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ABSTRACT

A method is developed to predict the dynamic behavior of a structure from experimental FRF data of the same system subjected to different constraints. In particular it is required that the new structure undergoes more restrained conditions, but any type of ideal constraint, involving either translational or rotational degrees of freedom, can be accounted for. Among several interesting applications, the method can be used to overcome typical experimental drawbacks on rigid tested structures and to estimate untestable FRF terms of constrained systems. Numerical and experimental results are provided to show the consistency of the method and the possible range of applications.

1. Introduction

Several problems exist in testing particular structures under given boundary conditions. Recently Barney, et al [1] considered the problem of designing a support device, capable of separating the rigid body modes of an unrestrained flexible structure from the structure's first flexural modes. In order to avoid the difficulty of testing the structure in free-free conditions, the algorithm developed in Ref. [1] identifies the free-free features of the considered structure from those of a multiply constrained system by means of a direct procedure which uses a force measure at the boundary points.

However, not only free-free structures present critical testing conditions. The measurement of any restrained system is often troublesome, too, and the experimental results differ significantly from the theoretical ones. In fact:

- experiments on rigidly constrained structures (e.g., clamped-clamped beams) produce sometimes unacceptable excitation conditions, as double peaks in impact excitation;
- ideal rigid boundary conditions are very hard to obtain, so that the response of the tested structure is highly affected by the modal behavior of the supporting system;
- the nature of the real constraints are generally very dissimilar from the designed ones, yielding inevitably impredictable results.

In order to avoid the above fixes, the behavior of the actual system can be predicted from experiments performed on the same structure, though subjected to unrestrained or less restrained conditions. This

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procedure is simple when accomplished on theoretical discrete systems, e.g., obtained by finite elements, provided that the mass and stiffness matrices of the unrestrained system are known. When, on the contrary, the less constrained structure is identified from its experimentally determined frequency response function (FRF), other procedures can be followed to determine the new FRF. The method of general constraints [2] provides the FRF of a structure, after introduction of any set of linear constraints, and some applications are being developed [3,4]. In this paper a new procedure is presented, which is in principle more simple than the previous one. The method may have interesting engineering applications.

- It can be used to check the effect of adding constraints on the modal behavior of the system or on the forced response of the structure subjected to external forces.
- It may be helpful to model the actual constraints acting on a tested structure, when the boundary conditions cannot be easily identified, e.g., when dealing with non-ideal constraints.
- It represents an effective way to obtain an optimal constraint location, when different solutions are available for the designer.
- Moreover, it basically represents a predictive method of structural modification that can be employed in developing an optimization procedure without requiring any modal identification on the original data. Modal identification, in fact, can lead to erroneous estimates of structural modifications, as it was shown by several authors in the last decade (see, e.g., Elliot and Mitchell [5] and Braun and Ram [6]).
- Finally, the method can be advantageously applied to estimate the FRF of coupled structures when the related substructure parameters are quantities difficult to determine experimentally.

Any type of ideal constraint, involving either translational or rotational degrees of freedom, can be considered. The method is developed here for any possible increment of constraints. It only requires that the FRF matrix of the original system be measured at the points where further constraints must be applied, along the whole set of degrees of freedom affected by the constraints.

2. Mathematical Formulation

The input-output relation for a linear, time invariant dynamic system can be expressed in the frequency domain as

$$\{\ddot{x}(\omega)\} = [H(\omega)]\{f(\omega)\} \tag{1}$$

Here $\{\ddot{x}(\omega)\}$ includes both linear ($\ddot{x}$) and angular ($\ddot{\theta}$) accelerations and $\{f\}$ includes both forces (F) and moments (M); consequently the elements of the $[H]$ matrix (FRF) involve translational as well as rotational degrees of freedom (DOFs). The experimental evaluation of rotational FRF elements is not very accurate because rotational accelerometers are only recently becoming feasible and there is not sufficient experience with them. Furthermore, lumped moments are not easily applicable. Different solutions have been proposed to compute the rotational terms [7,8,9]. Among them, the use of a finite difference scheme involving translational measurements [8] and modal curve fitting [7] are, up to now, valuable techniques used to compute the translational-rotational ($\ddot{\theta}/F$, $\ddot{x}/M$) and rotational-rotational ($\ddot{\theta}/M$) terms respectively, at least for beam-type structures. Based on these algorithms, these authors have recently developed a technique for predicting the assembled structure behavior, considered the sensitivity of the computed FRF elements to the finite difference spacing and the importance of low and high residuals on the accuracy of results [10].

In order to derive the FRF matrix of a structure subjected to additional constraints from the knowledge of the FRF of a less restrained structure, let us consider a system having $a + c$ DOFs, where c DOFs have to be constrained. A DOF constraint is meant here as a condition of null acceleration. Equation (1), written for the original system, can be partitioned as follows

$$\begin{Bmatrix} \{\ddot{x}_a\} \\ \{\ddot{x}_c\} \end{Bmatrix} = \begin{bmatrix} [H_{aa}] & [H_{ac}] \\ [H_{ca}] & [H_{cc}] \end{bmatrix} \begin{Bmatrix} \{f_a\} \\ \{f_c\} \end{Bmatrix} \tag{2}$$

The constraint condition on the c DOFs implies

$$\{O_c\} = [H_{ca}]\{f_a\} + [H_{cc}]\{f_c\} \tag{3}$$

from which the constraint forces can be determined

$$\{f_c\} = -[H_{cc}]^{-1}[H_{ca}]\{f_a\} \tag{4}$$

By substituting Eq. (4) into Eq. (2)

$$\{\ddot{x}_a\} = \left([H_{aa}] - [H_{ac}][H_{cc}]^{-1}[H_{ca}]\right) \cdot \{f_a\} \tag{5}$$

the new FRF matrix $[H^*_{aa}]$, with constraints on the c DOFs can be calculated, i.e.,

$$[H^*_{aa}] = \left([H_{aa}] - [H_{ac}][H_{cc}]^{-1}[H_{ca}]\right) \tag{6}$$

Therefore, by considering the dynamic effect of the constraints we obtain the new FRF of the more constrained structure. The procedure only involves simple matrix operations, besides a matrix inversion of order equal to the number of the degrees of freedom of the added constrains.

