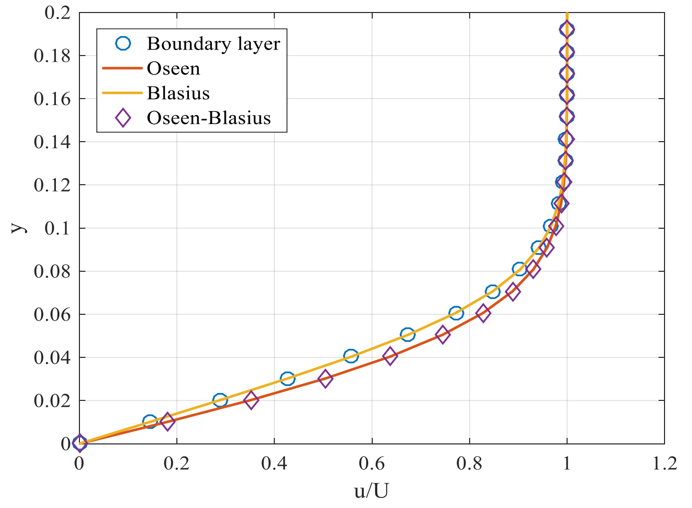


(a) The x-momentum velocity u/U with y-axis.

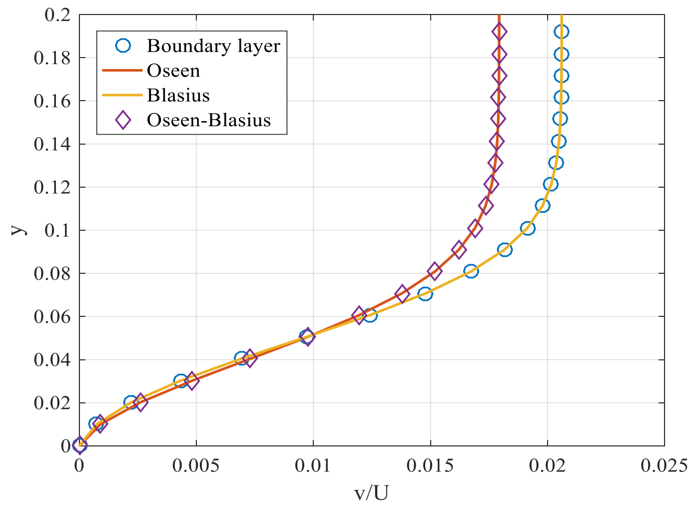


(b) The y-momentum velocity v/U with y-axis.

Figure 8.5 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $\mathrm{Re} = 10^3$, x=0.5. The Comparison is evaluated by Finite Difference Method of 2D flow over flat plate, the free stream velocity $U = 1$ with grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.



(a) The x-momentum velocity u/U with y-axis.



(b) The y-momentum velocity v/U with y-axis.

Figure 8.6 Comparison of Boundary Layer equations, Oseen Boundary Layer equations, Blasius equations and Oseen-Blasius equations at $\mathrm{Re} = 10^3$, x=1.
The Comparison is evaluated by Finite Difference Method of 2D flow over flat plate, the free stream velocity $U = 1$ with grid size 100×100, $\Delta x = \Delta y = 0.01$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$.

8.2 Comparison of Numerical and Analytical Solution of Oseen Blasius Solution

The numerical and analytical profile solutions of the x-momentum and y-momentum velocities and thickness of the Oseen-Blasius equation (3.6.1) are compared.

The comparison has performed at Reynolds Number $Re = 10^3$, $x = 1$ and the free stream velocity $U = 1$, where the analytical solution is

$$u(x,y) = \operatorname{erf}\left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right), \qquad v(x,y) = \frac{1}{\sqrt{\pi Re}}\frac{1}{\sqrt{x}}\left(1 - e^{-\frac{Re\, y^2}{4x}}\right),$$

and for numerical solution the finite different method (FDM) has been applied (see section 6.7 chapter 6) with grid size 500×500, $\Delta x = \Delta y = 0.002$ in the rectangle domain $\mathrm{D} = (0,1) \times (0,1)$,The boundary layer thickness δ is evaluated by $\mathrm{u} = 0.99\mathrm{U}$ and $\delta(x) = \frac{2(3.7)\sqrt{x}}{\sqrt{Re}}$, see Figures 8.7-8.9

The results shows that the numerical solution has well matched the analytic solution of Oseen-Blasius (linear partial differential) equation under same boundary condition.

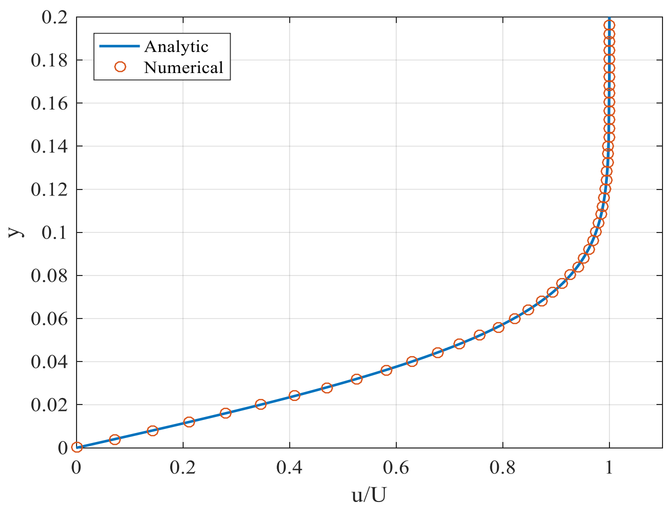


Figure 8.7 The x-momentum velocity u/U with y-axis of analytic and numerical solution of the Oseen-Blasius equation at $x = 1$.

