

An Optimization of the Risk Management using Derivatives

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Abstract. *This article aims to provide a process that can be used in financial risk management by resolving problems of minimizing the risk measure (VaR) using derivatives products, bonds and options. This optimization problem was formulated in the hedging situation of a portfolio formed by an active and a put option on this active, respectively a bond and an option on this bond. In the first optimization problem we will obtain the coverage ratio of the optimal price for the exertion of the option which is in fact the relative cost of the option's value. In the second optimization problem we obtained optimal exercise price for a put option which is to support a bond.*

Keywords: option; bond; risk management.

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Introduction

This article aims to provide a process that can be used in financial risk management by resolving problems of minimizing the risk measure (VaR) using derivatives products, bonds and options. This optimization problem was formulated in the hedging situation of a portfolio formed by an active and a put option on this active, respectively, a bond and an option on this bond.

The reasons for managing the risks are not lead by the firm's market risk magnitude, but rather by the magnitude at risk. More precisely, it is the probability and extent of the potential risks which determine speculation, especially in the case of hedging motivated by de costs of external finances and financial difficulties. An instrument used for measuring the risks is the Value-at-Risk. VaR is an estimation of the probability and scale of the loss potential which can be expected in a certain period of time. We will offer an analytical approach to the problem of optimal management of risks in a setting which is based on two key hypothesis. Firstly, the main criteria for risk management is VaR. Secondly, the coverage strategy implies the use of derived financial instruments. The problem is finding a strategy using the options that minimize VaR (given by a maximum of the coverage expenses), by determining an optimal compromise between the options that have the capacity of reducing the VaR level and the initial costs of these options. The analysis is carried out using the Black-Scholes formula and is, thus, better adapted to the problem of covering the exposure to the exchange rates and actions.

An approach to this issue is done and Dong Hyun Ahn in the article "Using Optimal risk management options" by The Journal of Finance, No. 1, 1999. This article presents an analytical approach to optimal risk management based on the assumption that financial institutions want to minimize the Value-at-Risk using options. Here it is shown that the most important factor is the conditional distribution of the underlying exposure, therefore, optimal exercise price is very sensitive to the relative size of the drift.

Considering the definition of VaR, Ahn used as risk measure

$$VaR_{t+\tau}^{\alpha} = S_t e^{r\tau} - S_t e^{\theta(\alpha)},$$

where

$$\theta(\alpha) = \left(\mu - \frac{1}{2} \sigma^2 \right) \tau + c(\alpha) \sigma \sqrt{\tau},$$

And $c(\cdot)$ is the cut-off point of cumulative distribution of standard normal.

Options situation

First, we will consider a financial active that checks a classic equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

where μ is the trend, σ is the active's volatility, and z_t is the brownian motion.

For a cover operation we will use a put option defined like this $P_t = P(S_t, K, r, \tau, \sigma)$, where τ is the contract period, K is the exercise price, and r interest rate.

Obviously, the option price is given by Black-Scholes model

$$P_t = Ke^{-r\tau}N(-d_2) - S_tN(-d_1)$$

$$d_1 = \frac{\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau},$$

and $N(d) = Prob(x \leq d)$ is the distribution function of the standardized normal law.

A way of using put options is by taking long positions $h_i, i = 1, \dots, n$ with n options whose exercise price $K_i, i=1, \dots, n$ so that the total price must be lower or equal with fixed C ,

$$\sum_{i=1}^n h_i P_{it} \leq C.$$

Additionally we will put the total overdraft condition $\sum_{i=1}^n h_i < 1$.

Considering that the exposure must be observed for the next τ periods, it will be necessary the measure risk characterization, VaR. We will define $VaR_{t+\tau}^\alpha$ as loss of $\alpha\%$ of a relative monetary unit to an institution exposure (financial) to invest at t moment in an active risk.