The previous relationship can be put in a more general expression which includes the change due to structural lumped modification, i.e., the addition (or subtraction) of mass, stiffness and damping on some points of the structure. This result could be particularly useful in developing a general optimization technique for vibration control, though the presence of external constraints is exclusively binary in contrast with structural modification, whose amount can be continuously graded.

Let a be the set of DOFs where the new FRF must be computed, b, the set of DOFs amenable to structural modification, and c the set of DOFs where further constraints must be imposed. It is obviously

$$\{b\} \subset \{a\} \tag{7}$$

and

$$\{c\} \cap (\{b\} \cup \{a\}) = \{0\} \tag{8}$$

Let $[B_{bb}]$ be a structural modification matrix (apparent mass, i.e., force over acceleration). The diagonal terms of it are inertial modifications or stiffners and/or dampers connected with a fixed point, while the off-diagonal terms represent stiffners and/or dampers between points of the structure [9]. We can rewrite Eq. (2) subdividing the DOFs according to the above stated groups as follows

$$
\begin{aligned}
\{\ddot{x}_a\} &= [H_{aa}]\{f_a\} + [H_{ab}]\{f_b\} + [H_{ab}][B_{bb}]\{x_b\} + [H_{ac}]\{f_c\} \\
\{\ddot{x}_b\} &= [H_{ba}]\{f_a\} + [H_{bb}]\{f_b\} + [H_{bb}][B_{bb}]\{x_b\} + [H_{bc}]\{f_c\} \\
\{O_c\} &= ([H_{ca}]\{f_a\} + [H_{cb}]\{f_b\} + [H_{cb}][B_{bb}]\{x_b\}) + [H_{cc}]\{f_c\}
\end{aligned} \tag{9}
$$

By deriving $\{f_c\}$ from the last equation of the previous system, the new FRF of the system undergoing further constraints and/or structural modifications can be written, after some algebraic manipulation, as

$$
\begin{aligned}
[H^*] = \begin{bmatrix} [H^*_{aa}] & [H^*_{ab}] \\ [H^*_{ba}] & [H^*_{bb}] \end{bmatrix} &= \begin{bmatrix} [H_{aa}] & [H_{ab}] \\ [H_{ba}] & [H_{bb}] \end{bmatrix} - \begin{bmatrix} [H_{ac}] \\ [H_{bc}] \end{bmatrix} [H_{cc}]^{-1} \left[[H_{ca}][H_{cb}] \right] + \\
&+ \begin{bmatrix} [H_{ab}] - [H_{ac}][H_{cc}]^{-1}[H_{cb}] \\ [H_{bb}] - [H_{bc}][H_{cc}]^{-1}[H_{cb}] \end{bmatrix} [B_{bb}] \times \left([I_{bb}] - [H_{bb}][B_{bb}] + [H_{bc}][H_{cc}]^{-1}[H_{cb}][B_{bb}] \right)^{-1} \\
&\times \left[[H_{ba}] - [H_{bc}][H_{cc}]^{-1}[H_{ca}] \quad [H_{bb}] - [H_{bc}][H_{cc}]^{-1}[H_{cb}] \right]
\end{aligned} \tag{10}
$$

$[H^*]$ is obviously a symmetric matrix. This expression computes the FRF matrix among the whole set of DOFs $a + b$, either amenable of structural modification or not. The FRF terms among constrained DOFs c are obviously meaningless.

The solution of Eq. (10) requires two inversions. The first is of order c, equal to the constrained DOFs, the second of order b, equal to the DOFs introduced by structural modifications.

3. Application of the Method to Optimize Design and Structural Coupling Problems

Two different applications of the method to structural problems are worthy of particular emphasis. The first one concerns the passive control of vibrating structures; the second, the FRF estimation of a structure from the knowledge of the experimental FRF of its break-down components.

A well developed technique for vibration control is structural modification. Structural modification involves three main related problems: prediction, sensitivity and optimization. The predictive approach determines the FRF of a structure once some kind of modification (either concentrated or distributed) is performed on it. The sensitivity approach can be used to estimate the optimal location on which to introduce an established modification. The optimization approach provides a set of optimal modification values and locations in order to obtain an established dynamic behavior. In Ref. [11] a non-linear optimization technique has been developed for lumped modification. The expression derived in Eq. (10), accounting for both lumped modifications and increment of constraints, could be advantageously used for design optimization. In fact, the combined action of these two procedures can provide a more effective vibration and/or acoustic control when suitable weights are introduced into the objective function which specifies the cost of the whole operation.

The second mentioned problem concerns the coupling of structures. When the FRF of a coupled structure must be determined from the knowledge of the experimental FRF of its break-down components, rotational FRF terms must be accounted for. The recent availability of angular accelerometers now makes the estimates of the rotational-translational FRFs more friendly but it is still difficult to measure the rotational-rotational quantities. A successful way to determine the rotational terms, even without using angular accelerometers, is performed through the computation of first and second order derivatives, corresponding to rotational-translational and rotational-rotational elements, respectively, [8]. A finite difference scheme that uses translational FRF data only can be employed to determine the first derivative, while a curve fitting procedure is numerically more efficient to estimate the second derivative [7]. However, in particular situations, e.g., when a structure must be welded to the pinned end of a beam, it is necessary to determine, for coupling purposes, the rotational FRF elements at the support of the beam.

In this case it is not possible to measure translational quantities at the support, and even very close to it, the FRF terms are unreliable because of the low coherence. In fact, close to the supports, the level of the response is usually very low, and the noise, always present in the experiments, makes the signal to noise ratio increasingly poor as the constrain position is approached. To overcome this drawback the problem can be solved in two steps, following the lines previously developed. First the support of the beam is eliminated and the rotational terms at this end are computed through finite difference and curve fitting, using the measured translational data. Then Eq. (6) is used to determine the rotational term at the support, which is necessary to compute the FRF of the whole system. A practical application of this procedure is exposed in the following section.

