Analytic Geometry and Calculus

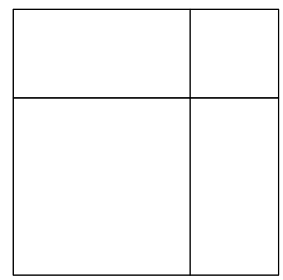
Modern mathematics is almost entirely *algebraic*: we trust equations and the rules of algebra more than pictures. For example, we consider that the expression $(x+y)^2 = x^2+2xy+y^2$ follows from the laws (axioms) of algebra:

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) && \text{(definition of `square')}\\
&= x(x+y)+y(x+y) && \text{(distributive law)}\\
&= x^2+xy+yx+y^2 && \text{(distributive law twice more)}\\
&= x^2+2xy+y^2 && \text{(commutativity)}
\end{aligned}$$

For most of mathematical history, this result would have been purely *geometric*: indeed it is Proposition 4 of Book II of Euclid's *Elements*:

> The square on two parts equals the squares on each part plus twice the rectangle on the parts.

The proof was geometric: staring at the picture should make it clear.



Algebra and algebraic notation were slowly slowly adopted in Renaissance Europe. While the utility of algebra for efficient calculation was noted, it was not initially considered acceptable to *prove* statements this way. Any algebraic calculation would have to be justified via a geometric proof. From our modern viewpoint this seems completely backwards: if a student were now asked to prove Euclid's proposition, they'd likely label the 'parts' x and y, before using the algebraic formula at the top of the page! Of course each of the lines in the algebraic proof has a geometric basis.

- Distributivity says that the rectangle on a side and two parts equals the sum of the rectangles on the side and each of the parts respectively.
- Commutativity says that a rectangle has the same area if rotated 90°.

The point is that we have converted geometric rules into algebraic ones and largely forgotten the geometric origin: a modern student will likely never have considered the geometric basis of something as basic as commutativity. This slow movement from geometry to algebra is one of the major revolutions of mathematical history: it has completely changed the way mathematicians *think.* More practically, the conversion to algebra has allowed for easy generalization: how would one geometrically justify an expression such as

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4 \ ?$$

Euclidean Geometry is often termed *synthetic:* it is based on purely geometric axioms without formulæ or co-ordinates. The revolution of *analytic geometry* was to marry algebra and geometry using *axes* and *co-ordinates.* Modern geometry is primarily analytic or, at an advanced level, described using algebra such as group theory. It is rare to find a modern mathematician working in synthetic geometry; algebra's triumph over geometry has been total! The critical step in this revolution was made almost simultaneously by Descartes and Fermat.

Pierre de Fermat (1601–1665) One of the most famous mathematicians of history, Fermat made great strides in several areas such as number theory, optics, probability, analytic geometry and early calculus. He approached mathematics as something of a dilettante: it was his pastime, not his profession.[1] Some of Fermat fame comes from his enigma: he published very little formally, and most of what we know of his work comes from letters to friends in which he rarely offers proofs. Indeed he would regularly challenge friends to prove results, and it is often unknown whether he had proofs himself, or merely suspected a general statement. Being outside the mainstream, his ideas were often ignored or downplayed. When he died, his notes and letters contained many unproven claims. Leonhard Euler (1707–1783) in particular expended much effort proving several of these.

Fermat's approach to analytic geometry was not dissimilar to that of Descartes which we shall describe below: he introduced a single axis which allowed the conversion of curves into algebraic equations. We shall return to Fermat when we discuss the beginnings of calculus when we see how he introduced an early notion of differentiation.

René Descartes (1596–1650) In his approach to mathematics and philosophy, Descartes is the chalk to Fermat's cheese, rigorously writing up everything. His defining work is 1637's *Discours de la méthode...*[2] While enormously influential in philosophy, *Discours* was intended to lay the groundwork for investigations of mathematics and the sciences; indeed Descartes finishes *Discours* by commenting on the necessity of experimentation in science and on his reluctance to publish due to the environment of hostility surrounding Galileo's prosecution.[3] The copious appendices to *Discours* contain Descartes' scientific work. It is in one of these, *La Géométrie,* that Descartes introduces axes and co-ordinates.