This definition must be translated into a formula, so we will consider $Y_t = \ln S_t$ and will apply lema Ito

$$dY_t = \left[\frac{1}{S_t} \mu S_t + \frac{1}{2} \left(-\frac{1}{S_t^2} \right) \sigma^2 S_t^2 \right] dt + \frac{1}{S_t} \sigma S_t dz_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz_t$$

That is

$$\ln S_t \sim N(m, \sigma^2 t), m = \ln S_i + \left(\mu - \frac{1}{2} \sigma^2 \right) \tau,$$

And we can state that, for a position without cover, we have

$$VaR_{t+\tau}^\alpha = S_t e^{r\tau} - S_t e^{\theta(\alpha)},$$

where

$$\theta(\alpha) = \left(\mu - \frac{1}{2} \sigma^2 \right) \tau + c(\alpha) \sigma \sqrt{\tau},$$

while $c(\cdot)$ is the separation point of the two regions of the cumulative normal standard distribution, *cut-off point*.

We see that the second term of the VaR formula can be interpreted as the asset's expectation of the active will return at the $\alpha\%$ level.

In order to make the calculus easier, we shall suppose that the put option will be "in money", so we obtain obtain the future value of such assets will be covered,

$$V_{t+\tau} = (1-h)S_{t+\tau} + hK - hP_t e^{r\tau}$$

$$VaR_{t+\tau}^\alpha = S_t e^{r\tau} - \left[(1 - \sum_{i=1}^n h_i) S_t e^{\theta(\alpha)} + \sum_{i=1}^n h_i K_i - (\sum_{i=1}^n h_i P_{it}) e^{r\tau} \right].$$

We now can formulate an optimization problem

$$\min_{h_i, K_i} VaR_{t+\tau}^\alpha$$

$$\sum_{i=1}^n h_i P_{it} \leq C,$$

$$\sum_{i=1}^n h_i < 1, h_i \geq 0,$$

This means that we want the minimization of the VaR, using long positions with put options and hedging cost restrictions.

As usual, in order to draw some conclusions that later we can generalize, eventually, we shall consider the above problem as having a sole long position on a put option. We so can re-write the minimum problem

$$\min_{h, K} VaR_{t+\tau}^\alpha = \min_{h, K} \left\{ S_t e^{r\tau} - \left[(1-h) S_t e^{\theta(\alpha)} + hK - hP_t e^{r\tau} \right] \right\}$$

$$C = hP_t, 0 \leq h < 1.$$

If we use the cost covering restriction, we shall obtain

$$\bar{K} = \arg \min_K \left\{ S_t e^{r\tau} - \left[\left(1 - \frac{C}{P_t} \right) S_t e^{\theta(\alpha)} + \frac{C}{P_t} K - C e^{r\tau} \right] \right\}.$$

We can re-write

$$\tilde{K} = \arg \min_K \left\{ S_t e^{r\tau} + C e^{r\tau} - S_t e^{\theta(\alpha)} - C \left[\frac{K - S_t e^{\theta(\alpha)}}{P_t} \right] \right\}$$

which leads us to

$$\tilde{K} = \arg \max_K C \left[\frac{K - S_t e^{\theta(\alpha)}}{P_t} \right] = \arg \max_K \left[\frac{K - S_t e^{\theta(\alpha)}}{P_t} \right]. (*)$$

Is noticeable that $VAR_{t+\tau}^\alpha$ is independent from C's selection and K optimum is determined according to the cash-flow of the active and the cover is adjusted depending of the cover's price. VaR is a linear function in relation with the expenditures with the cover, so each added monetary unit generates a decrease of the same level in VaR.

From the last relation we can deduce that the VaR's minimize is the same thing with the maximize of the difference between exertion's price and the level $\alpha\%$ of overdraft earnings relative at the price of the option put.

Intuitive, we may say that the objective function of the optimization problem can be interpreted as the rate between the cover's price and the cost of this operation. Furthermore, if the option's exercise price will decrease we will cover a larger part of distribution, but this option will become more expensive.