4. Numerical and Experimental Results

A first set of simulation tests was performed on an aluminum beam in order to check the consistency of the described approach. The beam characteristics were as follows: $E = 6.35 \cdot 10^{10}$ N/m^2 Young's modulus, $\rho = 2700$ kg/m^3 material density, $A = 3.2 \cdot 10^{-4}$ m^2 cross-area, $I = 1.7 \cdot 10^{-9}$ m^4 cross section moment of inertia, $l = 0.7$ m length of the beam.

In these tests, comparisons were made between FRF elements (translational and/or rotational) of the actual constrained beam, computed (*i*) directly from the homogeneous equation of motion, and (*ii*) determined through Eq. (6) which uses FRF data of a less constrained beam. For example, the FRF of a hinged-hinged beam may be determined theoretically or computed from Eq. (6), by employing the FRF of the same system with few constraint conditions, as a free-free or hinged-free beam.

A problem arising in comparing FRF theoretical results, obtained as a sum of contributions of vibrational modes, concerns the number of modes that must be considered. In the following tests the analyzed frequency range (0 - 800 Hz) includes the first 3 or 4 natural frequencies of the examined beams. The number of modes, used to compute the FRF elements theoretically, can affect considerably the estimated FRFs. As it will be seen later, mode truncation causes an overestimate of the natural frequencies, and this effect increases for the higher modes.