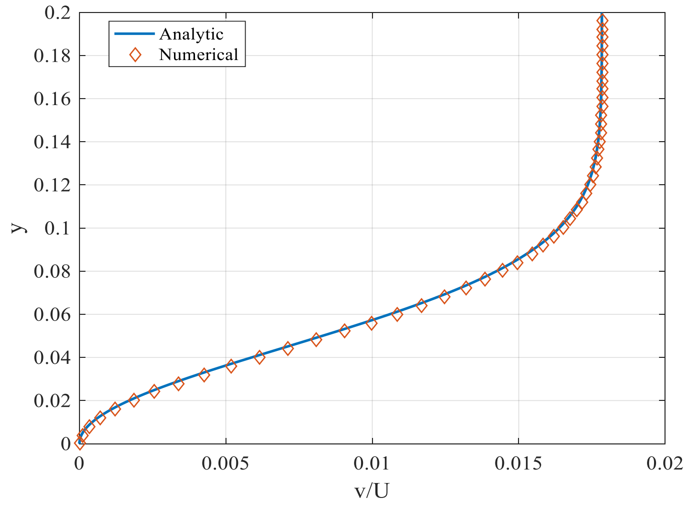


Figure 8.8 The x-momentum velocity v/U with y-axis of analytic and numerical solution of the Oseen-Blasius equation at $x = 1$.

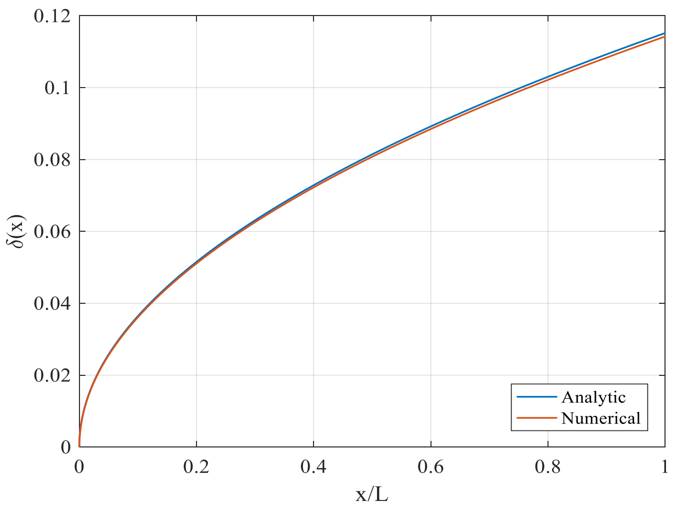


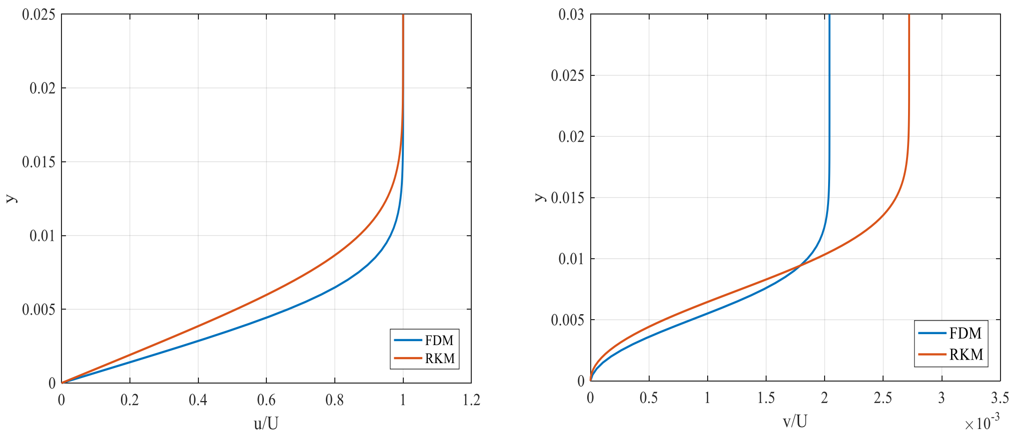
Figure 8.9 The Boundary Layer Thickness δ of analytic and numerical solution of the Oseen-Blasius equation.

8.3 Comparison of Finite Difference Method (FDM) solution and Runge-Kutta Method (RKM) of Blasius equation

In this section the Blasius solution by the Finite Difference method (FDM) has been compared for the Re $=10^5, 10^4$ and 10^3 with classic standard result of Blasius that have been reported previously in the literature review.

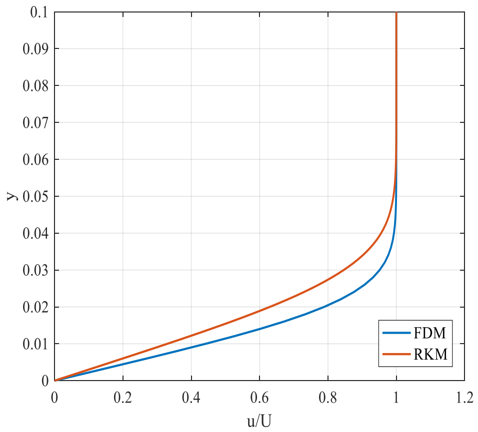
The Figures 8.10-8.8.12 have illustrated that the result of Finite Difference Method is different from Runge-Kutta method (Table 2.1 and Figure 2.1 in chapter 2) .

So unlike for Oseen, this method does not model perfectly the nonlinear partial differential equation.

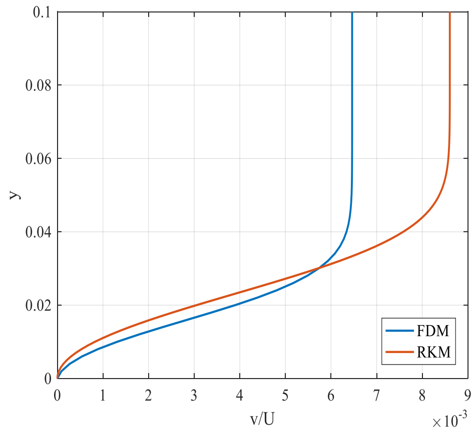


(a) Velocity u with y-axis **(b)** Velocity v with y-axis

Figure 8.10 Velocity of Blasius equation at x=1 at Re$=10^5$ by RKM and FDM.