We now think of Cartesian *axes* and *co-ordinates* as *plural.* Both Fermat and Descartes, however, only used one axis. Here is the rough idea of their approach.

- Draw a straight line (the axis) containing two fixed points (the origin and the location of 1).
- All points on the line are immediately identified with numbers x.
- To describe a curve in the plane, one draws a family of parallel lines intersecting the curve and the axis.
- The curve can then be thought of as a *function* f, where $f(x)$ is the distance from the axis to the curve measured along the corresponding line.
- While neither Descartes nor Fermat had a second axes, their approach implicitly imagines one: through the origin, parallel to the family of measuring lines. It therefore makes sense for us to speak of the *co-ordinates*[4] (x, y) of a point, where $y = f(x)$.

[1] He was wealthy but not aristocratic, attending the University of Orléans for three years where he trained as a lawyer.

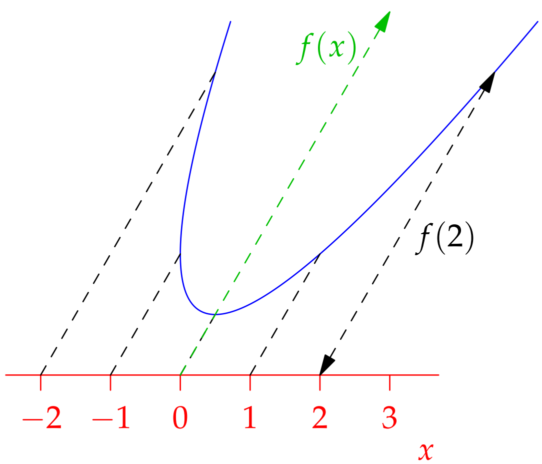
[2] ...*of rightly conducting one's reason and of seeking truth in the sciences.* Quite the mouthful. The primary part of this work is philosophical and contains the first use of his famous phrase *cogito egro sum* (*I think therefore I am*).

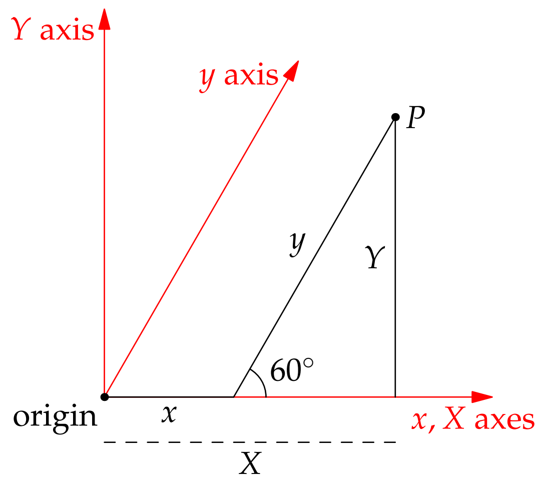
[3] At this time, France was still Catholic. Descartes had moved thence to Holland in part to pursue his work more freely. In 1649 Descartes moved to Sweden where he died the next year.

[4] The term *co-ordinates* suggests a symmetry of view when considering the point (x, y). The traditional terms are *abcissa* (for x) and *ordinate* (for y), stressing the idea that x is the *independent variable* and y is *dependent* on x.

Example: the parabola The function $f(x) = x^2$ describes the standard parabola in the usual way, where $f(x)$ measures the perpendicular distance from the axis to the curve.

Here is an alternative description of a parabola. This time the function is $f(x) = x^2 + 1$. Notice the slope: with only one axis, Descartes and Fermat could measure the distance to the curve using parallels inclined at whatever angle they liked. In a modern sense, this example has a second axis, drawn in green, inclined 30° to the vertical.