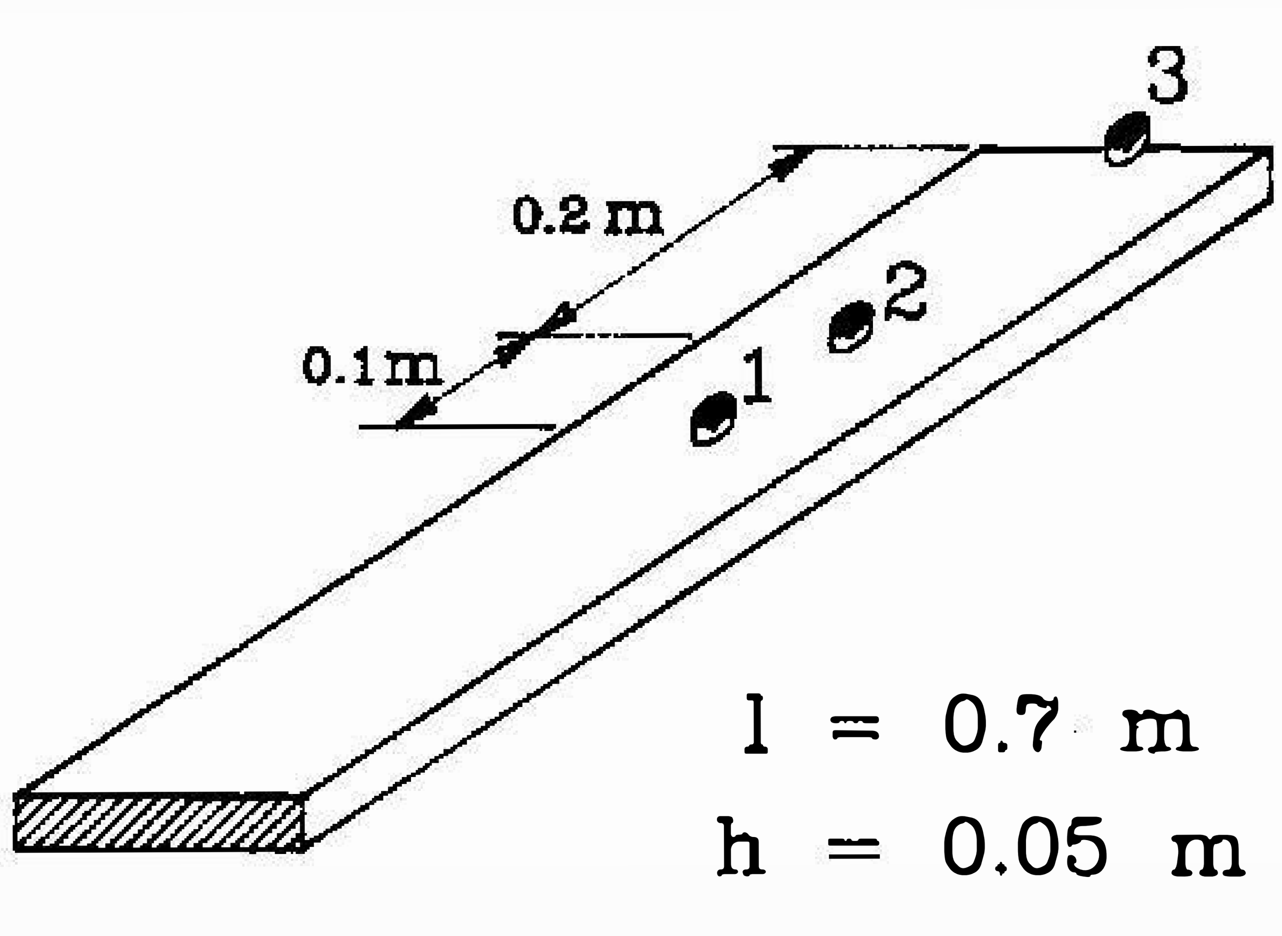

Fig. 1 Measurement point locations in the beam examined

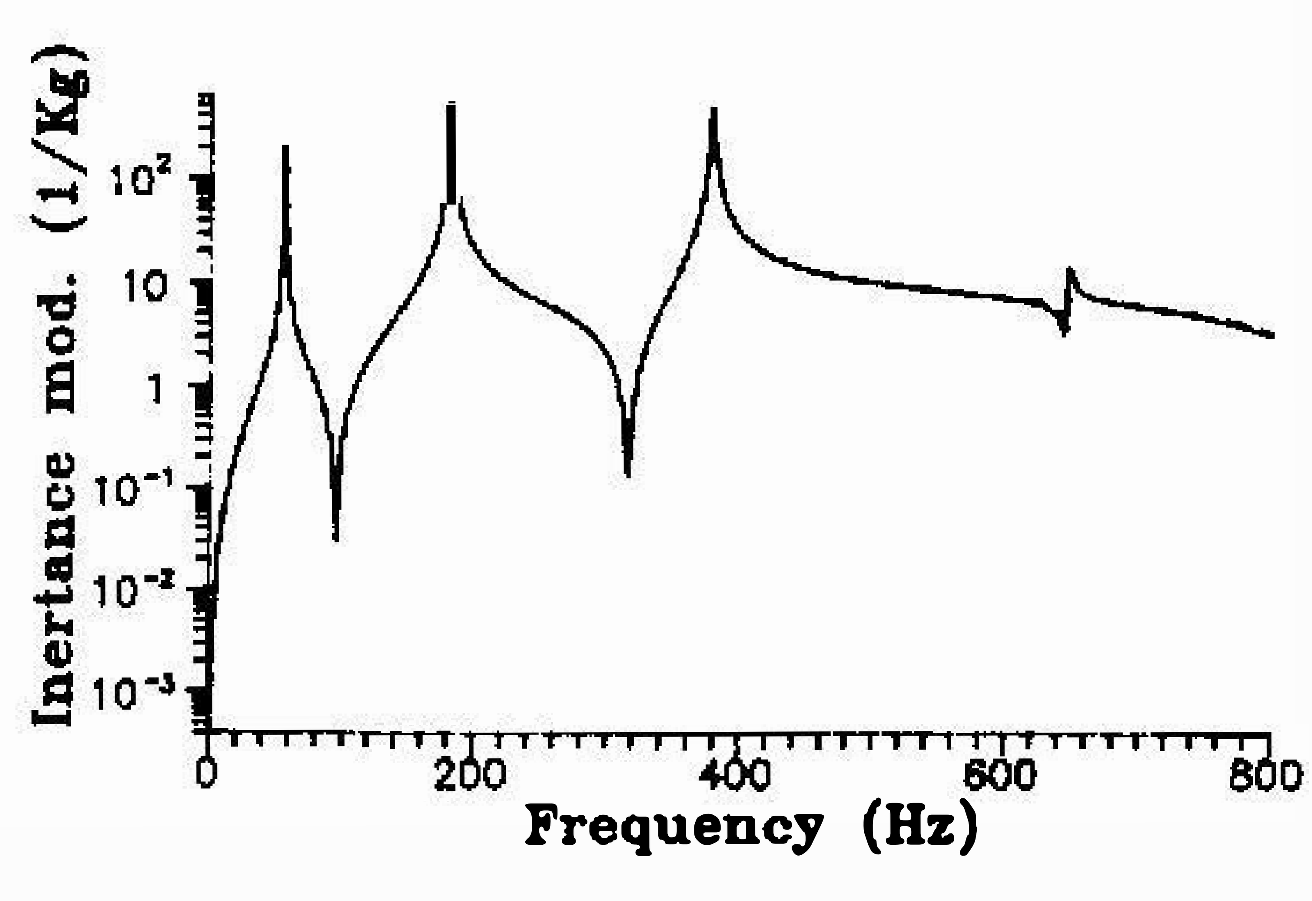

Fig. 2 Clamped-hinged beam: $\ddot{x}_1/F_2$
——— reference theoretical value
-------- derived from clamped-free beam

In Fig. 1 a general beam is sketched together with three typical points used in the following experiments. In Fig. 2 the theoretical translational inertance $\ddot{x}_1/F_2$ of a clamped-hinged beam (1 and 2 are two internal points on the beam) is compared with the same FRF determined by Eq. (6), with data derived from a clamped-free beam (1 DOF restrained) computed with 16 modes. Figure 3 compares the $\ddot{x}_1/F_2$ theoretical translational inertance of a clamped-clamped beam and the one obtained from a clamped-free beam (2 DOFs restrained). This second case is less accurate than the previous one, and can be easily explained by the truncation of modes. In fact, by using 8 modes instead of 16, the resonance shifts are much more relevant (see Fig. 4). This problem is more critical when the restrained DOFs are rotational, as in the previously examined case.

Figure 5 shows the translational-translational FRF of a hinged-hinged beam determined from a free-free beam. As in the case of Fig. 3, here again two DOFs are restrained, but no appreciable difference is observed between the two computations. A logical explanation of the above pitfall is the following: referring to the case of Fig. 3, the mode shapes present, at the clamped end, a zero slope. This geometrical condition cannot be satisfied by the modes of the clamped-free beam which presents a free rotation at the free end; therefore, a larger number of terms are required to approach the exact condition. Conversely, referring to the case of Fig. 5, at the hinged end the slope is still free and a lower number of modes with the same characteristics can be used. This is confirmed by Figs. 6 and 7. The first figure shows the rotational-rotational $\ddot{\theta}_1/M_2$ FRF of the hinged-hinged beam determined from the free-free one (2 DOFs restrained); again the effect of mode truncation is not appreciable. The second figure shows the