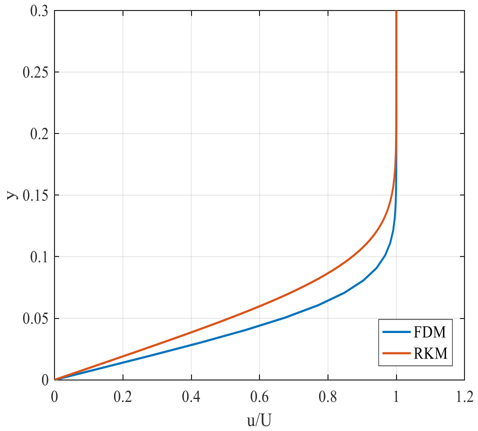


(a) Velocity u with y-axis

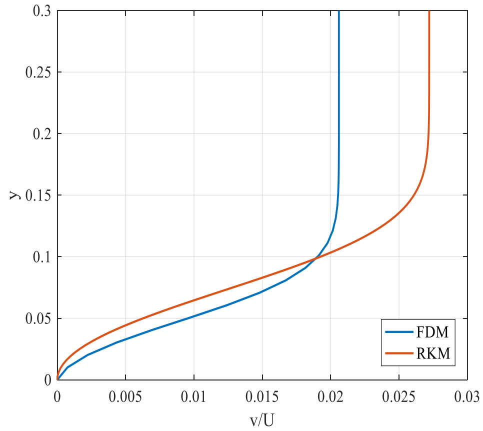


(b) Velocity v with y-axis

Figure 8.11 Velocity of Blasius Equation at x=1 at $Re = 10^4$ by RKM and FDM.



(a) Velocity u with y-axis



(b) Velocity v with y-axis

Figure 8.12 Velocity of Blasius Equation at x=1 at Re=10^3 by RKM and FDM.

8.4 Comparison of Oseen Boundary Layer and Blasius Boundary Layer

This section compares the Blasius and the Oseen-Blasius equation boundary layers. First, by comparing Blasius equation (3.5.16)

$$f'''(\eta) + 2f(\eta)f''(\eta) = 0,$$

with Oseen-Blasius equation (3.6.5)

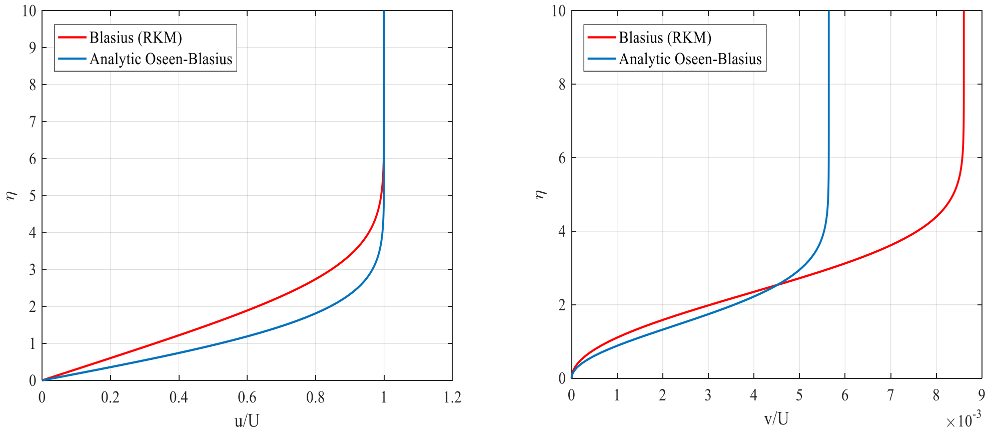
$$f'''(\eta) + 2\eta f''(\eta) = 0,$$

it could be easily noticed that the Blasius equation is approximated by

$$f(\eta) \approx \eta, \tag{8.4.1}$$

to give Oseen-Blasius equation on boundary layer. The numerical study discussed in chapter 6 supports that the x-momentum velocity profile of these two equations match each other in the far-field. Furthermore, this agrees with Sobey, (2000) and he stated that $f(\eta) \to \eta$ as $f(\eta) = \eta \to \infty$ with boundary condition $f(0) = f'(0) = 0$.

It is noted there are significant differences (approx. 30%) in the gradient in (a) and the asymptote in (b) (see Figure 8.13).



(a) Velocity $u$with η-axis. **(b)** Velocity $v$with η-axis.

Figure 8.13 Comparison between analytic solution of Oseen-Blasius equation and Blasius solution by Runge-Kutta method at x=1.

8.5 Comparison of Stokes, Oseen and Blasius Boundary Layer

The comparison will be performed of the solution for three approximation: Stokes boundary solution, Oseen-Blasius solution and Blasius solution.

First, we compare the solution of Stokes Boundary Layer $u = \frac{2}{\sqrt{\pi}}\eta$ with the solution of Oseen-Blasius $u = \operatorname{erf}(\eta)$, it could be noticed that, the *x*-momentum velocity u of Stokes equation in boundary layer matches the Oseen-Blasius solution (Figure 8.14) when η small enough i.e.

$$u(x,y) = \operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\eta, \qquad (\eta \to 0) \tag{8.5.1}$$

On the other hand, according to Sobey (2000, eq. 2.30), the function f of Blasius problem could be approximated by

$$f(s) \sim \frac{1}{2}\frac{s^2}{\sqrt{(\alpha_0)^3}}, \qquad (s \to 0) \tag{8.5.2}$$

where $s(x,y) = \frac{\sqrt{Re}}{\sqrt{2}}\frac{y}{\sqrt{x}}$.

Differentiating (8.5.2) with respect to s gives

$$u = f'(s) = \frac{s}{\sqrt{(\alpha_0)^3}}. \tag{8.5.3}$$

Since $\eta(x,y) = \frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}$ then $s = \sqrt{2}\,\eta$, substituting into (8.5.3) to obtain

$$u(x,y) = \frac{\sqrt{2}}{\sqrt{(\alpha_0)^3}}\,\eta(x,y).$$

From Sobey (2000, p.31, eq. 2.45), the solution of Blasius equation is obtained from Runge-Kutta Scheme and gives $\alpha_0 = 1.65519..$, then

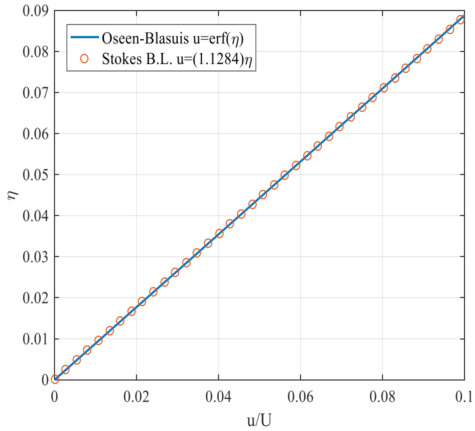
$$u(x,y) = (0.664115\,)\,\eta(x,y). \tag{8.5.4}$$

Therefore, in near field, the Blasius solution matches Stokes on boundary layer solution, (see Figure 8.15).