The optimization problem (*) requires a maxim condition

$$\frac{\partial \left(\frac{K - S_t e^{\theta(\alpha)}}{P_t} \right)}{\partial K} = 0$$

That is

$$\frac{P_t - (K - S_t e^{\theta(\alpha)}) \frac{\partial r_t}{\partial K}}{P_t^2} = 0.$$

From here

$$\tilde{K} - S_t e^{\theta(\alpha)} = \frac{P_t}{\frac{\partial P_t}{\partial K}} = \tilde{K} - S_t e^{r\tau} \frac{N(-d_1)}{N(-d_2)}$$

The last relation leads to

$$S_t e^{\theta(\alpha)} = S_t e^{r\tau} \frac{N(-d_1)}{N(-d_2)} (**)$$

Further

$$e^{\theta(\alpha)} = e^{r\tau} \frac{N(-d_1)}{N(-d_2)}$$

$$\theta(\alpha) - rT = \ln \frac{N(-d_1)}{N(-d_2)}$$

We can notice that due to the inequality $\frac{N(-d_1)}{N(-d_2)} < 1$ the solution's existence \tilde{K} is provided solely for $\theta(\alpha) - rT \leq 0$.

More, we obtain

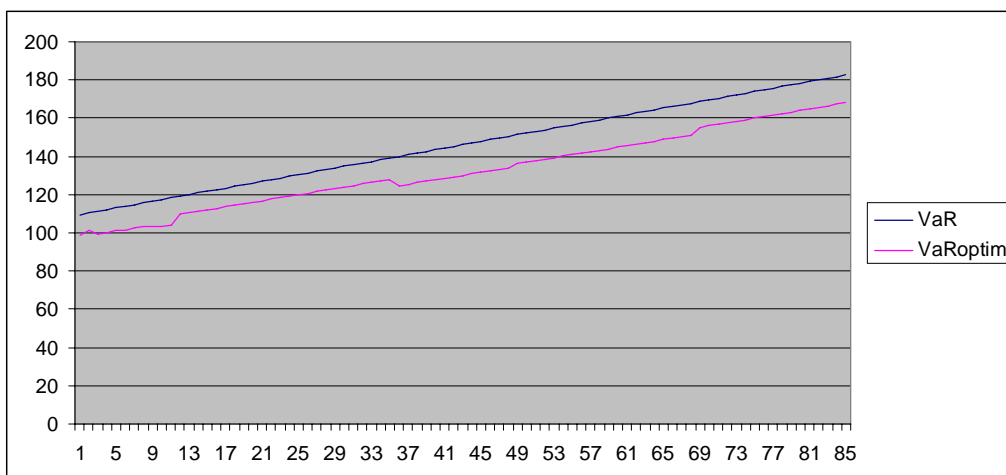
$$\tilde{h} = \frac{C}{P(\tilde{K})}$$

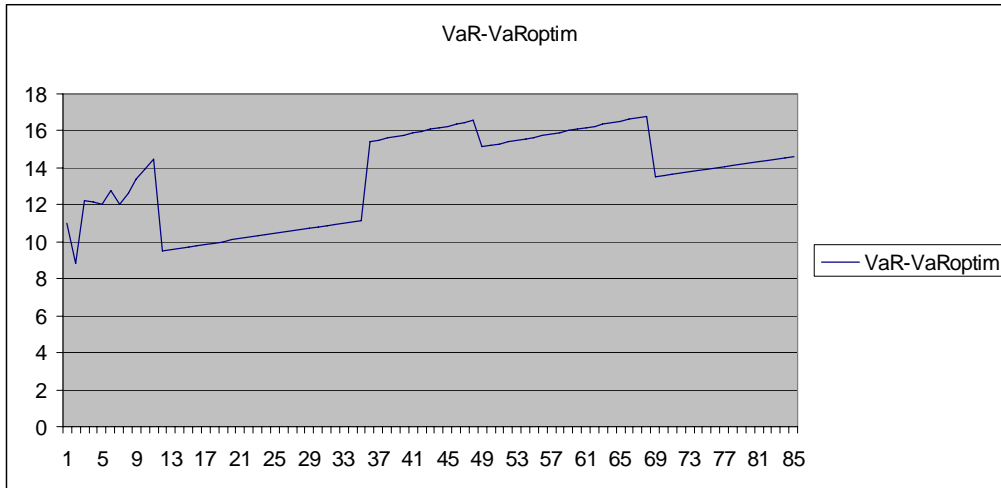
which means that we obtained the coverage ratio of the optimal exertion of the option price which is the relative cost to the option's value.