If this makes you nervous, you can perform a change of basis calculation from linear algebra: the point P in the second picture has co-ordinates (X, Y) relative to 'usual' orthogonal Cartesian axes; its co-ordinates are (x, y) relative to the slanted axes. It is easy to see that

$$\begin{cases} X = x + y\cos 60^\circ = x + \frac{1}{2}y \\ Y = y\sin 60^\circ = \frac{\sqrt{3}}{2}y \end{cases}$$

For any point on the curve, we then have

$$\begin{aligned} \sqrt{3}X - Y = \sqrt{3}x &\implies (\sqrt{3}X - Y)^2 = 3x^2 = 3(y-1) = 3\left(\frac{2}{\sqrt{3}}Y - 1\right) \\ &\implies 3X^2 - 2\sqrt{3}XY + Y^2 - 2\sqrt{3}Y + 3 = 0 \end{aligned}$$

which recovers the implicit equation for the parabola relative to the standard orthogonal axes.[5]

Other curves could be similarly described. Descartes was comfortable with curves having implicit equations. The standardized use of a second axis *orthogonal* to the first was instituted in 1649 by Frans van Schooten; this immediately gives us the modern notion of the *co-ordinates*.

Descartes used his method to solve several problems that had proved much more difficult synthetically such as finding complicated intersections. Given the novelty of his approach, he typically gave

[5] This really is a parabola, just rotated! If you've studied the topic, this is easily confirmed by computing the *discriminant*: a non-degenerate quadratic curve $aX^2 + bXY + cY^2 +$ linear terms is a parabola if the discriminant $b^2 - 4ac = 0$. A hyperbola has positive discriminant and an ellipse negative.

geometric proofs of all assertions to back up his algebraic work (similarly to how Islamic mathematicians had proceeded). He was not shy about his discovery however, stating that, once several examples were done, it wasn't necessary to draw physical lines and provide a geometric argument, *the algebra was the proof.* This point of view was controversial at the time, but over the following centuries it eventually won out.

As an example of the power of analytic geometry, consider the following result.

Theorem. *The medians of a triangle meet at a common point (the* centroid*), which lies a third of the way along each median.*

This can be done using pure Euclidean geometry, though it is somewhat involved. It is comparatively easy in analytic geometry.

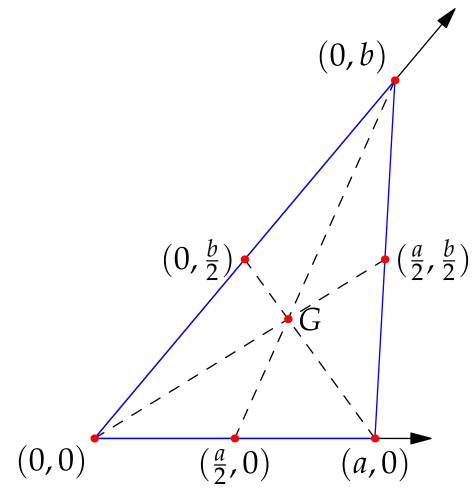
Proof. Choose axes pointing along two sides of the triangle with with the origin as one vertex.[6] If the side lengths are a and b, then the third side has equation $bx + ay = ab$ or $y = b - \frac{b}{a}x$. The midpoints now have co-ordinates:

$$\left(\frac{a}{2}, 0\right), \quad \left(0, \frac{b}{2}\right), \quad \left(\frac{a}{2}, \frac{b}{2}\right)$$

Now compute the point $1/3$ of the way along each median: for instance

$$\frac{2}{3}\left(\frac{a}{2}, 0\right) + \frac{1}{3}(0, b) = \frac{1}{3}(a, b)$$

One obtains the same result with the other medians. ∎



With the assistance of his notation, Descartes made many other mathematical breakthroughs. For instance, he was able to state a critical part of the Fundamental Theorem of Algebra, the *factor theorem*: if a is a root of a polynomial, then $x - a$ is a factor. He didn't give a complete proof of this fact as he thought it to be self-evident, perhaps because his notation made it so easy to work with polynomials. The full theorem[7] wasn't proved until 1821 (by Cauchy). The factor theorem is essentially a corollary of the division algorithm for polynomials: if $f(x), g(x)$ are polynomials, then there exist unique polynomials $q(x), r(x)$ for which

$$f(x) = q(x)g(x) + r(x) \qquad \deg r < \deg g$$

If $\deg g = 1$, then r is necessarily constant. Suppose $g(x) = x - a$ and $f(a) = 0$. Then $r = 0$.