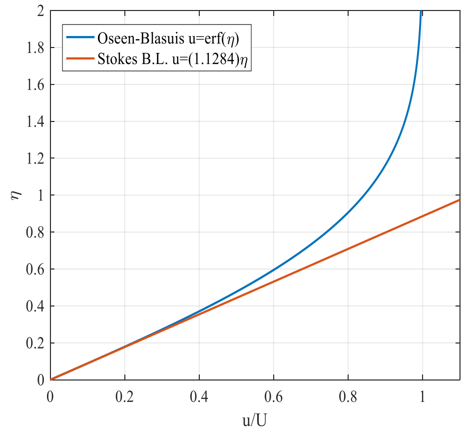
From above, we can use the constant of Oseen-Blasius $f''(0) = \frac{2}{\sqrt{\pi}}$ as a constant of general solution of Stokes Boundary Layer (7.11.6)

$$u(x, y) = \mathrm{A}\,\eta(x, y).$$

Whereas, the Blasius solution matches Stokes in near field when the Blasius constant (0.664115) used as a constant A in the general solution of Stokes Boundary Layer (7.11.6) (see Figure 8.16).



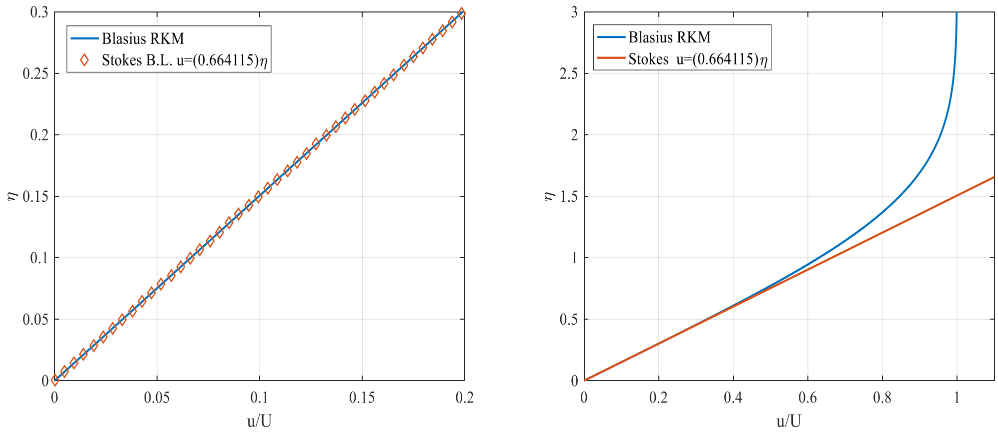
(a)



(b)

Figure 8.14 Comparison between Stokes Boundary Layer equations and Oseen-Blasius of Velocity u at x=1 and $Re = 10^3$ and $A = \frac{2}{\sqrt{\pi}} = 1.1284$

(a) Velocity u with η-axis in the small region.
(b) Velocity u with η-axis.



(a) **(b)**

Figure 8.15 Comparison Velocity *u* between Stokes near-field with *A*=0.664115 solution and Blasius solution by Runge-Kutta method at *x*=1 and *Re* = 10^3.

(a) Velocity u with η-axis in the small region.

(a) Velocity u with η-axis.

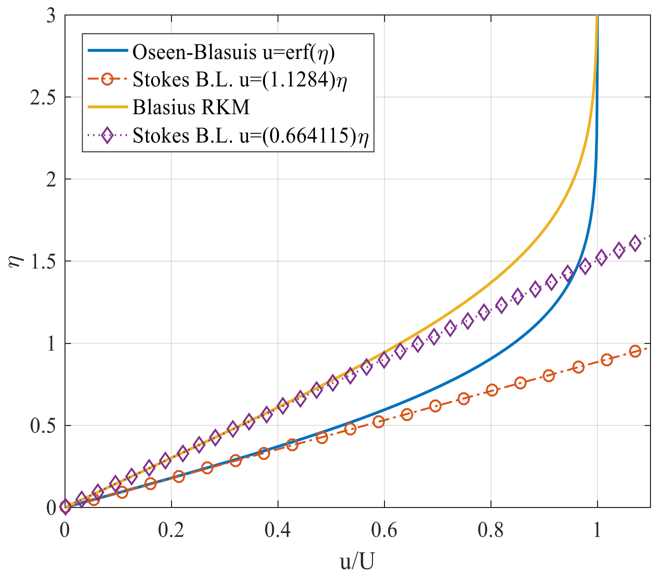


Figure 8.16 The profile velocity u solution of Oseen-Blasius, Blasius solution by Runge-Kutta method, Stokes with A=$2/\sqrt{\pi}$ and A=0.664115.

Chapter 9 A matched Oseen–Stokes Boundary Layer

This chapter study considers a matched Oseen–Stokes Boundary Layer by assuming a far-boundary layer Oseen flow matched to a near-boundary layer Stokes then show this solution is a good agreement with the Blasius solution.

9.1 The equations matched

Match Oseen-Blasius for boundary solution with Stokes near field in boundary layer. First, assume that

$$\eta - \eta_0 = \left(\frac{\sqrt{Re}}{2}\frac{y}{\sqrt{x}}\right), \tag{9.1.1}$$

where η_0 is the value to resize and make Oseen-Blasius solution $u(x,y) = \mathrm{erf}(\eta - \eta_0)$ matches Stokes solution.