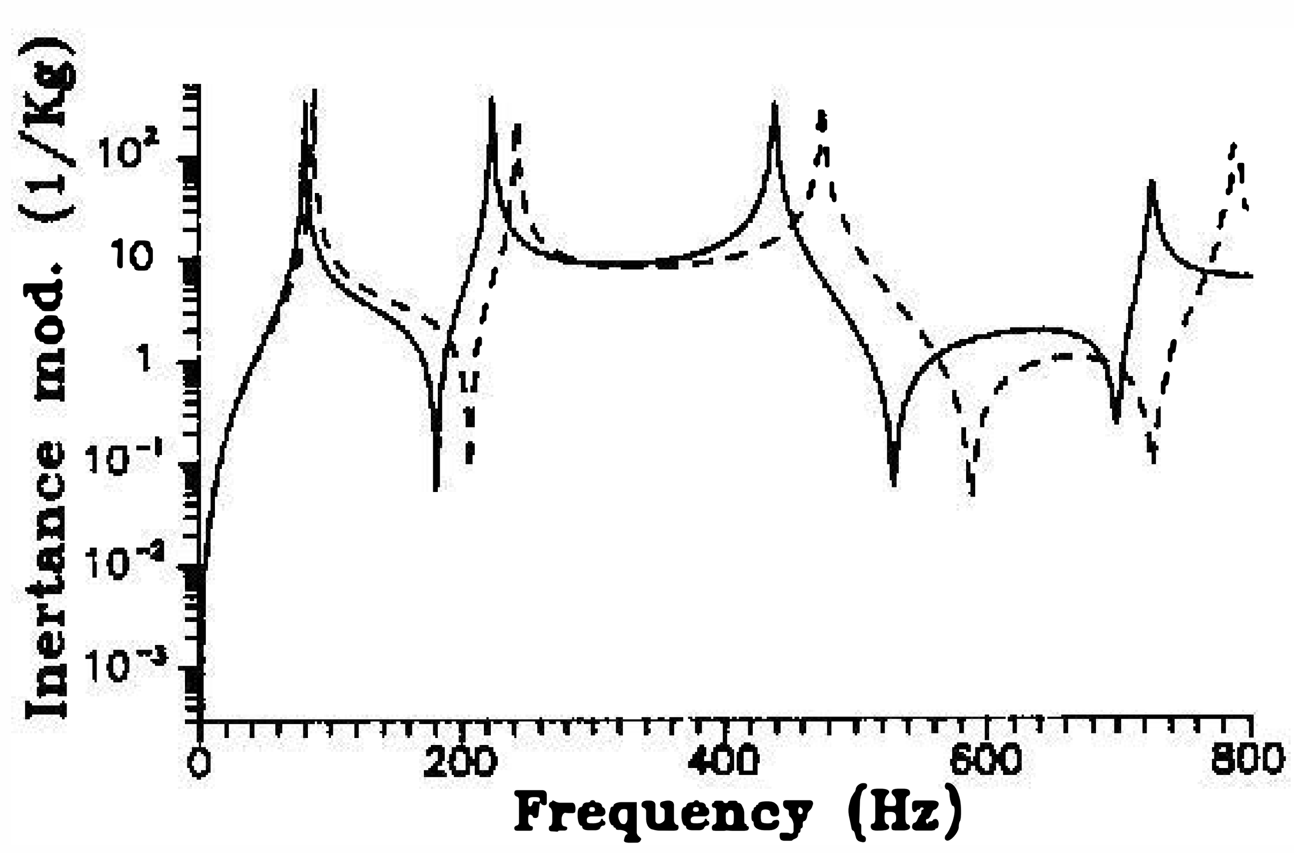

Fig. 3 Clamped-clamped beam: $\ddot{x}_1/F_2$
——— reference theoretical value
-------- derived from clamped-free beam

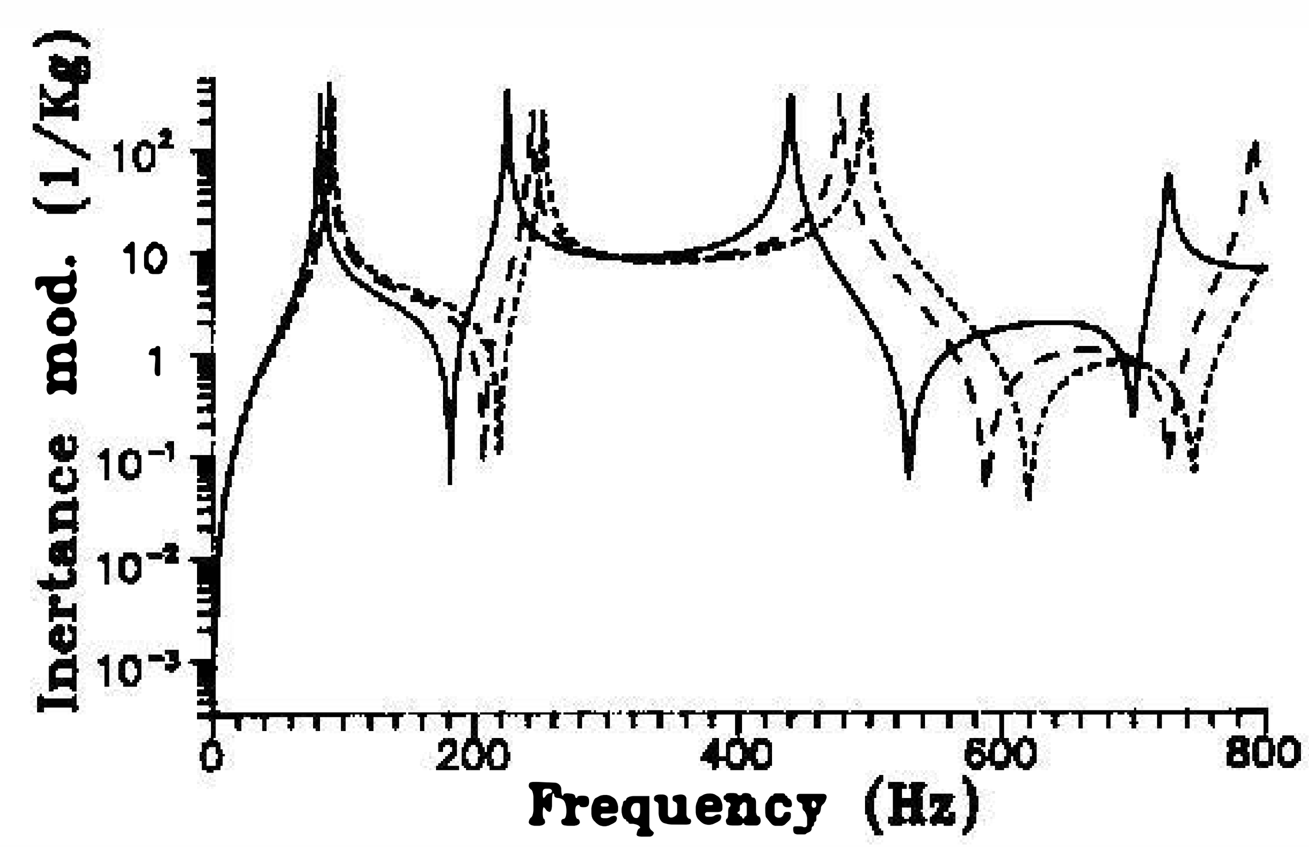

Fig. 4 Clamped-clamped beam: $\ddot{x}_1/F_2$
——— reference theoretical value
-------- derived from clamped-free beam (8 modes)
-------- derived from clamped-free beam (16 modes)