[6] This ability to *choose axes to fit the problem* is a critical advantage of analytic geometry. In one stroke, this dispenses with the tedious consideration of *congruence* in synthetic geometry.

[7] Every polynomial over $\mathbb{C}$ may be factorized completely over $\mathbb{C}$. This needs some heavier analysis to show that a root exists in the first place, then the factor theorem allows you to pull these out one at a time.

The Beginnings of Calculus

At the heart of calculus is the relationship between *velocity, displacement*, *rate of change* and *area.*

- The *instantaneous velocity* of a particle is the *rate of change* of its *displacement.*
- The *displacement* of a particle is the *net area* under its *velocity*-time graph.

To state such principles essentially requires graphs and some form of analytic geometry (*rate of change* means *slope*...). Once these appeared in the early 1600's, the rapid development of calculus was arguably inevitable. However, many of the basic ideas were in place prior to Descartes and Fermat.

In the context of the above, the *Fundamental Theorem of Calculus* intuitively states that complete knowledge of displacement is equivalent to complete knowledge of velocity. Of course, the modern statement is far more daunting:

Theorem. *1. If f is continuous on $[a, b]$, then $F(x) := \int_a^x f(x)\mathrm{d}x$ is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$.*

2. If F is continuous on $[a, b]$ with continuous derivative on (a, b), then $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$.

The triumph of the Fundamental Theorem is its *abstraction*: no longer must $f(x)$ describe the velocity of a particle at time x, and $F(x)$ its displacement. The challenge of *teaching*[8] and *proving* the Fundamental Theorem lies in understanding the meanings of *continuous* and *differentiable,* and why these concepts are necessary. The quest for good definitions of these concepts is the story of analysis in the 17 and 1800's. We begin with some older considerations of the velocity and area problems.

The Velocity Problem pre-1600

The concepts of *uniform* and *average* velocities are straightforward:

> Measure how far an object travels in a given time interval and divide one by the other.

Several ancient Greek mathematicians had thought about uniform velocity and even uniform acceleration, but neither were considered quantities that could be measured. Around 1200, Gerard of Brussels tried to *define* velocity as a ratio of two unlike quantities (distance : time), though this was not yet considered a numerical quantity in its own right.

Defining *instantaneous velocity* is more difficult: one must measure average velocity over smaller and smaller intervals before invoking the notion of *limit.* You are in good company if you find this challenging: Zeno's arrow paradox is essentially an objection to the very idea of instantaneous velocity! Even if one accepts the concept, its direct measurement, even in modern times, is essentially impossible.[9]

Gerard was credited in the 1330's by the Oxford/Merton Thinkers as influencing their investigations of instantaneous speed. They offered the following definition and made the first statement of the 'mean speed theorem.' Both are vague and logically dubious, but they are at least an attempt to approach this difficult notion.

[8] Calculus students can easily be taught the mechanics of calculus (the power law, chain rule, etc.) without having any idea of its meaning: witness both the power and the curse of analytic geometry and algebra!

[9] For instance, radar Doppler-shift (as used by the police to catch speeding motorists) still requires a measurement of the wavelength of a radar beam, which in turn requires a finite (albeit miniscule) time interval. Indeed quantum mechanics suggests that instantaneous velocity and precise location are possibly meaningless concepts. Thankfully mathematicians can choose to deal with idealized models of the universe rather than the real thing!

Definition. The *instantaneous velocity* of a particle at an instant will be measured as the uniform velocity along the path that would have been taken by the particle if it continued with that velocity.

This is really a convoluted idea of inertial motion.