By differentiating η with respect to *x* and y gives

$$\frac{\partial \eta}{\partial x} = -\frac{(\eta - \eta_0)}{2x}, \qquad \frac{\partial \eta}{\partial y} = \frac{\sqrt{Re}}{2}\frac{1}{\sqrt{x}},$$

respectively. By using the stream function $\psi(x,y)$ from the equation (7.2.9), the Oseen-Blasius partial differential equation is

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{1}{Re}\frac{\partial^3 \psi}{\partial y^3}.$$

Consider

$$\psi = \frac{2\sqrt{x}}{\sqrt{Re}} f_o(\eta). \tag{9.1.2}$$

Where o refer to Oseen-Blasius problem. In similar way in the Blasius derivation (Chapter 3, Section 3.3), we obtain

$$\frac{\partial \psi}{\partial y} = \frac{2\sqrt{x}}{\sqrt{Re}} f_o'(\eta)\frac{\partial \eta}{\partial y} = f_o'(\eta), \tag{9.1.3}$$

$$\frac{\partial^2 \psi}{\partial y^2} = f_o''(\eta)\frac{\partial \eta}{\partial y} = f_o''(\eta)\frac{\sqrt{Re}}{2}\frac{1}{\sqrt{x}},$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{\sqrt{Re}}{2}\frac{1}{\sqrt{x}} f_o'''(\eta)\frac{\partial \eta}{\partial y} = f_o'''(\eta)\frac{Re}{4x}. \tag{9.1.4}$$

Also

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = f_o''(\eta)\frac{\partial \eta}{\partial x} = -\frac{(\eta-\eta_0)}{2x} f_o''(\eta). \tag{9.1.5}$$

The governing the equations of (9.1.4) and (9.1.5), reduce to the Oseen-Blasius equation to

$$-\frac{(\eta-\eta_0)}{2x} f_o''(\eta) = \frac{1}{Re} f_o'''(\eta)\frac{Re}{4x}.$$

Simplifying, gives

$$f_o'''(\eta) + 2(\eta-\eta_0) f_o''(\eta) = 0. \tag{9.1.6}$$

So allowing Oseen to be far-field, the far-field Oseen solution is

$$u = f_o'(\eta-\eta_0) = c\,\mathrm{erf}(\eta-\eta_0) + b. \tag{9.1.7}$$

Applying boundary condition $u = 1,\ \eta \to \infty$, we have,

$$b = 1 - c. \tag{9.1.8}$$

Integrating (9.1.7), we obtain

$$f_o(\eta) = c\int_0^q \mathrm{erf}(q')\,dq' + (1-c)\eta + a, \tag{9.1.9}$$

where $q = \eta - \eta_0$. Also differentiating (9.1.7), gives

$$f_o''(\eta) = c\frac{2}{\sqrt{\pi}} e^{-(\eta-\eta_0)^2}. \tag{9.1.10}$$

Similarly, from (7.11.6) the general solution Stokes Boundary Layer is

$$u(x, y) = f_s'(\eta) = A\eta.$$

(where s refer to Stokes problem). So we have

$$f_s(\eta) = \frac{A\eta^2}{2} + C.$$

Setting the stream function ψ be zero on $\eta = 0$, from the equation (9.1.2), we obtain $f(0) = 0$ therefore $C = 0$ and

$$f_s(\eta) = \frac{A\eta^2}{2}. \tag{9.1.11}$$

So it could be summarized what obtained above by the following:

- Oseen-Blasius on Boundary layer

$$f_o(\eta) = c\int_0^q \operatorname{erf}(q')\, dq' + (1 - c)\eta + a, \tag{9.1.12}$$

$$f_o'(\eta) = c\, erf(\eta - \eta_0) + (1 - c), \tag{9.1.13}$$

$$f_o''(\eta) = c\frac{2}{\sqrt{\pi}} e^{-(\eta-\eta_0)^2}. \tag{9.1.14}$$

- Near-field Stokes Boundary Layer

$$f_s(\eta) \;= \frac{A}{2}\eta^2, \tag{9.1.15}$$

$$f_s'(\eta) \;= A\eta, \tag{9.1.16}$$

$$f_s''(\eta) = A. \tag{9.1.17}$$

Let the Oseen-Blasius match near-field Stokes boundary layer on

$$\eta = \eta_{match}\,. \tag{9.1.18}$$

So we have five unknown values $\eta_{match}, \eta_0, a, c$ and A. For A we use the Blasius constant from equation (8.5.4), so

$$A = 0.664115. \tag{9.1.19}$$

The governing the equations of (9.1.13), (9.1.16), (9.1.18) and (9.1.20) can be written the Oseen-Stokes matched solution as

$$u(x,y) = f_m'(\eta) = \begin{cases} 0.664115\eta\,, & \eta \le \eta_{match} \\ c\,\mathrm{erf}(\eta - \eta_0) + (1-c), & \eta \ge \eta_{match} \end{cases} \tag{9.1.20}$$

where m refers to Oseen-Stokes matched problem.

Now, from (9.1.14), (9.1.17), (9.1.18) and (9.1.19),

$$c\frac{2}{\sqrt{\pi}}e^{-(\eta-\eta_0)^2} \approx 0.664115. \tag{9.1.21}$$

Add and remove $\left[c(\eta-\eta_0) - \frac{c}{\sqrt{\pi}}\right]$ in (9.1.12) for the far field,

$$f_o(\eta) = c\left(\int_0^{\eta-\eta_0} \mathrm{erf}(q')\,dq' - (\eta-\eta_0) + \frac{1}{\sqrt{\pi}}\right) + c(\eta-\eta_0) - \frac{c}{\sqrt{\pi}} + (1-c)\eta + a$$

Since

$$\int_0^{\eta-\eta_0} \mathrm{erf}(q')\,dq' \approx (\eta-\eta_0) - \frac{1}{\sqrt{\pi}}, \qquad \text{as } \eta \to \infty,$$

therefore

$$f_o(\eta) \approx \eta - c\eta_0 - \frac{c}{\sqrt{\pi}} + a. \qquad\qquad \text{as } \eta \to \infty,$$

inserting into (9.1.2) to obtain

$$\psi = \frac{2\sqrt{x}}{\sqrt{Re}}\left(\eta - c\eta_0 - \frac{c}{\sqrt{\pi}} + a\right). \tag{9.1.22}$$