We observe that if $\tilde{h} < 1$ then solution is correct, if more than $\tilde{h} > 1$, then we consider $\tilde{h} = 1$ and $C = P(X)$.

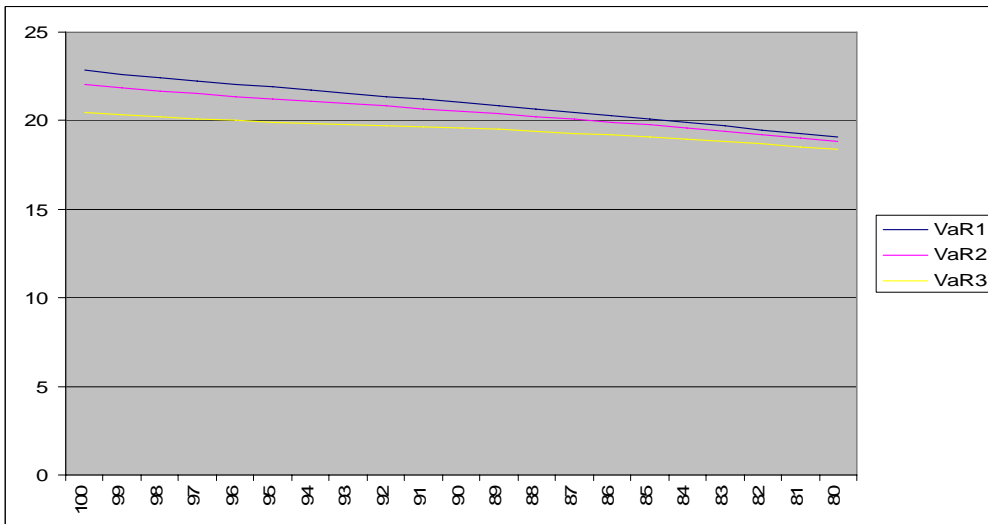
We will exemplify those stated above for a case in which: $S_t = 1000, \mu = 0.1, \sigma = 0.15, r = 0.05, \alpha = 2.5\%$. For these values we obtain $\tilde{K} = 876$ and it can be observed that the value of the option is by 12.4% more out-of-the-money. In case we do not effectuate any kind of risk coverage, the VaR value is 237, whereas with the help of hedging the value is reduced to 211. From what is stated above we can observe that in this case VaR is a linear function in relation to the expenses with the coverage of risk, and thus, for every unit, spent VaR will be reduced with 7.2.