Theorem. *If a particle is uniformly accelerated from rest to some velocity, it will travel half the distance it would have traveled over the same interval with the final velocity.*

For centuries it was thought that Galileo was the first to state such ideas, but the Oxford group beat him by 250 years. They had no algebra with which to prove their assertions.

In the 1350's, Nicolas Oresme (Paris) considered velocity geometrically by (essentially) drawing velocity-time graphs. As we saw previously, this is essentially the approach taken by Galileo. A major difference for Galileo is that he married mathematics to *observation*: uniform acceleration for Galileo was precisely the motion of a falling body.

The Area Problem pre-1600

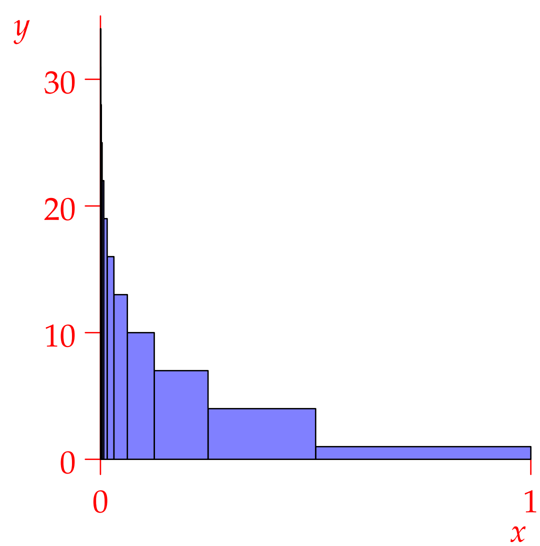
We've previously seen two situations in which calculus-like methods were used to describe areas.

- Archimedes computed/approximated the area of a circle and inside a parabola using infinitely many triangles. His 'cross-section' approach to finding area and volume also seems modern, though this work remained unknown until 1899.
- Kepler argued for his second law (equal areas in equal times) using infinitesimally small triangles to approximate segments of an ellipse. He also applied this method to several other problems, crediting Archimedes with the approach.

The modern notion of Riemann sums is just a special case of approximating an area using small rectangles: the philosophical challenge is again the notion of *limits* and infinitesimals.

In an early antecedent of Riemann sums, Oresme describes how to compute the distance travelled by a particle whose speed is constant on a sequence of intervals. For example:

Over the time interval $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$ a particle travels at speed $1+3n$. How far does it travel in 1 second?



Oresme drew boxes to compute areas and obtained

$$d = \sum_{n=0}^{\infty} (1+3n)2^{-n-1} = 4$$

The infinite sum was evaluated by spotting two patterns, similarly to how Archimedes had done things:

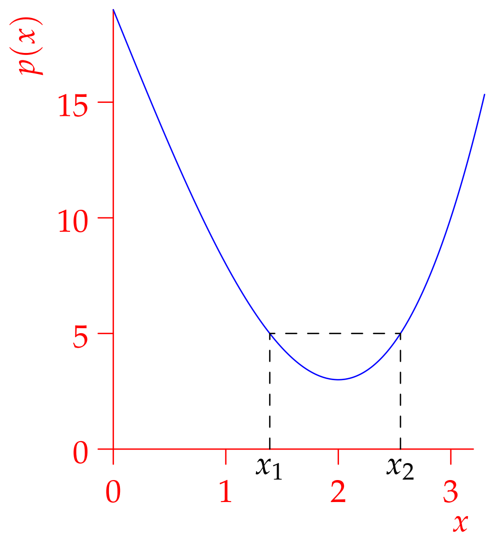
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^{n+1}} \qquad \frac{0}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}} = 1 - \frac{n+2}{2^{n+1}}$$

Of course Oresme had none of our notation, and certainly didn't have our (limit-dependent) notion of an infinite series! Oresme also worked with similar problems for uniform accelerations over intervals. These are not true Riemann sums, nor are they physical, for a particle cannot suddenly change speed!

Calculus à la Fermat & Descartes

The advent of analytic geometry allowed Fermat and Descartes to turn the computation of instantaneous velocity and related differentiation problems into algorithmic processes. In particular, the velocity of an object is identified with the slope of the displacement-time graph, which can be computed using variations on the modern method of *secant lines.* We discuss their competing methods.