From Sobey (2000, eq. 2.10, eq. 2.24, eq. 2.25, and eq.2.44), the stream function ψ can be approximated as

$$\begin{aligned} \psi &\sim \frac{\sqrt{2x}}{\sqrt{Re}}(s-\beta_0), \qquad \text{as } s \to \infty, \\ &= \frac{2\sqrt{x}}{\sqrt{Re}}\left(\eta - \frac{\beta_0}{\sqrt{2}}\right), \end{aligned} \tag{9.1.23}$$

where $\eta = s/\sqrt{2}$, and the constant value of potential out flow of Blasius is β_0 =1.21649.. (Sobey, 2000, p.31, eq. 2.46). By comparing (9.1.22) with (9.1.23), one can notice

$$\frac{\beta_0}{\sqrt{2}} = 0.860188 = c\eta_0 + \frac{c}{\sqrt{\pi}} - a. \tag{9.1.24}$$

9.2 The iteration scheme

An iteration scheme may be used to evaluate η_{match}, η_0, a and c. Set

$$q = \eta_{match} - \eta_0. \tag{9.2.1}$$

Then for determining c the equation (9.1.21) could be used as

$$c_i = (0.664115)\frac{\sqrt{\pi}}{2}e^{q_i}. \tag{9.2.2}$$

For determining η_{match}^i the equations (9.1.13) and (9.1.16) give

$$\eta_{match}^i = \frac{1}{0.664115}\left[c_i \operatorname{erf}(q_i) + 1 - c_i\ \right]. \tag{9.2.3}$$

Finally, for determining a_i the equations (9.1.12) and (9.1.15) gives

$$a_i = \frac{0.664115}{2}\left(\eta_{match}^i\ \right)^2 + c_i \int \operatorname{erf}(q_i)\, d\eta - (1 - c_i)\eta_{match}^i. \tag{9.2.4}$$

On the other hand, the equation (9.1.24) could be used to test the accuracy from

$$a_{test}^i = c_i\left(\eta_{match}^i - q_i\right) + \frac{c_i}{\sqrt{\pi}} - 0.860188. \tag{9.2.5}$$

This gives error

$$e_i = a_i - a_{test}^i. \tag{9.2.6}$$

Assume linear relation between e and q as

$$e = mq + n,$$

where m and n integer numbers, when $e = 0$ gives $= -n/m$. Also,

$$\left.\begin{aligned} e_i &= mq_i + n, \\ e_{i+1} &= mq_{i+1} + n, \end{aligned}\right\} \tag{9.2.7}$$

and

$$\left.\begin{aligned} e_i q_{i+1} &= mq_i q_{i+1} + nq_{i+1}, \\ e_{i+1} q_i &= mq_{i+1} q_i + nq_i\ . \end{aligned}\right\} \tag{9.2.8}$$

Now, from (9.2.7) and (9.2.8) give

$$m = \frac{e_i - e_{i+1}}{q_i - q_{i+1}}, \qquad n = \frac{e_i q_{i+1} - e_{i+1} q_i}{q_{i+1} - q_i},$$

respectively. Then for new guess of q_{i+2}

$$q_{i+2} = \frac{-n}{m} = \frac{(e_i q_{i+1}) - (e_{i+1} q_i)}{e_i - e_{i+1}}. \tag{9.2.9}$$

We can stop at $e_i/a_i < 10^{-16}$, and the start points, choose $q_0 = 0.1$ and $q_1 = 0.2$. This gives the following values after converging at 10 iteration steps:

$$\eta_{match} = 1.21129, \qquad c = 4.63560, \qquad \eta_0 = -0.225315, \qquad a = 0.710699.$$

Therefore, the Oseen-Stokes matched solution (9.1.20) may be rewritten as

$$u(x,y) = \begin{cases} 0.664115\eta\,, & \eta \le 1.21129, \\ (4.6356)\,\mathrm{erf}[\eta + 0.225315] - 3.6356, & \eta \ge 1.21129. \end{cases} \tag{9.2.10}$$

Figure 9.1 compares the boundary layer profile for Blasius, Oseen-Stokes matched, and it seen that Oseen-Stokes matched is a close approximation to Blasius.