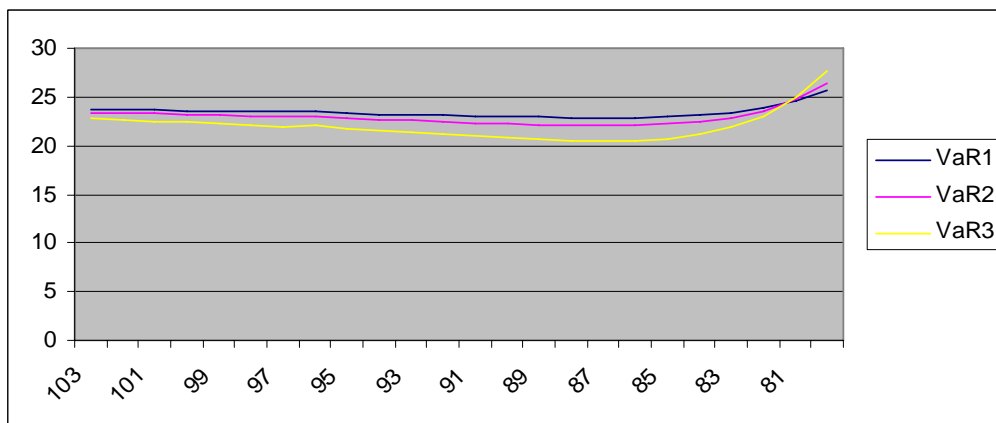
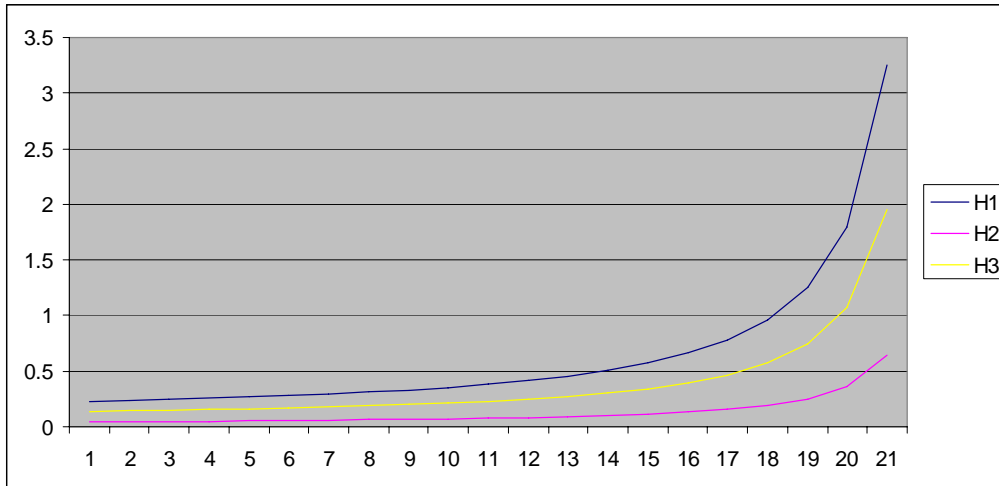
The figure below VaR and VaR optimal variation present after all the above calculated based on the above data rate, h, subunit.





If costs are $C1 = 0.2$ $C2 = 0.3$ $C3 = 0.5$ and for the above data we calculated VaR and compared the three results. The results are given in the following representations.





Bond situation

This time we will proceed almost the same, but we will use bonds.

We consider that we have a moment $t=0$, a zero-coupon bond which we can sell it at T moment.

If the interest rate will increase, the overdraft portfolio can lead to losses, therefore we can decide to do a cover of maximum C level. This cover can be made by buying a put option that is based on a bond, so, in the case of a strong decrease of the bond price, the option put can cover major losses. It remains to establish the choice of exercise price, price that can be chosen after the minimizing of VaR at a cover price of C .

Suppose we have an available bond $P(0, S)$ with the main $N=1$ and maturity at time S and we will cover this bond with a percentage $h(0 < h < 1)$ of a put option $BP(0, T, S, K)$ with the exercise price K at time $T \leq S$.

The bond price is given by

$$P(T, S) = A(T, S)e^{-B(T, S)r(T)},$$

where $r(T)$ is rate with the parameters independent of it, $A(T, S), B(T, S) > 0$.

We will consider, like usually done, the covered portfolio formed by P bond and BP option, and its value at T moment is

$$H_T = \max(hK + (1 - h)P(T, S), P(t, S)).$$

If the option ends the contract „in money”, worst case, the one that interests us, the portfolio value will be

$$H_T = hK + (1 - h)P(T, S).$$

We can express the value of the losses as

$$\begin{aligned} L &= L(r(T)) = P(0, S) + C - ((1 - h)P(T, S) + hK) = \\ &= P(0, S) + C - ((1 - h)A(T, S)e^{-B(T, S)r(T)} + hK) \end{aligned}$$

In case in which the option is “in money”

If we note

$$F_{r(T)}^{-1}(p) = \inf\{x \in \mathbb{R} | F_{r(T)}(x) \geq p\}, p \in [0, 1],$$

where $F_{r(T)}$ is the cumulative distribution of $r(T)$.