Fermat's method of adequation For example, the graph of $p(x) = x^3 - 12x + 19$ is drawn, where the minimum of p occurs at the x-value $m = 2$. Fermat argues that if x_1, x_2 are located near m such that $p(x_1) = p(x_2)$, then the polynomial $p(x_2) - p(x_1)$ (which equals zero!) is divisible by $x_2 - x_1$. Indeed



$$\begin{aligned}
0 &= \frac{p(x_2) - p(x_1)}{x_2 - x_1} \\
&= \frac{x_2^3 - 12x_2 + 19 - x_1^3 + 12x_1 - 19}{x_2 - x_1} \\
&= \frac{(x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2 - 12)}{x_2 - x_1} \\
&= x_2^2 + x_1x_2 + x_1^2 - 12
\end{aligned}$$

Since this holds for any x_1, x_2 with $p(x_1) = p(x_2)$, Fermat claims it also holds when $x_1 = x_2 = m$ (notice the assumption of continuity!), and he concludes

$$3m^2 - 12 = 0 \implies m = 2$$

By considering values of x near to m, it is clear to Fermat that he really has found a local minimum. We recognize the idea that the slope of the tangent line is zero at local extrema.

Fermat's approach dates from the 1620's and is similar to earlier work by Viète. Fermat proceeds to alter the method slightly: he considers the values $p(x)$ and $p(x+e)$ for a small value e (x is 'adequated' by e). The difference $p(x+e) - p(x)$ is more easily divided by e without nasty factorizations. Compared with the above, we obtain

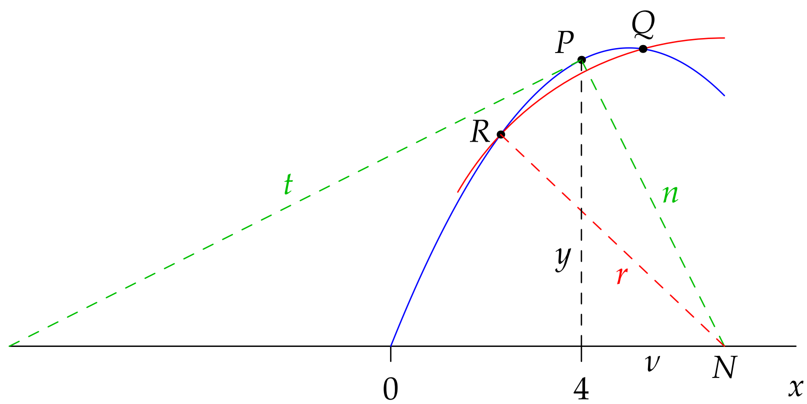
$$\begin{aligned}
0 &= \frac{p(x+e) - p(x)}{e} = \frac{x^3 + 3x^2e + 3xe^2 + e^3 - 12x - 12e + 19 - x^3 + 12x - 19}{e} \\
&= \frac{3x^2e + 3xe^2 + e^3 - 12e}{e} = 3x^2 - 12 + 3xe + e^2
\end{aligned}$$

He then sets e to zero and solves for x. Observe the derivative $p'(x) = 3x^2 - 12$ and how Fermat's e plays the same role as h in the modern expression

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

If you recall elementary calculus, Fermat's method is guaranteed to work for any polynomial: the concept of limit requires no more for polynomials than simple evaluation at $h = 0$. Fermat also extended his method to cover implicit curves and their tangents.

Descartes method of normals Descartes and Fermat are known to have corresponded regarding their methods. Descartes indeed seems to have felt somewhat challenged by Fermat, and engaged in some criticism of his approach. Descartes' method (in *La Géométrie*) relies on circles and repeated roots of polynomials in order to compute tangents. Here is an example where he calculates the slope of the curve $y = \frac{1}{4}(10x - x^2)$ at the point $P = (4, 6)$.