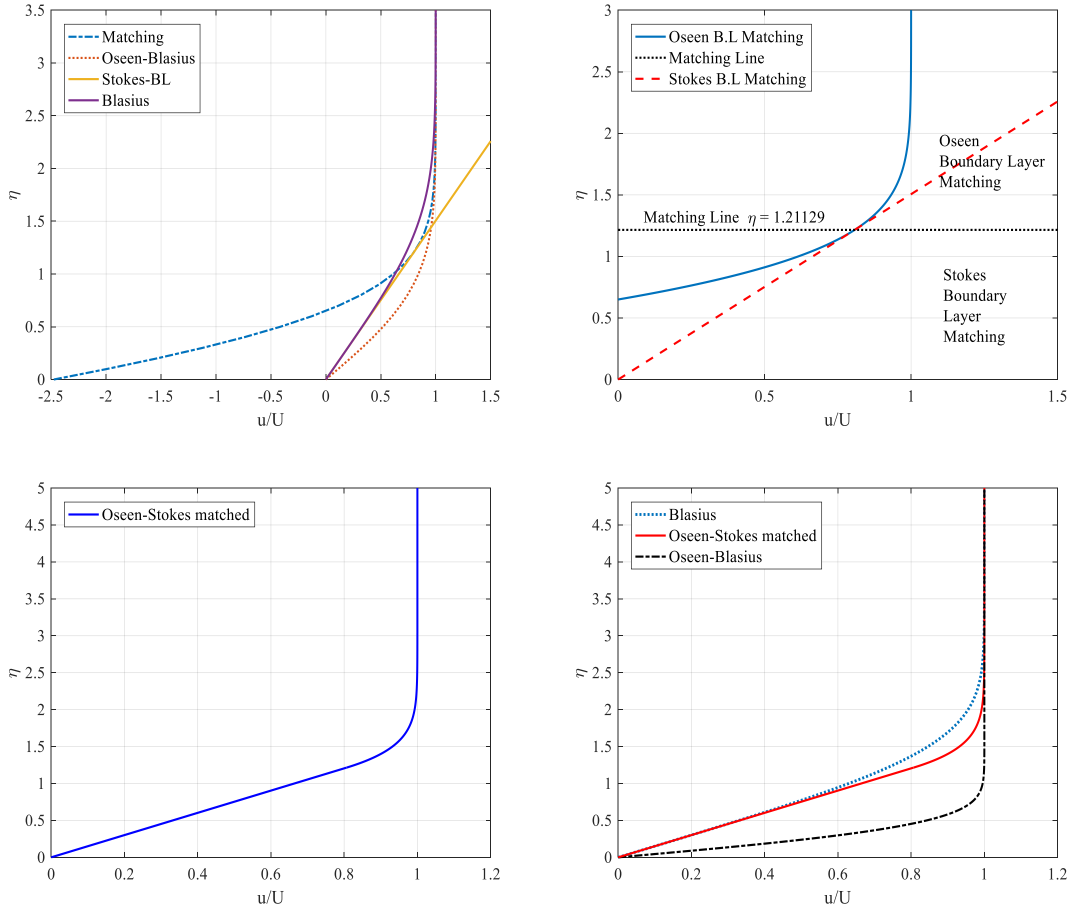


Figure 9.1 Oseen–Stokes matched solution

Chapter 10 Conclusion and Future Work

10.1 Conclusion

New research: To conclude, the new research presented in this thesis has:

Analytical:

- Given a complete description of the Wiener-Hopf technique in Oseen flow.
- Used Imai's approximation to show the equivalent of this description with Burgers' approximation (Burgers, 1930) and Kusukawa's approximation (Kusukawa, Suwa et al. 2014).
- Introduced an Oseen-Stokes matched boundary layer that agrees closely with Blasius.

Numerical:

- Introduced a finite difference scheme that agrees with analytical results for the Oseen boundary layer.

The thesis is on an investigation of the Oseen partial differential equations for the problem of laminar boundary layer flow for the steady two-dimensional case of an incompressible, viscous fluid with the boundary conditions that the velocity on the surface is zero and outside the boundary layer is the free stream velocity. In particular, focus is on flow past a semi-infinite flat plate.

First of all, the literature review was investigated, it shows that the velocity satisfying the Oseen partial differential equation (Oseen 1927) for the flow past a semi-infinite flat plate is equivalent to vector integral equations of the drag Oseenlet with Oseen strength function which satisfies the boundary condition (Olmstead 1965).

According to Noble (1958) and Polyanin and Manzhirov (2012) the integral equation is the Wiener–Hopf Equation of the first kind. Gautesen (1971b) derived the inverse of these integral equations and obtained the solution in the integral form by using the technique developed in Noble (1958, Ch. VI) in a manner similar to (Olmstead 1966) and determined the Oseen strength function σ. Also the study of Bhattacharya (1975) gives the same solution. However, many steps in Gautesen and Bhattacharya are missing and unexplained. In this thesis, all these steps have shown and given in full

In contrast, the study of Burgers, (1930) obtained an approximate solution of the problem where the drag Oseenlet has been simplified to give Abel's integral equation that was solved to obtain the strength or force function then the solution was derived and given in terms of the Error function.

In this thesis, for the first time, the relation between Burgers' and Gautesen's result are shown. We present in full details the derivation of the integral representation of Oseen flow past a semi-infinite flat plate found in Gautesen (1971) and in Bhattacharya (1975) and compare between them. In addition, we present the general solution of the integral equation of the Wiener-Hopf problem. Following the study of Noble and Peters (1961), determine this integral equation to reach the exact solution which agrees with the studies of Bhattacharya (1975), A. K. Gautesen (1971b), Lewis & Carrier (1949) and Olmstead & Gautesen (1976).

In this thesis, more details have been given, many complex integrals are calculated and explanations are introduced not present in the original studies about Wiener-Hopf technique to solve the Oseen integral representation. The main difference between what is discussed in this study and with Gautesen (Gautesen 1971) is we derive the solution for any kernel, this is different from the study of Gautesen (Gautesen 1971) who derived the solution when the kernel is the drag Oseenlet. Moreover, the Gautesen study did not mention how to calculate the Fourier transform of the kernel that is an alternative form of the drag Oseenlet function. So first, we proved this alternative function is equivalent to the drag Oseenlet. Then we compute the Fourier transform.

For the Blasius equation, there are numerous studies. This equation is a non-linear ordinary differential equation with boundary conditions. It is solved via numerical methods such as Runge-Kutta, Differential Transformation, Shooting and Finite Difference (Lien-Tsai and Cha'o-Kuang 1998, Cortell 2005, Puttkammer 2013), and the results are in agreement with the Howarth study (Howarth 1938) which is pioneering work for the flat-plate flow. In the same manner, there is an analytic study to obtain the solutions using an Iteration Method (Kusukawa, Suwa et al. 2014), and again the results are in good agreement the Howarth. It shows that the solution of the Oseen-Blasius equation in boundary layer is the first approximation solution of Blasius equation (Kusukawa, Suwa et al. 2014).

The numerical study is performed for the problem of the 2-D steady flow past a semi-infinite flat plate with various Reynolds Numbers Re =10^5,10^4, 10^3 and 10^2, where the Boundary Layer Theory and Oseen approximation is applied on the dimensionless Naiver-Stokes equations to obtain different equations: Boundary Layer equation, Oseen Boundary Layer equation, Blasius equation and Oseen-Blasius equation. The Finite Difference Method has been used with uniform grid to solve these conservation equations with boundary conditions that the velocity is zero on the flat plate and the free stream velocity outside boundary layer in different grid sizes according to Reynolds number.

Moreover, an analytic study is presented to consider the relationship between Oseen and Blasius approximation in the boundary layer. The Oseen strength function, is found by the Wiener-Hopf technique, and is employed to obtain the solution by utilizing the Imai approximation of the drag Oseenlet (Imai 1951) in the boundary layers (wake). This solution gives Burgers solution of Oseen equation (Burgers 1930), and also this solution is an agreement with the first approximation of Blasius problem according to Kusukawa's solution (Kusukawa et al., 2014). Furthermore, the Oseen-Blasius equation is derived and solved analytically in two ways, by ordinary and partial differential equation, and the same solution is obtained.