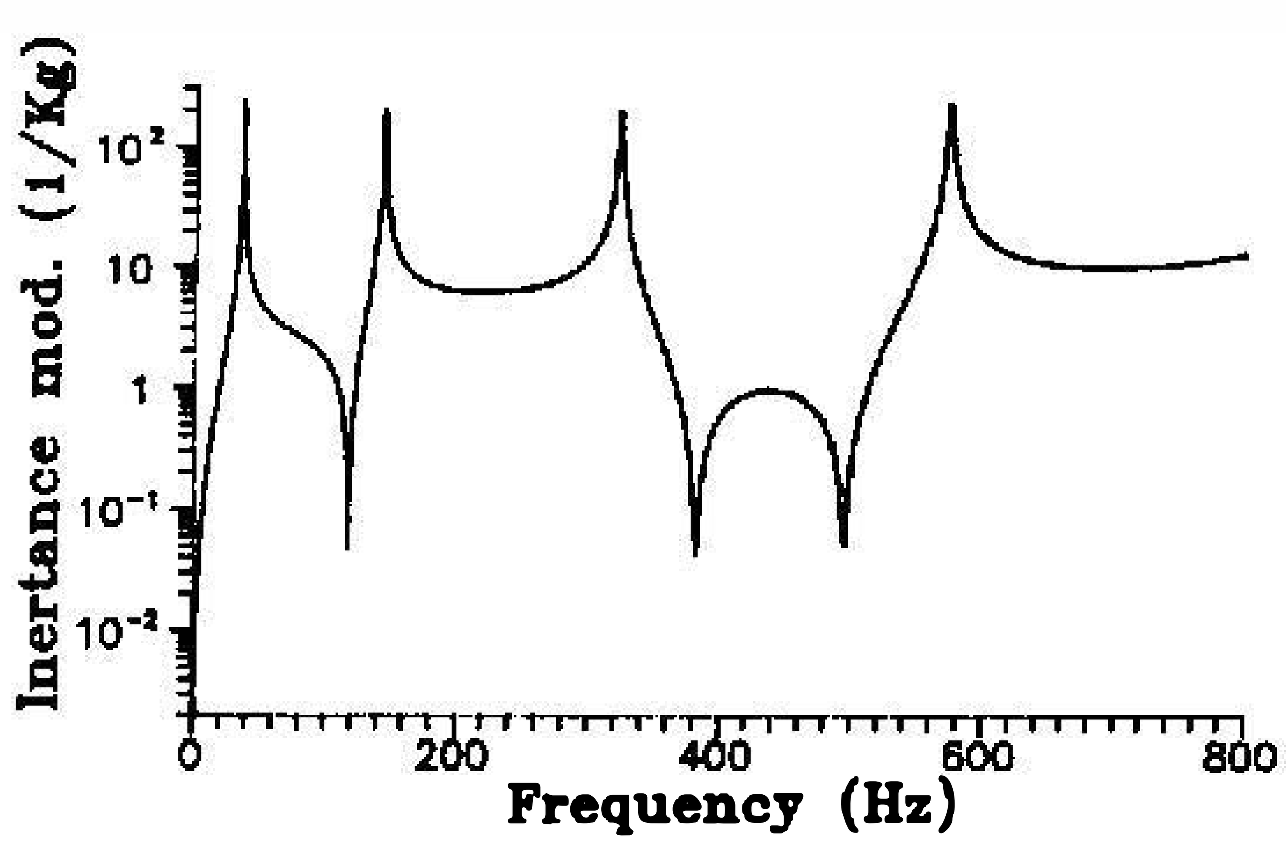

Fig. 5 Hinged-hinged beam: $\ddot{x}_1/F_2$
——— reference theoretical value
-------- derived from free-free beam