Let $N = (4 + v, 0)$ be the point where the normal to the curve intersects the x-axis.[10] Draw a circle radius r centered at N. If r is close to n, this intersects the curve in two points Q, R near to P. The line joining Q, R is clearly an approximation to the tangent line at P.

The co-ordinates of Q, R can be found by solving algebraic equations: substituting $y = \frac{1}{4}(10x - x^2)$ into the equation for the circle results in an equation with two known roots, namely the x-values of Q and R. By the factor theorem, we have

$$\begin{cases} \big(x - (4+v)\big)^2 + y^2 = r^2 \\ y = \frac{1}{4}(10x - x^2) \end{cases} \implies (x - Q_x)(x - R_x)f(x) = 0$$

where $f(x)$ is some polynomial (in this case quadratic). Rather than doing this explicitly, Descartes observes that if r is adjusted until it *equals* n, then Q and R coincide with P and the above equation has a double-root:

$$\begin{cases} \big(x - (4+v)\big)^2 + y^2 = n^2 \\ y = \frac{1}{4}(10x - x^2) \end{cases} \implies (x - P_x)^2 f(x) = (x-4)^2 f(x) = 0$$

Factorization can be done by hand using long-division (note that v and n are currently unknown!): substituting as above, we obtain

$$\begin{aligned} 0 &= x^4 - 20x^3 + 116x^2 - 32(4+v)x + 16(4+v)^2 - 16n^2 = 0 \\ &= (x-4)^2(x^2 - 12x + 4) + 32(3-v)x + 16(12 + 8v + v^2 - n^2) \end{aligned}$$

The remainder $32(3-v)x + 16(12 + 8v + v^2 - n^2)$ must be the zero polynomial, whence $v = 3$. By similar triangles, the slope of the curve at P is therefore (phew!)

$$\frac{y}{\sqrt{t^2 - y^2}} = \frac{v}{y} = \frac{1}{2}$$

[10] At the time, v was known as the *subnormal* and t the *tangent*.

Fermat and Area The previous methods essentially allow *differentiation*, albeit inefficiently. Fermat also approached the area problem in a manner similar to Oresme. Here is an example where we find the area under the curve $y = x^3$ between $x = 0$ and $x = a$.

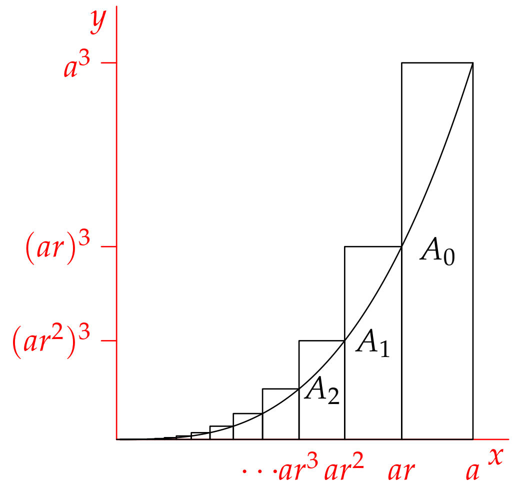
Let $0 < r < 1$ be constant. The area of the rectangle on the interval $[ar^{n+1}, ar^n]$ touching the curve at its upper right is

$$A_n = (ar^n - ar^{n+1}) \cdot (ar^n)^3 = a^4(1-r)r^{4n}$$

The sum of the areas is

$$\sum_{n=0}^{\infty} A_n = a^4(1-r)\sum_{n=0}^{\infty} r^{4n} = \frac{a^4(1-r)}{1-r^4}$$
$$= \frac{a^4}{1+r+r^2+r^3}$$

Setting $r = 1$ recovers the area under the curve: $\frac{1}{4}a^4$.



Fermat's geometric series approach was not rigorous by modern standards (he certainly didn't use the above notation), and he again implicitly invokes limits at the end by setting $r = 1$. Regardless, via Fermat's method one easily obtains the power law $\int_0^a x^n \, dx = \frac{1}{n+1}a^{n+1}$ for any positive integer n.