Additionally, for the potential flow, a Thin Body Theory is applied which is checked for the Laplacian Green function. The solution agrees with the Sobey outer flow solution (Sobey, 2000, p. 32, eq. 2.54).

Furthermore, the behaviour of the Stokes flow near field on the boundary layer is studied and it is found that the x-momentum velocity u matches the Oseen-Blasius boundary layer solution when the boundary layer variable η is small enough, and so the Stokes boundary layer can model the near field.

Then, several comparisons have been performed for numerical and analytical results. First, the comparisons by the Finite Difference method (FDM) of velocities $u, v,$ are made for Boundary Layer equation, Oseen Boundary Layer, Blasius and Oseen-Blasius equation for different Reynolds Number in rectangular domain. The numerical studies show that the Oseen-Blasius gives the same results as the Oseen Boundary layer equation. Therefore, it indicates that the pressure variation does not play a main role in the flow over a semi-infinite flat plate. Furthermore, it is concluded that the simplest way to formulate the Oseen problem is the Oseen-Blasius equation, where the study of Kusukawa (Kusukawa, Suwa et al. 2014) proves that the Oseen-Blasius solution is the first approximation to Blasius solution. Next, the results by Finite Difference Method of Blasius Boundary Layer is compared with classical result by Runge-Kutta Method. It shows the results are different, and so, the Finite Difference Method solution fails to give a good approximation for the Navier-Stokes equations allowing it gives excellent agreement for the Oseen equations.

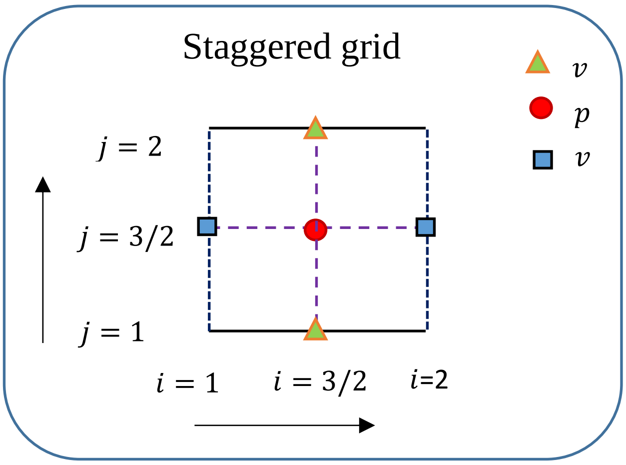
These results also show that the Oseen and Blasius models give significant. (around 30 %) differences in the streamwise velocity gradient giving the stream and the drag, and in the out flow/displacement due to the boundary layer. To overcome this, a near-boundary layer Stokes flow is matched to a far-boundary layer Oseen flow that enforces these equations to be the same instead of different, in this way, an accurate representation of boundary layer is obtained by analytic functions, this is useful in enabling mathematical analysis of the boundary layer, for example, stability analysis.

10.2 Future Work

In this section, we will illustrate the future work based on what has been discussed in this study of the flow past a semi-infinite flat plate. There are some suggested studies formulated as future works, on numerical and analytical studies to complete the framework in the thesis. These suggested studies are as follows:

I. Enhance the numerical results of Blasius problem by Finite Difference Method to achieve better agreement. This could be achieved through an iterative scheme or introduce a matrix solver.

II. Perform a numerical study to find the solution of the approximate equations of Navier-Stokes equations past a semi-infinite flat plate using a staggered grid

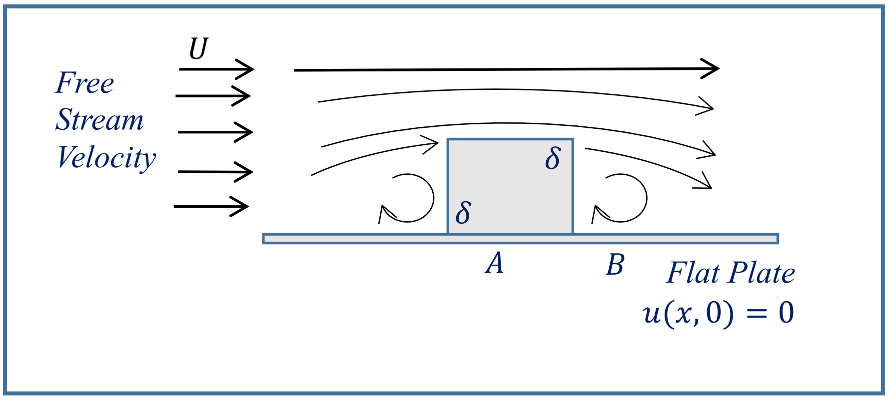


There are two suggested methods: Marker and Cell method (Hoffmann and Chiang 2000) , Projection method (Biringen and Chow 2011) .

III. Study Bhattacharya (1975) procedure to solve Oseen Integral representation. He use Reiman Hilbert boundary value problem. Then compares with wiener-Hopf techniuqe.

IV. Determine the second approximation in the Oseen-Stokes matched Boundary Layer formulation by using the method in Kusukawa (2014) and minimize this.

V. Study the problem of the flow over flat plat when there is an object on flat plate as the following



It could use the Finite Difference Method and the same Boundary conditions in Chapter 6 except on the flat plate it could be divided into 5 parts:

1. The horizontal line, in front of the object :

$$u(x,0) = 0 \ , \qquad v(x,0) = 0 \, , \qquad 0 < x < A \, .$$

2. The vertical line, the left side of the object

$$u(A,y) = 0 \, , \qquad v(A,y) = 0, \qquad 0 < y < \delta.$$

3. The horizontal line, on the object:

$$u(x,\delta) = 0 \, , \qquad v(x,\delta) = 0, \qquad A < x < B \, .$$

4. The vertical line, the right side of the object:

$$u(B,y) = 0 \, , \qquad v(B,y) = 0, \qquad 0 < y < \delta.$$

5. The horizontal line, behind the object:

$$u(x,0) = 0 \ , \qquad v(x,0) = 0, \qquad B < x < \infty \, .$$

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