With risk measure we can consider as we have pointed in those stated in the first part

$$VaR_{\alpha, T}(L) = \inf\{l \in \mathbb{R} | \Pr(L \geq l) \geq \alpha\} = \inf\{l \in \mathbb{R} | \Pr(L(r(T)) \geq l) \geq \alpha\}.$$

L function is an inverse function and strictly increasing, which leads us to

$$VaR_{\alpha, T}(L) = \inf\{l \in \mathbb{R} | \Pr(r(T) \geq L^{-1}(l)) \geq \alpha\}.$$

Considering dual equality we have

$$\begin{aligned} VaR_{\alpha, T}(L) &= \inf\{l \in \mathbb{R} | \Pr(r(t) \leq L^{-1}(l)) \geq 1 - \alpha\} = VaR_{\alpha, T}(L) \\ &= \inf\{l \in \mathbb{R} | F_{r(T)}(L^{-1}(l)) \geq 1 - \alpha\} = L\left(F_{r(T)}^{-1}(1 - \alpha)\right). \end{aligned}$$

From previous relations we may write

$$VaR_{\alpha,T}(L) = P(0,S) + C - ((1-h)A(T,S)e^{-B(T,S)F_r^{-1}(T)(1-\alpha)} + hK).$$

Like the situation treated in the first part of this article, we will state a minimum problem

$$\min_{h,K} VaR_{\alpha,T}(L)$$

$$C = hBP(0,T,S,K), h \in (0,1).$$

Considering function

$$\mathcal{L}(K, h, \lambda) = VaR_{\alpha,T}(L) - \lambda(C - hBP(0,T,S,K))$$

and we put the Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial K} = -h + h\lambda \frac{\partial BP}{\partial K}(0,T,S,K) = 0$$

$$\frac{\partial \mathcal{L}}{\partial h} = -\left(K - A(T,S)e^{-B(T,S)F_r^{-1}(T)(1-\alpha)}\right)\lambda BP(0,T,S,K) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C - hBP(0,T,S,K) = 0$$

$$0 < h < 1, \lambda > 0.$$

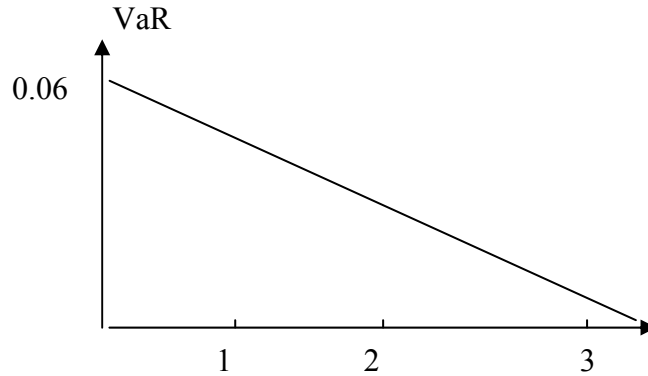
We deduce that

$$BP(0,T,S,K) - \left(K - A(T,S)e^{-B(T,S)F_r^{-1}(T)(1-\alpha)}\right) \frac{\partial BP}{\partial K}(0,T,S,K) = 0.$$

It's noticeable that the optimum exertion price K^* is independent from the coverage cost C , which means that VaR is a h linear function.

$$VaR_{\alpha,T}(L) = P(0,S) - A(T,S)e^{-B(T,S)F_r^{-1}(T)(1-\alpha)} +$$

$$+ h(BP(0,T,S,K^*) + A(T,S)e^{-B(T,S)F_r^{-1}(T)(1-\alpha)} - K^*)$$



As the function is decreasing, from the figure above, it will result that

$$\frac{\partial BP}{\partial X}(0, T, S, K^*) < 1$$

and

$$K^* - A(T, S)e^{-B(T, S)F_{r(t)}^{-1}(1-\alpha)} > BP(0, T, S, K^*).$$

From the last relation we notice that the price of exerting the optimal price K^* is greater than the maximum VaR, meaning

$$A(T, S)e^{-B(T, S)F_{r(t)}^{-1}(1-\alpha)} < K^*.$$

Conclusions

In the article's first part we obtained the coverage ratio of the optimal exercise price of options which is the relative cost at the option's value by an optimization problem of VaR in the situation of a portfolio consisting in a financial asset and a put option.

In the second part of this article we obtained the optimal exertion price of a put option in a portfolio in which is a bond and this option. And for this result it has been created a minimum problem of VaR and the obtained result leads to the idea that VaR linearly depends on the h percent of the bond covered by options.

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