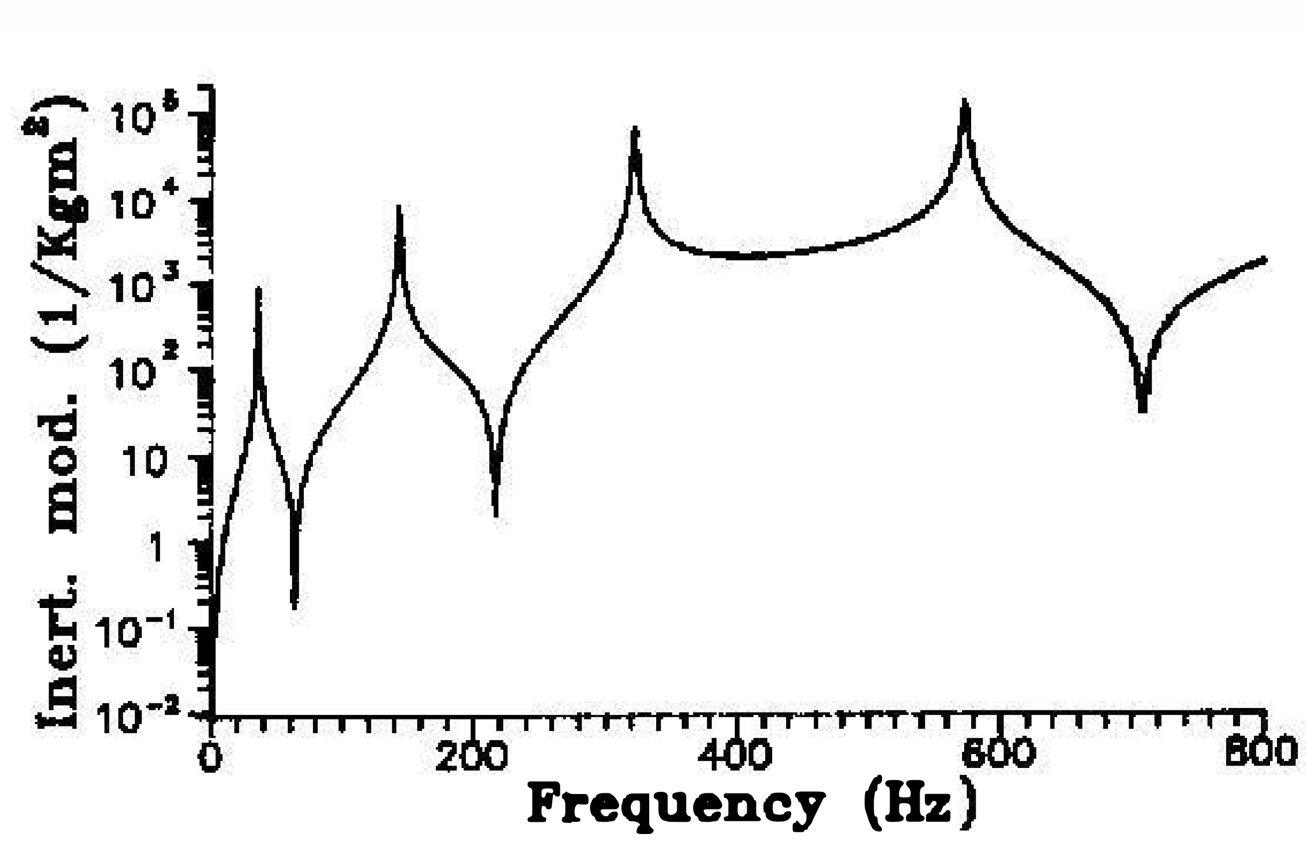

Fig. 6 Hinged-hinged beam: $\ddot{\theta}_1/M_2$
——— reference theoretical value
-------- derived from free-free beam

translational-translational FRF of a clamped-clamped beam, derived from free-free conditions (4 DOFs restrained) which presents again the observed resonance shifts. However, excluding the effect of mode truncation, the discussed simulated results confirm that the method is straighforward. Moreover the pitfall of modes truncation is only important when theoretical data are used. For experimental data, the FRF elements always account for high residuals, due to the effect of out-of-range modes, thus not affecting the derived results.

A second set of tests was then performed to verify the reliability of the procedure on experimental data. The drive point inertance element $\ddot{x}_3/F_3$ at the end of a *free-free* aluminum beam and other two translational elements, at points very close to the end, were measured. The inertance elements $\ddot{x}_3/M_3$ and $\ddot{\theta}_3/M_3$ were determined, using the experimental translational FRFs only. The first element was computed by means of the finite difference scheme; the second, through modal identification. With these data the rotational-rotational FRF element of a hinged-free beam, at the hinged end, was computed by means of Eq. (6).

The rotational-rotational FRF term of the *hinged-free* beam was then determined via translational theoretical data by applying the finite difference and identification algorithm described in Ref. [8]. In order to compute the term $\ddot{\theta}/M$, a forward finite difference was used on the translational data computed very close to the hinged end. (Note that the measurement of the FRF term at the hinged end is troublesome due to the difficulty of measuring translational accelerations with a good signal to noise ratio very close to a fixed end). The rotational-rotational FRFs obtained from these two approachs are compared in Fig. 8, together with the analytical solution. The advantage of the proposed procedure should be evident considering that the theoretical FRF $\ddot{\theta}/M$, computed in the proximity of the constraint, is surprisingly less accurate than the experimental one. This confirms the reliability of the proposed method, the possibility of using it for coupling purposes, as well as the difficulty of obtaining particular FRF terms from constrained systems.

5. Conclusions

A straightforward method is proposed here to compute FRF elements of any constrained structure from data derived from the same system, though less strictly constrained. Besides the ease and opportunity of testing more flexible systems, involving minor experimental inconveniences, several practical applications can be immediately carried out:

- determine the FRF matrix of a system, tested under different constraint conditions;
- predict the effect of adding constraints on mode shapes, natural frequencies, or on the forced response of a structure;

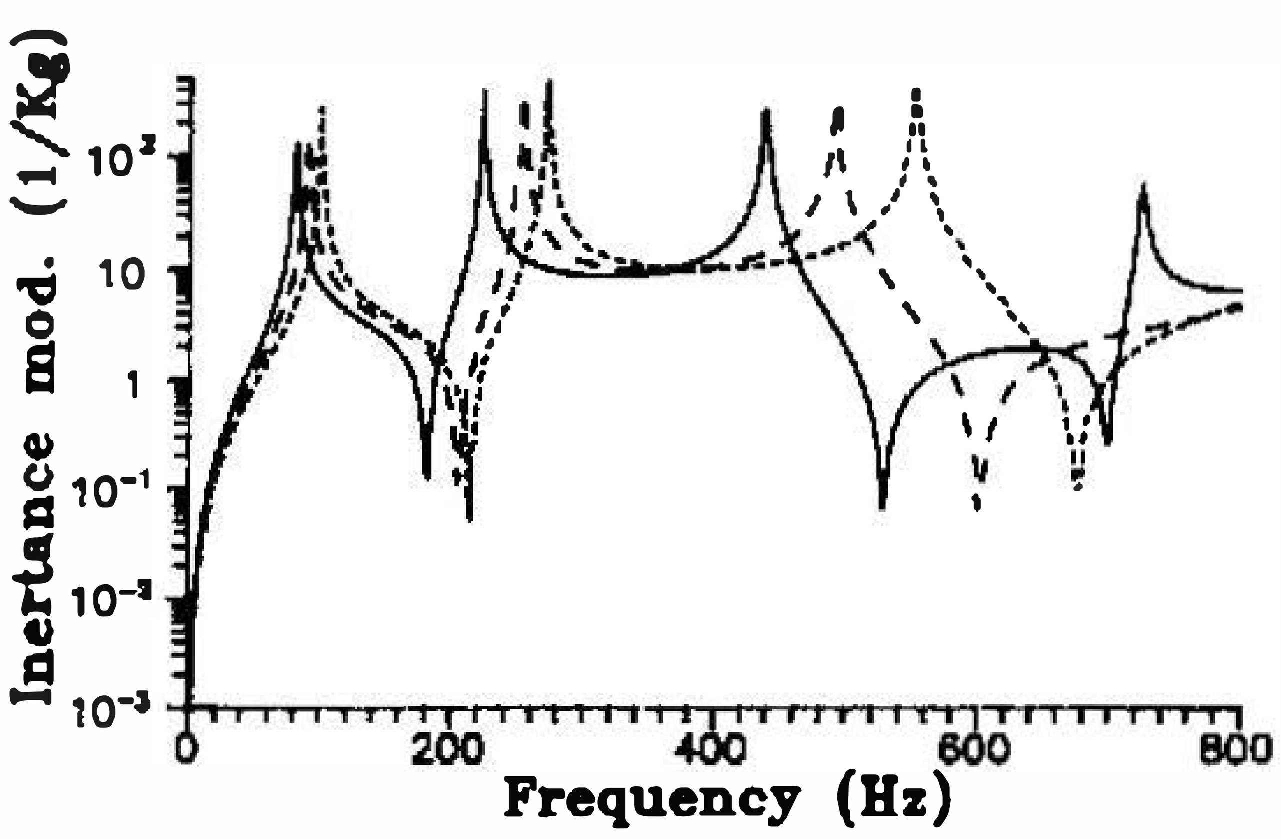

Fig. 7 Clamped-clamped beam: $\ddot{\theta}_1/F_2$
——— reference theoretical value
-------- derived from free-free beam (8 modes)
-------- derived from free-free beam (16 modes)

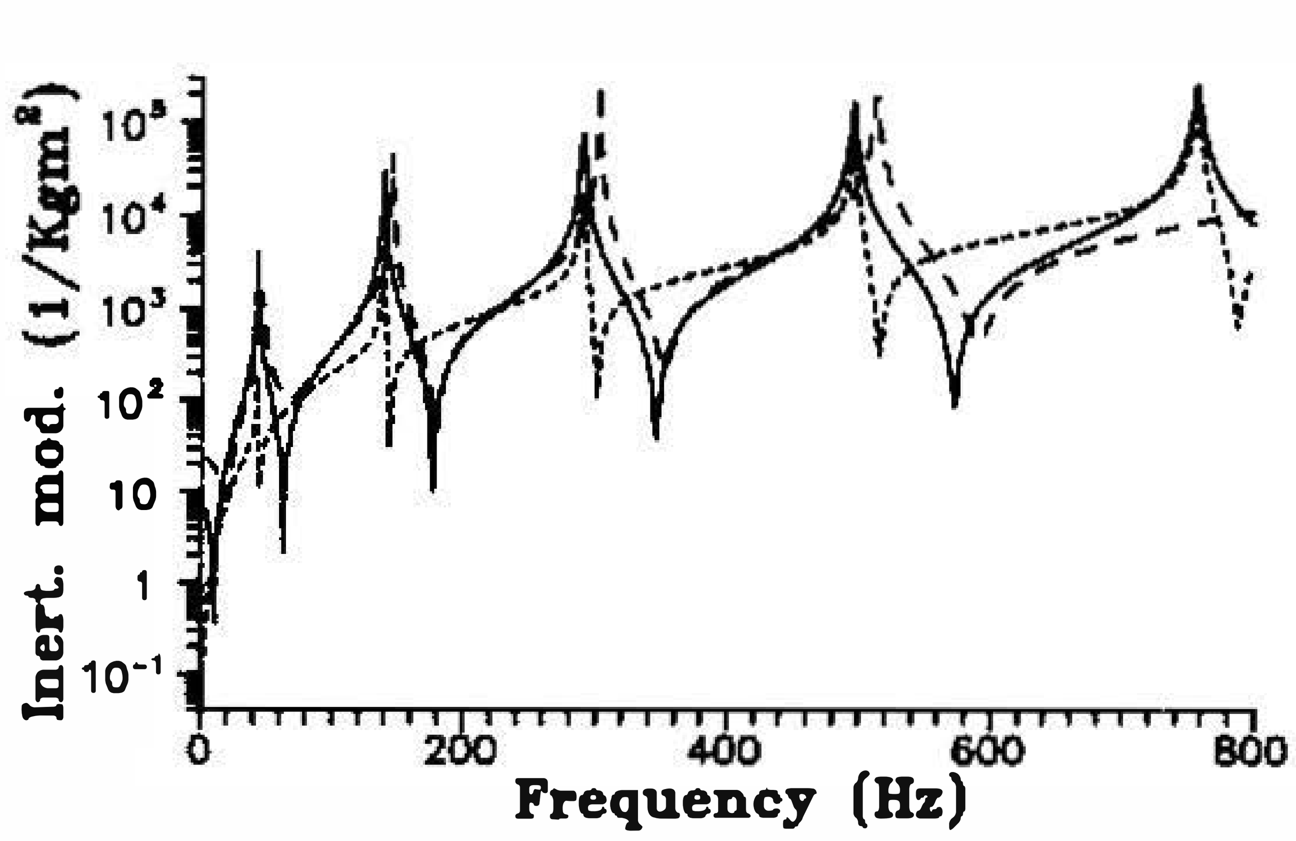

Fig. 8 Hinged-free beam: $\ddot{\theta}_3/M_3$
——— reference theoretical value
-------- experimental derived from free-free beam
-------- derived from translation inertance by Finite Difference

- formulate a general optimization procedure for passive vibration control accounting for a combined action of structural modification and constraint conditions;
- predict the FRF terms which are difficult to measure directly, though necessary to couple different structures;
- choose the best configuration of constraints.

The method is proved to be very effective and does not present general ill-conditioning limits as it was proven on the experimental data.

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